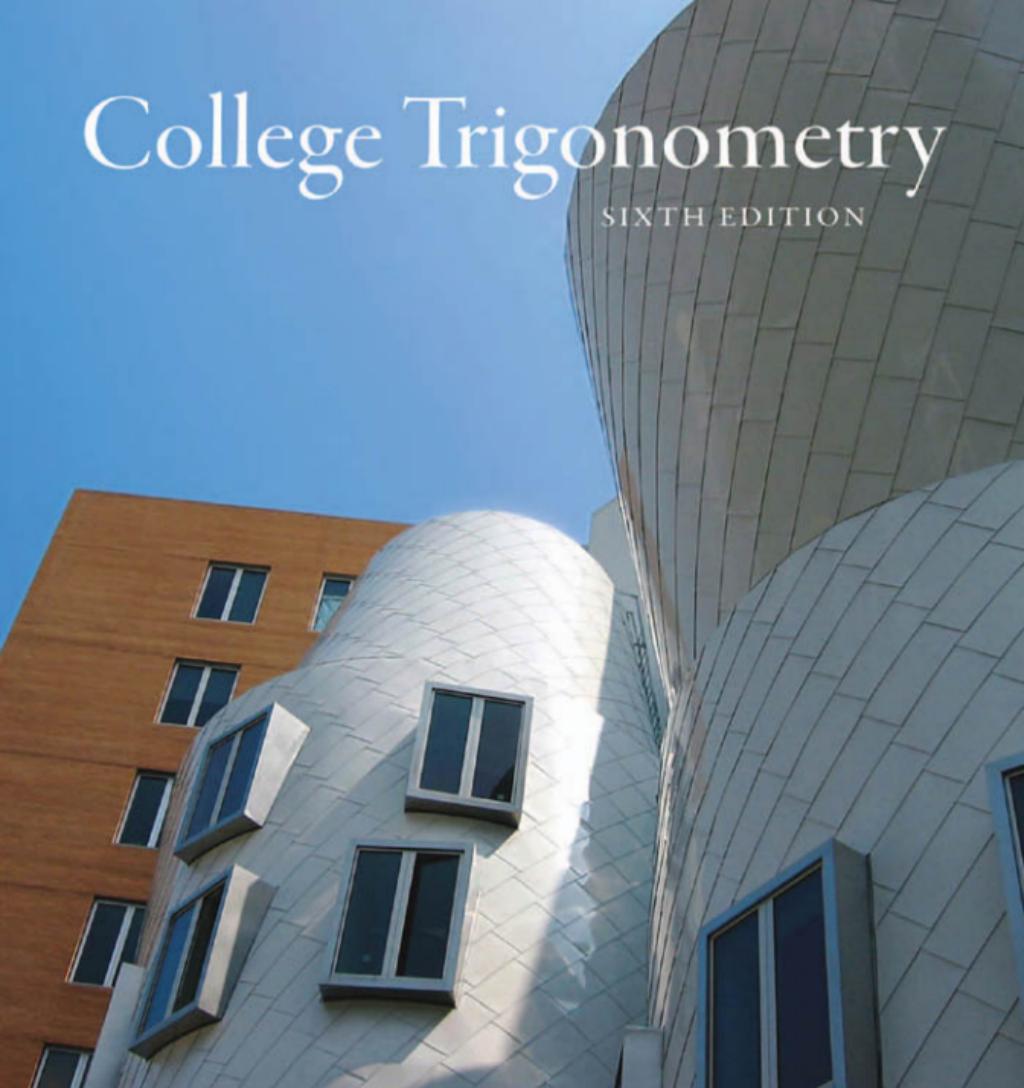


# College Trigonometry

SIXTH EDITION



AUFMANN

BARKER

NATION

# College Trigonometry

*This page intentionally left blank*

# College Trigonometry

SIXTH EDITION

---

**Richard N. Aufmann**

---

**Vernon C. Barker**

---

**Richard D. Nation**

*Palomar College*

HOUGHTON MIFFLIN COMPANY  
Boston New York

Publisher: Richard Stratton  
Senior Sponsoring Editor: Molly Taylor  
Senior Marketing Manager: Jennifer Jones  
Marketing Associate: Mary Legere  
Associate Editor: Noel Kamm  
Editorial Associate: Andrew Lipsett  
Senior Project Editor: Carol Merrigan  
Editorial Assistant: Anthony D'Aries  
Art and Design Manager: Gary Crespo  
Cover Design Manager: Anne S. Katzeff  
Photo Editor: Jennifer Meyer Dare  
Composition Buyer: Chuck Dutton

Cover photograph: © Dan Hill/cityofsound.com

#### PHOTO CREDITS

**Chapter 1:** p. 1: John Foxx/Stockbyte Silver/Getty Images. **Chapter 2:** p. 117: Andrew Brookes/CORBIS; p. 128: Bryan Mullennix/Iconica/Getty Images; p. 132: Courtesy of NASA and JPL; p. 145: Tony Craddock/Getty Images; p. 159: Reuters/New Media Inc./CORBIS. **Chapter 3:** p. 254: Courtesy of Richard Nation. **Chapter 4:** p. 307: McDuff/Everton/CORBIS. **Chapter 5:** p. 334: Dennis De Mars/Fractal Domains/www.fractaldomains.com; p. 334: Steve Allen/Alamy; p. 351: The Granger Collection. **Chapter 6:** p. 371: 1998 International Conference on Quality Control by Artificial Vision – QCAV '98, Kagawa Convention Center, Takamatsu, Kagawa, Japan, November 10–12, 1998, pp. 521–528; p. 372: Ian Morison/Jodrell Bank Conservatory; p. 374: Courtesy of Michael Levin, Opti-Gone International. Reprinted by permission; p. 389: Hugh Rooney/Eye Ubiquitous/CORBIS; p. 397: The Granger Collection. **Chapter 7:** p. 451: AP/Wide World Photos; p. 458: Bettmann/CORBIS; p. 463: Charles O'Rear/CORBIS; p. 464: David James/Getty Images; p. 467: Bettmann/CORBIS; p. 474: Jan Halaska/Index Stock Imagery/Jupiter Images; p. 513: Tom Brakefield/CORBIS; p. 521: Bettmann/CORBIS; p. 525: AP/Wide World.

Copyright © 2008 Houghton Mifflin Company. All rights reserved.

No part of this work may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photographing and recording, or by any information storage or retrieval system without the prior written permission of Houghton Mifflin Company unless such copying is expressly permitted by federal copyright law. Address inquiries to College Permissions, Houghton Mifflin Company, 222 Berkeley Street, Boston, MA 02116-3764.

Printed in the U.S.A.

Library of Congress Control Number: 2006933003

ISBN 10: 0-618-82507-X

ISBN 13: 978-0-618-82507-3

123456789-WC-10 09 08 07

# Contents

Preface ix

## 1 Functions and Graphs 1

---

- 1.1 Equations and Inequalities 2**
- 1.2 A Two-Dimensional Coordinate System and Graphs 15**
- 1.3 Introduction to Functions 30**
- 1.4 Properties of Graphs 53**
- 1.5 The Algebra of Functions 69**
- 1.6 Inverse Functions 81**
- 1.7 Modeling Data Using Regression 95**

**Exploring Concepts with Technology:**

- Graphing Piecewise Functions with a Graphing Calculator 108
- Chapter 1 Summary 110
- Chapter 1 Assessing Concepts 111
- Chapter 1 Review Exercises 112
- Quantitative Reasoning: Public Key Cryptography 114**
- Chapter 1 Test 116

## 2 Trigonometric Functions 117

---

- 2.1 Angles and Arcs 118**
- 2.2 Right Triangle Trigonometry 134**
- 2.3 Trigonometric Functions of Any Angle 147**
- 2.4 Trigonometric Functions of Real Numbers 155**
- 2.5 Graphs of the Sine and Cosine Functions 169**
- 2.6 Graphs of the Other Trigonometric Functions 180**
- 2.7 Graphing Techniques 192**
- 2.8 Harmonic Motion—An Application of the Sine and Cosine Functions 203**

**Exploring Concepts with Technology:**

- Sinusoidal Families 210
- Chapter 2 Summary 210
- Chapter 2 Assessing Concepts 212
- Chapter 2 Review Exercises 212
- Quantitative Reasoning: Find the Periods of Trigonometric Functions and Combined Musical Sound Tracks 214**
- Chapter 2 Test 214
- Cumulative Review Exercises 215

### 3 Trigonometric Identities and Equations 216

---

- 3.1 Verification of Trigonometric Identities 217
  - 3.2 Sum, Difference, and Cofunction Identities 225
  - 3.3 Double- and Half-Angle Identities 236
  - 3.4 Identities Involving the Sum of Trigonometric Functions 246
  - 3.5 Inverse Trigonometric Functions 255
  - 3.6 Trigonometric Equations 268
- 

**Exploring Concepts with Technology:**

- Approximate an Inverse Trigonometric Function with Polynomials 283
  - Chapter 3 Summary 284
  - Chapter 3 Assessing Concepts 286
  - Chapter 3 Review Exercises 286
  - Quantitative Reasoning: Basketball and Trigonometric Equations** 288
  - Chapter 3 Test 289
  - Cumulative Review Exercises 290
- 

### 4 Applications of Trigonometry 292

---

- 4.1 The Law of Sines 293
  - 4.2 The Law of Cosines and Area 302
  - 4.3 Vectors 312
- 

**Exploring Concepts with Technology:**

- Optimal Branching of Arteries 328
  - Chapter 4 Summary 328
  - Chapter 4 Assessing Concepts 329
  - Chapter 4 Review Exercises 329
  - Quantitative Reasoning: Trigonometry and Great Circle Routes** 330
  - Chapter 4 Test 332
  - Cumulative Review Exercises 332
- 

### 5 Complex Numbers 334

---

- 5.1 Complex Numbers 335
  - 5.2 Trigonometric Form of Complex Numbers 343
  - 5.3 De Moivre's Theorem 350
- 

**Exploring Concepts with Technology:**

- The Mandelbrot Iteration Procedure 356
- Chapter 5 Summary 357
- Chapter 5 Assessing Concepts 358
- Chapter 5 Review Exercises 358
- Quantitative Reasoning: Graphing the Mandelbrot Set** 359

|                             |     |
|-----------------------------|-----|
| Chapter 5 Test              | 360 |
| Cumulative Review Exercises | 361 |

---

## 6 Topics in Analytic Geometry 362

---

|            |                                   |     |
|------------|-----------------------------------|-----|
| <b>6.1</b> | Parabolas                         | 363 |
| <b>6.2</b> | Ellipses                          | 374 |
| <b>6.3</b> | Hyperbolas                        | 390 |
| <b>6.4</b> | Rotation of Axes                  | 403 |
| <b>6.5</b> | Introduction to Polar Coordinates | 412 |
| <b>6.6</b> | Polar Equations of the Conics     | 426 |
| <b>6.7</b> | Parametric Equations              | 432 |

---

### Exploring Concepts with Technology:

|   |     |
|---|-----|
| Using a Graphing Calculator to Find the $n$ th Roots of $z$ | 443 |
|---|-----|

|                   |     |
|-------------------|-----|
| Chapter 6 Summary | 444 |
|-------------------|-----|

|                              |     |
|------------------------------|-----|
| Chapter 6 Assessing Concepts | 445 |
|------------------------------|-----|

|                            |     |
|----------------------------|-----|
| Chapter 6 Review Exercises | 446 |
|----------------------------|-----|

|   |     |
|---|-----|
| <b>Quantitative Reasoning: The Mathematics of a Rotary Engine</b> | 447 |
|---|-----|

|                |     |
|----------------|-----|
| Chapter 6 Test | 449 |
|----------------|-----|

|                             |     |
|-----------------------------|-----|
| Cumulative Review Exercises | 450 |
|-----------------------------|-----|

---

## 7 Exponential and Logarithmic Functions 451

---

|            |  |     |
|------------|--|-----|
| <b>7.1</b> | Exponential Functions and Their Applications             | 452 |
| <b>7.2</b> | Logarithmic Functions and Their Applications             | 467 |
| <b>7.3</b> | Properties of Logarithms and Logarithmic Scales          | 480 |
| <b>7.4</b> | Exponential and Logarithmic Equations                    | 494 |
| <b>7.5</b> | Exponential Growth and Decay                             | 506 |
| <b>7.6</b> | Modeling Data with Exponential and Logarithmic Functions | 521 |

---

### Exploring Concepts with Technology:

|  |     |
|--|-----|
| Using a Semilog Graph to Model Exponential Decay | 535 |
|--|-----|

|                   |     |
|-------------------|-----|
| Chapter 7 Summary | 537 |
|-------------------|-----|

|                              |     |
|------------------------------|-----|
| Chapter 7 Assessing Concepts | 538 |
|------------------------------|-----|

|                            |     |
|----------------------------|-----|
| Chapter 7 Review Exercises | 539 |
|----------------------------|-----|

|                                      |     |
|--------------------------------------|-----|
| <b>Quantitative Reasoning: Sales</b> | 541 |
|--------------------------------------|-----|

|                |     |
|----------------|-----|
| Chapter 7 Test | 542 |
|----------------|-----|

|                             |     |
|-----------------------------|-----|
| Cumulative Review Exercises | 543 |
|-----------------------------|-----|

---

Solutions to the Try Exercises S1

Answers to Selected Exercises A1

Index I1

*This page intentionally left blank*

# Preface

*College Trigonometry, Sixth Edition* builds on the strong pedagogical features of the previous edition. Always with an eye toward supporting student success, we have increased our emphasis on conceptual understanding, quantitative reasoning, and applications.

## Applications

We have retained our basic philosophy, which is to deliver a comprehensive and mathematically sound treatment of the topics considered essential for a college algebra course. To help students master these concepts, we have maintained a dynamic balance among theory, application, modeling, and drill. Carefully developed mathematics is complemented by abundant, relevant, and contemporary applications, many of which feature real data and tables, graphs, and charts.

Ever mindful of the motivating influence that contemporary and appropriate applications have on students, we have included many new application exercises from a wide range of disciplines, and in new formats. For example, a new *Quantitative Reasoning* feature is found at the end of each chapter. Students are urged to investigate concepts and apply those concepts to a variety of contexts while testing the reasonableness of their answers. Applications require students to use problem-solving strategies and newly learned skills to solve practical problems, demonstrating the value of algebra. Many application exercises are accompanied by a diagram that helps students visualize the mathematics of the application.

## Technology

Technology is introduced very naturally to illustrate or enhance conceptual understanding of appropriate topics. We integrate technology into a discussion when it can be used to foster and promote better understanding of a concept. The optional graphing calculator exercises, optional *Integrating Technology* boxes, and optional *Exploring Concepts with Technology* features are designed to instill in students an appreciation for both the power and limitations of technology. Optional *Modeling* sections, which use real data, rely heavily on the use of a graphing calculator, and serve to motivate students, are incorporated throughout the text.

## Aufmann Interactive Method (AIM)

By incorporating many interactive learning techniques, including the key features outlined below, *College Trigonometry* helps students to understand concepts, work independently, and obtain greater mathematical fluency.

- *Try Exercise* references follow all worked examples. This feature encourages students to test their understanding by working an exercise similar to the worked example. An icon and a page reference are given below the example, making it easy for the student to navigate to the suggested exercise and back. The complete solution to the *Try Exercise* can be found in the *Solutions to the Try Exercises* appendix. This interaction among the examples, the *Try Exercises*, the *Solutions to the Try Exercises*, and the exercise sets serves as a checkpoint for students as they read the text, do their homework, and study a section.

- **Annotated Examples** are provided throughout each section, and are titled so that students can see at a glance the type of problem being illustrated. The annotated steps assist the student in moving from step to step, and help explain the solution.
- **Question/Answer** In each section, we pose at least one question that encourages the reader to pause and think about the current discussion. To ensure that the student does not miss important information, the answer to the question is provided as a footnote on the same page.

## CHANGES IN THE SIXTH EDITION

### Overall changes

- **NEW!** *Quantitative Reasoning* After each set of *Chapter Review Exercises*, a new *Quantitative Reasoning* scenario uses concepts from the chapter to explore an application in more depth or extend a mathematical concept from the chapter.
- **NEW!** Most definitions are now immediately followed by an example, to enhance conceptual understanding.
- **NEW!** A *Calculus Connection* icon alerts students to a connection between the current topic and calculus. This feature identifies topics that will be revisited in a subsequent calculus course or other advanced course.
- **Revised!** *Review Notes*, which help students recognize the prerequisite skills needed to understand new concepts, are featured more prominently throughout the text, encouraging students to use them more frequently. These example-specific notes direct students to the appropriate page(s) for review, thus decreasing student frustration and creating more opportunities for comprehension.
- **Revised!** We have thoroughly reviewed each exercise set. In addition to updating and adding contemporary applications, we have focused our revisions on providing a smooth progression from routine exercises to exercises that are more challenging.
- **Revised!** New chapter openers demonstrate how the mathematics developed in each chapter is applied.
- **Moved!** *Prepare for This Section* exercises, formerly called *Prepare for the Next Section* exercises, have been moved from the end of the section to the beginning. An up-front review gives students a chance to test their understanding of prerequisite skills and concepts before proceeding to a new topic.
- **Revised!** *Assessing Concepts* exercises, found at the end of each chapter, have been enhanced with more question types, including fill-in-the blank, multiple choice, matching, and true/false.
- **Revised!** We have highlighted more of the important points within the body of the text, to enhance conceptual understanding.
- **Enhanced!** Technology program

## Changes in each chapter

In addition to updating and adding new examples, applications, and exercises throughout, we have made a number of chapter-specific changes. Here are some of them:

### **Chapter 1: Functions and Graphs**

- Added concepts involving the solution of literal equations
- Added an introductory discussion of asymptotes

### **Chapter 2: Trigonometric Functions**

- Added exercises that involve finding a trigonometric function that can be used to model an application
- Rewrote the material on the linear and angular speed of a point moving on a circular path

### **Chapter 3: Trigonometric Identities and Equations**

- Expanded the coverage concerning the verification of trigonometric identities
- Expanded the coverage concerning the use of power-reducing identities

### **Chapter 4: Applications of Trigonometry**

- Added exercises that use the graph of a vector to find its components
- Added exercises that involve the equilibrium of forces

### **Chapter 5: Complex Numbers**

- Included introductory material that involves the use of complex numbers and iteration to produce fractal images
- Increased the coverage concerning the Mandelbrot iteration procedure and the Mandelbrot set

### **Chapter 6: Topics in Analytic Geometry**

- Included new figures to illustrate concepts involving the standard form of the equation of a parabola and its graph
- Included new exercises that involve matching the graph of a conic section with its equation
- Included a new example that illustrates a technique that can be used to write some polar equations in rectangular form
- Increased the parametric equation coverage concerning the use of time as a parameter and the simulation of motion

### **Chapter 7: Exponential and Logarithmic Functions**

- Added exercises that use translations and/or reflections to graph exponential functions
- Added exercises that involve the evaluation of a logarithm, without using a calculator

- Added exercises that involve finding the domain of a logarithmic function
- Included a proof of the Product Property of logarithms. The proof of the Quotient Property and the Power Property of logarithms are given as exercises.
- Provided additional coverage and additional exercises involving the expanding and condensing of logarithmic expressions
- Added application exercises that can be solved by using exponential or logarithmic equations
- Included guidelines for selecting the type of mathematical function that models a given application

## ACKNOWLEDGEMENTS

We would especially like to thank the users of the previous edition for their helpful suggestions on improving the text. Also, we sincerely appreciate the time, effort, and suggestions of the reviewers of this edition.

Jan Archibald – *Ventura College, CA*

Steve Armstrong – *LeTourneau University, TX*

Rick Bailey – *Midlands Technical College, SC*

Sandy Dieckman – *Northeast Community College, NE*

Robert Gardner – *East Tennessee State University, TN*

Jim Hendrickson – *Indiana University, IN*

Masoud F. Kazemi - *Lewis-Clark State College, ID*

Linda Kurosaki – *Erie Community College, NY*

Betty Larson – *South Dakota State University, SD*

Helen Medley – *Kent State University, OH*

Scott Metcalf – *Eastern Kentucky University, KY*

Suellen Robinson – *North Shore Community College, MA*

Gary Wardall – *University of Wisconsin, Green Bay, WI*

**College Trigonometry, Sixth Edition** is designed to enhance conceptual understanding and quantitative reasoning through its motivating opening features, its *Interactive Method* (AIM), its features for student success, its exercises, its contemporary applications, and its use of technology.

## Enhance Conceptual Understanding and Quantitative Reasoning: Using Motivating Features

### Revised! Chapter Openers

New **Chapter Openers** demonstrate how the mathematics developed in each chapter is applied.

The icons  SSG,  , and  **Online Study Center**

at the bottom of the page let students know of additional resources.

### Prepare for This Section

Each section opens with review exercises, titled **Prepare for This Section**, which gives students a chance to test their understanding of prerequisite skills and concepts before proceeding to a new topic. An outline of the section's contents is also provided in the margin as a study aid.

#### Section 4.2

#### The Law of Cosines and Area

##### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A20.

- PS1. Evaluate  $\sqrt{a^2 + b^2 - 2ab \cos C}$  for  $a = 10.0$ ,  $b = 15.0$ , and  $C = 110.0^\circ$ . Round your result to the nearest tenth. [2.3]
- PS2. Find the area of a triangle with a base of 6 inches and a height of 8.5 inches.
- PS3. Solve  $c^2 = a^2 + b^2 - 2ab \cos C$  for  $C$ . [3.5]
- PS4. The **semiperimeter** of a triangle is defined as one-half the perimeter of the triangle. Find the semiperimeter of a triangle with sides of 6 meters, 9 meters, and 10 meters.
- PS5. Evaluate  $\sqrt{s(s-a)(s-b)(s-c)}$  for  $a = 3$ ,  $b = 4$ ,  $c = 5$ , and  $s = \frac{a+b+c}{2}$ .
- PS6. State a relationship between the lengths  $a$ ,  $b$ , and  $c$  in the triangle shown at the right. [1.2]



292

**NEW! A Calculus Connection icon**  identifies topics that will be revisited in a subsequent calculus course.

##### Calculus Connection

##### The Difference Quotient

The expression

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$

is called the **difference quotient** of  $f$ . It enables us to study the manner in which a function changes in value as the independent variable changes.

## 4 Applications of Trigonometry

- 4.1 The Law of Sines
- 4.2 The Law of Cosines and Area
- 4.3 Vectors

### Trigonometry and Indirect Measurement

In Chapter 2 we used trigonometric functions to find the unknown length of a side of a given *right triangle*. In this chapter we develop theorems that can be used to find the length of a side or the measure of an angle of *any triangle*, even if it is not a right triangle. These theorems are often used in the areas of navigation, surveying, and building design. Meteorologists use these theorems to estimate the distance from an approaching hurricane to cities in the projected path of the hurricane. For instance, in the diagram on the left, the distance from the hurricane to Nags Head can be determined using the Law of Sines.

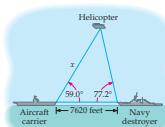
Sines, a theorem presented in this chapter.  
See Exercises 30 and 31 on page 299 for additional applications that can be solved by using the Law of Sines.



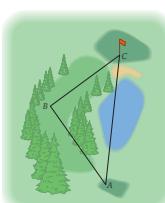
### Online Study Center

For online student resources, such as section quizzes, visit this website:  
[college.hmcbo.com/info/aufmannCAT](http://college.hmcbo.com/info/aufmannCAT)

30. **NAVAL MANEUVERS** The distance between an aircraft carrier and a Navy destroyer is 7620 feet. The angle of elevation from the destroyer to a helicopter is  $77.2^\circ$ , and the angle of elevation from the aircraft carrier to the helicopter is  $59.0^\circ$ . The helicopter is in the same vertical plane as the two ships, as shown in the following figure. Use this data to determine the distance  $x$  from the helicopter to the aircraft carrier.



31. **CHOOSING A GOLF STRATEGY** The following diagram shows two ways to play a golf hole. One is to hit the ball down the fairway for a first shot and then hit an approach shot to the green on your second shot. A second way is to hit directly toward the pin. Due to the water hazard, this is a more risky strategy. The distance  $AB$  is 165 yards,  $BC$  is 155 yards, and angle  $A = 42.0^\circ$ . Find the distance  $AC$  from the tee directly to the pin. Assume that angle  $B$  is an obtuse angle.



# Enhance Conceptual Understanding and Quantitative Reasoning: Using Contemporary Applications

## NEW! Quantitative Reasoning

After each set of **Chapter Review Exercises**, a new **Quantitative Reasoning** scenario explores an application in more depth or to extend a mathematical concept from the chapter.

**92 Chapter 1 Functions and Graphs**

35.  $f(x) = -2x + 5$

36.  $f(x) = -x + 3$

37.  $f(x) = \frac{2x}{x - 1}, x \neq 1$

38.  $f(x) = \frac{x}{x - 2^*}, x \neq 2$

39.  $f(x) = \frac{x - 1}{x + 1^*}, x \neq -1$

40.  $f(x) = \frac{2x - 1}{x + 3}, x \neq -3$

41.  $f(x) = x^2 + 1, x \geq 0$

42.  $f(x) = x^2 - 4, x \geq 0$

43.  $f(x) = \sqrt{x - 2}, x \geq 2$

44.  $f(x) = \sqrt[4]{x - 4}, x \leq 4$

45.  $f(x) = x^3 + 4x, x \geq -2$

46.  $f(x) = x^2 - 6x, x \leq 3$

47.  $f(x) = x^2 + 4x - 1, x \leq -2$

48.  $f(x) = x^2 - 6x + 1, x \geq 3$

49. **GEOMETRY** The volume of a cube is given by  $V(x) = x^3$ , where  $x$  is the measure of the length of a side of the cube. Find  $V^{-1}(x)$  and explain what it represents.

50. **UNIT CONVERSIONS** The function  $f(x) = 12x$  converts feet,  $x$ , into inches,  $f(x)$ . Find  $f^{-1}(x)$  and explain what it determines.

51. **FAHRENHEIT TO CELSIUS** The function  $f(x) = \frac{5}{9}(x - 32)$  is used to convert  $x$  degrees Fahrenheit to an equivalent Celsius temperature. Find  $f^{-1}$  and explain how it is used.

52. **RETAIL SALES** A clothing merchant uses the function  $S(x) = \frac{3}{2}x + 18$

to determine the retail selling price  $S$ , in dollars, of a winter coat for which she has paid a wholesale price of  $x$  dollars.

a. The merchant paid a wholesale price of \$96 for the winter coat. Use  $S$  to determine the retail selling price she will charge for this coat.

b. Find  $S^{-1}$  and use it to determine the merchant's wholesale price for a coat that retails at \$399.

53. **FASHION** The function  $s(x) = 2x + 24$  can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function  $s^{-1}(x)$  that can be used to convert an Italian women's shoe size to its equivalent U.S. shoe size.

54. **FASHION** The function  $K(x) = 1.3x - 47$  converts a men's shoe size in the United States to the equivalent shoe size in the United Kingdom. Determine the function  $K^{-1}(x)$  that can be used to convert a United Kingdom men's shoe size to its equivalent U.S. shoe size.

55. **COMPENSATION** The monthly earnings  $E(s)$ , in dollars, of a software sales executive is given by  $E(s) = 0.08s + 2500$ , where  $s$  is the value, in dollars, of the software sold by the executive during the month. Find  $E^{-1}(s)$  and explain how the executive could use this function?

56. **POSTAGE** Does the first-class postage rate function given below have an inverse function? Explain your answer.

| Weight (in ounces) | Cost   |
|--------------------|--------|
| $0 < w \leq 1$     | \$0.39 |
| $1 < w \leq 2$     | \$0.63 |
| $2 < w \leq 3$     | \$0.87 |
| $3 < w \leq 4$     | \$1.11 |

57. **THE BIRTHDAY PROBLEM** A famous problem called the *birthday problem* goes like this: Suppose there is a randomly selected group of  $n$  people in a room. What is the probability that at least two of the people have a birthday on the same day of the year? It may surprise you that for a group of 23 people, the probability that at least two of the people share a birthday is about 50.7%. The following graph can be used to estimate shared birthday probabilities for  $1 \leq n \leq 60$ .

## »»» Projects »»»

**MEDIAN-MEDIAN LINE** Another linear model of data is called the **median-median line**. This line employs *summary points* calculated using the medians of subsets of the independent and dependent variables. The **median** of a data set is the middle number or the average of the two middle numbers for a data set arranged in numerical order. For instance, to find the median of {8, 12, 6, 7, 9}, first arrange the data in numerical order.

6, 7, 8, 9, 12

The median is 8, the number in the middle. To find the median of {15, 12, 20, 9, 13, 10}, arrange the numbers in numerical order:

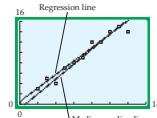
9, 10, 12, 13, 15, 20

The median is 12.5, the average of the two middle numbers.

$$\text{Median} = \frac{12 + 13}{2} = 12.5$$

The median-median line is determined by dividing a data set into three equal groups. (If the set cannot be divided into three equal groups, the first and third groups should be equal. For instance, if there are 11 data points, divide the set into groups of 4, 3, and 4.) The slope of the median-median line is the slope of the line through the  $x$ -medians and  $y$ -medians of the first and third sets of points. The median-median line passes through the average of the  $x$ -and  $y$ -medians of all three sets.

| x  | y  |
|----|----|
| 2  | 3  |
| 3  | 5  |
| 4  | 4  |
| 5  | 7  |
| 6  | 8  |
| 7  | 9  |
| 8  | 12 |
| 9  | 12 |
| 10 | 14 |
| 11 | 15 |
| 12 | 14 |



3. Consider the data set {(1, 3), (2, 5), (3, 7), (4, 9), (5, 11), (6, 13), (7, 15), (8, 17)}.

- a. Find the equation of the linear regression line for these data.
- b. Find the equation of the median-median line for these data.

- c. **What conclusion might you draw from the answers to a. and b.?**

## »»» Quantitative Reasoning: Public Key Cryptography »»»

As mentioned in the Chapter Opener, performing financial transactions over the Internet requires secure transmissions between two sites, the sender and the receiver. One method of creating secure transmissions is to use a **modular function**.

A modular function is one that gives, in integer form, the remainder when one number is divided by another. We write  $a \equiv b \pmod m$  to mean that  $a$  is the remainder when  $b$  is divided by  $m$ . Here are some examples.

$$4 \equiv 22 \pmod 6 \quad \text{because } 22 \div 6 = 3 \text{ remainder } 4.$$

$$1 \equiv 37 \pmod 4 \quad \text{because } 37 \div 4 = 9 \text{ remainder } 1.$$

$$0 \equiv 55 \pmod{11} \quad \text{because } 55 \div 11 = 5 \text{ remainder } 0.$$

$$17 \equiv 17 \pmod{31} \quad \text{because } 17 \div 31 = 0 \text{ remainder } 17.$$

- QR1.** Find the value of each expression.

$$a. 15 \pmod 4 \quad b. 37 \pmod 5 \quad c. 52 \pmod{321}$$

**Public key cryptography** uses a modular function to encrypt a message—say, a person's name or credit card number—so that only the receiver of the message can decrypt it. The message is decrypted by using the inverse of the modular function Section 1.6.

## Updated! Applications

Carefully developed mathematics is complemented by abundant, relevant, and contemporary applications, many of which feature real data and tables, graphs, and charts. Note that applications using real data are identified

by

Applications demonstrate to students the value of algebra and cover topics from a wide variety of disciplines—including agriculture, business, chemistry, construction, earth sciences, economics, education, manufacturing, medicine, nutrition, real estate, and sociology. Besides providing motivation to study mathematics, applications assist students in developing good problem-solving skills.

## Projects

Projects are designed to engage the student in mathematics. At the end of each section, students are asked to do one or more of the following types of projects:

- solve a more involved application problem
- investigate a concept in greater depth
- write a proof of a statement

With some projects, students are asked to chronicle the procedure used to solve it, and to suggest an extension to the project. These projects are ideal candidates for small group assignments.

## Enhance Conceptual Understanding and Quantitative Reasoning: Using Technology

328 Chapter 4 Applications of Trigonometry

### Exploring Concepts with Technology

#### Optimal Branching of Arteries

The physiologist Jean Louis Poiseuille (1799–1869) developed several laws concerning the flow of blood. One of his laws states that the resistance  $R$  of a blood vessel of length  $l$  and radius  $r$  is given by

$$R = k \frac{l}{r^4} \quad (1)$$

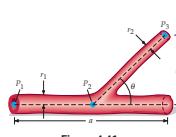


Figure 4.41

The number  $k$  is a variation constant that depends on the viscosity of the blood. Figure 4.41 shows a large artery with radius  $r_1$  and a smaller artery with radius  $r_2$ . The branching angle between the arteries is  $\theta$ . Make use of Poiseuille's Law, Equation (1), to show that the resistance  $R$  of the blood along the path  $P_1P_2P_3$  is

$$R = k \left( \frac{a - b \cot \theta}{(r_1)^4} + \frac{b \csc \theta}{(r_2)^4} \right) \quad (2)$$

Use a graphing utility to graph  $R$  with  $k = 0.0563$ ,  $a = 8$  centimeters,  $b = 4$  centimeters,  $r_1 = 0.4$  centimeter, and  $r_2 = \frac{3}{4}r_1 = 0.3$  centimeter. Then estimate (to the nearest degree) the angle  $\theta$  that minimizes  $R$ . By using calculus, it can be demonstrated that  $R$  is minimized when

$$\cos \theta = \left( \frac{r_2}{r_1} \right)^4 \quad (3)$$

This equation is remarkable because it is much simpler than Equation (2) and because it does not involve the distance  $a$  or  $b$ . Solve Equation (3) for  $\theta$ , with  $r_2 = \frac{3}{4}r_1$ . How does this value of  $\theta$  compare with the value of  $\theta$  you obtained by graphing?

### Exploring Concepts with Technology

#### Optional Exploring Concepts with Technology

problems extend ideas from the chapter, encouraging students to use calculators or computers to investigate solutions to computationally unpleasant problems. In this way, calculators and computers have expanded the limits of the types of problems that can be solved at this level. In addition, students are challenged to think about the pitfalls of computational solutions.

### Integrating Technology

Using optional **Integrating Technology** boxes, technology is integrated into a discussion when it can be used to foster and promote a better conceptual understanding of a concept. Additionally, optional graphing calculator



examples and exercises (identified by ) are presented throughout the text.

90 Chapter 1 Functions and Graphs

### Integrating Technology

Some graphing utilities can be used to draw the graph of the inverse of a function without the user having to find the inverse function. For instance, Figure 1.94 shows the graph of  $f(x) = 0.1x^3 - 4$ . The graphs of  $f$  and  $f^{-1}$  are both shown in Figure 1.95, along with the graph of  $y = x$ . Note that the graph of  $f^{-1}$  is the reflection of the graph of  $f$  with respect to the graph of  $y = x$ . The display shown in Figure 1.95 was produced on a TI-83/TI-83 Plus/TI-84 DRAW menu.

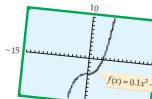


Figure 1.94

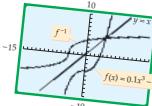


Figure 1.95

### Linear Regression Models

The data in the table below show the population of selected states and the number of professional sports teams (Major League Baseball, National Football League, National Basketball Association, Women's National Basketball Association, National Hockey League) in those states. A scatter diagram of the data is shown in Figure 1.96 on page 96.

Number of Professional Sports Teams for Selected States

| State      | Populations (in millions) | Number of Teams | State          | Populations (in millions) | Number of Teams |
|------------|---------------------------|-----------------|----------------|---------------------------|-----------------|
| Arizona    | 5.9                       | 5               | Minnesota      | 5.1                       | 5               |
| California | 36.1                      | 17              | New Jersey     | 8.7                       | 3               |
| Colorado   | 4.7                       | 4               | New York       | 19.3                      | 10              |
| Florida    | 17.8                      | 11              | North Carolina | 9.7                       | 3               |
| Illinois   | 12.8                      | 5               | Pennsylvania   | 12.4                      | 7               |
| Indiana    | 6.3                       | 3               | Texas          | 22.9                      | 9               |
| Michigan   | 10.1                      | 5               | Wisconsin      | 5.5                       | 3               |

### Modeling

Special modeling sections, which rely heavily on the use of a graphing calculator, are incorporated throughout the text. These optional sections introduce the idea of a mathematical model, using various real-world data sets that further motivate students and help them see the relevance of mathematical concepts.

# Enhance Conceptual Understanding and Quantitative Reasoning: Using the Aufmann Interactive Method (AIM)

By incorporating many interactive learning techniques, including the key features outlined below, **College Trigonometry** uses the proven **Aufmann Interactive Method (AIM)** to help students understand concepts, work independently, and obtain greater mathematical fluency.

**EXAMPLE 1** » Evaluate Functions

Let  $f(x) = x^2 - 1$ , and evaluate.

**take note**  
In Example 1, observe that  $f(3b) \neq 3f(b)$  and that  $f(a+3) \neq f(a) + f(3)$

**Solution**

a.  $f(-5) = (-5)^2 - 1 = 25 - 1 = 24$  • Substitute  $-5$  for  $x$ , and simplify.  
 b.  $f(3b) = (3b)^2 - 1 = 9b^2 - 1$  • Substitute  $3b$  for  $x$ , and simplify.  
 c.  $3f(b) = 3(b^2 - 1) = 3b^2 - 3$  • Substitute  $b$  for  $x$ , and simplify.  
 d.  $f(a+3) = (a+3)^2 - 1 = a^2 + 6a + 8$  • Substitute  $a+3$  for  $x$ .  
 e.  $f(a) + f(3) = (a^2 - 1) + (3^2 - 1) = a^2 + 7$  • Substitute  $a$  for  $x$ ; substitute  $3$  for  $x$ .  
 • Simplify.

» Try Exercise 2, page 46

**Piecewise-defined functions** are functions represented by more than one expression. For example,  $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$  is an example of a piecewise-defined function.

**In Exercises 1 to 8, evaluate each function.**

1. Given  $f(x) = 3x - 1$ , find  
 a.  $f(2)$       b.  $f(-1)$       c.  $f(0)$       d.  $f\left(\frac{2}{3}\right)$       e.  $f(k)$       f.  $f(k+2)$

2. Given  $g(x) = 2x^2 + 3$ , find  
 a.  $g(3)$       b.  $g(-1)$       c.  $g(0)$       d.  $g\left(\frac{1}{2}\right)$       e.  $g(c)$       f.  $g(c+5)$

3. Given  $A(w) = \sqrt{w^2 + 5}$ , find  
 a.  $A(0)$       b.  $A(2)$       c.  $A(-2)$       d.  $A(4)$       e.  $A(r+1)$       f.  $A(-c)$

4. Given  $J(t) = 3t^2 - t$ , find  
 a.  $J(-4)$       b.  $J(0)$       c.  $J\left(\frac{1}{3}\right)$       d.  $J(-c)$       e.  $J(x+1)$       f.  $J(x+h)$

5. Given  $T(x) = \frac{1}{|x|}$ , find  
 a.  $T(2)$       b.  $T(-2)$       c.  $T(0)$       d.  $T(2) + T(-2)$       e.  $T(c^2 + 4)$

6. Given  $T(x) = 5$ , find  
 a.  $T(-3)$       b.  $T(0)$       c.  $T(3) + T(1)$       d.  $T(x+h)$

7. Given  $s(x) = \frac{x}{|x|}$ , find  
 a.  $s(4)$       b.  $s(5)$       c.  $s(-4)$       d.  $s(-3)$       e.  $s(t)$

8. Given  $r(x) = \frac{x}{x+4}$ , find  
 a.  $r(0)$       b.  $r(-4)$       c.  $r\left(\frac{1}{2}\right)$       d.  $r\left(-\frac{1}{2}\right)$       e.  $r(0,3)$

Exercise Set 1.3, page 46

2. Given  $g(x) = 2x^2 + 3$   
 a.  $g(3) = 2(3)^2 + 3 = 18 + 3 = 21$   
 b.  $g(-1) = 2(-1)^2 + 3 = 2 + 3 = 5$   
 c.  $g(0) = 2(0)^2 + 3 = 0 + 3 = 3$   
 d.  $g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 + 3 = \frac{1}{2} + 3 = \frac{7}{2}$   
 e.  $g(c) = 2(c)^2 + 3 = 2c^2 + 3$   
 f.  $g(c+5) = 2(c+5)^2 + 3 = 2c^2 + 20c + 50 + 3 = 2c^2 + 20c + 53$

For the population/sports team data,  $r^2 \approx 0.87$ . This means that approximately 87% of the total variation in the dependent variable (number of teams) can be attributed to the state population. This also means that population alone does not predict with certainty the number of sports teams. Other factors, such as climate, are also involved in the number of sports teams.

**QUESTION** What is the coefficient of determination for the odometer reading/trade-in value data (see page 99), and what is its significance?

**Quadratic Regression Models**

To this point our focus has been *linear* regression equations. However, there may be a nonlinear relationship between two quantities. The scatter diagram to the left suggests that a quadratic function might be a better model of the data than a linear model.

**ANSWER**  $r^2 = 0.964$ . This means that about 96.4% of the total variation in trade-in value can be attributed to the odometer reading.

## Annotated Examples

**Annotated Examples** are provided throughout each section and are titled. The annotated steps assist the student in moving from step to step and help explain the solution.

## Try Exercises

References to **Try Exercises** follow all worked examples, encouraging students to test their understanding by working an exercise similar to the worked example. An icon and a page reference are given below the example, making it easy for the student to navigate to the suggested exercise and back.

## Solutions to Try Exercises

The complete solution to the **Try Exercises** can be found in the **Solutions to the Try Exercises** appendix. This interaction among the examples, the **Try Exercises**, the **Solutions to the Try Exercises**, and the exercise sets serves as a checkpoint for students as they read the text, do their homework, and study a section.

## Question/Answer

In each section, the authors pose at least one question that encourages students to pause and think about the concepts presented in the current discussion. To ensure that students do not miss important information, the answer to the question is provided as a footnote on the same page.

# Enhance Conceptual Understanding and Quantitative Reasoning: Using Features for Student Success

**1.2 A Two-Dimensional Coordinate System and Graphs** **25**

circle with center  $(h, k)$  and radius  $r$ . The point  $(x, y)$  is on the circle if and only if it is a distance of  $r$  units from the center  $(h, k)$ . Thus  $(x, y)$  is on the circle if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{• Square each side.}$$

**Standard Form of the Equation of a Circle**

The standard form of the equation of a circle with center at  $(h, k)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 = r^2$$

**Example**

The equation  $(x - 2)^2 + (y - 4)^2 = 3^2$  is in standard form, where  $C(2, 4)$  and radius 3.

The equation  $(x - 3)^2 + (y + 1)^2 = 25$  can be written in standard form as  $(x - 3)^2 + (y - (-1))^2 = 5^2$ . Note that in the standard form,  $(x - h)^2$  and  $(y - k)^2$  are written using subtraction. Because  $(y + 1)^2$  is written using addition, the expression is rewritten as  $(y - (-1))^2$ . The graph of this equation is a circle with center  $C(3, -1)$  and radius 5.

The equation  $(x + 4)^2 + (y - 2)^2 = 10$  can be written in standard form as  $(x - (-4))^2 + (y - 2)^2 = (\sqrt{10})^2$ . The graph of this equation is a circle with center  $C(-4, 2)$  and radius  $\sqrt{10}$ .

If a circle is centered at the origin  $(0, 0)$ , then  $h = 0$  and  $k = 0$  and the standard form of the equation of the circle simplifies to

$$x^2 + y^2 = r^2 \quad \text{• Equation of a circle with center at the origin and radius } r.$$

For instance,  $x^2 + y^2 = 9$  is the equation of the circle with center at the origin and radius  $\sqrt{9} = 3$ .

**QUESTION** What are the radius and the coordinates of the center of the circle with equation  $x^2 + (y - 2)^2 = 30$ ?

**EXAMPLE 6** Find the Standard Form of the Equation of a Circle

Find the standard form of the equation of the circle that has center  $C(-4, -2)$  and contains the point  $P(-1, 2)$ .

**ANSWER** The radius is  $\sqrt{30}$  and the coordinates of the center are  $(0, 2)$ .

Continued ▶

## Updated! To Review Note

To Review notes help students recognize the prerequisite skills needed to understand new concepts. These notes direct students to the appropriate page(s) for review, thus decreasing student frustration and creating more opportunities for comprehension.

**TO REVIEW**  
Axis of Symmetry  
See page 000.

This is the standard form of the equation of a parabola with vertex at the origin and the  $y$ -axis as its axis of symmetry. The standard form of the equation of a parabola with vertex at the origin and the  $x$ -axis as its axis of symmetry is derived in a similar manner.

### Standard Forms of the Equation of a Parabola with Vertex at the Origin

#### Axis of Symmetry Is the $y$ -Axis

The standard form of the equation of a parabola with vertex  $(0, 0)$  and the  $y$ -axis as its axis of symmetry is

$$x^2 = 4py$$

The focus is  $(0, p)$ , and the equation of the directrix is  $y = -p$ . If  $p > 0$ , the graph of the parabola opens up. See Figure 6.4a. If  $p < 0$ , the graph of the parabola opens down. See Figure 6.4b.

#### Axis of Symmetry Is the $x$ -Axis

The standard form of the equation of a parabola with vertex  $(0, 0)$  and the  $x$ -axis as its axis of symmetry is

$$y^2 = 4px$$

The focus is  $(p, 0)$ , and the equation of the directrix is  $x = -p$ . If  $p > 0$ , the graph of the parabola opens to the right. See Figure 6.4c. If  $p < 0$ , the graph of the parabola opens to the left. See Figure 6.4d.

#### take note

The tests for  $y$ -axis and  $x$ -axis symmetry can be used to verify these statements and provide connections to earlier topics on symmetry.

## NEW! Immediate Examples of Definitions and Concepts

Immediate examples of many definitions enhance understanding.

## Margin Notes

**Take Notes** alert students to a point requiring special attention or are used to amplify the concept under discussion. And **Math Matters** contain interesting sidelights about mathematics, its history, or its application.

#### take note

$f^{-1}(x)$  does not mean  $\frac{1}{f(x)}$ . For  $f(x) = 2x$ ,  $f^{-1}(x) = \frac{1}{2}x$  but  $\frac{1}{f(x)} = \frac{1}{2x}$ .

## Visualize the Solution

For appropriate examples, both algebraic and graphical solutions are provided to help the student visualize the mathematics of the example and to create a link between the algebraic and visual components of a solution.

### EXAMPLE 3 Find Cube Roots by De Moivre's Theorem

Find the three cube roots of 27.

#### ALGEBRAIC SOLUTION

Write 27 in trigonometric form:  $27 = 27 \operatorname{cis} 0^\circ$ . Then, from De Moivre's Theorem for finding roots, the cube roots of 27 are

$$w_k = 27^{1/3} \operatorname{cis} \frac{0^\circ + 360^\circ k}{3} \text{ for } k = 0, 1, 2$$

Substitute for  $k$  to find the three cube roots of 27.

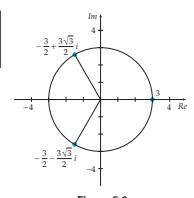
$$\begin{aligned} w_0 &= 27^{1/3} \operatorname{cis} 0^\circ & k = 0; \frac{0^\circ + 360^\circ (0)}{3} = 0^\circ \\ &= 3(\cos 0^\circ + i \sin 0^\circ) \\ &= 3 \end{aligned}$$

$$\begin{aligned} w_1 &= 27^{1/3} \operatorname{cis} 120^\circ & k = 1; \frac{0^\circ + 360^\circ (1)}{3} = 120^\circ \\ &= 3(\cos 120^\circ + i \sin 120^\circ) \\ &= -\frac{3}{2} + \frac{3\sqrt{3}}{2}i \end{aligned}$$

$$\begin{aligned} w_2 &= 27^{1/3} \operatorname{cis} 240^\circ & k = 2; \frac{0^\circ + 360^\circ (2)}{3} = 240^\circ \\ &= 3(\cos 240^\circ + i \sin 240^\circ) \\ &= -\frac{3}{2} - \frac{3\sqrt{3}}{2}i \end{aligned}$$

For  $k = 3$ ,  $\frac{0^\circ + 360^\circ (3)}{3} = 360^\circ$ . The angles start repeating; thus there are only three cube roots of 27. The three cube roots are graphed in Figure 5.9.

#### VISUALIZE THE SOLUTION



Note that the arguments of the three cube roots of 27 are  $0^\circ$ ,  $120^\circ$ , and  $240^\circ$  and that  $|w_0| = |w_1| = |w_2| = 3$ . In geometric terms, this means that the three cube roots of 27 are equally spaced on a circle centered at the origin with a radius of 3.

### Try Exercise 28, page 355

## Enhance Conceptual Understanding and Quantitative Reasoning: Using Well-Developed Exercise Sets

### Connecting Concepts

Each end-of-section exercise set features **Connecting Concepts** problems which include material from previous sections, are extensions of topics in the section, require data analysis, and offer challenge problems or problems of the form “prove or disprove.”

### Revised! Assessing Concepts

**Assessing Concepts** exercises, found at the end of each chapter, have been enhanced with more question types—including fill-in-the-blank, multiple choice, matching, and true/false.

### Topics for Discussion

Each section ends with **Topics for Discussion** exercises to be used for group discussion or as writing exercises. These exercises are frequently conceptual and focus on key ideas in the section.

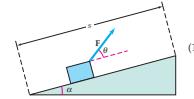
### Extensive Exercises

The authors thoroughly reviewed each exercise set to update applications and to ensure a smooth progression from routine exercises to exercises that are more challenging. The exercises illustrate the many facets of topics discussed in the text. Each exercise set emphasizes skill building, skill maintenance, conceptual understanding, quantitative reasoning, and, as appropriate, applications. Each exercise set is directly preceded by **Topics for Discussion** exercises and directly followed by **Connecting Concepts** exercises and **Projects**.

Each chapter ends with **Assessing Concepts** exercises, **Chapter Review Exercises**, and a **Chapter Test**. Each chapter, except chapter P, includes a **Cumulative Review Exercise Set**. Answers to all exercises in the **Review Exercises**, the **Chapter Test**, and the **Cumulative Review Exercises** are included in the student answer section. Along with the answer, there is a reference to the section that pertains to each exercise.

#### Connecting Concepts

83. For  $\mathbf{u} = (-1, 1)$ ,  $\mathbf{v} = (2, 3)$ , and  $\mathbf{w} = (5, 5)$ , find the sum of the three vectors geometrically by using the triangle method of adding vectors.
84. For  $\mathbf{u} = (1, 2)$ ,  $\mathbf{v} = (3, -2)$ , and  $\mathbf{w} = (-1, 4)$ , find  $\mathbf{u} + \mathbf{v} - \mathbf{w}$  geometrically by using the triangle method of adding vectors.
85. Find a vector that has initial point  $(3, -1)$  and is equivalent to  $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ .
86. Find a vector that has initial point  $(-2, 4)$  and is equivalent to  $\mathbf{v} = (-1, 3)$ .
87. If  $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$  and  $\mathbf{w} = 5\mathbf{i} + 2\mathbf{j}$  have the same initial point, is  $\mathbf{v}$  perpendicular to  $\mathbf{w}$ ? Why or why not?
88. If  $\mathbf{v} = (5, 6)$  and  $\mathbf{w} = (6, 5)$  have the same initial point, is  $\mathbf{v}$  perpendicular to  $\mathbf{w}$ ? Why or why not?
89. Let  $\mathbf{v} = (-2, 7)$ . Find a vector perpendicular to  $\mathbf{v}$ .
90. Let  $\mathbf{w} = 4\mathbf{i} + \mathbf{j}$ . Find a vector perpendicular to  $\mathbf{w}$ .



#### Chapter 4 Assessing Concepts

1. What is an oblique triangle?
2. In triangle ABC,  $a = 4.5$ ,  $b = 6.2$ , and  $C = 107^\circ$ . Which law, the Law of Sines or the Law of Cosines, can be used to find  $c$ ?
3. Which of the following cases, ASA, AAS, SSA, SSS, or SAS is known as the ambiguous case of the Law of Sines?
4. In Heron's formula, what does the variable  $s$  represent?
5. Is the dot product of two vectors a vector or a scalar?
6. Let  $\mathbf{v}$  and  $\mathbf{w}$  be nonzero vectors. Is  $\text{proj}_{\mathbf{w}} \mathbf{v}$  a vector or a scalar?
7. True or false: The vector  $\left(\frac{12}{13}, -\frac{5}{13}\right)$  is a unit vector.
8. True or false:  $\mathbf{i} \cdot \mathbf{j} = 0$ .
9. True or false: The Law of Sines can be used to solve any triangle, given two angles and any side.
10. True or false: If two nonzero vectors are orthogonal, then their dot product is 0.

#### Topics for Discussion

1. Discuss the meaning of symmetry of a graph with respect to a line. How do you determine whether a graph has symmetry with respect to the  $x$ -axis? with respect to the  $y$ -axis?
2. Discuss the meaning of symmetry of a graph with respect to a point. How do you determine whether a graph has symmetry with respect to the origin?
3. What does it mean to reflect a graph across the  $x$ -axis or across the  $y$ -axis?
4. Explain how the graphs of  $y > 2x^3 - x^2$  and  $y > 2(-x)^3 - (-x)^2$  are related.
5. Given the graph of the function  $y > f(x)$ , explain how to obtain the graph of the function  $y > f(x - 3) + 1$ .
6. The graph of the step function  $y > \|x\|$  has steps that are 1 unit wide. Determine how wide the steps are for the graph of  $y > \left[\frac{1}{3}x\right]$ .



## Additional Resources • Get More Value from Your Textbook!

[college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

### Instructor Resources

#### Online Teaching Center



#### Online Teaching Center

This website offers instructors prep and chapter tests, instructor's solutions, and more.

#### HM Testing™ (powered by Diploma®)

**"Testing the way you want it"** HM Testing offers all the tools needed to create, author, deliver, and customize multiple types of tests—including authoring and editing algorithmic questions.

#### Instructional DVDs



Hosted by Dana Moseley, these text-specific DVDs cover all sections of the text and provide explanations of key concepts, examples, exercises, and applications in a lecture-based format. DVDs are now closed-captioned for the hearing-impaired.

#### Eduspace® (powered by Blackboard®)



Eduspace is a web-based learning system that provides instructors with powerful course management tools and students with text-specific content to support all of their online teaching and learning needs. Eduspace makes it easy to deliver all or part of a course online. Resources such as algorithmic, automatically-graded homework exercises, tutorials, instructional video clips, an online multimedia eBook, live, online tutoring with SMARTHINKING™, and additional study materials all come ready-to-use. Instructors can choose to use the content as is, modify it, or even add their own.

- Visit [www.eduspace.com](http://www.eduspace.com) for more information.

#### SMARTHINKING™ Live, Online Tutoring



SMARTHINKING provides an easy-to-use and effective online, text-specific tutoring service. A dynamic whiteboard and graphing calculator function enables students and e-structors to collaborate easily. SMARTHINKING offers three levels of service:

- Text-specific Tutoring provides real-time, one-on-one instruction with a specially qualified e-structor.
- Questions Any Time allows students to submit questions to the tutor outside the scheduled hours and receive a reply within 24 hours.
- Independent Study Resources connect students with around-the-clock access to additional educational services, including interactive websites, diagnostic tests, and Frequently Asked Questions posed to SMARTHINKING e-structors.

#### Online Course Content for Blackboard®, WebCT®, and eCollege®

Deliver program or text-specific Houghton Mifflin content online using your institution's local course management system. Houghton Mifflin offers homework, tutorials, videos, and other resources formatted for Blackboard®, WebCT®, eCollege®, and other course management systems. Add to an existing online course or create a new one by selecting from a wide range of powerful learning and instructional materials.

For more information, visit [college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

or contact your Houghton Mifflin sales representative.

### Student Resources

#### Student Study Guide



Contains complete solutions to all odd-numbered exercises and all of the solutions to the end-of-chapter material.

#### Online Study Center



This free website gives students access to ACE practice tests, and other study resources.

*This page intentionally left blank*

# 1

# Functions and Graphs



- 
- 1.1 Equations and Inequalities
  - 1.2 A Two-Dimensional Coordinate System and Graphs
  - 1.3 Introduction to Functions
  - 1.4 Properties of Graphs
  - 1.5 The Algebra of Functions
  - 1.6 Inverse Functions
  - 1.7 Modeling Data Using Regression
- 

## The Internet, the World Wide Web, and Modular Functions

In 1965, Lawrence Roberts connected two computers, one at MIT and one at UCLA, using a telephone line. This computer connection was

the first demonstration of the feasibility of computer networks and led to the eventual establishment of the Internet, first known as ARPANET after the Advanced Research Projects Agency (ARPA), a group founded by the Department of Defense. Originally, ARPANET connected computers at four universities: the University of California, Los Angeles; Stanford University; the University of California, Santa Barbara; and the University of Utah. The first message was sent from UCLA to Stanford in 1969.

To expand the Internet, computer programs had to be written that allowed the interaction of many computers. This set of programs, called protocols, was created by Robert Kahn and Vint Cerf. The programs were called TCP/IP (Transmission Control Protocol/Internet Protocol). TCP/IP was universally adopted in 1983 and is still used today to control communication among computers on the Internet. A rough analog to TCP/IP programs is our telephone system. Each telephone has a unique telephone number. Similarly, each computer on the Internet has a unique IP (Internet Protocol) address. TCP programs are similar to telephone services such as call waiting and caller ID. They enable a computer to know when other computers are trying to communicate with it, and what kind of communication is incoming.

In 1991, British physicist Tim Berners-Lee developed a program that allowed physicists to exchange information in an efficient manner. His creation was called the World Wide Web. Two functions that can be performed using the Internet and the World Wide Web are credit card purchases and banking transactions. These transactions must be carried out in a secure mode using programs that encrypt credit card numbers and bank account numbers. One method of encryption makes use of a *modular function* and the *inverse* of that function. Modular functions are discussed in the **Quantitative Reasoning feature on page 114**. A basic example of the use of inverse functions to encrypt and decrypt messages is given in **Exercise 59 on page 93**.



### Online Study Center

For online student resources, such as section quizzes, visit this website:  
[college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

## Section 1.1

- The Real Numbers
- Absolute Value and Distance
- Linear and Quadratic Equations
- Inequalities
- Solving Inequalities by the Critical Value Method
- Absolute Value Inequalities

### Math Matters

Archimedes (c. 287–212 B.C.) was the first to calculate  $\pi$  with any degree of precision. He was able to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

from which we get the approximation

$3\frac{1}{7} \approx \pi$ . The use of the symbol  $\pi$

for this quantity was introduced by Leonhard Euler (1707–1783) in 1739, approximately 2000 years after Archimedes.

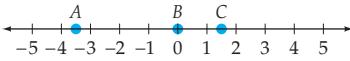


Figure 1.1

#### take note

The interval notation  $[1, \infty)$  represents all real numbers greater than or equal to 1. The interval notation  $(-\infty, 4)$  represents all real numbers less than 4.

# Equations and Inequalities

## The Real Numbers

The real numbers are used extensively in mathematics. The set of real numbers is quite comprehensive and contains several unique sets of numbers.

The **integers** are the set of numbers

$$\{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Recall that the brace symbols, { }, are used to identify a set. The positive integers are called **natural numbers**.

The **rational numbers** are the set of numbers of the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b \neq 0$ . Thus the rational numbers include  $-\frac{3}{4}$  and  $\frac{5}{2}$ . Because

each integer can be expressed in the form  $\frac{a}{b}$  with denominator  $b = 1$ , the integers are included in the set of rational numbers. Every rational number can be written as either a terminating or a repeating decimal.

A number written in decimal form that does not repeat or terminate is called an **irrational number**. Some examples of irrational numbers are  $0.141141114\dots$ ,  $\sqrt{2}$ , and  $\pi$ . These numbers cannot be expressed as quotients of integers. The set of **real numbers** is the union of the sets of rational and irrational numbers.

A real number can be represented geometrically on a **coordinate axis** called a **real number line**. Each point on this line is associated with a real number called the **coordinate** of the point. Conversely, each real number can be associated with a point on a real number line. In **Figure 1.1**, the coordinate of  $A$  is  $-\frac{7}{2}$ , the coordinate of  $B$  is 0, and the coordinate of  $C$  is  $\sqrt{2}$ .

Given any two real numbers  $a$  and  $b$ , we say that  $a$  is **less than**  $b$ , denoted by  $a < b$ , if  $a - b$  is a negative number. Similarly, we say that  $a$  is **greater than**  $b$ , denoted by  $a > b$ , if  $a - b$  is a positive number. When  $a$  **equals**  $b$ ,  $a - b$  is zero. The symbols  $<$  and  $>$  are called **inequality symbols**. Two other inequality symbols,  $\leq$  (less than or equal to) and  $\geq$  (greater than or equal to), are also used.

The inequality symbols can be used to designate sets of real numbers. If  $a < b$ , the **interval notation**  $(a, b)$  is used to indicate the set of real numbers between  $a$  and  $b$ . This set of numbers also can be described using **set-builder notation**:

$$(a, b) = \{x | a < x < b\}$$

When reading a set written in set-builder notation, we read  $\{x\}$  as “the set of  $x$  such that.” The expression that follows the vertical bar designates the elements in the set.

The set  $(a, b)$  is called an **open interval**. The graph of the open interval consists of all the points on the real number line between  $a$  and  $b$ , not including  $a$  and  $b$ . A **closed interval**, denoted by  $[a, b]$ , consists of all points between  $a$  and  $b$ , including  $a$  and  $b$ . We can also discuss **half-open intervals**. An example of each type of interval is shown in **Figure 1.2**.

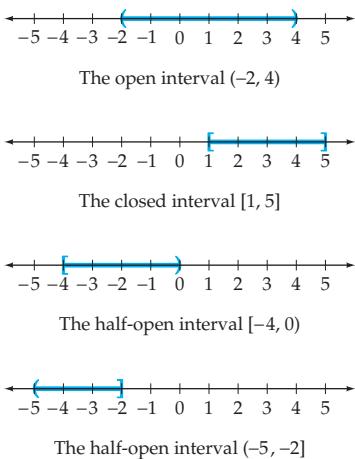


Figure 1.2

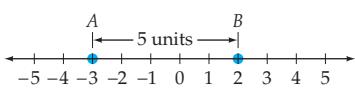


Figure 1.3

## Absolute Value and Distance

The *absolute value* of a real number is a measure of the distance from zero to the point associated with the number on a real number line. Therefore, the absolute value of a real number is always positive or zero. We now give a more formal definition of absolute value.

### Definition of Absolute Value

For a real number  $a$ , the **absolute value** of  $a$ , denoted by  $|a|$ , is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

The distance  $d$  between the points  $A$  and  $B$  with coordinates  $-3$  and  $2$ , respectively, on a real number line is the absolute value of the difference between the coordinates. See **Figure 1.3**.

$$d = |2 - (-3)| = 5$$

Because the absolute value is used, we could also write

$$d = |(-3) - 2| = 5$$

In general, we define the *distance* between any two points  $A$  and  $B$  on a real number line as the absolute value of the difference between the coordinates of the points.

### Definition of the Distance Between Two Points on a Real Number Line

Let  $a$  and  $b$  be the coordinates of the points  $A$  and  $B$ , respectively, on a real number line. Then the **distance** between  $A$  and  $B$ , denoted by  $d(A, B)$ , is

$$d(A, B) = |a - b|$$

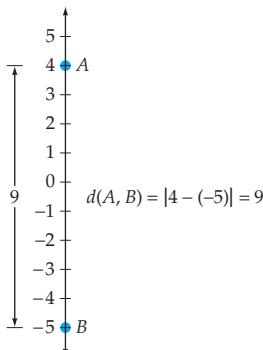


Figure 1.4

This formula applies to any real number line. It can be used to find the distance between two points on a vertical real number line, as shown in **Figure 1.4**.

## Linear and Quadratic Equations

An **equation** is a statement about the equality of two expressions. Examples of equations follow.

$$7 = 2 + 5 \quad x^2 = 4x + 5 \quad 3x - 2 = 2(x + 1) + 3$$

The values of the variable that make an equation a true statement are the **roots** or **solutions** of the equation. To **solve** an equation means to find the solutions of the equation. The number 2 is said to **satisfy** the equation  $2x + 1 = 5$  because substituting 2 for  $x$  produces  $2(2) + 1 = 5$ , which is a true statement.

### Definition of a Linear Equation

A **linear equation** in the single variable  $x$  is an equation of the form  $ax + b = 0$ , where  $a \neq 0$ .

To solve a linear equation in one variable, isolate the variable on one side of the equals sign.

### EXAMPLE 1 » Solve a Linear Equation

Solve:  $3x - 5 = 2$

#### Solution

$$\begin{aligned} 3x - 5 &= 2 \\ 3x - 5 + 5 &= 2 + 5 && \bullet \text{Add 5 to each side of the equation.} \\ 3x &= 7 \\ \frac{3x}{3} &= \frac{7}{3} && \bullet \text{Divide each side of the equation by 3.} \\ x &= \frac{7}{3} \end{aligned}$$

The solution is  $\frac{7}{3}$ .

### » Try Exercise 6, page 12

An equation may contain more than one variable. For these equations, called **literal equations**, we may choose to solve for any one of the variables.

### EXAMPLE 2 » Solve a Literal Equation

Solve  $\frac{by}{c - y} = ax$  for  $y$ .

**Solution**

$$\begin{aligned} \frac{by}{c-y} &= ax \\ \frac{by}{c-y}(c-y) &= ax(c-y) && \bullet \text{ Multiply each side of the equation by } c-y. \\ by &= axc - axy && \bullet \text{ Simplify.} \\ by + axy &= axc && \bullet \text{ Add } axy \text{ to each side of the equation.} \\ y(b + ax) &= axc && \bullet \text{ Factor } y \text{ from the left side of the equation.} \\ y = \frac{axc}{b + ax} & && \bullet \text{ Divide each side of the equation by } b + ax. \end{aligned}$$

» Try Exercise 20, page 12

**Math Matters**

The term *quadratic* is derived from the Latin word *quadrāre*, which means “to make square.” Because the area of a square that measures  $x$  units on each side is  $x^2$ , we refer to equations that can be written in the form  $ax^2 + bx + c = 0$  as equations that are quadratic in  $x$ .

**Definition of a Quadratic Equation**

An equation of the form  $ax^2 + bx + c = 0$ ,  $a \neq 0$ , is a **quadratic equation** in  $x$ .

A quadratic equation can be solved by using the **quadratic formula**.

**The Quadratic Formula**

The solution of the quadratic equation  $ax^2 + bx + c = 0$ ,  $a \neq 0$ , is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**QUESTION** For  $2x^2 - 3x - 1 = 0$ , what are the values of  $a$ ,  $b$ , and  $c$ ?

**Math Matters**

There exists a general procedure to solve “by radicals” the general cubic

$$ax^3 + bx^2 + cx + d = 0$$

and the general quartic

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

However, it has been proved that there are no general procedures that can be used to solve “by radicals” general equations of degree 5 or larger.

**EXAMPLE 3 » Solve a Quadratic Equation**

Solve by using the quadratic formula:  $2x^2 - 4x + 1 = 0$

**Solution**

We have  $a = 2$ ,  $b = -4$ , and  $c = 1$ .

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)} = \frac{4 \pm \sqrt{16 - 8}}{4} \\ &= \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2} \end{aligned}$$

The solutions are  $\frac{2 + \sqrt{2}}{2}$  and  $\frac{2 - \sqrt{2}}{2}$ .

» Try Exercise 30, page 12

**ANSWER**  $a = 2$ ,  $b = -3$ ,  $c = -1$

Although every quadratic equation can be solved using the quadratic formula, it is sometimes easier to factor and use the **zero product principle**.

### Zero Product Principle

If  $a$  and  $b$  are algebraic expressions, then  $ab = 0$  if and only if  $a = 0$  or  $b = 0$ .

#### Example

To solve  $2x^2 + x - 6 = 0$ , first factor the polynomial.

$$2x^2 + x - 6 = 0$$

$$(2x - 3)(x + 2) = 0$$

$$2x - 3 = 0 \quad \text{or} \quad x + 2 = 0 \quad \bullet \text{ Zero product principle}$$

$$x = \frac{3}{2} \qquad \qquad x = -2$$

The solutions are  $\frac{3}{2}$  and  $-2$ .

### EXAMPLE 4 »» Solve by Using the Zero Product Principle

Solve:  $(2x - 1)(x - 3) = x^2 + x - 4$

#### Solution

$$(2x - 1)(x - 3) = x^2 + x - 4$$

$$2x^2 - 7x + 3 = x^2 + x - 4$$

$$x^2 - 8x + 7 = 0$$

$$(x - 7)(x - 1) = 0$$

$$x - 7 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 7 \qquad \qquad x = 1$$

• Expand the binomial product.

• Write as  $ax^2 + bx + c = 0$ .

• Factor.

• Apply the zero product principle.

The solutions are 1 and 7.

»» Try Exercise 44, page 12

### Inequalities

A statement that contains the symbol  $<$ ,  $>$ ,  $\leq$ , or  $\geq$  is called an **inequality**. An inequality expresses the relative order of two mathematical expressions. The **solution set of an inequality** is the set of real numbers each of which, when substituted for the variable, results in a true inequality. The inequality  $x > 4$  is true for any

value of  $x$  greater than 4. For instance, 5,  $\sqrt{21}$ , and  $\frac{17}{3}$  are all solutions of  $x > 4$ .

The solution set of the inequality can be written in set-builder notation as  $\{x | x > 4\}$  or in interval notation as  $(4, \infty)$ .

**Equivalent inequalities** have the same solution set. We solve an inequality by producing *simpler* but equivalent inequalities until the solutions are found. To produce these simpler but equivalent inequalities, we apply the following properties.

### Properties of Inequalities

Let  $a$ ,  $b$ , and  $c$  be real numbers.

1. **Addition Property** Adding the same real number to each side of an inequality preserves the direction of the inequality symbol.

$a < b$  and  $a + c < b + c$  are equivalent inequalities.

2. **Multiplication Properties**

- Multiplying each side of an inequality by the same *positive* real number *preserves* the direction of the inequality symbol.

If  $c > 0$ , then  $a < b$  and  $ac < bc$  are equivalent inequalities.

- Multiplying each side of an inequality by the same *negative* real number *changes* the direction of the inequality symbol.

If  $c < 0$ , then  $a < b$  and  $ac > bc$  are equivalent inequalities.

Note the difference between Property 2a and Property 2b. Property 2a states that an equivalent inequality is produced when each side of a given inequality is multiplied by the same *positive* real number and that the direction of the inequality symbol is *not* changed. By contrast, Property 2b states that when each side of a given inequality is multiplied by a *negative* real number, we must *reverse* the direction of the inequality symbol to produce an equivalent inequality.

For instance,  $-2b < 6$  and  $b > -3$  are equivalent inequalities. (We multiplied each side of the first inequality by  $-\frac{1}{2}$ , and we changed the “less than” symbol to a “greater than” symbol.)

Because subtraction is defined in terms of addition, subtracting the same real number from each side of an inequality does not change the direction of the inequality symbol.

Because division is defined in terms of multiplication, dividing each side of an inequality by the same *positive* real number does *not* change the direction of the inequality symbol, and dividing each side of an inequality by a *negative* real number *changes* the direction of the inequality symbol.

**take note**

Solutions of inequalities are often stated using set-builder notation or interval notation. For instance, the real numbers that are solutions of the inequality in Example 5 can be written in set notation as  $\{x | x > -2\}$  or in interval notation as  $(-2, \infty)$ .

**EXAMPLE 5** » Solve an Inequality

Solve  $2(x + 3) < 4x + 10$ . Write the solution set in set-builder notation.

**Solution**

$$2(x + 3) < 4x + 10$$

$$2x + 6 < 4x + 10$$

$$-2x < 4$$

$$x > -2$$

- Use the distributive property.

- Subtract  $4x$  and  $6$  from each side of the inequality.

- Divide each side by  $-2$  and reverse the inequality symbol.

The solution set is  $\{x | x > -2\}$ .

## » Try Exercise 58, page 12

**Solving Inequalities by the Critical Value Method**

Any value of  $x$  that causes a polynomial in  $x$  to equal zero is called a **zero of the polynomial**. For example,  $-4$  and  $1$  are both zeros of the polynomial  $x^2 + 3x - 4$ , because  $(-4)^2 + 3(-4) - 4 = 0$  and  $1^2 + 3 \cdot 1 - 4 = 0$ .

**A Sign Property of Polynomials**

Nonzero polynomials in  $x$  have the property that for any value of  $x$  between two consecutive real zeros, either all values of the polynomial are positive or all values of the polynomial are negative.

In our work with inequalities that involve polynomials, the real zeros of the polynomial are also referred to as **critical values of the inequality**, because on a number line they separate the real numbers that make the inequality true from those that make it false. In Example 6 we use critical values and the sign property of polynomials to solve an inequality.

**EXAMPLE 6** » Solve a Polynomial Inequality

Solve:  $x^2 + 3x - 4 < 0$

**Solution**

Factoring the polynomial  $x^2 + 3x - 4$  produces the equivalent inequality

$$(x + 4)(x - 1) < 0$$

Thus the zeros of the polynomial  $x^2 + 3x - 4$  are  $-4$  and  $1$ . They are the critical values of the inequality  $x^2 + 3x - 4 < 0$ . They separate the real number line into the three intervals shown in **Figure 1.5**.

To determine the intervals on which  $x^2 + 3x - 4 < 0$ , pick a number called a **test value** from each of the three intervals and then determine



**Figure 1.5**

whether  $x^2 + 3x - 4 < 0$  for each of these test values. For example, in the interval  $(-\infty, -4)$ , pick a test value of, say,  $-5$ . Then

$$x^2 + 3x - 4 = (-5)^2 + 3(-5) - 4 = 6$$

Because 6 is not less than 0, by the sign property of polynomials, no number in the interval  $(-\infty, -4)$  makes  $x^2 + 3x - 4 < 0$ .

Now pick a test value from the interval  $(-4, 1)$ , say, 0. When  $x = 0$ ,

$$x^2 + 3x - 4 = 0^2 + 3(0) - 4 = -4$$

Because  $-4$  is less than 0, by the sign property of polynomials, all numbers in the interval  $(-4, 1)$  make  $x^2 + 3x - 4 < 0$ .

If we pick a test value of 2 from the interval  $(1, \infty)$ , then

$$x^2 + 3x - 4 = (2)^2 + 3(2) - 4 = 6$$

Because 6 is not less than 0, by the sign property of polynomials, no number in the interval  $(1, \infty)$  makes  $x^2 + 3x - 4 < 0$ .

The following table is a summary of our work.

| Interval        | Test Value $x$ | $x^2 + 3x - 4 < 0$                        |
|-----------------|----------------|---|
| $(-\infty, -4)$ | -5             | $(-5)^2 + 3(-5) - 4 < 0$<br>$6 < 0$ False |
| $(-4, 1)$       | 0              | $(0)^2 + 3(0) - 4 < 0$<br>$-4 < 0$ True   |
| $(1, \infty)$   | 2              | $(2)^2 + 3(2) - 4 < 0$<br>$6 < 0$ False   |



Figure 1.6

In interval notation, the solution set of  $x^2 + 3x - 4 < 0$  is  $(-4, 1)$ . The solution set is graphed in Figure 1.6. Note that in this case the critical values  $-4$  and  $1$  are not included in the solution set because they do not make  $x^2 + 3x - 4$  less than 0.

### » Try Exercise 66, page 12

To avoid the arithmetic in Example 6, we often use a *sign diagram*. For example, note that the factor  $(x + 4)$  is negative for all  $x < -4$  and positive for all  $x > -4$ . The factor  $(x - 1)$  is negative for all  $x < 1$  and positive for all  $x > 1$ . These results are shown in Figure 1.7.

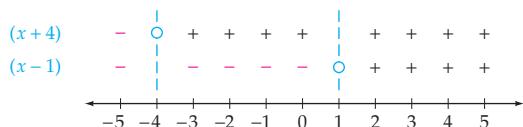


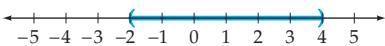
Figure 1.7

To determine on which interval(s) the product  $(x + 4)(x - 1)$  is negative, we examine the sign diagram to see where the factors have opposite signs. This occurs only on the interval  $(-4, 1)$ , where  $(x + 4)$  is positive and  $(x - 1)$  is negative. Thus the original inequality is true only on the interval  $(-4, 1)$ .

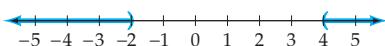
Following is a summary of the steps used to solve polynomial inequalities by the critical value method.

### Solving a Polynomial Inequality by the Critical Value Method

1. Write the inequality so that one side of the inequality is a nonzero polynomial and the other side is 0.
2. Find the real zeros of the polynomial. They are the critical values of the original inequality.
3. Use test values to determine which of the intervals formed by the critical values are to be included in the solution set.
4. Any critical value that satisfies the original inequality is an element of the solution set.



**Figure 1.8**



**Figure 1.9**

### Absolute Value Inequalities

The solution set of the absolute value inequality  $|x - 1| < 3$  is the set of all real numbers whose distance from 1 is *less than* 3. Therefore, the solution set consists of all numbers between  $-2$  and  $4$ . See **Figure 1.8**. In interval notation, the solution set is  $(-2, 4)$ .

The solution set of the absolute value inequality  $|x - 1| > 3$  is the set of all real numbers whose distance from 1 is *greater than* 3. Therefore, the solution set consists of all real numbers less than  $-2$  or greater than  $4$ . See **Figure 1.9**. In interval notation, the solution set is  $(-\infty, -2) \cup (4, \infty)$ .

The following properties are used to solve absolute value inequalities.

### Properties of Absolute Value Inequalities

For any variable expression  $E$  and any nonnegative real number  $k$ ,

$$\begin{aligned} |E| \leq k &\quad \text{if and only if} \quad -k \leq E \leq k \\ |E| \geq k &\quad \text{if and only if} \quad E \leq -k \quad \text{or} \quad E \geq k \end{aligned}$$

### EXAMPLE 7 » Solve an Absolute Value Inequality

Solve:  $|2 - 3x| < 7$

#### Solution

$|2 - 3x| < 7$  implies  $-7 < 2 - 3x < 7$ . Solve this compound inequality.

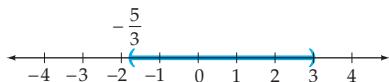
$$-7 < 2 - 3x < 7$$

$$-9 < -3x < 5$$

• Subtract 2 from each of the three parts of the inequality.

$$3 > x > -\frac{5}{3}$$

• Multiply each part of the inequality by  $-\frac{1}{3}$  and reverse the inequality symbols.



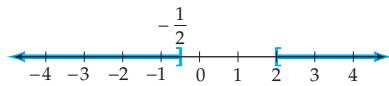
**Figure 1.10**

In interval notation, the solution set is given by  $\left(-\frac{5}{3}, 3\right)$ . See **Figure 1.10**.

» Try Exercise 80, page 13

### take note

Some inequalities have a solution set that consists of all real numbers. For example,  $|x + 9| \geq 0$  is true for all values of  $x$ . Because an absolute value is always nonnegative, the equation is always true.



**Figure 1.11**

### EXAMPLE 8 » Solve an Absolute Value Inequality

Solve:  $|4x - 3| \geq 5$

#### Solution

$|4x - 3| \geq 5$  implies  $4x - 3 \leq -5$  or  $4x - 3 \geq 5$ . Solving each of these inequalities produces

$$\begin{array}{ll} 4x - 3 \leq -5 & \text{or} \\ 4x \leq -2 & \\ x \leq -\frac{1}{2} & \end{array} \quad \begin{array}{ll} 4x - 3 \geq 5 \\ 4x \geq 8 \\ x \geq 2 \end{array}$$

Therefore, the solution set is  $\left(-\infty, -\frac{1}{2}\right] \cup [2, \infty)$ . See **Figure 1.11**.

» Try Exercise 78, page 12



### Topics for Discussion

1. Discuss the similarities and differences among natural numbers, integers, rational numbers, and real numbers.
2. Discuss the differences among an equation, an inequality, and an expression.
3. Is it possible for an equation to have no solution? If not, explain why. If so, give an example of an equation with no solution.
4. Is the statement  $|x| = -x$  ever true? Explain why or why not.
5. How do quadratic equations in one variable differ from linear equations in one variable? Explain how the method used to solve an equation depends on whether it is a linear or a quadratic equation.



## Exercise Set 1.1

In Exercises 1 to 18, solve and check each equation.

1.  $2x + 10 = 40$

2.  $-3y + 20 = 2$

3.  $5x + 2 = 2x - 10$

4.  $4x - 11 = 7x + 20$

5.  $2(x - 3) - 5 = 4(x - 5)$

6.  $6(5s - 11) - 12(2s + 5) = 0$

7.  $\frac{3}{4}x + \frac{1}{2} = \frac{2}{3}$

8.  $\frac{x}{4} - 5 = \frac{1}{2}$

9.  $\frac{2}{3}x - 5 = \frac{1}{2}x - 3$

10.  $\frac{1}{2}x + 7 - \frac{1}{4}x = \frac{19}{2}$

11.  $0.2x + 0.4 = 3.6$

12.  $0.04x - 0.2 = 0.07$

13.  $\frac{3}{5}(n + 5) - \frac{3}{4}(n - 11) = 0$

14.  $-\frac{5}{7}(p + 11) + \frac{2}{5}(2p - 5) = 0$

15.  $3(x + 5)(x - 1) = (3x + 4)(x - 2)$

16.  $5(x + 4)(x - 4) = (x - 3)(5x + 4)$

17.  $0.08x + 0.12(4000 - x) = 432$

18.  $0.075y + 0.06(10,000 - y) = 727.50$

In Exercises 19 to 26, solve each equation for the indicated variable.

19.  $x + 2y = 8; y$

20.  $3x - 5y = 15; y$

21.  $2x + 5y = 10; x$

22.  $5x - 4y = 10; x$

23.  $ay - by = c; y$

24.  $ax + by = c; y$

25.  $x = \frac{y}{1 - y}; y$

26.  $x = \frac{2y - 3}{y - 1}; y$

In Exercises 27 to 40, solve by using the quadratic formula.

27.  $x^2 - 2x - 15 = 0$

28.  $x^2 - 5x - 24 = 0$

29.  $x^2 + x - 1 = 0$

30.  $x^2 + x - 2 = 0$

31.  $2x^2 + 4x + 1 = 0$

32.  $2x^2 + 4x - 1 = 0$

33.  $3x^2 - 5x - 3 = 0$

34.  $3x^2 - 5x - 4 = 0$

35.  $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$

36.  $\frac{2}{3}x^2 - 5x + \frac{1}{2} = 0$

37.  $\sqrt{2}x^2 + 3x + \sqrt{2} = 0$

38.  $2x^2 + \sqrt{5}x - 3 = 0$

39.  $x^2 = 3x + 5$

40.  $-x^2 = 7x - 1$

In Exercises 41 to 48, solve each quadratic equation by factoring and applying the zero product property.

41.  $x^2 - 2x - 15 = 0$

42.  $y^2 + 3y - 10 = 0$

43.  $8y^2 + 189y - 72 = 0$

44.  $12w^2 - 41w + 24 = 0$

45.  $3x^2 - 7x = 0$

46.  $5x^2 = -8x$

47.  $(x - 5)^2 - 9 = 0$

48.  $(3x + 4)^2 - 16 = 0$

In Exercises 49 to 58, use the properties of inequalities to solve each inequality. Write answers using interval notation.

49.  $2x + 3 < 11$

50.  $3x - 5 > 16$

51.  $x + 4 > 3x + 16$

52.  $5x + 6 < 2x + 1$

53.  $-6x + 1 \geq 19$

54.  $-5x + 2 \leq 37$

55.  $-3(x + 2) \leq 5x + 7$

56.  $-4(x - 5) \geq 2x + 15$

57.  $-4(3x - 5) > 2(x - 4)$

58.  $3(x + 7) \leq 5(2x - 8)$

In Exercises 59 to 66, use the critical value method to solve each inequality. Use interval notation to write each solution set.

59.  $x^2 + 7x > 0$

60.  $x^2 - 5x \leq 0$

61.  $x^2 + 7x + 10 < 0$

62.  $x^2 + 5x + 6 < 0$

63.  $x^2 - 3x \geq 28$

64.  $x^2 < -x + 30$

65.  $6x^2 - 4 \leq 5x$

66.  $12x^2 + 8x \geq 15$

In Exercises 67 to 84, use interval notation to express the solution set of each inequality.

67.  $|x| < 4$

68.  $|x| > 2$

69.  $|x - 1| < 9$

70.  $|x - 3| < 10$

71.  $|x + 3| > 30$

72.  $|x + 4| < 2$

73.  $|2x - 1| > 4$

74.  $|2x - 9| < 7$

75.  $|x + 3| \geq 5$

76.  $|x - 10| \geq 2$

77.  $|3x - 10| \leq 14$

78.  $|2x - 5| \geq 1$

79.  $|4 - 5x| \geq 24$

80.  $|3 - 2x| \leq 5$

81.  $|x - 5| \geq 0$

82.  $|x - 7| \geq 0$

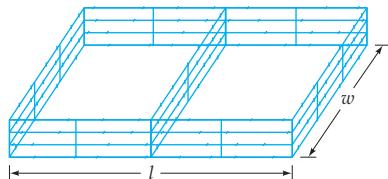
83.  $|x - 4| \leq 0$

84.  $|2x + 7| \leq 0$

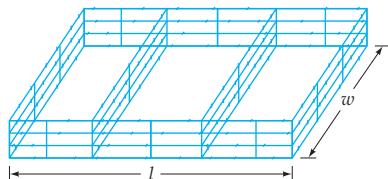
- 85. GEOMETRY** The perimeter of a rectangle is 27 centimeters, and its area is 35 square centimeters. Find the length and width of the rectangle.

- 86. GEOMETRY** The perimeter of a rectangle is 34 feet and its area is 60 square feet. Find the length and width of the rectangle.

- 87. RECTANGULAR ENCLOSURE** A gardener wishes to use 600 feet of fencing to enclose a rectangular region and subdivide the region into two smaller rectangles. The total enclosed area is 15,000 square feet. Find the dimensions of the enclosed region.



- 88. RECTANGULAR ENCLOSURE** A farmer wishes to use 400 yards of fencing to enclose a rectangular region and subdivide the region into three smaller rectangles. If the total enclosed area is 400 square yards, find the dimensions of the enclosed region.



- 89. PERSONAL FINANCE** A bank offers two checking account plans. The monthly fee and charge per check for each plan are shown in the table at the top of the next column. Under what conditions is it less expensive to use the LowCharge plan?

| Account Plan | Monthly Fee | Charge per Check |
|--------------|-------------|------------------|
| LowCharge    | \$5.00      | \$.01            |
| FeeSaver     | \$1.00      | \$.08            |

- 90. PERSONAL FINANCE** You can rent a car for the day from company A for \$29.00 plus \$0.12 a mile. Company B charges \$22.00 plus \$0.21 a mile. Find the number of miles  $m$  (to the nearest mile) per day for which it is cheaper to rent from company A.

- 91. PERSONAL FINANCE** A sales clerk has a choice between two payment plans. Plan A pays \$100.00 a week plus \$8.00 a sale. Plan B pays \$250.00 a week plus \$3.50 a sale. How many sales per week must be made for plan A to yield the greater paycheck?

- 92. PERSONAL FINANCE** A video store offers two rental plans. The yearly membership fee and the daily charge per video are shown below. How many videos can be rented per year if the No-fee plan is to be the less expensive of the plans?

| THE VIDEO STORE |            |                        |
|-----------------|------------|------------------------|
| Rental Plan     | Yearly Fee | Daily Charge per Video |
| Low-rate        | \$15.00    | \$1.49                 |
| No-fee          | None       | \$1.99                 |

- 93. AVERAGE TEMPERATURES** The average daily minimum-to-maximum temperature range for the city of Palm Springs during the month of September is  $68^{\circ}\text{F}$  to  $104^{\circ}\text{F}$ . What is the corresponding temperature range measured on the Celsius temperature scale? (Hint: Let  $F$  be the average daily temperature. Then  $68 \leq F \leq 104$ . Now substitute  $\frac{9}{5}C + 32$  for  $F$  and solve the resulting inequality for  $C$ .)

- 94. A GOLDEN RECTANGLE** The ancient Greeks defined a rectangle as a *golden rectangle* if its length  $l$  and its width  $w$  satisfied the equation

$$\frac{l}{w} = \frac{w}{l-w}$$

- a. Solve this formula for  $w$ .
  - b. If the length of a golden rectangle is 101 feet, determine its width. Round to the nearest hundredth of a foot.

- 95. SUM OF NATURAL NUMBERS** The sum  $S$  of the first  $n$  natural numbers  $1, 2, 3, \dots, n$  is given by the formula

$$S = \frac{n}{2}(n + 1)$$

How many consecutive natural numbers starting with 1 produce a sum of 253?

- 96. NUMBER OF DIAGONALS** The number of diagonals  $D$  of a polygon with  $n$  sides is given by the formula

$$D = \frac{n}{2}(n - 3)$$

- a. Determine the number of sides of a polygon with 464 diagonals.
  - b. Can a polygon have 12 diagonals? Explain.

- 97. REVENUE** The monthly revenue  $R$  for a product is given by  $R = 420x - 2x^2$ , where  $x$  is the price in dollars of each unit produced. Find the interval in terms of  $x$  for which the monthly revenue is greater than zero.

- 98. ABSOLUTE VALUE INEQUALITIES** Write an absolute value inequality to represent all real numbers within

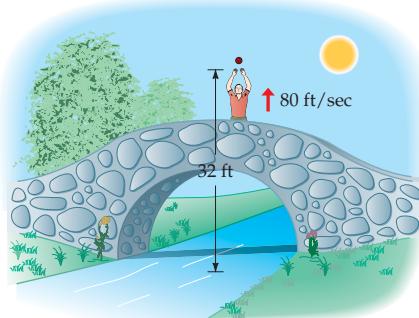
- a. 8 units of 3
  - b.  $k$  units of  $j$  (assume  $k > 0$ )

- 99. HEIGHT OF A PROJECTILE** The equation

$$s = -16t^2 + v_0 t + s_0$$

gives the height  $s$ , in feet above ground level, at the time  $t$  seconds after an object is thrown directly upward from a height  $s_0$  feet above the ground with an initial velocity of  $v_0$  feet per second. A ball is thrown directly upward from ground level with an initial velocity of 64 feet per second. Find the time interval during which the ball attains a height of more than 48 feet.

- 100. HEIGHT OF A PROJECTILE** A ball is thrown directly upward from a height of 32 feet above a stream with an initial velocity of 80 feet per second. Find the time interval during which the ball will be more than 96 feet above the stream. (*Hint:* See Exercise 99.)



- 101. GEOMETRY** The length of the side of a square has been measured accurately to within 0.01 foot. This measured length is 4.25 feet.

- a. Write an absolute value inequality that describes the relationship between the actual length of each side of the square  $s$  and its measured length.
  - b. Solve the absolute value inequality you found in a. for  $s$ .

- 1.**  **TEACHING MATHEMATICS** Prepare a lesson that you could use to explain to someone how to solve linear and quadratic equations. Be sure to include an explanation of the differences between these two types of equations and the different methods that are used to solve them.

- 2.**  **CUBIC EQUATIONS** Write an essay on the development of the solution of the cubic equation. An excellent source of information is the chapter "Cardano and the Solution of the Cubic" in *Journey Through Genius* by William Dunham (New York: Wiley, 1990). Another excellent source is *A History of Mathematics: An Introduction* by Victor J. Katz (New York: Harper Collins, 1993).

## Section 1.2

- Cartesian Coordinate Systems
- The Distance and Midpoint Formulas
- Graph of an Equation
- Intercepts
- Circles, Their Equations, and Their Graphs

# A Two-Dimensional Coordinate System and Graphs

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A1.

**PS1.** Evaluate  $\frac{x_1 + x_2}{2}$  when  $x_1 = 4$  and  $x_2 = -7$ .

**PS2.** Simplify  $\sqrt{50}$ .

**PS3.** Is  $y = 3x - 2$  a true equation when  $y = 5$  and  $x = -1$ ? [1.1]

**PS4.** If  $y = x^2 - 3x - 2$ , find  $y$  when  $x = -3$ . [1.1]

**PS5.** Evaluate  $| -x - y |$  when  $x = 3$  and  $y = -1$ . [1.1]

**PS6.** Evaluate  $\sqrt{b^2 - 4ac}$  when  $a = -2$ ,  $b = -3$ , and  $c = 2$ .

## Cartesian Coordinate Systems

### take note

Abcissa comes from the same root word as scissors. An open pair of scissors looks like an x.

### Math Matters

The concepts of *analytic geometry* developed over an extended period of time, culminating in 1637 with the publication of two works: *Discourse on the Method for Rightly Directing One's Reason and Searching for Truth in the Sciences* by René Descartes (1596–1650) and *Introduction to Plane and Solid Loci* by Pierre de Fermat. Each of these works was an attempt to integrate the study of geometry with the study of algebra. Of the two mathematicians, Descartes is usually given most of the credit for developing analytic geometry. In fact, Descartes became so famous in La Haye, the city in which he was born, that it was renamed La Haye-Descartes.

Each point on a coordinate axis is associated with a number called its **coordinate**. Each point on a flat, two-dimensional surface, called a **coordinate plane** or *xy*-plane, is associated with an **ordered pair** of numbers called **coordinates** of the point. Ordered pairs are denoted by  $(a, b)$ , where the real number  $a$  is the **x-coordinate** or **abscissa** and the real number  $b$  is the **y-coordinate** or **ordinate**.

The coordinates of a point are determined by the point's position relative to a horizontal coordinate axis called the **x-axis** and a vertical coordinate axis called the **y-axis**. The axes intersect at the point  $(0, 0)$ , called the **origin**. In **Figure 1.12**, the axes are labeled such that positive numbers appear to the right of the origin on the *x*-axis and above the origin on the *y*-axis. The four regions formed by the axes are called **quadrants** and are numbered counterclockwise. This two-dimensional coordinate system is referred to as a **Cartesian coordinate system** in honor of René Descartes.

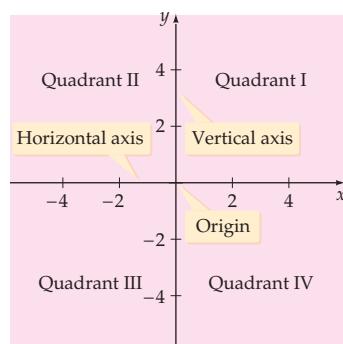


Figure 1.12

**take note**

The notation  $(a, b)$  was used earlier to denote an interval on a one-dimensional number line. In this section,  $(a, b)$  denotes an ordered pair in a two-dimensional plane. This should not cause confusion in future sections because as each mathematical topic is introduced, it will be clear whether a one-dimensional or a two-dimensional coordinate system is involved.

To **plot a point**  $P(a, b)$  means to draw a dot at its location in the coordinate plane. In **Figure 1.13** we have plotted the points  $(4, 3)$ ,  $(-3, 1)$ ,  $(-2, -3)$ ,  $(3, -2)$ ,  $(0, 1)$ ,  $(1, 3)$ , and  $(3, 1)$ . Note that  $(1, 3)$  and  $(3, 1)$  are not the same point. The order in which the coordinates of an ordered pair are listed is important.

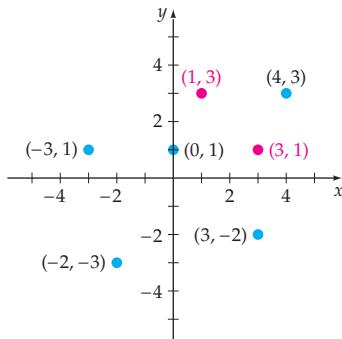


Figure 1.13



Data often are displayed in visual form as a set of points called a **scatter diagram** or **scatter plot**. For instance, the scatter diagram in **Figure 1.14** shows the current and projected revenues of Web-filtering software vendors. (Web-filtering software allows businesses to control which Internet sites are available to employees while at work.) The point whose coordinates are approximately  $(2005, 520)$  means that in the year 2005, approximately \$520 million in revenues were generated by companies that supplied this software. The line segments that connect the points in **Figure 1.14** help illustrate trends.

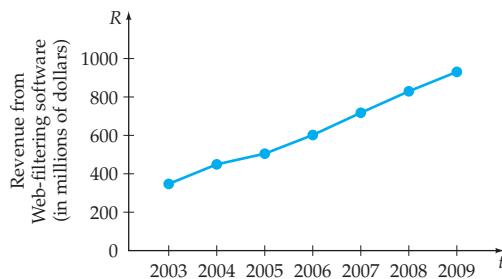


Figure 1.14

Source: IDC, 2005

**QUESTION** From **Figure 1.14**, will the revenues from Web-filtering software in 2009 be more or less than twice the revenues in 2003?

In some instances, it is important to know when two ordered pairs are equal.

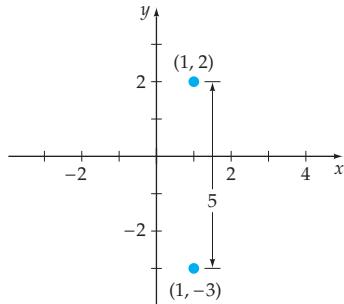
**ANSWER** More. The revenue in 2003 was about \$350 million. The projected revenue in 2009 is about \$925 million, more than twice \$350 million.

### Definition of the Equality of Ordered Pairs

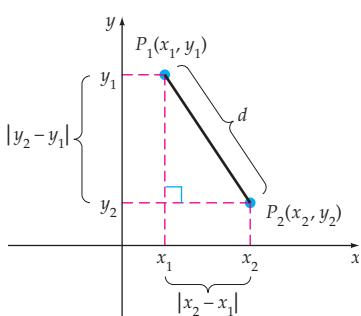
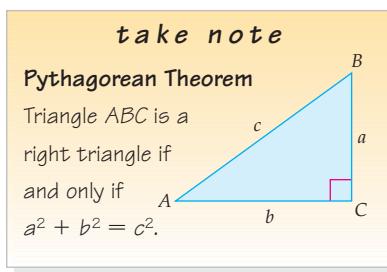
The ordered pairs  $(a, b)$  and  $(c, d)$  are equal if and only if  $a = c$  and  $b = d$ .

#### Example

If  $(3, y) = (x, -2)$ , then  $x = 3$  and  $y = -2$ .



**Figure 1.15**



**Figure 1.16**

$$\begin{aligned}d^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\d &= \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\&= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\end{aligned}$$

- Take the square root of each side of the equation. Because  $d$  is nonnegative, the negative root is not listed.
- Because  $|x_2 - x_1|^2 = (x_2 - x_1)^2$  and  $|y_2 - y_1|^2 = (y_2 - y_1)^2$

Thus we have established the following theorem.

### The Distance Formula

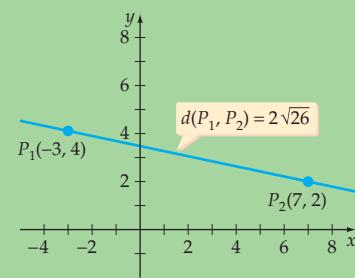
The distance  $d(P_1, P_2)$  between the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

#### Example

The distance between  $P_1(-3, 4)$  and  $P_2(7, 2)$  is given by

$$\begin{aligned}d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\&= \sqrt{[7 - (-3)]^2 + (2 - 4)^2} \\&= \sqrt{10^2 + (-2)^2} \\&= \sqrt{104} = 2\sqrt{26} \approx 10.2\end{aligned}$$



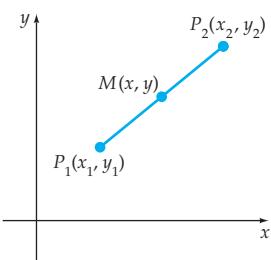


Figure 1.17

The **midpoint**  $M$  of a line segment is the point on the line segment that is equidistant from the endpoints  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  of the segment. See **Figure 1.17**.

### The Midpoint Formula

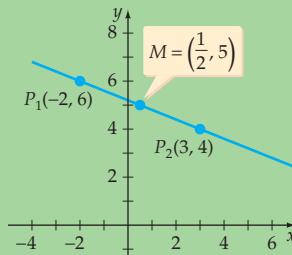
The midpoint  $M$  of the line segment from  $P_1(x_1, y_1)$  to  $P_2(x_2, y_2)$  is given by

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

#### Example

The midpoint of the line segment between  $P_1(-2, 6)$  and  $P_2(3, 4)$  is given by

$$\begin{aligned} M &= \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\ &= \left( \frac{(-2) + 3}{2}, \frac{6 + 4}{2} \right) \\ &= \left( \frac{1}{2}, 5 \right) \end{aligned}$$



The midpoint formula states that the  $x$ -coordinate of the midpoint of a line segment is the *average* of the  $x$ -coordinates of the endpoints of the line segment and that the  $y$ -coordinate of the midpoint of a line segment is the *average* of the  $y$ -coordinates of the endpoints of the line segment.

### EXAMPLE 1 » Find the Midpoint and Length of a Line Segment

Find the midpoint and the length of the line segment connecting the points whose coordinates are  $P_1(-4, 3)$  and  $P_2(4, -2)$ .

#### Solution

$$\begin{aligned} \text{Midpoint} &= \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\ &= \left( \frac{-4 + 4}{2}, \frac{3 + (-2)}{2} \right) \\ &= \left( 0, \frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-4))^2 + (-2 - 3)^2} = \sqrt{(8)^2 + (-5)^2} \\ &= \sqrt{64 + 25} = \sqrt{89} \end{aligned}$$

» Try Exercise 6, page 28

## Graph of an Equation

The equations below are equations in two variables.

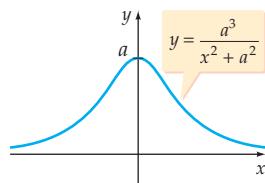
$$y = 3x^3 - 4x + 2 \quad x^2 + y^2 = 25 \quad y = \frac{x}{x+1}$$

### Math Matters

Maria Agnesi (1718–1799) wrote *Foundations of Analysis for the Use of Italian Youth*, one of the most successful textbooks of the 18th century. The French Academy authorized a translation into French in 1749, noting that “there is no other book, in any language, which would enable a reader to penetrate as deeply, or as rapidly, into the fundamental concepts of analysis.” A curve that she discusses in her text is given by the equation

$$y = \frac{a^3}{x^2 + a^2}.$$

Unfortunately, due to a translation error from Italian to English, the curve became known as the “witch of Agnesi.”



The solution of an equation in two variables is an ordered pair  $(x, y)$  whose coordinates satisfy the equation. For instance, the ordered pairs  $(3, 4)$ ,  $(4, -3)$ , and  $(0, 5)$  are some of the solutions of  $x^2 + y^2 = 25$ . Generally, there are an infinite number of solutions of an equation in two variables. These solutions can be displayed in a graph.

### Definition of the Graph of an Equation

The **graph of an equation** in the two variables  $x$  and  $y$  is the set of all points  $(x, y)$  whose coordinates satisfy the equation.

Consider  $y = 2x - 1$ . Substituting various values of  $x$  into the equation and solving for  $y$  produces some of the ordered pairs that satisfy the equation. It is convenient to record the results in a table similar to the one shown below. The graph of the ordered pairs is shown in **Figure 1.18**.

| $x$ | $y = 2x - 1$ | $y$ | $(x, y)$   |
|-----|--------------|-----|------------|
| -2  | $2(-2) - 1$  | -5  | $(-2, -5)$ |
| -1  | $2(-1) - 1$  | -3  | $(-1, -3)$ |
| 0   | $2(0) - 1$   | -1  | $(0, -1)$  |
| 1   | $2(1) - 1$   | 1   | $(1, 1)$   |
| 2   | $2(2) - 1$   | 3   | $(2, 3)$   |

Choosing some noninteger values of  $x$  produces more ordered pairs to graph, such as  $\left(-\frac{3}{2}, -4\right)$  and  $\left(\frac{5}{2}, 4\right)$ , as shown in **Figure 1.19**. Using still other values of  $x$  would result in more and more ordered pairs to graph. The result would be so many dots that the graph would appear as the straight line shown in **Figure 1.20**, which is the graph of  $y = 2x - 1$ .

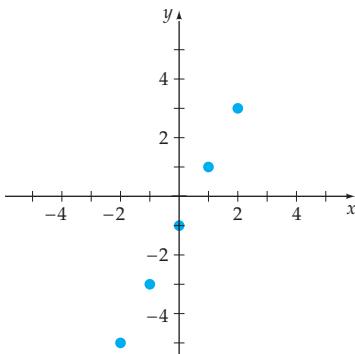


Figure 1.18

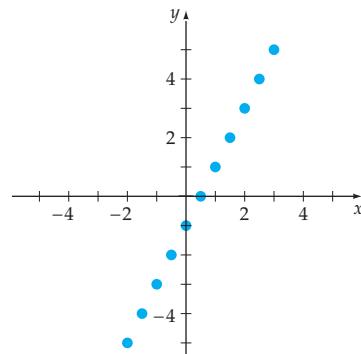


Figure 1.19

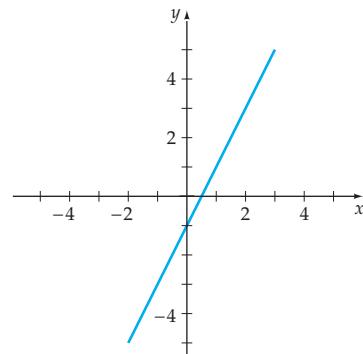


Figure 1.20

**E X A M P L E 2 »» Draw a Graph by Plotting Points**

Graph:  $-x^2 + y = 1$

**Solution**

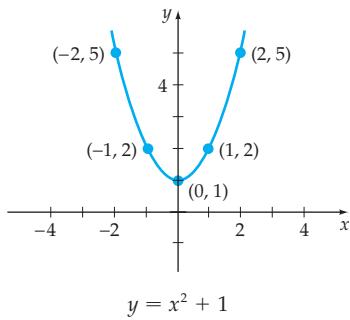
Solve the equation for  $y$ .

$$-x^2 + y = 1$$

$$y = x^2 + 1$$

• Add  $x^2$  to each side.

Select values of  $x$  and use the equation to calculate  $y$ . Choose enough values of  $x$  so that an accurate graph can be drawn. Plot the points and draw a smooth curve through them. See **Figure 1.21**.



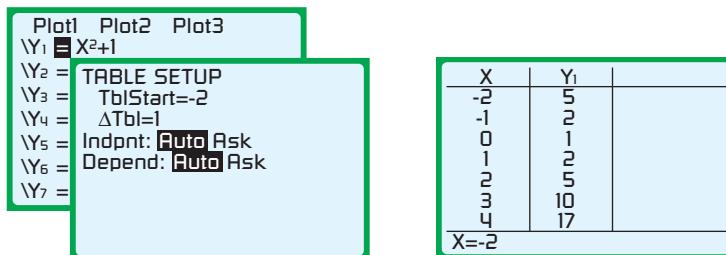
**Figure 1.21**

| $x$ | $y = x^2 + 1$ | $y$ | $(x, y)$ |
|-----|---------------|-----|----------|
| -2  | $(-2)^2 + 1$  | 5   | (-2, 5)  |
| -1  | $(-1)^2 + 1$  | 2   | (-1, 2)  |
| 0   | $(0)^2 + 1$   | 1   | (0, 1)   |
| 1   | $(1)^2 + 1$   | 2   | (1, 2)   |
| 2   | $(2)^2 + 1$   | 5   | (2, 5)   |

**»» Try Exercise 26, page 28**

**Integrating Technology**

Some graphing calculators, such as the TI-83/TI-83 Plus/TI-84 Plus, have a TABLE feature that allows you to create a table similar to the one shown in Example 2. Enter the equation to be graphed, the first value for  $x$ , and the increment (the difference between successive values of  $x$ ). For instance, entering  $y_1 = x^2 + 1$ , an initial value of  $x$  of  $-2$ , and an increment of  $1$  yields a display similar to the one in **Figure 1.22**. Changing the initial value to  $-6$  and the increment to  $2$  gives the table in **Figure 1.23**.



**Figure 1.22**

| TABLE SETUP    |    |  |
|----------------|----|--|
| TblStart=-6    | 37 |  |
| $\Delta Tbl=2$ | 17 |  |
| Indpt: Auto    | 5  |  |
| Depend: Auto   | 1  |  |
|                | 5  |  |
|                | 17 |  |
|                | 37 |  |
| X=-6           |    |  |

Figure 1.23

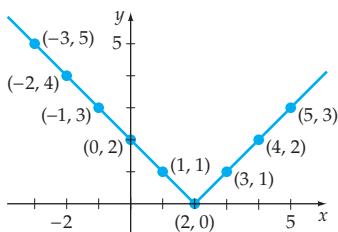
With some calculators, you can scroll through the table by using the up- or down-arrow keys. In this way, you can determine many more ordered pairs of the graph.

### EXAMPLE 3 » Graph by Plotting Points

Graph:  $y = |x - 2|$

#### Solution

This equation is already solved for  $y$ , so start by choosing an  $x$  value and using the equation to determine the corresponding  $y$  value. For example, if  $x = -3$ , then  $y = |(-3) - 2| = |-5| = 5$ . Continuing in this manner produces the following table.



$$y = |x - 2|$$

Figure 1.24

| When $x$ is | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|-------------|----|----|----|---|---|---|---|---|---|
| $y$ is      | 5  | 4  | 3  | 2 | 1 | 0 | 1 | 2 | 3 |

Now plot the points listed in the table. Connecting the points forms a V shape, as shown in **Figure 1.24**.

» Try Exercise 30, page 28

### EXAMPLE 4 » Graph by Plotting Points

Graph:  $y^2 = x$

#### Solution

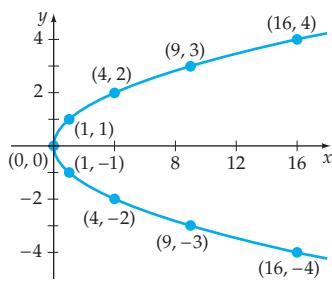
Solve the equation for  $y$ .

$$y^2 = x$$

$$y = \pm\sqrt{x}$$

• Take the square root of each side.

Continued ►



$$y^2 = x$$

**Figure 1.25**

Choose several  $x$  values, and use the equation to determine the corresponding  $y$  values.

|                               |   |         |         |         |         |
|-------------------------------|---|---------|---------|---------|---------|
| <b>When <math>x</math> is</b> | 0 | 1       | 4       | 9       | 16      |
| <b><math>y</math> is</b>      | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ |

Plot the points as shown in **Figure 1.25**. The graph is a *parabola*.

» Try Exercise 32, page 28



### Integrating Technology

A graphing calculator or computer graphing software can be used to draw the graphs in Examples 3 and 4. These graphing utilities graph a curve in much the same way as you would, by selecting values of  $x$  and calculating the corresponding values of  $y$ . A curve is then drawn through the points.

If you use a graphing utility to graph  $y = |x - 2|$ , you will need to use the *absolute value* function that is built into the utility. The equation you enter will look similar to  $Y_1 = \text{abs}(X - 2)$ .

To graph the equation in Example 4, you will enter two equations. The equations you enter will be similar to

$$Y_1 = \sqrt{|X|}$$

$$Y_2 = -\sqrt{|X|}$$

The graph of the first equation will be the top half of the parabola; the graph of the second equation will be the bottom half.

### Intercepts

Any point that has an  $x$ - or a  $y$ -coordinate of zero is called an **intercept** of the graph of an equation because it is at these points that the graph intersects the  $x$ - or the  $y$ -axis.

#### Definitions of $x$ -Intercepts and $y$ -Intercepts

If  $(x_1, 0)$  satisfies an equation, then the point  $(x_1, 0)$  is called an  **$x$ -intercept** of the graph of the equation.

If  $(0, y_1)$  satisfies an equation, then the point  $(0, y_1)$  is called a  **$y$ -intercept** of the graph of the equation.

To find the  $x$ -intercepts of the graph of an equation, let  $y = 0$  and solve the equation for  $x$ . To find the  $y$ -intercepts of the graph of an equation, let  $x = 0$  and solve the equation for  $y$ .

**EXAMPLE 5** » Find  $x$ - and  $y$ -Intercepts

Find the  $x$ - and  $y$ -intercepts of the graph of  $y = x^2 - 2x - 3$ .

**ALGEBRAIC SOLUTION**

To find the  $y$ -intercept, let  $x = 0$  and solve for  $y$ .

$$y = 0^2 - 2(0) - 3 = -3$$

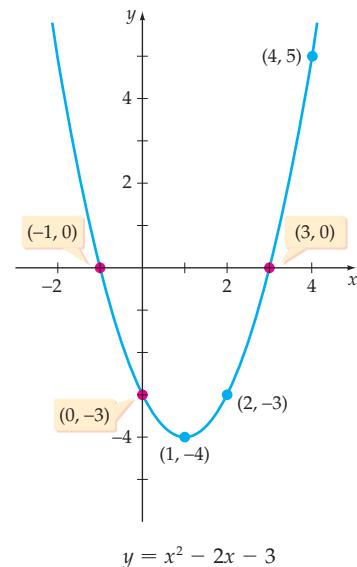
To find the  $x$ -intercepts, let  $y = 0$  and solve for  $x$ .

$$\begin{aligned} 0 &= x^2 - 2x - 3 \\ 0 &= (x - 3)(x + 1) \\ (x - 3) &= 0 \quad \text{or} \quad (x + 1) = 0 \\ x &= 3 \quad \text{or} \quad x = -1 \end{aligned}$$

Because  $y = -3$  when  $x = 0$ ,  $(0, -3)$  is a  $y$ -intercept. Because  $x = 3$  or  $-1$  when  $y = 0$ ,  $(3, 0)$  and  $(-1, 0)$  are  $x$ -intercepts. **Figure 1.26** confirms that these three points are intercepts.

**VISUALIZE THE SOLUTION**

The graph of  $y = x^2 - 2x - 3$  is shown below. Observe that the graph intersects the  $x$ -axis at  $(-1, 0)$  and  $(3, 0)$ , the  $x$ -intercepts. The graph also intersects the  $y$ -axis at  $(0, -3)$ , the  $y$ -intercept.



**Figure 1.26**

» Try Exercise 40, page 28

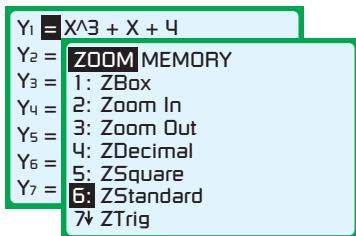
**Integrating Technology**

In Example 5 it was possible to find the  $x$ -intercepts by solving a quadratic equation. In some instances, however, solving an equation to find the intercepts may be very difficult. In these cases, a graphing calculator can be used to estimate the  $x$ -intercepts.

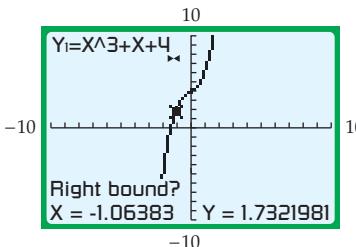
The  $x$ -intercepts of the graph of  $y = x^3 + x + 4$  can be estimated using the ZERO feature of a TI-83/TI-83 Plus/TI-84 Plus calculator. The keystrokes and some sample screens for this procedure are shown on page 24.

**Continued ▶**

Press **[Y=]**. Now enter  $X^3+X+4$ . Press **[ZOOM]** and select the standard viewing window. Press **[ENTER]**.



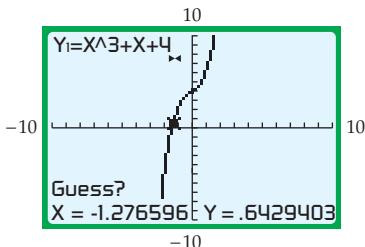
The “Right Bound?” shown on the bottom of the screen means to move the cursor until it is to the right of the desired  $x$ -intercept. Press **[ENTER]**.



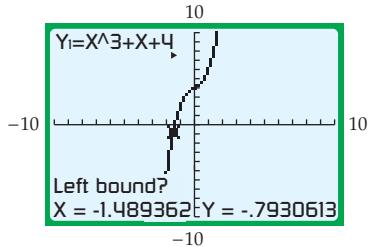
Press **[2nd]** CALC to access the CALCULATE menu. The  $y$ -coordinate of an  $x$ -intercept is zero. Therefore, select 2:zero. Press **[ENTER]**.



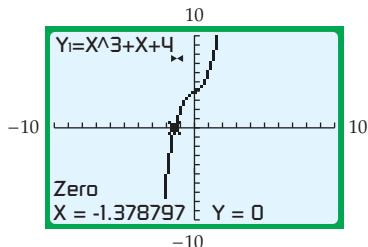
“Guess?” is shown on the bottom of the screen. Move the cursor until it is approximately on the  $x$ -intercept. Press **[ENTER]**.



The “Left Bound?” shown on the bottom of the screen means to move the cursor until it is to the left of an  $x$ -intercept. Press **[ENTER]**.



The “Zero” shown on the bottom of the screen means that the value of  $y$  is 0 when  $x = -1.378797$ . The  $x$ -intercept is about  $(-1.378797, 0)$ .



## Circles, Their Equations, and Their Graphs

Frequently you will sketch graphs by plotting points. However, some graphs can be sketched merely by recognizing the form of the equation. A *circle* is an example of a curve whose graph you can sketch after you have inspected its equation.

### Definition of a Circle

A **circle** is the set of points in a plane that are a fixed distance from a specified point. The fixed distance is the **radius** of the circle, and the specified point is the **center** of the circle.

The standard form of the equation of a circle is derived by using this definition. To derive the standard form, we use the distance formula. **Figure 1.27** is a

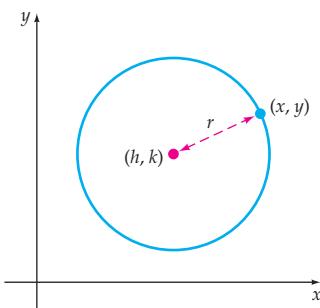


Figure 1.27

circle with center  $(h, k)$  and radius  $r$ . The point  $(x, y)$  is on the circle if and only if it is a distance of  $r$  units from the center  $(h, k)$ . Thus  $(x, y)$  is on the circle if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \quad \bullet \text{ Square each side.}$$

### Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** with center at  $(h, k)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 = r^2$$

#### Example

- The equation  $(x - 2)^2 + (y - 4)^2 = 3^2$  is in standard form, where  $h = 2$ ,  $k = 4$ , and  $r = 3$ . The graph of this equation is a circle with center  $C(2, 4)$  and radius 3.
- The equation  $(x - 3)^2 + (y + 1)^2 = 25$  can be written in standard form as  $(x - 3)^2 + (y - (-1))^2 = 5^2$ . Note that in the standard form,  $(x - h)^2$  and  $(y - k)^2$  are written using subtraction. Because  $(y + 1)^2$  is written using addition, the expression is rewritten as  $(y - (-1))^2$ . The graph of this equation is a circle with center  $C(3, -1)$  and radius 5.
- The equation  $(x + 4)^2 + (y + 2)^2 = 10$  can be written in standard form as  $(x - (-4))^2 + (y - (-2))^2 = (\sqrt{10})^2$ . The graph of this equation is a circle with center  $C(-4, -2)$  and radius  $\sqrt{10}$ .

If a circle is centered at the origin  $(0, 0)$ , then  $h = 0$  and  $k = 0$  and the standard form of the equation of the circle simplifies to

$$x^2 + y^2 = r^2 \quad \bullet \text{ Equation of a circle with center at the origin and radius } r.$$

For instance,  $x^2 + y^2 = 9$  is the equation of the circle with center at the origin and radius  $\sqrt{9} = 3$ .

**QUESTION** What are the radius and the coordinates of the center of the circle with equation  $x^2 + (y - 2)^2 = 30$ ?

### EXAMPLE 6 » Find the Standard Form of the Equation of a Circle

Find the standard form of the equation of the circle that has center  $C(-4, -2)$  and contains the point  $P(-1, 2)$ .

Continued ►

**ANSWER** The radius is  $\sqrt{30}$  and the coordinates of the center are  $(0, 2)$ .

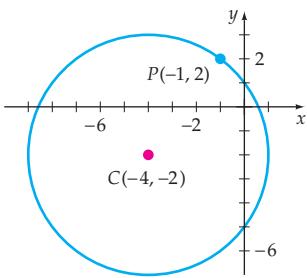


Figure 1.28

**Solution**

See the graph of the circle in **Figure 1.28**. Because the point  $P$  is on the circle, the radius  $r$  of the circle must equal the distance from  $C$  to  $P$ . Thus

$$\begin{aligned} r &= \sqrt{(-1 - (-4))^2 + (2 - (-2))^2} \\ &= \sqrt{9 + 16} = \sqrt{25} = 5 \end{aligned}$$

Using the standard form with  $h = -4$ ,  $k = -2$ , and  $r = 5$ , we obtain

$$(x + 4)^2 + (y + 2)^2 = 5^2$$

**» Try Exercise 64, page 28**

If we rewrite  $(x + 4)^2 + (y + 2)^2 = 5^2$  by squaring and combining like terms, we produce

$$\begin{aligned} x^2 + 8x + 16 + y^2 + 4y + 4 &= 25 \\ x^2 + y^2 + 8x + 4y - 5 &= 0 \end{aligned}$$

This form of the equation is known as the **general form of the equation of a circle**. By completing the square, it is always possible to write the general form  $x^2 + y^2 + Ax + By + C = 0$  in the standard form

$$(x - h)^2 + (y - k)^2 = s$$

for some number  $s$ . If  $s > 0$ , the graph is a circle with radius  $r = \sqrt{s}$ . If  $s = 0$ , the graph is the point  $(h, k)$ . If  $s < 0$ , the equation has no real solutions and there is no graph.

**E X A M P L E 7 » Find the Center and Radius of a Circle by Completing the Square**

Find the center and radius of the circle given by

$$x^2 + y^2 - 6x + 4y - 3 = 0$$

**Solution**

First rearrange and group the terms as shown.

$$(x^2 - 6x) + (y^2 + 4y) = 3$$

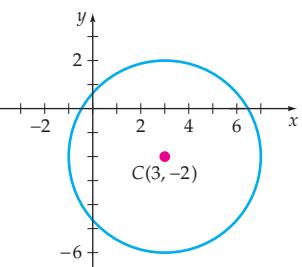
Complete the square of  $(x^2 - 6x)$  by adding 9, and complete the square of  $(y^2 + 4y)$  by adding 4.

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 3 + 9 + 4$$

$$(x - 3)^2 + (y + 2)^2 = 16$$

**• Add 9 and 4 to each side of the equation.**

$$(x - 3)^2 + (y - (-2))^2 = 4^2$$



$$x^2 + y^2 - 6x + 4y - 3 = 0$$

Figure 1.29

This equation is the standard form of the equation of a circle and indicates that the graph of the original equation is a circle centered at  $(3, -2)$  with radius 4. See **Figure 1.29**.

**» Try Exercise 66, page 29**



## Topics for Discussion

- The distance formula states that the distance  $d$  between the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ . Can the distance formula also be written as follows? Explain.
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
- Does the equation  $(x - 3)^2 + (y + 4)^2 = -6$  have a graph that is a circle? Explain.
- Explain why the graph of  $|x| + |y| = 1$  does not contain any points that have
  - a  $y$ -coordinate that is greater than 1 or less than  $-1$
  - an  $x$ -coordinate that is greater than 1 or less than  $-1$
- Discuss the graph of  $xy = 0$ .
- Explain how to determine the  $x$ - and  $y$ -intercepts of a graph defined by an equation (without using the graph).

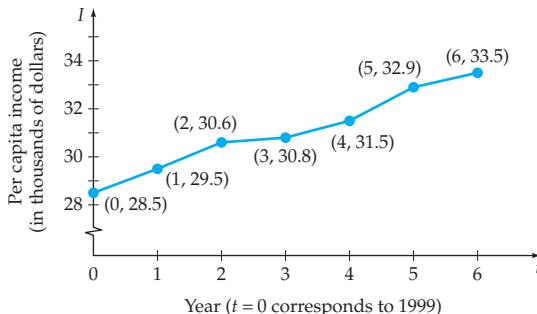


## Exercise Set 1.2

In Exercises 1 and 2, plot the points whose coordinates are given on a Cartesian coordinate system.

- $(2, 4), (0, -3), (-2, 1), (-5, -3)$
- $(-3, -5), (-4, 3), (0, 2), (-2, 0)$

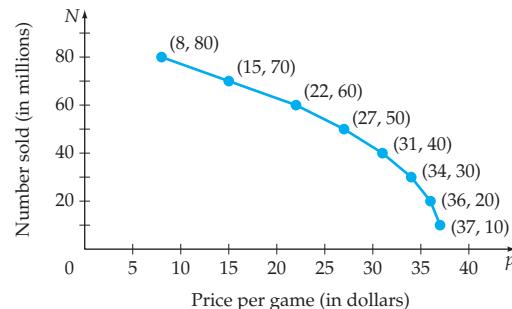
- PER CAPITA INCOME** The following graph, based on data from the Bureau of Economic Analysis, shows the annual per capita income (total income earned divided by population) in the United States for selected years.



- From the graph, what was the per capita income in 2003?
- If the increase in the per capita income from 2005 to 2006 were the same as the increase from 2004 to 2005, what would be the per capita income in 2006?

- If the percent increase in the per capita income from 2005 to 2006 were the same as the percent increase in the per capita income from 2004 to 2005, what would be the per capita income in 2006?

- COMPUTER GAMES** The graph below shows the results of market research conducted by a developer of computer games. It shows the projected numbers of sales  $N$ , in millions, of a game for selected selling prices  $p$  in dollars per game.



- Explain the meaning of the ordered pair  $(22, 60)$  in the context of this problem.
- Based on the graph, does the projected numbers of sales increase or decrease as the price of this game increases?

(Continued)

- c. The product of the coordinates of the ordered pairs,  $R = p \cdot N$ , indicates the revenue  $R$  to the company generated by the sale of  $N$  games at  $p$  dollars per game. Create a scatter diagram of  $(p, R)$ .

- d. Based on the scatter diagram in c., what happens to the revenue as the price of the game increases?

**In Exercises 5 to 16, find the distance between the points whose coordinates are given.**

5.  $(6, 4), (-8, 11)$       6.  $(-5, 8), (-10, 14)$   
 7.  $(-4, -20), (-10, 15)$       8.  $(40, 32), (36, 20)$   
 9.  $(5, -8), (0, 0)$       10.  $(0, 0), (5, 13)$   
 11.  $(\sqrt{3}, \sqrt{8}), (\sqrt{12}, \sqrt{27})$       12.  $(\sqrt{125}, \sqrt{20}), (6, 2\sqrt{5})$   
 13.  $(a, b), (-a, -b)$       14.  $(a - b, b), (a, a + b)$   
 15.  $(x, 4x), (-2x, 3x)$ , given that  $x < 0$   
 16.  $(x, 4x), (-2x, 3x)$ , given that  $x > 0$   
 17. Find all points on the  $x$ -axis that are 10 units from the point  $(4, 6)$ . (Hint: First write the distance formula with  $(4, 6)$  as one of the points and  $(x, 0)$  as the other point.)  
 18. Find all points on the  $y$ -axis that are 12 units from the point  $(5, -3)$ .

**In Exercises 19 to 24, find the midpoint of the line segment having the given endpoints.**

19.  $(1, -1), (5, 5)$       20.  $(-5, -2), (6, 10)$   
 21.  $(6, -3), (6, 11)$       22.  $(4, 7), (-10, 7)$   
 23.  $(1.75, 2.25), (-3.5, 5.57)$       24.  $(-8.2, 10.1), (-2.4, -5.7)$

**In Exercises 25 to 38, graph each equation by plotting points that satisfy the equation.**

25.  $x - y = 4$       26.  $2x + y = -1$   
 27.  $y = 0.25x^2$       28.  $3x^2 + 2y = -4$   
 29.  $y = -2|x - 3|$       30.  $y = |x + 3| - 2$   
 31.  $y = x^2 - 3$       32.  $y = x^2 + 1$   
 33.  $y = \frac{1}{2}(x - 1)^2$       34.  $y = 2(x + 2)^2$

35.  $y = x^2 + 2x - 8$       36.  $y = x^2 - 2x - 8$   
 37.  $y = -x^2 + 2$       38.  $y = -x^2 - 1$

**In Exercises 39 to 48, find the  $x$ - and  $y$ -intercepts of the graph of each equation. Use the intercepts and additional points as needed to draw the graph of the equation.**

39.  $2x + 5y = 12$       40.  $3x - 4y = 15$   
 41.  $x = -y^2 + 5$       42.  $x = y^2 - 6$   
 43.  $x = |y| - 4$       44.  $x = y^3 - 2$   
 45.  $x^2 + y^2 = 4$       46.  $x^2 = y^2$   
 47.  $|x| + |y| = 4$       48.  $|x - 4y| = 8$

**In Exercises 49 to 56, determine the center and radius of the circle with the given equation.**

49.  $x^2 + y^2 = 36$       50.  $x^2 + y^2 = 49$   
 51.  $(x - 1)^2 + (y - 3)^2 = 49$       52.  $(x - 2)^2 + (y - 4)^2 = 25$   
 53.  $(x + 2)^2 + (y + 5)^2 = 25$   
 54.  $(x + 3)^2 + (y + 5)^2 = 121$   
 55.  $(x - 8)^2 + y^2 = \frac{1}{4}$       56.  $x^2 + (y - 12)^2 = 1$

**In Exercises 57 to 64, find an equation of a circle that satisfies the given conditions. Write your answer in standard form.**

57. Center  $(4, 1)$ , radius  $r = 2$   
 58. Center  $(5, -3)$ , radius  $r = 4$   
 59. Center  $\left(\frac{1}{2}, \frac{1}{4}\right)$ , radius  $r = \sqrt{5}$   
 60. Center  $\left(0, \frac{2}{3}\right)$ , radius  $r = \sqrt{11}$   
 61. Center  $(0, 0)$ , passing through  $(-3, 4)$   
 62. Center  $(0, 0)$ , passing through  $(5, 12)$   
 63. Center  $(1, 3)$ , passing through  $(4, -1)$   
 64. Center  $(-2, 5)$ , passing through  $(1, 7)$

**In Exercises 65 to 72, find the center and radius of the graph of the circle. The equations of the circles are written in general form.**

**65.**  $x^2 + y^2 - 6x + 5 = 0$

**66.**  $x^2 + y^2 - 6x - 4y + 12 = 0$

**67.**  $x^2 + y^2 - 14x + 8y + 56 = 0$

**68.**  $x^2 + y^2 - 10x + 2y + 25 = 0$

**69.**  $4x^2 + 4y^2 + 4x - 63 = 0$

**70.**  $9x^2 + 9y^2 - 6y - 17 = 0$

**71.**  $x^2 + y^2 - x + 3y - \frac{15}{4} = 0$

**72.**  $x^2 + y^2 + 3x - 5y + \frac{25}{4} = 0$

**In Exercises 77 to 86, graph the set of all points whose  $x$ - and  $y$ -coordinates satisfy the given conditions.**

77.  $x = 1, y \geq 1$

**78.**  $y = -3, x \geq -2$

79.  $v \leq 3$

**80.**  $x \geq 2$

81.  $x\nu \geqslant 0$

$$82. |y| \geq 1, \frac{x}{y} \leq 0$$

**83.**  $|x| = 2, |y| = 3$

**84.**  $|x| = 4$ ,  $|y| = 1$

**85.**  $|x| \leq 2, y \geq 2$

**86.**  $x \geq 1, |y| \leq 3$

**In Exercises 87 to 90, find the other endpoint of the line segment that has the given endpoint and midpoint.**

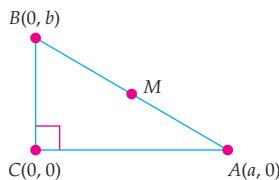
**87.** Endpoint  $(5, 1)$ , midpoint  $(9, 3)$

**88.** Endpoint  $(4, -6)$ , midpoint  $(-2, 11)$

**89.** Endpoint  $(-3, -8)$ , midpoint  $(2, -7)$

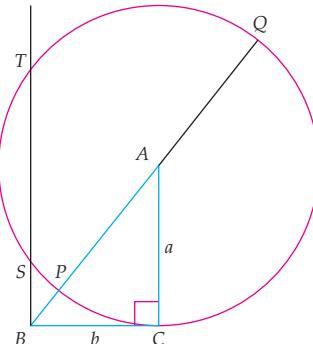
## »»» Projects

- 1. VERIFY A GEOMETRIC THEOREM** Use the midpoint formula and the distance formula to prove that the midpoint  $M$  of the hypotenuse of a right triangle is equidistant from each of the vertices of the triangle. (*Hint:* Label the vertices of the triangle as shown in the figure at the right.)



- 2. SOLVE A QUADRATIC EQUATION GEOMETRICALLY** In the 17th century, Descartes (and others) solved equations by using both algebra and geometry. This project outlines the method Descartes used to solve certain quadratic equations.

- Consider the equation  $x^2 = 2ax + b^2$ . Construct a right triangle  $ABC$  with  $AC = a$  and  $CB = b$ . Now draw a circle with center at  $A$  and radius  $a$ . Let  $P$  be the point at which the circle intersects the hypotenuse and  $Q$  the point at which an extension of the hypotenuse intersects the circle. Your drawing should be similar to the one at the right.
- Show that  $QB$  is a solution of the equation  $x^2 = 2ax + b^2$ .
- Show that  $PB$  is a solution of the equation  $x^2 = -2ax + b^2$ .
- Construct a line parallel to line segment  $AC$  and passing through  $B$ . Let  $S$  and  $T$  be the points at which the line intersects the circle. Show that  $SB$  and  $TB$  are solutions of the equation  $x^2 = 2ax - b^2$ .



## Section 1.3

- Relations
- Functions
- Function Notation
- Identifying Functions
- Graphs of Functions
- The Greatest Integer Function (Floor Function)
- Applications

## Introduction to Functions

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A2.

- PS1.** Let  $y = 3x + 12$ . Find  $x$  when  $y = 0$ . [1.1]
- PS2.** Let  $y = x^2 - 4x + 3$ . Find  $x$  when  $y = 0$ . [1.1]
- PS3.** Let  $(x, y)$  be ordered pairs for  $y^2 = x$ . Determine values of  $a$  and  $b$  for the ordered pairs  $(9, a)$  and  $(9, b)$ . [1.2]
- PS4.** Find the length of the line segment connecting  $P_1(-4, 1)$  and  $P_2(3, -2)$ . [1.2]
- PS5.** What is the greatest integer that is less than  $-3.2$ ? [1.1]
- PS6.** Suppose  $y = -2x + 6$ . As the values of  $x$  increase, do the values of  $y$  increase or decrease? [1.2]

## ■ Relations

In many situations in science, business, and mathematics, a correspondence exists between two sets. The correspondence is often defined by a *table*, an *equation*, or a *graph*, each of which can be viewed from a mathematical perspective as a set of ordered pairs. In mathematics, any set of ordered pairs is called a **relation**.

**Table 1.1** defines a correspondence between a set of percent scores and a set of letter grades. For each score from 0 to 100, there corresponds only one letter grade. The score 94% corresponds to the letter grade of A. Using ordered-pair notation, we record this correspondence as (94, A).

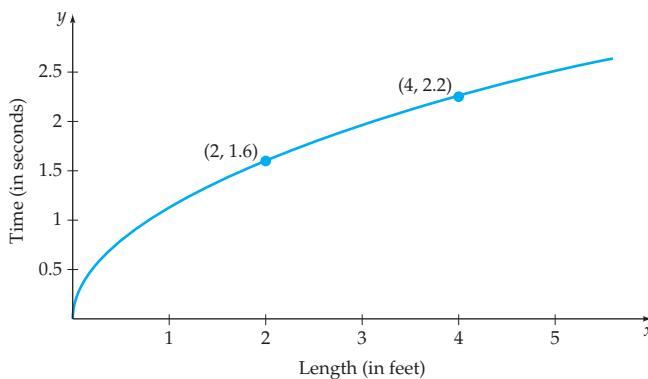
The *equation*  $d = 16t^2$  indicates that the distance  $d$  that a rock falls (neglecting air resistance) corresponds to the time  $t$  that it has been falling. For each nonnegative value  $t$ , the equation assigns only one value for the distance  $d$ . According to this equation, in 3 seconds a rock will fall 144 feet, which we record as (3, 144). Some of the other ordered pairs determined by  $d = 16t^2$  are (0, 0), (1, 16), (2, 64), and (2.5, 100).

$$\text{Equation: } d = 16t^2$$

$$\text{If } t = 3, \text{ then } d = 16(3)^2 = 144$$

The *graph* in **Figure 1.30** defines a correspondence between the length of a pendulum and the time it takes the pendulum to complete one oscillation. For each nonnegative pendulum length, the graph yields only one time. According to the graph, a pendulum length of 2 feet yields an oscillation time of 1.6 seconds, and a length of 4 feet yields an oscillation time of 2.2 seconds, where the time is measured to the nearest tenth of a second. These results can be recorded as the ordered pairs (2, 1.6) and (4, 2.2).

**Graph:** A pendulum's oscillation time



**Figure 1.30**

## ■ Functions

The preceding table, equation, and graph each determines a special type of relation called a **function**.

**Math Matters**

It is generally agreed among historians that Leonhard Euler (1707–1783) was the first person to use the word *function*. His definition of function occurs in his book *Introduction to Analysis of the Infinite*, published in 1748. Euler contributed to many areas of mathematics and was one of the most prolific expositors of mathematics.

**Definition of a Function**

A **function** is a set of ordered pairs in which no two ordered pairs have the same first coordinate and different second coordinates.

Although every function is a relation, not every relation is a function. For instance, consider (94, A) from the grading correspondence. The first coordinate, 94, is paired with a second coordinate of A. It would not make sense to have 94 paired with A, (94, A), and 94 paired with B, (94, B). The same first coordinate would be paired with two different second coordinates. This would mean that two students with the same score received different grades, one student an A and the other a B!

Functions may have ordered pairs with the same second coordinate. For instance, (94, A) and (95, A) are both ordered pairs that belong to the function defined by **Table 1.1**. A function may have different first coordinates and the same second coordinate.

The equation  $d = 16t^2$  represents a function because for each value of  $t$  there is only one value of  $d$ . Not every equation, however, represents a function. For instance,  $y^2 = 25 - x^2$  does not represent a function. The ordered pairs  $(-3, 4)$  and  $(-3, -4)$  are both solutions of the equation. However, these ordered pairs do not satisfy the definition of a function: there are two ordered pairs with the same first coordinate but *different* second coordinates.

**QUESTION** Does the set  $\{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)\}$  define a function?

The **domain** of a function is the set of all the first coordinates of the ordered pairs. The **range** of a function is the set of all the second coordinates. In the function determined by the grading correspondence in **Table 1.1**, the domain is the interval  $[0, 100]$ . The range is  $\{A, B, C, D, F\}$ . In a function, each domain element is paired with one and only one range element.

If a function is defined by an equation, the variable that represents elements of the domain is the **independent variable**. The variable that represents elements of the range is the **dependent variable**. For the free-fall of a rock situation, we used the equation  $d = 16t^2$ . The elements of the domain represented the time the rock fell, and the elements of the range represented the distance the rock fell. Thus, in  $d = 16t^2$ , the independent variable is  $t$  and the dependent variable is  $d$ .

The specific letters used for the independent and dependent variables are not important. For example,  $y = 16x^2$  represents the same function as  $d = 16t^2$ . Traditionally,  $x$  is used for the independent variable and  $y$  for the dependent variable. Anytime we use the phrase “ $y$  is a function of  $x$ ” or a similar phrase with different letters, the variable that follows “function of” is the independent variable.

**ANSWER** Yes. There are no two ordered pairs with the same first coordinate and different second coordinates.

## Function Notation

Functions can be named by using a letter or a combination of letters, such as  $f$ ,  $g$ ,  $A$ ,  $\log$ , or  $\tan$ . If  $x$  is an element of the domain of  $f$ , then  $f(x)$ , which is read “ $f$  of  $x$ ” or “the value of  $f$  at  $x$ ,” is the element in the range of  $f$  that corresponds to the domain element  $x$ . The notation “ $f$ ” and the notation “ $f(x)$ ” mean different things. “ $f$ ” is the name of the function, whereas “ $f(x)$ ” is the value of the function at  $x$ . Finding the value of  $f(x)$  is referred to as *evaluating*  $f$  at  $x$ . To evaluate  $f(x)$  at  $x = a$ , substitute  $a$  for  $x$  and simplify.

### EXAMPLE 1 » Evaluate Functions

Let  $f(x) = x^2 - 1$ , and evaluate.

- a.  $f(-5)$
- b.  $f(3b)$
- c.  $3f(b)$
- d.  $f(a + 3)$
- e.  $f(a) + f(3)$

#### Solution

- a.  $f(-5) = (-5)^2 - 1 = 25 - 1 = 24$  • Substitute **-5** for  $x$ , and simplify.
- b.  $f(3b) = (3b)^2 - 1 = 9b^2 - 1$  • Substitute  **$3b$**  for  $x$ , and simplify.
- c.  $3f(b) = 3(b^2 - 1) = 3b^2 - 3$  • Substitute  **$b$**  for  $x$ , and simplify.
- d.  $f(a + 3) = (a + 3)^2 - 1$  • Substitute  **$a + 3$**  for  $x$ .
- $= a^2 + 6a + 8$  • Simplify.
- e.  $f(a) + f(3) = (a^2 - 1) + (3^2 - 1)$  • Substitute  **$a$**  for  $x$ ; substitute **3** for  $x$ .
- $= a^2 + 7$  • Simplify.

#### take note

In Example 1, observe that

$$f(3b) \neq 3f(b)$$

and that

$$f(a + 3) \neq f(a) + f(3)$$

### » Try Exercise 2, page 46

**Piecewise-defined functions** are functions represented by more than one expression. The function shown below is an example of a piecewise-defined function.

$$f(x) = \begin{cases} 2x, & x < -2 \\ x^2, & -2 \leq x < 1 \\ 4 - x, & x \geq 1 \end{cases}$$

- This function is made up of different pieces,  $2x$ ,  $x^2$ , and  $4 - x$ , depending on the value of  $x$ .

The expression that is used to evaluate this function depends on the value of  $x$ . For instance, to find  $f(-3)$ , we note that  $-3 < -2$  and therefore use the expression  $2x$  to evaluate the function.

$$f(-3) = 2(-3) = -6 \quad \bullet \text{When } x < -2, \text{ use the expression } 2x.$$

Here are some additional instances of evaluating this function:

$$f(-1) = (-1)^2 = 1$$

- When  $x$  satisfies  $-2 \leq x < 1$ , use the expression  $x^2$ .

$$f(4) = 4 - 4 = 0$$

- When  $x \geq 1$ , use the expression  $4 - x$ .

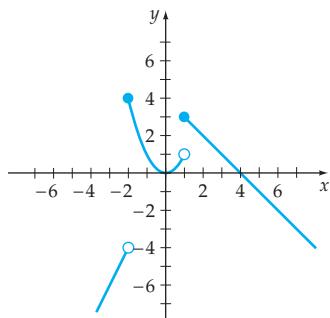


Figure 1.31

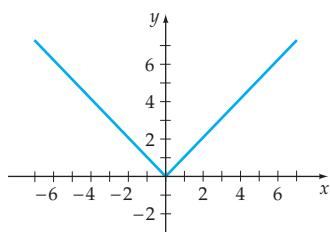


Figure 1.32

The graph of this function is shown in **Figure 1.31**. Note the use of the open and closed circles at the endpoints of the intervals. These circles are used to show the evaluation of the function at the endpoints of each interval. For instance, because  $-2$  is in the interval  $-2 \leq x < 1$ , the value of the function at  $-2$  is  $4$  [ $f(-2) = (-2)^2 = 4$ ]. Therefore a closed dot is placed at  $(-2, 4)$ . Similarly, when  $x = 1$ , because  $1$  is in the interval  $x \geq 1$ , the value of the function at  $1$  is  $3$  [ $f(1) = 4 - 1 = 3$ ].

**QUESTION** Evaluate the function  $f$  defined at the bottom of page 33 when  $x = 0.5$ .

The absolute value function is another example of a piecewise-defined function. Below is the definition of this function, which is sometimes abbreviated  $\text{abs}(x)$ . Its graph (**Figure 1.32**) is shown at the left.

$$\text{abs}(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

### EXAMPLE 2 » Evaluate a Piecewise-Defined Function

 **Figure 1.33** shows the distance from home plate to the base of the outfield fence of PETCO Park, the stadium for the San Diego Padres baseball team.

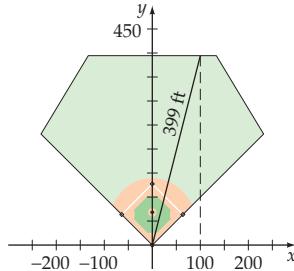


Figure 1.33

The distance  $T(x)$ , in feet, from home plate to the base of the outfield fences can be approximated by the piecewise function

$$T(x) = \begin{cases} \sqrt{4.298x^2 - 2332.11x + 412,253}, & 141 \leq x < 228 \\ \sqrt{1.000x^2 - 6.3208x + 149,886}, & -104 \leq x < 141 \\ \sqrt{2.356x^2 + 1169.88x + 257,818}, & -236 \leq x < -104 \end{cases}$$

where  $x$  is the horizontal distance between the vertical axis and a place at the base of the outfield fence.

**ANSWER**  $0.5$  is in the interval  $-2 \leq x < 1$ . Therefore,  $f(0.5) = 0.5^2 = 0.25$ .

For instance, when  $x = 100$ , the distance to the base of the fence is approximately 399 feet. Use this function to find, to the nearest foot, the distance from home plate to the outfield fence for the following values of  $x$ .

- a.  $x = -100$  feet      b.  $x = 200$  feet

### Solution

- a. Because  $-100$  is in the interval  $-104 \leq x < 141$ , evaluate

$$T(x) = \sqrt{1.000x^2 - 6.3208x + 149,886} \text{ when } x = -100.$$

$$T(-100) = \sqrt{1.000(-100)^2 - 6.3208(-100) + 149,886} \approx 401$$

The distance to the base of the fence is approximately 401 feet.

- b. Because  $200$  is in the interval  $141 \leq x < 228$ , evaluate

$$T(x) = \sqrt{4.298x^2 - 2332.11x + 412,253} \text{ when } x = 200.$$

$$T(200) = \sqrt{4.298(200)^2 - 2332.11(200) + 412,253} \approx 343$$

The distance to the base of the fence is approximately 343 feet.

» Try Exercise 10, page 47

## Identifying Functions

Recall that although every function is a relation, not every relation is a function. In the next example we examine four relations to determine which are functions.

### EXAMPLE 3 » Identify Functions

State whether the relation defines  $y$  as a function of  $x$ .

- a.  $\{(2, 3), (4, 1), (4, 5)\}$     b.  $3x + y = 1$     c.  $-4x^2 + y^2 = 9$   
d. The correspondence between the  $x$  values and the  $y$  values in **Figure 1.34** on page 36.

### Solution

- a. There are two ordered pairs,  $(4, 1)$  and  $(4, 5)$ , with the same first coordinate and different second coordinates. This set does not define  $y$  as a function of  $x$ .
- b. Solving  $3x + y = 1$  for  $y$  yields  $y = -3x + 1$ . Because  $-3x + 1$  is a unique real number for each  $x$ , this equation defines  $y$  as a function of  $x$ .

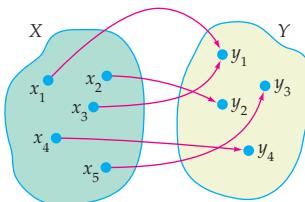


Figure 1.34

- c. Solving  $-4x^2 + y^2 = 9$  for  $y$  yields  $y = \pm\sqrt{4x^2 + 9}$ . The right side  $\pm\sqrt{4x^2 + 9}$  produces two values of  $y$  for each value of  $x$ . For example, when  $x = 0$ ,  $y = 3$  or  $y = -3$ . Thus  $-4x^2 + y^2 = 9$  does not define  $y$  as a function of  $x$ .
- d. Each  $x$  is paired with one and only one  $y$ . The correspondence in Figure 1.34 defines  $y$  as a function of  $x$ .

» Try Exercise 14, page 47

**take note**

You may indicate the domain of a function using set notation or interval notation. For instance, the domain of  $f(x) = \sqrt{x - 3}$  may be given in each of the following ways:

Set notation:  $\{x | x \geq 3\}$   
 Interval notation:  $[3, \infty)$

Sometimes the domain of a function is stated explicitly. For example, each of  $f$ ,  $g$ , and  $h$  below is given by an equation followed by a statement that indicates the domain of the function.

$$f(x) = x^2, x > 0 \quad g(t) = \frac{1}{t^2 + 4}, 0 \leq t \leq 5 \quad h(x) = x^2, x = 1, 2, 3$$

Although  $f$  and  $h$  have the same equation, they are different functions because they have different domains. If the domain of a function is not explicitly stated, then its domain is determined by the following convention.

### Domain of a Function

Unless otherwise stated, the domain of a function is the set of all real numbers for which the function makes sense and yields real numbers.

### EXAMPLE 4 » Determine the Domain of a Function

Determine the domain of each function.

- a.  $G(t) = \frac{1}{t - 4}$       b.  $f(x) = \sqrt{x + 1}$   
 c.  $A(s) = s^2$ , where  $A(s)$  is the area of a square whose sides are  $s$  units.

### Solution

- a. The number 4 is not an element of the domain because  $G$  is undefined when the denominator  $t - 4$  equals 0. The domain of  $G$  is all real numbers except 4. In interval notation the domain is  $(-\infty, 4) \cup (4, \infty)$ .
- b. The radical  $\sqrt{x + 1}$  is a real number only when  $x + 1 \geq 0$  or when  $x \geq -1$ . Thus, in set notation, the domain of  $f$  is  $\{x | x \geq -1\}$ .
- c. Because  $s$  represents the length of a side of a square,  $s$  must be positive. In interval notation the domain of  $A$  is  $(0, \infty)$ .

» Try Exercise 28, page 47

## Graphs of Functions

If  $a$  is an element of the domain of a function  $f$ , then  $(a, f(a))$  is an ordered pair that belongs to the function.

### Definition of the Graph of a Function

The **graph of a function** is the graph of all the ordered pairs that belong to the function.

### EXAMPLE 5 » Graph a Function by Plotting Points

Graph each function. State the domain and the range of each function.

a.  $f(x) = |x - 1|$       b.  $n(x) = \begin{cases} 2, & \text{if } x \leq 1 \\ x, & \text{if } x > 1 \end{cases}$

#### Solution

- a. The domain of  $f$  is the set of all real numbers. Write the function as  $y = |x - 1|$ . Evaluate the function for several domain values. We have used  $x = -3, -2, -1, 0, 1, 2, 3$ , and 4.

|               |    |    |    |   |   |   |   |   |
|---------------|----|----|----|---|---|---|---|---|
| $x$           | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
| $y =  x - 1 $ | 4  | 3  | 2  | 1 | 0 | 1 | 2 | 3 |

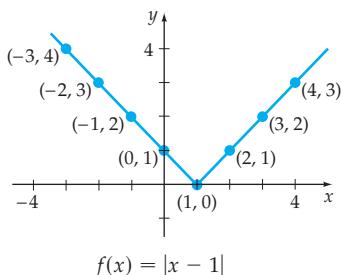


Figure 1.35



#### Integrating Technology

A graphing utility can be used to draw the graph of a function. For instance, to graph  $f(x) = x^2 - 1$ , enter the equation  $Y_1 = x^2 - 1$ . The graph is shown in **Figure 1.37**.

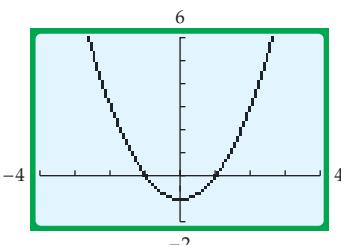


Figure 1.37

Plot the points determined by the ordered pairs. Connect the points to form the graph in **Figure 1.35**.

Because  $|x - 1| \geq 0$ , we can conclude that the graph of  $f$  extends from a height of 0 upward, so the range is  $\{y | y \geq 0\}$ .

- b. The domain is the union of the inequalities  $x \leq 1$  and  $x > 1$ . Thus the domain of  $n$  is the set of all real numbers. For  $x \leq 1$ , graph  $n(x) = 2$ . This results in the horizontal ray in **Figure 1.36**. The solid circle indicates that the point  $(1, 2)$  is part of the graph. For  $x > 1$ , graph  $n(x) = x$ . This produces the second ray in **Figure 1.36**. The open circle indicates that the point  $(1, 1)$  is not part of the graph.

Examination of the graph shows that it includes only points whose  $y$  values are greater than 1. Thus the range of  $n$  is  $\{y | y > 1\}$ .

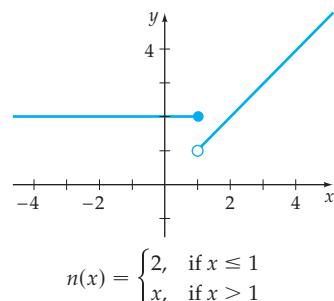


Figure 1.36

### » Try Exercise 40, page 47

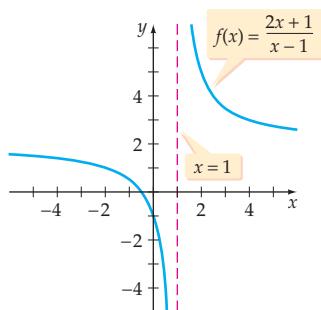


Figure 1.38

**take note**

We can also write, “As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow -\infty$ .” This is read, “As  $x$  approaches 1 from the left,  $f(x)$  approaches negative infinity.”

The graph of  $f(x) = \frac{2x + 1}{x - 1}$  is shown in **Figure 1.38**. The domain of  $f$  is all real numbers except 1. Because  $f(1)$  is undefined, there is no point on the graph when  $x = 1$ . However, there are points on the graph for values of  $x$  close to 1. The table below shows some values of the function when  $x$  is *less than* 1 but close to 1.

|                          |     |      |      |       |       |         |
|--------------------------|-----|------|------|-------|-------|---------|
| <b><math>x</math></b>    | 0.9 | 0.95 | 0.99 | 0.995 | 0.999 | 0.9999  |
| <b><math>f(x)</math></b> | -28 | -58  | -298 | -598  | -2998 | -29,998 |

It appears from the graph and the table that as  $x$  gets closer and closer to 1,  $f(x)$  becomes smaller and smaller. That is, as  $x$  approaches 1 using values of  $x$  that are less than 1,  $f(x)$  approaches  $-\infty$ . The notation

$$f(x) \rightarrow -\infty \text{ as } x \rightarrow 1^-$$

is used to describe this situation. This notation is read “ $f(x)$  approaches negative infinity as  $x$  approaches 1 from the left.” The negative superscript tells us to use values of  $x$  that are less than 1—that is, to the left of 1 on the  $x$ -axis.

Next we focus on values of  $x$  that are close to 1 but *greater than* 1.

|                          |     |      |      |       |       |        |
|--------------------------|-----|------|------|-------|-------|--------|
| <b><math>x</math></b>    | 1.1 | 1.05 | 1.01 | 1.005 | 1.001 | 1.0001 |
| <b><math>f(x)</math></b> | 32  | 62   | 302  | 602   | 3002  | 30,002 |

This time, from **Figure 1.38** and the table, it appears that as  $x$  gets closer and closer to 1,  $f(x)$  becomes larger and larger. That is, as  $x$  approaches 1 using values of  $x$  that are greater than 1,  $f(x)$  approaches  $\infty$ . The notation

$$f(x) \rightarrow \infty \text{ as } x \rightarrow 1^+$$

is used to describe this situation. This notation is read “ $f(x)$  approaches infinity as  $x$  approaches 1 from the right.” The positive superscript tells us to use values of  $x$  that are greater than 1—that is, to the right of 1 on the  $x$ -axis.

The preceding analysis shows that the graph of  $f(x) = \frac{2x + 1}{x - 1}$  approaches, but does not cross, the vertical line  $x = 1$ . The dashed line in **Figure 1.38** is called a *vertical asymptote* of the graph.

**take note**

Generally, we show the asymptotes of the graph of a function by using dashed lines.

**Definition of Vertical Asymptote**

The line  $x = a$  is a **vertical asymptote** of the graph of a function  $f$  provided

$$f(x) \rightarrow -\infty \quad \text{or} \quad f(x) \rightarrow \infty$$

as  $x$  approaches  $a$  from either the left,  $x \rightarrow a^-$ , or from the right,  $x \rightarrow a^+$ .

As an example of this definition, let  $r(x) = \frac{5x}{(x - 2)^2}$ . The domain of  $r$  is all real numbers except 2. The table below shows the behavior of  $r(x)$  as  $x \rightarrow 2^-$ .

|                          |     |      |        |         |
|--------------------------|-----|------|--------|---------|
| <b><math>x</math></b>    | 1.9 | 1.95 | 1.99   | 1.995   |
| <b><math>r(x)</math></b> | 950 | 3900 | 99,500 | 399,000 |

From the table, it appears that  $r(x) \rightarrow \infty$  as  $x \rightarrow 2^-$ .

The following table shows the behavior of  $r(x)$  as  $x \rightarrow 2^+$ .

|                          |      |      |         |         |
|--------------------------|------|------|---------|---------|
| <b><math>x</math></b>    | 2.1  | 2.05 | 2.01    | 2.005   |
| <b><math>r(x)</math></b> | 1050 | 4100 | 100,500 | 401,000 |

From the table, it appears that  $r(x) \rightarrow \infty$  as  $x \rightarrow 2^+$ .

Because  $r(x) \rightarrow \infty$  as  $x \rightarrow 2^-$  and as  $x \rightarrow 2^+$ , the line  $x = 2$  is a vertical asymptote. The graph of  $r(x) = \frac{5x}{(x - 2)^2}$ , along with the vertical asymptote, is shown in **Figure 1.39**. In this case, the value of the function is approaching positive infinity on both sides of the vertical asymptote.

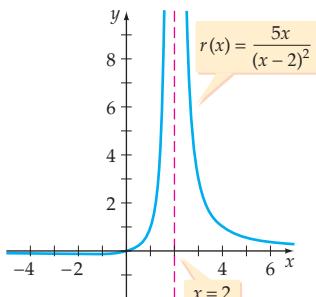


Figure 1.39

The definition of a function as a set of ordered pairs in which no two ordered pairs that have the same first coordinate have different second coordinates implies that any vertical line intersects the graph of a function at no more than one point. This is known as the *vertical line test*.

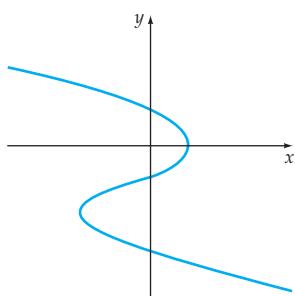
### The Vertical Line Test for Functions

A graph is the graph of a function if and only if no vertical line intersects the graph at more than one point.

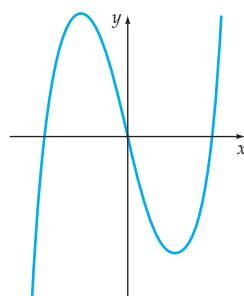
### EXAMPLE 6 » Apply the Vertical Line Test

State whether the graph is the graph of a function.

a.



b.

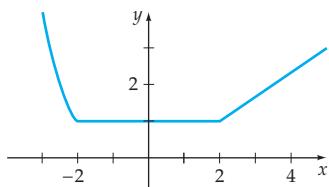


Continued ►

**Solution**

- This graph is *not* the graph of a function because some vertical lines intersect the graph in more than one point.
- This graph is the graph of a function because every vertical line intersects the graph in at most one point.

» Try Exercise 56, page 48



**Figure 1.40**

Consider the graph in **Figure 1.40**. As a point on the graph moves from left to right, this graph falls for values of  $x \leq -2$ , remains the same height from  $x = -2$  to  $x = 2$ , and rises for  $x \geq 2$ . The function represented by the graph is said to be *decreasing* on the interval  $(-\infty, -2]$ , *constant* on the interval  $[-2, 2]$ , and *increasing* on the interval  $[2, \infty)$ .

**Definition of Increasing, Decreasing, and Constant Functions**

If  $a$  and  $b$  are elements of an interval  $I$  that is a subset of the domain of a function  $f$ , then

- $f$  is **increasing** on  $I$  if  $f(a) < f(b)$  whenever  $a < b$ .
- $f$  is **decreasing** on  $I$  if  $f(a) > f(b)$  whenever  $a < b$ .
- $f$  is **constant** on  $I$  if  $f(a) = f(b)$  for all  $a$  and  $b$ .

Recall that a function is a relation in which no two ordered pairs that have the same first coordinate have different second coordinates. This means that given any  $x$ , there is only one  $y$  that can be paired with that  $x$ . A **one-to-one function** satisfies the additional condition that given any  $y$ , there is only one  $x$  that can be paired with that given  $y$ . In a manner similar to applying the vertical line test, we can apply a *horizontal line test* to identify one-to-one functions.

**Horizontal Line Test for a One-To-One Function**

If every horizontal line intersects the graph of a function at most once, then the graph is the graph of a one-to-one function.

For example, some horizontal lines intersect the graph in **Figure 1.41** at more than one point. It is *not* the graph of a one-to-one function. Every horizontal line intersects the graph in **Figure 1.42** at most once. This is the graph of a one-to-one function.

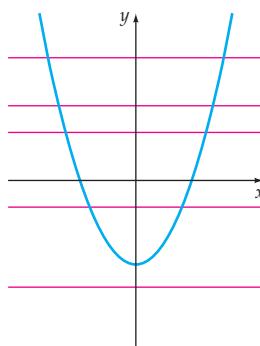


Figure 1.41

Some horizontal lines intersect this graph at more than one point. It is *not* the graph of a one-to-one function.

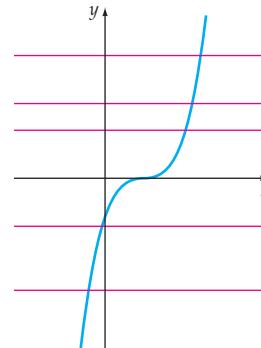


Figure 1.42

Every horizontal line intersects this graph at most once. It is the graph of a one-to-one function.

## The Greatest Integer Function (Floor Function)

### take note

The greatest integer function is an important function that is often used in advanced mathematics and computer science.

The graphs of some functions do not have any breaks or gaps. These functions, whose graphs can be drawn without lifting the pencil off the paper, are called *continuous functions*. The graphs of other functions do have breaks or *discontinuities*. One such function is the **greatest integer function** or **floor function**. This function is denoted by various symbols such as  $\lfloor x \rfloor$ ,  $\lfloor x \rfloor$ , and  $\text{int}(x)$ .

The value of the greatest integer function at  $x$  is the greatest integer that is less than or equal to  $x$ . For instance,

$$\lfloor -1.1 \rfloor = -2 \quad \lfloor -3 \rfloor = -3 \quad \text{int}\left(\frac{5}{2}\right) = 2 \quad \lfloor 5 \rfloor = 5 \quad \lfloor \pi \rfloor = 3$$

**QUESTION** Evaluate. a.  $\text{int}\left(-\frac{3}{2}\right)$  b.  $\lfloor 2 \rfloor$

To graph the floor function, first observe that the value of the floor function is constant between any two consecutive integers. For instance, between the integers 1 and 2, we have

$$\text{int}(1.1) = 1 \quad \text{int}(1.35) = 1 \quad \text{int}(1.872) = 1 \quad \text{int}(1.999) = 1$$

Between  $-3$  and  $-2$ , we have

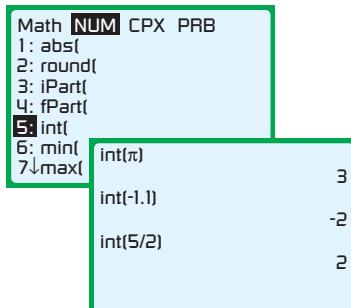
$$\text{int}(-2.98) = -3 \quad \text{int}(-2.4) = -3 \quad \text{int}(-2.35) = -3 \quad \text{int}(-2.01) = -3$$

Using this property of the floor function, we can create a table of values and then graph the floor function. See **Figure 1.43** on page 42.



### Integrating Technology

Many graphing calculators use the notation  $\text{int}(x)$  for the greatest integer function. Here are screens from a TI-83 Plus/TI-84 Plus.



**ANSWER** a. Because  $-2$  is the greatest integer less than or equal to  $-\frac{3}{2}$ ,  $\text{int}\left(-\frac{3}{2}\right) = -2$ .  
b. Because  $2$  is the greatest integer less than or equal to  $2$ ,  $\lfloor 2 \rfloor = 2$ .

| $x$              | $y = \text{int}(x)$ |
|------------------|---------------------|
| $-5 \leq x < -4$ | -5                  |
| $-4 \leq x < -3$ | -4                  |
| $-3 \leq x < -2$ | -3                  |
| $-2 \leq x < -1$ | -2                  |
| $-1 \leq x < 0$  | -1                  |
| $0 \leq x < 1$   | 0                   |
| $1 \leq x < 2$   | 1                   |
| $2 \leq x < 3$   | 2                   |
| $3 \leq x < 4$   | 3                   |
| $4 \leq x < 5$   | 4                   |

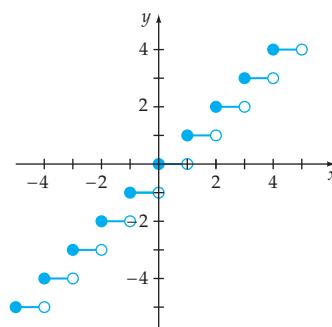


Figure 1.43

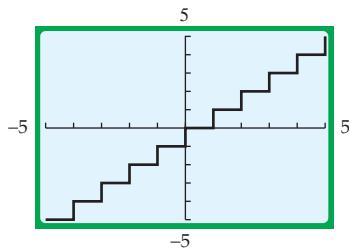
The graph of the floor function has discontinuities (breaks) whenever  $x$  is an integer. The domain of the floor function is the set of real numbers; the range is the set of integers. Because the graph appears to be a series of steps, sometimes the floor function is referred to as a **step function**.



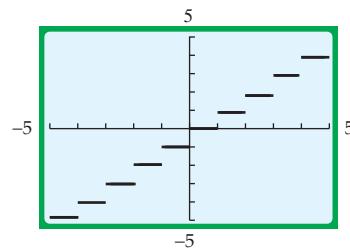
### Integrating Technology

Many graphing calculators use the notation  $\text{int}(x)$  for the floor function. The screens below are from a TI-83 Plus/TI-84 Plus graphing calculator. The graph in **Figure 1.44** was drawn in “connected” mode. This graph does not show the discontinuities that occur whenever  $x$  is an integer.

The graph in **Figure 1.45** was constructed by graphing the floor function in “dot” mode. In this case the discontinuities at the integers are apparent.



$y = \text{int}(x)$   
connected mode



$y = \text{int}(x)$   
dot mode

Figure 1.44

Figure 1.45

One application of the floor function is rounding numbers. For instance, suppose a credit card company charges 1.5% interest on an unpaid monthly balance of \$237.84. Then the interest charge  $I$  is

$$I = 237.84 \cdot 0.015 = 3.5676$$

Thus the interest charge is \$3.57. Note that the result was rounded to the nearest cent, or hundredth. The computer program that determines the interest charge uses the floor function to calculate the rounding. To round a number  $N$  to the nearest  $k$ th decimal place, we use the following formula.

**take note**

Observe the effect of adding 0.5. In this case, it increases the units digit that will occupy the hundredths place.

Now suppose that the monthly balance was \$237.57. Then  $100(237.57 \cdot 0.015) + 0.5$  is  $356.355 + 0.5 = 356.855$ . In this case, the units digit is not changed and the final interest charge is \$3.56.

$$N \text{ to the nearest } k\text{th decimal} = \frac{\text{int}[10^k(N) + 0.5]}{10^k}$$

Here is the calculation for the interest owed. In this case,  $N = 237.84 \cdot 0.015$  and  $k = 2$  (round to the second decimal place).

$$\begin{aligned} I &= \frac{\text{int}[10^2(237.84 \cdot 0.015) + 0.5]}{10^2} \\ &= \frac{\text{int}[100(3.5676) + 0.5]}{100} \\ &= \frac{\text{int}[356.76 + 0.5]}{100} \\ &= \frac{\text{int}[357.26]}{100} \\ &= \frac{357}{100} = 3.57 \end{aligned}$$

Example 7 gives another application of the floor function.

### EXAMPLE 7 » Use the Greatest Integer Function to Model Expenses

The cost of parking in a garage is \$3 for the first hour or any part of the hour and \$2 for each additional hour or any part of an hour thereafter. If  $x$  is the time in hours that you park your car, then the cost is given by

$$C(x) = 3 - 2 \text{ int}(1 - x), \quad x > 0$$

- a. Evaluate  $C(2)$  and  $C(2.5)$ .    b. Graph  $y = C(x)$  for  $0 < x \leq 5$ .

#### Solution

$$\begin{array}{ll} \text{a. } C(2) = 3 - 2 \text{ int}(1 - 2) & C(2.5) = 3 - 2 \text{ int}(1 - 2.5) \\ = 3 - 2 \text{ int}(-1) & = 3 - 2 \text{ int}(-1.5) \\ = 3 - 2(-1) & = 3 - 2(-2) \\ = \$5 & = \$7 \end{array}$$

- b. To graph  $C(x)$  for  $0 < x \leq 5$ , consider the value of  $\text{int}(1 - x)$  for each of the intervals  $0 < x \leq 1$ ,  $1 < x \leq 2$ ,  $2 < x \leq 3$ ,  $3 < x \leq 4$ , and  $4 < x \leq 5$ . For instance, when  $0 < x \leq 1$ ,  $0 \leq 1 - x < 1$ . Thus  $\text{int}(1 - x) = 0$  when  $0 < x \leq 1$ . Now consider  $1 < x \leq 2$ . When  $1 < x \leq 2$ ,  $-1 \leq 1 - x < 0$ . Thus  $\text{int}(1 - x) = -1$  when  $1 < x \leq 2$ . Applying the same reasoning to each of the other intervals gives the following table of values and the graph of  $C$  shown in Figure 1.46.

| $x$            | $C(x) = 3 - 2 \text{ int}(1 - x)$ |
|----------------|-----------------------------------|
| $0 < x \leq 1$ | 3                                 |
| $1 < x \leq 2$ | 5                                 |
| $2 < x \leq 3$ | 7                                 |
| $3 < x \leq 4$ | 9                                 |
| $4 < x \leq 5$ | 11                                |

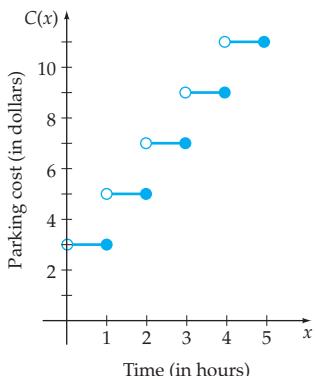
 $C(x) = 3 - 2 \text{ int}(1 - x)$ 

Figure 1.46

Because  $C(1) = 3$ ,  $C(2) = 5$ ,  $C(3) = 7$ ,  $C(4) = 9$ , and  $C(5) = 11$ , we can use a solid circle at the right endpoint of each “step” and an open circle at each left endpoint.

» Try Exercise 54, page 48



### Integrating Technology

The function graphed in Example 7 is an example of a function for which a graphing calculator may not produce a graph that is a good representation of the function. You may be required to *make adjustments* in the MODE, SET UP, or WINDOW of the graphing calculator so that it will produce a better representation of the function. A graph may also require some *fine tuning*, such as open or solid circles at particular points, to accurately represent the function.

## Applications

### EXAMPLE 8 » Solve an Application

A car was purchased for \$16,500. Assuming the car depreciates at a constant rate of \$2200 per year (*straight-line depreciation*) for the first 7 years, write the value  $v$  of the car as a function of time, and calculate the value of the car 3 years after purchase.

#### Solution

Let  $t$  represent the number of years that have passed since the car was purchased. Then  $2200t$  is the amount by which the value of the car has depreciated after  $t$  years. The value of the car at time  $t$  is given by

$$v(t) = 16,500 - 2200t, \quad 0 \leq t \leq 7$$

When  $t = 3$ , the value of the car is

$$v(3) = 16,500 - 2200(3) = 16,500 - 6600 = \$9900$$

» Try Exercise 72, page 49

Often in applied mathematics, formulas are used to determine the functional relationship that exists between two variables.

**EXAMPLE 9** » Solve an Application

A lighthouse is 2 miles south of a port. A ship leaves port and sails east at a rate of 7 mph. Express the distance  $d$  between the ship and the lighthouse as a function of time, given that the ship has been sailing for  $t$  hours.

**Solution**

Draw a diagram and label it as shown in **Figure 1.47**. Note that because distance = (rate)(time) and the rate is 7, in  $t$  hours the ship has sailed a distance of  $7t$ .

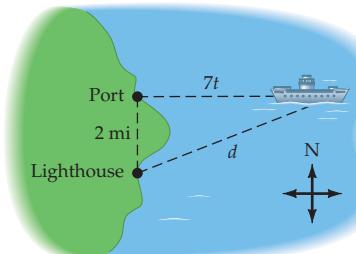


Figure 1.47

$$[d(t)]^2 = (7t)^2 + 2^2$$

• The Pythagorean Theorem

$$[d(t)]^2 = 49t^2 + 4$$

$$d(t) = \sqrt{49t^2 + 4}$$

• The ± sign is not used because the distance  $d(t)$  must be nonnegative.

» Try Exercise 78, page 50

**EXAMPLE 10** » Solve an Application

An open box is to be made from a square piece of cardboard that measures 40 inches on each side. To construct the box, squares that measure  $x$  inches on each side are cut from each corner of the cardboard as shown in **Figure 1.48**.

- Express the volume  $V$  of the box as a function of  $x$ .
- Determine the domain of  $V$ .

**Solution**

- The length  $l$  of the box is  $40 - 2x$ . The width  $w$  is also  $40 - 2x$ . The height of the box is  $x$ . The volume  $V$  of a box is the product of its length, its width, and its height. Thus

$$V = (40 - 2x)^2x$$

- The squares that are cut from each corner require  $x$  to be larger than 0 inches but less than 20 inches. Thus the domain is  $\{x | 0 < x < 20\}$ .

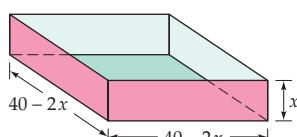
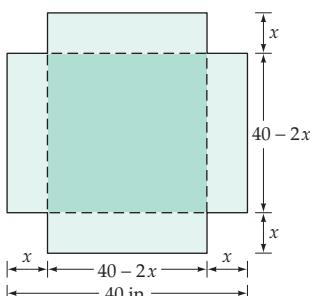


Figure 1.48

» Try Exercise 74, page 49



## Topics for Discussion

1. Discuss the definition of *function*. Give some examples of relationships that are functions and some that are not functions.
2. What is the difference between the domain and range of a function?
3. How many  $y$ -intercepts can a function have? How many  $x$ -intercepts can a function have?
4. Discuss how the vertical line test is used to determine whether or not a graph is the graph of a function. Explain why the vertical line test works.
5. What is the domain of  $f(x) = \frac{\sqrt{1-x}}{x^2 - 9}$ ? Explain.
6. Is 2 in the range of  $g(x) = \frac{6x - 5}{3x + 1}$ ? Explain the process you used to make your decision.
7. Suppose that  $f$  is a function and that  $f(a) = f(b)$ . Does this imply that  $a = b$ ? Explain your answer.

## Exercise Set 1.3

In Exercises 1 to 8, evaluate each function.

1. Given  $f(x) = 3x - 1$ , find

- |                                       |                   |                      |
|---------------------------------------|-------------------|----------------------|
| <b>a.</b> $f(2)$                      | <b>b.</b> $f(-1)$ | <b>c.</b> $f(0)$     |
| <b>d.</b> $f\left(\frac{2}{3}\right)$ | <b>e.</b> $f(k)$  | <b>f.</b> $f(k + 2)$ |

2. Given  $g(x) = 2x^2 + 3$ , find

- |                                       |                   |                      |
|---------------------------------------|-------------------|----------------------|
| <b>a.</b> $g(3)$                      | <b>b.</b> $g(-1)$ | <b>c.</b> $g(0)$     |
| <b>d.</b> $g\left(\frac{1}{2}\right)$ | <b>e.</b> $g(c)$  | <b>f.</b> $g(c + 5)$ |

3. Given  $A(w) = \sqrt{w^2 + 5}$ , find

- |                  |                      |                   |
|------------------|----------------------|-------------------|
| <b>a.</b> $A(0)$ | <b>b.</b> $A(2)$     | <b>c.</b> $A(-2)$ |
| <b>d.</b> $A(4)$ | <b>e.</b> $A(r + 1)$ | <b>f.</b> $A(-c)$ |

4. Given  $J(t) = 3t^2 - t$ , find

- |                   |                      |                                       |
|-------------------|----------------------|---------------------------------------|
| <b>a.</b> $J(-4)$ | <b>b.</b> $J(0)$     | <b>c.</b> $J\left(\frac{1}{3}\right)$ |
| <b>d.</b> $J(-c)$ | <b>e.</b> $J(x + 1)$ | <b>f.</b> $J(x + h)$                  |

5. Given  $f(x) = \frac{1}{|x|}$ , find

- |                          |                        |  |
|--------------------------|------------------------|--|
| <b>a.</b> $f(2)$         | <b>b.</b> $f(-2)$      | <b>c.</b> $f\left(\frac{-3}{5}\right)$ |
| <b>d.</b> $f(2) + f(-2)$ | <b>e.</b> $f(c^2 + 4)$ | <b>f.</b> $f(2 + h)$                   |

6. Given  $T(x) = 5$ , find

- |                         |                      |                                       |
|-------------------------|----------------------|---------------------------------------|
| <b>a.</b> $T(-3)$       | <b>b.</b> $T(0)$     | <b>c.</b> $T\left(\frac{2}{7}\right)$ |
| <b>d.</b> $T(3) + T(1)$ | <b>e.</b> $T(x + h)$ | <b>f.</b> $T(3k + 5)$                 |

7. Given  $s(x) = \frac{x}{|x|}$ , find

- |                   |                         |                         |
|-------------------|-------------------------|-------------------------|
| <b>a.</b> $s(4)$  | <b>b.</b> $s(5)$        | <b>c.</b> $s(-2)$       |
| <b>d.</b> $s(-3)$ | <b>e.</b> $s(t), t > 0$ | <b>f.</b> $s(t), t < 0$ |

8. Given  $r(x) = \frac{x}{x + 4}$ , find

- |                                       |                    |                       |
|---------------------------------------|--------------------|-----------------------|
| <b>a.</b> $r(0)$                      | <b>b.</b> $r(-1)$  | <b>c.</b> $r(-3)$     |
| <b>d.</b> $r\left(\frac{1}{2}\right)$ | <b>e.</b> $r(0.1)$ | <b>f.</b> $r(10,000)$ |

**In Exercises 9 and 10, evaluate each piecewise-defined function for the indicated values.**

9.  $P(x) = \begin{cases} 3x + 1, & \text{if } x < 2 \\ -x^2 + 11, & \text{if } x \geq 2 \end{cases}$

a.  $P(-4)$

b.  $P(\sqrt{5})$

c.  $P(c), c < 2$

d.  $P(k + 1), k \geq 1$

10.  $Q(t) = \begin{cases} 4, & \text{if } 0 \leq t \leq 5 \\ -t + 9, & \text{if } 5 < t \leq 8 \\ \sqrt{t - 7}, & \text{if } 8 < t \leq 11 \end{cases}$

a.  $Q(0)$

b.  $Q(a), 6 < a < 7$

c.  $Q(n), 1 < n < 2$

d.  $Q(m^2 + 7), 1 < m \leq 2$

**In Exercises 11 to 20, state whether the equation defines  $y$  as a function of  $x$ .**

11.  $2x + 3y = 7$

12.  $5x + y = 8$

13.  $-x + y^2 = 2$

14.  $x^2 - 2y = 2$

15.  $y = 4 \pm \sqrt{x}$

16.  $x^2 + y^2 = 9$

17.  $y = \sqrt[3]{x}$

18.  $y = |x| + 5$

19.  $y^2 = x^2$

20.  $y^3 = x^3$

**In Exercises 21 to 26, state whether the set of ordered pairs  $(x, y)$  defines  $y$  as a function of  $x$ .**

21.  $\{(2, 3), (5, 1), (-4, 3), (7, 11)\}$

22.  $\{(5, 10), (3, -2), (4, 7), (5, 8)\}$

23.  $\{(4, 4), (6, 1), (5, -3)\}$

24.  $\{(2, 2), (3, 3), (7, 7)\}$

25.  $\{(1, 0), (2, 0), (3, 0)\}$

26.  $\left\{ \left( -\frac{1}{3}, \frac{1}{4} \right), \left( -\frac{1}{4}, \frac{1}{3} \right), \left( \frac{1}{4}, \frac{2}{3} \right) \right\}$

**In Exercises 27 to 38, determine the domain of the function represented by the given equation.**

27.  $f(x) = 3x - 4$

28.  $f(x) = -2x + 1$

29.  $f(x) = x^2 + 2$

30.  $f(x) = 3x^2 + 1$

31.  $f(x) = \frac{4}{x+2}$

32.  $f(x) = \frac{6}{x-5}$

33.  $f(x) = \sqrt{7+x}$

34.  $f(x) = \sqrt{4-x}$

35.  $f(x) = \sqrt{4-x^2}$

36.  $f(x) = \sqrt{12-x^2}$

37.  $f(x) = \frac{1}{\sqrt{x+4}}$

38.  $f(x) = \frac{1}{\sqrt{5-x}}$

**In Exercises 39 to 46, graph each function. Insert solid circles or open circles where necessary to indicate the true nature of the function.**

39.  $f(x) = \begin{cases} |x|, & \text{if } x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$

40.  $g(x) = \begin{cases} -4, & \text{if } x \leq 0 \\ x^2 - 4, & \text{if } 0 < x \leq 1 \\ -x, & \text{if } x > 1 \end{cases}$

41.  $J(x) = \begin{cases} 4, & \text{if } x \leq -1 \\ x^2, & \text{if } -1 < x < 1 \\ -x + 5, & \text{if } x \geq 1 \end{cases}$

42.  $K(x) = \begin{cases} 1, & \text{if } x \leq -2 \\ x^2 - 4, & \text{if } -2 < x < 2 \\ \frac{1}{2}x, & \text{if } x \geq 2 \end{cases}$

43.  $L(x) = \left\lfloor \frac{1}{3}x \right\rfloor \quad \text{for } -6 \leq x \leq 6$

44.  $M(x) = \lceil x \rceil + 2 \quad \text{for } 0 \leq x \leq 4$

45.  $N(x) = \text{int}(-x) \quad \text{for } -3 \leq x \leq 3$

46.  $P(x) = \text{int}(x) + x \quad \text{for } 0 \leq x \leq 4$

**In Exercises 47 to 52, use the floor function to write and then evaluate an expression that can be used to round the given number to the given place value.**

47.  $N = 2.3458$ ; hundredths

48.  $N = 34.567$ ; tenths

49.  $N = 34.05622$ ; thousandths

50.  $N = 109.83$ ; whole number

51.  $N = 0.08951$ ; ten-thousandths

52.  $N = 2.98245$ ; thousandths

- 53. FIRST-CLASS MAIL** In 2006, the cost to mail a first-class letter was given by

$$C(w) = 0.39 - 0.34 \operatorname{int}(1 - w), \quad w > 0$$

where  $C$  is in dollars and  $w$  is the weight of the letter in ounces.

- a. What was the cost (in 2006) to mail a letter that weighed 2.8 ounces?

- b. Graph  $C$  for  $0 < w \leq 5$ .

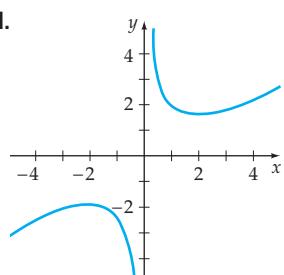
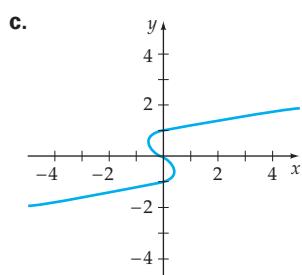
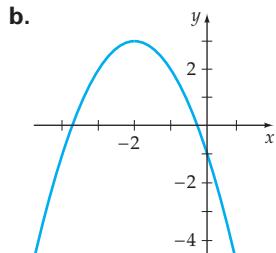
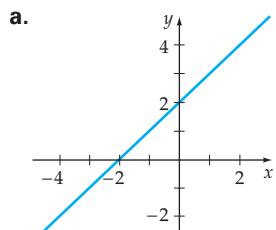
- 54. INCOME TAX** The amount of federal income tax  $T(x)$  a person owed in 2006 is given by

$$T(x) = \begin{cases} 0.10x, & 0 \leq x < 7550 \\ 0.15(x - 7550) + 755, & 7550 \leq x < 30,650 \\ 0.25(x - 30,650) + 4220, & 30,650 \leq x < 74,200 \\ 0.28(x - 74,200) + 15,107.50, & 74,200 \leq x < 154,800 \\ 0.33(x - 154,800) + 37,675.50, & 154,800 \leq x < 336,550 \\ 0.35(x - 336,550) + 97,653, & x \geq 336,550 \end{cases}$$

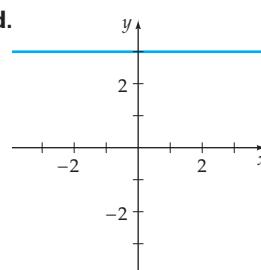
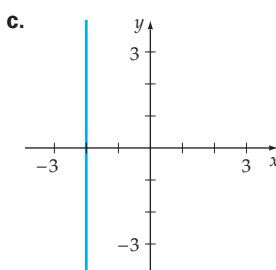
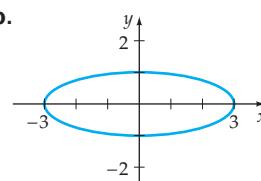
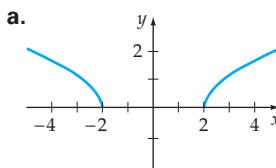
where  $x$  is the adjusted gross income tax of the taxpayer.

- a. What is the domain of this function?  
 b. Find the income tax owed by a taxpayer whose adjusted gross income was \$31,250.  
 c. Find the income tax owed by a taxpayer whose adjusted gross income was \$78,900.

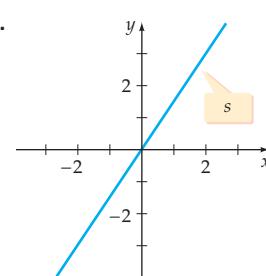
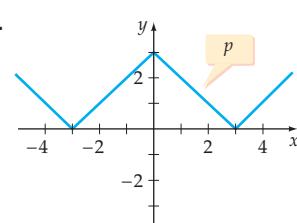
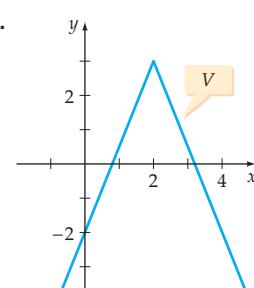
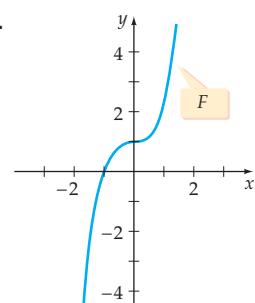
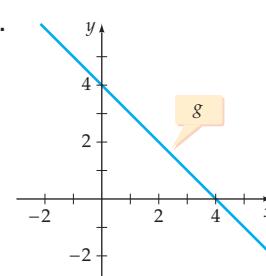
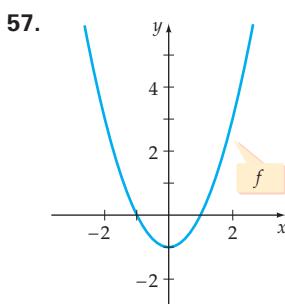
- 55.** Use the vertical line test to determine which of the following graphs are graphs of functions.



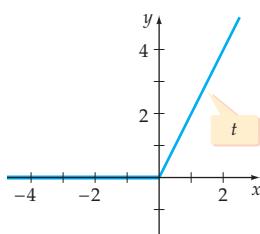
- 56.** Use the vertical line test to determine which of the following graphs are graphs of functions.



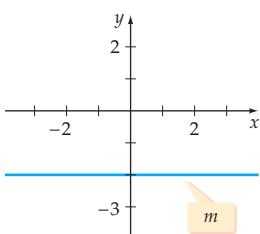
In Exercises 57 to 66, use the indicated graph to identify the intervals over which the function is increasing, constant, or decreasing.



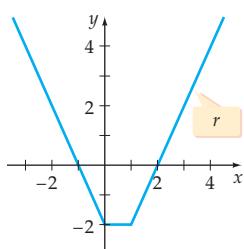
63.



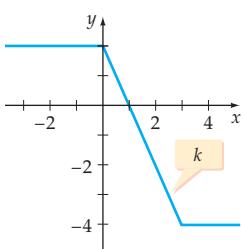
64.



65.



66.



67. Use the horizontal line test to determine which of the following functions are one-to-one.

*f* as shown in Exercise 57  
*g* as shown in Exercise 58  
*F* as shown in Exercise 59  
*V* as shown in Exercise 60  
*p* as shown in Exercise 61

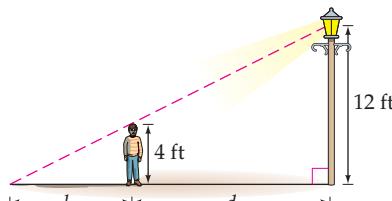
68. Use the horizontal line test to determine which of the following functions are one-to-one.

*s* as shown in Exercise 62  
*t* as shown in Exercise 63  
*m* as shown in Exercise 64  
*r* as shown in Exercise 65  
*k* as shown in Exercise 66

69. **GEOMETRY** A rectangle has a length of  $l$  feet and a perimeter of 50 feet.

- Write the width  $w$  of the rectangle as a function of its length.
- Write the area  $A$  of the rectangle as a function of its length.

70. **LENGTH OF A SHADOW** A child 4 feet tall is standing near a street lamp that is 12 feet high. The light from the lamp casts a shadow as shown in the following diagram.



- a. Find the length  $l$  of the shadow as a function of the distance  $d$  of the child from the lamppost. (Suggestion: Use the fact that the ratios of corresponding sides of similar triangles are equal.)

- b. What is the domain of the function?

- c. What is the length of the shadow when the child is 8 feet from the base of the lamppost?

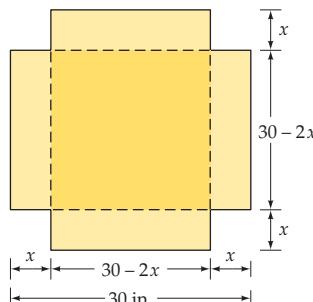
71. **DEPRECIATION** A bus was purchased for \$80,000. Assuming the bus depreciates at a rate of \$6500 per year (*straight-line depreciation*) for the first 10 years, write the value  $v$  of the bus as a function of the time  $t$  (measured in years) for  $0 \leq t \leq 10$ .

72. **DEPRECIATION** A boat was purchased for \$44,000. Assuming the boat depreciates at a rate of \$4200 per year (*straight-line depreciation*) for the first 8 years, write the value  $v$  of the boat as a function of the time  $t$  (measured in years) for  $0 \leq t \leq 8$ .

73. **COST, REVENUE, AND PROFIT** A manufacturer produces a product at a cost of \$22.80 per unit. The manufacturer has a fixed cost of \$400.00 per day. Each unit retails for \$37.00. Let  $x$  represent the number of units produced in a 5-day period.

- Write the total cost  $C$  as a function of  $x$ .
- Write the revenue  $R$  as a function of  $x$ .
- Write the profit  $P$  as a function of  $x$ . (Hint: The profit function is given by  $P(x) = R(x) - C(x)$ .)

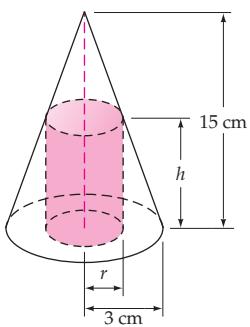
74. **VOLUME OF A BOX** An open box is to be made from a square piece of cardboard having dimensions 30 inches by 30 inches by cutting out squares of area  $x^2$  from each corner, as shown in the figure.



- a. Express the volume  $V$  of the box as a function of  $x$ .

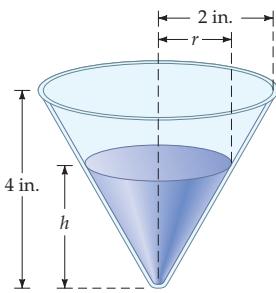
- b. State the domain of  $V$ .

- 75. HEIGHT OF AN INSCRIBED CYLINDER** A cone has an altitude of 15 centimeters and a radius of 3 centimeters. A right circular cylinder of radius  $r$  and height  $h$  is inscribed in the cone as shown in the figure. Use similar triangles to write  $h$  as a function of  $r$ .

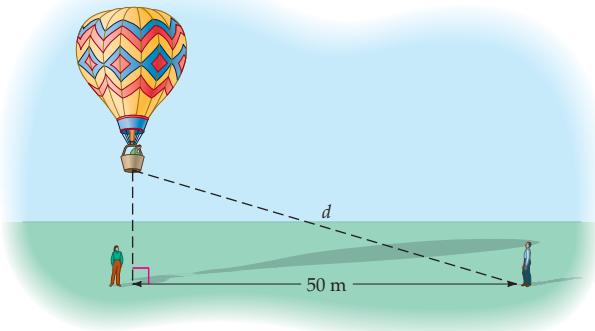


- 76. VOLUME OF WATER** Water is flowing into a conical drinking cup that has an altitude of 4 inches and a radius of 2 inches, as shown in the figure.

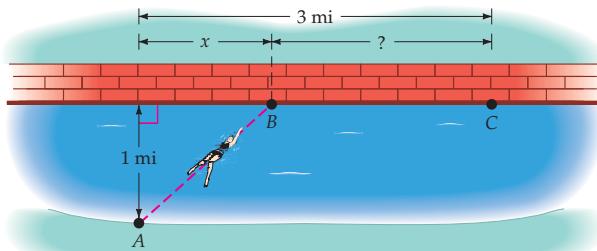
- Write the radius  $r$  of the surface of the water as a function of its depth  $h$ .
- Write the volume  $V$  of the water as a function of its depth  $h$ .



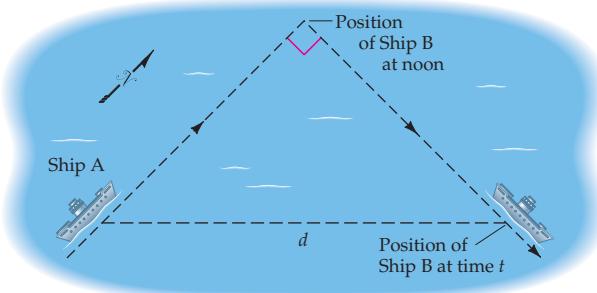
- 77. DISTANCE FROM A BALLOON** For the first minute of flight, a hot air balloon rises vertically at a rate of 3 meters per second. If  $t$  is the time in seconds that the balloon has been airborne, write the distance  $d$  between the balloon and a point on the ground 50 meters from the point of lift-off as a function of  $t$ .



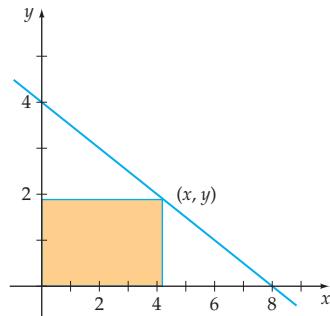
- 78. TIME FOR A SWIMMER** An athlete swims from point  $A$  to point  $B$  at a rate of 2 mph and runs from point  $B$  to point  $C$  at a rate of 8 mph. Use the dimensions in the figure to write the time  $t$  required to reach point  $C$  as a function of  $x$ .



- 79. DISTANCE BETWEEN SHIPS** At 12:00 noon Ship A is 45 miles due south of Ship B and is sailing north at a rate of 8 mph. Ship B is sailing east at a rate of 6 mph. Write the distance  $d$  between the ships as a function of the time  $t$ , where  $t = 0$  represents 12:00 noon.



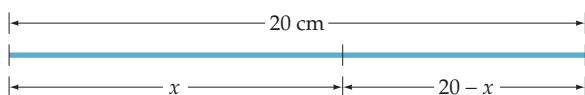
- 80. AREA** A rectangle is bounded by the  $x$ - and  $y$ -axes and the graph of  $y = -\frac{1}{2}x + 4$ .



- Find the area of the rectangle as a function of  $x$ .
- Complete the following table.

| <b>x</b> | <b>Area</b> |
|----------|-------------|
| 1        |             |
| 2        |             |
| 4        |             |
| 6        |             |
| 7        |             |

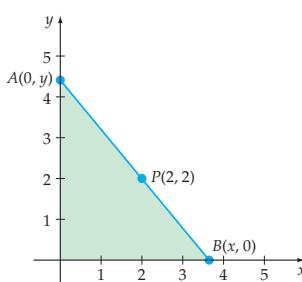
- c.** What is the domain of this function?
- 81. AREA** A piece of wire 20 centimeters long is cut at a point  $x$  centimeters from the left end. The left-hand piece is formed into the shape of a circle and the right-hand piece is formed into a square.



- a.** Find the area enclosed by the two figures as a function of  $x$ .
- b.** Complete the following table. Round the area to the nearest hundredth of a square centimeter.

| <b>x</b> | <b>Total Area Enclosed</b> |
|----------|----------------------------|
| 0        |                            |
| 4        |                            |
| 8        |                            |
| 12       |                            |
| 16       |                            |
| 20       |                            |

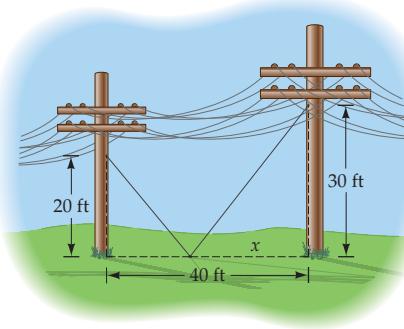
- c.** What is the domain of this function?
- 82. AREA** A triangle is bounded by the  $x$ - and  $y$ -axes and must pass through  $P(2, 2)$ , as shown below.



- a.** Find the area of the triangle as a function of  $x$ . (*Suggestion:* Let  $C$  be the point  $(0, 2)$  and  $D$  be the point  $(2, 0)$ . Use the fact that  $ACP$  and  $PDB$  are similar triangles.)

- b.** What is the domain of the function you found in a.?

- 83. LENGTH** Two guy wires are attached to utility poles that are 40 feet apart, as shown in the following diagram.



- a.** Find the total length of the two guy wires as a function of  $x$ .

- b.** Complete the following table. Round the length to the nearest hundredth of a foot.

| <b>x</b> | <b>Total Length of Wires</b> |
|----------|------------------------------|
| 0        |                              |
| 10       |                              |
| 20       |                              |
| 30       |                              |
| 40       |                              |

- c.** What is the domain of this function?

- 84. SALES VS. PRICE** A business finds that the number of feet  $f$  of pipe it can sell per week is a function of the price  $p$  in cents per foot as given by

$$f(p) = \frac{320,000}{p + 25}, \quad 40 \leq p \leq 90$$

Complete the following table by evaluating  $f$  (to the nearest hundred feet) for the indicated values of  $p$ .

| <b>p</b>    | 40 | 50 | 60 | 75 | 90 |
|-------------|----|----|----|----|----|
| <b>f(p)</b> |    |    |    |    |    |

- 85. MODEL YIELD** The yield  $Y$  of apples per tree is related to the amount  $x$  of a particular type of fertilizer applied (in pounds per year) by the function

$$Y(x) = 400[1 - 5(x - 1)^{-2}], \quad 5 \leq x \leq 20$$



A **fixed point** of a function is a number  $a$  such that  $f(a) = a$ . In Exercises 105 and 106, find all fixed points for the given function.

105.  $f(x) = x^2 + 3x - 3$

106.  $g(x) = \frac{x}{x+5}$

In Exercises 107 and 108, sketch the graph of the piecewise-defined function.

107.  $s(x) = \begin{cases} 1, & \text{if } x \text{ is an integer} \\ 2, & \text{if } x \text{ is not an integer} \end{cases}$

108.  $v(x) = \begin{cases} 2x - 2, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}$

## »»» Projects

1.  **DAY OF THE WEEK** A formula known as Zeller's Congruence makes use of the greatest integer function  $\lfloor x \rfloor$  to determine the day of the week on which a given day fell or will fall. To use Zeller's Congruence, we first compute the integer  $z$  given by

$$z = \left\lfloor \frac{13m - 1}{5} \right\rfloor + \left\lfloor \frac{y}{4} \right\rfloor + \left\lfloor \frac{c}{4} \right\rfloor + d + y - 2c$$

The variables  $c$ ,  $y$ ,  $d$ , and  $m$  are defined as follows:

$c$  = the century

$y$  = the year of the century

$d$  = the day of the month

$m$  = the month, using 1 for March, 2 for April, ..., 10 for December. January and February are assigned the values 11 and 12 of the previous year.

For example, for the date September 12, 2001, we use  $c = 20$ ,  $y = 1$ ,  $d = 12$ , and  $m = 7$ . The remainder of  $z$  divided by 7 gives the day of the week. A remainder of 0 represents a Sunday, a remainder of 1 a Monday, ..., and a remainder of 6 a Saturday.

- a. Verify that December 7, 1941 was a Sunday.
- b. Verify that January 1, 2010 will fall on a Friday.
- c. Determine on what day of the week Independence Day (July 4, 1776) fell.
- d. Determine on what day of the week you were born.

## Section 1.4

- Symmetry
- Even and Odd Functions
- Translations of Graphs
- Reflections of Graphs
- Compressing and Stretching of Graphs

## Properties of Graphs

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A3.

**PS1.** Graph  $f(x) = x^2$  and  $g(x) = x^2 + 1$  on the same coordinate grid. How is the graph of  $g$  related to the graph of  $f$ ? [1.2]

**PS2.** For  $f(x) = \frac{3x^4}{x^2 + 1}$ , show that  $f(-3) = f(3)$ . [1.3]

**PS3.** For  $f(x) = 2x^3 - 5x$ , show that  $f(-2) = -f(2)$ . [1.3]

**PS4.** Let  $f(x) = x^2$  and  $g(x) = x + 3$ . Find  $f(a) - g(a)$  for  $a = -2, -1, 0, 1, 2$ . [1.3]

**PS5.** What is the midpoint of the line segment between  $P(-a, b)$  and  $Q(a, b)$ ? [1.2]

**PS6.** What is the midpoint of the line segment between  $P(-a, -b)$  and  $Q(a, b)$ ? [1.2]

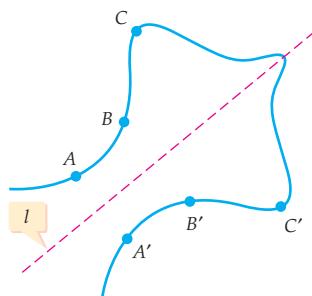


Figure 1.49

## Symmetry

The graph in **Figure 1.49** is symmetric with respect to the line  $l$ . Note that the graph has the property that if the paper is folded along the dotted line  $l$ , the point  $A'$  will coincide with the point  $A$ , the point  $B'$  will coincide with the point  $B$ , and the point  $C'$  will coincide with the point  $C$ . One part of the graph is a *mirror image* of the rest of the graph across the line  $l$ .

A graph is **symmetric with respect to the  $y$ -axis** if, whenever the point given by  $(x, y)$  is on the graph, then  $(-x, y)$  is also on the graph. The graph in **Figure 1.50** is symmetric with respect to the  $y$ -axis. A graph is **symmetric with respect to the  $x$ -axis** if, whenever the point given by  $(x, y)$  is on the graph, then  $(x, -y)$  is also on the graph. The graph in **Figure 1.51** is symmetric with respect to the  $x$ -axis.

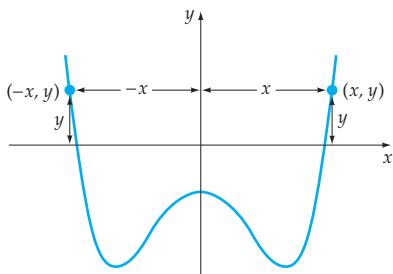


Figure 1.50

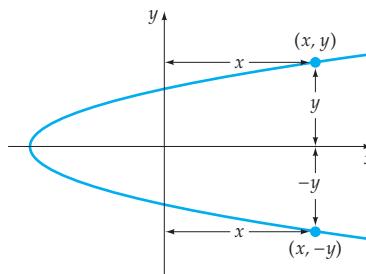
Symmetry with respect to the  $y$ -axis

Figure 1.51

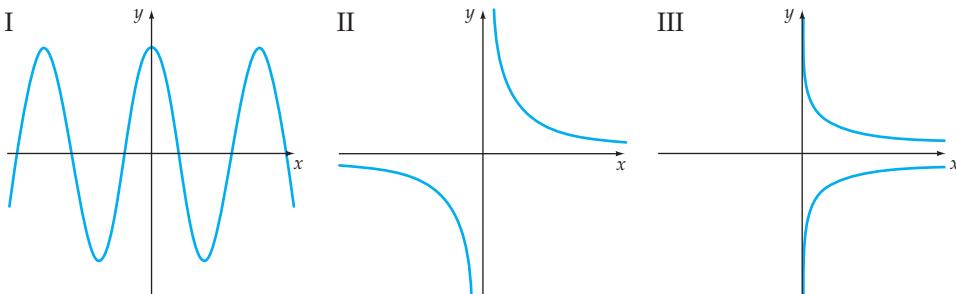
Symmetry with respect to the  $x$ -axis

### Tests for Symmetry with Respect to a Coordinate Axis

The graph of an equation is symmetric with respect to

- the  $y$ -axis if the replacement of  $x$  with  $-x$  leaves the equation unaltered.
- the  $x$ -axis if the replacement of  $y$  with  $-y$  leaves the equation unaltered.

**QUESTION** Which of the graphs below, I, II, or III, is **a.** symmetric with respect to the  $x$ -axis? **b.** symmetric with respect to the  $y$ -axis?



**ANSWER** **a.** III is symmetric with respect to the  $x$ -axis.  
**b.** I is symmetric with respect to the  $y$ -axis.

**EXAMPLE 1** » Determine Symmetries of a Graph

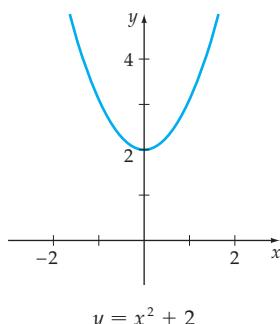
Determine whether the graph of the given equation has symmetry with respect to either the  $x$ - or the  $y$ -axis.

- a.  $y = x^2 + 2$     b.  $x = |y| - 2$

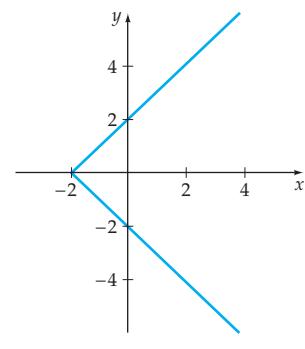
**Solution**

a. The equation  $y = x^2 + 2$  is *unaltered* by the replacement of  $x$  with  $-x$ . That is, the simplification of  $y = (-x)^2 + 2$  yields the original equation  $y = x^2 + 2$ . Thus the graph of  $y = x^2 + 2$  is symmetric with respect to the  $y$ -axis. However, the equation  $y = x^2 + 2$  is *altered* by the replacement of  $y$  with  $-y$ . That is, the simplification of  $-y = x^2 + 2$ , which is  $y = -x^2 - 2$ , does not yield the original equation  $y = x^2 + 2$ . The graph of  $y = x^2 + 2$  is not symmetric with respect to the  $x$ -axis. See **Figure 1.52**.

b. The equation  $x = |y| - 2$  is *altered* by the replacement of  $x$  with  $-x$ . That is, the simplification of  $-x = |y| - 2$ , which is  $x = -|y| + 2$ , does not yield the original equation  $x = |y| - 2$ . This implies that the graph of  $x = |y| - 2$  is not symmetric with respect to the  $y$ -axis. However, the equation  $x = |y| - 2$  is *unaltered* by the replacement of  $y$  with  $-y$ . That is, the simplification of  $x = |-y| - 2$  yields the original equation  $x = |y| - 2$ . The graph of  $x = |y| - 2$  is symmetric with respect to the  $x$ -axis. See **Figure 1.53**.



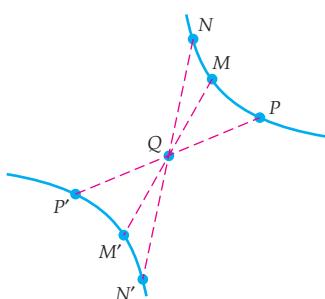
**Figure 1.52**



**Figure 1.53**

**» Try Exercise 14, page 64**
**Definition of Symmetry with Respect to a Point**

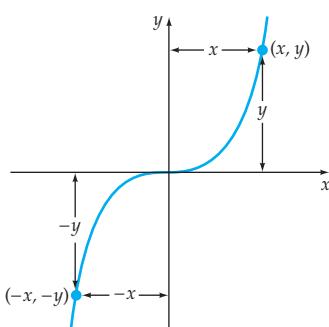
A graph is **symmetric with respect to a point**  $Q$  if for each point  $P$  on the graph there is a point  $P'$  on the graph such that  $Q$  is the midpoint of the line segment  $PP'$ .



**Figure 1.54**

The graph in **Figure 1.54** is symmetric with respect to the point  $Q$ . For any point  $P$  on the graph, there exists a point  $P'$  on the graph such that  $Q$  is the midpoint of  $PP'$ .

When we discuss symmetry with respect to a point, we frequently use the origin. A graph is symmetric with respect to the origin if, whenever the point given by  $(x, y)$  is on the graph, then  $(-x, -y)$  is also on the graph. The graph in **Figure 1.55** is symmetric with respect to the origin.



**Figure 1.55**

Symmetry with respect to the origin

**Test for Symmetry with Respect to the Origin**

The graph of an equation is symmetric with respect to the origin if the replacement of  $x$  with  $-x$  and of  $y$  with  $-y$  leaves the equation unaltered.

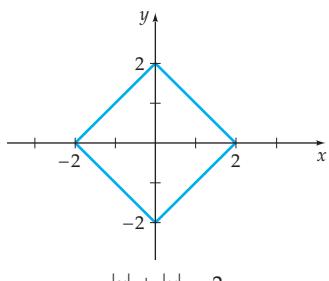
**EXAMPLE 2** » Determine Symmetry with Respect to the Origin

Determine whether the graph of each equation has symmetry with respect to the origin.

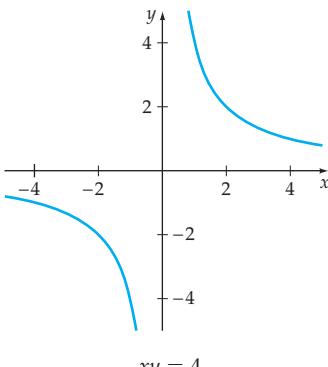
a.  $xy = 4$     b.  $y = x^3 + 1$

**Solution**

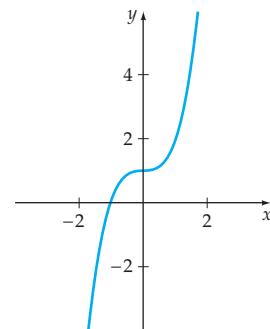
- a. The equation  $xy = 4$  is unaltered by the replacement of  $x$  with  $-x$  and  $y$  with  $-y$ . That is, the simplification of  $(-x)(-y) = 4$  yields the original equation  $xy = 4$ . Thus the graph of  $xy = 4$  is symmetric with respect to the origin. See **Figure 1.56**.
- b. The equation  $y = x^3 + 1$  is altered by the replacement of  $x$  with  $-x$  and  $y$  with  $-y$ . That is, the simplification of  $-y = (-x)^3 + 1$ , which is  $y = x^3 - 1$ , does not yield the original equation  $y = x^3 + 1$ . Thus the graph of  $y = x^3 + 1$  is not symmetric with respect to the origin. See **Figure 1.57**.



**Figure 1.58**



**Figure 1.56**



**Figure 1.57**

## » Try Exercise 24, page 64

Some graphs have more than one symmetry. For example, the graph of  $|x| + |y| = 2$  has symmetry with respect to the  $x$ -axis, the  $y$ -axis, and the origin. **Figure 1.58** is the graph of  $|x| + |y| = 2$ .

**Even and Odd Functions**

Some functions are classified as either *even* or *odd*.

**Definition of Even and Odd Functions**

The function  $f$  is an **even function** if

$$f(-x) = f(x) \quad \text{for all } x \text{ in the domain of } f$$

The function  $f$  is an **odd function** if

$$f(-x) = -f(x) \quad \text{for all } x \text{ in the domain of } f$$

**EXAMPLE 3** » Identify Even or Odd Functions

Determine whether each function is even, odd, or neither.

a.  $f(x) = x^3$     b.  $F(x) = |x|$     c.  $h(x) = x^4 + 2x$

**Solution**

Replace  $x$  with  $-x$  and simplify.

- a.  $f(-x) = (-x)^3 = -x^3 = -(x^3) = -f(x)$   
Because  $f(-x) = -f(x)$ , this function is an odd function.
- b.  $F(-x) = |-x| = |x| = F(x)$   
Because  $F(-x) = F(x)$ , this function is an even function.
- c.  $h(-x) = (-x)^4 + 2(-x) = x^4 - 2x$   
This function is neither an even nor an odd function because

$$h(-x) = x^4 - 2x,$$

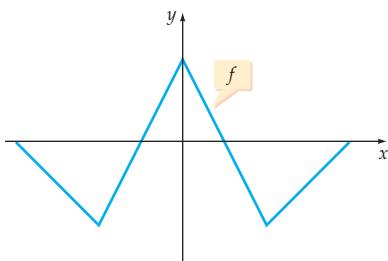
which is not equal to either  $h(x)$  or  $-h(x)$ .

## » Try Exercise 44, page 64

The following properties are results of the tests for symmetry:

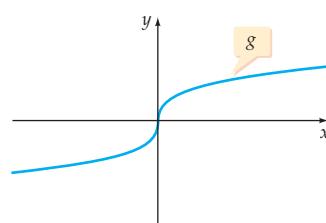
- The graph of an even function is symmetric with respect to the  $y$ -axis.
- The graph of an odd function is symmetric with respect to the origin.

The graph of  $f$  in **Figure 1.59** is symmetric with respect to the  $y$ -axis. It is the graph of an even function. The graph of  $g$  in **Figure 1.60** is symmetric with respect to the origin. It is the graph of an odd function. The graph of  $h$  in **Figure 1.61** is not symmetric with respect to the  $y$ -axis and is not symmetric with respect to the origin. It is neither an even nor an odd function.



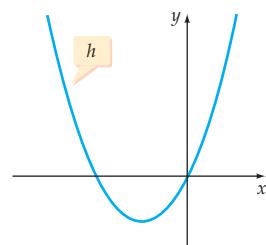
**Figure 1.59**

The graph of an even function is symmetric with respect to the  $y$ -axis.



**Figure 1.60**

The graph of an odd function is symmetric with respect to the origin.



**Figure 1.61**

The graph of a function that is neither even nor odd is not symmetric with respect to the  $y$ -axis or the origin.

## Translations of Graphs

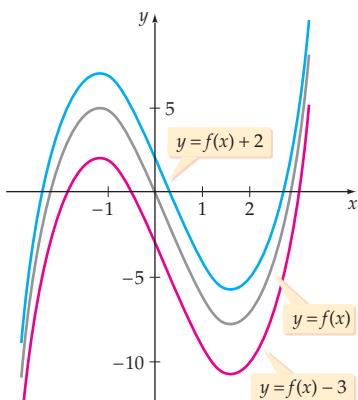


Figure 1.62

The shape of a graph may be exactly the same as the shape of another graph; only their positions in the  $xy$ -plane may differ. For example, the graph of  $y = f(x) + 2$  is the graph of  $y = f(x)$  with each point moved up vertically 2 units. The graph of  $y = f(x) - 3$  is the graph of  $y = f(x)$  with each point moved down vertically 3 units. See **Figure 1.62**.

The graphs of  $y = f(x) + 2$  and  $y = f(x) - 3$  in **Figure 1.62** are called *vertical translations* of the graph of  $y = f(x)$ .

### Vertical Translations

If  $f$  is a function and  $c$  is a positive constant, then the graph of

- $y = f(x) + c$  is the graph of  $y = f(x)$  shifted up *vertically*  $c$  units.
- $y = f(x) - c$  is the graph of  $y = f(x)$  shifted down *vertically*  $c$  units.

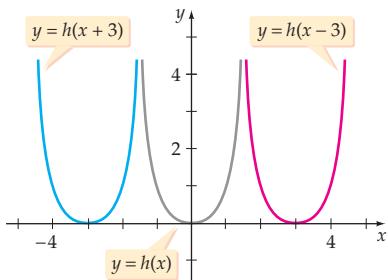


Figure 1.63

In **Figure 1.63**, the graph of  $y = h(x + 3)$  is the graph of  $y = h(x)$  with each point shifted to the left horizontally 3 units. Similarly, the graph of  $y = h(x - 3)$  is the graph of  $y = h(x)$  with each point shifted to the right horizontally 3 units.

The graphs of  $y = h(x + 3)$  and  $y = h(x - 3)$  in **Figure 1.63** are called *horizontal translations* of the graph of  $y = h(x)$ .

### Horizontal Translations

If  $f$  is a function and  $c$  is a positive constant, then the graph of

- $y = f(x + c)$  is the graph of  $y = f(x)$  shifted left *horizontally*  $c$  units.
- $y = f(x - c)$  is the graph of  $y = f(x)$  shifted right *horizontally*  $c$  units.



### Integrating Technology

A graphing calculator can be used to draw the graphs of a *family* of functions. For instance,  $f(x) = x^2 + c$  constitutes a family of functions with **parameter**  $c$ . The only feature of the graph that changes is the value of  $c$ .

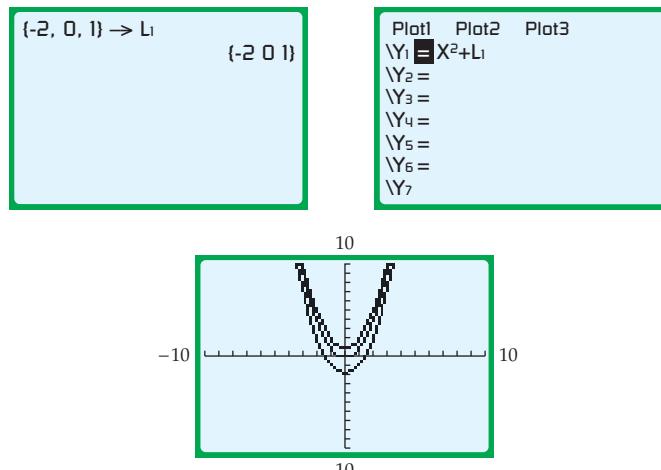
A graphing calculator can be used to produce the graphs of a family of curves for specific values of the parameter. The LIST feature of the calculator can be used. For instance, to graph  $f(x) = x^2 + c$  for  $c = -2, 0$ , and  $1$ , we will create a list and use that list to produce the family of curves. The keystrokes for a TI-83/TI-83 Plus/TI-84 Plus calculator are given below.

2nd { -2 , 0 , 1 2nd } STO 2nd L1

Now use the **[Y=]** key to enter

**Y= X x<sup>2</sup> + 2nd L1 ZOOM 6**

Sample screens for the keystrokes and graphs are shown here. You can use similar keystrokes for Exercises 75–82 of this section.



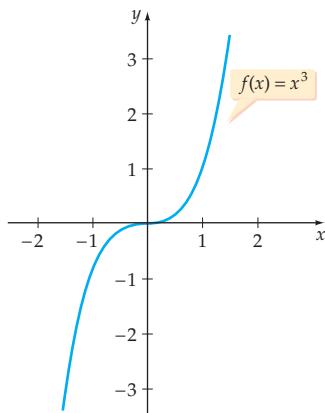
### EXAMPLE 4 » Graph by Using Translations

Use vertical and horizontal translations of the graph of  $f(x) = x^3$ , shown in **Figure 1.64**, to graph

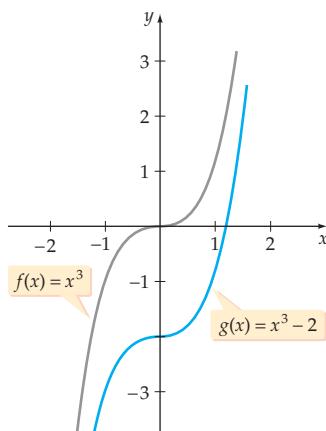
- a.  $g(x) = x^3 - 2$     b.  $h(x) = (x + 1)^3$

#### Solution

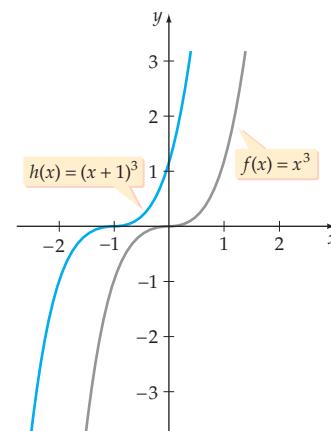
- a. The graph of  $g(x) = x^3 - 2$  is the graph of  $f(x) = x^3$  shifted down vertically 2 units. See **Figure 1.65**.
- b. The graph of  $h(x) = (x + 1)^3$  is the graph of  $f(x) = x^3$  shifted to the left horizontally 1 unit. See **Figure 1.66**.



**Figure 1.64**



**Figure 1.65**



**Figure 1.66**

» Try Exercise 58, page 65

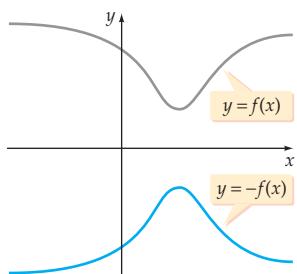


Figure 1.67

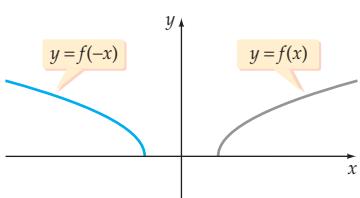


Figure 1.68

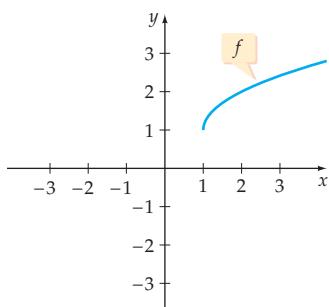


Figure 1.69

## ■ Reflections of Graphs

The graph of  $y = -f(x)$  cannot be obtained from the graph of  $y = f(x)$  by a combination of vertical and/or horizontal shifts. **Figure 1.67** illustrates that the graph of  $y = -f(x)$  is the reflection of the graph of  $y = f(x)$  across the  $x$ -axis.

The graph of  $y = f(-x)$  is the reflection of the graph of  $y = f(x)$  across the  $y$ -axis, as shown in **Figure 1.68**.

### Reflections

The graph of

- $y = -f(x)$  is the graph of  $y = f(x)$  reflected across the  $x$ -axis.
- $y = f(-x)$  is the graph of  $y = f(x)$  reflected across the  $y$ -axis.

### EXAMPLE 5 » Graph by Using Reflections

Use reflections of the graph of  $f(x) = \sqrt{x-1} + 1$ , shown in **Figure 1.69**, to graph

a.  $g(x) = -(\sqrt{x-1} + 1)$     b.  $h(x) = \sqrt{-x-1} + 1$

#### Solution

- Because  $g(x) = -f(x)$ , the graph of  $g$  is the graph of  $f$  reflected across the  $x$ -axis. See **Figure 1.70**.
- Because  $h(x) = f(-x)$ , the graph of  $h$  is the graph of  $f$  reflected across the  $y$ -axis. See **Figure 1.71**.

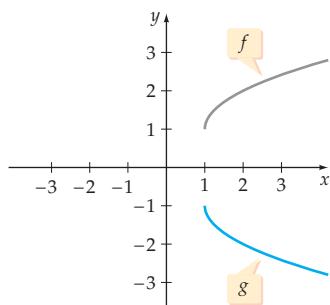


Figure 1.70

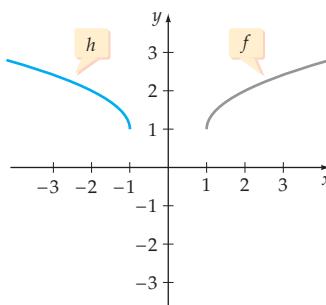


Figure 1.71

### » Try Exercise 68, page 66

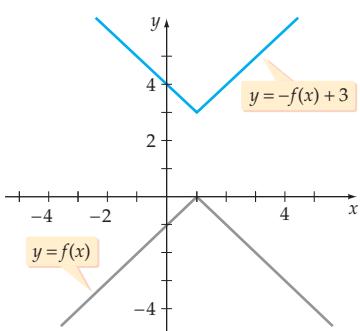


Figure 1.72

Some graphs of functions can be constructed by using a combination of translations and reflections. For instance, the graph of  $y = -f(x) + 3$  in **Figure 1.72** was obtained by reflecting the graph of  $y = f(x)$  in **Figure 1.72** across the  $x$ -axis and then shifting that graph up vertically 3 units.

## Compressing and Stretching of Graphs

The graph of the equation  $y = c \cdot f(x)$  for  $c \neq 1$  vertically compresses or stretches the graph of  $y = f(x)$ . To determine the points on the graph of  $y = c \cdot f(x)$ , multiply the  $y$ -coordinate of each point on the graph of  $y = f(x)$  by  $c$ . For example,

**Figure 1.73** shows that the graph of  $y = \frac{1}{2}|x|$  can be obtained by plotting points that have a  $y$ -coordinate that is one-half of the  $y$ -coordinate of those points that make up the graph of  $y = |x|$ .

If  $0 < c < 1$ , then the graph of  $y = c \cdot f(x)$  is obtained by *compressing* the graph of  $y = f(x)$ . **Figure 1.73** illustrates the vertical compressing of the graph of  $y = |x|$  toward the  $x$ -axis to form the graph of  $y = \frac{1}{2}|x|$ .

If  $c > 1$ , then the graph of  $y = c \cdot f(x)$  is obtained by *stretching* the graph of  $y = f(x)$ . For example, if  $f(x) = |x|$ , then we obtain the graph of

$$y = 2f(x) = 2|x|$$

by stretching the graph of  $f$  away from the  $x$ -axis. See **Figure 1.74**.

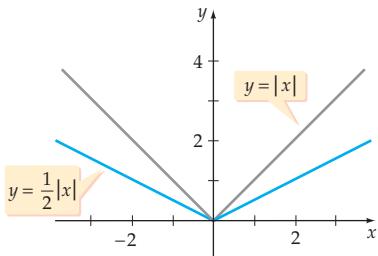


Figure 1.73

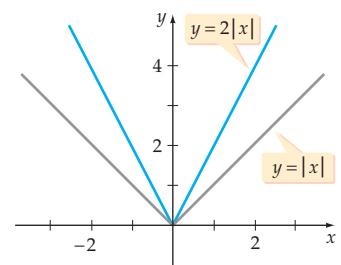


Figure 1.74

### Vertical Stretching and Compressing of Graphs

If  $f$  is a function and  $c$  is a positive constant, then

- if  $c > 1$ , the graph of  $y = c \cdot f(x)$  is the graph of  $y = f(x)$  stretched vertically away from the  $x$ -axis by a factor of  $c$ .
- if  $0 < c < 1$ , the graph of  $y = c \cdot f(x)$  is the graph of  $y = f(x)$  compressed vertically toward the  $x$ -axis by a factor of  $c$ .

### EXAMPLE 6 » Graph by Using Vertical Compressing and Shifting

Graph:  $H(x) = \frac{1}{4}|x| - 3$

#### Solution

The graph of  $y = |x|$  has a V shape that has its lowest point at  $(0, 0)$  and passes through  $(4, 4)$  and  $(-4, 4)$ . The graph of  $y = \frac{1}{4}|x|$  is a compression of the graph of  $y = |x|$ . The  $y$ -coordinates of the ordered pairs  $(0, 0)$ ,  $(4, 1)$ , and  $(-4, 1)$  are obtained by multiplying the  $y$ -coordinates of the ordered pairs  $(0, 0)$ ,  $(4, 4)$ , and  $(-4, 4)$  by  $\frac{1}{4}$ . To find the points on the graph of  $H$ , we still need to subtract 3 from each  $y$ -coordinate. Thus the graph of  $H$  is a V shape that has its lowest point at  $(0, -3)$  and passes through  $(4, -2)$  and  $(-4, -2)$ . See **Figure 1.75**.

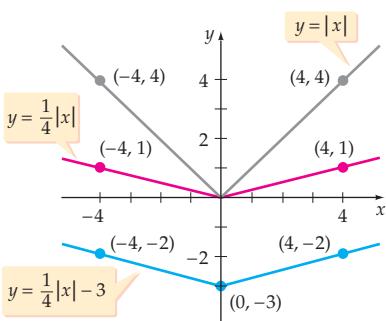


Figure 1.75

» Try Exercise 70, page 66

Some functions can be graphed by using a horizontal compressing or stretching of a given graph. The procedure makes use of the following concept.

### Horizontal Compressing and Stretching of Graphs

If  $f$  is a function and  $c$  is a positive constant, then

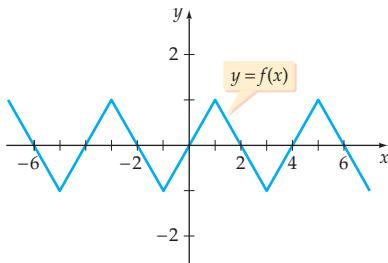
- if  $c > 1$ , the graph of  $y = f(c \cdot x)$  is the graph of  $y = f(x)$  compressed horizontally toward the  $y$ -axis by a factor of  $\frac{1}{c}$ .
- if  $0 < c < 1$ , the graph of  $y = f(c \cdot x)$  is the graph of  $y = f(x)$  stretched horizontally away from the  $y$ -axis by a factor of  $\frac{1}{c}$ .

If the point  $(x, y)$  is on the graph of  $y = f(x)$ , then the graph of  $y = f(cx)$  will contain the point  $\left(\frac{1}{c}x, y\right)$ .

### EXAMPLE 7 » Graph by Using Horizontal Compressing and Stretching

Use the graph of  $y = f(x)$ , shown in **Figure 1.76**, to graph

a.  $y = f(2x)$     b.  $y = f\left(\frac{1}{3}x\right)$



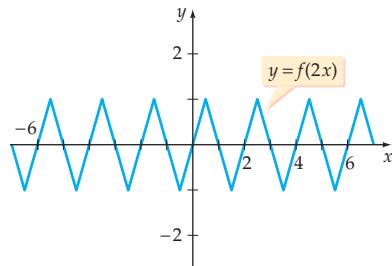
**Figure 1.76**

#### Solution

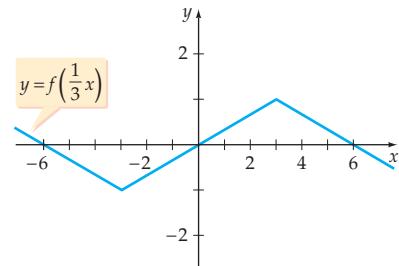
- Because  $2 > 1$ , the graph of  $y = f(2x)$  is a horizontal compression of the graph of  $y = f(x)$  by a factor of  $\frac{1}{2}$ . For example, the point  $(2, 0)$  on the graph of  $y = f(x)$  becomes the point  $(1, 0)$  on the graph of  $y = f(2x)$ . See **Figure 1.77**.
- Since  $0 < \frac{1}{3} < 1$ , the graph of  $y = f\left(\frac{1}{3}x\right)$  is a horizontal stretching of the graph of  $y = f(x)$  by a factor of 3. For example, the point  $(1, 1)$  on the

graph of  $y = f(x)$  becomes the point  $(3, 1)$  on the graph of  $y = f\left(\frac{1}{3}x\right)$ .

See **Figure 1.78**.



**Figure 1.77**



**Figure 1.78**

» Try Exercise 72, page 66



## Topics for Discussion

1. Discuss the meaning of symmetry of a graph with respect to a line. How do you determine whether a graph has symmetry with respect to the  $x$ -axis? with respect to the  $y$ -axis?
2. Discuss the meaning of symmetry of a graph with respect to a point. How do you determine whether a graph has symmetry with respect to the origin?
3. What does it mean to reflect a graph across the  $x$ -axis or across the  $y$ -axis?
4. Explain how the graphs of  $y = 2x^3 - x^2$  and  $y = 2(-x)^3 - (-x)^2$  are related.
5. Given the graph of the function  $y = f(x)$ , explain how to obtain the graph of the function  $y = f(x - 3) + 1$ .
6. The graph of the *step function*  $y = \llbracket x \rrbracket$  has steps that are 1 unit wide. Determine how wide the steps are for the graph of  $y = \left\llbracket \frac{1}{3}x \right\rrbracket$ .

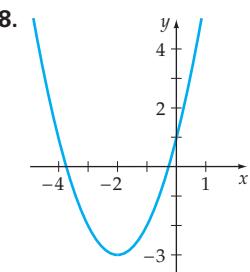
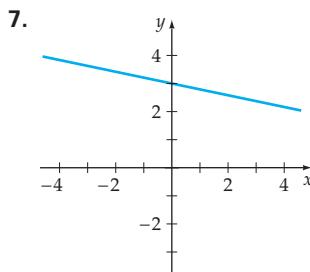
## Exercise Set 1.4

In Exercises 1 to 6, plot the image of the given point with respect to

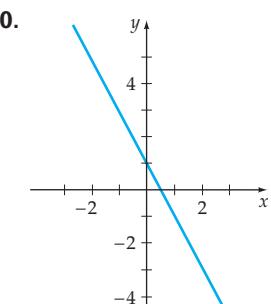
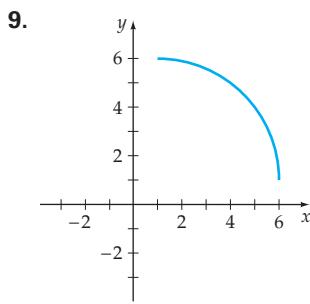
- the  $y$ -axis. Label this point  $A$ .
- the  $x$ -axis. Label this point  $B$ .
- the origin. Label this point  $C$ .

- $P(5, -3)$
- $Q(-4, 1)$
- $R(-2, 3)$
- $S(-5, 3)$
- $T(-4, -5)$
- $U(5, 1)$

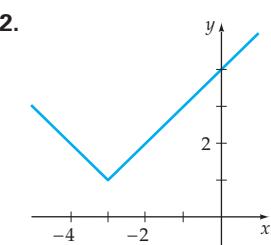
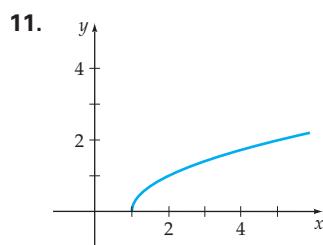
In Exercises 7 and 8, sketch a graph that is symmetric to the given graph with respect to the  $x$ -axis.



In Exercises 9 and 10, sketch a graph that is symmetric to the given graph with respect to the  $y$ -axis.



In Exercises 11 and 12, sketch a graph that is symmetric to the given graph with respect to the origin.



In Exercises 13 to 21, determine whether the graph of each equation is symmetric with respect to the a.  $x$ -axis, b.  $y$ -axis.

- |                    |                     |                      |
|--------------------|---------------------|----------------------|
| 13. $y = 2x^2 - 5$ | 14. $x = 3y^2 - 7$  | 15. $y = x^3 + 2$    |
| 16. $y = x^5 - 3x$ | 17. $x^2 + y^2 = 9$ | 18. $x^2 - y^2 = 10$ |
| 19. $x^2 = y^4$    | 20. $xy = 8$        | 21. $ x  -  y  = 6$  |

In Exercises 22 to 30, determine whether the graph of each equation is symmetric with respect to the origin.

- |                     |                         |                      |
|---------------------|-------------------------|----------------------|
| 22. $y = x + 1$     | 23. $y = 3x - 2$        | 24. $y = x^3 - x$    |
| 25. $y = -x^3$      | 26. $y = \frac{9}{x}$   | 27. $x^2 + y^2 = 10$ |
| 28. $x^2 - y^2 = 4$ | 29. $y = \frac{x}{ x }$ | 30. $ y  =  x $      |

In Exercises 31 to 42, graph the given equation. Label each intercept. Use the concept of symmetry to confirm that the graph is correct.

- |                         |                         |
|-------------------------|-------------------------|
| 31. $y = x^2 - 1$       | 32. $x = y^2 - 1$       |
| 33. $y = x^3 - x$       | 34. $y = -x^3$          |
| 35. $xy = 4$            | 36. $xy = -8$           |
| 37. $y = 2 x - 4 $      | 38. $y =  x - 2  - 1$   |
| 39. $y = (x - 2)^2 - 4$ | 40. $y = (x - 1)^2 - 4$ |
| 41. $y = x -  x $       | 42. $ y  =  x $         |

In Exercises 43 to 56, determine whether the given function is an even function, an odd function, or neither.

- |                             |                             |
|-----------------------------|-----------------------------|
| 43. $g(x) = x^2 - 7$        | 44. $h(x) = x^2 + 1$        |
| 45. $F(x) = x^5 + x^3$      | 46. $G(x) = 2x^5 - 10$      |
| 47. $H(x) = 3 x $           | 48. $T(x) =  x  + 2$        |
| 49. $f(x) = 1$              | 50. $k(x) = 2 + x + x^2$    |
| 51. $r(x) = \sqrt{x^2 + 4}$ | 52. $u(x) = \sqrt{3 - x^2}$ |

53.  $s(x) = 16x^2$

54.  $v(x) = 16x^2 + x$

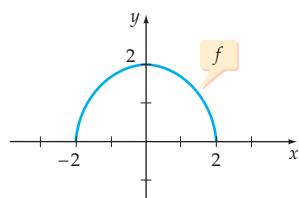
55.  $w(x) = 4 + \sqrt[3]{x}$

56.  $z(x) = \frac{x^3}{x^2 + 1}$

57. Use the graph of  $f$  to sketch the graph of

a.  $y = f(x) + 3$

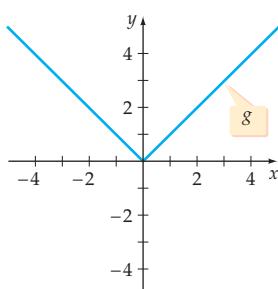
b.  $y = f(x - 3)$



» 58. Use the graph of  $g$  to sketch the graph of

a.  $y = g(x) - 2$

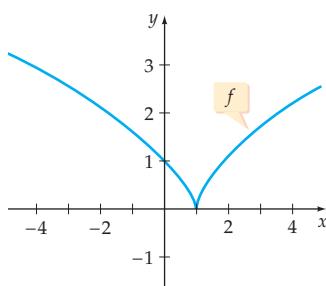
b.  $y = g(x - 3)$



59. Use the graph of  $f$  to sketch the graph of

a.  $y = f(x + 2)$

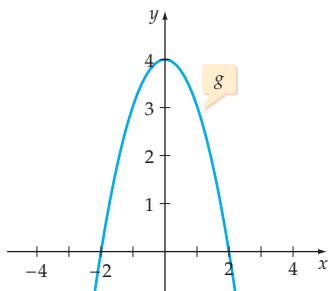
b.  $y = f(x) + 2$



60. Use the graph of  $g$  to sketch the graph of

a.  $y = g(x - 1)$

b.  $y = g(x) - 1$



61. Let  $f$  be a function such that  $f(-2) = 5$ ,  $f(0) = -2$ , and  $f(1) = 0$ . Give the coordinates of three points on the graph of

a.  $y = f(x + 3)$

b.  $y = f(x) + 1$

62. Let  $g$  be a function such that  $g(-3) = -1$ ,  $g(1) = -3$ , and  $g(4) = 2$ . Give the coordinates of three points on the graph of

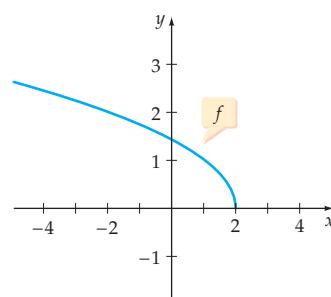
a.  $y = g(x - 2)$

b.  $y = g(x) - 2$

63. Use the graph of  $f$  to sketch the graph of

a.  $y = f(-x)$

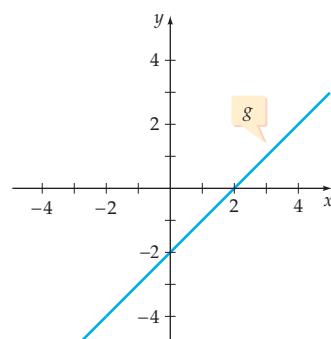
b.  $y = -f(x)$



64. Use the graph of  $g$  to sketch the graph of

a.  $y = -g(x)$

b.  $y = g(-x)$



65. Let  $f$  be a function such that  $f(-1) = 3$  and  $f(2) = -4$ . Give the coordinates of two points on the graph of

a.  $y = f(-x)$

b.  $y = -f(x)$

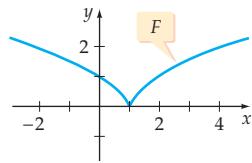
66. Let  $g$  be a function such that  $g(4) = -5$  and  $g(-3) = 2$ . Give the coordinates of two points on the graph of

a.  $y = -g(x)$

b.  $y = g(-x)$

67. Use the graph of  $F$  to sketch the graph of

a.  $y = -F(x)$

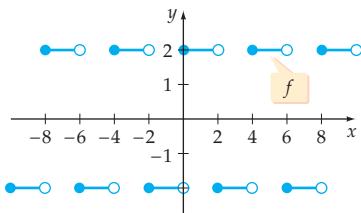


b.  $y = F(-x)$

71. Use the graph of  $y = f(x)$  to sketch the graph of

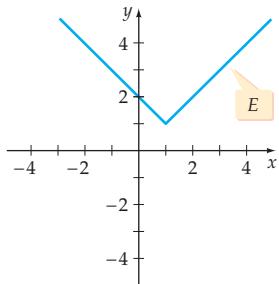
a.  $y = f(2x)$

b.  $y = f\left(\frac{1}{3}x\right)$



68. Use the graph of  $E$  to sketch the graph of

a.  $y = -E(x)$

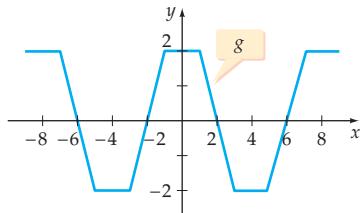


b.  $y = E(-x)$

72. Use the graph of  $y = g(x)$  to sketch the graph of

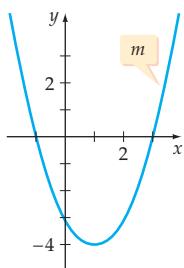
a.  $y = g(2x)$

b.  $y = g\left(\frac{1}{2}x\right)$



69. Use the graph of  $m(x) = x^2 - 2x - 3$  to sketch the graph

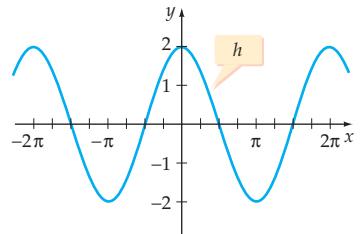
of  $y = -\frac{1}{2}m(x) + 3$ .



73. Use the graph of  $y = h(x)$  to sketch the graph of

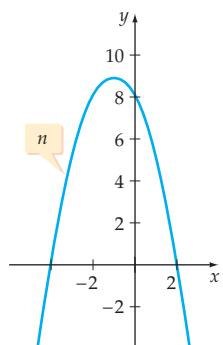
a.  $y = h(2x)$

b.  $y = h\left(\frac{1}{2}x\right)$



70. Use the graph of  $n(x) = -x^2 - 2x + 8$  to sketch the graph

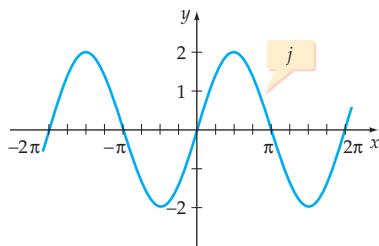
of  $y = \frac{1}{2}n(x) + 1$ .



74. Use the graph of  $y = j(x)$  to sketch the graph of

a.  $y = j(2x)$

b.  $y = j\left(\frac{1}{3}x\right)$





**In Exercises 75 to 82, use a graphing utility.**

75. On the same coordinate axes, graph

$$G(x) = \sqrt[3]{x} + c$$

for  $c = 0, -1$ , and  $3$ .

76. On the same coordinate axes, graph

$$H(x) = \sqrt[3]{x + c}$$

for  $c = 0, -1$ , and  $3$ .

77. On the same coordinate axes, graph

$$J(x) = |2(x + c) - 3| - |x + c|$$

for  $c = 0, -1$ , and  $2$ .

78. On the same coordinate axes, graph

$$K(x) = |x - 1| - |x| + c$$

for  $c = 0, -1$ , and  $2$ .

79. On the same coordinate axes, graph

$$L(x) = cx^2$$

for  $c = 1, \frac{1}{2}$ , and  $2$ .

80. On the same coordinate axes, graph

$$M(x) = c\sqrt{x^2 - 4}$$

for  $c = 1, \frac{1}{3}$ , and  $3$ .

81. On the same coordinate axes, graph

$$S(x) = c(|x - 1| - |x|)$$

for  $c = 1, \frac{1}{4}$ , and  $4$ .

82. On the same coordinate axes, graph

$$T(x) = c\left(\frac{x}{|x|}\right)$$

for  $c = 1, \frac{2}{3}$ , and  $\frac{3}{2}$ .

83. Graph  $V(x) = \llbracket cx \rrbracket, 0 \leq x \leq 6$ , for each value of  $c$ .

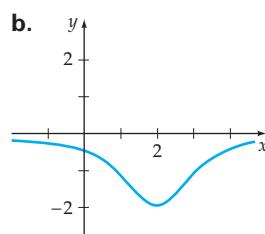
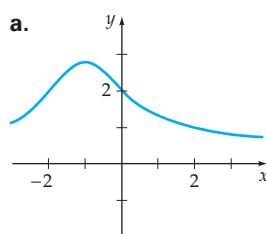
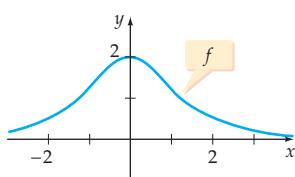
- a.  $c = 1$       b.  $c = \frac{1}{2}$       c.  $c = 2$

84. Graph  $W(x) = \llbracket cx \rrbracket - cx, 0 \leq x \leq 6$ , for each value of  $c$ .

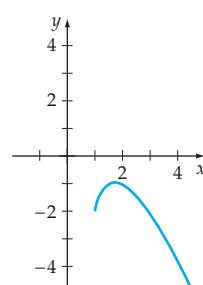
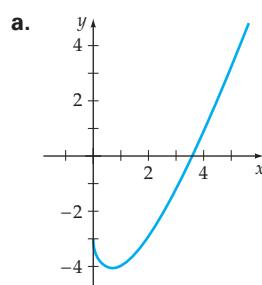
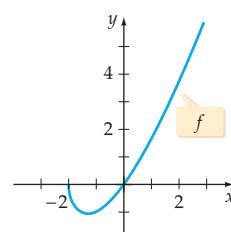
- a.  $c = 1$       b.  $c = \frac{1}{3}$       c.  $c = 3$

## »»» Connecting Concepts »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»»

85. Use the graph of  $f(x) = \frac{2}{x^2 + 1}$  to determine an equation for each graph.



86. Use the graph of  $f(x) = x\sqrt{2 + x}$  to determine an equation for each graph.



## »»» Projects

- 1. DIRICHLET FUNCTION** We owe our present-day definition of a function to the German mathematician Peter Gustav Dirichlet (1805–1859). He created the following unusual function, which is now known as the *Dirichlet function*.

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is a rational number} \\ 1, & \text{if } x \text{ is an irrational number} \end{cases}$$

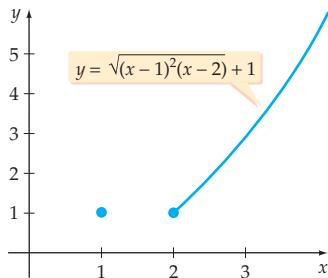
Answer the following questions about the Dirichlet function.

- a. What is its domain?
- b. What is its range?
- c. What are its  $x$ -intercepts?
- d. What is its  $y$ -intercept?
- e. Is it an even or an odd function?
- f.  Explain why a graphing calculator cannot be used to produce an accurate graph of the Dirichlet function.
- g.  Write a sentence or two that describes the graph of the Dirichlet function.

- 2. ISOLATED POINT** Consider the function given by the equation

$$y = \sqrt{(x-1)^2(x-2)} + 1$$

Verify that the point  $(1, 1)$  is a solution of the equation. Now use a graphing utility to graph the function. Does your graph include the isolated point at  $(1, 1)$ , as shown below? If the graphing utility you used failed to include the point  $(1, 1)$ , explain at least one reason for the omission of this isolated point.



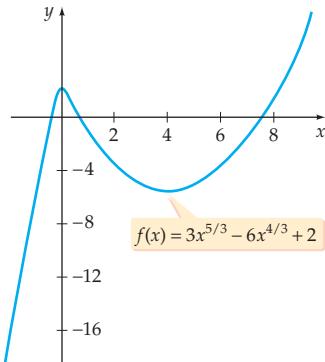
- A LINE WITH A HOLE** The graph of the function

$$f(x) = \frac{(x-2)(x+1)}{(x-2)}$$

is a line with a  $y$ -intercept of 1, a slope of 1, and a hole at  $(2, 3)$ . Use a graphing utility to graph  $f$ . Explain why a graphing utility might not show the hole at  $(2, 3)$ .



- FINDING A COMPLETE GRAPH** Use a graphing utility to graph the function  $f(x) = 3x^{5/3} - 6x^{4/3} + 2$  for  $-2 \leq x \leq 10$ . Compare your graph with the graph below. Does your graph include the part to the left of the  $y$ -axis? If not, how might you enter the function so that the graphing utility would include this part?



## Section 1.5

- Operations on Functions
- The Difference Quotient
- Composition of Functions

# The Algebra of Functions

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A5.

**PS1.** If  $f(x) = x^2 + 3x + 1$  and  $g(x) = 4x + 5$ , find  $f(3) - g(3)$ . [1.3]

**PS2.** If  $f(x) = 3x^2 - x - 4$  and  $g(x) = 2x - 5$ , find  $f(-2) \cdot g(-2)$ . [1.3]

In Exercises PS3 and PS4, find each of the following for  $f(x) = 2x^2 - 5x + 2$ .

**PS3.**  $f(3a)$  [1.3]

**PS4.**  $f(2 + h)$  [1.3]

In Exercises PS5 and PS6, find the domain of each function.

**PS5.**  $F(x) = \frac{x}{x - 1}$  [1.3]

**PS6.**  $r(x) = \sqrt{2x - 8}$  [1.3]

## Operations on Functions

Functions can be defined in terms of other functions. For example, the function defined by  $h(x) = x^2 + 8x$  is the sum of

$$f(x) = x^2 \quad \text{and} \quad g(x) = 8x$$

Thus, if we are given any two functions  $f$  and  $g$ , we can define the four new functions  $f + g$ ,  $f - g$ ,  $fg$ , and  $\frac{f}{g}$  as follows.

### Definitions of Operations on Functions

If  $f$  and  $g$  are functions with domains  $D_f$  and  $D_g$ , then we define the sum, difference, product, and quotient of  $f$  and  $g$  as

|            |   |                                     |
|------------|---|-------------------------------------|
| Sum        | $(f + g)(x) = f(x) + g(x)$                        | Domain: $D_f \cap D_g$              |
| Difference | $(f - g)(x) = f(x) - g(x)$                        | Domain: $D_f \cap D_g$              |
| Product    | $(f \cdot g)(x) = f(x) \cdot g(x)$                | Domain: $D_f \cap D_g$              |
| Quotient   | $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ | Domain: $D_f \cap D_g, g(x) \neq 0$ |

### Example

Let  $f(x) = 3x - 2$  and  $g(x) = x^2 + 6$ . Then

$$(f + g)(x) = f(x) + g(x) = (3x - 2) + (x^2 + 6) = x^2 + 3x + 4$$

$$(f - g)(x) = f(x) - g(x) = (3x - 2) - (x^2 + 6) = -x^2 + 3x - 8$$

$$(f \cdot g)(x) = f(x) \cdot g(x) = (3x - 2)(x^2 + 6) = 3x^3 - 2x^2 + 18x - 12$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x - 2}{x^2 + 6}$$

**EXAMPLE 1** » Determine the Domain of a Function

If  $f(x) = \sqrt{x - 1}$  and  $g(x) = x^2 - 4$ , find the domains of  $f + g$ ,  $f - g$ ,  $fg$ , and  $\frac{f}{g}$ .

**Solution**

Note that  $f$  has the domain  $\{x | x \geq 1\}$  and  $g$  has the domain of all real numbers. Therefore, the domain of  $f + g$ ,  $f - g$ , and  $fg$  is  $\{x | x \geq 1\}$ . Because  $g(x) = 0$  when  $x = -2$  or  $x = 2$ , neither  $-2$  nor  $2$  is in the domain of  $\frac{f}{g}$ . The domain of  $\frac{f}{g}$  is  $\{x | x \geq 1 \text{ and } x \neq 2\}$ .

» Try Exercise 10, page 77

**EXAMPLE 2** » Evaluate Functions

Let  $f(x) = x^2 - 9$  and  $g(x) = 2x + 6$ . Find

- a.  $(f + g)(5)$     b.  $(fg)(-1)$     c.  $\left(\frac{f}{g}\right)(4)$

**Solution**

a.  $(f + g)(x) = f(x) + g(x) = (x^2 - 9) + (2x + 6) = x^2 + 2x - 3$   
Therefore,  $(f + g)(5) = (5)^2 + 2(5) - 3 = 25 + 10 - 3 = 32$ .

b.  $(fg)(x) = f(x) \cdot g(x) = (x^2 - 9)(2x + 6) = 2x^3 + 6x^2 - 18x - 54$   
Therefore,  $(fg)(-1) = 2(-1)^3 + 6(-1)^2 - 18(-1) - 54$   
 $= -2 + 6 + 18 - 54 = -32$ .

c.  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 9}{2x + 6} = \frac{(x+3)(x-3)}{2(x+3)} = \frac{x-3}{2}, \quad x \neq -3$   
Therefore,  $\left(\frac{f}{g}\right)(4) = \frac{4-3}{2} = \frac{1}{2}$ .

» Try Exercise 14, page 77

**The Difference Quotient**

The expression

$$\frac{f(x + h) - f(x)}{h}, \quad h \neq 0$$

is called the **difference quotient** of  $f$ . It enables us to study the manner in which a function changes in value as the independent variable changes.

**EXAMPLE 3** » Determine a Difference Quotient

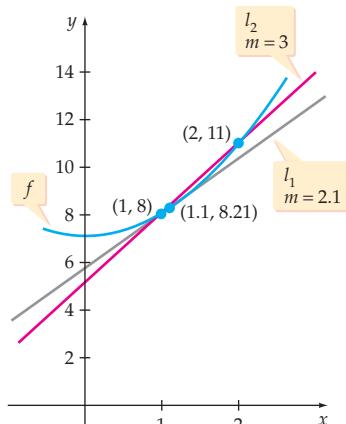
Determine the difference quotient of  $f(x) = x^2 + 7$ .

**Solution**

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 + 7] - [x^2 + 7]}{h} \\ &= \frac{x^2 + 2xh + h^2 + 7 - x^2 - 7}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{h(2x + h)}{h} = 2x + h \end{aligned}$$

- Apply the difference quotient.

## » Try Exercise 30, page 77



**Figure 1.79**

The difference quotient  $2x + h$  of  $f(x) = x^2 + 7$  from Example 3 is the slope of the secant line through the points

$$(x, f(x)) \quad \text{and} \quad (x + h, f(x + h))$$

For instance, let  $x = 1$  and  $h = 1$ . Then the difference quotient is

$$2x + h = 2(1) + 1 = 3$$

This is the slope of the secant line  $l_2$  through  $(1, 8)$  and  $(2, 11)$ , as shown in Figure 1.79. If we let  $x = 1$  and  $h = 0.1$ , then the difference quotient is

$$2x + h = 2(1) + 0.1 = 2.1$$

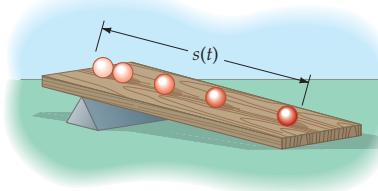
This is the slope of the secant line  $l_1$  through  $(1, 8)$  and  $(1.1, 8.21)$ .

The difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

can be used to compute *average velocities*. In such cases it is traditional to replace  $f$  with  $s$  (for distance), the variable  $x$  with the variable  $a$  (for the time at the start of an observed interval of time), and the variable  $h$  with  $\Delta t$  (read as “delta  $t$ ”), where  $\Delta t$  is the difference between the time at the end of an interval and the time at the start of the interval. For example, if an experiment is observed over the time interval from  $t = 3$  seconds to  $t = 5$  seconds, then the time interval is denoted as  $[3, 5]$  with  $a = 3$  and  $\Delta t = 5 - 3 = 2$ . Thus if the distance traveled by a ball that rolls down a ramp is given by  $s(t)$ , where  $t$  is the time in seconds after the ball is released (see Figure 1.80), then the **average velocity** of the ball over the interval  $t = a$  to  $t = a + \Delta t$  is the difference quotient

$$\frac{s(a + \Delta t) - s(a)}{\Delta t}$$



**Figure 1.80**

**E X A M P L E 4** » Evaluate Average Velocities

The distance traveled by a ball rolling down a ramp is given by  $s(t) = 4t^2$ , where  $t$  is the time in seconds after the ball is released, and  $s(t)$  is measured in feet. Evaluate the average velocity of the ball for each time interval.

- a. [3, 5]    b. [3, 4]    c. [3, 3.5]    d. [3, 3.01]

**Solution**

- a. In this case,  $a = 3$  and  $\Delta t = 5 - 3 = 2$ . Thus the average velocity over this interval is

$$\begin{aligned}\frac{s(a + \Delta t) - s(a)}{\Delta t} &= \frac{s(3 + 2) - s(3)}{2} = \frac{s(5) - s(3)}{2} = \frac{100 - 36}{2} \\ &= 32 \text{ feet per second}\end{aligned}$$

- b. Let  $a = 3$  and  $\Delta t = 4 - 3 = 1$ .

$$\begin{aligned}\frac{s(a + \Delta t) - s(a)}{\Delta t} &= \frac{s(3 + 1) - s(3)}{1} = \frac{s(4) - s(3)}{1} = \frac{64 - 36}{1} \\ &= 28 \text{ feet per second}\end{aligned}$$

- c. Let  $a = 3$  and  $\Delta t = 3.5 - 3 = 0.5$ .

$$\frac{s(a + \Delta t) - s(a)}{\Delta t} = \frac{s(3 + 0.5) - s(3)}{0.5} = \frac{49 - 36}{0.5} = 26 \text{ feet per second}$$

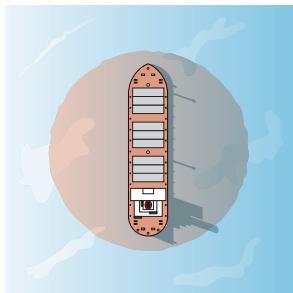
- d. Let  $a = 3$  and  $\Delta t = 3.01 - 3 = 0.01$ .

$$\begin{aligned}\frac{s(a + \Delta t) - s(a)}{\Delta t} &= \frac{s(3 + 0.01) - s(3)}{0.01} = \frac{36.2404 - 36}{0.01} \\ &= 24.04 \text{ feet per second}\end{aligned}$$

**» Try Exercise 72, page 79**
**Composition of Functions**

*Composition of functions* is another way in which functions can be combined. This method of combining functions uses the output of one function as the input for a second function.

Suppose that the spread of oil from a leak in a tanker can be approximated by a circle with the tanker at its center. The radius  $r$  (in feet) of the spill  $t$  hours after the leak begins is given by  $r(t) = 150\sqrt{t}$ . The area of the spill is the area of a circle and is given by the formula  $A(r) = \pi r^2$ . To find the area of the spill 4 hours after the leak begins, we first find the radius of the spill and then use that number to find the area of the spill.



$$\begin{aligned}r(t) &= 150\sqrt{t} & A(r) &= \pi r^2 \\ r(4) &= 150\sqrt{4} & \bullet t = 4 \text{ hours} & A(300) = \pi(300^2) & \bullet r = 300 \text{ feet} \\ &= 150(2) & & = 90,000\pi & \\ &= 300 & & \approx 283,000 &\end{aligned}$$

The area of the spill after 4 hours is approximately 283,000 square feet.

There is an alternative way to solve this problem. Because the area of the spill depends on the radius and the radius depends on the time, there is a relationship between area and time. We can determine this relationship by evaluating the formula for the area of a circle using  $r(t) = 150\sqrt{t}$ . This will give the area of the spill as a function of time.

$$\begin{aligned} A(r) &= \pi r^2 \\ A[r(t)] &= \pi[r(t)]^2 && \bullet \text{Replace } r \text{ by } r(t). \\ &= \pi[150\sqrt{t}]^2 && \bullet r(t) = 150\sqrt{t} \\ A(t) &= 22,500\pi t && \bullet \text{Simplify.} \end{aligned}$$

The area of the spill as a function of time is  $A(t) = 22,500\pi t$ . To find the area of the oil spill after 4 hours, evaluate this function at  $t = 4$ .

$$\begin{aligned} A(t) &= 22,500\pi t \\ A(4) &= 22,500\pi(4) && \bullet t = 4 \text{ hours} \\ &= 90,000\pi \\ &\approx 283,000 \end{aligned}$$

This is the same result we calculated earlier.

The function  $A(t) = 22,500\pi t$  is referred to as the *composition* of  $A$  with  $r$ . The notation  $A \circ r$  is used to denote this composition of functions. That is,

$$(A \circ r)(t) = 22,500\pi t$$

### take note

The requirement in the definition of the composition of two functions that  $g(x)$  be in the domain of  $f$  for all  $x$  in the domain of  $g$  is important. For instance, let

$$f(x) = \frac{1}{x-1} \quad \text{and}$$

$$g(x) = 3x - 5$$

When  $x = 2$ ,

$$g(2) = 3(2) - 5 = 1$$

$$\begin{aligned} f[g(2)] &= f(1) \\ &= \frac{1}{1-1} \\ &= \frac{1}{0} && \bullet \text{Undefined} \end{aligned}$$

In this case,  $g(2)$  is not in the domain of  $f$ . Thus the composition  $(f \circ g)(x)$  is not defined at 2.

### Definition of the Composition of Two Functions

Let  $f$  and  $g$  be two functions such that  $g(x)$  is in the domain of  $f$  for all  $x$  in the domain of  $g$ . Then the composition of the two functions, denoted by  $f \circ g$ , is the function whose value at  $x$  is given by  $(f \circ g)(x) = f[g(x)]$ .

The function defined by  $(f \circ g)(x)$  is also called the *composite* of  $f$  and  $g$ . We read  $(f \circ g)(x)$  as “ $f$  circle  $g$  of  $x$ ” and  $f[g(x)]$  as “ $f$  of  $g$  of  $x$ .”

Consider the functions  $f(x) = 2x - 1$  and  $g(x) = x^2 - 3$ . The expression  $(f \circ g)(-1)$  (or, equivalently,  $f[g(-1)]$ ) means to evaluate the function  $f$  at  $g(-1)$ .

$$\begin{aligned} g(x) &= x^2 - 3 \\ g(-1) &= (-1)^2 - 3 && \bullet \text{Evaluate } g \text{ at } -1. \\ &= -2 \\ f(x) &= 2x - 1 \\ f[g(-1)] &= 2[g(-1)] - 1 && \bullet \text{Replace } x \text{ by } g(-1). \\ &= 2(-2) - 1 = -5 && \bullet g(-1) = -2 \end{aligned}$$

A graphical depiction of the composition  $(f \circ g)(-1)$  would look something like **Figure 1.81** on page 74.

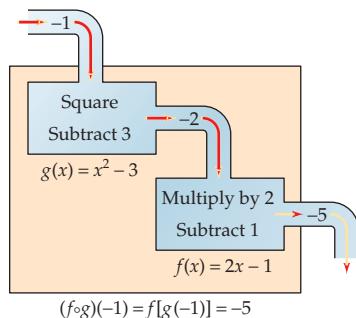


Figure 1.81

We can find a general expression for  $f[g(x)]$  by evaluating  $f$  at  $g(x)$ .

$$\begin{aligned} f(x) &= 2x - 1 \\ f[g(x)] &= 2[g(x)] - 1 && \bullet \text{ Replace } x \text{ by } g(x). \\ &= 2[x^2 - 3] - 1 && \bullet \text{ Replace } g(x) \text{ by } x^2 - 3. \\ &= 2x^2 - 7 && \bullet \text{ Simplify.} \end{aligned}$$

Thus  $f[g(x)] = 2x^2 - 7$ . If we evaluate this function at  $-1$ , we have

$$\begin{aligned} f[g(x)] &= 2x^2 - 7 \\ f[g(-1)] &= 2(-1)^2 - 7 = 2 - 7 = -5 \end{aligned}$$

This is the same result that is obtained in **Figure 1.81**.

In general, the composition of functions is not a commutative operation. That is,  $(f \circ g)(x) \neq (g \circ f)(x)$ . To verify this, we will compute the composition  $(g \circ f)(x) = g[f(x)]$ , again using the functions  $f(x) = 2x - 1$  and  $g(x) = x^2 - 3$ .

$$\begin{aligned} g(x) &= x^2 - 3 \\ g[f(x)] &= [f(x)]^2 - 3 && \bullet \text{ Replace } x \text{ by } f(x). \\ &= [2x - 1]^2 - 3 && \bullet \text{ Replace } f(x) \text{ by } 2x - 1. \\ &= 4x^2 - 4x - 2 && \bullet \text{ Simplify.} \end{aligned}$$

Thus  $g[f(x)] = 4x^2 - 4x - 2$ , which is not equal to  $f[g(x)] = 2x^2 - 7$ . Therefore,  $(f \circ g)(x) \neq (g \circ f)(x)$  and composition is not a commutative operation.

**QUESTION** Let  $f(x) = x - 1$  and  $g(x) = x + 1$ . Then  $f[g(x)] = g[f(x)]$ . (You should verify this statement.) Does this contradict the statement we made that composition is not a commutative operation?

### EXAMPLE 5 » Form Composite Functions

If  $f(x) = x^2 - 3x$  and  $g(x) = 2x + 1$ , find

- a.  $(g \circ f)(x)$     b.  $(f \circ g)(x)$

**ANSWER** No. When we say that composition is not a commutative operation, we mean that generally, given any two functions,  $(f \circ g)(x) \neq (g \circ f)(x)$ . However, there may be particular instances in which  $(f \circ g)(x) = (g \circ f)(x)$ . It turns out that these particular instances are quite important, as we shall see later.

**Solution**

a. 
$$\begin{aligned}(g \circ f)(x) &= g[f(x)] = 2(f(x)) + 1 \\&= 2(x^2 - 3x) + 1 \\&= 2x^2 - 6x + 1\end{aligned}$$

- Substitute  $f(x)$  for  $x$  in  $g$ .
- $f(x) = x^2 - 3x$

b. 
$$\begin{aligned}(f \circ g)(x) &= f[g(x)] = (g(x))^2 - 3(g(x)) \\&= (2x + 1)^2 - 3(2x + 1) \\&= 4x^2 + 4x + 1 - 6x - 3 \\&= 4x^2 - 2x - 2\end{aligned}$$

- Substitute  $g(x)$  for  $x$  in  $f$ .
- $g(x) = 2x + 1$

» Try Exercise 38, page 78

**Caution** Some care must be used when forming the composition of functions. For instance, if  $f(x) = x + 1$  and  $g(x) = \sqrt{x - 4}$ , then

$$(g \circ f)(2) = g[f(2)] = g(3) = \sqrt{3 - 4} = \sqrt{-1}$$

which is not a real number. We can avoid this problem by imposing suitable restrictions on the domain of  $f$  so that the range of  $f$  is part of the domain of  $g$ . If the domain of  $f$  is restricted to  $[3, \infty)$ , then the range of  $f$  is  $[4, \infty)$ . But this is precisely the domain of  $g$ . Note that  $2 \notin [3, \infty)$ , and thus we avoid the problem of  $(g \circ f)(2)$  not being a real number.

To evaluate  $(f \circ g)(c)$  for some constant  $c$ , you can use either of the following methods.

**Method 1** First evaluate  $g(c)$ . Then substitute this result for  $x$  in  $f(x)$ .

**Method 2** First determine  $f[g(x)]$  and then substitute  $c$  for  $x$ .

### EXAMPLE 6 » Evaluate a Composite Function

Evaluate  $(f \circ g)(3)$ , where  $f(x) = 2x - 7$  and  $g(x) = x^2 + 4$ .

**Solution**

**Method 1** 
$$\begin{aligned}(f \circ g)(3) &= f[g(3)] \\&= f[(3)^2 + 4] \\&= f(13) \\&= 2(13) - 7 = 19\end{aligned}$$

- Evaluate  $g(3)$ .
- Substitute 13 for  $x$  in  $f$ .

**Method 2** 
$$\begin{aligned}(f \circ g)(x) &= 2[g(x)] - 7 \\&= 2[x^2 + 4] - 7 \\&= 2x^2 + 1 \\(f \circ g)(3) &= 2(3)^2 + 1 = 19\end{aligned}$$

- Form  $f[g(x)]$ .
- Substitute 3 for  $x$ .

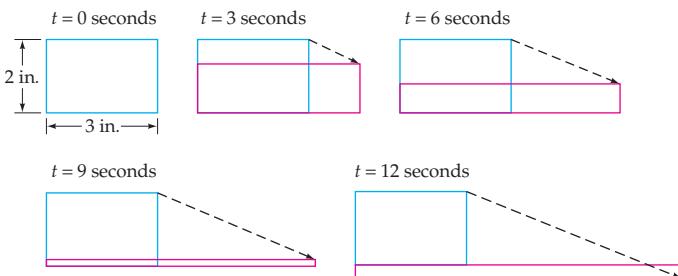
**take note**

In Example 6, both Method 1 and Method 2 produce the same result. Although Method 2 is longer, it is the better method if you must evaluate  $(f \circ g)(x)$  for several values of  $x$ .

» Try Exercise 50, page 78

**EXAMPLE 7** » Use a Composite Function to Solve an Application

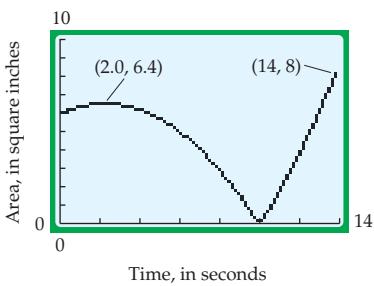

A graphic artist has drawn a 3-inch by 2-inch rectangle on a computer screen. The artist has been scaling the size of the rectangle for  $t$  seconds in such a way that the upper right corner of the original rectangle is moving to the right at the rate of 0.5 inch per second and downward at the rate of 0.2 inch per second. See **Figure 1.82**.



**Figure 1.82**

- Write the length  $l$  and the width  $w$  of the scaled rectangle as functions of  $t$ .
- Write the area  $A$  of the scaled rectangle as a function of  $t$ .
- Find the intervals on which  $A$  is an increasing function for  $0 \leq t \leq 14$ . Also find the intervals on which  $A$  is a decreasing function.
- Find the value of  $t$  (where  $0 \leq t \leq 14$ ) that maximizes  $A(t)$ .

### Solution



$$A = (3 + 0.5t)|2 - 0.2t|$$

**Figure 1.83**

- Because  $distance = rate \cdot time$ , we see that the change in  $l$  is given by  $0.5t$ . Therefore, the length at any time  $t$  is  $l = 3 + 0.5t$ . For  $0 \leq t \leq 10$ , the width is given by  $w = 2 - 0.2t$ . For  $10 < t \leq 14$ , the width is  $w = -2 + 0.2t$ . In either case the width can be determined by finding  $w = |2 - 0.2t|$ . (The absolute value symbol is needed to keep the width positive for  $10 < t \leq 14$ .)
- $A = lw = (3 + 0.5t)|2 - 0.2t|$
- Use a graphing utility to determine that  $A$  is increasing on  $[0, 2]$  and on  $[10, 14]$  and that  $A$  is decreasing on  $[2, 10]$ . See **Figure 1.83**.
- The highest point on the graph of  $A$  occurs when  $t = 14$  seconds. See **Figure 1.83**.

**» Try Exercise 66, page 78**

You may be inclined to think that if the area of a rectangle is decreasing, then its perimeter is also decreasing, but this is not always the case. For example, the area of the scaled rectangle in Example 7 was shown to decrease on  $[2, 10]$  even though its perimeter is always increasing. See Exercise 68 in Exercise Set 1.5.



## Topics for Discussion

- The domain of  $f + g$  consists of all real numbers formed by the *union* of the domain of  $f$  and the domain of  $g$ . Do you agree?
- Given  $f(x) = 3x - 2$  and  $g(x) = \frac{1}{3}x + \frac{2}{3}$ , determine  $f \circ g$  and  $g \circ f$ . Do your results show that composition of functions is a commutative operation?
- A tutor states that the difference quotient of  $f(x) = x^2$  and the difference quotient of  $g(x) = x^2 + 4$  are the same. Do you agree?
- A classmate states that the difference quotient of any linear function  $f(x) = mx + b$  is always  $m$ . Do you agree?
- When we use a difference quotient to determine an average velocity, we generally replace the variable  $h$  with the variable  $\Delta t$ . What does  $\Delta t$  represent?



## Exercise Set 1.5

In Exercises 1 to 12, use the given functions  $f$  and  $g$  to find  $f + g$ ,  $f - g$ ,  $fg$ , and  $\frac{f}{g}$ . State the domain of each.

1.  $f(x) = x^2 - 2x - 15$ ,  $g(x) = x + 3$

2.  $f(x) = x^2 - 25$ ,  $g(x) = x - 5$

3.  $f(x) = 2x + 8$ ,  $g(x) = x + 4$

4.  $f(x) = 5x - 15$ ,  $g(x) = x - 3$

5.  $f(x) = x^3 - 2x^2 + 7x$ ,  $g(x) = x$

6.  $f(x) = x^2 - 5x - 8$ ,  $g(x) = -x$

7.  $f(x) = 4x - 7$ ,  $g(x) = 2x^2 + 3x - 5$

8.  $f(x) = 6x + 10$ ,  $g(x) = 3x^2 + x - 10$

9.  $f(x) = \sqrt{x - 3}$ ,  $g(x) = x$

10.  $f(x) = \sqrt{x - 4}$ ,  $g(x) = -x$

11.  $f(x) = \sqrt{4 - x^2}$ ,  $g(x) = 2 + x$

12.  $f(x) = \sqrt{x^2 - 9}$ ,  $g(x) = x - 3$

In Exercises 13 to 28, evaluate the indicated function, where  $f(x) = x^2 - 3x + 2$  and  $g(x) = 2x - 4$ .

13.  $(f + g)(5)$

14.  $(f + g)(-7)$

15.  $(f + g)\left(\frac{1}{2}\right)$

16.  $(f + g)\left(\frac{2}{3}\right)$

17.  $(f - g)(-3)$

18.  $(f - g)(24)$

19.  $(f - g)(-1)$

20.  $(f - g)(0)$

21.  $(fg)(7)$

22.  $(fg)(-3)$

23.  $(fg)\left(\frac{2}{5}\right)$

24.  $(fg)(-100)$

25.  $\left(\frac{f}{g}\right)(-4)$

26.  $\left(\frac{f}{g}\right)(11)$

27.  $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$

28.  $\left(\frac{f}{g}\right)\left(\frac{1}{4}\right)$

In Exercises 29 to 36, find the difference quotient of the given function.

29.  $f(x) = 2x + 4$

30.  $f(x) = 4x - 5$

31.  $f(x) = x^2 - 6$

32.  $f(x) = x^2 + 11$

33.  $f(x) = 2x^2 + 4x - 3$

34.  $f(x) = 2x^2 - 5x + 7$

35.  $f(x) = -4x^2 + 6$

36.  $f(x) = -5x^2 - 4x$

In Exercises 37 to 48, find  $(g \circ f)(x)$  and  $(f \circ g)(x)$  for the given functions  $f$  and  $g$ .

37.  $f(x) = 3x + 5, \quad g(x) = 2x - 7$

38.  $f(x) = 2x - 7, \quad g(x) = 3x + 2$

39.  $f(x) = x^2 + 4x - 1, \quad g(x) = x + 2$

40.  $f(x) = x^2 - 11x, \quad g(x) = 2x + 3$

41.  $f(x) = x^3 + 2x, \quad g(x) = -5x$

42.  $f(x) = -x^3 - 7, \quad g(x) = x + 1$

43.  $f(x) = \frac{2}{x+1}, \quad g(x) = 3x - 5$

44.  $f(x) = \sqrt{x+4}, \quad g(x) = \frac{1}{x}$

45.  $f(x) = \frac{1}{x^2}, \quad g(x) = \sqrt{x-1}$

46.  $f(x) = \frac{6}{x-2}, \quad g(x) = \frac{3}{5x}$

47.  $f(x) = \frac{3}{|5-x|}, \quad g(x) = -\frac{2}{x}$

48.  $f(x) = |2x+1|, \quad g(x) = 3x^2 - 1$

In Exercises 49 to 64, evaluate each composite function, where  $f(x) = 2x + 3$ ,  $g(x) = x^2 - 5x$ , and  $h(x) = 4 - 3x^2$ .

49.  $(g \circ f)(4)$

50.  $(f \circ g)(4)$

51.  $(f \circ g)(-3)$

52.  $(g \circ f)(-1)$

53.  $(g \circ h)(0)$

54.  $(h \circ g)(0)$

55.  $(f \circ f)(8)$

56.  $(f \circ f)(-8)$

57.  $(h \circ g)\left(\frac{2}{5}\right)$

58.  $(g \circ h)\left(-\frac{1}{3}\right)$

59.  $(g \circ f)(\sqrt{3})$

60.  $(f \circ g)(\sqrt{2})$

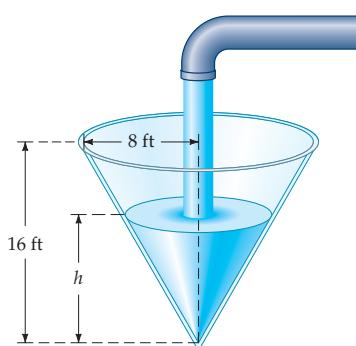
61.  $(g \circ f)(2c)$

62.  $(f \circ g)(3k)$

63.  $(g \circ h)(k+1)$

64.  $(h \circ g)(k-1)$

65. **WATER TANK** A water tank has the shape of a right circular cone with height 16 feet and radius 8 feet. Water is running into the tank so that the radius  $r$  (in feet) of the surface of the water is given by  $r = 1.5t$ , where  $t$  is the time (in minutes) that the water has been running.



- a. The area  $A$  of the surface of the water is  $A = \pi r^2$ . Find  $A(t)$  and use it to determine the area of the surface of the water when  $t = 2$  minutes.

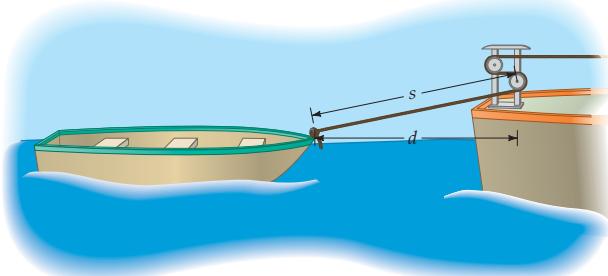
- b. The volume  $V$  of the water is given by  $V = \frac{1}{3} \pi r^2 h$ .

Find  $V(t)$  and use it to determine the volume of the water when  $t = 3$  minutes. (Hint: The height of the water in the cone is always twice the radius of the surface of the water.)



66. **SCALING A RECTANGLE** Rework Example 7 of this section with the scaling as follows. The upper right corner of the original rectangle is pulled to the left at 0.5 inch per second and downward at 0.2 inch per second.

67. **TOWING A BOAT** A boat is towed by a rope that runs through a pulley that is 4 feet above the point where the rope is tied to the boat. The length (in feet) of the rope from the boat to the pulley is given by  $s = 48 - t$ , where  $t$  is the time in seconds that the boat has been in tow. The horizontal distance from the pulley to the boat is  $d$ .



- a. Find  $d(t)$ .      b. Evaluate  $s(35)$  and  $d(35)$ .

- 68.**  PERIMETER OF A SCALED RECTANGLE Show by a graph that the perimeter

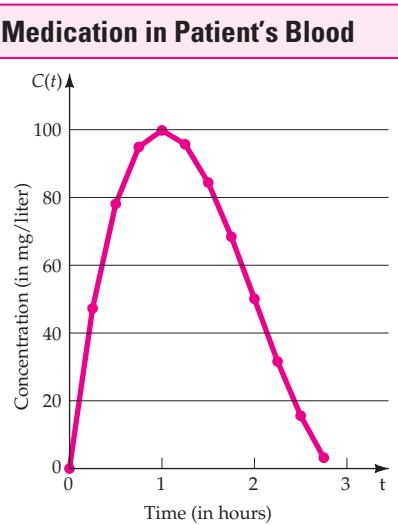
$$P = 2(3 + 0.5t) + 2|2 - 0.2t|$$

of the scaled rectangle in Example 7 of this section is an increasing function over  $0 \leq t \leq 14$ .

- 69. CONVERSION FUNCTIONS** The function  $F(x) = \frac{x}{12}$  converts  $x$  inches to feet. The function  $Y(x) = \frac{x}{3}$  converts  $x$  feet to yards. Explain the meaning of  $(Y \circ F)(x)$ .

- 70. CONVERSION FUNCTIONS** The function  $F(x) = 3x$  converts  $x$  yards to feet. The function  $I(x) = 12x$  converts  $x$  feet to inches. Explain the meaning of  $(I \circ F)(x)$ .

- 71.**  **CONCENTRATION OF A MEDICATION** The concentration  $C(t)$  (in milligrams per liter) of a medication in a patient's blood is given by the data in the following table.



The **average rate of change** of the concentration over the time interval from  $t = a$  to  $t = a + \Delta t$  is

$$\frac{C(a + \Delta t) - C(a)}{\Delta t}$$

Use the data in the table to evaluate the average rate of change for each of the following time intervals.

- a. [0, 1] (Hint: In this case,  $a = 0$  and  $\Delta t = 1$ .) Compare this result to the slope of the line through  $(0, C(0))$  and  $(1, C(1))$ .

b. [0, 0.5]    c. [1, 2]    d. [1, 1.5]    e. [1, 1.25]

f. The data in the table can be modeled by the function  $Con(t) = 25t^3 - 150t^2 + 225t$ . Use  $Con(t)$  to verify that the average rate of change over  $[1, 1 + \Delta t]$  is  $-75(\Delta t) + 25(\Delta t)^2$ . What does the average rate of change over  $[1, 1 + \Delta t]$  seem to approach as  $\Delta t$  approaches 0?



- 2. BALL ROLLING ON A RAMP** The distance traveled by a ball rolling down a ramp is given by  $s(t) = 6t^2$ , where  $t$  is the time in seconds after the ball is released, and  $s(t)$  is measured in feet. The ball travels 6 feet in 1 second and 24 feet in 2 seconds. Use the difference quotient for average velocity given on page 71 to evaluate the average velocity for each of the following time intervals.

- a. [2, 3] (Hint: In this case,  $a = 2$  and  $\Delta t = 1$ .) Compare this result to the slope of the line through  $(2, s(2))$  and  $(3, s(3))$ .

b. [2, 2.5]    c. [2, 2.1]    d. [2, 2.01]    e. [2, 2.001]

f. Verify that the average velocity over  $[2, 2 + \Delta t]$  is  $24 + 6(\Delta t)$ . What does the average velocity seem to approach as  $\Delta t$  approaches 0?

**In Exercises 73 to 76, show that  $(f \circ g)(x) = (g \circ f)(x)$ .**

**73.**  $f(x) = 2x + 3$ ;  $g(x) = 5x + 12$

**74.**  $f(x) = 4x - 2$ ;  $g(x) = 7x - 4$

**75.**  $f(x) = \frac{6x}{x - 1}; \quad g(x) = \frac{5x}{x - 2}$

$$76. f(x) = \frac{5x}{x+3}; \quad g(x) = -\frac{2x}{x-4}$$

**In Exercises 77 to 82, show that**

$$(g \circ f)(x) = x \quad \text{and} \quad (f \circ g)(x) = x$$

$$77. f(x) = 2x + 3, \quad g(x) = \frac{x - 3}{2}$$

**78.**  $f(x) = 4x - 5$ ,  $g(x) = \frac{x + 5}{4}$

$$79. f(x) = \frac{4}{x+1}, \quad g(x) = \frac{4-x}{x}$$

**80.**  $f(x) = \frac{2}{1-x}$ ,  $g(x) = \frac{x-2}{x}$

**81.**  $f(x) = x^3 - 1$ ,  $g(x) = \sqrt[3]{x + 1}$

**82.**  $f(x) = -x^3 + 2$ ,  $g(x) = \sqrt[3]{2-x}$

- 1.**   **A GRAPHING UTILITY PROJECT** For any two different real numbers  $x$  and  $y$ , the larger of the two numbers is given by

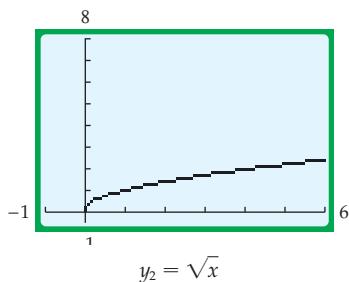
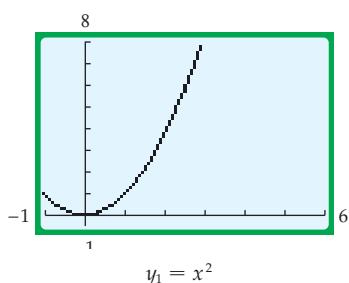
$$\text{Maximum}(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2} \quad (1)$$

- a. Verify Equation (1) for  $x = 5$  and  $y = 9$ .
  - b. Verify Equation (1) for  $x = 201$  and  $y = 80$ .

For any two different function values  $f(x)$  and  $g(x)$ , the larger of the two is given by

$$\text{Maximum}(f(x), g(x)) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2} \quad (2)$$

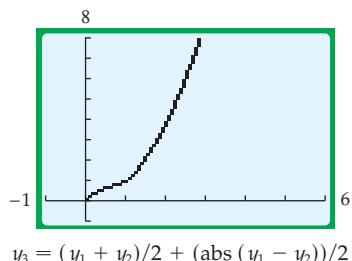
To illustrate how we might make use of Equation (2), consider the functions  $y_1 = x^2$  and  $y_2 = \sqrt{x}$  on the interval from  $X_{\min} = -1$  to  $X_{\max} = 6$ . The graphs of  $y_1$  and  $y_2$  are shown below.



Now consider the function

$$y_3 = (y_1 + y_2)/2 + (\text{abs}(y_1 - y_2))/2$$

where "abs" represents the absolute value function. The graph of  $y_3$  is shown below.



- c.  Write a sentence or two that explains why the graph of  $y_3$  is as shown.

d.  What is the domain of  $y_1$ ? of  $y_2$ ? of  $y_3$ ? Write a sentence that explains how to determine the domain of  $y_3$ , given the domain of  $y_1$  and the domain of  $y_2$ .

e. Determine a formula for the function

2.   THE NEVER-NEGATIVE FUNCTION The author J. D. Murray describes a function  $f_+$  that is defined in the following manner.<sup>1</sup>

Minimum( $f(x), g(x)$ )

$$f_+ = \begin{cases} f, & \text{if } f \geq 0 \\ 0, & \text{if } f < 0 \end{cases}$$

We will refer to this function as a **never-negative** function. Never-negative functions can be graphed by using Equation (2) in Project 1. For example, if we let  $g(x) = 0$ , then Equation (2) simplifies to

$$\text{Maximum}(f(x), 0) = \frac{f(x)}{2} + \frac{|f(x)|}{2} \quad (3)$$

<sup>1</sup>*Mathematical Biology* (New York: Springer-Verlag, 1989), p. 101.

The graph of  $y = \text{Maximum}(f(x), 0)$  is the graph of  $y = f(x)$  provided that  $f(x) \geq 0$ , and it is the graph of  $y = 0$  provided that  $f(x) < 0$ .

**An Application** The mosquito population per acre of a large resort is controlled by spraying on a monthly basis. A biologist has determined that the mosquito population can be approximated by the never-negative function  $M_+$  with

$$M(t) = -35,400(t - \text{int}(t))^2 + 35,400(t - \text{int}(t)) - 4000$$

Here  $t$  represents the month, and  $t = 0$  corresponds to June 1, 2004.

- a. Use a graphing utility to graph  $M$  for  $0 \leq t \leq 3$ .
- b. Use a graphing utility to graph  $M_+$  for  $0 \leq t \leq 3$ .
- c.  Write a sentence or two that explains how the graph of  $M_+$  differs from the graph of  $M$ .
- d. What is the maximum mosquito population per acre for  $0 \leq t \leq 3$ ? When does this maximum mosquito population occur?
- e.  Explain when would be the best time to visit the resort, provided that you wished to minimize your exposure to mosquitos.

## Section 1.6

- Introduction to Inverse Functions
- Graphs of Inverse Functions
- Composition of a Function and Its Inverse
- Find an Inverse Function

## Inverse Functions

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A6.

**PS1.** Solve  $2x + 5y = 15$  for  $y$ . [1.1]

**PS2.** Solve  $x = \frac{y+1}{y}$  for  $y$ . [1.1]

**PS3.** Given  $f(x) = \frac{2x^2}{x-1}$ , find  $f(-1)$ . [1.3]

**PS4.** If  $f$  is a function and  $f(3) = 7$ , write an ordered pair of the function. [1.3]

**PS5.** Let  $f(x) = 3x - 7$ . What is the domain of  $f$ ? [1.3]

**PS6.** Let  $f(x) = \sqrt{x+2}$ . What is the domain of  $f$ ? [1.3]

### ■ Introduction to Inverse Functions

Consider the “doubling function”  $f(x) = 2x$  that doubles every input. Some of the ordered pairs of this function are

$$\left\{ (-4, -8), (-1.5, -3), (1, 2), \left(\frac{5}{3}, \frac{10}{3}\right), (7, 14) \right\}$$

Now consider the “halving function”  $g(x) = \frac{1}{2}x$  that takes one-half of every input. Some of the ordered pairs of this function are

$$\left\{ (-8, -4), (-3, -1.5), (2, 1), \left(\frac{10}{3}, \frac{5}{3}\right), (14, 7) \right\}$$

**take note**

In this section our primary interest concerns finding the inverse of a function; however, we can also find the inverse of a relation. Recall that a relation  $r$  is any set of ordered pairs. The inverse of  $r$  is the set of ordered pairs formed by reversing the order of the coordinates of the ordered pairs in  $r$ .

Observe that the ordered pairs of  $g$  are the ordered pairs of  $f$  with the order of the coordinates reversed. The following two examples illustrate this concept.

$$f(5) = 2(5) = 10$$

Ordered pair: (5, 10)

$$g(10) = \frac{1}{2}(10) = 5$$

Ordered pair: (10, 5)

$$f(a) = 2(a) = 2a$$

Ordered pair: (a, 2a)

$$g(2a) = \frac{1}{2}(2a) = a$$

Ordered pair: (2a, a)

The function  $g$  is said to be the *inverse function* of  $f$ .

**Definition of an Inverse Function**

If the ordered pairs of a function  $g$  are the ordered pairs of a function  $f$  with the order of the coordinates reversed, then  $g$  is the **inverse function** of  $f$ .

Consider a function  $f$  and its inverse function  $g$ . Because the ordered pairs of  $g$  are the ordered pairs of  $f$  with the order of the coordinates reversed, the domain of the inverse function  $g$  is the range of  $f$ , and the range of  $g$  is the domain of  $f$ .

Not all functions have an inverse that is a function. Consider, for instance, the “square function”  $S(x) = x^2$ . Some of the ordered pairs of  $S$  are

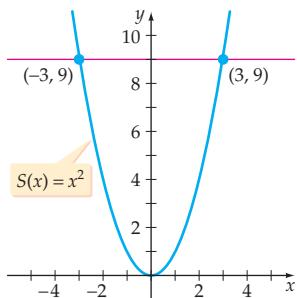
$$\{(-3, 9), (-1, 1), (0, 0), (1, 1), (3, 9), (5, 25)\}$$

If we reverse the coordinates of the ordered pairs, we have

$$\{(9, -3), (1, -1), (0, 0), (1, 1), (9, 3), (25, 5)\}$$

This set of ordered pairs is not a function because there are ordered pairs, for instance  $(9, -3)$  and  $(9, 3)$ , with the same first coordinate and different second coordinates. In this case,  $S$  has an inverse *relation* but not an inverse *function*.

A graph of  $S$  is shown in **Figure 1.84**. Note that  $x = -3$  and  $x = 3$  produce the same value of  $y$ . Thus the graph of  $S$  fails the horizontal line test, and therefore  $S$  is not a one-to-one function. This observation is used in the following theorem.



**Figure 1.84**

**Condition for an Inverse Function**

A function  $f$  has an inverse function if and only if  $f$  is a one-to-one function.

Recall that increasing functions and decreasing functions are one-to-one functions. Thus we can state the following theorem.

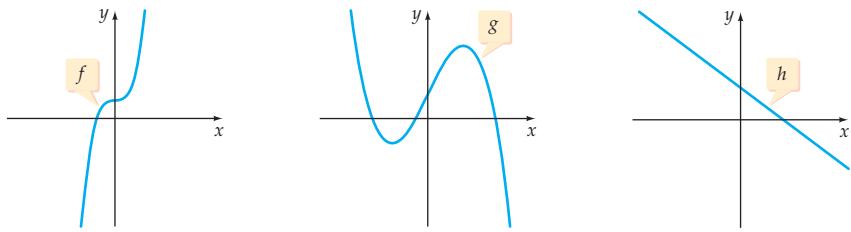
**Alternative Condition for an Inverse Function**

If  $f$  is an increasing or a decreasing function, then  $f$  has an inverse function.

**QUESTION** Which of the functions graphed below has an inverse function?

**take note**

$f^{-1}(x)$  does not mean  $\frac{1}{f(x)}$ . For  $f(x) = 2x$ ,  $f^{-1}(x) = \frac{1}{2}x$  but  $\frac{1}{f(x)} = \frac{1}{2x}$ .



If a function  $g$  is the inverse of a function  $f$ , we usually denote the inverse function by  $f^{-1}$  rather than  $g$ . For the doubling and halving functions  $f$  and  $g$  discussed on page 81, we write

$$f(x) = 2x \quad f^{-1}(x) = \frac{1}{2}x$$

## Graphs of Inverse Functions

Because the coordinates of the ordered pairs of the inverse of a function  $f$  are the ordered pairs of  $f$  with the order of the coordinates reversed, we can use them to create a graph of  $f^{-1}$ .

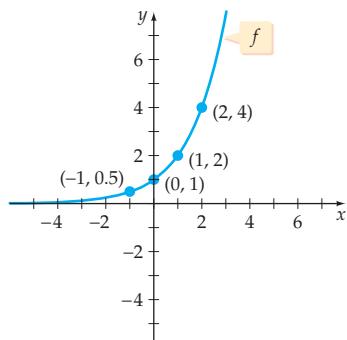


Figure 1.85

### EXAMPLE 1 >> Sketch the Graph of the Inverse of a Function

Sketch the graph of  $f^{-1}$  given that  $f$  is the function shown in **Figure 1.85**.

#### Solution

Because the graph of  $f$  passes through  $(-1, 0.5)$ ,  $(0, 1)$ ,  $(1, 2)$ , and  $(2, 4)$ , the graph of  $f^{-1}$  must pass through  $(0.5, -1)$ ,  $(1, 0)$ ,  $(2, 1)$ , and  $(4, 2)$ . Plot the points and then draw a smooth curve through the points, as shown in **Figure 1.86**.

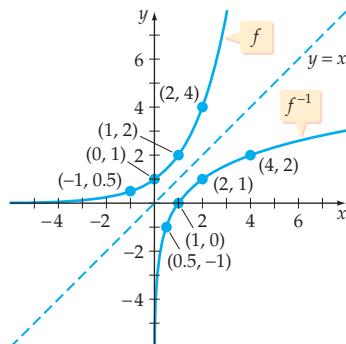


Figure 1.86

### >> Try Exercise 10, page 91

**ANSWER** The graph of  $f$  is the graph of an increasing function. Therefore,  $f$  is a one-to-one function and has an inverse function. The graph of  $h$  is the graph of a decreasing function. Therefore,  $h$  is a one-to-one function and has an inverse function. The graph of  $g$  is not the graph of a one-to-one function.  $g$  does not have an inverse function.

The graph from the solution to Example 1 is shown again in **Figure 1.87**. Note that the graph of  $f^{-1}$  is symmetric to the graph of  $f$  with respect to the graph of  $y = x$ . If the graph were folded along the dashed line, the graph of  $f$  would lie on top of the graph of  $f^{-1}$ . This is a characteristic of all graphs of functions and their inverses. In **Figure 1.88**, although  $S$  does not have an inverse that is a function, the graph of the inverse relation  $S^{-1}$  is symmetric to  $S$  with respect to the graph of  $y = x$ .

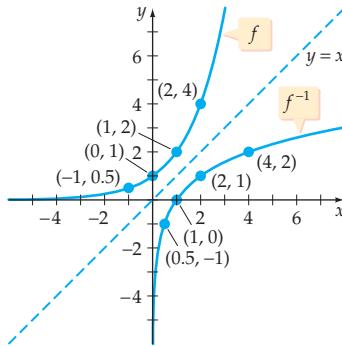


Figure 1.87

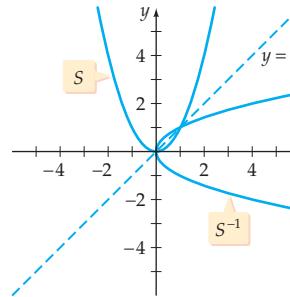


Figure 1.88

## Composition of a Function and Its Inverse

Observe the effect of forming the composition of  $f(x) = 2x$  and  $g(x) = \frac{1}{2}x$ .

$$f(x) = 2x$$

$$f[g(x)] = 2\left[\frac{1}{2}x\right]$$

$$f[g(x)] = x$$

$$g(x) = \frac{1}{2}x$$

$$g[f(x)] = \frac{1}{2}[2x]$$

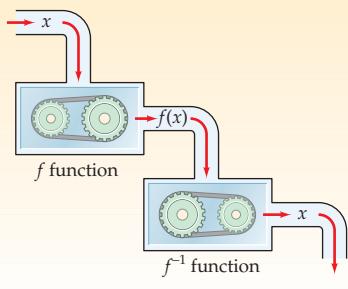
$$g[f(x)] = x$$

• Replace  $x$   
by  $g(x)$ .

• Replace  $x$   
by  $f(x)$ .

**take note**

If we think of a function as a machine, then the Composition of Inverse Functions Property can be represented as shown below. Take any input  $x$  for  $f$ . Use the output of  $f$  as the input for  $f^{-1}$ . The result is the original input,  $x$ .



### Composition of Inverse Functions Property

If  $f$  is a one-to-one function, then  $f^{-1}$  is the inverse function of  $f$  if and only if

$$(f \circ f^{-1})(x) = f[f^{-1}(x)] = x \quad \text{for all } x \text{ in the domain of } f^{-1}$$

and

$$(f^{-1} \circ f)(x) = f^{-1}[f(x)] = x \quad \text{for all } x \text{ in the domain of } f.$$

**EXAMPLE 2** » **Use the Composition of Inverse Functions Property**

Use composition of functions to show that  $f^{-1}(x) = 3x - 6$  is the inverse function of  $f(x) = \frac{1}{3}x + 2$ .

**Solution**

We must show that  $f[f^{-1}(x)] = x$  and  $f^{-1}[f(x)] = x$ .

$$f(x) = \frac{1}{3}x + 2 \quad f^{-1}(x) = 3x - 6$$

$$f[f^{-1}(x)] = \frac{1}{3}[3x - 6] + 2 \quad f^{-1}[f(x)] = 3\left[\frac{1}{3}x + 2\right] - 6$$

$$f[f^{-1}(x)] = x \quad f^{-1}[f(x)] = x$$

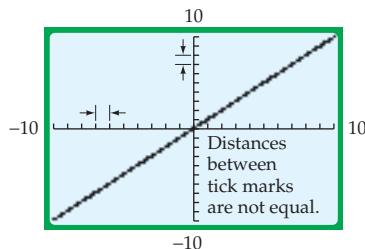
» Try Exercise 20, page 91


**Integrating Technology**

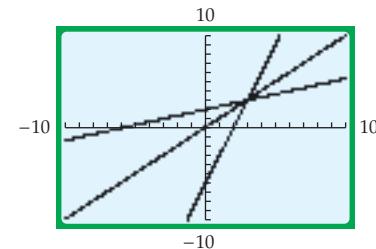
In the standard viewing window of a calculator, the distance between two tick marks on the  $x$ -axis is not equal to the distance between two tick marks on the  $y$ -axis. As a result, the graph of  $y = x$  does not appear to bisect the first and third quadrants. See **Figure 1.89**. This anomaly is important if a graphing calculator is being used to check whether two functions are inverses of one another. Because the graph of  $y = x$  does not appear to bisect the first and third quadrants, the graphs of  $f$  and  $f^{-1}$  will not appear to be symmetric

about the graph of  $y = x$ . The graphs of  $f(x) = \frac{1}{3}x + 2$  and  $f^{-1}(x) = 3x - 6$

from Example 2 are shown in **Figure 1.90**. Notice that the graphs do not appear to be quite symmetric about the graph of  $y = x$ .



$y = x$  in the standard viewing window



$f, f^{-1}$ , and  $y = x$  in the standard viewing window

**Figure 1.89**

**Figure 1.90**

Continued ►

To get a better view of a function and its inverse, it is necessary to use the **SQUARE** viewing window, as in **Figure 1.91**. In this window, the distance between two tick marks on the  $x$ -axis is equal to the distance between two tick marks on the  $y$ -axis.

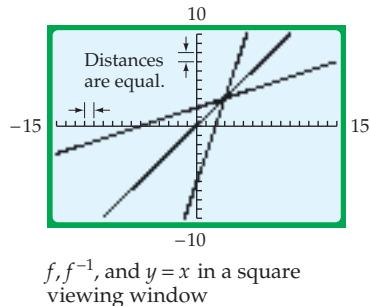


Figure 1.91

### Find an Inverse Function

If a one-to-one function  $f$  is defined by an equation, then we can use the following method to find the equation for  $f^{-1}$ .

#### take note

If the ordered pairs of  $f$  are given by  $(x, y)$ , then the ordered pairs of  $f^{-1}$  are given by  $(y, x)$ . That is,  $x$  and  $y$  are interchanged. This is the reason for Step 2 at the right.

#### Steps for Finding the Inverse of a Function

To find the equation of the inverse  $f^{-1}$  of the one-to-one function  $f$ :

1. Substitute  $y$  for  $f(x)$ .
2. Interchange  $x$  and  $y$ .
3. Solve, if possible, for  $y$  in terms of  $x$ .
4. Substitute  $f^{-1}(x)$  for  $y$ .

### EXAMPLE 3 » Find the Inverse of a Function

Find the inverse of  $f(x) = 3x + 8$ .

#### Solution

$$f(x) = 3x + 8$$

$$y = 3x + 8$$

$$x = 3y + 8$$

$$x - 8 = 3y$$

• Replace  $f(x)$  by  $y$ .

• Interchange  $x$  and  $y$ .

• Solve for  $y$ .

$$\frac{x - 8}{3} = y$$

$$\frac{1}{3}x - \frac{8}{3} = f^{-1}(x)$$

• Replace  $y$  by  $f^{-1}$ .

The inverse function is given by  $f^{-1}(x) = \frac{1}{3}x - \frac{8}{3}$ .

» Try Exercise 32, page 91

In the next example we find the inverse of a rational function.

#### EXAMPLE 4 » Find the Inverse of a Function

Find the inverse of  $f(x) = \frac{2x + 1}{x}, x \neq 0$ .

##### Solution

$$f(x) = \frac{2x + 1}{x}$$

$$y = \frac{2x + 1}{x}$$

• Replace  $f(x)$  by  $y$ .

$$x = \frac{2y + 1}{y}$$

• Interchange  $x$  and  $y$ .

$$xy = 2y + 1$$

• Solve for  $y$ .

$$xy - 2y = 1$$

$$y(x - 2) = 1$$

• Factor the left side.

$$y = \frac{1}{x - 2}$$

$$f^{-1}(x) = \frac{1}{x - 2}, x \neq 2$$

• Replace  $y$  by  $f^{-1}$ .

» Try Exercise 38, page 92

**QUESTION** If  $f$  is a one-to-one function and  $f(4) = 5$ , what is  $f^{-1}(5)$ ?

The graph of  $f(x) = x^2 + 4x + 3$  is shown in **Figure 1.92a**. The function  $f$  is not a one-to-one function and therefore does not have an inverse function. However,

---

**ANSWER** Because  $(4, 5)$  is an ordered pair of  $f$ ,  $(5, 4)$  must be an ordered pair of  $f^{-1}$ . Therefore,  $f^{-1}(5) = 4$ .

the function given by  $G(x) = x^2 + 4x + 3$ , shown in **Figure 1.92b**, for which the domain is restricted to  $\{x | x \geq -2\}$ , is a one-to-one function and has an inverse function  $G^{-1}$ . This is shown in Example 5.

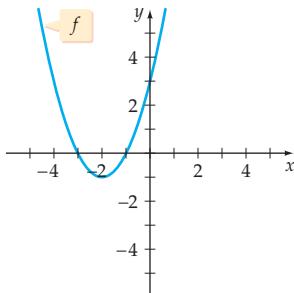


Figure 1.92a

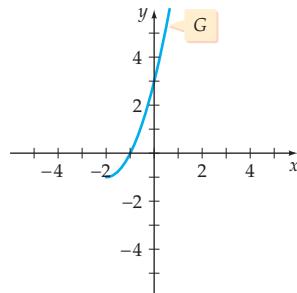


Figure 1.92b

### EXAMPLE 5 » Find the Inverse of a Function with a Restricted Domain

Find the inverse of  $G(x) = x^2 + 4x + 3$ , where the domain of  $G$  is  $\{x | x \geq -2\}$ .

#### take note

Recall that the range of a function  $f$  is the domain of  $f^{-1}$ , and the domain of  $f$  is the range of  $f^{-1}$ .

#### Solution

$$\begin{aligned} G(x) &= x^2 + 4x + 3 \\ y &= x^2 + 4x + 3 \\ x &= y^2 + 4y + 3 \\ x &= (y^2 + 4y + 4) - 4 + 3 \end{aligned}$$

$$\begin{aligned} x &= (y + 2)^2 - 1 \\ x + 1 &= (y + 2)^2 \end{aligned}$$

$$\sqrt{x + 1} = \sqrt{(y + 2)^2}$$

$$\pm\sqrt{x + 1} = y + 2$$

$$\pm\sqrt{x + 1} - 2 = y$$

- Replace  $G(x)$  by  $y$ .
- Interchange  $x$  and  $y$ .
- Solve for  $y$  by completing the square of  $y^2 + 4y$ .
- Factor.
- Add 1 to each side of the equation.
- Take the square root of each side of the equation.
- Recall that if  $a^2 = b$ , then  $a = \pm\sqrt{b}$ .

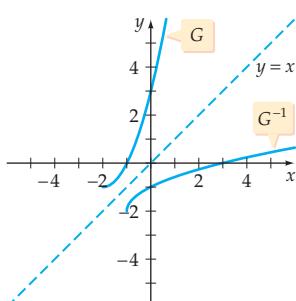


Figure 1.93

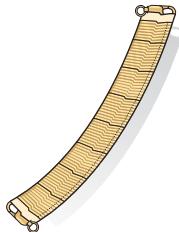
Because the domain of  $G$  is  $\{x | x \geq -2\}$ , the range of  $G^{-1}$  is  $\{y | y \geq -2\}$ . This means that we must choose the positive value of  $\pm\sqrt{x + 1}$ . Thus  $G^{-1}(x) = \sqrt{x + 1} - 2$ . See **Figure 1.93**.

#### » Try Exercise 44, page 92

In Example 6b, we use an inverse function to determine the wholesale price of a gold bracelet for which we know the retail price.

**EXAMPLE 6** » **Solve an Application**

A merchant uses the function



$$S(x) = \frac{4}{3}x + 100$$

to determine the retail selling price  $S$ , in dollars, of a gold bracelet for which she has paid a wholesale price of  $x$  dollars.

- The merchant paid a wholesale price of \$672 for a gold bracelet. Use  $S$  to determine the retail selling price of this bracelet.
- Find  $S^{-1}$  and use it to determine the merchant's wholesale price for a gold bracelet that retails at \$1596.

**Solution**

a.  $S(672) = \frac{4}{3}(672) + 100 = 896 + 100 = 996$

The merchant charges \$996 for a bracelet that has a wholesale price of \$672.

- b. To find  $S^{-1}$ , begin by substituting  $y$  for  $S(x)$ .

$$S(x) = \frac{4}{3}x + 100$$

$$y = \frac{4}{3}x + 100 \quad \bullet \text{ Replace } S(x) \text{ with } y.$$

$$x = \frac{4}{3}y + 100 \quad \bullet \text{ Interchange } x \text{ and } y.$$

$$x - 100 = \frac{4}{3}y \quad \bullet \text{ Solve for } y.$$

$$\frac{3}{4}(x - 100) = y$$

$$\frac{3}{4}x - 75 = y$$

Using inverse notation, the above equation can be written as

$$S^{-1}(x) = \frac{3}{4}x - 75$$

Substitute 1596 for  $x$  to determine the wholesale price.

$$\begin{aligned} S^{-1}(1596) &= \frac{3}{4}(1596) - 75 \\ &= 1197 - 75 \\ &= 1122 \end{aligned}$$

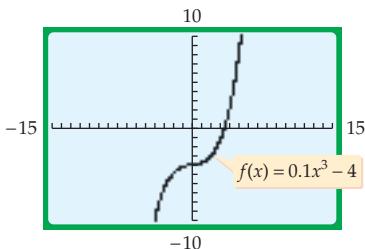
A gold bracelet that the merchant retails at \$1596 has a wholesale price of \$1122.

» Try Exercise 54, page 92

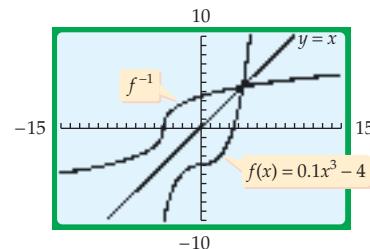


### Integrating Technology

Some graphing utilities can be used to draw the graph of the inverse of a function without the user having to find the inverse function. For instance, **Figure 1.94** shows the graph of  $f(x) = 0.1x^3 - 4$ . The graphs of  $f$  and  $f^{-1}$  are both shown in **Figure 1.95**, along with the graph of  $y = x$ . Note that the graph of  $f^{-1}$  is the reflection of the graph of  $f$  with respect to the graph of  $y = x$ . The display shown in **Figure 1.95** was produced on a TI-83/TI-83 Plus/TI-84 Plus graphing calculator by using the DrawInv command, which is in the DRAW menu.



**Figure 1.94**



**Figure 1.95**



### Topics for Discussion

1. If  $f(x) = 3x + 1$ , what are the values of  $f^{-1}(2)$  and  $[f(2)]^{-1}$ ?
2. How are the domain and range of a one-to-one function  $f$  related to the domain and range of the inverse function of  $f$ ?
3. How is the graph of the inverse of a function  $f$  related to the graph of  $f$ ?
4. The function  $f(x) = -x$  is its own inverse. Find two other functions that are their own inverses.
5. What are the steps in finding the inverse of a one-to-one function?

## Exercise Set 1.6

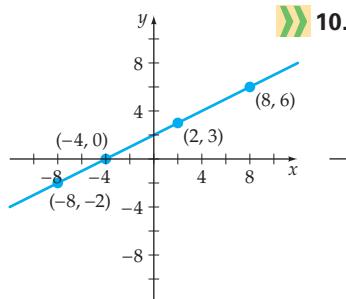
In Exercises 1 to 4, assume that the given function has an inverse function.

1. Given  $f(3) = 7$ , find  $f^{-1}(7)$ .
2. Given  $g(-3) = 5$ , find  $g^{-1}(5)$ .
3. Given  $h^{-1}(-3) = -4$ , find  $h(-4)$ .
4. Given  $f^{-1}(7) = 0$ , find  $f(0)$ .

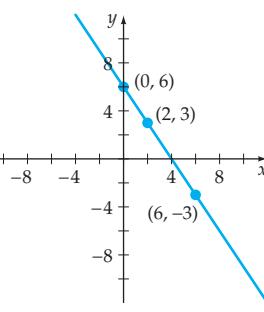
5. If 3 is in the domain of  $f^{-1}$ , find  $f[f^{-1}(3)]$ .
6. If  $f$  is a one-to-one function and  $f(0) = 5$ ,  $f(1) = 2$ , and  $f(2) = 7$ , find:
  - a.  $f^{-1}(5)$
  - b.  $f^{-1}(2)$
7. The domain of the inverse function  $f^{-1}$  is the \_\_\_\_\_ of  $f$ .
8. The range of the inverse function  $f^{-1}$  is the \_\_\_\_\_ of  $f$ .

In Exercises 9 to 16, draw the graph of the inverse relation. Is the inverse relation a function?

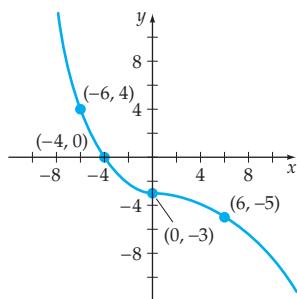
9.



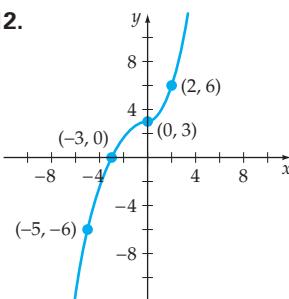
10.



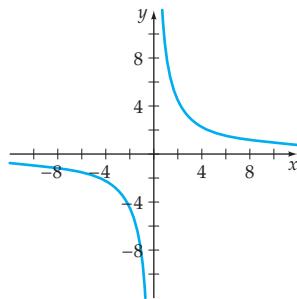
11.



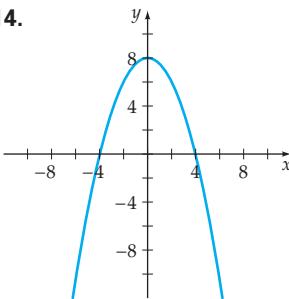
12.



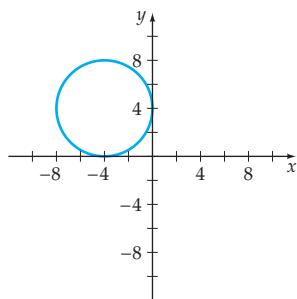
13.



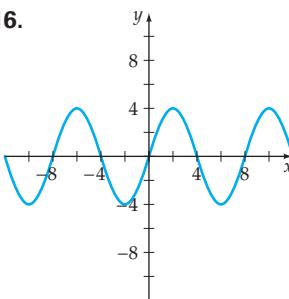
14.



15.



16.



In Exercises 17 to 26, use composition of functions to determine whether  $f$  and  $g$  are inverses of one another.

17.  $f(x) = 4x; g(x) = \frac{x}{4}$

18.  $f(x) = 3x; g(x) = \frac{1}{3x}$

19.  $f(x) = 4x - 1; g(x) = \frac{1}{4}x + \frac{1}{4}$

20.  $f(x) = \frac{1}{2}x - \frac{3}{2}; g(x) = 2x + 3$

21.  $f(x) = -\frac{1}{2}x - \frac{1}{2}; g(x) = -2x + 1$

22.  $f(x) = 3x + 2; g(x) = \frac{1}{3}x - \frac{2}{3}$

23.  $f(x) = \frac{5}{x-3}; g(x) = \frac{5}{x} + 3$

24.  $f(x) = \frac{2x}{x-1}; g(x) = \frac{x}{x-2}$

25.  $f(x) = x^3 + 2; g(x) = \sqrt[3]{x-2}$

26.  $f(x) = (x+5)^3; g(x) = \sqrt[3]{x} - 5$

In Exercises 27 to 30, find the inverse of the function. If the function does not have an inverse function, write "no inverse function."

27.  $\{(-3, 1), (-2, 2), (1, 5), (4, -7)\}$

28.  $\{(-5, 4), (-2, 3), (0, 1), (3, 2), (7, 11)\}$

29.  $\{(0, 1), (1, 2), (2, 4), (3, 8), (4, 16)\}$

30.  $\{(1, 0), (10, 1), (100, 2), (1000, 3), (10,000, 4)\}$

In Exercises 31 to 48, find  $f^{-1}(x)$ . State any restrictions on the domain of  $f^{-1}(x)$ .

31.  $f(x) = 2x + 4$

32.  $f(x) = 4x - 8$

33.  $f(x) = 3x - 7$

34.  $f(x) = -3x - 8$

35.  $f(x) = -2x + 5$

36.  $f(x) = -x + 3$

37.  $f(x) = \frac{2x}{x-1}, \quad x \neq 1$

38.  $f(x) = \frac{x}{x-2}, \quad x \neq 2$

39.  $f(x) = \frac{x-1}{x+1}, \quad x \neq -1$

40.  $f(x) = \frac{2x-1}{x+3}, \quad x \neq -3$

41.  $f(x) = x^2 + 1, \quad x \geq 0$

42.  $f(x) = x^2 - 4, \quad x \geq 0$

43.  $f(x) = \sqrt{x-2}, \quad x \geq 2$

44.  $f(x) = \sqrt{4-x}, \quad x \leq 4$

45.  $f(x) = x^2 + 4x, \quad x \geq -2$

46.  $f(x) = x^2 - 6x, \quad x \leq 3$

47.  $f(x) = x^2 + 4x - 1, \quad x \leq -2$

48.  $f(x) = x^2 - 6x + 1, \quad x \geq 3$

49.  **GEOMETRY** The volume of a cube is given by  $V(x) = x^3$ , where  $x$  is the measure of the length of a side of the cube. Find  $V^{-1}(x)$  and explain what it represents.

50.  **UNIT CONVERSIONS** The function  $f(x) = 12x$  converts feet,  $x$ , into inches,  $f(x)$ . Find  $f^{-1}(x)$  and explain what it determines.

51. **FAHRENHEIT TO CELSIUS** The function

$$f(x) = \frac{5}{9}(x - 32)$$

is used to convert  $x$  degrees Fahrenheit to an equivalent Celsius temperature. Find  $f^{-1}$  and explain how it is used.

52. **RETAIL SALES** A clothing merchant uses the function

$$S(x) = \frac{3}{2}x + 18$$

to determine the retail selling price  $S$ , in dollars, of a winter coat for which she has paid a wholesale price of  $x$  dollars.

a. The merchant paid a wholesale price of \$96 for the winter coat. Use  $S$  to determine the retail selling price she will charge for this coat.

b. Find  $S^{-1}$  and use it to determine the merchant's wholesale price for a coat that retails at \$399.

53.  **FASHION** The function

$s(x) = 2x + 24$  can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function  $s^{-1}(x)$  that can be used to convert an Italian women's shoe size to its equivalent U.S. shoe size.



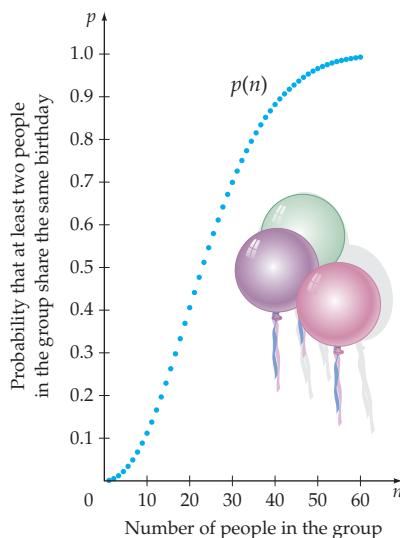
54.  **FASHION** The function  $K(x) = 1.3x - 4.7$  converts a men's shoe size in the United States to the equivalent shoe size in the United Kingdom. Determine the function  $K^{-1}(x)$  that can be used to convert a United Kingdom men's shoe size to its equivalent U.S. shoe size.

55.  **COMPENSATION** The monthly earnings  $E(s)$ , in dollars, of a software sales executive is given by  $E(s) = 0.05s + 2500$ , where  $s$  is the value, in dollars, of the software sold by the executive during the month. Find  $E^{-1}(s)$  and explain how the executive could use this function.

56.   **POSTAGE** Does the first-class postage rate function given below have an inverse function? Explain your answer.

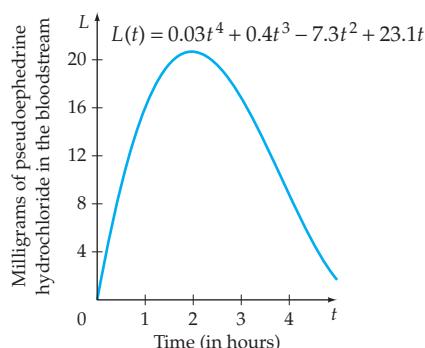
| Weight (in ounces) | Cost    |
|--------------------|---------|
| $0 < w \leq 1$     | \$ .39  |
| $1 < w \leq 2$     | \$ .63  |
| $2 < w \leq 3$     | \$ .87  |
| $3 < w \leq 4$     | \$ 1.11 |

57. **THE BIRTHDAY PROBLEM** A famous problem called the *birthday problem* goes like this: Suppose there is a randomly selected group of  $n$  people in a room. What is the probability that at least two of the people have a birthday on the same day of the year? It may surprise you that for a group of 23 people, the probability that at least two of the people share a birthday is about 50.7%. The following graph can be used to estimate shared birthday probabilities for  $1 \leq n \leq 60$ .



- a. Use the graph of  $p$  to estimate  $p(10)$  and  $p(30)$ .
- b. Consider the function  $p$  with  $1 \leq n \leq 60$ , as shown in the graph. Explain how you can tell that  $p$  has an inverse that is a function.
- c. Write a sentence that explains the meaning of  $p^{-1}(0.223)$  in the context of this application.

58. **MEDICATION LEVEL** The function  $L$  shown in the following graph models the level of pseudoephedrine hydrochloride, in milligrams, in the bloodstream of a patient  $t$  hours after 30 milligrams of the medication have been administered.



- a. Use the graph of  $L$  to estimate two different values of  $t$  for which the pseudoephedrine hydrochloride levels are the same.
- b. Does  $L$  have an inverse that is a function? Explain.

59. **CRYPTOLOGY** Cryptology is the study of making and breaking secret codes. Secret codes are often used to send messages over the Internet. By devising a code that is difficult to break, the sender hopes to prevent the messages from being read by an unauthorized person.

In practice, very complicated one-to-one functions and their inverses are used to encode and decode messages. The following procedure uses the simple function  $f(x) = 2x - 1$  to illustrate the basic concepts that are involved.

Assign to each letter of the alphabet, and a blank space, a two-digit numerical value, as shown below.

|   |    |   |    |   |    |   |    |
|---|----|---|----|---|----|---|----|
| A | 10 | H | 17 | O | 24 | V | 31 |
| B | 11 | I | 18 | P | 25 | W | 32 |
| C | 12 | J | 19 | Q | 26 | X | 33 |
| D | 13 | K | 20 | R | 27 | Y | 34 |
| E | 14 | L | 21 | S | 28 | Z | 35 |
| F | 15 | M | 22 | T | 29 |   | 36 |
| G | 16 | N | 23 | U | 30 |   |    |

**Note:** A blank space is represented by the numerical value 36.

Using these numerical values, the message MEET YOU AT NOON would be represented by

22 14 14 29 36 34 24 30 36 10 29 36 23 24 24 23

Let  $f(x) = 2x - 1$  define a coding function. The above message can be encoded by finding  $f(22)$ ,  $f(14)$ ,  $f(14)$ ,  $f(29)$ ,  $f(36)$ ,  $f(34)$ ,  $f(24)$ , ...,  $f(23)$ , which yields

43 27 27 57 71 67 47 59 71 19 57 71 45 47 47 45

The inverse of  $f$ , which is

$$f^{-1}(x) = \frac{x+1}{2}$$

is used by the receiver of the message to decode the message. For instance,

$$f^{-1}(43) = \frac{43+1}{2} = 22$$

which represents M, and

$$f^{-1}(27) = \frac{27+1}{2} = 14$$

which represents E.

- a. Use the above coding procedure to encode the message DO YOUR HOMEWORK.
- b. Use  $f^{-1}(x)$  as defined above to decode the message 49 33 47 45 27 71 33 47 43 27.
- c. Explain why it is important to use a one-to-one function to encode a message.

- 60. CRYPTOGRAPHY** A friend is using the letter-number correspondence in Exercise 59 and the coding function  $g(x) = 2x + 3$ . Your friend sends you the coded message

59 31 39 73 31 75 61 37 31 75 29 23 71

Use  $g^{-1}(x)$  to decode this message.

**In Exercises 61 to 66, answer the question without finding the equation of the linear function.**

- 61.** Suppose that  $f$  is a linear function,  $f(2) = 7$ , and  $f(5) = 12$ . If  $f(4) = c$ , then is  $c$  less than 7, between 7 and 12, or greater than 12? Explain your answer.

**62.** Suppose that  $f$  is a linear function,  $f(1) = 13$ , and  $f(4) = 9$ . If  $f(3) = c$ , then is  $c$  less than 9, between 9 and 13, or greater than 13? Explain your answer.

- 67.** Consider the linear function  $f(x) = mx + b$ ,  $m \neq 0$ . The graph of  $f$  has a slope of  $m$  and a  $y$ -intercept of  $(0, b)$ . What are the slope and  $y$ -intercept of the graph of  $f^{-1}$ ?

**68.** Find the inverse of  $f(x) = ax^2 + bx + c$ ,  $a \neq 0$ ,  $x \geq -\frac{b}{2a}$ .

**69.** Use a graph of  $f(x) = -x + 3$  to explain why  $f$  is its own inverse.

**70.** Use a graph of  $f(x) = \sqrt{16 - x^2}$ , with  $0 \leq x \leq 4$ , to explain why  $f$  is its own inverse.

**Only one-to-one functions have inverses that are functions.**  
**In Exercises 71 to 74, determine if the given function is a one-to-one function.**

**71.**  $p(t) = \sqrt{9 - t}$

**72.**  $v(t) = \sqrt{16 + t}$

**73.**  $F(x) = |x| + x$

**74.**  $T(x) = |x^2 - 6|$ ,  $x \geq 0$

-  **INTERSECTION POINTS FOR THE GRAPHS OF  $f$  AND  $f^{-1}$ .** For each of the following, graph  $f$  and its inverse.

  - i.  $f(x) = 2x - 4$
  - ii.  $f(x) = -x + 2$
  - iii.  $f(x) = x^3 + 1$
  - iv.  $f(x) = x - 3$
  - v.  $f(x) = -3x + 2$
  - vi.  $f(x) = \frac{1}{x}$

- a. Do the graphs of a function and its inverse always intersect?
  - b. If the graphs of a function and its inverse intersect at one point, what is true about the coordinates of the point of intersection?
  - c. Can the graphs of a function and its inverse intersect at more than one point?

## Section 1.7

- Linear Regression Models
- Correlation Coefficient
- Quadratic Regression Models

# Modeling Data Using Regression

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A6.

- PS1.** Find the slope and  $y$ -intercept of the graph of  $y = -\frac{3}{4}x + 4$ .
- PS2.** Find the slope and  $y$ -intercept of the graph of  $3x - 4y = 12$ .
- PS3.** Find an equation of the line that has slope  $-0.45$  and  $y$ -intercept  $(0, 2.3)$ .
- PS4.** If  $f(x) = 3x^2 + 4x - 1$ , find  $f(2)$ . [1.3]
- PS5.** Find the distance between  $P_1(3, 5)$  and  $P_2(3, f(3))$ , where  $f(x) = 2x + 5$ . [1.2/1.3]
- PS6.** You are given  $P_1(2, -1)$  and  $P_2(4, 14)$ . If  $f(x) = x^2 - 3$ , find  $|f(x_1) - y_1| + |f(x_2) - y_2|$ . [1.3]

## Linear Regression Models

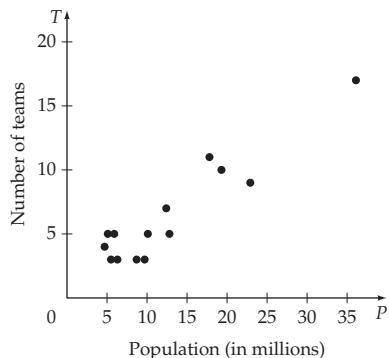
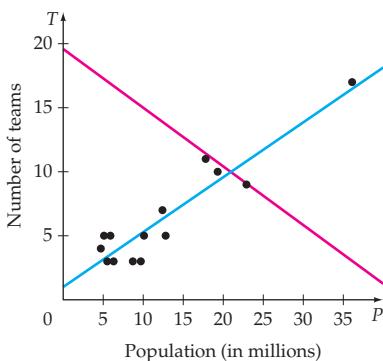
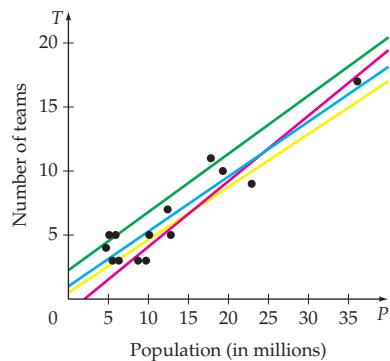


The data in the table below show the population of selected states and the number of professional sports teams (Major League Baseball, National Football League, National Basketball Association, Women's National Basketball Association, National Hockey League) in those states. A scatter diagram of the data is shown in **Figure 1.96** on page 96.

Number of Professional Sports Teams for Selected States

| State      | Populations<br>(in millions) | Number of<br>Teams | State          | Populations<br>(in millions) | Number of<br>Teams |
|------------|------------------------------|--------------------|----------------|------------------------------|--------------------|
| Arizona    | 5.9                          | 5                  | Minnesota      | 5.1                          | 5                  |
| California | 36.1                         | 17                 | New Jersey     | 8.7                          | 3                  |
| Colorado   | 4.7                          | 4                  | New York       | 19.3                         | 10                 |
| Florida    | 17.8                         | 11                 | North Carolina | 9.7                          | 3                  |
| Illinois   | 12.8                         | 5                  | Pennsylvania   | 12.4                         | 7                  |
| Indiana    | 6.3                          | 3                  | Texas          | 22.9                         | 9                  |
| Michigan   | 10.1                         | 5                  | Wisconsin      | 5.5                          | 3                  |

Although there is no one line that passes through every point, we could find an approximate linear model for these data. For instance, the line shown in **Figure 1.97** in blue approximates the data better than the line shown in red. However, as **Figure 1.98** shows, there are many other lines we could draw that seem to approximate the data.

**Figure 1.96****Figure 1.97****Figure 1.98**

To find the line that “best” approximates the data, **regression analysis** is used. This type of analysis produces the linear function whose graph is called the **line of best fit** or the **least-squares regression line**.<sup>2</sup>

#### Definition of the Least-Squares Regression Line

The **least-squares regression line** is the line that minimizes the sum of the squares of the vertical deviations of all data points from the line.

To help understand this definition, consider the data set

$$S = \{(1, 2), (2, 3), (3, 3), (4, 4), (5, 7)\}$$

as shown in **Figure 1.99**. As we will show later, the least-squares line for this data set is  $y = 1.1x + 0.5$ , also shown in **Figure 1.99**. If we evaluate this linear function at the  $x$ -coordinates of the data set  $S$ , we obtain the set of ordered pairs  $T = \{(1, 1.6), (2, 2.7), (3, 3.8), (4, 4.9), (5, 6)\}$ . The vertical deviations are the differences between the  $y$ -coordinates in  $S$  and the  $y$ -coordinates in  $T$ . From the definition, we must calculate the sum of the squares of these deviations.

$$(2 - 1.6)^2 + (3 - 2.7)^2 + (3 - 3.8)^2 + (4 - 4.9)^2 + (7 - 6)^2 = 2.7$$

Because  $y = 1.1x + 0.5$  is the least squares regression line, for no other line is the sum of the squares of the deviations less than 2.7. For instance, if we consider the equation  $y = 1.25x + 0.75$ , which is the equation of the line through the two

<sup>2</sup>The least-squares regression line is also called the *least-squares line* and the *regression line*.

points  $P_1(1, 2)$  and  $P_2(5, 7)$  of the data set, the sum of the squared deviations is larger than 2.7. See **Figure 1.100**.

$$(2 - 2)^2 + (3 - 3.25)^2 + (3 - 4.5)^2 + (4 - 5.75)^2 + (7 - 7)^2 = 5.375$$

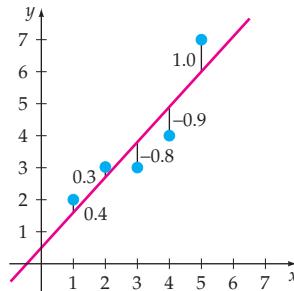


Figure 1.99

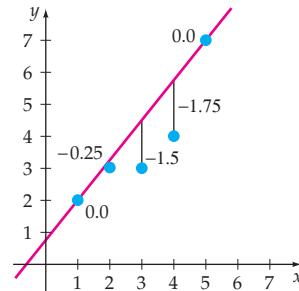


Figure 1.100



### Integrating Technology

The equations used to calculate a regression line are somewhat cumbersome. Fortunately, these equations are preprogrammed into most graphing calculators. We will now illustrate the technique for a TI-83/TI-83 Plus/TI-84 Plus calculator using data set S given on page 96.

Press **STAT**. Select EDIT.  
Press **ENTER**.

If necessary, delete any data in L1 and L2.  
Enter the given data.

Press **STAT**. Select CALC.  
Select 4, LinReg(ax+b).  
Press **ENTER**.  
Press **VARS**.

```
EDIT CALC TESTS
1:Edit...
2:SortA(
3:SortD(
4:ClrList
5:SetUpEditor
```

| L1    | L2    | L3 | 2 |
|-------|-------|----|---|
| 1     | 2     |    |   |
| 2     | 3     |    |   |
| 3     | 3     |    |   |
| 4     | 4     |    |   |
| 5     | 7     |    |   |
| ----- | ----- |    |   |

L2(6) =

```
EDIT CALC TESTS
1:1-Var Stats
2:1-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
```

LinReg(ax+b)

Select Y-VARS.  
Press **ENTER**.

Select 1. Press **ENTER**.

Press **ENTER**.

View the results.

```
VARS Y-VARS
1:Function...
2:Parametric...
3:Polar...
4:On/Off...
```

```
FUNCTION
1:Y1
2:Y2
3:Y3
4:Y4
5:Y5
6:Y6
7:Y7
```

LinReg(ax+b) Y1

```
LinReg
y=ax+b
a=1.1
b=.5
r²=.8175675676
r=.9041944302
```

From the last screen, the equation of the regression line is  $y = 1.1x + 0.5$ . Your last screen may not look exactly like ours. The information provided on our screen requires that DiagnosticOn be enabled. This is accomplished using the following keystrokes:

**2ND CATALOG** (Scroll to DiagnosticOn) **ENTER** **ENTER**

With DiagnosticOn enabled, in addition to the values for the regression equation, two other values,  $r^2$  and  $r$ , are given. We will discuss these values later in this section.

If you used the keystrokes we have shown here, the regression line will be stored in  $Y_1$ . This is helpful if you wish to graph the regression line. However, if it is not necessary to graph the regression line, then instead of pressing **VARS** at step 4, just press **ENTER**. The result will be the last screen showing the results of the regression calculations.

### EXAMPLE 1 >> Find a Regression Equation



Find the regression equation for the data on the population of a state and the number of professional sports teams in that state. How many sports teams are predicted for Indiana, whose population is approximately 6.3 million? Round to the nearest whole number.

#### Solution

Using your calculator, enter the data from the table on page 95. Then have the calculator produce the values for the regression equation. Your results should be similar to those shown in **Figure 1.101**. The equation of the regression line is

```
LinReg
y=ax+b
a=.4285944776
b=1.000728509
r²=.8725655709
r=.9341121833
```

**Figure 1.101**



#### Integrating Technology

If you followed the steps we gave on page 97 and stored the regression equation from Example 1 in  $Y_1$ , then you can evaluate the regression equation using the following keystrokes:

**VARS** **►** **ENTER** **ENTER**

**[** 6.3 **]** **ENTER**

$$y = 0.4285944776x + 1.000728509$$

To find the number of sports teams the regression equation predicts for Indiana, evaluate the regression equation for  $x = 6.3$ .

$$\begin{aligned} y &= 0.4285944776x + 1.000728509 \\ &= 0.4285944776(6.3) + 1.000728509 \\ &\approx 3.7008737 \end{aligned}$$

The equation predicts that Indiana should have four sports teams.

>> Try Exercise 18, page 104

## Correlation Coefficient

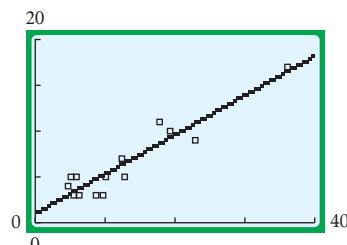


Figure 1.102

The scatter diagram of a state's population and the corresponding number of professional sports teams is shown in **Figure 1.102**, along with the graph of the regression line. Note that the slope of the regression line is positive. This indicates that as a state's population increases, the number of teams increases. Note also that for these data the value of  $r$  on the regression calculation screen was positive,  $r \approx 0.9341$ .

Now consider the data in the table below, which shows the trade-in value of a 2003 Corvette for various odometer readings. The graph in **Figure 1.103** shows the scatter diagram and the regression line.

### take note

The data for the Corvette were created assuming that the condition of the car was excellent. The only variable that changed was the odometer reading.

```

LinReg
y=ax+b
a=-150.745614
b=32869.07895
r2=.9635182628
r=-.9815896611
  
```

Figure 1.104

### Trade-in Value of 2003 Corvette Coupe, January 2006

| Odometer Reading, in thousands of miles | Trade-in Value in dollars |
|---|---------------------------|
| 25                                      | 29,100                    |
| 30                                      | 28,000                    |
| 40                                      | 27,175                    |
| 45                                      | 26,450                    |
| 55                                      | 24,225                    |

Source: Kelley Blue Book website, January 2006.

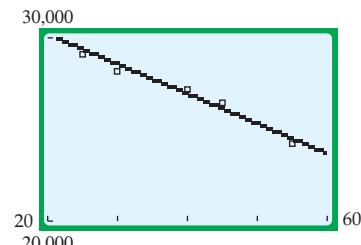
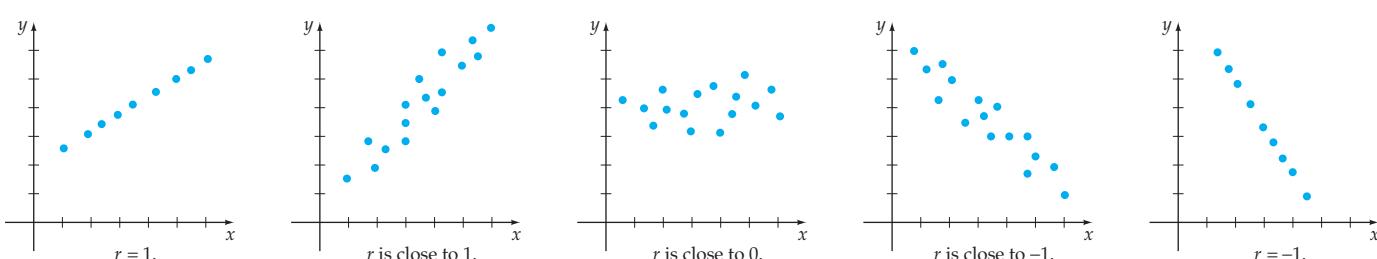


Figure 1.103

In this case the slope of the regression line is negative. This means that as the odometer reading increases, the trade-in value of the car decreases. Note also that the value of  $r$  is negative,  $r \approx -0.9816$ . See **Figure 1.104**.

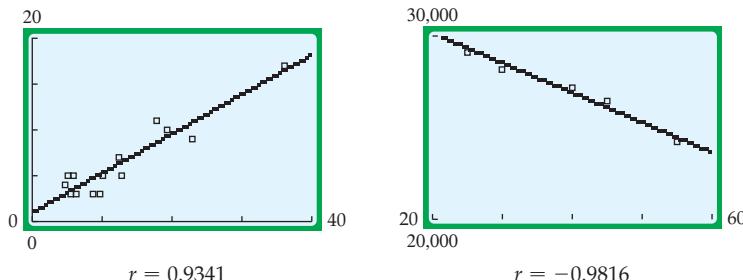
### Linear Correlation Coefficient

The **linear correlation coefficient  $r$**  is a measure of how close the points of a data set can be modeled by a straight line. If  $r = 1$ , then the points of the data set can be modeled *exactly* by a straight line with positive slope. If  $r = -1$ , then the data set can be modeled *exactly* by a straight line with negative slope. For all data sets,  $-1 \leq r \leq 1$ .



If  $r \neq 1$  or  $r \neq -1$ , then the data set *cannot* be modeled exactly by a straight line. The further the value of  $r$  is from 1 or  $-1$  (the closer the value of  $r$  to zero), the more the ordered pairs of the data set deviate from a straight line.

The graphs below show the points of the data sets and the graphs of the regression lines for the state population/sports teams data and the odometer reading/trade-in data. Note the values of  $r$  and the closeness of the data points to the regression lines.



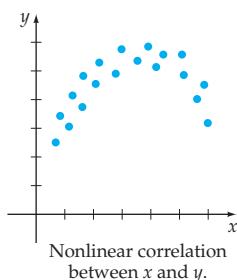
A researcher calculates a regression line to determine the relationship between two variables. The researcher wants to know whether a change in one variable produces a predictable change in the second variable. The value of  $r^2$  tells the researcher the extent of that relationship.

### Coefficient of Determination

The **coefficient of determination  $r^2$**  measures the proportion of the variation in the dependent variable that is explained by the regression line.

For the population/sports team data,  $r^2 \approx 0.87$ . This means that approximately 87% of the total variation in the dependent variable (number of teams) can be attributed to the state population. This also means that population alone does not predict with certainty the number of sports teams. Other factors, such as climate, are also involved in the number of sports teams.

**QUESTION** What is the coefficient of determination for the odometer reading/trade-in value data (see page 99), and what is its significance?



### Quadratic Regression Models

To this point our focus has been *linear* regression equations. However, there may be a nonlinear relationship between two quantities. The scatter diagram to the left suggests that a quadratic function might be a better model of the data than a linear model.

**ANSWER**  $r^2 \approx 0.964$ . This means that about 96.4% of the total variation in trade-in value can be attributed to the odometer reading.

**EXAMPLE 2** » Find a Quadratic Regression Model

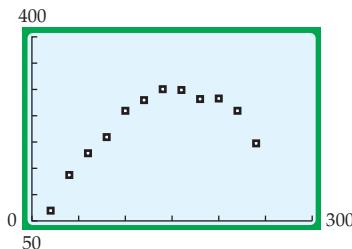

The data in the table below were collected on five successive Saturdays. They show the average number of cars entering a shopping center parking lot. The value of  $t$  is the number of minutes after 9:00 A.M. The value of  $N$  is the number of cars that entered the parking lot in the 10 minutes prior to the value of  $t$ . Find a regression model for this data.

**Average Number of Cars Entering a Shopping Center Parking Lot**

| $t$ | $N$ | $t$ | $N$ |
|-----|-----|-----|-----|
| 20  | 70  | 140 | 301 |
| 40  | 135 | 160 | 298 |
| 60  | 178 | 180 | 284 |
| 80  | 210 | 200 | 286 |
| 100 | 260 | 220 | 260 |
| 120 | 280 | 240 | 195 |

**Solution**

1. **Construct a scatter diagram for these data.** Enter the data into your calculator as explained on page 97.



From the scatter diagram, it appears that there is a nonlinear relationship between the variables.

2. **Find the regression equation.** Try a quadratic regression model. For a TI-83/TI-83 Plus/TI-84 Plus calculator, press **STAT** ▶ 5 **ENTER**.

**take note**

For nonlinear regression calculations, the value of  $r$  is not shown on a TI-83/TI-83 Plus/TI-84 Plus graphing calculator. In such cases, the coefficient of determination is used to determine how well the data fit the model.

```

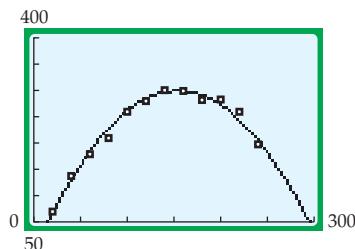
QuadReg
y=ax2+bx+c
a=-.0124881369
b=3.904433067
c=-7.25
R2=.9840995401

```

3. **Examine the coefficient of determination.** The coefficient of determination is approximately 0.984. Because this number is fairly close to 1, the regression equation  $y = -0.0124881369x^2 + 3.904433067x - 7.25$  provides a good model of the data.

Continued ►

The following calculator screen shows the scatter diagram and the parabola that is the graph of the regression equation from page 101.



» Try Exercise 32, page 107

```
LinReg
y=ax+b
a=.6575174825
b=144.2727273
r²=.4193509866
r=.6475731515
```

In Example 2, we could have calculated the *linear* regression model for the data. The results are shown at the left. Note that the coefficient of determination for this calculation is approximately 0.419. Because this number is less than the coefficient of determination for the quadratic model, we choose a quadratic model of the data rather than a linear model.

A final note: The regression line equation does not *prove* that the changes in the dependent variable are *caused* by the independent variable. For instance, suppose various cities throughout the United States were randomly selected and the numbers of gas stations (independent variable) and restaurants (dependent variable) were recorded in a table. If we calculated the regression equation for these data, we would find that  $r$  would be close to 1. However, this does not mean that gas stations *cause* restaurants to be built. The primary cause is that there are fewer gas stations and restaurants in cities with small populations and greater numbers of gas stations and restaurants in cities with large populations.



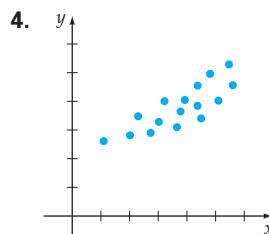
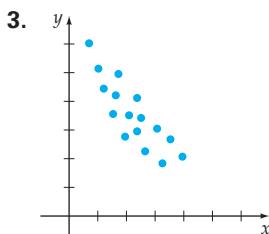
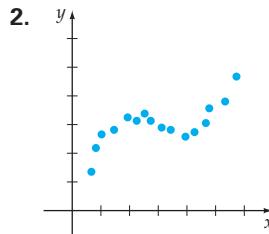
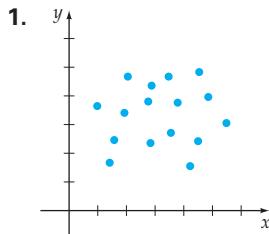
### Topics for Discussion

- What is the purpose of calculating the equation of a regression line?
- Discuss the implications of the following correlation coefficients:  $r = -1$ ,  $r = 0$ , and  $r = 1$ .
- Discuss the coefficient of determination and what its value says about a data set.
- What are the implications of  $r^2 = 1$  for a quadratic regression equation?

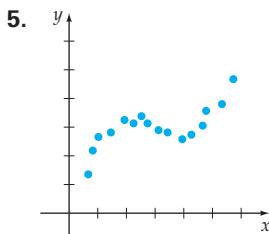

**Exercise Set 1.7**


Use a graphing calculator for this Exercise Set.

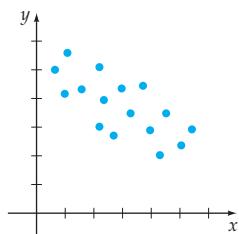
In Exercises 1 to 4, determine whether the scatter diagram suggests a linear relationship between  $x$  and  $y$ , a nonlinear relationship between  $x$  and  $y$ , or no relationship between  $x$  and  $y$ .



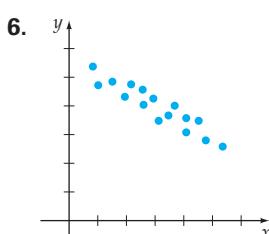
In Exercises 5 and 6, determine for which scatter diagram, A or B, the coefficient of determination is closer to 1.



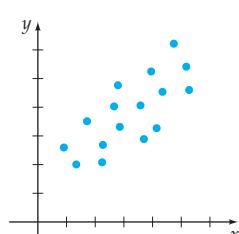
**Figure A**



**Figure B**



**Figure A**



**Figure B**

In Exercises 7 to 12, find the linear regression equation for the given set.

7.  $\{(2, 6), (3, 6), (4, 8), (6, 11), (8, 18)\}$
8.  $\{(2, -3), (3, -4), (4, -9), (5, -10), (7, -12)\}$
9.  $\{(-3, 11.8), (-1, 9.5), (0, 8.6), (2, 8.7), (5, 5.4)\}$
10.  $\{(-7, -11.7), (-5, -9.8), (-3, -8.1), (1, -5.9), (2, -5.7)\}$
11.  $\{(1.3, -4.1), (2.6, -0.9), (5.4, 1.2), (6.2, 7.6), (7.5, 10.5)\}$
12.  $\{(-1.5, 8.1), (-0.5, 6.2), (3.0, -2.3), (5.4, -7.1), (6.1, -9.6)\}$

In Exercises 13 to 16, find a quadratic model for the given data.

13.  $\{(1, -1), (2, 1), (4, 8), (5, 14), (6, 25)\}$
14.  $\{(-2, -5), (-1, 0), (0, 1), (1, 4), (2, 4)\}$
15.  $\{(1.5, -2.2), (2.2, -4.8), (3.4, -11.2), (5.1, -20.6), (6.3, -28.7)\}$
16.  $\{(-2, -1), (-1, -3.1), (0, -2.9), (1, 0.8), (2, 6.8), (3, 15.9)\}$

17. ARCHEOLOGY The data below show the length of the humerus and the wingspan, in centimeters, of several pterosaurs, which are extinct flying reptiles of the order Pterosauria. (Source: Southwest Educational Development Laboratory.)

**Pterosaur Data**

| Humerus (cm) | Wingspan (cm) |
|--------------|---------------|
| 24           | 600           |
| 32           | 750           |
| 22           | 430           |
| 17           | 370           |
| 13           | 270           |
| 4.4          | 68            |
| 3.2          | 53            |
| 1.5          | 24            |

| Humerus (cm) | Wingspan (cm) |
|--------------|---------------|
| 20           | 500           |
| 27           | 570           |
| 15           | 300           |
| 15           | 310           |
| 9            | 240           |
| 4.4          | 55            |
| 2.9          | 50            |

- a. Compute the linear regression equation for these data.

- b. On the basis of this model, what is the projected wingspan of the pterosaur *Quetzalcoatlus northropi*, which is thought to have been the largest of the prehistoric birds, if its humerus is 54 centimeters? Round to the nearest centimeter.

-  18.  **SPORTS** The data in the table below show the distance, in feet, a ball travels for various swing speeds, in miles per hour, of a bat.

| Bat speed (mph) | Distance (ft) |
|-----------------|---------------|
| 40              | 200           |
| 45              | 213           |
| 50              | 242           |
| 60              | 275           |
| 70              | 297           |
| 75              | 326           |
| 80              | 335           |

- a. Find the linear regression equation for these data.  
 b. Using the regression model, what is the expected distance a ball will travel when the bat speed is 58 miles per hour? Round to the nearest foot.
19.  **SPORTS** The table below shows the number of strokes per minute that a rower makes and the speed of the boat in meters per second.

| Strokes per minute | Speed (m/s) |
|--------------------|-------------|
| 30                 | 4.1         |
| 31                 | 4.2         |
| 33                 | 4.4         |
| 34                 | 4.5         |
| 36                 | 4.7         |
| 39                 | 5.1         |

- a. Find the linear regression equation for these data.  
 b. Using the regression model, what is the expected speed of the boat when the rowing rate is 32 strokes per minute? Round to the nearest tenth of a meter per second.
20.  **BIOLOGY** A medical researcher wanted to determine the effect of pH (a measure of alkalinity or acidity,

with pure water having a pH of 7) on the growth of a bacteria culture. The table below gives the measurements of different cultures, in thousands of bacteria, after 8 hours.

| pH | Number of Bacteria (in thousands) |
|----|-----------------------------------|
| 4  | 116                               |
| 5  | 120                               |
| 6  | 131                               |
| 7  | 136                               |
| 8  | 141                               |
| 9  | 151                               |
| 10 | 148                               |
| 11 | 163                               |

- a. Find the linear regression equation for these data.  
 b. Using the regression model, what is the expected number of bacteria when the pH is 7.5? Round to the nearest thousand bacteria.
21.  **HEALTH** The body mass index (BMI) of a person is a measure of the person's ideal body weight. The table below shows the BMI for different weights for a person 5 feet 6 inches tall. (Source: San Diego Union-Tribune, May 31, 2000)

**BMI Data for Person 5' 6" Tall**

| Weight (lb) | BMI | Weight (lb) | BMI |
|-------------|-----|-------------|-----|
| 110         | 17  | 160         | 25  |
| 120         | 19  | 170         | 27  |
| 125         | 20  | 180         | 29  |
| 135         | 21  | 190         | 30  |
| 140         | 22  | 200         | 32  |
| 145         | 23  | 205         | 33  |
| 150         | 24  | 215         | 34  |

- a. Compute the linear regression equation for these data.  
 b. On the basis of the model, what is the estimated BMI for a person 5 feet 6 inches tall whose weight is 158 pounds?
22.  **BIOLOGY** The BMI (see Exercise 21) of a person depends on height as well as weight. The table

below shows the changes in BMI for a 150-pound person as height (in inches) changes. (Source: San Diego Union-Tribune, May 31, 2000)

#### BMI Data for 150-Pound Person

| Height (in.) | BMI | Height (in.) | BMI |
|--------------|-----|--------------|-----|
| 60           | 29  | 71           | 21  |
| 62           | 27  | 72           | 20  |
| 64           | 25  | 73           | 19  |
| 66           | 24  | 74           | 19  |
| 67           | 23  | 75           | 18  |
| 68           | 23  | 76           | 18  |
| 70           | 21  |              |     |

- a. Compute the linear regression equation for these data.
- b. On the basis of the model, what is the estimated BMI for a 150-pound person who is 5 feet 8 inches tall?
23. **INDUSTRIAL ENGINEERING** Permanent-magnet direct-current motors are used in a variety of industrial applications. For these motors to be effective, there must be a strong linear relationship between the current (in amps, A) supplied to the motor and the resulting torque (in newton-centimeters, N-cm) produced by the motor. A randomly selected motor is chosen from a production line and tested, with the following results.

#### Direct-Current Motor Data at 12 Volts

| Current, in A | Torque, in N-cm | Current, in A | Torque, in N-cm |
|---------------|-----------------|---------------|-----------------|
| 7.3           | 9.4             | 8.5           | 8.6             |
| 11.9          | 2.8             | 7.9           | 4.3             |
| 5.6           | 5.6             | 14.5          | 9.5             |
| 14.2          | 4.9             | 12.7          | 8.3             |
| 7.9           | 7.0             | 10.6          | 4.7             |

-  Based on the data in this table, is the chosen motor effective? Explain.
24.  **HEALTH SCIENCES** The average remaining lifetime for men in the United States is given in the following table. (Source: National Institutes of Health.)

#### Average Remaining Lifetime for Men

| Age | Years | Age | Years |
|-----|-------|-----|-------|
| 0   | 73.6  | 65  | 15.9  |
| 15  | 59.4  | 75  | 9.9   |
| 35  | 40.8  |     |       |
|     |       |     |       |

 Based on the data in this table, is there a strong correlation between a man's age and the average remaining lifetime for that man? Explain.

25.  **HEALTH SCIENCES** The average remaining lifetime for women in the United States is given in the following table. (Source: National Institutes of Health.)

#### Average Remaining Lifetime for Women

| Age | Years | Age | Years |
|-----|-------|-----|-------|
| 0   | 79.4  | 65  | 19.2  |
| 15  | 65.1  | 75  | 12.1  |
| 35  | 45.7  |     |       |
|     |       |     |       |

-  Based on the data in this table, is there a strong correlation between a woman's age and the average remaining lifetime for that woman?
- b. Compute the linear regression equation for these data.
- c. On the basis of the model, what is the estimated remaining lifetime of a woman of age 25?

26.  **BIOLOGY** The table below gives the body lengths, in centimeters, and the highest observed flying speeds, in meters per second, of various animals.

| Species        | Length (cm) | Flying speed (m/s) |
|----------------|-------------|--------------------|
| Horsefly       | 1.3         | 6.6                |
| Hummingbird    | 8.1         | 11.2               |
| Dragonfly      | 8.5         | 10.0               |
| Willow warbler | 11          | 12.0               |
| Common pintail | 56          | 22.8               |

Based on these data, what is the flying speed of a Whimbrel whose length is 41 centimeters? Round to the nearest meter per second. (Source: Based on data from Leiva, *Algebra 2: Explorations and Applications*, p. 76, McDougal Littell, Boston; copyright 1997.)

- 27. HEALTH** The table below shows the number of calories burned in 1 hour when running at various speeds.

| Running Speed (mph) | Calories Burned |
|---------------------|-----------------|
| 10                  | 1126            |
| 10.9                | 1267            |
| 5                   | 563             |
| 5.2                 | 633             |
| 6                   | 704             |
| 6.7                 | 774             |
| 7                   | 809             |
| 8                   | 950             |
| 8.6                 | 985             |
| 9                   | 1056            |
| 7.5                 | 880             |

- a.** Are the data positively or negatively correlated?
- b.** How many calories does this model predict a person who runs at 9.5 mph for 1 hour will burn? Round to the nearest calorie.
- 28. TRAFFIC SAFETY** A traffic safety institute measured the braking distance, in feet, of a car traveling at certain speeds in miles per hour. The data from one of those tests are given in the following table.

| Speed (mph) | Breaking Distance (ft) |
|-------------|------------------------|
| 20          | 23.9                   |
| 30          | 33.7                   |
| 40          | 40.0                   |
| 50          | 41.7                   |
| 60          | 46.8                   |
| 70          | 48.9                   |
| 80          | 49.0                   |

- a.** Find the quadratic regression equation for these data.
- b.** Using the regression model, what is the expected braking distance when a car is traveling at 65 mph? Round to the nearest tenth of a foot.
- 29. BIOLOGY** The survival of certain larvae after hatching depends on the temperature (in degrees Celsius) of the surrounding environment. The following table shows the number of larvae that survive at various temperatures. Find a quadratic model for these data.

### Larvae Surviving for Various Temperatures

| Temp. (°C) | Number Surviving | Temp. (°C) | Number Surviving |
|------------|------------------|------------|------------------|
| 20         | 40               | 26         | 68               |
| 21         | 47               | 27         | 67               |
| 22         | 52               | 28         | 64               |
| 23         | 61               | 29         | 62               |
| 24         | 64               | 30         | 61               |
| 25         | 64               |            |                  |

- 30. METEOROLOGY** The temperature at various times on a summer day at a resort in southern California is given in the following table. The variable  $t$  is the number of minutes after 6:00 A.M., and the variable  $T$  is the temperature in degrees Fahrenheit.

### Temperatures at a Resort

| $t$ (min) | $T$ (°F) | $t$ (min) | $T$ (°F) |
|-----------|----------|-----------|----------|
| 20        | 59       | 240       | 86       |
| 40        | 65       | 280       | 88       |
| 80        | 71       | 320       | 86       |
| 120       | 78       | 360       | 85       |
| 160       | 81       | 400       | 80       |
| 200       | 83       |           |          |

- a.** Find a quadratic model for these data.
- b.** Use the model to predict the temperature at 1:00 P.M. Round to the nearest tenth of a degree.
- 31. AUTOMOTIVE ENGINEERING** The fuel efficiency, in miles per gallon, for a certain midsize car at various speeds, in miles per hour, is given in the table below.

### Fuel Efficiency of a Midsize Car

| mph | mpg | mph | mpg |
|-----|-----|-----|-----|
| 25  | 29  | 55  | 31  |
| 30  | 32  | 60  | 28  |
| 35  | 33  | 65  | 24  |
| 40  | 35  | 70  | 19  |
| 45  | 34  | 75  | 17  |
| 50  | 33  |     |     |

- a.** Find a quadratic model for these data.
- b.** Use the model to predict the fuel efficiency of this car when it is traveling at a speed of 50 mph.

-  **32. BIOLOGY** The data in the table at the right show the oxygen consumption, in milliliters per minute, of a bird flying level at various speeds in kilometers per hour.

- a. Find a quadratic model for these data.
  - b. Use the model to predict the oxygen consumption of a bird flying at 40 kilometers per hour. Round to the nearest milliliter per minute.

| Oxygen Consumption |                         |
|--------------------|-------------------------|
| Speed<br>(km/h)    | Consumption<br>(ml/min) |
| 20                 | 32                      |
| 25                 | 27                      |
| 28                 | 22                      |
| 35                 | 21                      |
| 42                 | 26                      |
| 50                 | 34                      |

## »»» Connecting Concepts »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»»

- 33. PHYSICS** Galileo (1564–1642) studied acceleration due to gravity by allowing balls of various weights to roll down an incline. This allowed him to time the descent of a ball more accurately than by just dropping the ball. The data in the table show some possible results of such an experiment using balls of different masses. Time  $t$  is measured in seconds; distance  $s$  is measured in centimeters.

## Distance Traveled for Balls of Various Weights

| 5-Pound Ball |          | 10-Pound Ball |          | 15-Pound Ball |          |
|--------------|----------|---------------|----------|---------------|----------|
| <i>t</i>     | <i>s</i> | <i>t</i>      | <i>s</i> | <i>t</i>      | <i>s</i> |
| 2            | 2        | 3             | 5        | 3             | 5        |
| 4            | 10       | 6             | 22       | 5             | 15       |
| 6            | 22       | 9             | 49       | 7             | 30       |
| 8            | 39       | 12            | 87       | 9             | 49       |
| 10           | 61       | 15            | 137      | 11            | 75       |
| 12           | 86       | 18            | 197      | 13            | 103      |
| 14           | 120      |               |          | 15            | 137      |
| 16           | 156      |               |          |               |          |

- a. Find a quadratic model for each of the balls.

b.  On the basis of a similar experiment, Galileo concluded that if air resistance is excluded, all falling objects fall with the same acceleration. Explain how one could make such a conclusion from the regression equations.

- 34.**  **ASTRONOMY** In 1929, Edwin Hubble published a paper that revolutionized astronomy ("A Relationship Between Distance and Radial Velocity Among Extra-Galactic Nebulae," *Proceedings of the National Academy of*

*Science*, 168). His paper dealt with the distance an extragalactic nebula was from the Milky Way galaxy and the nebula's velocity with respect to the Milky Way. The data are given in the table below. Distance is measured in megaparsecs (1 megaparsec equals  $1.918 \times 10^{19}$  miles), and velocity (called the *recession velocity*) is measured in kilometers per second. A negative velocity means the nebula is moving toward the Milky Way; a positive velocity means the nebula is moving away from the Milky Way.

## Recession Velocities

| <b>Distance</b> | <b>Velocity</b> | <b>Distance</b> | <b>Velocity</b> |
|-----------------|-----------------|-----------------|-----------------|
| 0.032           | 170             | 0.9             | 650             |
| 0.034           | 290             | 0.9             | 150             |
| 0.214           | -130            | 0.9             | 500             |
| 0.263           | -70             | 1.0             | 920             |
| 0.275           | -185            | 1.1             | 450             |
| 0.275           | -220            | 1.1             | 500             |
| 0.45            | 200             | 1.4             | 500             |
| 0.5             | 290             | 1.7             | 960             |
| 0.5             | 270             | 2.0             | 500             |
| 0.63            | 200             | 2.0             | 850             |
| 0.8             | 300             | 2.0             | 800             |
| 0.9             | -30             | 2.0             | 1090            |

- a. Find the linear regression model for these data.
  - b. On the basis of this model, what is the recession velocity of a nebula that is 1.5 megaparsecs from the Milky Way?

## »»» Projects



**MEDIAN-MEDIAN LINE** Another linear model of data is called the **median-median line**. This line employs *summary points* calculated using the medians of subsets of the independent and dependent variables. The **median** of a data set is the middle number or the average of the two middle numbers for a data set arranged in numerical order. For instance, to find the median of {8, 12, 6, 7, 9}, first arrange the data in numerical order.

$$6, 7, 8, 9, 12$$

The median is 8, the number in the middle. To find the median of {15, 12, 20, 9, 13, 10}, arrange the numbers in numerical order.

$$9, 10, 12, 13, 15, 20$$

The median is 12.5, the average of the two middle numbers.

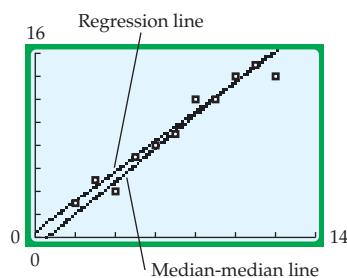
$$\text{Median} = \frac{12 + 13}{2} = 12.5$$

The median-median line is determined by dividing a data set into three equal groups. (If the set cannot be divided into three equal groups, the first and third groups should be equal. For instance, if there are 11 data points, divide the set into groups of 4, 3, and 4.) The slope of the median-median line is the slope of the line through the  $x$ -medians and  $y$ -medians of the first and third sets of points. The median-median line passes through the average of the  $x$ -and  $y$ -medians of all three sets.

A graphing calculator can be used to find the equation of the median-median line. This line, along with the linear regression line, is shown in the next column for the data in the accompanying table.

- Find the equation of the median-median line for the data in Exercise 17 on page 103.
- Find the equation of the median-median line for the data in Exercise 18 on page 104.

| <b>x</b> | <b>y</b> |
|----------|----------|
| 2        | 3        |
| 3        | 5        |
| 4        | 4        |
| 5        | 7        |
| 6        | 8        |
| 7        | 9        |
| 8        | 12       |
| 9        | 12       |
| 10       | 14       |
| 11       | 15       |
| 12       | 14       |



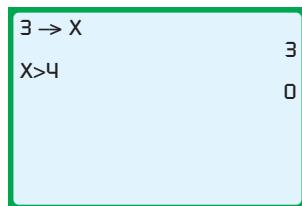
- Consider the data set {(1, 3), (2, 5), (3, 7), (4, 9), (5, 11), (6, 13), (7, 15), (8, 17)}.
  - Find the equation of the linear regression line for these data.
  - Find the equation of the median-median line for these data.
  - What conclusion might you draw from the answers to a. and b.?
- For this exercise, use the data in the table above.
  - Calculate the equation of the median-median line and the linear regression line.
  - Change the entry (12, 14) to (12, 1) and then recalculate the equations of the median-median line and the linear regression line.
  - Explain why there is more change in the linear regression line than in the median-median line.



### Exploring Concepts with Technology

#### Graphing Piecewise Functions with a Graphing Calculator

A graphing calculator can be used to graph piecewise functions by including as part of the function the interval on which each piece of the function is defined. The method is based on the fact that a graphing calculator “evaluates” inequalities. For purposes of this Exploration, we will use keystrokes for a TI-83/TI-83 Plus/TI-84 Plus calculator.

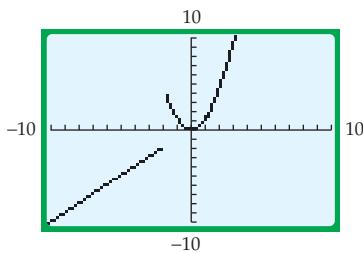


For instance, store 3 in  $X$  by pressing 3 **STO** **X,T,Θ,n** **ENTER**. Now enter the inequality  $x > 4$  by pressing **X,T,Θ,n** **2ND TEST** 3 4 **ENTER**. Your screen should look like the one at the left. Note that the value of the inequality on the screen is 0. This occurs because the calculator replaced  $X$  by 3 and then determined whether the inequality  $3 > 4$  was true or false. The calculator expresses the fact that the inequality is false by placing a zero on the screen. If we repeat the sequence of steps above, except that we store 5 in  $X$  instead of 3, the calculator will determine that the inequality is true and place a 1 on the screen.

This property of calculators is used to graph piecewise functions. Graphs of these functions work best when Dot mode rather than Connected mode is used. To switch to Dot mode, select **MODE**, use the arrow keys to highlight **DOT**, and then press **ENTER**.

Next we will graph the piecewise function defined by

$$f(x) = \begin{cases} x, & x \leq -2 \\ x^2, & x > -2 \end{cases}$$



Enter the function<sup>3</sup> as  $Y_1=X*(X\leq-2)+X^2*(X>-2)$  and graph it in the standard viewing window. Note that you are multiplying each piece of the function by its domain. The graph will appear as shown at the left.

To understand how the graph is drawn, we will consider two values of  $x$ ,  $-8$  and  $2$ , and evaluate  $Y_1$  for each of these values.

$$\begin{aligned} Y_1 &= X*(X\leq-2)+X^2*(X>-2) \\ &= -8(-8 \leq -2) + (-8)^2(-8 > -2) \\ &= -8(1) + 64(0) = -8 \end{aligned} \quad \bullet \text{ When } x = -8, \text{ the value assigned to } X \leq -2 \text{ is 1; the value assigned to } X > -2 \text{ is 0.}$$

$$\begin{aligned} Y_1 &= X*(X\leq-2)+X^2*(X>-2) \\ &= 2(2 \leq -2) + 2^2(2 > -2) \\ &= 2(0) + 4(1) = 4 \end{aligned} \quad \bullet \text{ When } x = 2, \text{ the value assigned to } X \leq -2 \text{ is 0; the value assigned to } X > -2 \text{ is 1.}$$

In a similar manner, for any value of  $x$ ,  $x \leq -2$ , the value assigned to  $(X \leq -2)$  is 1 and the value assigned to  $(X > -2)$  is 0. Thus  $Y_1=X*1+X^2*0=X$  on that interval. This means that only the  $f(x) = x$  piece of the function is graphed. When  $x > -2$ , the value assigned to  $(X \leq -2)$  is 0 and the value assigned to  $(X > -2)$  is 1. Thus  $Y_1=X*0+X^2*1=X^2$  on that interval. This means that only the  $f(x) = x^2$  piece of the function is graphed on that interval.

- |  |  |
|--|--|
| <ol style="list-style-type: none"> <li>Graph: <math>f(x) = \begin{cases} x^2, &amp; x &lt; 2 \\ -x, &amp; x \geq 2 \end{cases}</math></li> </ol>           | <ol style="list-style-type: none"> <li>Graph: <math>f(x) = \begin{cases} x^2 - x, &amp; x &lt; 2 \\ -x + 4, &amp; x \geq 2 \end{cases}</math></li> </ol>       |
| <ol style="list-style-type: none"> <li>Graph: <math>f(x) = \begin{cases} -x^2 + 1, &amp; x &lt; 0 \\ x^2 - 1, &amp; x \geq 0 \end{cases}</math></li> </ol> | <ol style="list-style-type: none"> <li>Graph: <math>f(x) = \begin{cases} x^3 - 4x, &amp; x &lt; 1 \\ x^2 - x + 2, &amp; x \geq 1 \end{cases}</math></li> </ol> |

<sup>3</sup>Note that pressing **2ND TEST** will display the inequality menu.

## Chapter 1 Summary

### 1.1 Equations and Inequalities

- The integers are the set  $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ . The positive integers are called *natural numbers*. The rational numbers are  $\left\{ \frac{a}{b} \mid a, b \text{ integers and } b \neq 0 \right\}$ . Irrational numbers are nonterminating and nonrepeating decimals. The *real numbers* are the set of rational and irrational numbers.
- The interval of real numbers  $(a, b) = \{x \mid a < x < b\}$ .
- A *linear equation* is one that can be written in the form  $ax + b = 0$ ,  $a \neq 0$ . A *quadratic equation* is one that can be written in the form  $ax^2 + bx + c = 0$ ,  $a \neq 0$ .
- An *inequality* is a statement that contains the symbol  $<$ ,  $\leq$ ,  $>$ , or  $\geq$ .

### 1.2 A Two-Dimensional Coordinate System and Graphs

- The Distance Formula* The distance  $d$  between the points represented by  $(x_1, y_1)$  and  $(x_2, y_2)$  is
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
- The midpoint of the line segment from  $P_1(x_1, y_1)$  to  $P_2(x_2, y_2)$  is

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

- The standard form of the equation of a circle with center at  $(h, k)$  and radius  $r$  is  $(x - h)^2 + (y - k)^2 = r^2$ .

### 1.3 Introduction to Functions

- Definition of a Function* A function is a set of ordered pairs in which no two ordered pairs that have the same first coordinate have different second coordinates.
- A graph is the graph of a function if and only if no vertical line intersects the graph at more than one point. If every horizontal line intersects the graph of a function at most once, then the graph is the graph of a one-to-one function.

### 1.4 Properties of Graphs

- The graph of an equation is symmetric with respect to the  $y$ -axis if the replacement of  $x$  with  $-x$  leaves the equation unaltered.  
the  $x$ -axis if the replacement of  $y$  with  $-y$  leaves the equation unaltered.  
the origin if the replacement of  $x$  with  $-x$  and  $y$  with  $-y$  leaves the equation unaltered.

- If  $f$  is a function and  $c$  is a positive constant, then the graph of  $y = f(x) + c$  is the graph of  $y = f(x)$  shifted up *vertically*  $c$  units  
 $y = f(x) - c$  is the graph of  $y = f(x)$  shifted down *vertically*  $c$  units  
 $y = f(x + c)$  is the graph of  $y = f(x)$  shifted left *horizontally*  $c$  units  
 $y = f(x - c)$  is the graph of  $y = f(x)$  shifted right *horizontally*  $c$  units
- The graph of  
 $y = -f(x)$  is the graph of  $y = f(x)$  reflected across the  $x$ -axis.  
 $y = f(-x)$  is the graph of  $y = f(x)$  reflected across the  $y$ -axis.
- If  $c > 1$ , then the graph of  $y = c \cdot f(x)$  is a vertical stretching of the graph of  $y = f(x)$ .
- If  $0 < c < 1$ , then the graph of  $y = c \cdot f(x)$  is a vertical compression of the graph of  $y = f(x)$ .
- If  $c > 1$ , then the graph of  $y = f(cx)$  is a horizontal compression of the graph of  $y = f(x)$ .
- If  $0 < c < 1$ , then the graph of  $y = f(cx)$  is a horizontal stretching of the graph of  $y = f(x)$ .

### 1.5 The Algebra of Functions

- For all values of  $x$  for which both  $f(x)$  and  $g(x)$  are defined, we define the following functions.
- |            |  |
|------------|--|
| Sum        | $(f + g)(x) = f(x) + g(x)$   |
| Difference | $(f - g)(x) = f(x) - g(x)$   |
| Product    | $(f \cdot g)(x) = f(x) \cdot g(x)$                                     |
| Quotient   | $\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$ |
- The expression

$$\frac{f(x + h) - f(x)}{h}, \quad h \neq 0$$

is called the difference quotient of  $f$ . The difference quotient can be used to compute the *average velocity* of a moving object over a given time interval.

- For the functions  $f$  and  $g$ , the composition of the two functions is given by  $(f \circ g)(x) = f[g(x)]$  for all  $x$  in the domain of  $g$  such that  $g(x)$  is in the domain of  $f$ .

### 1.6 Inverse Functions

- If  $f$  is a one-to-one function with domain  $X$  and range  $Y$ , and  $g$  is a function with domain  $Y$  and range  $X$ , then  $g$  is the inverse function of  $f$  if and only if  $(f \circ g)(x) = x$  for all  $x$  in the domain of  $g$  and  $(g \circ f)(x) = x$  for all  $x$  in the domain of  $f$ .

- A function  $f$  has an inverse function if and only if it is a one-to-one function. The graph of a function  $f$  and the graph of its inverse function  $f^{-1}$  are symmetric with respect to the line given by  $y = x$ .

### 1.7 Modeling Data Using Regression

- Regression analysis is used to find a mathematical model of collected data.
- The least-squares regression line is the line that minimizes the sum of the squares of the vertical deviations of all data points from the line.

- The linear correlation coefficient  $r$  is a measure of how closely the points of a data set can be modeled by a straight line. If  $r = -1$ , then the points of the data set can be modeled *exactly* by a straight line with negative slope. If  $r = 1$ , then the data set can be modeled *exactly* by a straight line with positive slope. For all data sets,  $-1 \leq r \leq 1$ .

- The coefficient of determination is  $r^2$ . It measures the percent of the total variation in the dependent variable that is explained by the regression line.
- It is possible to find both linear and nonlinear mathematical models of data.

## Chapter 1 Assessing Concepts

- A function  $f$  has the property that for two real numbers  $a$  and  $b$ , with  $a < b$ ,  $f(a) > f(b)$ . Which of the following are true statements?
  - $f$  is a one-to-one function.
  - $f$  is an increasing function.
  - $f$  is a decreasing function.
  - $f$  has an inverse function.
  - $f$  could be a linear function.
  - $f$  could be a quadratic function.
- Let  $f(x) = 3x + 8$  and  $g(x) = 2x + 4$ . Show that
$$f[g(x)] = g[f(x)]$$
Are  $f$  and  $g$  inverse functions?
- If  $f(x) = f(x + 4)$  for all real numbers  $x$ , and  $f(2) = 3$ , find  $f(18)$ .
- Use absolute value notation to describe the set of points that are less than 3 units from the point whose coordinate is  $-2$  on the real number line.
- If  $a$  and  $b$  are two different numbers in the domain of a function  $f$ , explain the meaning of  $\frac{f(b) - f(a)}{b - a}$ .
- Suppose  $f$  is a one-to-one function and  $f(-2) = 3$ . Find the coordinates of an ordered pair on the graph of the inverse of  $f$ .
- Suppose  $(2, 3)$  are the coordinates of a point on the graph of  $y = f(x)$ . Give the coordinates of a point on the graph of  $y = f(x - 5)$ .
- Suppose  $(-3, 5)$  are the coordinates of a point on the graph of  $y = f(x)$ . Give the coordinates of a point on the graph of  $y = f(x) + 1$ .
- Suppose  $(6, 4)$  are the coordinates of a point on the graph of  $y = f(x)$ . Give the coordinates of a point on the graph of  $y = f(2x)$ .
- Suppose that the correlation coefficient for a linear regression equation is  $-0.86$ . Does this information enable you to determine anything about the slope of the regression line?



## Chapter 1 Review Exercises

In Exercises 1 to 14, solve each equation or inequality.

1.  $3 - 4z = 12$

2.  $4y - 3 = 6y + 5$

3.  $2x - 3(2 - 3x) = 14x$

4.  $5 - 2(3m + 2) = 3(1 - m)$

5.  $y^2 - 3y - 18 = 0$

6.  $2z^2 - 9z + 4 = 0$

7.  $3v^2 + v = 1$

8.  $3s = 4 - 2s^2$

9.  $3c - 5 \leq 5c + 7$

10.  $7a > 5 - 2(3a - 4)$

11.  $x^2 - x - 12 \geq 0$

12.  $2x^2 - x < 1$

13.  $|2x - 5| > 3$

14.  $|1 - 3x| \leq 4$

In Exercises 15 to 18, use the Pythagorean Theorem,  $c^2 = a^2 + b^2$ , to find the unknown. The letters  $a$  and  $b$  represent the lengths of the legs of a right triangle, and  $c$  represents the length of the hypotenuse.

15.  $a = 6, b = 8$ . Find  $c$ .

16.  $a = 11, c = 20$ . Find  $b$ .

17.  $b = 12, c = 13$ . Find  $a$ .

18.  $a = 7, b = 14$ . Find  $c$ .

In Exercises 19 and 20, find the distance between the points whose coordinates are given.

19.  $(-3, 2)$     $(7, 11)$

20.  $(5, -4)$     $(-3, -8)$

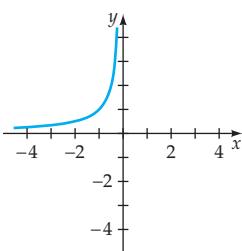
In Exercises 21 and 22, find the midpoint of the line segment with the given endpoints.

21.  $(2, 8)$     $(-3, 12)$

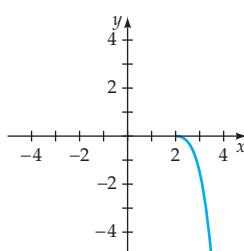
22.  $(-4, 7)$     $(8, -11)$

In Exercises 23 and 24, complete the graph so that it is symmetric with respect to a. the  $x$ -axis, b. the  $y$ -axis, and c. the origin.

23.



24.



In Exercises 25 to 32, determine whether the graph of each equation is symmetric with respect to the  $x$ -axis, the  $y$ -axis, or the origin.

25.  $y = x^2 - 7$

26.  $x = y^2 + 3$

27.  $y = x^3 - 4x$

28.  $y^2 = x^2 + 4$

29.  $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

30.  $xy = 8$

31.  $|y| = |x|$

32.  $|x + y| = 4$

In Exercises 33 and 34, determine the center and radius of the circle with the given equation.

33.  $(x - 3)^2 + (y + 4)^2 = 81$

34.  $x^2 + y^2 + 10x + 4y + 20 = 0$

In Exercises 35 and 36, find the equation in standard form of the a circle that satisfies the given conditions.

35. Center  $C = (2, -3)$ , radius  $r = 5$

36. Center  $C = (-5, 1)$ , passing through  $(3, 1)$

37. If  $f(x) = 3x^2 + 4x - 5$ , find

- |                      |                   |                   |
|----------------------|-------------------|-------------------|
| <b>a.</b> $f(1)$     | <b>b.</b> $f(-3)$ | <b>c.</b> $f(t)$  |
| <b>d.</b> $f(x + h)$ | <b>e.</b> $3f(t)$ | <b>f.</b> $f(3t)$ |

38. If  $g(x) = \sqrt{64 - x^2}$ , find

- |                   |                   |                   |
|-------------------|-------------------|-------------------|
| <b>a.</b> $g(3)$  | <b>b.</b> $g(-5)$ | <b>c.</b> $g(8)$  |
| <b>d.</b> $g(-x)$ | <b>e.</b> $2g(t)$ | <b>f.</b> $g(2t)$ |

39. If  $f(x) = x^2 + 4x$  and  $g(x) = x - 8$ , find

- |                            |                             |
|----------------------------|-----------------------------|
| <b>a.</b> $(f \circ g)(3)$ | <b>b.</b> $(g \circ f)(-3)$ |
| <b>c.</b> $(f \circ g)(x)$ | <b>d.</b> $(g \circ f)(x)$  |

40. If  $f(x) = 2x^2 + 7$  and  $g(x) = |x - 1|$ , find

- |                             |                             |
|-----------------------------|-----------------------------|
| <b>a.</b> $(f \circ g)(-5)$ | <b>b.</b> $(g \circ f)(-5)$ |
| <b>c.</b> $(f \circ g)(x)$  | <b>d.</b> $(g \circ f)(x)$  |

41. If  $f(x) = 4x^2 - 3x - 1$ , find the difference quotient

$$\frac{f(x + h) - f(x)}{h}$$

- 42.** If  $g(x) = x^3 - x$ , find the difference quotient

$$\frac{g(x+h) - g(x)}{h}$$

**In Exercises 43 to 46, determine the domain of the function represented by the given equation.**

**43.**  $f(x) = -2x^2 + 3$

**44.**  $f(x) = \sqrt{6-x}$

**45.**  $f(x) = \sqrt{25-x^2}$

**46.**  $f(x) = \frac{3}{x^2 - 2x - 15}$

**In Exercises 47 to 52, sketch the graph of  $f$ . Find the interval(s) on which  $f$  is increasing, constant, or decreasing.**

**47.**  $f(x) = |x - 3| - 2$

**48.**  $f(x) = x^2 - 5$

**49.**  $f(x) = |x + 2| - |x - 2|$

**50.**  $f(x) = \llbracket x + 3 \rrbracket$

**51.**  $f(x) = \frac{1}{2}x - 3$

**52.**  $f(x) = \sqrt[3]{x}$

**In Exercises 53 to 58, sketch the graph of  $g$ . a. Find the domain and range of  $g$ . b. State whether  $g$  is even, odd, or neither.**

**53.**  $g(x) = -x^2 + 4$

**54.**  $g(x) = -2x - 4$

**55.**  $g(x) = |x - 2| + |x + 2|$

**56.**  $g(x) = \sqrt{16 - x^2}$

**57.**  $g(x) = x^3 - x$

**58.**  $g(x) = 2\llbracket x \rrbracket$

**In Exercises 59 and 60, use the given functions  $f$  and  $g$  to find  $f + g$ ,  $f - g$ ,  $fg$ , and  $\frac{f}{g}$ . State the domain of each.**

**59.**  $f(x) = x^2 - 9$ ,  $g(x) = x + 3$

**60.**  $f(x) = x^3 + 8$ ,  $g(x) = x^2 - 2x + 4$

**In Exercises 61 to 64, determine whether the given functions are inverses.**

**61.**  $F(x) = 2x - 5$      $G(x) = \frac{x+5}{2}$

**62.**  $h(x) = \sqrt{x}$      $k(x) = x^2$ ,  $x \geq 0$

**63.**  $l(x) = \frac{x+3}{x}$      $m(x) = \frac{3}{x-1}$

**64.**  $p(x) = \frac{x-5}{2x}$      $q(x) = \frac{2x}{x-5}$

**In Exercises 65 to 68, find the inverse of the function. Sketch the graph of the function and its inverse on the same set of coordinate axes.**

**65.**  $f(x) = 3x - 4$

**66.**  $g(x) = -2x + 3$

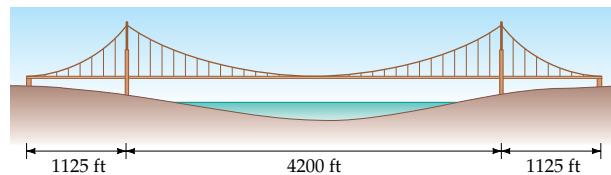
**67.**  $h(x) = -\frac{1}{2}x - 2$

**68.**  $k(x) = \frac{1}{x}$

**69. HEIGHT OF A FALLING OBJECT** The roadway of the Golden Gate Bridge is 220 feet above the water. The height  $h$  of a rock dropped from the bridge is a function of the time  $t$  it has been falling. If  $h$  is measured in feet and  $t$  is measured in seconds, then the function is given by  $h(t) = -16t^2 + 220$ . Use this function to find the time it will take the rock to hit the water.

**70. BRIDGE CABLES** The suspension cables of the Golden Gate Bridge approximate the shape of a parabola. If  $h$  is the height of the cables above the roadway in feet and  $|x|$  is the distance in feet from the center of the bridge, then the parabolic shape of the cables is represented by

$$h(x) = \frac{1}{8820}x^2 + 25, \quad -2100 \leq x \leq 2100$$



- a.** Find the height of the cables 1050 feet from the center of the bridge.
- b.** The towers that support the cables are 2100 feet from the center of the bridge. Find the height (above the roadway) of the towers that support the cables.

- 71.** **COMPUTER SCIENCE** A test of an Internet service provider showed the following download times (in seconds) for files of various sizes (in kilobytes).

#### Download Times

| Size | Time |
|------|------|
| 10.5 | 0.20 |
| 12.9 | 0.24 |
| 15   | 0.27 |
| 20   | 0.36 |
| 60   | 1.09 |
| 75   | 1.42 |

| Size | Time |
|------|------|
| 110  | 2.01 |
| 156  | 2.68 |
| 163  | 2.87 |
| 175  | 3.10 |
| 200  | 3.64 |
| 250  | 4.61 |

- a.** Find a linear regression model for these data.

- b.**  From the value of  $r$ , is a linear model of the data a reasonable model? Explain.
- c.** On the basis of the model, what is the expected download time for a file that is 100 kilobytes in size?
- 72.**  **PHYSICS** The rate at which water will escape from a hole in the bottom of a can depends on a number of factors, including the height of the water, the size of the hole, and the diameter of the can. The table below shows the height (in millimeters) of water in a can after  $t$  seconds.

#### Water Escaping a Ruptured Can

| Height | Time |
|--------|------|
| 0      | 180  |
| 10     | 163  |
| 20     | 147  |
| 30     | 133  |
| 40     | 118  |
| 50     | 105  |

| Height | Time |
|--------|------|
| 60     | 93   |
| 70     | 81   |
| 80     | 70   |
| 90     | 60   |
| 100    | 50   |
| 110    | 48   |

- a.** Find a quadratic regression model for these data.
- b.** On the basis of this model, will the can ever empty?
- c.**  Explain why there seems to be a contradiction between the model and reality in that we know the can will eventually run out of water.

### »»» Quantitative Reasoning: *Public Key Cryptography* »»»

As mentioned in the Chapter Opener, performing financial transactions over the Internet requires secure transmissions between two sites, the sender and the receiver. One method of creating secure transmissions is to use a *modular function*.

A modular function is one that gives, in integer form, the remainder when one number is divided by another. We write  $a \equiv b \pmod{m}$  to mean that  $a$  is the remainder when  $b$  is divided by  $m$ . Here are some examples.

$$4 \equiv 22 \pmod{6} \quad \text{because } 22 \div 6 = 3 \text{ remainder } 4.$$

$$1 \equiv 37 \pmod{4} \quad \text{because } 37 \div 4 = 9 \text{ remainder } 1.$$

$$0 \equiv 55 \pmod{11} \quad \text{because } 55 \div 11 = 5 \text{ remainder } 0.$$

$$17 \equiv 17 \pmod{31} \quad \text{because } 17 \div 31 = 0 \text{ remainder } 17.$$

#### take note

To encrypt a message means to use a secret code to change the message so that it cannot be understood by an unauthorized user. To decrypt a message means to change a coded message back to its original form.

**QR1.** Find the value of each expression.

a.  $15 \pmod{4}$     b.  $37 \pmod{5}$     c.  $52 \pmod{321}$

*Public key cryptography* uses a modular function to encrypt a message—say, a person's name or credit card number—so that only the receiver of the message can decrypt it. The message is decrypted by using the inverse of the modular function that was used to encrypt the message. Inverse functions were discussed in Section 1.6.

The unique feature of public key cryptography is that knowing the modular function (called the *public key*) used to encrypt a message does not allow one to easily determine the inverse modular function (called the *private key*) that is required to decrypt the message. For instance, Alice could give many people her public key to encrypt a message sent to her, but none of those people could decrypt a message written by another person and sent to Alice.

**QR2.** You will need to do some research on public key cryptography to complete this exercise. Suppose a computer hacker wanted to decrypt a message that was encrypted using public key cryptography. What mathematical operation would the hacker need to be able to complete in order to find the inverse of the modular function that was used to encrypt the message?

**take note**

Numbers can be assigned to letters in many ways. One common method is to use the American Standard Code for Information Interchange (ASCII). This system assigns a number to every letter, number, and punctuation mark. For instance, "A" is 65 and "a" is 97.

Computing powers of a modular function is difficult without a calculator. The program **POWERMOD** can be used for these calculations. This program, intended for use with a TI-83/TI-83 Plus/TI-84 Plus calculator, can be found at our website [college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT).

To encrypt a message, first assign a number to each letter of the alphabet. For instance, let  $A = 10, B = 11, C = 12, \dots, Z = 36$ . The word MATH becomes 22102917.

Once a message has been converted to numbers, it is divided into *blocks*. We will use a block of 4 numbers so that our message is now 2210|2917. Each block is encrypted using the public key. We will use a modulus of 3233, but in practice this number would be very large—at least 200 digits. The following are our encryption and decryption functions.

*Encryption function:*  $E = M^{17} \pmod{3233}$ , where  $M$  is the message to be coded and  $E$  is the coded message.

*Decryption function:*  $M = E^{2753} \pmod{3233}$ , where  $E$  is the message to be decoded and  $M$  is the original message.

Encrypt block 1:

$$2210^{17} \pmod{3233} = 1085$$

Encrypt block 2:

$$2917^{17} \pmod{3233} = 904$$

Instead of 2210|2917, the encrypted message 1085|0904 is sent (zeros are usually inserted to keep the blocks equal in length).

The person receiving the message would use the decryption function to obtain the original message.

Decrypt block 1:

$$1085^{2753} \pmod{3233} = 2210$$

Decrypt block 2:

$$0904^{2753} \pmod{3233} = 2917$$

The decrypted message is 2210|2917, or MATH.

**QR3.** Explain how public key cryptography could be used to ensure that an email you received was sent by a particular person and not just someone using that person's computer. If you have difficulty with this assignment, research the topic *digital signatures*.



## Chapter 1 Test

1. Solve:  $4x - 2(2 - x) = 5 - 3(2x + 1)$
  2. Solve:  $6 - 3x \geq 3 - 4(2 - 2x)$
  3. Solve:  $2x^2 - 3x = 2$
  4. Solve:  $3x^2 - x = 2$
  5. Solve:  $|4 - 5x| > 6$
  6. Find the distance between the points  $(-2, 5)$  and  $(4, -2)$ .
  7. Find the midpoint and the length of the line segment with endpoints  $(-2, 3)$  and  $(4, -1)$ .
  8. Determine the  $x$ - and  $y$ -intercepts of, and then graph, the equation  $x = 2y^2 - 4$ .
  9. Graph the equation  $y = |x + 2| + 1$ .
  10. Find the center and radius of the circle that has the general form  $x^2 - 4x + y^2 + 2y - 4 = 0$ .
  11. Given  $f(x) = -\sqrt{25 - x^2}$ , evaluate  $f(-3)$ .
  12. Determine the domain of the function defined by  

$$f(x) = -\sqrt{x^2 - 16}$$
  13. Graph  $f(x) = -2|x - 2| + 1$ . Identify the intervals over which the function is increasing, constant, or decreasing.
  14. Use the graph of  $f(x) = |x|$  to graph  $y = -f(x + 2) - 1$ .
  15. Which of the following define odd functions?
    - $f(x) = x^4 - x^2$
    - $f(x) = x^3 - x$
    - $f(x) = x - 1$
  16. Let  $f(x) = x^2 - 1$  and  $g(x) = x - 2$ . Find  $(f + g)$  and  $\left(\frac{f}{g}\right)$ .
  17. Find the difference quotient of the function  

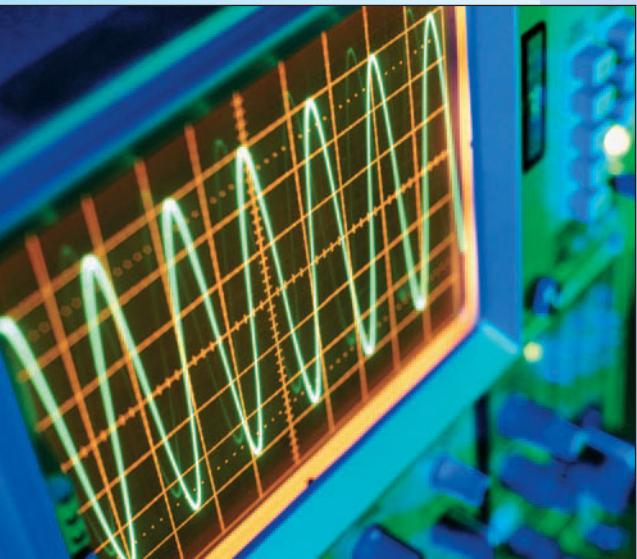
$$f(x) = x^2 + 1$$
  18. Evaluate  $(f \circ g)$ , where  

$$f(x) = x^2 - 2x + 1 \quad \text{and} \quad g(x) = \sqrt{x - 2}$$
  19. Find the inverse of the function given by the equation  

$$f(x) = \frac{x}{x + 1}$$
  20. **CALORIE CONTENT** The table below shows the percent of water and the number of calories in various canned soups to which 100 grams of water are added.
- Percent Water in Soups**
- | % Water | Calories |
|---------|----------|
| 93.2    | 28       |
| 92.3    | 26       |
| 91.9    | 39       |
| 89.5    | 56       |
- | % Water | Calories |
|---------|----------|
| 89.6    | 56       |
| 90.5    | 36       |
| 91.9    | 32       |
| 91.7    | 32       |
- Find the equation of the linear regression line for these data.
  - Using the linear model from a., find the expected number of calories in a soup that is 89% water.

# 2

# Trigonometric Functions



- 2.1 Angles and Arcs
- 2.2 Right Triangle Trigonometry
- 2.3 Trigonometric Functions of Any Angle
- 2.4 Trigonometric Functions of Real Numbers
- 2.5 Graphs of the Sine and Cosine Functions
- 2.6 Graphs of the Other Trigonometric Functions
- 2.7 Graphing Techniques
- 2.8 Harmonic Motion—An Application of the Sine and Cosine Functions

## Applications of Trigonometric Functions

In this chapter we introduce an important group of functions called the *trigonometric functions*. These functions are often used in applications involving relationships among the sides and angles of triangles. [Exercise 59 on page 143](#) illustrates this aspect of trigonometry.

In the seventeenth century, a unit circle approach was used to create *trigonometric functions of real numbers* or *circular functions*. These functions enable us to solve a wider variety of application problems. For instance, in [Exercise 63 on page 178](#), a trigonometric function of a real number is used to model a sound wave and find the frequency of the sound wave.

An application that involves finding the time it takes for the music produced by two sound tracks to repeat is given in the [Quantitative Reasoning exercises on page 214](#).

| Tracks        |           | 1         | 2         | 3         | 4         | 5         | 6         | 7         | 8         |
|---------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Guitar        | - - - - - | - - - - - | - - - - - | - - - - - | - - - - - | - - - - - | - - - - - | - - - - - | - - - - - |
| Electric Bass | — — — —   | — — — —   | — — — —   | — — — —   | — — — —   | — — — —   | — — — —   | — — — —   | — — — —   |

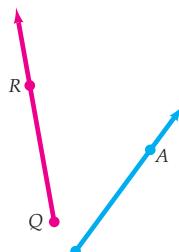


## Online Study Center

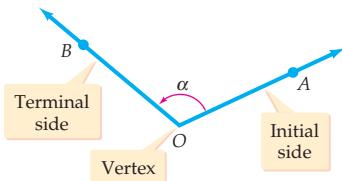
For online student resources, such as section quizzes, visit this website:  
[college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

## Section 2.1

- Degree Measure
- Radian Measure
- Arcs and Arc Length
- Linear and Angular Speed



**Figure 2.1**



**Figure 2.2**

## Angles and Arcs

A point  $P$  on a line separates the line into two parts, each of which is called a **half-line**. The union of point  $P$  and the half-line formed by  $P$  that includes point  $A$  is called a **ray**, and it is represented as  $\overrightarrow{PA}$ . The point  $P$  is the **endpoint** of ray  $\overrightarrow{PA}$ . **Figure 2.1** shows the ray  $\overrightarrow{PA}$  and a second ray  $\overrightarrow{QR}$ .

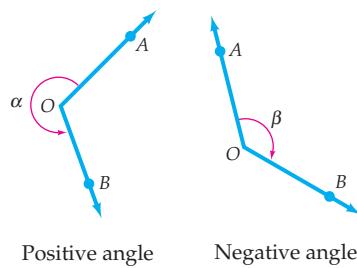
In geometry, an **angle** is defined simply as the union of two rays that have a common endpoint. In trigonometry and many advanced mathematics courses, it is beneficial to define an angle in terms of a rotation.

### Definition of an Angle

An **angle** is formed by rotating a given ray about its endpoint to some terminal position. The original ray is the **initial side** of the angle, and the second ray is the **terminal side** of the angle. The common endpoint is the **vertex** of the angle.

There are several methods used to name an angle. One way is to employ Greek letters. For example, the angle shown in **Figure 2.2** can be designated as  $\alpha$  or as  $\angle\alpha$ . It also can be named  $\angle O$ ,  $\angle AOB$ , or  $\angle BOA$ . If you name an angle by using three points, such as  $\angle AOB$ , it is traditional to list the vertex point between the other two points.

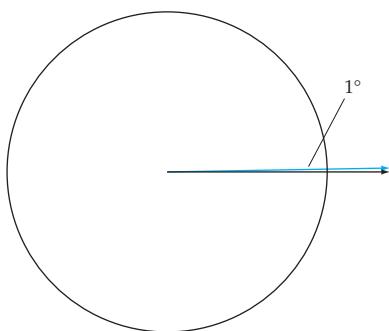
Angles formed by a counterclockwise rotation are considered **positive angles**, and angles formed by a clockwise rotation are considered **negative angles**. See **Figure 2.3**.



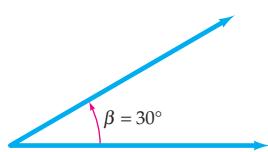
**Figure 2.3**

### Degree Measure

The **measure** of an angle is determined by the amount of rotation of the initial side. An angle formed by rotating the initial side counterclockwise exactly once until it coincides with itself (one complete revolution) is defined to have a measure of 360 degrees, which is abbreviated as  $360^\circ$ .

**Figure 2.4**

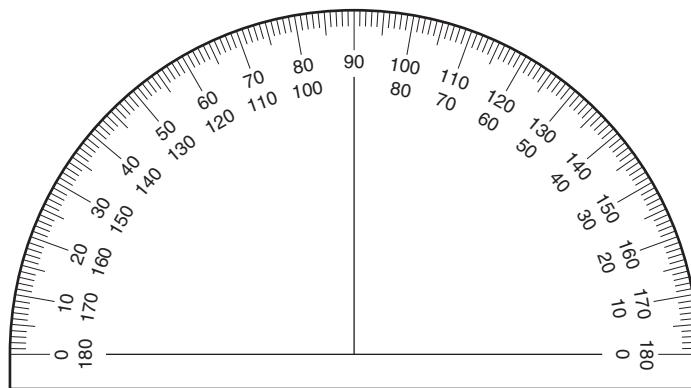
$$1^\circ = \frac{1}{360} \text{ of a revolution}$$

**Figure 2.5**

### Definition of Degree

One **degree** is the measure of an angle formed by rotating a ray  $\frac{1}{360}$  of a complete revolution. The symbol for degree is  $^\circ$ .

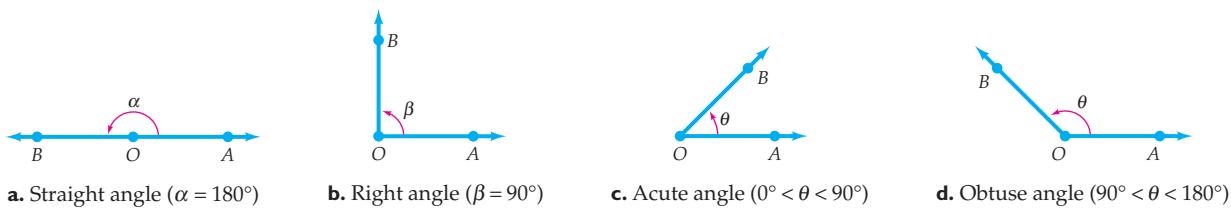
The angle shown in **Figure 2.4** has a measure of  $1^\circ$ . The angle  $\beta$  shown in **Figure 2.5** has a measure of  $30^\circ$ . We will use the notation  $\beta = 30^\circ$  to denote that the measure of angle  $\beta$  is  $30^\circ$ . The protractor shown in **Figure 2.6** can be used to measure an angle in degrees or to draw an angle with a given degree measure.

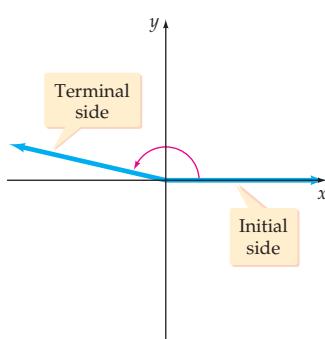
**Figure 2.6**

Protractor for measuring angles in degrees

Angles are often classified according to their measure.

- $180^\circ$  angles are **straight angles**. See **Figure 2.7a**.
- $90^\circ$  angles are **right angles**. See **Figure 2.7b**.
- Angles that have a measure greater than  $0^\circ$  but less than  $90^\circ$  are **acute angles**. See **Figure 2.7c**.
- Angles that have a measure greater than  $90^\circ$  but less than  $180^\circ$  are **obtuse angles**. See **Figure 2.7d**.

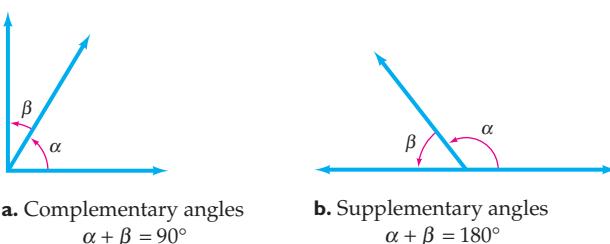
**Figure 2.7**



**Figure 2.8**  
An angle in standard position

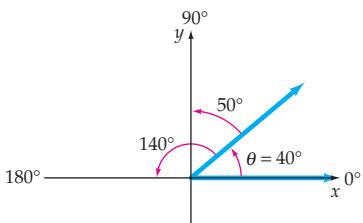
An angle superimposed in a Cartesian coordinate system is in **standard position** if its vertex is at the origin and its initial side is on the positive  $x$ -axis. See **Figure 2.8**.

Two positive angles are **complementary angles** (**Figure 2.9a**) if the sum of the measures of the angles is  $90^\circ$ . Each angle is the *complement* of the other angle. Two positive angles are **supplementary angles** (**Figure 2.9b**) if the sum of the measures of the angles is  $180^\circ$ . Each angle is the *supplement* of the other angle.



**Figure 2.9**

### EXAMPLE 1 » Find the Measure of the Complement and the Supplement of an Angle



**Figure 2.10**

For each angle, find the measure (if possible) of its complement and of its supplement.

- a.  $\theta = 40^\circ$     b.  $\theta = 125^\circ$

#### Solution

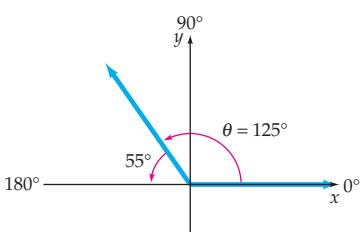
- a. **Figure 2.10** shows  $\angle \theta = 40^\circ$  in standard position. The measure of its complement is  $90^\circ - 40^\circ = 50^\circ$ . The measure of its supplement is  $180^\circ - 40^\circ = 140^\circ$ .

- b. **Figure 2.11** shows  $\angle \theta = 125^\circ$  in standard position. Angle  $\theta$  **does not have a complement** because there is no positive number  $x$  such that

$$x^\circ + 125^\circ = 90^\circ$$

The measure of its supplement is  $180^\circ - 125^\circ = 55^\circ$ .

» Try Exercise 2, page 130



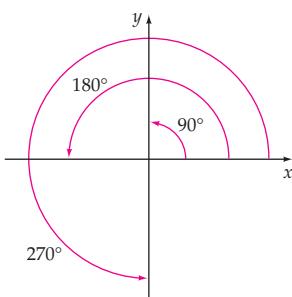
**Figure 2.11**

- ?
- QUESTION** Are the two acute angles of any right triangle complementary angles? Explain.

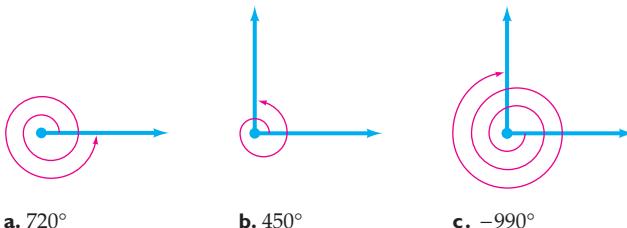
Some angles have a measure greater than  $360^\circ$ . See **Figure 2.12a** and **Figure 2.12b**. The angle shown in **Figure 2.12c** has a measure less than  $-360^\circ$ ,

- ?
- ANSWER** Yes. The sum of the measures of the angles of any triangle is  $180^\circ$ . The right angle has a measure of  $90^\circ$ . Thus the sum of the measures of the two acute angles must be  $180^\circ - 90^\circ = 90^\circ$ .

because it is formed by a clockwise rotation of more than one revolution of the initial side.



**Figure 2.13**

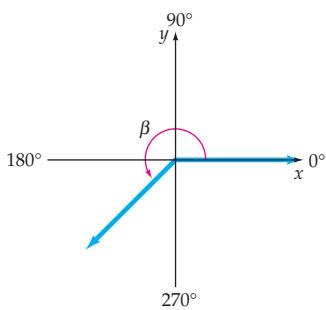


**Figure 2.12**

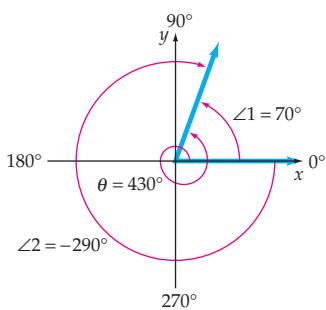
If the terminal side of an angle in standard position lies on a coordinate axis, then the angle is classified as a **quadrantal angle**. For example, the  $90^\circ$  angle, the  $180^\circ$  angle, and the  $270^\circ$  angle shown in **Figure 2.13** are all quadrantal angles.

If the terminal side of an angle in standard position does not lie on a coordinate axis, then the angle is classified according to the quadrant that contains the terminal side. For example,  $\angle\beta$  in **Figure 2.14** is a Quadrant III angle.

Angles in standard position that have the same terminal sides are **coterminal angles**. Every angle has an unlimited number of coterminal angles. **Figure 2.15** shows  $\angle\theta$  and two of its coterminal angles, labeled  $\angle 1$  and  $\angle 2$ .



**Figure 2.14**



**Figure 2.15**

### Measures of Coterminal Angles

Given  $\angle\theta$  in standard position with measure  $x^\circ$ , then the measures of the angles that are coterminal with  $\angle\theta$  are given by

$$x^\circ + k \cdot 360^\circ$$

where  $k$  is an integer.

This theorem states that the measures of any two coterminal angles differ by an integer multiple of  $360^\circ$ . For instance, in **Figure 2.15**,  $\theta = 430^\circ$ ,

$$\angle 1 = 430^\circ + (-1) \cdot 360^\circ = 70^\circ, \text{ and}$$

$$\angle 2 = 430^\circ + (-2) \cdot 360^\circ = -290^\circ$$

If we add positive multiples of  $360^\circ$  to  $430^\circ$ , we find that the angles with measures  $790^\circ, 1150^\circ, 1510^\circ, \dots$  are also coterminal with  $\angle\theta$ .

### EXAMPLE 2 » Classify by Quadrant and Find a Coterminal Angle

Assume the following angles are in standard position. Determine the measure of the positive angle with measure less than  $360^\circ$  that is coterminal with the given angle and then classify the angle by quadrant.

- a.  $\alpha = 550^\circ$     b.  $\beta = -225^\circ$     c.  $\gamma = 1105^\circ$

**Continued ►**

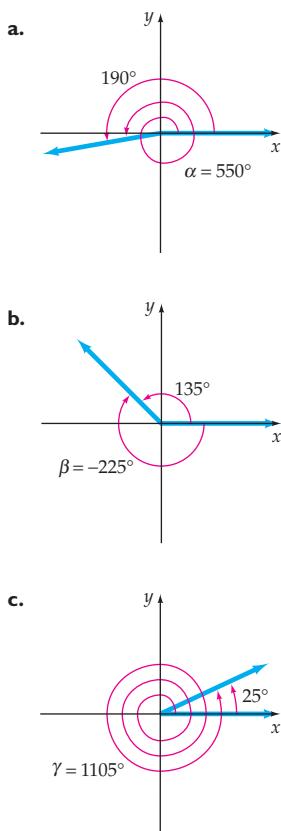


Figure 2.16

**Solution**

a. Because  $550^\circ = 190^\circ + 360^\circ$ ,  $\angle\alpha$  is coterminal with an angle that has a measure of  $190^\circ$ .  $\angle\alpha$  is a Quadrant III angle. See **Figure 2.16a.**

b. Because  $-225^\circ = 135^\circ + (-1) \cdot 360^\circ$ ,  $\angle\beta$  is coterminal with an angle that has a measure of  $135^\circ$ .  $\angle\beta$  is a Quadrant II angle. See **Figure 2.16b.**

c.  $1105^\circ \div 360^\circ = 3\frac{5}{72}$ . Thus  $\angle\gamma$  is an angle formed by three complete

counterclockwise rotations, plus  $\frac{5}{72}$  of a rotation. To convert  $\frac{5}{72}$  of a rotation to degrees, multiply  $\frac{5}{72}$  times  $360^\circ$ .

$$\frac{5}{72} \cdot 360^\circ = 25^\circ$$

Thus  $1105^\circ = 25^\circ + 3 \cdot 360^\circ$ . Hence  $\angle\gamma$  is coterminal with an angle that has a measure of  $25^\circ$ .  $\angle\gamma$  is a Quadrant I angle. See **Figure 2.16c.**

» Try Exercise 14, page 130

There are two popular methods for representing a fractional part of a degree. One is the **decimal degree method**. For example, the measure  $29.76^\circ$  is a decimal degree. It means

$$29^\circ \text{ plus } 76 \text{ hundredths of } 1^\circ$$

A second method of measurement is known as the **DMS** (Degree, Minute, Second) **method**. In the DMS method, a degree is subdivided into 60 equal parts, each of which is called a *minute*, denoted by '. Thus  $1^\circ = 60'$ . Furthermore, a minute is subdivided into 60 equal parts, each of which is called a *second*, denoted by ". Thus  $1' = 60''$  and  $1^\circ = 3600''$ . The fractions

$$\frac{1^\circ}{60'} = 1, \quad \frac{1'}{60''} = 1, \quad \text{and} \quad \frac{1^\circ}{3600''} = 1$$

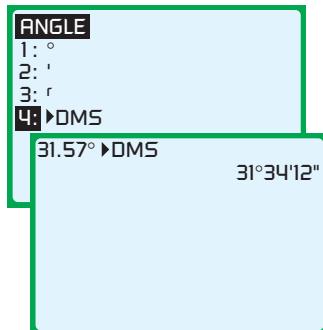
are another way of expressing the relationships among degrees, minutes, and seconds. Each of the fractions is known as a **unit fraction** or a **conversion factor**. Because all conversion factors are equal to 1, you can multiply a numerical value by a conversion factor and not change the numerical value, even though you change the units used to express the numerical value. The following illustrates the process of multiplying by conversion factors to write  $126^\circ 12' 27''$  as a decimal degree.

$$\begin{aligned} 126^\circ 12' 27'' &= 126^\circ + 12' + 27'' \\ &= 126^\circ + 12'\left(\frac{1^\circ}{60'}\right) + 27''\left(\frac{1^\circ}{3600''}\right) \\ &= 126^\circ + 0.2^\circ + 0.0075^\circ = 126.2075^\circ \end{aligned}$$

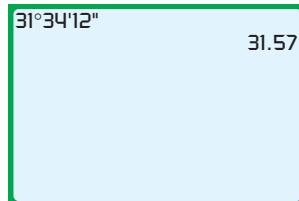


## Integrating Technology

Many graphing calculators can be used to convert a decimal degree measure to its equivalent DMS measure, and vice versa. For instance, **Figure 2.17** shows that  $31.57^\circ$  is equivalent to  $31^\circ 34'12''$ . On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, the degree symbol,  $^\circ$ , and the DMS function are in the ANGLE menu.

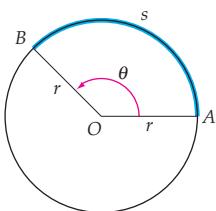


**Figure 2.17**



**Figure 2.18**

To convert a DMS measure to its equivalent decimal degree measure, enter the DMS measure and press **ENTER**. The calculator screen in **Figure 2.18** shows that  $31^\circ 34'12''$  is equivalent to  $31.57^\circ$ . A TI-83/TI-83 Plus/TI-84 Plus calculator needs to be in degree mode to produce the results displayed in **Figures 2.17** and **2.18**. On a TI-83/TI-83 Plus/TI-84 Plus calculator, the degree symbol,  $^\circ$ , and the minute symbol, ', are both in the ANGLE menu; however, the second symbol, ", is entered by pressing **ALPHA** +.



**Figure 2.19**

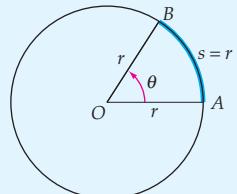
**Calculus Connection**

## Radian Measure

Another commonly used angle measurement is the *radian*. To define a radian, first consider a circle of radius  $r$  and two radii  $OA$  and  $OB$ . The angle  $\theta$  formed by the two radii is a **central angle**. The portion of the circle between  $A$  and  $B$  is an **arc** of the circle and is written  $\widehat{AB}$ . We say that  $\widehat{AB}$  subtends the angle  $\theta$ . The length of  $\widehat{AB}$  is  $s$  (see **Figure 2.19**).

### Definition of a Radian

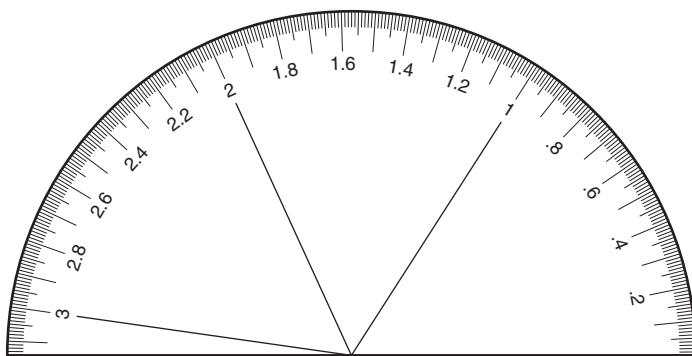
One **radian** is the measure of the central angle subtended by an arc of length  $r$  on a circle of radius  $r$ . See **Figure 2.20**.



**Figure 2.20**

Central angle  $\theta$  has a measure of 1 radian.

**Figure 2.21** shows a protractor that can be used to measure angles in radians or to construct angles given in radian measure.

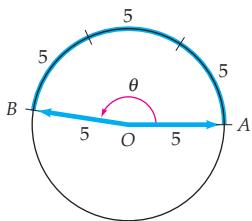


**Figure 2.21**

Protractor for measuring angles in radians

### Definition of Radian Measure

Given an arc of length  $s$  on a circle of radius  $r$ , the measure of the central angle subtended by the arc is  $\theta = \frac{s}{r}$  radians.



**Figure 2.22**

Central angle  $\theta$  has a measure of 3 radians.

As an example, consider that an arc of length 15 centimeters on a circle with a radius of 5 centimeters subtends an angle of 3 radians, as shown in **Figure 2.22**. The same result can be found by dividing 15 centimeters by 5 centimeters.

To find the measure in radians of any central angle  $\theta$ , divide the length  $s$  of the arc that subtends  $\theta$  by the length of the radius of the circle. Using the formula for radian measure, we find that an arc of length 12 centimeters on a circle of radius 8 centimeters subtends a central angle  $\theta$  whose measure is

$$\theta = \frac{s}{r} \text{ radians} = \frac{12 \text{ centimeters}}{8 \text{ centimeters}} \text{ radians} = \frac{3}{2} \text{ radians}$$

Note that the centimeter units are *not* part of the final result. The radian measure of a central angle formed by an arc of length 12 miles on a circle of radius 8 miles would be the same,  $\frac{3}{2}$  radians. If an angle has a measure of  $t$  radians, where  $t$  is a

real number, then the measure of the angle is often stated as  $t$  instead of  $t$  radians. For instance, if an angle  $\theta$  has a measure of 2 radians, we can simply write  $\theta = 2$  instead of  $\theta = 2$  radians. There will be no confusion concerning whether an angle measure is in degrees or radians, because the degree symbol is *always* used for angle measurements that are in degrees.

Recall that the circumference of a circle is given by the equation  $C = 2\pi r$ . The radian measure of the central angle  $\theta$  subtended by the circumference is  $\theta = \frac{2\pi r}{r} = 2\pi$ . In degree measure, the central angle  $\theta$  subtended by the circumference is  $360^\circ$ . Thus we have the relationship  $360^\circ = 2\pi$  radians. Dividing each side of the equation by 2 gives  $180^\circ = \pi$  radians. From this last equation, we can establish the following conversion factors.



### Integrating Technology

A calculator shows that

$$1 \text{ radian} \approx 57.29577951^\circ$$

and

$$1^\circ \approx 0.017453293 \text{ radian}$$

### Radian-Degree Conversion

- To convert from radians to degrees, multiply by  $\left(\frac{180^\circ}{\pi \text{ radians}}\right)$ .
- To convert from degrees to radians, multiply by  $\left(\frac{\pi \text{ radians}}{180^\circ}\right)$ .

#### EXAMPLE 3 » Convert from Degrees to Radians

Convert each angle in degrees to radians.

a.  $60^\circ$     b.  $315^\circ$     c.  $-150^\circ$

#### Solution

Multiply each degree measure by  $\left(\frac{\pi \text{ radians}}{180^\circ}\right)$  and simplify. In each case, the degree units in the numerator cancel with the degree units in the denominator.

$$\begin{aligned} \text{a. } 60^\circ &= 60^\circ \left( \frac{\pi \text{ radians}}{180^\circ} \right) = \frac{60\pi}{180} \text{ radians} = \frac{\pi}{3} \text{ radians} \\ \text{b. } 315^\circ &= 315^\circ \left( \frac{\pi \text{ radians}}{180^\circ} \right) = \frac{315\pi}{180} \text{ radians} = \frac{7\pi}{4} \text{ radians} \\ \text{c. } -150^\circ &= -150^\circ \left( \frac{\pi \text{ radians}}{180^\circ} \right) = -\left( \frac{150\pi}{180} \right) \text{ radians} = -\frac{5\pi}{6} \text{ radians} \end{aligned}$$

» Try Exercise 32, page 130

#### EXAMPLE 4 » Convert from Radians to Degrees

Convert each angle in radians to degrees.

a.  $\frac{3\pi}{4}$  radians    b. 1 radian    c.  $-\frac{5\pi}{2}$  radians

#### Solution

Multiply each radian measure by  $\left(\frac{180^\circ}{\pi \text{ radians}}\right)$  and simplify. In each case, the radian units in the numerator cancel with the radian units in the denominator.

$$\begin{aligned} \text{a. } \frac{3\pi}{4} \text{ radians} &= \left( \frac{3\pi \text{ radians}}{4} \right) \left( \frac{180^\circ}{\pi \text{ radians}} \right) = \frac{3 \cdot 180^\circ}{4} = 135^\circ \\ \text{b. } 1 \text{ radian} &= (1 \text{ radian}) \left( \frac{180^\circ}{\pi \text{ radians}} \right) = \frac{180^\circ}{\pi} \approx 57.3^\circ \\ \text{c. } -\frac{5\pi}{2} \text{ radians} &= \left( -\frac{5\pi \text{ radians}}{2} \right) \left( \frac{180^\circ}{\pi \text{ radians}} \right) = -\frac{5 \cdot 180^\circ}{2} = -450^\circ \end{aligned}$$

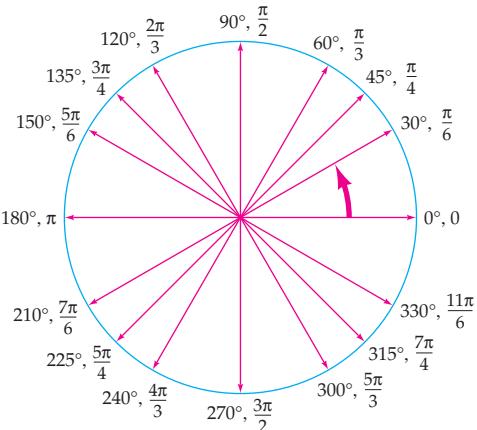
» Try Exercise 44, page 130

**Table 2.1**

| Degrees | Radians   |
|---------|-----------|
| 0       | 0         |
| 30      | $\pi/6$   |
| 45      | $\pi/4$   |
| 60      | $\pi/3$   |
| 90      | $\pi/2$   |
| 120     | $2\pi/3$  |
| 135     | $3\pi/4$  |
| 150     | $5\pi/6$  |
| 180     | $\pi$     |
| 210     | $7\pi/6$  |
| 225     | $5\pi/4$  |
| 240     | $4\pi/3$  |
| 270     | $3\pi/2$  |
| 300     | $5\pi/3$  |
| 315     | $7\pi/4$  |
| 330     | $11\pi/6$ |
| 360     | $2\pi$    |

**Figure 2.24****Figure 2.25**

**Table 2.1** lists the degree and radian measures of selected angles. **Figure 2.23** illustrates each angle listed in the table as measured from the positive  $x$ -axis.

**Figure 2.23**

Degree and radian measures of selected angles



### Integrating Technology

A graphing calculator can convert degree measure to radian measure, and vice versa. For example, the calculator display in **Figure 2.24** shows that  $100^\circ$  is approximately 1.74533 radians. The calculator must be in radian mode to convert from degrees to radians. The display in **Figure 2.25** shows that 2.2 radians is approximately  $126.051^\circ$ . The calculator must be in degree mode to convert from radians to degrees.

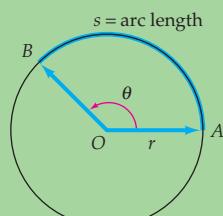
On a TI-83/TI-83 Plus/TI-84 Plus calculator, the symbol for radian measure is  $r$ , and it is in the ANGLE menu.

## ■ Arcs and Arc Length

Consider a circle of radius  $r$ . By solving the formula  $\theta = \frac{s}{r}$  for  $s$ , we have an equation for arc length.

### Arc Length Formula

Let  $r$  be the length of the radius of a circle and  $\theta$  the nonnegative radian measure of a central angle of the circle. Then the length of the arc  $s$  that subtends the central angle is  $s = r\theta$ . See **Figure 2.26**.

**Figure 2.26**

$$s = r\theta$$

**EXAMPLE 5** » Find the Length of an Arc

Find the length of an arc that subtends a central angle of  $120^\circ$  in a circle of radius 10 centimeters.

**Solution****take note**

The formula  $s = r\theta$  is valid only when  $\theta$  is expressed in radians.

The formula  $s = r\theta$  requires that  $\theta$  be expressed in radians. We first convert  $120^\circ$  to radian measure and then use the formula  $s = r\theta$ .

$$\theta = 120^\circ = 120^\circ \left( \frac{\pi \text{ radians}}{180^\circ} \right) = \frac{2\pi}{3} \text{ radians} = \frac{2\pi}{3}$$

$$s = r\theta = (10 \text{ centimeters}) \left( \frac{2\pi}{3} \right) = \frac{20\pi}{3} \text{ centimeters}$$

» Try Exercise 68, page 130

**EXAMPLE 6** » Solve an Application

A pulley with a radius of 10 inches uses a belt to drive a pulley with a radius of 6 inches. Find the angle through which the smaller pulley turns as the 10-inch pulley makes one revolution. State your answer in radians and also in degrees.

**Solution**

Use the formula  $s = r\theta$ . As the 10-inch pulley turns through an angle  $\theta_1$ , a point on the rim of that pulley moves  $s_1$  inches, where  $s_1 = 10\theta_1$ . See

**Figure 2.27.** At the same time, the 6-inch pulley turns through an angle of  $\theta_2$  and a point on the rim of that pulley moves  $s_2$  inches, where  $s_2 = 6\theta_2$ .

Assuming that the belt does not slip on the pulleys, we have  $s_1 = s_2$ . Thus

$$10\theta_1 = 6\theta_2$$

$$10(2\pi) = 6\theta_2 \quad \bullet \text{Solve for } \theta_2 \text{ when } \theta_1 = 2\pi \text{ radians.}$$

$$\frac{10}{3}\pi = \theta_2$$

The 6-inch pulley turns through an angle of  $\frac{10}{3}\pi$  radians, or  $600^\circ$ .



Figure 2.27

» Try Exercise 72, page 131

## Linear and Angular Speed

A ride at an amusement park has an inner ring of swings and an outer ring of swings. During each complete revolution, the swings in the outer ring travel a greater distance than the swings in the inner ring. We can say that the swings in the outer ring have a greater *linear speed* than the swings in the inner ring. Interestingly enough, all of the swings complete the same number of revolutions



during any given ride. We say that all of the swings have the same *angular speed*. In the following definitions,  $v$  denotes linear speed and  $\omega$  (omega) denotes angular speed.

### Definition of Linear and Angular Speed of a Point Moving on a Circular Path

A point moves on a circular path with radius  $r$  at a constant rate of  $\theta$  radians per unit of time  $t$ . Its **linear speed** is

$$v = \frac{s}{t}$$

where  $s$  is the distance the point travels, given by  $s = r\theta$ . The point's **angular speed** is

$$\omega = \frac{\theta}{t}$$

Some common units of angular speed are revolutions per second, revolutions per minute, radians per second, and radians per minute.

### EXAMPLE 7 » Convert an Angular Speed

A hard disk in a computer rotates at 3600 revolutions per minute. Find the angular speed of the disk in radians per second.

#### Solution

As a point on the disk rotates 1 revolution, the angle through which the point moves is  $2\pi$  radians. Thus  $\frac{2\pi \text{ radians}}{1 \text{ revolution}}$  will be the conversion factor we will use to convert from revolutions to radians. To convert from minutes to seconds, use the conversion factor  $\frac{1 \text{ minute}}{60 \text{ seconds}}$ .

$$\begin{aligned} 3600 \text{ revolutions/minute} &= \frac{3600 \text{ revolutions}}{1 \text{ minute}} \left( \frac{2\pi \text{ radians}}{1 \text{ revolution}} \right) \left( \frac{1 \text{ minute}}{60 \text{ seconds}} \right) \\ &= 120\pi \text{ radians/second} && \bullet \text{Exact answer} \\ &\approx 377 \text{ radians/second} && \bullet \text{Approximate answer} \end{aligned}$$

### » Try Exercise 74, page 131

We can establish an important relationship between linear speed and angular speed. We start with the linear speed formula and then substitute  $r\theta$  for  $s$ , as shown below.

$$v = \frac{s}{t} = \frac{r\theta}{t} = r \frac{\theta}{t} = r\omega$$

Thus the linear speed of a point moving on a circular path is the product of the radius of the circle and the angular speed of the point.

### The Linear Speed–Angular Speed Relationship

The linear speed  $v$  and the angular speed  $\omega$ , in radians per unit of time, of a point moving on a circular path with radius  $r$  are related by

$$v = r\omega$$

The equation  $v = r\omega$  gives the linear speed of a point moving on a circular path in terms of distance  $r$  from the center of the circle and the angular speed  $\omega$ , provided  $\omega$  is in radians per unit of time.

### EXAMPLE 8 » Find a Linear Speed

A wind machine is used to generate electricity. The wind machine has propeller blades that are 12 feet in length (see **Figure 2.28**). If the propeller is rotating at 3 revolutions per second, what is the linear speed in feet per second of the tips of the blades?

#### Solution

Convert the angular speed  $\omega = 3$  revolutions per second into radians per second, and then use the formula  $v = r\omega$ .

$$\omega = \frac{3 \text{ revolutions}}{1 \text{ second}} = \left( \frac{3 \text{ revolutions}}{1 \text{ second}} \right) \left( \frac{2\pi \text{ radians}}{1 \text{ revolution}} \right) = \frac{6\pi \text{ radians}}{1 \text{ second}}$$

Thus

$$\begin{aligned} v &= r\omega = (12 \text{ feet}) \left( \frac{6\pi \text{ radians}}{1 \text{ second}} \right) \\ &= 72\pi \text{ feet per second} \approx 226 \text{ feet per second} \end{aligned}$$

**Figure 2.28**

» Try Exercise 80, page 131



### Topics for Discussion

1. The measure of a radian differs depending on the length of the radius of the circle used. Do you agree? Explain.
2. What are the necessary conditions for an angle to be in standard position?
3. Is the supplement of an obtuse angle an acute angle?
4. Do all acute angles have a positive measure?

 **Exercise Set 2.1**

In Exercises 1 to 12, find the measure (if possible) of the complement and the supplement of each angle.

1.  $15^\circ$

2.  $87^\circ$

3.  $70^\circ 15'$

46.  $-\frac{2\pi}{3}$

47.  $\frac{\pi}{6}$

48.  $\frac{\pi}{9}$

4.  $22^\circ 43'$

5.  $56^\circ 33' 15''$

6.  $19^\circ 42' 05''$

49.  $\frac{3\pi}{8}$

50.  $\frac{11\pi}{18}$

51.  $\frac{11\pi}{3}$

7. 1

8. 0.5

9.  $\frac{\pi}{4}$

52.  $\frac{6\pi}{5}$

53.  $-\frac{5\pi}{12}$

54.  $-\frac{4\pi}{5}$

10.  $\frac{\pi}{3}$

11.  $\frac{2\pi}{5}$

12.  $\frac{\pi}{6}$

In Exercises 13 to 18, determine the measure of the positive angle with measure less than  $360^\circ$  that is coterminal with the given angle and then classify the angle by quadrant. Assume the angles are in standard position.

13.  $\alpha = 610^\circ$

14.  $\alpha = 765^\circ$

15.  $\alpha = -975^\circ$

16.  $\alpha = -872^\circ$

17.  $\alpha = 2456^\circ$

18.  $\alpha = -3789^\circ$



In Exercises 19 to 24, use a calculator to convert each decimal degree measure to its equivalent DMS measure.

19.  $24.56^\circ$

20.  $110.24^\circ$

21.  $64.158^\circ$

22.  $18.96^\circ$

23.  $3.402^\circ$

24.  $224.282^\circ$



In Exercises 25 to 30, use a calculator to convert each DMS measure to its equivalent decimal degree measure.

25.  $25^\circ 25' 12''$

26.  $63^\circ 29' 42''$

27.  $183^\circ 33' 36''$

28.  $141^\circ 6' 9''$

29.  $211^\circ 46' 48''$

30.  $19^\circ 12' 18''$

In Exercises 31 to 42, convert the degree measure to exact radian measure.

31.  $30^\circ$

32.  $-45^\circ$

33.  $90^\circ$

65.  $r = 8 \text{ inches}, \theta = \frac{\pi}{4}$

34.  $15^\circ$

35.  $165^\circ$

36.  $315^\circ$

66.  $r = 3 \text{ feet}, \theta = \frac{7\pi}{2}$

37.  $420^\circ$

38.  $630^\circ$

39.  $585^\circ$

67.  $r = 25 \text{ centimeters}, \theta = 42^\circ$

40.  $135^\circ$

41.  $-9^\circ$

42.  $-110^\circ$

68.  $r = 5 \text{ meters}, \theta = 144^\circ$

In Exercises 43 to 54, convert the radian measure to exact degree measure.

43.  $\frac{7\pi}{3}$

44.  $\frac{\pi}{4}$

45.  $\frac{\pi}{5}$

69. Find the number of radians in  $1\frac{1}{2}$  revolutions.

In Exercises 43 to 54, convert the radian measure to exact degree measure.

70. Find the number of radians in  $\frac{3}{8}$  revolution.

**71. ANGULAR ROTATION OF TWO PULLEYS** A pulley with a radius of 14 inches uses a belt to drive a pulley with a radius of

28 inches. The 14-inch pulley turns through an angle of  $150^\circ$ . Find the angle through which the 28-inch pulley turns.

- 72. ANGULAR ROTATION OF TWO PULLEYS** A pulley with a diameter of 1.2 meters uses a belt to drive a pulley with a diameter of 0.8 meter. The 1.2-meter pulley turns through an angle of  $240^\circ$ . Find the angle through which the 0.8-meter pulley turns.

- 73. ANGULAR SPEED** Find the angular speed, in radians per second, of the second hand on a clock.

- 74. ANGULAR SPEED** Find the angular speed, in radians per second, of a point on the equator of the earth.

- 75. ANGULAR SPEED** A wheel is rotating at 50 revolutions per minute. Find the angular speed in radians per second.

- 76. ANGULAR SPEED** A wheel is rotating at 200 revolutions per minute. Find the angular speed in radians per second.

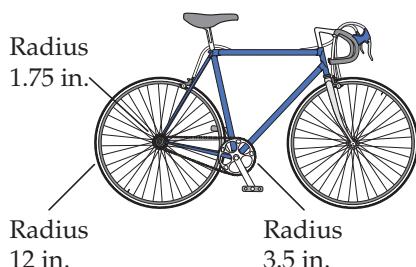
- 77. ANGULAR SPEED** The turntable of a record player turns at  $33\frac{1}{3}$  revolutions per minute. Find the angular speed in radians per second.

- 78. ANGULAR SPEED** A car with a wheel of radius 14 inches is moving with a speed of 55 mph. Find the angular speed of the wheel in radians per second.

- 79. LINEAR SPEED OF A CAR** Each tire on a car has a radius of 15 inches. The tires are rotating at 450 revolutions per minute. Find the speed of the automobile to the nearest mile per hour.

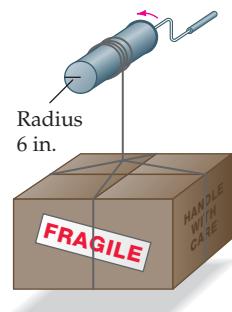
- 80. LINEAR SPEED OF A TRUCK** Each tire on a truck has a radius of 18 inches. The tires are rotating at 500 revolutions per minute. Find the speed of the truck to the nearest mile per hour.

- 81. BICYCLE GEARS** The chain wheel of Emma's bicycle has a radius of 3.5 inches. The rear gear has a radius of 1.75 inches, and the back tire has a radius of 12 inches. If Emma pedals for 150 revolutions of the chain wheel, how far will she travel? Round to the nearest foot.



**82. ROTATION VERSUS LIFT DISTANCE**

A winch with a 6-inch radius is used to lift a container. The winch is designed so that as it is rotated, the cable stays in contact with the surface of the winch. That is, the cable does not wrap on top of itself.



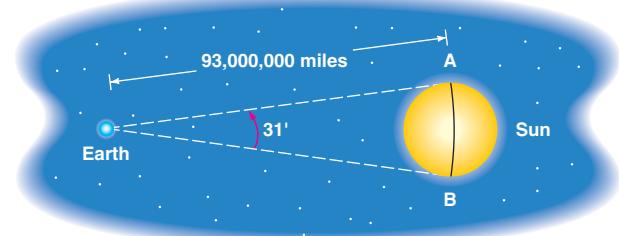
- a. Find the distance the container is lifted as the winch is rotated through an angle of  $\frac{5\pi}{6}$  radians.

- b. Determine the angle, in radians, through which the winch must be rotated to lift the container a distance of 2 feet.

- 83. AMUSEMENT PARK RIDE** A ride at an amusement park consists of two circular rings of swings. At full speed the swings in the inner ring travel on a circular path with a radius of 32 feet and the swings in the outer ring travel on a circular path with a radius of 38 feet. Each swing makes one complete revolution every 3.75 seconds. How much greater, in miles per hour, is the linear speed of the swings in the outer ring than the linear speed of the swings in the inner ring? Round to the nearest tenth of a mile per hour.

- 84. HORSE RACING** The semicircular turns of a horse race track each have a radius of 200 feet. During the first turn of a race, the lead horse is running near the inside rail on a path with a 202.0-foot radius, at a constant rate of 24.4 feet per second. A second horse is rounding the same turn on a path with a 206.5-foot radius. At what constant rate does the second horse need to run to keep pace with the lead horse during this turn? Round to the nearest tenth of a foot per second.

- 85. ASTRONOMY** At a time when Earth was 93,000,000 miles from the sun, you observed through a tinted glass that the diameter of the sun occupied an arc of  $31'$ . Determine, to the nearest ten thousand miles, the diameter of the sun.



(Hint: Because the radius of arc  $AB$  is large and its central angle is small, the length of the diameter of the sun is approximately the length of the arc  $AB$ .)

- 86. ANGLE OF ROTATION AND DISTANCE** The minute hand on the clock atop city hall measures 6 feet 3 inches from its tip to its axle.

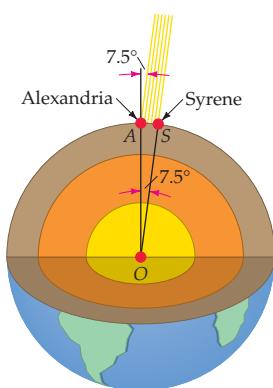
- Through what angle (in radians) does the minute hand pass between 9:12 A.M. and 9:48 A.M.?
- What distance, to the nearest tenth of a foot, does the tip of the minute hand travel during this period?

- 87. VELOCITY OF THE HUBBLE SPACE TELESCOPE**

On April 25, 1990, the Hubble Space Telescope (HST) was deployed into a circular orbit 625 kilometers above the surface of the earth. The HST completes an Earth orbit every 1.61 hours.

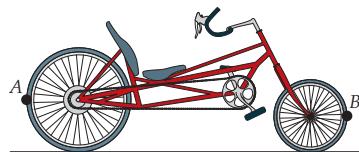


- Find the angular velocity, with respect to the center of Earth, of the HST. Round your answer to the nearest 0.1 radian per hour.
  - Find the linear velocity of the HST. (*Hint:* The radius of Earth is about 6370 kilometers.) Round your answer to the nearest 100 kilometers per hour.
- 88. ESTIMATING THE RADIUS OF EARTH** Eratosthenes, the fifth librarian of Alexandria (230 B.C.), was able to estimate the radius of Earth from the following data: The distance between the Egyptian cities of Alexandria and Syrene was 5000 stadia (520 miles). Syrene was located directly south of Alexandria. One summer, at noon, the sun was directly overhead at Syrene, whereas at the same time in Alexandria, the sun was at a  $7.5^\circ$  angle from the zenith.



Eratosthenes reasoned that because the sun is far away, the rays of sunlight that reach Earth must be nearly parallel. From this assumption he concluded that the measure of  $\angle AOS$  in the accompanying figure must be  $7.5^\circ$ . Use this information to estimate the radius (to the nearest 10 miles) of Earth.

- 89. VELOCITY COMPARISONS** Assume that the bicycle in the figure is moving forward at a constant rate. Point A is on the edge of the 30-inch rear tire, and point B is on the edge of the 20-inch front tire.



- Which point (A or B) has the greater angular velocity?
  - Which point (A or B) has the greater linear velocity?
- 90.** Given that  $s$ ,  $r$ ,  $\theta$ ,  $t$ ,  $v$ , and  $\omega$  are as defined in Section 2.1, determine which of the following formulas are valid.

$$\begin{aligned}s &= r\theta & r &= \frac{s}{\theta} & v &= \frac{r\theta}{t} \\v &= r\omega & v &= \frac{s}{t} & \omega &= \frac{\theta}{t}\end{aligned}$$

- 91. NAUTICAL MILES AND STATUTE MILES** A **nautical mile** is the length of an arc, on Earth's equator, that subtends a  $1'$  central angle. The equatorial radius of Earth is about 3960 **statute miles**.

- Convert 1 nautical mile to statute miles. Round to the nearest hundredth of a statute mile.
- Determine what percent (to the nearest 1 percent) of Earth's circumference is covered by a trip from Los Angeles, California to Honolulu, Hawaii (a distance of 2217 nautical miles).

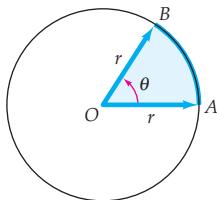
- 92. PHOTOGRAPHY** The field of view for a camera with a 200-millimeter lens is  $12^\circ$ . A photographer takes a photograph of a large building that is 485 feet in front of the camera. What is the approximate width, to the nearest foot, of the building that will appear in the photograph? (*Hint:* If the radius of an arc  $AB$  is large and its central angle is small, then the length of the line segment  $AB$  is approximately the length of the arc  $AB$ .)

## »»» Connecting Concepts

A sector of a circle is the region bounded by radii  $OA$  and  $OB$  and the *intercepted arc*  $AB$ . See the following figure. The area of the sector is given by

$$A = \frac{1}{2} r^2 \theta$$

where  $r$  is the radius of the circle and  $\theta$  is the measure of the central angle in radians.



**In Exercises 93 to 96, find the area, to the nearest square unit, of the sector of a circle with the given radius and central angle.**

**93.**  $r = 5$  inches,  $\theta = \frac{\pi}{3}$  radians

**94.**  $r = 2.8$  feet,  $\theta = \frac{5\pi}{2}$  radians

**95.**  $r = 120$  centimeters,  $\theta = 0.65$  radian

**96.**  $r = 30$  feet,  $\theta = 62^\circ$

**Latitude describes the position of a point on Earth's surface in relation to the equator. A point on the equator has a latitude of 0°.**

## »»» Projects

- 1. CURRENCY CONVERSION** You are traveling in a foreign country. You discover that the currency used consists of dollars, mollarz, nollars, and tollars.

5 dollars = 1 mollar

3 dollars = 5 nollars

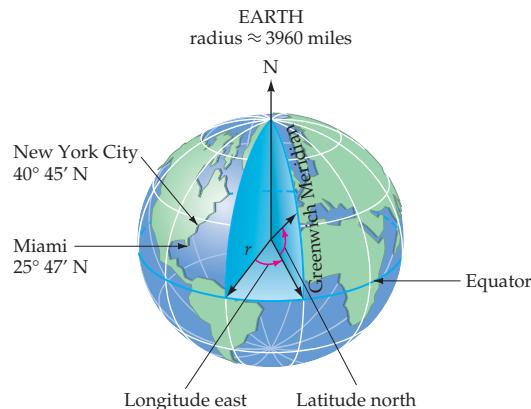
$$4 \text{ dollars} = 7 \text{ tollars}$$

The fare for a taxi is 14 tollars.

- a. How much is the fare in dollars? (*Hint:* Make use of conversion factors to convert from tollars to dollars.)
  - b. If you have only tollars and lollars, how many of each could you use to pay the fare?
  - c.  Explain how you knew which of the following conversion factors should be used in the conversion in a.

$$\left( \begin{array}{c} 4 \text{ dollars} \\ 7 \text{ dollars} \end{array} \right) \quad \left( \begin{array}{c} 7 \text{ dollars} \\ 4 \text{ dollars} \end{array} \right)$$

tude of  $0^\circ$ . The north pole has a latitude of  $90^\circ$ . The radius of Earth is approximately 3960 miles.



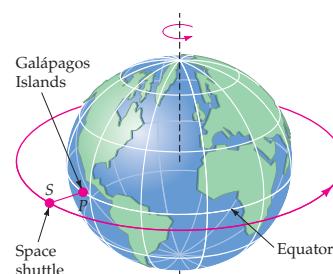
- 97. GEOGRAPHY** The city of Miami has a latitude of  $25^{\circ}47'N$ . How far north, to the nearest 10 miles, is it from the equator? Use 3960 miles as the radius of Earth.

**98. GEOGRAPHY** New York City has a latitude of  $40^{\circ}45'N$ . How far north, to the nearest 10 miles, is it from the equator? Use 3960 miles as the radius of Earth.

- d. If you wish to convert  $x$  degrees to radians, you need to multiply  $x$  by which of the following conversion factors?

$$\left( \frac{\pi}{180^\circ} \right) \quad \left( \frac{180^\circ}{\pi} \right)$$

- 2. SPACE SHUTTLE** The rotational period of Earth is 23.933 hours. A space shuttle revolves around Earth's equator every 2.231 hours. Both are rotating in the same direction. At the present time, the space shuttle is directly above the Galápagos Islands. How long will it take for the space shuttle to circle Earth and return to a position directly above the Galápagos Islands?



## Section 2.2

- The Six Trigonometric Functions
- Trigonometric Functions of Special Angles
- Applications Involving Right Triangles

# Right Triangle Trigonometry

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A9.

**PS1.** Rationalize the denominator of  $\frac{1}{\sqrt{3}}$ .

**PS2.** Rationalize the denominator of  $\frac{2}{\sqrt{2}}$ .

**PS3.** Simplify:  $a \div \left(\frac{a}{2}\right)$

**PS4.** Simplify:  $\left(\frac{a}{2}\right) \div \left(\frac{\sqrt{3}}{2}a\right)$

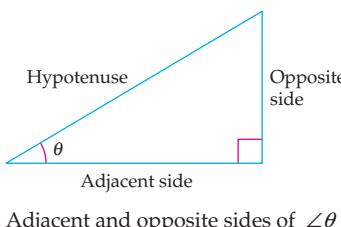
**PS5.** Solve  $\frac{\sqrt{2}}{2} = \frac{x}{5}$  for  $x$ . Round your answer to the nearest hundredth. [1.1]

**PS6.** Solve  $\frac{\sqrt{3}}{3} = \frac{x}{18}$  for  $x$ . Round your answer to the nearest hundredth. [1.1]

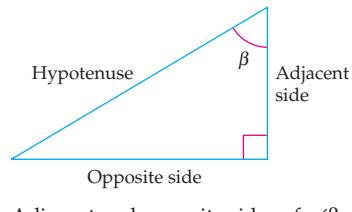
## The Six Trigonometric Functions

The study of trigonometry, which means “triangle measurement,” began more than 2000 years ago, partially as a means of solving surveying problems. Early trigonometry used the length of a line segment between two points of a circle as the value of a *trigonometric function*. In the sixteenth century, right triangles were used to define a trigonometric function. We will use a modification of this approach.

When working with right triangles, it is convenient to refer to the side *opposite* an angle or the side *adjacent* to (next to) an angle. **Figure 2.29** shows the sides opposite and adjacent to the angle  $\theta$ . **Figure 2.30** shows the sides opposite and adjacent to the angle  $\beta$ . In both cases, the hypotenuse remains the same.



Adjacent and opposite sides of  $\angle\theta$



Adjacent and opposite sides of  $\angle\beta$

Figure 2.29

Figure 2.30

Six ratios can be formed by using two lengths of the three sides of a right triangle. Each ratio defines a value of a trigonometric function of a given acute angle  $\theta$ . The functions are **sine** (sin), **cosine** (cos), **tangent** (tan), **cosecant** (csc), **secant** (sec), and **cotangent** (cot).

### Definitions of Trigonometric Functions of an Acute Angle

Let  $\theta$  be an acute angle of a right triangle. See **Figure 2.29**. The values of the six trigonometric functions of  $\theta$  are

$$\sin \theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

$$\tan \theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

$$\sec \theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$$

$$\cos \theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

$$\cot \theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$$

$$\csc \theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$$

We will write opp, adj, and hyp as abbreviations for *the length of the opposite side*, *adjacent side*, and *hypotenuse*, respectively.

### EXAMPLE 1 » Evaluate Trigonometric Functions

Find the values of the six trigonometric functions of  $\theta$  for the triangle given in **Figure 2.31**.

#### Solution

Use the Pythagorean Theorem to find the length of the hypotenuse.

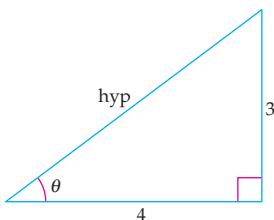
$$\text{hyp} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

From the definitions of the trigonometric functions,

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5} \quad \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}$$

$$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{4}{3} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{4} \quad \csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{3}$$

» Try Exercise 6, page 142



**Figure 2.31**

Given the value of one trigonometric function of the acute angle  $\theta$ , it is possible to find the value of any of the remaining trigonometric functions of  $\theta$ .

### EXAMPLE 2 » Find the Value of a Trigonometric Function

Given that  $\theta$  is an acute angle and  $\cos \theta = \frac{5}{8}$ , find  $\tan \theta$ .

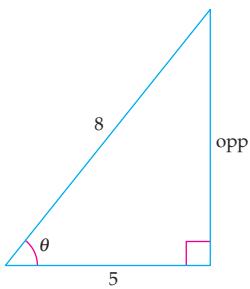
Continued ►

**Solution**

$$\cos \theta = \frac{5}{8} = \frac{\text{adj}}{\text{hyp}}$$

Sketch a right triangle with one leg of length 5 units and a hypotenuse of length 8 units. Label as  $\theta$  the acute angle that has the leg of length 5 units as its adjacent side (see **Figure 2.32**). Use the Pythagorean Theorem to find the length of the opposite side.

$$\begin{aligned} (\text{opp})^2 + 5^2 &= 8^2 \\ (\text{opp})^2 + 25 &= 64 \\ (\text{opp})^2 &= 39 \\ \text{opp} &= \sqrt{39} \end{aligned}$$

**Figure 2.32**

$$\text{Therefore, } \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{39}}{5}.$$

» Try Exercise 18, page 142

## Trigonometric Functions of Special Angles

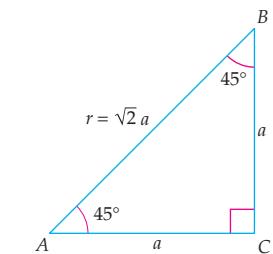
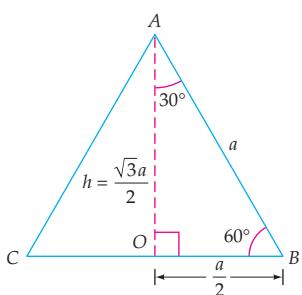
In Example 1, the lengths of the legs of the triangle were given, and you were asked to find the values of the six trigonometric functions of the angle  $\theta$ . Often we will want to find the value of a trigonometric function when we are given *the measure of an angle* rather than the measure of the sides of a triangle. For most angles, advanced mathematical methods are required to evaluate a trigonometric function. For some *special angles*, however, the value of a trigonometric function can be found by geometric methods. These special acute angles are  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$ .

First, we will find the values of the six trigonometric functions of  $45^\circ$ . (This discussion is based on angles measured in degrees. Radian measure could have been used without changing the results.) **Figure 2.33** shows a right triangle with angles  $45^\circ$ ,  $45^\circ$ , and  $90^\circ$ . Because  $\angle A = \angle B$ , the lengths of the sides opposite these angles are equal. Let the length of each equal side be denoted by  $a$ . From the Pythagorean Theorem,

$$\begin{aligned} r^2 &= a^2 + a^2 = 2a^2 \\ r &= \sqrt{2a^2} = \sqrt{2}a \end{aligned}$$

The values of the six trigonometric functions of  $45^\circ$  are

$$\begin{array}{ll} \sin 45^\circ = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} & \cos 45^\circ = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \tan 45^\circ = \frac{a}{a} = 1 & \cot 45^\circ = \frac{a}{a} = 1 \\ \sec 45^\circ = \frac{\sqrt{2}a}{a} = \sqrt{2} & \csc 45^\circ = \frac{\sqrt{2}a}{a} = \sqrt{2} \end{array}$$

**Figure 2.33****Figure 2.34**

The values of the trigonometric functions of the special angles  $30^\circ$  and  $60^\circ$  can be found by drawing an equilateral triangle and bisecting one of the angles, as **Figure 2.34** shows. The angle bisector also bisects one of the sides. Thus the length of the side opposite the  $30^\circ$  angle is one-half the length of the hypotenuse of triangle  $OAB$ .

Let  $a$  denote the length of the hypotenuse. Then the length of the side opposite the  $30^\circ$  angle is  $\frac{a}{2}$ . The length of the side adjacent to the  $30^\circ$  angle,  $h$ , is found by using the Pythagorean Theorem.

$$a^2 = \left(\frac{a}{2}\right)^2 + h^2$$

$$a^2 = \frac{a^2}{4} + h^2$$

$$\frac{3a^2}{4} = h^2$$

• Subtract  $\frac{a^2}{4}$  from each side.

$$h = \frac{\sqrt{3}a}{2}$$

• Solve for  $h$ .

The values of the six trigonometric functions of  $30^\circ$  are

$$\sin 30^\circ = \frac{a/2}{a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}a/2}{a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{a/2}{\sqrt{3}a/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot 30^\circ = \frac{\sqrt{3}a/2}{a/2} = \sqrt{3}$$

$$\sec 30^\circ = \frac{a}{\sqrt{3}a/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc 30^\circ = \frac{a}{a/2} = 2$$

The values of the trigonometric functions of  $60^\circ$  can be found by again using Figure 2.34. The length of the side opposite the  $60^\circ$  angle is  $\frac{\sqrt{3}a}{2}$ , and the length of the side adjacent to the  $60^\circ$  angle is  $\frac{a}{2}$ . The values of the trigonometric functions of  $60^\circ$  are

$$\sin 60^\circ = \frac{\sqrt{3}a/2}{a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{a/2}{a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{\sqrt{3}a/2}{a/2} = \sqrt{3}$$

$$\cot 60^\circ = \frac{a/2}{\sqrt{3}a/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec 60^\circ = \frac{a}{\sqrt{3}a/2} = 2$$

$$\csc 60^\circ = \frac{a}{a/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

**Table 2.2** summarizes the values of the trigonometric functions of the special angles  $30^\circ$  ( $\pi/6$ ),  $45^\circ$  ( $\pi/4$ ), and  $60^\circ$  ( $\pi/3$ ).

### take note

Memorizing the values given in Table 2.2 will prove to be extremely useful in the remaining trigonometry sections.

**Table 2.2 Trigonometric Functions of Special Angles**

| $\theta$                  | $\sin \theta$        | $\cos \theta$        | $\tan \theta$        | $\csc \theta$         | $\sec \theta$         | $\cot \theta$        |
|---------------------------|----------------------|----------------------|----------------------|-----------------------|-----------------------|----------------------|
| $30^\circ; \frac{\pi}{6}$ | $\frac{1}{2}$        | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | 2                     | $\frac{2\sqrt{3}}{3}$ | $\sqrt{3}$           |
| $45^\circ; \frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1                    | $\sqrt{2}$            | $\sqrt{2}$            | 1                    |
| $60^\circ; \frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$        | $\sqrt{3}$           | $\frac{2\sqrt{3}}{3}$ | 2                     | $\frac{\sqrt{3}}{3}$ |

**QUESTION** What is the measure, in degrees, of the acute angle  $\theta$  for which  $\sin \theta = \cos \theta$ ,  $\tan \theta = \cot \theta$ , and  $\sec \theta = \csc \theta$ ?

### EXAMPLE 3 » Evaluate a Trigonometric Expression

Find the *exact* value of  $\sin^2 45^\circ + \cos^2 60^\circ$ .

Note:  $\sin^2 \theta = (\sin \theta)(\sin \theta) = (\sin \theta)^2$  and  $\cos^2 \theta = (\cos \theta)(\cos \theta) = (\cos \theta)^2$ .

#### Solution

Substitute the values of  $\sin 45^\circ$  and  $\cos 60^\circ$  and simplify.

$$\sin^2 45^\circ + \cos^2 60^\circ = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$$

» Try Exercise 34, page 142

#### take note

The patterns in the following chart can be used to memorize the sine and cosine of  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$ .

|                                      |                                      |
|--------------------------------------|--------------------------------------|
| $\sin 30^\circ = \frac{\sqrt{1}}{2}$ | $\cos 30^\circ = \frac{\sqrt{3}}{2}$ |
| $\sin 45^\circ = \frac{\sqrt{2}}{2}$ | $\cos 45^\circ = \frac{\sqrt{2}}{2}$ |
| $\sin 60^\circ = \frac{\sqrt{3}}{2}$ | $\cos 60^\circ = \frac{\sqrt{1}}{2}$ |

From the definition of the sine and cosecant functions,

$$(\sin \theta)(\csc \theta) = \frac{\text{opp}}{\text{hyp}} \cdot \frac{\text{hyp}}{\text{opp}} = 1 \quad \text{or} \quad (\sin \theta)(\csc \theta) = 1$$

By rewriting the last equation, we find

$$\sin \theta = \frac{1}{\csc \theta} \quad \text{and} \quad \csc \theta = \frac{1}{\sin \theta}, \text{ provided } \sin \theta \neq 0$$

The sine and cosecant functions are called **reciprocal functions**. The cosine and secant are also reciprocal functions, as are the tangent and cotangent functions. **Table 2.3** shows each trigonometric function and its reciprocal. These relationships hold for all values of  $\theta$  for which both of the functions are defined.

**Table 2.3 Trigonometric Functions and Their Reciprocals**

|                                       |                                       |                                       |
|---------------------------------------|---------------------------------------|---------------------------------------|
| $\sin \theta = \frac{1}{\csc \theta}$ | $\cos \theta = \frac{1}{\sec \theta}$ | $\tan \theta = \frac{1}{\cot \theta}$ |
| $\csc \theta = \frac{1}{\sin \theta}$ | $\sec \theta = \frac{1}{\cos \theta}$ | $\cot \theta = \frac{1}{\tan \theta}$ |

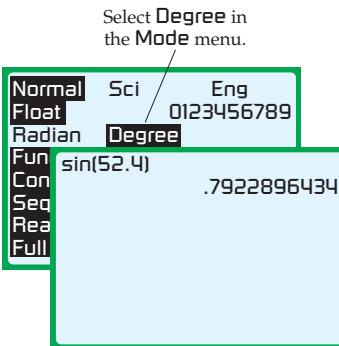
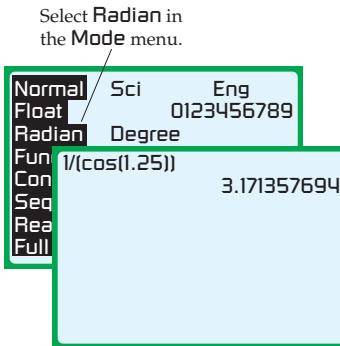


### Integrating Technology

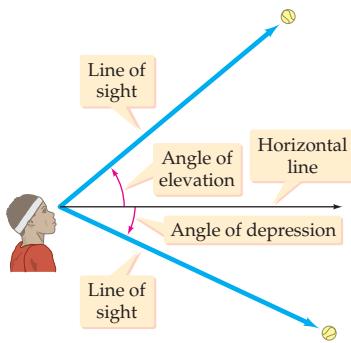
The values of the trigonometric functions of the special angles  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$  shown in **Table 2.2** are exact values. If an angle is not one of these special angles, then a graphing calculator often is used to approximate the value of a trigonometric function. For instance, to find  $\sin 52.4^\circ$  on a TI-83/TI-83 Plus/TI-84 Plus calculator, first check that the calculator is in degree mode. Then use the sine function key [**SIN**] to key in  $\sin(52.4)$  and press [**ENTER**]. See **Figure 2.35**.

**ANSWER**  $45^\circ$

To find  $\sec 1.25$ , first check that the calculator is in radian mode. A TI-83/TI-83 Plus/TI-84 Plus calculator does not have a secant function key, but because the secant function is the reciprocal of the cosine function, we can evaluate  $\sec 1.25$  by evaluating  $1/(\cos 1.25)$ . See **Figure 2.36**.

**Figure 2.35****Figure 2.36**

When you evaluate a trigonometric function with a calculator, be sure the calculator is in the correct mode. Many errors are made because the correct mode has not been selected.

**Figure 2.37**

## Applications Involving Right Triangles

Some applications concern an observer looking at an object. In these applications, angles of elevation or angles of depression are formed by a line of sight and a horizontal line. If the object being observed is above the observer, the acute angle formed by the line of sight and the horizontal line is an **angle of elevation**. If the object being observed is below the observer, the acute angle formed by the line of sight and the horizontal line is an **angle of depression**. See **Figure 2.37**.

### EXAMPLE 4 Solve an Angle-of-Elevation Application

From a point 115 feet from the base of a redwood tree, the angle of elevation to the top of the tree is  $64.3^\circ$ . Find the height of the tree to the nearest foot.

#### Solution

From **Figure 2.38**, the length of the adjacent side of the angle is known (115 feet). Because we need to determine the height of the tree (length of the opposite side), we use the tangent function. Let  $h$  represent the length of the opposite side.

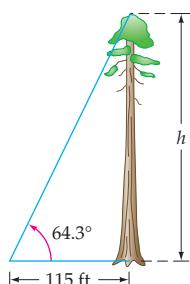
$$\tan 64.3^\circ = \frac{\text{opp}}{\text{adj}} = \frac{h}{115}$$

$$h = 115 \tan 64.3^\circ \approx 238.952$$

• Use a calculator to evaluate  $\tan 64.3^\circ$ .

The height of the tree is approximately 239 feet.

### Try Exercise 56, page 143

**Figure 2.38**

**take note**

The significant digits of an approximate number are

- every nonzero digit.
- the digit 0, provided it is between two nonzero digits or it is to the right of a nonzero digit in a number which includes a decimal point.

For example, the approximate number:

502 has 3 significant digits.

3700 has 2 significant digits.

47.0 has 3 significant digits.

0.0023 has 2 significant digits.

0.00840 has 3 significant digits.

Because the cotangent function involves the sides adjacent to and opposite an angle, we could have solved Example 4 by using the cotangent function. The solution would have been

$$\cot 64.3^\circ = \frac{\text{adj}}{\text{opp}} = \frac{115}{h}$$

$$h = \frac{115}{\cot 64.3^\circ} \approx 238.952 \text{ feet}$$

The accuracy of a calculator is sometimes beyond the limits of measurement. In Example 4 the distance from the base of the tree was given as 115 feet (three significant digits), whereas the height of the tree was shown to be 238.952 feet (six significant digits). When using approximate numbers, we will use the conventions given below for calculating with trigonometric functions.

### A Rounding Convention: Significant Digits for Trigonometric Calculations

#### *Angle Measure to the Nearest*

Degree

#### *Significant Digits of the Lengths*

Two

Tenth of a degree

Three

Hundredth of a degree

Four

### EXAMPLE 5 » Solve an Angle-of-Depression Application

DME (Distance Measuring Equipment) is standard avionic equipment on a commercial airplane. This equipment measures the distance from a plane to a radar station. If the distance from a plane to a radar station is 160 miles and the angle of depression is  $33^\circ$ , find the number of ground miles from a point directly below the plane to the radar station.

#### Solution

From **Figure 2.39**, the length of the hypotenuse is known (160 miles). The length of the side opposite the angle of  $57^\circ$  is unknown. The sine function involves the hypotenuse and the opposite side,  $x$ , of the  $57^\circ$  angle.

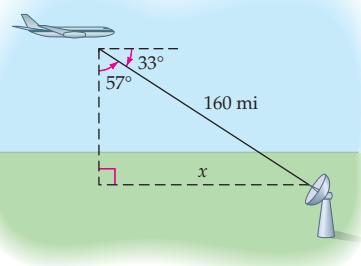
$$\sin 57^\circ = \frac{x}{160}$$

$$x = 160 \sin 57^\circ \approx 134.1873$$

Rounded to two significant digits, the plane is 130 ground miles from the radar station.

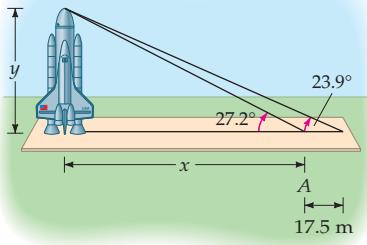
**Figure 2.39**

» Try Exercise 58, page 143



**EXAMPLE 6** » Solve an Angle-of-Elevation Application

An observer notes that the angle of elevation from point  $A$  to the top of a space shuttle is  $27.2^\circ$ . From a point 17.5 meters further from the space shuttle, the angle of elevation is  $23.9^\circ$ . Find the height of the space shuttle.



**Figure 2.40**

**take note**

The intermediate calculations in Example 6 were not rounded off. This ensures better accuracy for the final result. Using the rounding convention stated on page 140, we round off only the last result.

$$(1) \tan 27.2^\circ = \frac{y}{x} \quad \text{and} \quad (2) \tan 23.9^\circ = \frac{y}{x + 17.5}$$

Solving Equation (1) for  $x$ ,  $x = \frac{y}{\tan 27.2^\circ} = y \cot 27.2^\circ$ , and substituting into Equation (2), we have

$$\begin{aligned} \tan 23.9^\circ &= \frac{y}{y \cot 27.2^\circ + 17.5} \\ y &= (\tan 23.9^\circ)(y \cot 27.2^\circ + 17.5) \quad \bullet \text{Solve for } y. \\ y - y \tan 23.9^\circ \cot 27.2^\circ &= (\tan 23.9^\circ)(17.5) \\ y(1 - \tan 23.9^\circ \cot 27.2^\circ) &= (\tan 23.9^\circ)(17.5) \\ y &= \frac{(\tan 23.9^\circ)(17.5)}{1 - \tan 23.9^\circ \cot 27.2^\circ} \approx 56.2993 \end{aligned}$$

To three significant digits, the height of the space shuttle is 56.3 meters.

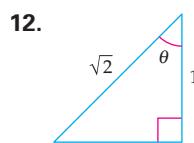
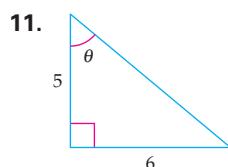
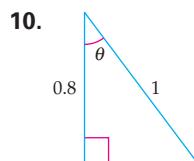
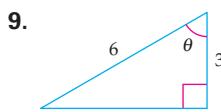
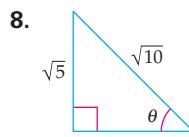
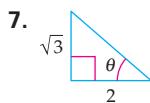
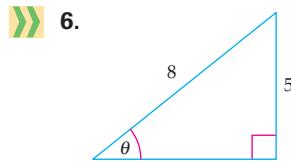
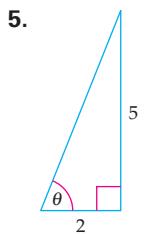
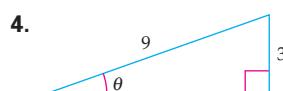
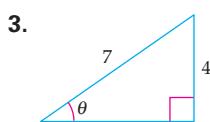
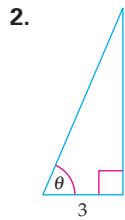
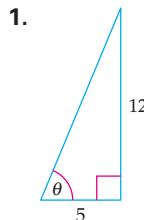
## » Try Exercise 66, page 145


**Topics for Discussion**

- If  $\theta$  is an acute angle of a right triangle for which  $\cos \theta = \frac{3}{8}$ , then it must be the case that  $\sin \theta = \frac{5}{8}$ . Do you agree? Explain.
- A tutor claims that  $\tan 30^\circ = \cot 60^\circ$ . Do you agree?
- Does  $\sin 2\theta = 2 \sin \theta$ ? Explain.
- How many significant digits are in each of the following measurements?
  - 0.0042 inches
  - 5.03 inches
  - 62.00 inches

## Exercise Set 2.2

In Exercises 1 to 12, find the values of the six trigonometric functions of  $\theta$  for the right triangle with the given sides.



In Exercises 13 to 15, let  $\theta$  be an acute angle of a right triangle for which  $\sin \theta = \frac{3}{5}$ . Find

13.  $\tan \theta$

14.  $\sec \theta$

15.  $\cos \theta$

In Exercises 16 to 18, let  $\theta$  be an acute angle of a right triangle for which  $\tan \theta = \frac{4}{3}$ . Find

16.  $\sin \theta$

17.  $\cot \theta$

18.  $\sec \theta$

In Exercises 19 to 21, let  $\beta$  be an acute angle of a right triangle for which  $\sec \beta = \frac{13}{12}$ . Find

19.  $\cos \beta$

20.  $\cot \beta$

21.  $\csc \beta$

In Exercises 22 to 24, let  $\theta$  be an acute angle of a right triangle for which  $\cos \theta = \frac{2}{3}$ . Find

22.  $\sin \theta$

23.  $\sec \theta$

24.  $\tan \theta$

In Exercises 25 to 38, find the exact value of each expression.

25.  $\sin 45^\circ + \cos 45^\circ$

26.  $\csc 45^\circ - \sec 45^\circ$

27.  $\sin 30^\circ \cos 60^\circ - \tan 45^\circ$

28.  $\csc 60^\circ \sec 30^\circ + \cot 45^\circ$

29.  $\sin 30^\circ \cos 60^\circ + \tan 45^\circ$

30.  $\sec 30^\circ \cos 30^\circ - \tan 60^\circ \cot 60^\circ$

31.  $\sin \frac{\pi}{3} + \cos \frac{\pi}{6}$

32.  $\csc \frac{\pi}{6} - \sec \frac{\pi}{3}$

33.  $\sin \frac{\pi}{4} + \tan \frac{\pi}{6}$

34.  $\sin \frac{\pi}{3} \cos \frac{\pi}{4} - \tan \frac{\pi}{4}$

35.  $\sec \frac{\pi}{3} \cos \frac{\pi}{3} - \tan \frac{\pi}{6}$

36.  $\cos \frac{\pi}{4} \tan \frac{\pi}{6} + 2 \tan \frac{\pi}{3}$

37.  $2 \csc \frac{\pi}{4} - \sec \frac{\pi}{3} \cos \frac{\pi}{6}$

38.  $3 \tan \frac{\pi}{4} + \sec \frac{\pi}{6} \sin \frac{\pi}{3}$



In Exercises 39 to 50, use a calculator to find the value of the trigonometric function to four decimal places.

39.  $\tan 32^\circ$

40.  $\sec 88^\circ$

41.  $\cos 63^\circ 20'$

42.  $\cot 55^\circ 50'$

43.  $\cos 34.7^\circ$

44.  $\tan 81.3^\circ$

45.  $\sec 5.9^\circ$

46.  $\sin \frac{\pi}{5}$

47.  $\tan \frac{\pi}{7}$

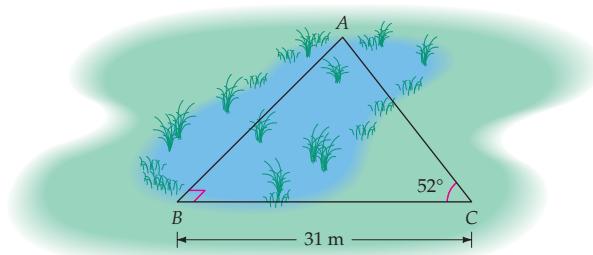
48.  $\sec \frac{3\pi}{8}$

49.  $\csc 1.2$

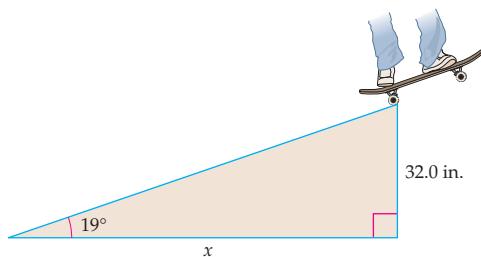
50.  $\sin 0.45$

- 51. VERTICAL HEIGHT FROM SLANT HEIGHT** A 12-foot ladder is resting against a wall and makes an angle of  $52^\circ$  with the ground. Find the height to which the ladder will reach on the wall.

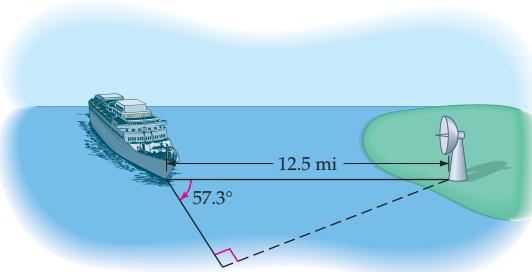
- 52. DISTANCE ACROSS A MARSH** Find the distance  $AB$  across the marsh shown in the accompanying figure.



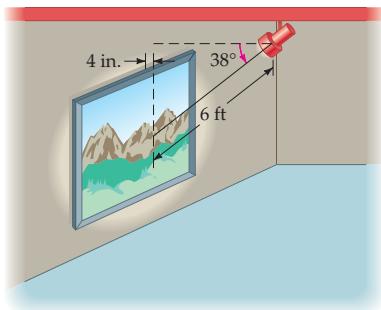
- 53. WIDTH OF A RAMP** A skateboarder wishes to build a jump ramp that is inclined at a  $19^\circ$  angle and that has a maximum height of 32.0 inches. Find the horizontal width  $x$  of the ramp.



- 54. TIME OF CLOSEST APPROACH** At 3:00 P.M., a boat is 12.5 miles due west of a radar station and traveling at 11 mph in a direction that is  $57.3^\circ$  south of an east-west line. At what time will the boat be closest to the radar station?



- 55. PLACEMENT OF A LIGHT** For best illumination of a piece of art, a lighting specialist for an art gallery recommends that a ceiling-mounted light be 6 feet from the piece of art and that the angle of depression of the light be  $38^\circ$ . How far from a wall should the light be placed so that the recommendations of the specialist are met? Notice that the art extends outward 4 inches from the wall.

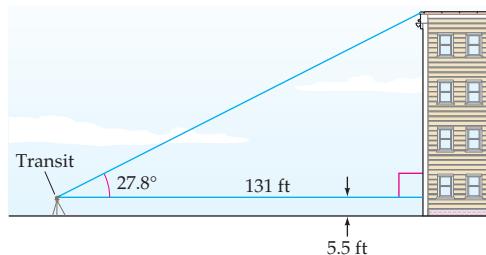


- 56. HEIGHT OF THE EIFFEL TOWER** The angle of elevation from a point 116 meters from the base of the Eiffel Tower to the top of the tower is  $68.9^\circ$ . Find the approximate height of the tower.

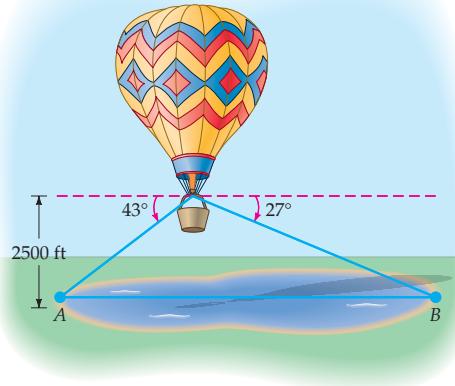
- 57. DISTANCE OF A DESCENT** An airplane traveling at 240 mph is descending at an angle of depression of  $6^\circ$ . How many miles will the plane descend in 4 minutes?

- 58. TIME OF A DESCENT** A submarine traveling at 9.0 mph is descending at an angle of depression of  $5^\circ$ . How many minutes, to the nearest tenth, does it take the submarine to reach a depth of 80 feet?

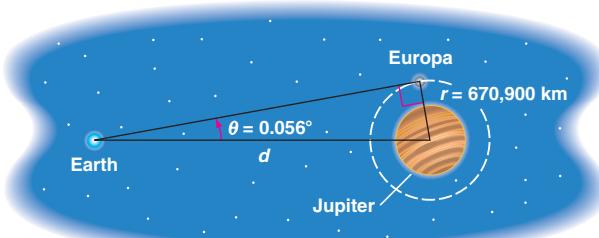
- 59. HEIGHT OF A BUILDING** A surveyor determines that the angle of elevation from a transit to the top of a building is  $27.8^\circ$ . The transit is positioned 5.5 feet above ground level and 131 feet from the building. Find the height of the building to the nearest tenth of a foot.



- 60. WIDTH OF A LAKE** The angle of depression to one side of a lake, measured from a balloon 2500 feet above the lake as shown in the accompanying figure, is  $43^\circ$ . The angle of depression to the opposite side of the lake is  $27^\circ$ . Find the width of the lake.

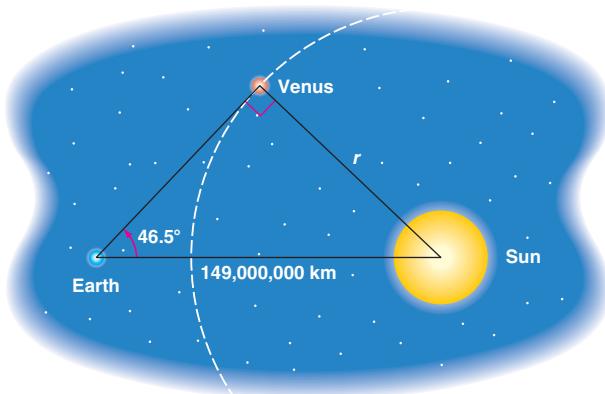


- 61. ASTRONOMY** The moon Europa rotates in a nearly circular orbit around Jupiter. The orbital radius of Europa is approximately 670,900 kilometers. During a revolution of Europa around Jupiter, an astronomer found that the maximum value of the angle  $\theta$  formed by Europa, Earth, and Jupiter was  $0.056^\circ$ . Find the distance  $d$  between Earth and Jupiter at the time the astronomer found the maximum value of  $\theta$ . Round to the nearest million kilometers.

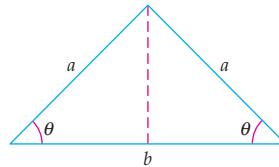


Not drawn to scale.

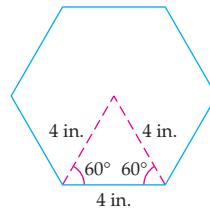
- 62. ASTRONOMY** Venus rotates in a nearly circular orbit around the sun. The largest angle formed by Venus, Earth, and the sun is  $46.5^\circ$ . The distance from Earth to the sun is approximately 149,000,000 kilometers. See the following figure. What is the orbital radius  $r$  of Venus? Round to the nearest million kilometers.



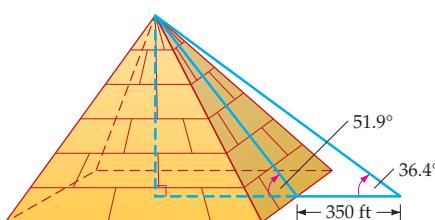
- 63. AREA OF AN ISOSCELES TRIANGLE** Consider the following isosceles triangle. The length of each of the two equal sides of the triangle is  $a$ , and each of the base angles has a measure of  $\theta$ . Verify that the area of the triangle is  $A = a^2 \sin \theta \cos \theta$ .



- 64. AREA OF A HEXAGON** Find the area of the hexagon. (*Hint:* The area consists of six isosceles triangles. Use the formula from Exercise 63 to compute the area of one of the triangles and multiply by 6.)

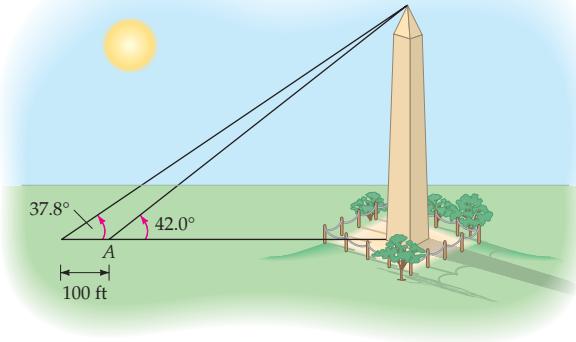


- 65. HEIGHT OF A PYRAMID** The angle of elevation to the top of the Egyptian pyramid of Cheops is  $36.4^\circ$ , measured from a point 350 feet from the base of the pyramid. The angle of elevation from the base of a face of the pyramid is  $51.9^\circ$ . Find the height of the Cheops pyramid.

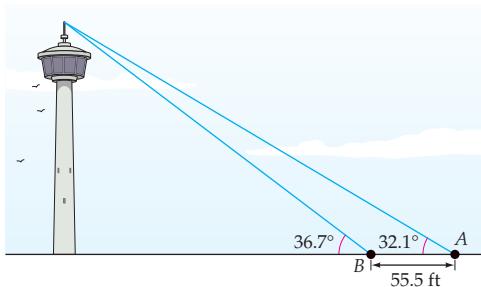


- 66. HEIGHT OF A BUILDING** Two buildings are 240 feet apart. The angle of elevation from the top of the shorter building to the top of the other building is  $22^\circ$ . If the shorter building is 80 feet high, how high is the taller building?

- 67. HEIGHT OF THE WASHINGTON MONUMENT** From a point  $A$  on a line from the base of the Washington Monument, the angle of elevation to the top of the monument is  $42.0^\circ$ . From a point 100 feet away from  $A$  and on the same line, the angle to the top is  $37.8^\circ$ . Find the height, to the nearest foot, of the Washington Monument.



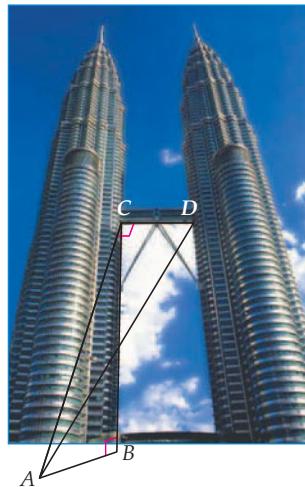
- 68. HEIGHT OF A TOWER** The angle of elevation from a point  $A$  to the top of a tower is  $32.1^\circ$ . From point  $B$ , which is on the same line but 55.5 feet closer to the tower, the angle of elevation is  $36.7^\circ$ . Find the height of the tower.



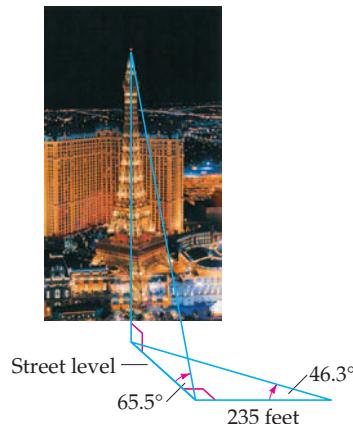
- 69. THE PETRONAS TOWERS** The Petronas Towers in Kuala Lumpur, Malaysia, are the world's tallest twin towers. Each tower is 1483 feet in height. The towers are

connected by a skybridge at the forty-first floor. Note the information given in the accompanying figure.

- Determine the height of the skybridge.
- Determine the length of the skybridge.



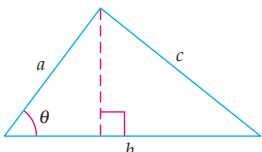
- 70. AN EIFFEL TOWER REPLICA** Use the information in the accompanying figure to estimate the height of the Eiffel Tower replica that stands in front of the Paris Las Vegas Hotel in Las Vegas, Nevada.



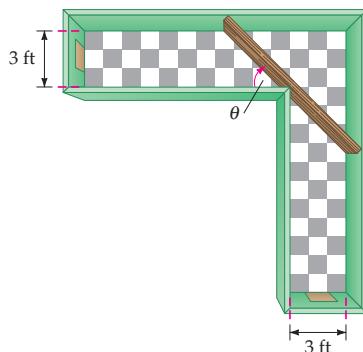
- 71.** A circle is inscribed in a regular hexagon with each side 6.0 meters long. Find the radius of the circle.

- 72.** Show that the area  $A$  of the triangle given in the figure is  

$$A = \frac{1}{2} ab \sin \theta.$$



- 73. FIND A MAXIMUM LENGTH** Find the length of the longest piece of wood that can be slid around the corner of the hallway in the figure in the column at right. Round to the nearest tenth of a foot.



- 74. FIND A MAXIMUM LENGTH** In Exercise 73, suppose that the hall is 8 feet high. Find the length of the longest piece of wood that can be taken around the corner. Round to the nearest tenth of a foot.

- 1. a. PERIMETER OF A REGULAR  $n$ -GON** Show that the perimeter  $P$  of a regular  $n$ -sided polygon ( $n$ -gon) inscribed in a circle of radius 1 is  $P = 2n \sin \frac{180^\circ}{n}$ .

- b.** Let  $P_n$  denote the perimeter of a regular  $n$ -gon inscribed in a circle of radius 1. Use the result from a. to complete the following table.

| $n$   | 10 | 50 | 100 | 1000 | 10,000 |
|-------|----|----|-----|------|--------|
| $P_n$ |    |    |     |      |        |

Write a few sentences explaining why  $P_n$  approaches  $2\pi$  as  $n$  increases without bound.

- $$A = \frac{n}{2} \sin \frac{360^\circ}{n}.$$

- b.**  Let  $A_n$  denote the area of a regular  $n$ -gon inscribed in a circle of radius 1. Use the result from a. to complete the following table.

| $n$   | 10 | 50 | 100 | 1000 | 10,000 |
|-------|----|----|-----|------|--------|
| $A_n$ |    |    |     |      |        |

Write a few sentences explaining why  $A_n$  approaches  $\pi$  as  $n$  increases without bound.

## Section 2.3

- Trigonometric Functions of Any Angle
- Trigonometric Functions of Quadrantal Angles
- Signs of Trigonometric Functions
- The Reference Angle

# Trigonometric Functions of Any Angle

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A9.

**PS1.** Find the reciprocal of  $-\frac{3}{4}$ .

**PS2.** Find the reciprocal of  $\frac{2\sqrt{5}}{5}$ .

**PS3.** Evaluate:  $|120 - 180|$  [1.1]

**PS4.** Simplify:  $2\pi - \frac{9\pi}{5}$

**PS5.** Simplify:  $\frac{3}{2}\pi - \frac{\pi}{2}$

**PS6.** Simplify:  $\sqrt{(-3)^2 + (-5)^2}$  [1.2]

## Trigonometric Functions of Any Angle

The applications of trigonometry would be quite limited if all angles had to be acute angles. Fortunately, this is not the case. In this section we extend the definition of a trigonometric function to include any angle.

Consider angle  $\theta$  in **Figure 2.41** in standard position and a point  $P(x, y)$  on the terminal side of the angle. We define the trigonometric functions of any angle according to the following definitions.

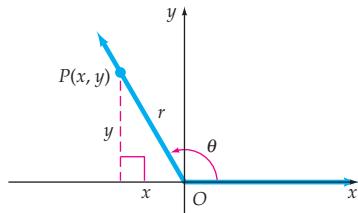


Figure 2.41

### Definitions of the Trigonometric Functions of Any Angle

Let  $P(x, y)$  be any point, except the origin, on the terminal side of an angle  $\theta$  in standard position. Let  $r = d(O, P)$ , the distance from the origin to  $P$ . The six trigonometric functions of  $\theta$  are

$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \tan \theta = \frac{y}{x}, \quad x \neq 0$$

$$\csc \theta = \frac{r}{y}, \quad y \neq 0 \quad \sec \theta = \frac{r}{x}, \quad x \neq 0 \quad \cot \theta = \frac{x}{y}, \quad y \neq 0$$

where  $r = \sqrt{x^2 + y^2}$ .

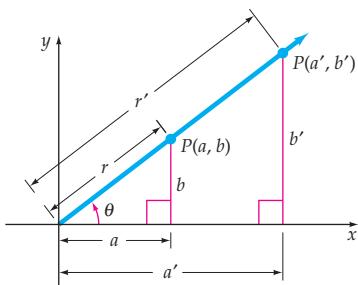


Figure 2.42

The value of a trigonometric function is independent of the point chosen on the terminal side of the angle. Consider any two points on the terminal side of an angle  $\theta$  in standard position, as shown in **Figure 2.42**. The right triangles formed are similar triangles, so the ratios of the corresponding sides are equal. Thus, for example,  $\frac{b}{a} = \frac{b'}{a'}$ . Because  $\tan \theta = \frac{b}{a} = \frac{b'}{a'}$ , we have  $\tan \theta = \frac{b'}{a'}$ . Therefore, the value of the tangent function is independent of the point chosen on the terminal side of the angle. By a similar argument, we can show that the value of any trigonometric function is independent of the point chosen on the terminal side of the angle.

Any point in a rectangular coordinate system (except the origin) can determine an angle in standard position. For example,  $P(-4, 3)$  in **Figure 2.43** is a point in the second quadrant and determines an angle  $\theta$  in standard position with  $r = \sqrt{(-4)^2 + 3^2} = 5$ .

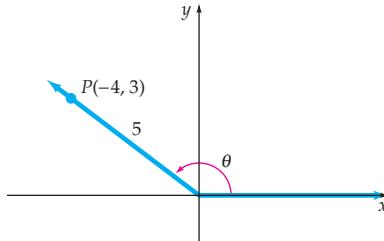


Figure 2.43

The values of the trigonometric functions of  $\theta$  as shown in **Figure 2.43** are

$$\begin{aligned}\sin \theta &= \frac{3}{5} & \cos \theta &= \frac{-4}{5} = -\frac{4}{5} & \tan \theta &= \frac{3}{-4} = -\frac{3}{4} \\ \csc \theta &= \frac{5}{3} & \sec \theta &= \frac{5}{-4} = -\frac{5}{4} & \cot \theta &= \frac{-4}{3} = -\frac{4}{3}\end{aligned}$$

### EXAMPLE 1 » Evaluate Trigonometric Functions

Find the exact value of each of the six trigonometric functions of an angle  $\theta$  in standard position whose terminal side contains the point  $P(-3, -2)$ .

#### Solution

The angle is sketched in **Figure 2.44**. Find  $r$  by using the equation  $r = \sqrt{x^2 + y^2}$ , where  $x = -3$  and  $y = -2$ .

$$r = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$$

Now use the definitions of the trigonometric functions.

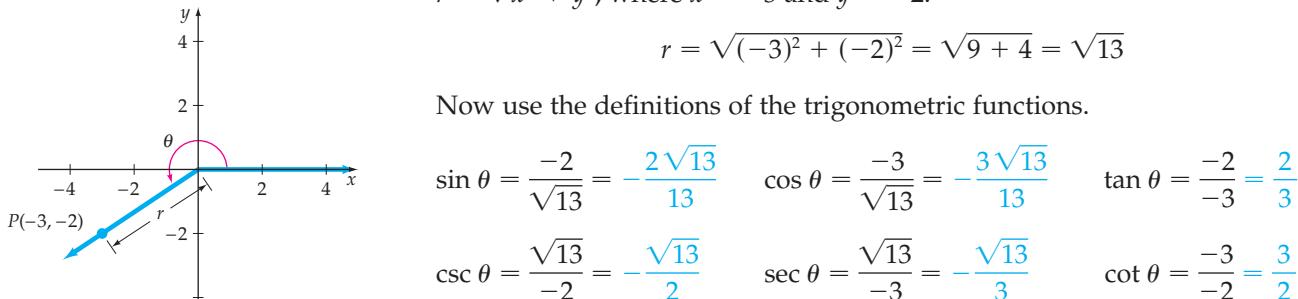


Figure 2.44

### » Try Exercise 6, page 153

### ■ Trigonometric Functions of Quadrantal Angles

Recall that a quadrantal angle is an angle whose terminal side coincides with the  $x$ - or  $y$ -axis. The value of a trigonometric function of a quadrantal angle can be found by choosing any point on the terminal side of the angle and then applying the definition of that trigonometric function.

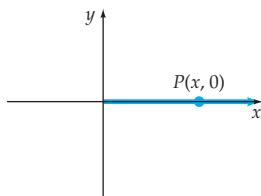


Figure 2.45

The terminal side of  $0^\circ$  coincides with the positive  $x$ -axis. Let  $P(x, 0)$ ,  $x > 0$ , be a point on the  $x$ -axis, as shown in **Figure 2.45**. Then  $y = 0$  and  $r = x$ . The values of the six trigonometric functions of  $0^\circ$  are

$$\sin 0^\circ = \frac{0}{r} = 0 \quad \cos 0^\circ = \frac{x}{r} = \frac{x}{x} = 1 \quad \tan 0^\circ = \frac{0}{x} = 0$$

$$\csc 0^\circ \text{ is undefined.} \quad \sec 0^\circ = \frac{r}{x} = \frac{x}{x} = 1 \quad \cot 0^\circ \text{ is undefined.}$$

**QUESTION** Why are  $\csc 0^\circ$  and  $\cot 0^\circ$  undefined?

In like manner, the values of the trigonometric functions of the other quadrantal angles can be found. The results are shown in **Table 2.4**.

**Table 2.4 Values of Trigonometric Functions of Quadrantal Angles**

| $\theta$    | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | $\csc \theta$ | $\sec \theta$ | $\cot \theta$ |
|-------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $0^\circ$   | 0             | 1             | 0             | undefined     | 1             | undefined     |
| $90^\circ$  | 1             | 0             | undefined     | 1             | undefined     | 0             |
| $180^\circ$ | 0             | -1            | 0             | undefined     | -1            | undefined     |
| $270^\circ$ | -1            | 0             | undefined     | -1            | undefined     | 0             |

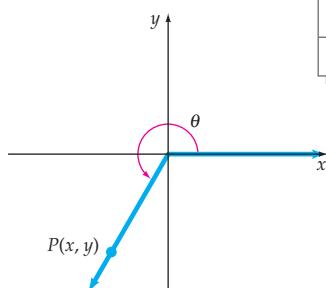


Figure 2.46

### Signs of Trigonometric Functions

The sign of a trigonometric function depends on the quadrant in which the terminal side of the angle lies. For example, if  $\theta$  is an angle whose terminal side lies in Quadrant III and  $P(x, y)$  is on the terminal side of  $\theta$ , then both  $x$  and  $y$  are negative, and therefore  $\frac{y}{x}$  and  $\frac{x}{y}$  are positive. See **Figure 2.46**. Because  $\tan \theta = \frac{y}{x}$

and  $\cot \theta = \frac{x}{y}$ , the values of the tangent and cotangent functions are positive for any Quadrant III angle. The values of the other four trigonometric functions of any Quadrant III angle are all negative.

**Table 2.5** lists the signs of the six trigonometric functions in each quadrant. **Figure 2.47** is a graphical display of the contents of **Table 2.5**.

|                                |                            |
|--------------------------------|----------------------------|
| Sine and cosecant positive     | All functions positive     |
| Tangent and cotangent positive | Cosine and secant positive |

Figure 2.47

**Table 2.5 Signs of the Trigonometric Functions**

| Sign of                         | Terminal Side of $\theta$ in Quadrant |          |          |          |
|---------------------------------|---------------------------------------|----------|----------|----------|
|                                 | I                                     | II       | III      | IV       |
| $\sin \theta$ and $\csc \theta$ | positive                              | positive | negative | negative |
| $\cos \theta$ and $\sec \theta$ | positive                              | negative | negative | positive |
| $\tan \theta$ and $\cot \theta$ | positive                              | negative | positive | negative |

**ANSWER**  $P(x, 0)$  is a point on the terminal side of a  $0^\circ$  angle in standard position. Thus  $\csc 0^\circ = \frac{r}{0}$ , which is undefined. Similarly,  $\cot 0^\circ = \frac{x}{0}$ , which is undefined.

In the next example we are asked to evaluate two trigonometric functions of the angle  $\theta$ . A key step is to use our knowledge about trigonometric functions and their signs to determine that  $\theta$  is a Quadrant IV angle.

### EXAMPLE 2 » Evaluate Trigonometric Functions

Given  $\tan \theta = -\frac{7}{5}$  and  $\sin \theta < 0$ , find  $\cos \theta$  and  $\csc \theta$ .

#### Solution

The terminal side of angle  $\theta$  must lie in Quadrant IV; that is the only quadrant in which  $\sin \theta$  and  $\tan \theta$  are both negative. Because

$$\tan \theta = -\frac{7}{5} = \frac{y}{x} \quad (1)$$

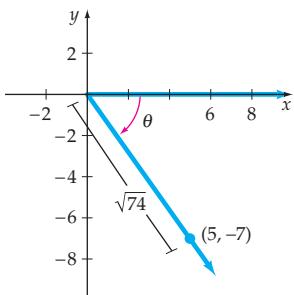
and the terminal side of  $\theta$  is in Quadrant IV, we know that  $y$  must be negative and  $x$  must be positive. Thus Equation (1) is true for  $y = -7$  and  $x = 5$ . Now  $r = \sqrt{5^2 + (-7)^2} = \sqrt{74}$ . See **Figure 2.48**. Hence

$$\cos \theta = \frac{x}{r} = \frac{5}{\sqrt{74}} = \frac{5\sqrt{74}}{74}$$

and

$$\csc \theta = \frac{r}{y} = \frac{\sqrt{74}}{-7} = -\frac{\sqrt{74}}{7}$$

» Try Exercise 30, page 153



**Figure 2.48**

#### take note

In this section  $r$  is a distance and hence nonnegative.

#### take note

The reference angle is a very important concept that will be used time and time again in the remaining trigonometry sections.

### ■ The Reference Angle

We will often find it convenient to evaluate trigonometric functions by making use of the concept of a *reference angle*.

#### Definition of a Reference Angle

Given  $\angle\theta$  in standard position, its **reference angle**  $\theta'$  is the acute angle formed by the terminal side of  $\angle\theta$  and the  $x$ -axis.

**Figure 2.49** shows  $\angle\theta$  and its reference angle  $\theta'$  for four cases. In every case the reference angle  $\theta'$  is formed by the terminal side of  $\angle\theta$  and the  $x$ -axis (never the  $y$ -axis). The process of determining the measure of  $\angle\theta'$  varies according to which quadrant contains the terminal side of  $\angle\theta$ .

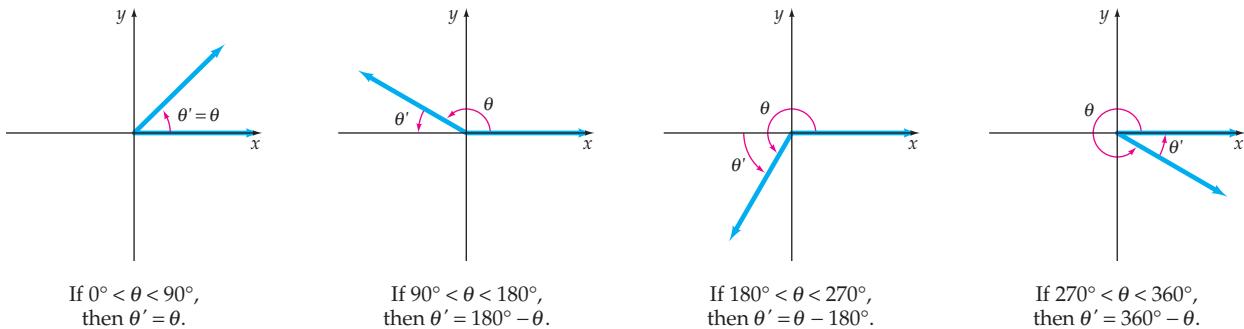


Figure 2.49

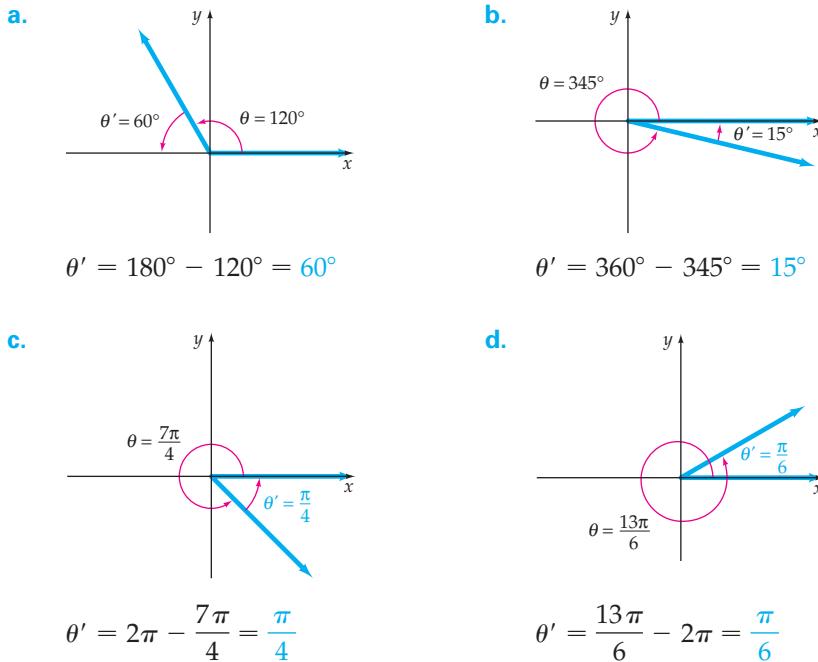
**EXAMPLE 3** » Find the Measure of a Reference Angle

Find the measure of the reference angle  $\theta'$  for each angle.

$$\text{a. } \theta = 120^\circ \quad \text{b. } \theta = 345^\circ \quad \text{c. } \theta = \frac{7\pi}{4} \quad \text{d. } \theta = \frac{13\pi}{6}$$

**Solution**

For any angle in standard position, the measure of its reference angle is the measure of the acute angle formed by its terminal side and the  $x$ -axis.

**Integrating Technology**

A TI-83/TI-83 Plus/TI-84 Plus graphing calculator program is available to compute the measure of the reference angle for a given angle. This program, REFANG, can be found on our website at [college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT).

» Try Exercise 38, page 153

The following theorem states an important relationship that exists between  $\sin \theta$  and  $\sin \theta'$ , where  $\theta'$  is the reference angle for angle  $\theta$ .

**take note**

The Reference Angle Theorem is also valid if the sine function is replaced by any other trigonometric function.

**Reference Angle Theorem**

To evaluate  $\sin \theta$ , determine  $\sin \theta'$ . Then use either  $\sin \theta'$  or its opposite as the answer, depending on which has the correct sign.

In the following example, we illustrate how to evaluate a trigonometric function of  $\theta$  by first evaluating the trigonometric function of  $\theta'$ .

**EXAMPLE 4**  **Use the Reference Angle Theorem to Evaluate Trigonometric Functions**

Determine the exact value of each function.

a.  $\sin 210^\circ$     b.  $\cos 405^\circ$     c.  $\tan \frac{5\pi}{3}$

**Solution**

- a. We know that  $\sin 210^\circ$  is negative (the sign chart is given in **Table 2.5** on page 149). The reference angle for  $\theta = 210^\circ$  is  $\theta' = 30^\circ$ . By the Reference Angle Theorem, we know that  $\sin 210^\circ$  equals either

$$\sin 30^\circ = \frac{1}{2} \quad \text{or} \quad -\sin 30^\circ = -\frac{1}{2}$$

Thus  $\sin 210^\circ = -\frac{1}{2}$ .

- b. Because  $\theta = 405^\circ$  is a Quadrant I angle, we know that  $\cos 405^\circ > 0$ . The reference angle for  $\theta = 405^\circ$  is  $\theta' = 45^\circ$ . By the reference angle theorem,  $\cos 405^\circ$  equals either

$$\cos 45^\circ = \frac{\sqrt{2}}{2} \quad \text{or} \quad -\cos 45^\circ = -\frac{\sqrt{2}}{2}$$

Thus  $\cos 405^\circ = \frac{\sqrt{2}}{2}$ .

- c. Because  $\theta = \frac{5\pi}{3}$  is a Quadrant IV angle,  $\tan \frac{5\pi}{3} < 0$ . The reference angle for  $\theta = \frac{5\pi}{3}$  is  $\theta' = \frac{\pi}{3}$ . Hence  $\tan \frac{5\pi}{3}$  equals either

$$\tan \frac{\pi}{3} = \sqrt{3} \quad \text{or} \quad -\tan \frac{\pi}{3} = -\sqrt{3}$$

Thus  $\tan \frac{5\pi}{3} = -\sqrt{3}$ .

 **Try Exercise 50, page 154**



### Topics for Discussion

1. Is every reference angle an acute angle? Explain.
2. If  $\theta'$  is the reference angle for the angle  $\theta$ , then  $\sin \theta = \sin \theta'$ . Do you agree? Explain.
3. If  $\sin \theta < 0$  and  $\cos \theta > 0$ , then the terminal side of the angle  $\theta$  lies in which quadrant?
4. Explain how to find the measure of the reference angle  $\theta'$  for the angle  $\theta = \frac{19}{5}\pi$ .

## Exercise Set 2.3

In Exercises 1 to 8, find the value of each of the six trigonometric functions for the angle whose terminal side passes through the given point.

1.  $P(2, 3)$
2.  $P(3, 7)$
3.  $P(-2, 3)$
4.  $P(-3, 5)$
5.  $P(-8, -5)$
6.  $P(-6, -9)$
7.  $P(-5, 0)$
8.  $P(0, 2)$

In Exercises 9 to 20, evaluate the trigonometric function of the quadrant angle, or state that the function is undefined.

9.  $\sin 180^\circ$
10.  $\cos 270^\circ$
11.  $\tan 180^\circ$
12.  $\sec 90^\circ$
13.  $\csc 90^\circ$
14.  $\cot 90^\circ$
15.  $\cos \frac{\pi}{2}$
16.  $\sin \frac{3\pi}{2}$
17.  $\tan \frac{\pi}{2}$
18.  $\cot \pi$
19.  $\sin \frac{\pi}{2}$
20.  $\cos \pi$

In Exercises 21 to 26, let  $\theta$  be an angle in standard position. State the quadrant in which the terminal side of  $\theta$  lies.

21.  $\sin \theta > 0, \cos \theta > 0$
22.  $\tan \theta < 0, \sin \theta < 0$
23.  $\cos \theta > 0, \tan \theta < 0$
24.  $\sin \theta < 0, \cos \theta > 0$
25.  $\sin \theta < 0, \cos \theta < 0$
26.  $\tan \theta < 0, \cos \theta < 0$

In Exercises 27 to 36, find the exact value of each expression.

27.  $\sin \theta = -\frac{1}{2}, 180^\circ < \theta < 270^\circ$ ; find  $\tan \theta$ .
28.  $\cot \theta = -1, 90^\circ < \theta < 180^\circ$ ; find  $\cos \theta$ .

29.  $\csc \theta = \sqrt{2}, \frac{\pi}{2} < \theta < \pi$ ; find  $\cot \theta$ .

30.  $\sec \theta = \frac{2\sqrt{3}}{3}, \frac{3\pi}{2} < \theta < 2\pi$ ; find  $\sin \theta$ .

31.  $\sin \theta = -\frac{1}{2}$  and  $\cos \theta > 0$ ; find  $\tan \theta$ .

32.  $\tan \theta = 1$  and  $\sin \theta < 0$ ; find  $\cos \theta$ .

33.  $\cos \theta = \frac{1}{2}$  and  $\tan \theta = \sqrt{3}$ ; find  $\csc \theta$ .

34.  $\tan \theta = 1$  and  $\sin \theta = \frac{\sqrt{2}}{2}$ ; find  $\sec \theta$ .

35.  $\cos \theta = -\frac{1}{2}$  and  $\sin \theta = \frac{\sqrt{3}}{2}$ ; find  $\cot \theta$ .

36.  $\sec \theta = \frac{2\sqrt{3}}{3}$  and  $\sin \theta = -\frac{1}{2}$ ; find  $\cot \theta$ .

In Exercises 37 to 48, find the measure of the reference angle  $\theta'$  for the given angle  $\theta$ .

37.  $\theta = 160^\circ$

38.  $\theta = 255^\circ$

39.  $\theta = 351^\circ$

40.  $\theta = 48^\circ$

41.  $\theta = \frac{11}{5}\pi$

42.  $\theta = -6$

43.  $\theta = \frac{8}{3}$

44.  $\theta = \frac{18}{7}\pi$

45.  $\theta = 1406^\circ$

46.  $\theta = 840^\circ$

47.  $\theta = -475^\circ$

48.  $\theta = -650^\circ$



-  **FIND SUMS OR PRODUCTS** Determine the following sums or products. Do not use a calculator. (*Hint:* The Reference Angle Theorem may be helpful.) Explain to a classmate how you know you are correct.

  - $\cos 0^\circ + \cos 1^\circ + \cos 2^\circ + \dots + \cos 178^\circ + \cos 179^\circ + \cos 180^\circ$
  - $\sin 0^\circ + \sin 1^\circ + \sin 2^\circ + \dots + \sin 358^\circ + \sin 359^\circ + \sin 360^\circ$
  - $\cot 1^\circ + \cot 2^\circ + \cot 3^\circ + \dots + \cot 177^\circ + \cot 178^\circ + \cot 179^\circ$
  - $(\cos 1^\circ)(\cos 2^\circ)(\cos 3^\circ) \dots (\cos 177^\circ)(\cos 178^\circ)(\cos 179^\circ)$
  - $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \dots + \cos^2 357^\circ + \cos^2 358^\circ + \cos^2 359^\circ$

## Section 2.4

- The Wrapping Function
  - Trigonometric Functions of Real Numbers
  - Properties of Trigonometric Functions of Real Numbers
  - Trigonometric Identities

# Trigonometric Functions of Real Numbers

## PREPARE FOR THIS SECTION

**Prepare for this section by completing the following exercises. The answers can be found on page A10.**

- PS1.** Determine whether the point  $(0, 1)$  is a point on the circle defined by  $x^2 + y^2 = 1$ . [1.1/1.2]

**PS2.** Determine whether the point  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$  is a point on the circle defined by  $x^2 + y^2 = 1$ . [1.1/1.2]

**PS3.** Determine whether the point  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)$  is a point on the circle defined by  $x^2 + y^2 = 1$ . [1.1/1.2]

**PS4.** Determine the circumference of a circle with a radius of 1. [2.1]

**PS5.** Determine whether  $f(x) = x^2 - 3$  is an even function, an odd function, or neither. [1.4]

**PS6.** Determine whether  $f(x) = x^3 - x^2$  is an even function, an odd function, or neither. [1.4]

## ■ The Wrapping Function

In the seventeenth century, applications of trigonometry were extended to problems in physics and engineering. These kinds of problems required trigonometric functions whose domains were sets of real numbers rather than sets of angles. This extension of the definitions of trigonometric functions to include real numbers was accomplished by using a correspondence between an angle and the length of an arc on a *unit circle*.

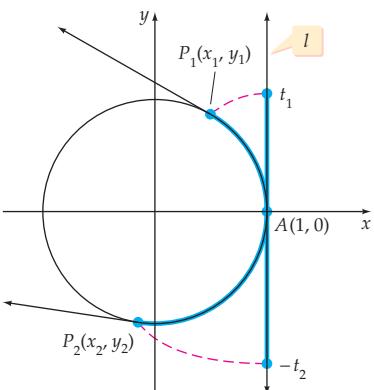


Figure 2.50

Consider a circle given by the equation  $x^2 + y^2 = 1$ , called a **unit circle**, and a vertical coordinate line  $l$  tangent to the unit circle at  $(1, 0)$ . We define a function  $W$  that pairs a real number  $t$  on the coordinate line with a point  $P(x, y)$  on the unit circle. This function  $W$  is called the *wrapping function* because it is analogous to wrapping a line around a circle.

As shown in **Figure 2.50**, the positive part of the coordinate line is wrapped around the unit circle in a counterclockwise direction. The negative part of the coordinate line is wrapped around the circle in a clockwise direction. The wrapping function is defined by the equation  $W(t) = P(x, y)$ , where  $t$  is a real number and  $P(x, y)$  is the point on the unit circle that corresponds to  $t$ .

Through the wrapping function, each real number  $t$  defines an arc  $\widehat{AP}$  that subtends a central angle with a measure of  $\theta$  radians. The length of the arc  $\widehat{AP}$  is  $t$  (see **Figure 2.51**). From the equation  $s = r\theta$  for the arc length of a circle, we have (with  $t = s$ )  $t = r\theta$ . For a unit circle,  $r = 1$ , and the equation becomes  $t = \theta$ . Thus, on a unit circle, *the measure of a central angle and the length of its arc can be represented by the same real number  $t$* .

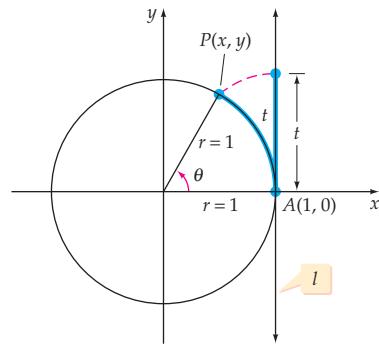


Figure 2.51

### EXAMPLE 1 » Evaluate the Wrapping Function

$$\text{Evaluate } W\left(\frac{\pi}{3}\right).$$

#### Solution

The point  $\frac{\pi}{3}$  on line  $l$  is shown in **Figure 2.52**. From the wrapping function,  $W\left(\frac{\pi}{3}\right)$  is the point  $P$  on the unit circle for which arc  $\widehat{AP}$  subtends an angle  $\theta$ , the measure of which is  $\frac{\pi}{3}$  radians. The coordinates of  $P$  can be determined from the definitions of  $\cos \theta$  and  $\sin \theta$  given in Section 2.3 and from **Table 2.2**, page 137.

$$\cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r}$$

$$\cos \frac{\pi}{3} = \frac{x}{1} = x \quad \sin \frac{\pi}{3} = \frac{y}{1} = y \quad \bullet \theta = \frac{\pi}{3}; r = 1$$

$$\frac{1}{2} = x \quad \frac{\sqrt{3}}{2} = y \quad \bullet \cos \frac{\pi}{3} = \frac{1}{2}; \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

From these equations,  $x = \frac{1}{2}$  and  $y = \frac{\sqrt{3}}{2}$ . Therefore,  $W\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ .

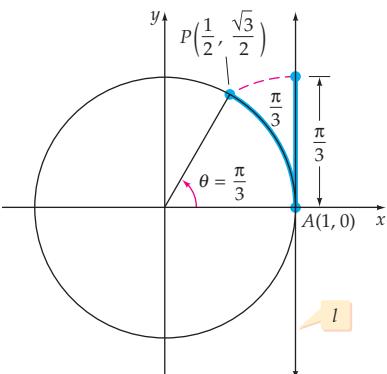


Figure 2.52

» Try Exercise 10, page 166

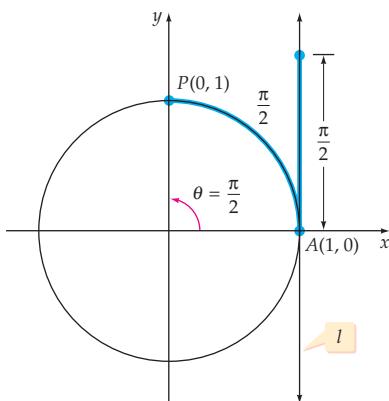


Figure 2.53

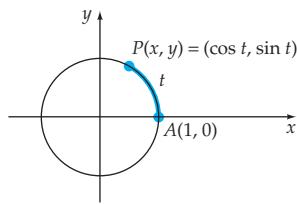


Figure 2.54

To determine  $W\left(\frac{\pi}{2}\right)$ , recall that the circumference of a unit circle is  $2\pi$ . One-fourth the circumference is  $\frac{1}{4}(2\pi) = \frac{\pi}{2}$  (see **Figure 2.53**). Thus  $W\left(\frac{\pi}{2}\right) = P(0, 1)$ .

Note from the last two examples that for the given real number  $t$ ,  $\cos t = x$  and  $\sin t = y$ . That is, for a real number  $t$  and  $W(t) = P(x, y)$ , the value of the cosine of  $t$  is the  $x$ -coordinate of  $P$ , and the value of the sine of  $t$  is the  $y$ -coordinate of  $P$ . See **Figure 2.54**.

**QUESTION** What is the point defined by  $W\left(\frac{\pi}{4}\right)$ ?

## Trigonometric Functions of Real Numbers

The following definition makes use of the wrapping function  $W(t)$  to define trigonometric functions of real numbers.

### Definitions of the Trigonometric Functions of Real Numbers

Let  $W$  be the wrapping function,  $t$  be a real number, and  $W(t) = P(x, y)$ . Then

$$\begin{array}{lll} \sin t = y & \cos t = x & \tan t = \frac{y}{x}, \quad x \neq 0 \\ \csc t = \frac{1}{y}, \quad y \neq 0 & \sec t = \frac{1}{x}, \quad x \neq 0 & \cot t = \frac{x}{y}, \quad y \neq 0 \end{array}$$

Trigonometric functions of real numbers are frequently called *circular functions* to distinguish them from trigonometric functions of angles.

The *trigonometric functions of real numbers* (or circular functions) look remarkably like the trigonometric functions defined in the last section. The difference between the two is that of domain: In one case, the domains are sets of *real numbers*; in the other case, the domains are sets of *angle measurements*. However, there are similarities between the two functions.

Consider an angle  $\theta$  (measured in radians) in standard position, as shown in **Figure 2.55**. Let  $P(x, y)$  and  $P'(x', y')$  be two points on the terminal side of  $\theta$ , where  $x^2 + y^2 = 1$  and  $(x')^2 + (y')^2 = r^2$ . Let  $t$  be the length of the arc from  $A(1, 0)$  to  $P(x, y)$ . Then

$$\sin \theta = \frac{y'}{r} = \frac{y}{1} = \sin t$$

Thus the value of the sine function of  $\theta$ , measured in radians, is equal to the value of the sine of the real number  $t$ . Similar arguments can be given to show cor-

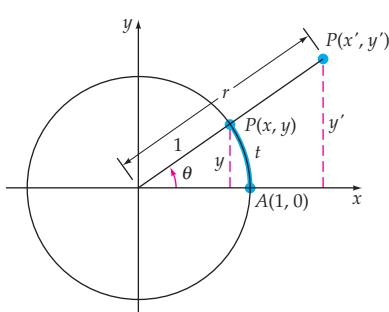


Figure 2.55

**ANSWER**  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

responding results for the other five trigonometric functions. With this in mind, we can assert that the value of a trigonometric function at the real number  $t$  is its value at an angle of  $t$  radians.

### EXAMPLE 2 »» Evaluate Trigonometric Functions of Real Numbers

Find the exact value of each function.

$$\text{a. } \cos \frac{\pi}{4} \quad \text{b. } \sin \left(-\frac{7\pi}{6}\right) \quad \text{c. } \tan \left(-\frac{5\pi}{4}\right) \quad \text{d. } \sec \frac{5\pi}{3}$$

#### Solution

The value of a trigonometric function at the real number  $t$  is its value at an angle of  $t$  radians. Using **Table 2.2** on page 137, we have

$$\text{a. } \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\text{b. } \sin \left(-\frac{7\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$$

- Reference angle for  $-\frac{7\pi}{6}$  is  $\frac{\pi}{6}$
- and  $\sin t > 0$  in Quadrant II.

$$\text{c. } \tan \left(-\frac{5\pi}{4}\right) = -\tan \frac{\pi}{4} = -1$$

- Reference angle for  $-\frac{5\pi}{4}$  is  $\frac{\pi}{4}$
- and  $\tan t < 0$  in Quadrant II.

$$\text{d. } \sec \frac{5\pi}{3} = \sec \frac{\pi}{3} = 2$$

- Reference angle for  $\frac{5\pi}{3}$  is  $\frac{\pi}{3}$
- and  $\sec t > 0$  in Quadrant IV.

»» Try Exercise 16, page 166

In Example 3 we evaluate a trigonometric function of a real number to solve an application.

### EXAMPLE 3 »» Determine a Height as a Function of Time

The Millennium Wheel, in London, is the world's largest Ferris wheel. It has a diameter of 450 feet. When the Millennium Wheel is in uniform motion, it completes one revolution every 30 minutes. The height  $h$ , in feet above the Thames River, of a person riding on the Millennium Wheel can be estimated by

$$h(t) = 255 - 225 \cos\left(\frac{\pi}{15}t\right)$$

where  $t$  is the time in minutes since the person started the ride.



The Millennium Wheel, on the banks of the Thames River, London.



- How high is the person at the start of the ride ( $t = 0$ )?
- How high is the person after 18.0 minutes? Round to the nearest foot.

### Solution

$$\begin{aligned} \text{a. } h(0) &= 255 - 225 \cos\left(\frac{\pi}{15} \cdot 0\right) & \text{b. } h(18.0) &= 255 - 225 \cos\left(\frac{\pi}{15} \cdot 18.0\right) \\ &= 255 - 225 & & \approx 255 - (-182) \\ &= 30 & & = 437 \end{aligned}$$

At the start of the ride, the person is 30 feet above the Thames.

After 18.0 minutes, the person is about 437 feet above the Thames.

» Try Exercise 82, page 167

## Properties of Trigonometric Functions of Real Numbers

The domain and range of the trigonometric functions of real numbers can be found from the definitions of these functions. If  $t$  is any real number and  $P(x, y)$  is the point corresponding to  $W(t)$ , then by definition  $\cos t = x$  and  $\sin t = y$ . Thus the domain of the sine and cosine functions is the set of real numbers.

Because the radius of the unit circle is 1, we have

$$-1 \leq x \leq 1 \quad \text{and} \quad -1 \leq y \leq 1$$

Therefore, with  $x = \cos t$  and  $y = \sin t$ , we have

$$-1 \leq \cos t \leq 1 \quad \text{and} \quad -1 \leq \sin t \leq 1$$

The range of the cosine and sine functions is  $[-1, 1]$ .

Using the definitions of tangent and secant,

$$\tan t = \frac{y}{x} \quad \text{and} \quad \sec t = \frac{1}{x}$$

The domain of the tangent function is all real numbers  $t$  except those for which the  $x$ -coordinate of  $W(t)$  is zero. The  $x$ -coordinate is zero when  $t = \pm \frac{\pi}{2}$ ,  $t = \pm \frac{3\pi}{2}$ ,

$t = \pm \frac{5\pi}{2}$ , and in general when  $t = \frac{(2n+1)\pi}{2}$ , where  $n$  is an integer.

Thus the domain of the tangent function is the set of all real numbers  $t$  except  $t = \frac{(2n+1)\pi}{2}$ , where  $n$  is an integer. The range of the tangent function is all real numbers.

Similar methods can be used to find the domain and range of the cotangent, secant, and cosecant functions. The results are summarized in **Table 2.6** on the next page.

**Table 2.6 Domain and Range of the Trigonometric Functions of Real Numbers ( $n$  is an integer)**

| Function     | Domain   | Range                             |
|--------------|--|-----------------------------------|
| $y = \sin t$ | $\{t \mid -\infty < t < \infty\}$  | $\{y \mid -1 \leq y \leq 1\}$     |
| $y = \cos t$ | $\{t \mid -\infty < t < \infty\}$  | $\{y \mid -1 \leq y \leq 1\}$     |
| $y = \tan t$ | $\left\{t \mid -\infty < t < \infty, t \neq \frac{(2n+1)\pi}{2}\right\}$ | $\{y \mid -\infty < y < \infty\}$ |
| $y = \csc t$ | $\{t \mid -\infty < t < \infty, t \neq n\pi\}$                           | $\{y \mid y \geq 1, y \leq -1\}$  |
| $y = \sec t$ | $\left\{t \mid -\infty < t < \infty, t \neq \frac{(2n+1)\pi}{2}\right\}$ | $\{y \mid y \geq 1, y \leq -1\}$  |
| $y = \cot t$ | $\{t \mid -\infty < t < \infty, t \neq n\pi\}$                           | $\{y \mid -\infty < y < \infty\}$ |

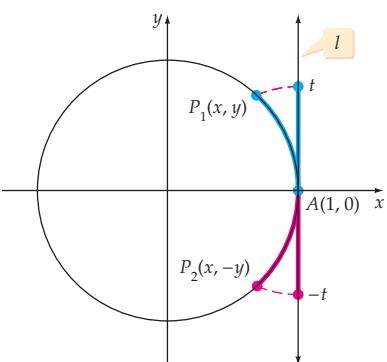


Figure 2.56

Consider the points  $t$  and  $-t$  on the coordinate line  $l$  tangent to the unit circle at the point  $(1, 0)$ . The points  $W(t)$  and  $W(-t)$  are symmetric with respect to the  $x$ -axis. Therefore, if  $P_1(x, y)$  are the coordinates of  $W(t)$ , then  $P_2(x, -y)$  are the coordinates of  $W(-t)$ . See **Figure 2.56**.

From the definitions of the trigonometric functions, we have

$$\sin t = y \quad \text{and} \quad \sin(-t) = -y \quad \text{and} \quad \cos t = x \quad \text{and} \quad \cos(-t) = x$$

Substituting  $\sin t$  for  $y$  and  $\cos t$  for  $x$  yields

$$\sin(-t) = -\sin t \quad \text{and} \quad \cos(-t) = \cos t$$

Thus the sine is an odd function and the cosine is an even function. Because  $\csc t = \frac{1}{\sin t}$  and  $\sec t = \frac{1}{\cos t}$ , it follows that

$$\csc(-t) = -\csc t \quad \text{and} \quad \sec(-t) = \sec t$$

These equations show that the cosecant is an odd function and the secant is an even function.

From the definition of the tangent function, we have  $\tan t = \frac{y}{x}$  and  $\tan(-t) = -\frac{y}{x}$ . Substituting  $\tan t$  for  $\frac{y}{x}$  yields  $\tan(-t) = -\tan t$ . Because  $\cot t = \frac{1}{\tan t}$ , it follows that  $\cot(-t) = -\cot t$ . Thus the tangent and cotangent functions are odd functions.

### Even and Odd Trigonometric Functions

The odd trigonometric functions are  $y = \sin t$ ,  $y = \csc t$ ,  $y = \tan t$ , and  $y = \cot t$ . The even trigonometric functions are  $y = \cos t$  and  $y = \sec t$ . Thus for all  $t$  in their domain,

$$\sin(-t) = -\sin t \quad \cos(-t) = \cos t \quad \tan(-t) = -\tan t$$

$$\csc(-t) = -\csc t \quad \sec(-t) = \sec t \quad \cot(-t) = -\cot t$$

**TO REVIEW**  
Odd and Even Functions  
See page 56.

**EXAMPLE 4** » Determine Whether a Function Is Even, Odd, or Neither

Is  $f(x) = x - \tan x$  an even function, an odd function, or neither?

**Solution**

Find  $f(-x)$  and compare it to  $f(x)$ .

$$\begin{aligned} f(-x) &= (-x) - \tan(-x) = -x + \tan x && \bullet \tan(-x) = -\tan x \\ &= -(x - \tan x) \\ &= -f(x) \end{aligned}$$

The function  $f(x) = x - \tan x$  is an **odd function**.

## » Try Exercise 44, page 166

We encounter many recurring patterns in everyday life. For instance, the time of day repeats every 24 hours. If  $f(t)$  represents the present time of day, then 24 hours later the time of day will be exactly the same. Using mathematical notation, we can express this concept as

$$f(t + 24) = f(t)$$

The function  $f$  is said to be **periodic** in that it repeats itself over and over. The **period** of  $f$  is 24 hours, the time it takes to complete one full cycle.

**Definition of a Periodic Function**

A function  $f$  is **periodic** if there exists a positive constant  $p$  such that

$$f(t + p) = f(t)$$

for all  $t$  in the domain of  $f$ . The smallest such positive number  $p$  for which  $f$  is periodic is called the **period** of  $f$ .

The unit circle can be used to show that the sine and cosine functions are periodic functions. First note that the circumference of the unit circle is  $2\pi$ . Thus, if we start at any point  $P(x, y) = P(\cos t, \sin t)$  on the unit circle and travel a distance of  $2\pi$  units around the circumference, we will be back at point  $P$ . Hence  $(\cos(t + 2\pi), \sin(t + 2\pi)) = (\cos t, \sin t)$ . Equating the first components gives us  $\cos(t + 2\pi) = \cos t$  and equating the second components yields  $\sin(t + 2\pi) = \sin t$ .

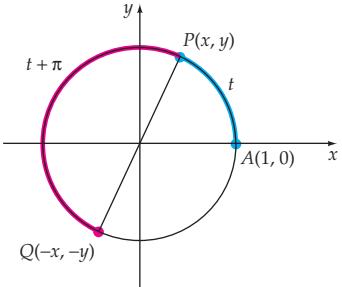
The following equations illustrate that the secant and cosecant functions are also periodic functions.

$$\begin{aligned} \sec(t + 2\pi) &= \frac{1}{\cos(t + 2\pi)} = \frac{1}{\cos t} = \sec t \\ \csc(t + 2\pi) &= \frac{1}{\sin(t + 2\pi)} = \frac{1}{\sin t} = \csc t \end{aligned}$$

### Period of the Sine, Cosine, Secant, and Cosecant Functions

The sine, cosine, secant, and cosecant functions are periodic functions with a period of  $2\pi$ .

$$\begin{array}{ll} \sin(t + 2\pi) = \sin t & \cos(t + 2\pi) = \cos t \\ \sec(t + 2\pi) = \sec t & \csc(t + 2\pi) = \csc t \end{array}$$



**Figure 2.57**

Although it is true that  $\tan(t + 2\pi) = \tan t$ , the period of the tangent function is not  $2\pi$ . Recall that the period of a function is the *smallest* value of  $p$  for which  $f(t + p) = f(t)$ . Examine **Figure 2.57**, which shows that if you start at any point  $P(x, y)$  on the unit circle and travel a distance of  $\pi$  units around the circumference, you will arrive at the point  $(-x, -y)$ . By definition,

$$\tan t = \frac{y}{x} \quad \text{and} \quad \tan(t + \pi) = \frac{-y}{-x} = \frac{y}{x} = \tan t$$

Thus we know that  $\tan(t + \pi) = \tan t$  for all  $t$ . A similar argument can be used to show that  $\cot(t + \pi) = \cot t$  for all  $t$ .

### Period of the Tangent and Cotangent Functions

The tangent and cotangent functions are periodic functions with a period of  $\pi$ .

$$\tan(t + \pi) = \tan t \quad \text{and} \quad \cot(t + \pi) = \cot t$$

The following theorem illustrates the repetitive nature of each of the trigonometric functions of a real number.

### Periodic Properties of the Trigonometric Functions of a Real Number

For any real number  $t$  and integer  $k$ ,

$$\begin{array}{ll} \sin(t + 2k\pi) = \sin t & \cos(t + 2k\pi) = \cos t \\ \sec(t + 2k\pi) = \sec t & \csc(t + 2k\pi) = \csc t \\ \tan(t + k\pi) = \tan t & \cot(t + k\pi) = \cot t \end{array}$$

## Trigonometric Identities

Recall that any equation that is true for every number in the domain of the equation is an identity. The statement

$$\csc t = \frac{1}{\sin t}, \quad \sin t \neq 0$$

is an identity because the two expressions produce the same result for all values of  $t$  for which both functions are defined.

The **ratio identities** are obtained by writing the tangent and cotangent functions in terms of the sine and cosine functions.

$$\tan t = \frac{y}{x} = \frac{\sin t}{\cos t} \quad \text{and} \quad \cot t = \frac{x}{y} = \frac{\cos t}{\sin t} \quad \bullet x = \cos t \text{ and } y = \sin t$$

The **Pythagorean identities** are based on the equation of a unit circle,  $x^2 + y^2 = 1$ , and on the definitions of the sine and cosine functions.

$$x^2 + y^2 = 1 \\ \cos^2 t + \sin^2 t = 1 \quad \bullet \text{Replace } x \text{ by } \cos t \text{ and } y \text{ by } \sin t.$$

Dividing each term of  $\cos^2 t + \sin^2 t = 1$  by  $\cos^2 t$ , we have

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \quad \bullet \cos t \neq 0 \\ 1 + \tan^2 t = \sec^2 t \quad \bullet \frac{\sin t}{\cos t} = \tan t$$

Dividing each term of  $\cos^2 t + \sin^2 t = 1$  by  $\sin^2 t$ , we have

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \quad \bullet \sin t \neq 0 \\ \cot^2 t + 1 = \csc^2 t \quad \bullet \frac{\cos t}{\sin t} = \cot t$$

Here is a summary of the Fundamental Trigonometric Identities:

### Fundamental Trigonometric Identities

The reciprocal identities are

$$\sin t = \frac{1}{\csc t} \quad \cos t = \frac{1}{\sec t} \quad \tan t = \frac{1}{\cot t}$$

The ratio identities are

$$\tan t = \frac{\sin t}{\cos t} \quad \cot t = \frac{\cos t}{\sin t}$$

The Pythagorean identities are

$$\cos^2 t + \sin^2 t = 1 \quad 1 + \tan^2 t = \sec^2 t \quad 1 + \cot^2 t = \csc^2 t$$

**E X A M P L E 5 »» Use the Unit Circle to Verify an Identity**

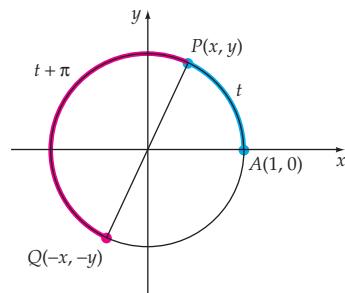
Use the unit circle and the definitions of the trigonometric functions to show that  $\sin(t + \pi) = -\sin t$ .

**Solution**

Sketch the unit circle and let  $P$  be the point on the unit circle such that  $W(t) = P(x, y)$ , as shown in **Figure 2.58**. Draw a diameter from  $P$  and label the endpoint  $Q$ . For any line through the origin, if  $P(x, y)$  is a point on the line, then  $Q(-x, -y)$  is also a point on the line. Because line segment  $PQ$  is a diameter, the length of the arc from  $P$  to  $Q$  is  $\pi$ . Thus the length of the arc from  $A$  through  $P$  to  $Q$  is  $t + \pi$ . Therefore,  $W(t + \pi) = Q(-x, -y)$ . From the definition of  $\sin t$ , we have

$$\sin t = y \quad \text{and} \quad \sin(t + \pi) = -y$$

Thus  $\sin(t + \pi) = -\sin t$ .

**Figure 2.58****» Try Exercise 56, page 167**

Using identities and basic algebra concepts, we can rewrite trigonometric expressions in different forms.

**E X A M P L E 6 »» Simplify a Trigonometric Expression**

Write the expression  $\frac{1}{\sin^2 t} + \frac{1}{\cos^2 t}$  as a single term.

**take note**

Because

$$\begin{aligned}\frac{1}{\sin^2 t \cos^2 t} &= \frac{1}{\sin^2 t} \cdot \frac{1}{\cos^2 t} \\ &= (\csc^2 t)(\sec^2 t)\end{aligned}$$

we could have written the answer to Example 6 in terms of the cosecant and secant functions.

**Solution**

Express each fraction in terms of a common denominator. The common denominator is  $\sin^2 t \cos^2 t$ .

$$\begin{aligned}\frac{1}{\sin^2 t} + \frac{1}{\cos^2 t} &= \frac{1}{\sin^2 t} \cdot \frac{\cos^2 t}{\cos^2 t} + \frac{1}{\cos^2 t} \cdot \frac{\sin^2 t}{\sin^2 t} \\ &= \frac{\cos^2 t + \sin^2 t}{\sin^2 t \cos^2 t} = \frac{1}{\sin^2 t \cos^2 t} \quad \bullet \cos^2 t + \sin^2 t = 1\end{aligned}$$

**» Try Exercise 72, page 167**

**EXAMPLE 7** » Write a Trigonometric Expression in Terms of a Given Function

For  $\frac{\pi}{2} < t < \pi$ , write  $\tan t$  in terms of  $\sin t$ .

**Solution**

Write  $\tan t = \frac{\sin t}{\cos t}$ . Now solve  $\cos^2 t + \sin^2 t = 1$  for  $\cos t$ .

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = 1 - \sin^2 t$$

$$\cos t = \pm\sqrt{1 - \sin^2 t}$$

Because  $\frac{\pi}{2} < t < \pi$ ,  $\cos t$  is negative. Therefore,  $\cos t = -\sqrt{1 - \sin^2 t}$ .

Thus

$$\tan t = \frac{\sin t}{\cos t} = -\frac{\sin t}{\sqrt{1 - \sin^2 t}} \quad \cdot \frac{\pi}{2} < t < \pi$$

## » Try Exercise 78, page 167

**Topics for Discussion**

1. Is  $W(t)$  a number? Explain.
2. Explain why the equation  $\cos^2 t + \sin^2 t = 1$  is called a Pythagorean identity.
3. Is  $f(x) = \cos^3 x$  an even function or an odd function? Explain how you made your decision.
4. Explain how to make use of a unit circle to show that  $\sin(-t) = -\sin t$ .

 **Exercise Set 2.4**

In Exercises 1 to 12, evaluate  $W(t)$  for each given  $t$ .

1.  $t = \frac{\pi}{6}$

2.  $t = \frac{\pi}{4}$

3.  $t = \frac{7\pi}{6}$

4.  $t = \frac{4\pi}{3}$

5.  $t = \frac{5\pi}{3}$

6.  $t = -\frac{\pi}{6}$

7.  $t = \frac{11\pi}{6}$

8.  $t = 0$

9.  $t = \pi$

**10.**  $t = -\frac{7\pi}{4}$

**11.**  $t = -\frac{2\pi}{3}$

**12.**  $t = -\pi$

In Exercises 13 to 22, find the exact value of each function.

13.  $\tan\left(\frac{11\pi}{6}\right)$

14.  $\cot\left(\frac{2\pi}{3}\right)$

15.  $\cos\left(-\frac{2\pi}{3}\right)$

**16.**  $\sec\left(-\frac{5\pi}{6}\right)$

17.  $\csc\left(-\frac{\pi}{3}\right)$

18.  $\tan(12\pi)$

19.  $\sin\left(\frac{3\pi}{2}\right)$

20.  $\cos\left(\frac{7\pi}{3}\right)$

21.  $\sec\left(-\frac{7\pi}{6}\right)$

22.  $\sin\left(-\frac{5\pi}{3}\right)$



In Exercises 23 to 32, use a calculator to find an approximate value of each function. Round your answers to the nearest ten-thousandth.

23.  $\sin 1.22$

24.  $\cos 4.22$

25.  $\csc(-1.05)$

26.  $\sin(-0.55)$

27.  $\tan\left(\frac{11\pi}{12}\right)$

28.  $\cos\left(\frac{2\pi}{5}\right)$

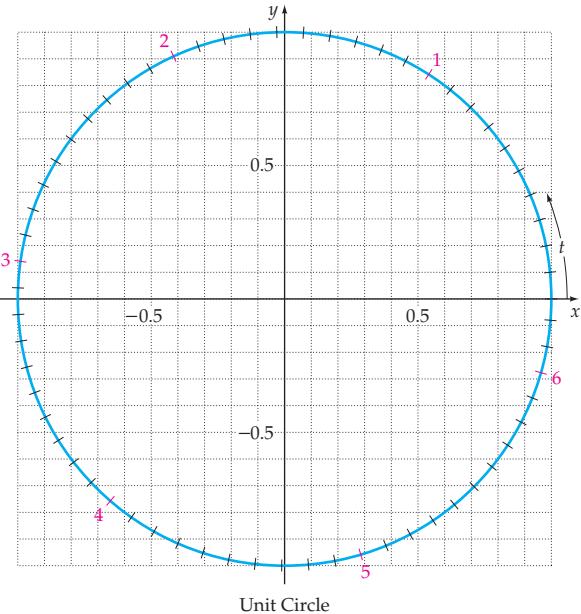
29.  $\cos\left(-\frac{\pi}{5}\right)$

30.  $\csc 8.2$

31.  $\sec 1.55$

32.  $\cot 2.11$

In Exercises 33 to 40, use the unit circle to estimate the following values to the nearest tenth.



33. a.  $\sin 2$       b.  $\cos 2$

34. a.  $\sin 3$       b.  $\cos 3$

35. a.  $\sin 5.4$       b.  $\cos 5.4$

36. a.  $\sin 4.1$       b.  $\cos 4.1$

37. All real numbers  $t$  between 0 and  $2\pi$  for which  $\sin t = 0.4$

38. All real numbers  $t$  between 0 and  $2\pi$  for which  $\cos t = 0.8$

39. All real numbers  $t$  between 0 and  $2\pi$  for which  $\sin t = -0.3$

40. All real numbers  $t$  between 0 and  $2\pi$  for which  $\cos t = -0.7$

In Exercises 41 to 48, determine whether the function is even, odd, or neither.

41.  $f(x) = -4 \sin x$

42.  $f(x) = -2 \cos x$

43.  $G(x) = \sin x + \cos x$

**44.**  $F(x) = \tan x + \sin x$

45.  $S(x) = \frac{\sin x}{x}, x \neq 0$

46.  $C(x) = \frac{\cos x}{x}, x \neq 0$

47.  $v(x) = 2 \sin x \cos x$

48.  $w(x) = x \tan x$

In Exercises 49 to 54, state the period of each function.

49.  $f(t) = \sin t$

50.  $f(t) = \cos t$

51.  $f(t) = \tan t$

52.  $f(t) = \cot t$

53.  $f(t) = \sec t$

54.  $f(t) = \csc t$

In Exercises 55 to 60, use the unit circle to verify each identity.

55.  $\cos(-t) = \cos t$

56.  $\tan(t - \pi) = \tan t$

57.  $\cos(t + \pi) = -\cos t$

58.  $\sin(-t) = -\sin t$

59.  $\sin(t - \pi) = -\sin t$

60.  $\sec(-t) = \sec t$

In Exercises 61 to 76, use trigonometric identities to write each expression in terms of a single trigonometric function or a constant. Answers may vary.

61.  $\tan t \cos t$

62.  $\cot t \sin t$

63.  $\frac{\csc t}{\cot t}$

64.  $\frac{\sec t}{\tan t}$

65.  $1 - \sec^2 t$

66.  $1 - \csc^2 t$

67.  $\tan t - \frac{\sec^2 t}{\tan t}$

68.  $\frac{\csc^2 t}{\cot t} - \cot t$

69.  $\frac{1 - \cos^2 t}{\tan^2 t}$

70.  $\frac{1 - \sin^2 t}{\cot^2 t}$

71.  $\frac{1}{1 - \cos t} + \frac{1}{1 + \cos t}$

72.  $\frac{1}{1 - \sin t} + \frac{1}{1 + \sin t}$

73.  $\frac{\tan t + \cot t}{\tan t}$

74.  $\frac{\csc t - \sin t}{\csc t}$

75.  $\sin^2 t(1 + \cot^2 t)$

76.  $\cos^2 t(1 + \tan^2 t)$

77. Write  $\sin t$  in terms of  $\cos t$ ,  $0 < t < \frac{\pi}{2}$ .

78. Write  $\tan t$  in terms of  $\sec t$ ,  $\frac{3\pi}{2} < t < 2\pi$ .

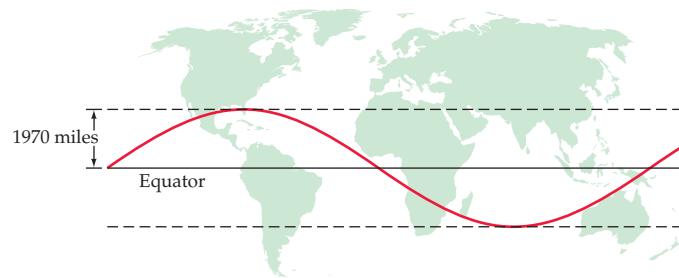
79. Write  $\csc t$  in terms of  $\cot t$ ,  $\frac{\pi}{2} < t < \pi$ .

80. Write  $\sec t$  in terms of  $\tan t$ ,  $\pi < t < \frac{3\pi}{2}$ .

81. **PATH OF A SATELLITE** A satellite is launched into space from Cape Canaveral. The directed distance, in miles, that the satellite is north or south of the equator is

$$d(t) = 1970 \cos\left(\frac{\pi}{64}t\right)$$

where  $t$  is the number of minutes since liftoff. A negative  $d$  value indicates that the satellite is south of the equator.



What distance, to the nearest 10 miles, is the satellite north of the equator 24 minutes after liftoff?

82. **AVERAGE HIGH TEMPERATURE** The average high temperature  $T$ , in degrees Fahrenheit, for Fairbanks, Alaska, is given by

$$T(t) = -41 \cos\left(\frac{\pi}{6}t\right) + 36$$

where  $t$  is the number of months after January 5. Use the formula to estimate (to the nearest tenth of a degree Fahrenheit) the average high temperature in Fairbanks for March 5 and July 20.

In Exercises 83 to 94, perform the indicated operation and simplify.

83.  $\cos t - \frac{1}{\cos t}$

84.  $\tan t + \frac{1}{\tan t}$

85.  $\cot t + \frac{1}{\cot t}$

86.  $\sin t - \frac{1}{\sin t}$

87.  $(1 - \sin t)^2$

88.  $(1 - \cos t)^2$

89.  $(\sin t - \cos t)^2$

90.  $(\sin t + \cos t)^2$

91.  $(1 - \sin t)(1 + \sin t)$

92.  $(1 - \cos t)(1 + \cos t)$

93.  $\frac{\sin t}{1 + \cos t} + \frac{1 + \cos t}{\sin t}$

94.  $\frac{1 - \sin t}{\cos t} - \frac{1}{\tan t + \sec t}$

**In Exercises 95 to 100, factor the expression.**

**95.**  $\cos^2 t - \sin^2 t$

**96.**  $\sec^2 t - \csc^2 t$

**97.**  $\tan^2 t - \tan t - 6$

**98.**  $\cos^2 t + 3 \cos t - 4$

**99.**  $2 \sin^2 t - \sin t - 1$

**100.**  $4 \cos^2 t + 4 \cos t + 1$

**In Exercises 101 to 104, use trigonometric identities to find the value of the function.**

**101.** Given  $\csc t = \sqrt{2}$ ,  $0 < t < \frac{\pi}{2}$ , find  $\cos t$ .

**102.** Given  $\cos t = \frac{1}{2}$ ,  $\frac{3\pi}{2} < t < 2\pi$ , find  $\sin t$ .

**103.** Given  $\sin t = \frac{1}{2}$ ,  $\frac{\pi}{2} < t < \pi$ , find  $\tan t$ .

**104.** Given  $\cot t = \frac{\sqrt{3}}{3}$ ,  $\pi < t < \frac{3\pi}{2}$ , find  $\cos t$ .

**In Exercises 105 to 108, simplify the first expression to the second expression.**

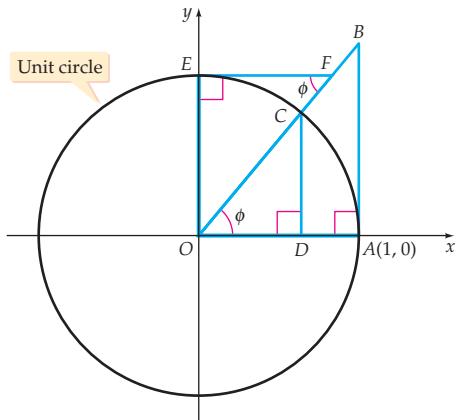
$$105. \frac{\sin^2 t + \cos^2 t}{\sin^2 t}; \csc^2 t$$

**106.**  $\frac{\sin^2 t + \cos^2 t}{\cos^2 t}; \sec^2 t$

**107.**  $(\cos t - 1)(\cos t + 1); -\sin^2 t$

**108.**  $(\sec t - 1)(\sec t + 1); \tan^2 t$

1.   VISUAL INSIGHT

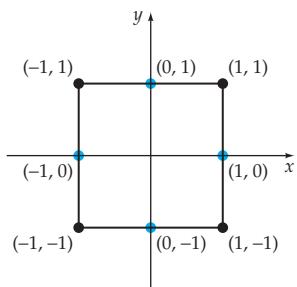


$$OD = \cos \phi \quad DC = \sin \phi$$

Make use of the above circle and similar triangles to explain why

- a.  $AB$  is equal to  $\tan \phi$
  - b.  $EF$  is equal to  $\cot \phi$
  - c.  $OB$  is equal to  $\sec \phi$
  - d.  $OF$  is equal to  $\csc \phi$

2. Consider a square as shown. Start at the point  $(1, 0)$  and travel counterclockwise around the square for a distance  $t$  ( $t \geq 0$ ).



Let  $WSQ(t) = P(x, y)$  be the point on the square determined by traveling counterclockwise a distance of  $t$  units from  $(1, 0)$ . For instance,

$$WSO(0.5) = (1, 0.5)$$

$$WSO(1.75) = (0.25, 1)$$

Find WSQ(4.2) and WSQ(6.4). We define the square sine of  $t$ , denoted by  $\text{ssin } t$ , to be the  $y$ -value of point  $P$ . The square cosine of  $t$ , denoted by  $\text{scos } t$ , is defined to be

the  $x$ -value of point  $P$ . For example,

$$\begin{array}{ll} \text{ssin } 0.4 = 0.4 & \text{scos } 0.4 = 1 \\ \text{scos } 1.2 = 0.8 & \text{scos } 5.3 = -0.7 \end{array}$$

The square tangent of  $t$ , denoted by stan  $t$ , is defined as

$$\text{stan } t = \frac{\text{ssin } t}{\text{scos } t}, \quad \text{scos } t \neq 0$$

Find each of the following.

- a. ssin 3.2
- b. scos 4.4
- c. stan 5.5
- d. ssin 11.2
- e. scos  $-5.2$
- f. stan  $-6.5$

## Section 2.5

- The Graph of the Sine Function
- The Graph of the Cosine Function

# Graphs of the Sine and Cosine Functions

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A10.

**PS1.**  Estimate, to the nearest tenth,  $\sin \frac{3\pi}{4}$ . [2.4]

**PS2.**  Estimate, to the nearest tenth,  $\cos \frac{5\pi}{4}$ . [2.4]

**PS3.** Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = -f(x)$ . [1.4]

**PS4.** Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = f(2x)$ . [1.4]

**PS5.** Simplify:  $\frac{2\pi}{1/3}$

**PS6.** Simplify:  $\frac{2\pi}{2/5}$

## The Graph of the Sine Function

The trigonometric functions can be graphed on a rectangular coordinate system by plotting the points whose coordinates belong to the function. We begin with the graph of the sine function.

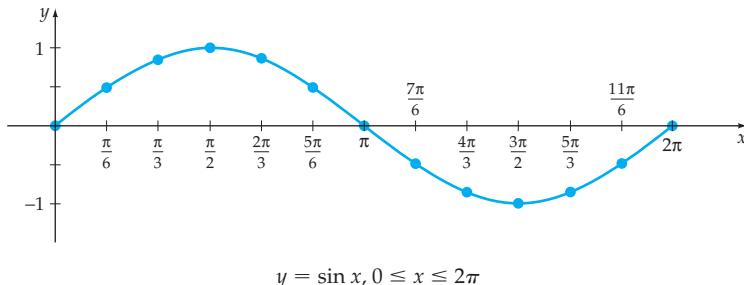
**Table 2.7** lists some ordered pairs  $(x, y)$  of the graph of  $y = \sin x$  for  $0 \leq x \leq 2\pi$ .

**Table 2.7 Ordered Pairs of the Graph of  $y = \sin x$**

| $x$          | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{3}$      | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$     | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{4\pi}{3}$      | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$      | $\frac{11\pi}{6}$ | $2\pi$ |
|--------------|---|-----------------|----------------------|-----------------|----------------------|------------------|-------|------------------|-----------------------|------------------|-----------------------|-------------------|--------|
| $y = \sin x$ | 0 | $\frac{1}{2}$   | $\frac{\sqrt{3}}{2}$ | 1               | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$    | 0     | $-\frac{1}{2}$   | $-\frac{\sqrt{3}}{2}$ | -1               | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$    | 0      |

In **Figure 2.59**, the points from the table are plotted and a smooth curve is drawn through the points. We could use decimal approximations for  $\pi$  on the  $x$ -axis, but it is more convenient to simply label the tick marks on the  $x$ -axis in terms of  $\pi$ . Note:

The  $y$ -value  $\frac{\sqrt{3}}{2} \approx 0.87$ .

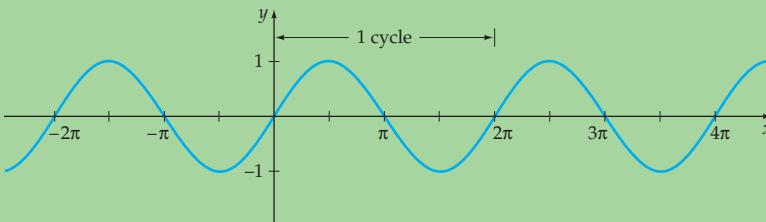


$$y = \sin x, 0 \leq x \leq 2\pi$$

**Figure 2.59**

Because the domain of the sine function is the real numbers and the period is  $2\pi$ , the graph of  $y = \sin x$  is drawn by repeating the portion shown in **Figure 2.59**. Any part of the graph that corresponds to one period ( $2\pi$ ) is one **cycle** of the graph of  $y = \sin x$  (see **Figure 2.60**).

### The Graph of $y = \sin x$



**Figure 2.60**

### Basic Properties

- **Domain:** All real numbers
- **Range:**  $\{y \mid -1 \leq y \leq 1\}$
- **Period:**  $2\pi$
- **Symmetry:** With respect to the origin
- **$x$ -intercepts:** At multiples of  $\pi$

The maximum value  $M$  reached by  $\sin x$  is 1, and the minimum value  $m$  is -1. The **amplitude** of the graph of  $y = \sin x$  is given by

$$\text{Amplitude} = \frac{1}{2}(M - m)$$

**QUESTION** What is the amplitude of  $y = \sin x$ ?

Recall that the graph of  $y = a \cdot f(x)$  is obtained by *stretching* ( $|a| > 1$ ) or *shrinking* ( $0 < |a| < 1$ ) the graph of  $y = f(x)$ . **Figure 2.61** shows the graph of  $y = 3 \sin x$

---

**ANSWER** Amplitude  $= \frac{1}{2}(M - m) = \frac{1}{2}[1 - (-1)] = \frac{1}{2}(2) = 1$

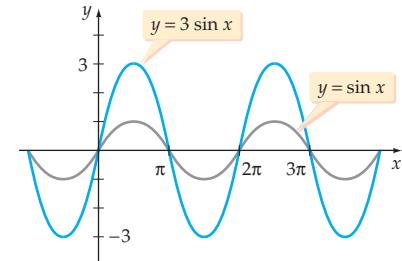
that was drawn by stretching the graph of  $y = \sin x$ . The amplitude of  $y = 3 \sin x$  is 3 because

$$\text{Amplitude} = \frac{1}{2}(M - m) = \frac{1}{2}[3 - (-3)] = 3$$

Note that for  $y = \sin x$  and  $y = 3 \sin x$ , the amplitude of the graph is the coefficient of  $\sin x$ . This suggests the following theorem.

### take note

The amplitude is defined to be half the difference between the maximum height and the minimum height. It may not be equal to the maximum height. For example, the graph of  $y = 4 + \sin x$  has a maximum height of 5 and an amplitude of 1.



**Figure 2.61**

### Amplitude of $y = a \sin x$

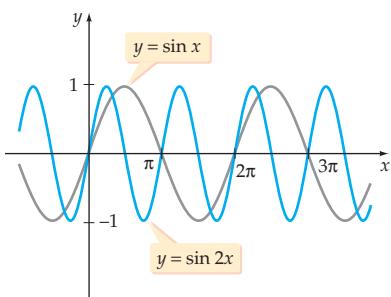
The amplitude of  $y = a \sin x$  is  $|a|$ .

### EXAMPLE 1 » Graph $y = a \sin x$

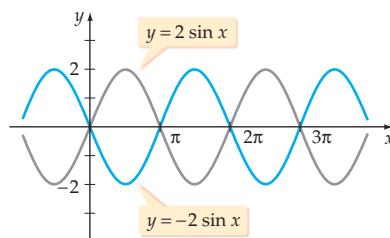
Graph:  $y = -2 \sin x$

#### Solution

The amplitude of  $y = -2 \sin x$  is 2. The graph of  $y = -f(x)$  is a reflection across the  $x$ -axis of  $y = f(x)$ . Thus the graph of  $y = -2 \sin x$  is a reflection across the  $x$ -axis of  $y = 2 \sin x$ . See the following figure.



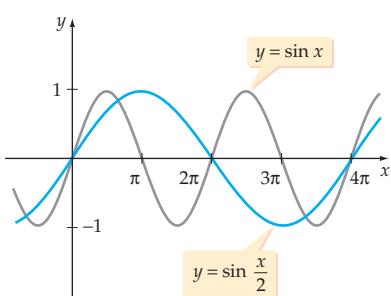
**Figure 2.62**



### » Try Exercise 22, page 177

The graphs of  $y = \sin x$  and  $y = \sin 2x$  are shown in **Figure 2.62**. Because one cycle of the graph of  $y = \sin 2x$  is completed in an interval of length  $\pi$ , the period of  $y = \sin 2x$  is  $\pi$ .

The graphs of  $y = \sin x$  and  $y = \sin \frac{x}{2}$  are shown in **Figure 2.63**. Because one cycle of the graph of  $y = \sin \frac{x}{2}$  is completed in an interval of length  $4\pi$ , the period of  $y = \sin \frac{x}{2}$  is  $4\pi$ .



**Figure 2.63**

Generalizing the last two examples, one cycle of  $y = \sin bx$ ,  $b > 0$ , is completed as  $bx$  varies from 0 to  $2\pi$ . Therefore,

$$0 \leq bx \leq 2\pi$$

$$0 \leq x \leq \frac{2\pi}{b}$$

The length of the interval,  $\frac{2\pi}{b}$ , is the period of  $y = \sin bx$ . Now we consider the case when the coefficient of  $x$  is negative. If  $b > 0$ , then using the fact that the sine function is an odd function, we have  $y = \sin(-bx) = -\sin bx$ , and thus the period is still  $\frac{2\pi}{b}$ . This gives the following theorem.

### Period of $y = \sin bx$

The period of  $y = \sin bx$  is  $\frac{2\pi}{|b|}$ .

**Table 2.8** gives the amplitude and period of several sine functions.

**Table 2.8**

| Function  | $y = a \sin bx$    | $y = 3 \sin(-2x)$      | $y = -\sin \frac{x}{3}$   | $y = -2 \sin \frac{3x}{4}$          |
|-----------|--------------------|------------------------|---------------------------|-------------------------------------|
| Amplitude | $ a $              | $ 3  = 3$              | $ -1  = 1$                | $ -2  = 2$                          |
| Period    | $\frac{2\pi}{ b }$ | $\frac{2\pi}{2} = \pi$ | $\frac{2\pi}{1/3} = 6\pi$ | $\frac{2\pi}{3/4} = \frac{8\pi}{3}$ |

### EXAMPLE 2 » Graph $y = \sin bx$

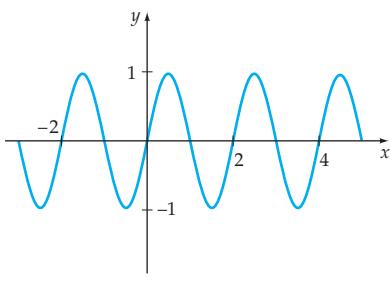
Graph:  $y = \sin \pi x$

#### Solution

$$\text{Amplitude} = 1 \quad \text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{|\pi|} = 2 \quad \bullet b = \pi$$

The graph is sketched in **Figure 2.64**.

» Try Exercise 32, page 177



**Figure 2.64**

**Figure 2.65** shows the graph of  $y = a \sin bx$  for both  $a$  and  $b$  positive. Note from the graph the following properties of the function  $y = a \sin bx$ .

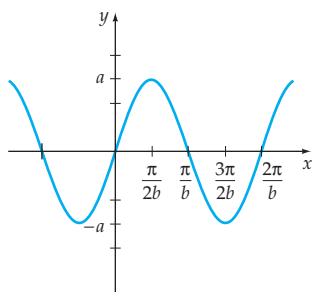


Figure 2.65

- The amplitude is  $a$ .
- The period is  $\frac{2\pi}{b}$ .
- For  $0 \leq x \leq \frac{2\pi}{b}$ , the zeros are  $0, \frac{\pi}{b}$ , and  $\frac{2\pi}{b}$ .
- The maximum value is  $a$  when  $x = \frac{\pi}{2b}$ .
- The minimum value is  $-a$  when  $x = \frac{3\pi}{2b}$ .
- If  $a < 0$ , the graph is reflected across the  $x$ -axis.

### EXAMPLE 3 » Graph $y = a \sin bx$

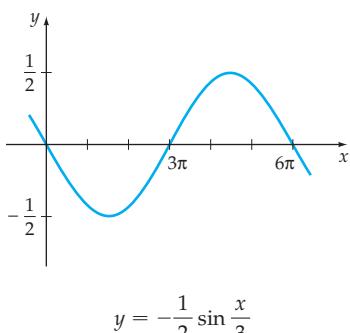


Figure 2.66

$$\text{Graph: } y = -\frac{1}{2} \sin \frac{x}{3}$$

#### Solution

$$\text{Amplitude} = \left| -\frac{1}{2} \right| = \frac{1}{2} \quad \text{Period} = \frac{2\pi}{|1/3|} = 6\pi \quad \bullet b = \frac{1}{3}$$

The zeros in the interval  $0 \leq x \leq 6\pi$  are  $0, \frac{\pi}{1/3} = 3\pi$ , and  $\frac{2\pi}{1/3} = 6\pi$ , so the graph has  $x$ -intercepts at  $(0, 0)$ ,  $(3\pi, 0)$ , and  $(6\pi, 0)$ . Because  $-\frac{1}{2} < 0$ , the graph is the graph of  $y = \frac{1}{2} \sin \frac{x}{3}$  reflected across the  $x$ -axis, as shown in **Figure 2.66**.

» Try Exercise 40, page 178

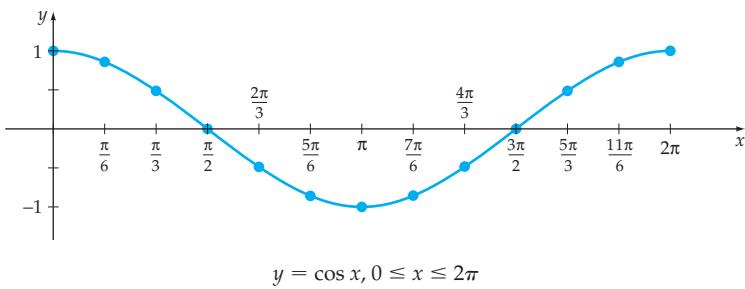
### ■ The Graph of the Cosine Function

**Table 2.9** lists some ordered pairs of the graph of  $y = \cos x$  for  $0 \leq x \leq 2\pi$ .

**Table 2.9 Ordered Pairs of the Graph of  $y = \cos x$**

| $x$          | 0 | $\frac{\pi}{6}$      | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$      | $\pi$ | $\frac{7\pi}{6}$      | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{11\pi}{6}$    | $2\pi$ |
|--------------|---|----------------------|-----------------|-----------------|------------------|-----------------------|-------|-----------------------|------------------|------------------|------------------|----------------------|--------|
| $y = \cos x$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$   | 0               | $-\frac{1}{2}$   | $-\frac{\sqrt{3}}{2}$ | -1    | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$   | 0                | $\frac{1}{2}$    | $\frac{\sqrt{3}}{2}$ | 0      |

In **Figure 2.67** on page 174, the points from the table are plotted and a smooth curve is drawn through the points.

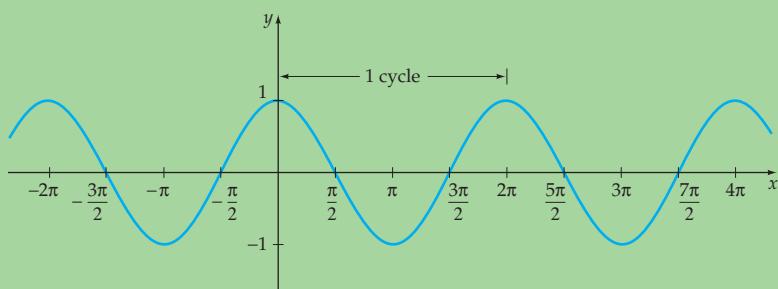


$$y = \cos x, 0 \leq x \leq 2\pi$$

**Figure 2.67**

Because the domain of  $y = \cos x$  is the real numbers and the period is  $2\pi$ , the graph of  $y = \cos x$  is drawn by repeating the portion shown in **Figure 2.67**. Any part of the graph corresponding to one period ( $2\pi$ ) is one cycle of  $y = \cos x$  (see **Figure 2.68**).

### The Graph of $y = \cos x$



**Figure 2.68**

#### Basic Properties

- **Domain:** All real numbers
- **Range:**  $\{y | -1 \leq y \leq 1\}$
- **Period:**  $2\pi$
- **Symmetry:** With respect to the  $y$ -axis
- **$x$ -intercepts:** At odd multiples of  $\frac{\pi}{2}$

The following two theorems concerning cosine functions can be developed using methods that are analogous to those we used to determine the amplitude and period of a sine function.

#### Amplitude of $y = a \cos x$

The amplitude of  $y = a \cos x$  is  $|a|$ .

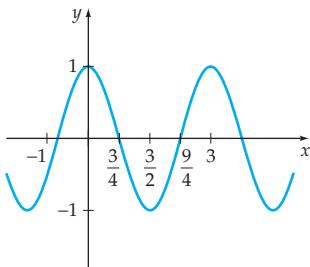
#### Period of $y = \cos bx$

The period of  $y = \cos bx$  is  $\frac{2\pi}{|b|}$ .

**Table 2.10** gives the amplitude and period of some cosine functions.

**Table 2.10**

|                  |                    |                  |                            |
|------------------|--------------------|------------------|----------------------------|
| <b>Function</b>  | $y = a \cos bx$    | $y = 2 \cos 3x$  | $y = -3 \cos \frac{2x}{3}$ |
| <b>Amplitude</b> | $ a $              | $ 2  = 2$        | $ -3  = 3$                 |
| <b>Period</b>    | $\frac{2\pi}{ b }$ | $\frac{2\pi}{3}$ | $\frac{2\pi}{2/3} = 3\pi$  |



**Figure 2.69**

### EXAMPLE 4 » Graph $y = \cos bx$

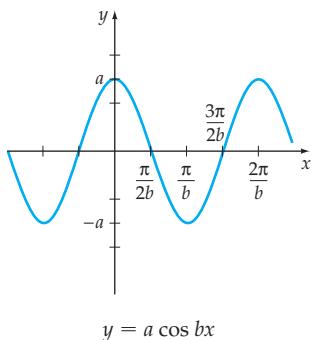
Graph:  $y = \cos \frac{2\pi}{3} x$

#### Solution

$$\text{Amplitude} = 1 \quad \text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{2\pi/3} = 3 \quad \bullet b = \frac{2\pi}{3}$$

The graph is shown in **Figure 2.69**.

### » Try Exercise 34, page 177



**Figure 2.70**

**Figure 2.70** shows the graph of  $y = a \cos bx$  for both  $a$  and  $b$  positive. Note from the graph the following properties of the function  $y = a \cos bx$ .

- The amplitude is  $a$ .
- The period is  $\frac{2\pi}{b}$ .
- For  $0 \leq x \leq \frac{2\pi}{b}$ , the zeros are  $\frac{\pi}{2b}$  and  $\frac{3\pi}{2b}$ .
- The maximum value is  $a$  when  $x = 0$ .
- The minimum value is  $-a$  when  $x = \frac{\pi}{b}$ .
- If  $a < 0$ , then the graph is reflected across the  $x$ -axis.

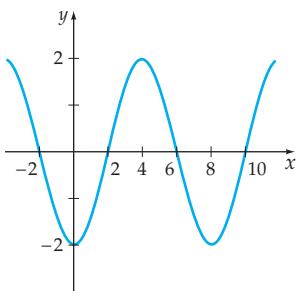
### EXAMPLE 5 » Graph a Cosine Function

Graph:  $y = -2 \cos \frac{\pi x}{4}$

#### Solution

$$\text{Amplitude} = |-2| = 2 \quad \text{Period} = \frac{2\pi}{|\pi/4|} = 8 \quad \bullet b = \frac{\pi}{4}$$

Continued ►



$$y = -2 \cos \frac{\pi x}{4}$$

Figure 2.71

The zeros in the interval  $0 \leq x \leq 8$  are  $\frac{\pi}{2\pi/4} = 2$  and  $\frac{3\pi}{2\pi/4} = 6$ , so the graph has  $x$ -intercepts at  $(2, 0)$  and  $(6, 0)$ . Because  $-2 < 0$ , the graph is the graph of  $y = 2 \cos \frac{\pi x}{4}$  reflected across the  $x$ -axis, as shown in **Figure 2.71**.

» Try Exercise 48, page 178

### EXAMPLE 6 » Graph the Absolute Value of the Cosine Function

Graph  $y = |\cos x|$ , where  $0 \leq x \leq 2\pi$ .

#### Solution

Because  $|\cos x| \geq 0$ , the graph of  $y = |\cos x|$  is drawn by reflecting the negative portion of the graph of  $y = \cos x$  across the  $x$ -axis. The graph is the one shown in purple and light blue in **Figure 2.72**.

» Try Exercise 54, page 178

In Example 7 we determine an equation for a given graph.

### EXAMPLE 7 » Find an Equation of a Graph

The graph at the right shows one cycle of the graph of a sine or cosine function. Find an equation for the graph.

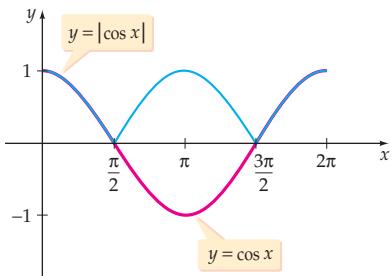
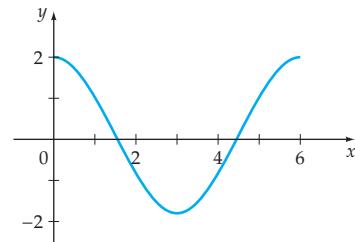


Figure 2.72



#### Solution

Because the graph obtains its maximum value at  $x = 0$ , start with an equation of the form  $y = a \cos bx$ . The graph completes one cycle in 6 units.

Thus the period is 6. Use the equation  $6 = \frac{2\pi}{|b|}$  to determine  $b$ .

$$6 = \frac{2\pi}{|b|}$$

$6|b| = 2\pi$  • Multiply each side by  $|b|$ .

$$|b| = \frac{\pi}{3}$$
 • Divide each side by 6.

$$b = \pm \frac{\pi}{3}$$

We can use either  $\frac{\pi}{3}$  or  $-\frac{\pi}{3}$  as the  $b$  value. The graph has a maximum height of 2 and a minimum height of -2. Thus the amplitude is  $a = 2$ .

Substituting 2 for  $a$  and  $\frac{\pi}{3}$  for  $b$  in  $y = a \cos bx$  produces  $y = 2 \cos \frac{\pi}{3}x$ .

» Try Exercise 60, page 178



### Topics for Discussion

- Is the graph of  $f(x) = |\sin x|$  the same as the graph of  $y = \sin|x|$ ? Explain.
- Explain how the graph of  $y = \cos 2x$  differs from the graph of  $y = \cos x$ .
- Does the graph of  $y = \sin(-2x)$  have the same period as the graph of  $y = \sin 2x$ ? Explain.
- The function  $h(x) = a \sin bt$  has an amplitude of 3 and a period of 4. What are the possible values of  $a$ ? What are the possible values of  $b$ ?

## Exercise Set 2.5

In Exercises 1 to 18, state the amplitude and period of the function defined by each equation.

1.  $y = 2 \sin x$

2.  $y = -\frac{1}{2} \sin x$

3.  $y = \sin 2x$

4.  $y = \sin \frac{2x}{3}$

5.  $y = \frac{1}{2} \sin 2\pi x$

6.  $y = 2 \sin \frac{\pi x}{3}$

7.  $y = -2 \sin \frac{x}{2}$

8.  $y = -\frac{1}{2} \sin \frac{x}{2}$

9.  $y = \frac{1}{2} \cos x$

10.  $y = -3 \cos x$

11.  $y = \cos \frac{x}{4}$

12.  $y = \cos 3x$

13.  $y = 2 \cos \frac{\pi x}{3}$

14.  $y = \frac{1}{2} \cos 2\pi x$

15.  $y = -3 \cos \frac{2x}{3}$

16.  $y = \frac{3}{4} \cos 4x$

17.  $y = 4.7 \sin 0.8\pi t$

18.  $y = 2.3 \cos 0.005\pi t$

In Exercises 19 to 56, graph one full period of the function defined by each equation.

19.  $y = \frac{1}{2} \sin x$

20.  $y = \frac{3}{2} \cos x$

» 22.  $y = -\frac{3}{2} \sin x$

23.  $y = -\frac{7}{2} \cos x$

24.  $y = 3 \sin x$

25.  $y = -4 \sin x$

26.  $y = -5 \cos x$

27.  $y = \cos 3x$

28.  $y = \sin 4x$

29.  $y = \sin \frac{3x}{2}$

30.  $y = \cos \pi x$

» 32.  $y = \sin \frac{3\pi}{4}x$

31.  $y = \cos \frac{\pi}{2}x$

33.  $y = \sin 2\pi x$

» 34.  $y = \cos 3\pi x$

35.  $y = 4 \cos \frac{x}{2}$

36.  $y = 2 \cos \frac{3x}{4}$

37.  $y = -2 \cos \frac{x}{3}$

38.  $y = -\frac{4}{3} \cos 3x$

39.  $y = 2 \sin \pi x$

40.  $\gg y = \frac{1}{2} \sin \frac{\pi x}{3}$

41.  $y = \frac{3}{2} \cos \frac{\pi x}{2}$

42.  $y = \cos \frac{\pi x}{3}$

43.  $y = 4 \sin \frac{2\pi x}{3}$

44.  $y = 3 \cos \frac{3\pi x}{2}$

45.  $y = 2 \cos 2x$

46.  $y = \frac{1}{2} \sin 2.5x$

47.  $y = -2 \sin 1.5x$

48.  $\gg y = -\frac{3}{4} \cos 5x$

49.  $y = \left| 2 \sin \frac{x}{2} \right|$

50.  $y = \left| \frac{1}{2} \sin 3x \right|$

51.  $y = |-2 \cos 3x|$

52.  $y = \left| -\frac{1}{2} \cos \frac{x}{2} \right|$

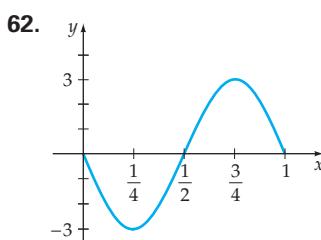
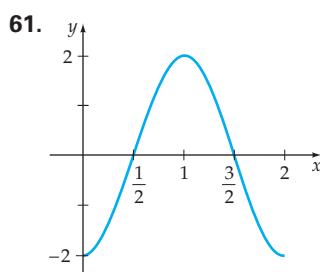
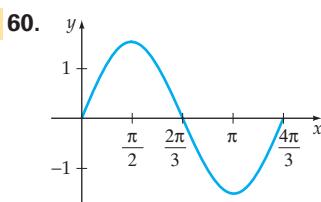
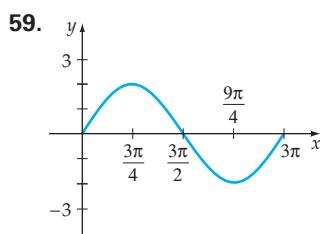
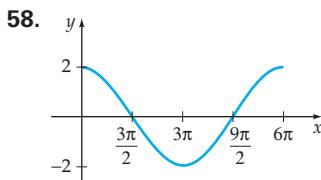
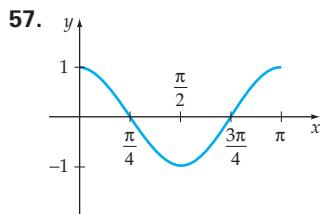
53.  $y = -\left| 2 \sin \frac{x}{3} \right|$

54.  $\gg y = -\left| 3 \sin \frac{2x}{3} \right|$

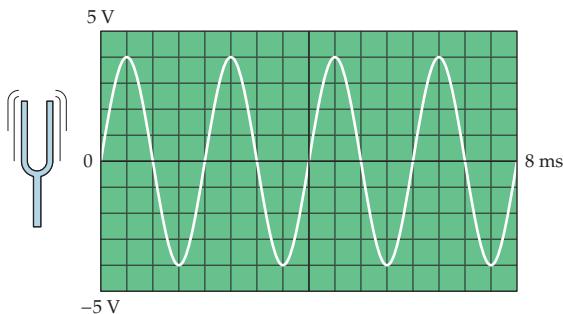
55.  $y = -|3 \cos \pi x|$

56.  $y = -\left| 2 \cos \frac{\pi x}{2} \right|$

In Exercises 57 to 62, one cycle of the graph of a sine or cosine function is shown. Find an equation of each graph.

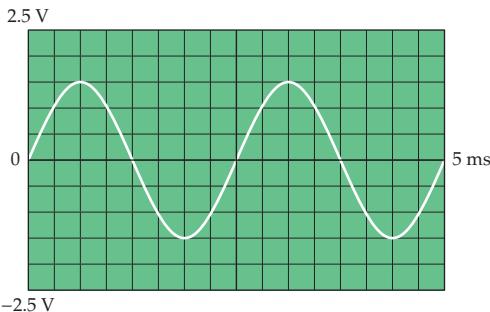


63. **MODEL A SOUND WAVE** The following oscilloscope screen displays the signal generated by the sound of a tuning fork during a time span of 8 milliseconds.



- Write an equation of the form  $V = a \sin bt$  that models the signal, where  $V$  is in volts and  $t$  is in milliseconds.
- The **frequency of a sound wave** is defined as the reciprocal of the period of the sound wave. What is the frequency of the sound wave that produced the signal shown above? State your answer in cycles per millisecond.

64. **MODEL A SOUND WAVE** The following oscilloscope screen displays the signal generated by the sound of a tuning fork during a time span of 5 milliseconds.



- Write an equation of the form  $V = a \sin bt$  that models the signal, where  $V$  is in volts and  $t$  is in milliseconds.
- What is the frequency of the sound wave that produced the signal shown above? (Hint: See Exercise 63b). State your answer in cycles per millisecond.

65. Sketch the graph of  $y = 2 \sin \frac{2x}{3}$ ,  $-3\pi \leq x \leq 6\pi$ .

66. Sketch the graph of  $y = -3 \cos \frac{3x}{4}$ ,  $-2\pi \leq x \leq 4\pi$ .

67. Sketch the graphs of

$$y_1 = 2 \cos \frac{x}{2} \quad \text{and} \quad y_2 = 2 \cos x$$

on the same set of axes for  $-2\pi \leq x \leq 4\pi$ .

- 68.** Sketch the graphs of

$$y_1 = \sin 3\pi x \quad \text{and} \quad y_2 = \sin \frac{\pi x}{3}$$

on the same set of axes for  $-2 \leq x \leq 4$ .



**In Exercises 69 to 72, use a graphing utility to graph each function.**

**69.**  $y = \frac{1}{2} x \sin x$

**70.**  $y = \frac{1}{2}x + \sin x$

**71.**  $y = -x \cos x$

**72.**  $y = -x + \cos x$

- 73.**  Graph  $y = e^{\sin x}$ , where  $e \approx 2.71828$ . What is the maximum value of  $e^{\sin x}$ ? What is the minimum value of  $e^{\sin x}$ ? Is the function defined by  $y = e^{\sin x}$  a periodic function? If so, what is the period?

- 74.**  Graph  $y = e^{\cos x}$ , where  $e \approx 2.71828$ . What is the maximum value of  $e^{\cos x}$ ? What is the minimum value of  $e^{\cos x}$ ? Is the function defined by  $y = e^{\cos x}$  a periodic function? If so, what is the period?

**In Exercises 75 to 77, write an equation for a sine function using the given information.**

**75.** Amplitude = 2; period =  $3\pi$

**76.** Amplitude = 4; period = 2

**77.** Amplitude = 2.5; period = 3.2

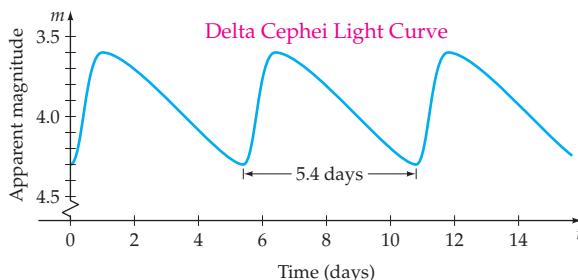
**In Exercises 78 to 80, write an equation for a cosine function using the given information.**

**78.** Amplitude = 3; period =  $\frac{\pi}{2}$

79. Amplitude = 3; period = 2.5

**80.** Amplitude  $\equiv 4.2$ ; period  $\equiv 1$

- 1.**  **CEPHEID VARIABLE STARS AND THE PERIOD-LUMINOSITY RELATIONSHIP** A *variable star* is a star whose brightness increases and decreases in a periodic fashion. Variable stars known as *Cepheid variables* have very regular periods. The *light curve* below is for the star Delta Cephei. It shows how Delta Cephei's brightness varies every 5.4 days.



- a. During the first decade of the 1900s, the astronomer Henrietta Leavitt (1868–1921) discovered the *period-luminosity relationship* for Cepheid variable stars. Write a few sentences that describe this relationship.
  
  - b. What is the major application of Leavitt's discovery? (Note: The Internet is a good source of information on the period-luminosity relationship.)

## Section 2.6

- The Graph of the Tangent Function
- The Graph of the Cotangent Function
- The Graph of the Cosecant Function
- The Graph of the Secant Function

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A11.

**PS1.**  Estimate, to the nearest tenth,  $\tan \frac{\pi}{3}$ . [2.2]

**PS2.**  Estimate, to the nearest tenth,  $\cot \frac{\pi}{3}$ . [2.2]

**PS3.** Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = 2f(x)$ . [1.4]

**PS4.** Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = f(x - 2) + 3$ . [1.4]

**PS5.** Simplify:  $\frac{\pi}{1/2}$

**PS6.** Simplify:  $\left| -\frac{3}{4} \right|$

### The Graph of the Tangent Function

**Table 2.11** lists some ordered pairs  $(x, y)$  of the graph of  $y = \tan x$  for  $0 \leq x < \frac{\pi}{2}$ .

**Table 2.11 Ordered Pairs of the Graph of  $y = \tan x$ ,  $0 \leq x < \frac{\pi}{2}$**

| $x$          | 0 | $\frac{\pi}{12}$ | $\frac{\pi}{6}$                   | $\frac{\pi}{4}$ | $\frac{\pi}{3}$         | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ |
|--------------|---|------------------|-----------------------------------|-----------------|-------------------------|-------------------|-----------------|
| $y = \tan x$ | 0 | $\approx 0.27$   | $\frac{\sqrt{3}}{3} \approx 0.58$ | 1               | $\sqrt{3} \approx 1.73$ | $\approx 3.7$     | undefined       |

In **Figure 2.73**, the points from the table are plotted and a smooth curve is drawn

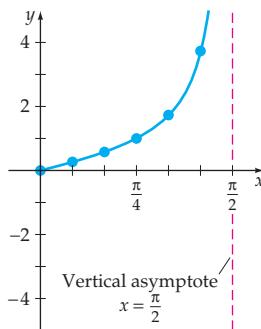
through the points. Notice that as  $x$  increases on  $\left[0, \frac{\pi}{2}\right)$ ,  $y = \tan x$  increases on  $[0, \infty)$ . The  $y$  values increase slowly at first and then more rapidly as  $x \rightarrow \frac{\pi}{2}$  from the left. The line given by  $x = \frac{\pi}{2}$  is a vertical asymptote of the graph.

Because the tangent function is an odd function, its graph is symmetric with respect to the origin. We have used this property to produce the graph of  $y = \tan x$

for  $-\frac{\pi}{2} < x < \frac{\pi}{2}$  shown in **Figure 2.74**.

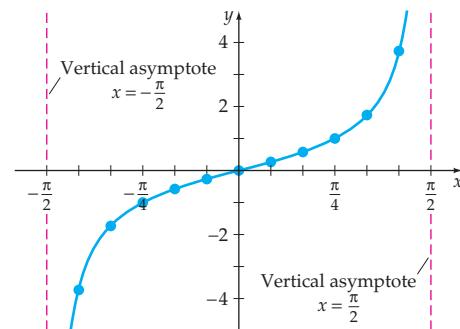
### TO REVIEW

Vertical Asymptote  
See page 38.



$$y = \tan x, 0 \leq x \leq \frac{\pi}{2}$$

Figure 2.73



$$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Figure 2.74

Recall from Section 2.4 that the period of  $y = \tan x$  is  $\pi$ . Thus a complete graph of  $y = \tan x$  can be produced by replicating the graph in **Figure 2.74** to the right and left, as shown in **Figure 2.75**.

### The Graph of $y = \tan x$

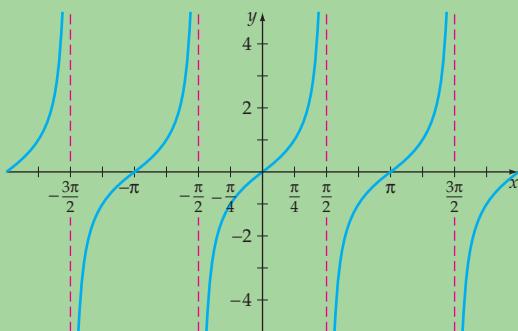


Figure 2.75

#### Basic Properties

- **Domain:** All real numbers except odd multiples of  $\frac{\pi}{2}$
- **Range:** All real numbers
- **Period:**  $\pi$
- **Symmetry:** With respect to the origin
- **$x$ -intercepts:** At multiples of  $\pi$
- **Vertical asymptotes:** At odd multiples of  $\frac{\pi}{2}$
- **Key points:**  $\left(\frac{\pi}{4} + k\pi, 1\right)$  and  $\left(-\frac{\pi}{4} + k\pi, -1\right)$  when  $k$  is an integer.

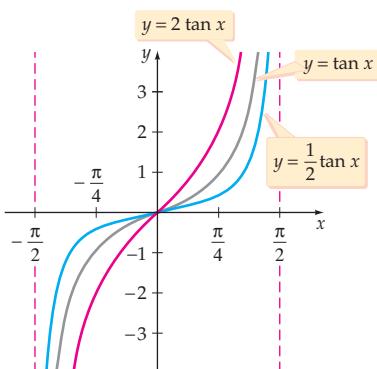


Figure 2.76

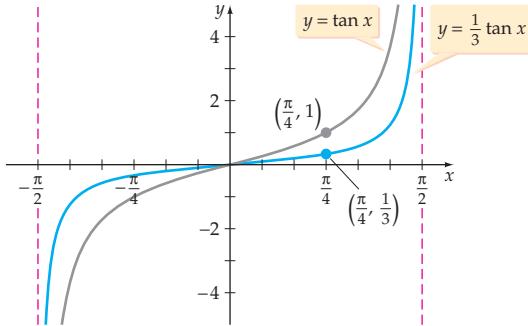
Because the tangent function is unbounded, it has no amplitude. The graph of  $y = a \tan x$  can be drawn by stretching ( $|a| > 1$ ) or shrinking ( $|a| < 1$ ) the graph of  $y = \tan x$ . **Figure 2.76** shows the graph of three tangent functions. Because the point  $\left(\frac{\pi}{4}, 1\right)$  is on the graph of  $y = \tan x$ , we can see that the point  $\left(\frac{\pi}{4}, a\right)$  must be on the graph of  $y = a \tan x$ .

**EXAMPLE 1** » Graph  $y = a \tan x$ 

Graph one period of the function  $y = \frac{1}{3} \tan x$ .

**Solution**

The graph of  $y = \frac{1}{3} \tan x$  can be produced by shrinking the graph of  $y = \tan x$  toward the  $x$ -axis by a factor of  $\frac{1}{3}$ . The point  $(\frac{\pi}{4}, 1)$  is on the graph of  $y = \tan x$ . Thus we know that  $(\frac{\pi}{4}, \frac{1}{3})$  is on the graph of  $y = \frac{1}{3} \tan x$ .



## » Try Exercise 24, page 189

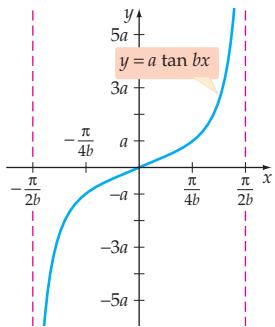
The period of  $y = \tan x$  is  $\pi$ , and the graph completes one cycle on the interval  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ . The period of  $y = \tan bx$  ( $b > 0$ ) is  $\frac{\pi}{b}$ . The graph of  $y = \tan bx$  completes one cycle on the interval  $(-\frac{\pi}{2b}, \frac{\pi}{2b})$ .

**Period of  $y = \tan bx$** 

The period of  $y = \tan bx$  is  $\frac{\pi}{|b|}$ .

**QUESTION** What is the period of the graph of  $y = \tan \pi x$ ?

**ANSWER**  $\frac{\pi}{|b|} = \frac{\pi}{\pi} = 1$

**Figure 2.77**

**Figure 2.77** shows one cycle of the graph of  $y = a \tan bx$  for both  $a$  and  $b$  positive. Note from the graph the following properties of the function  $y = a \tan bx$ .

- The period is  $\frac{\pi}{b}$ .
- $x = 0$  is a zero.
- The graph passes through  $\left(-\frac{\pi}{4b}, -a\right)$  and  $\left(\frac{\pi}{4b}, a\right)$ .
- If  $a < 0$ , the graph is reflected across the  $x$ -axis.

In Example 2 we graph a function of the form  $y = a \tan bx$ .

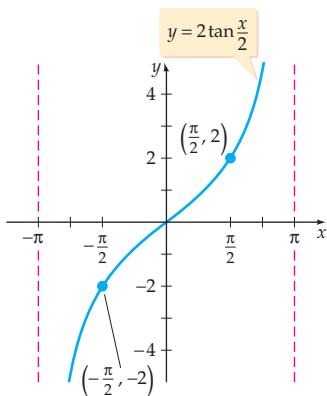
### EXAMPLE 2 »» Graph $y = a \tan bx$

Graph one period of the function  $y = 2 \tan \frac{x}{2}$ .

#### Solution

The function  $y = 2 \tan \frac{x}{2}$  is of the form  $y = a \tan bx$ , with  $a = 2$  and  $b = \frac{1}{2}$ .

The period is given by  $\frac{\pi}{|b|} = \frac{\pi}{1/2} = 2\pi$ . Thus one period of the graph will be displayed on any interval of length  $2\pi$ . In the graph below, we have chosen to sketch the function over the interval  $-\pi < x < \pi$ . The graph passes through  $\left(-\frac{\pi}{4b}, -a\right) = \left(-\frac{\pi}{4(1/2)}, -2\right) = \left(-\frac{\pi}{2}, -2\right)$  and  $\left(\frac{\pi}{4b}, a\right) = \left(\frac{\pi}{4(1/2)}, 2\right) = \left(\frac{\pi}{2}, 2\right)$ . The function has a zero at  $x = 0$ , so the graph has an  $x$ -intercept at  $(0, 0)$ .

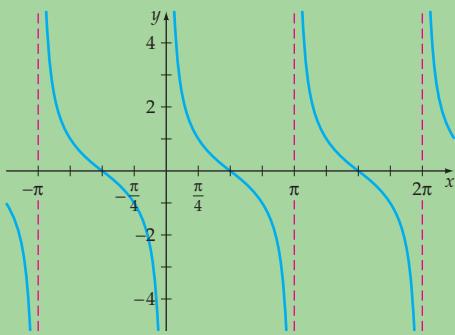


»» Try Exercise 32, page 190

## The Graph of the Cotangent Function

A graph of the cotangent function  $y = \cot x$  is shown in **Figure 2.78**. Notice that its graph is similar to the graph of  $y = \tan x$  in that it has a period of  $\pi$  and is symmetric with respect to the origin.

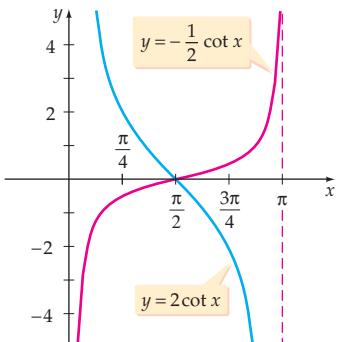
**The Graph of  $y = \cot x$**



**Figure 2.78**

### Basic Properties

- **Domain:** All real numbers except multiples of  $\pi$
- **Range:** All real numbers
- **Period:**  $\pi$
- **Symmetry:** With respect to the origin
- **x-intercepts:** At odd multiples of  $\frac{\pi}{2}$
- **Vertical asymptotes:** At multiples of  $\pi$
- **Key points:**  $\left(\frac{\pi}{4} + k\pi, 1\right)$  and  $\left(-\frac{\pi}{4} + k\pi, -1\right)$  where  $k$  is an integer



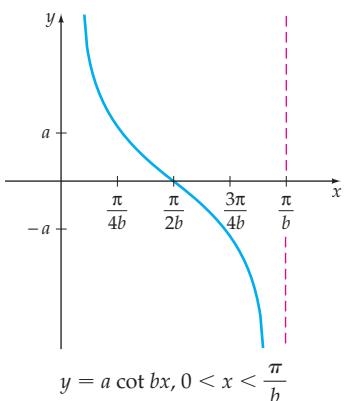
**Figure 2.79**

The graph of  $y = a \cot x$  is drawn by stretching ( $|a| > 1$ ) or shrinking ( $|a| < 1$ ) the graph of  $y = \cot x$ . The graph is reflected across the  $x$ -axis when  $a < 0$ . **Figure 2.79** shows the graphs of two cotangent functions.

The period of  $y = \cot x$  is  $\pi$ , and the period of  $y = \cot bx$  is  $\frac{\pi}{|b|}$ . One cycle of the graph of  $y = \cot bx$  is completed on the interval  $\left(0, \frac{\pi}{b}\right)$ .

### Period of $y = \cot bx$

The period of  $y = \cot bx$  is  $\frac{\pi}{|b|}$ .



**Figure 2.80**

**Figure 2.80** shows one cycle of the graph of  $y = a \cot bx$  for both  $a$  and  $b$  positive. Note from the graph the following properties of the function  $y = a \cot bx$ .

- The period is  $\frac{\pi}{b}$ .
- $x = \frac{\pi}{2b}$  is a zero.
- The graph passes through  $\left(\frac{\pi}{4b}, a\right)$  and  $\left(\frac{3\pi}{4b}, -a\right)$ .
- If  $a < 0$ , the graph is reflected across the  $x$ -axis.

In Example 3 we graph a function of the form  $y = a \cot bx$ .

### EXAMPLE 3 » Graph $y = a \cot bx$

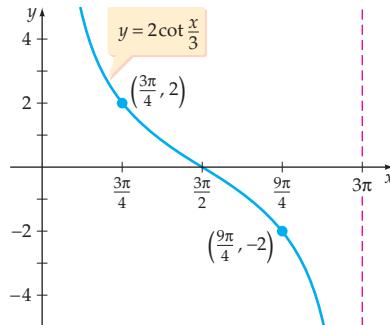
Graph one period of the function  $y = 2 \cot \frac{x}{3}$ .

#### Solution

The function  $y = 2 \cot \frac{x}{3}$  is of the form  $y = a \cot bx$ , with  $a = 2$  and  $b = \frac{1}{3}$ .

The period is given by  $\frac{\pi}{|b|} = \frac{\pi}{1/3} = 3\pi$ . Thus one period of the graph will be displayed on any interval of length  $3\pi$ . In the graph below, we have chosen to sketch the function over the interval  $0 < x < 3\pi$ . The graph

passes through  $\left(\frac{\pi}{4b}, a\right) = \left(\frac{\pi}{4(1/3)}, 2\right) = \left(\frac{3\pi}{4}, 2\right)$  and  $\left(\frac{3\pi}{4b}, -a\right) = \left(\frac{3\pi}{4(1/3)}, -2\right) = \left(\frac{9\pi}{4}, -2\right)$ . The graph has an  $x$ -intercept at  $x = \frac{\pi}{2b} = \frac{\pi}{2(1/3)} = \frac{3}{2}\pi$ .



» Try Exercise 34, page 190

### ■ The Graph of the Cosecant Function

Because  $\csc x = \frac{1}{\sin x}$ , the value of  $\csc x$  is the reciprocal of the value of  $\sin x$ .

Therefore,  $\csc x$  is undefined when  $\sin x = 0$  or when  $x = k\pi$ , where  $k$  is an integer. The graph of  $y = \csc x$  has vertical asymptotes at  $k\pi$ . Because  $y = \csc x$  has period  $2\pi$ , the graph will be repeated along the  $x$ -axis every  $2\pi$  units. A graph of  $y = \csc x$  is shown in **Figure 2.81** on page 186.

The graph of  $y = \sin x$  is also shown in **Figure 2.81**. Note the relationships among the  $x$ -intercepts of  $y = \sin x$  and the asymptotes of  $y = \csc x$ .

### The Graph of $y = \csc x$

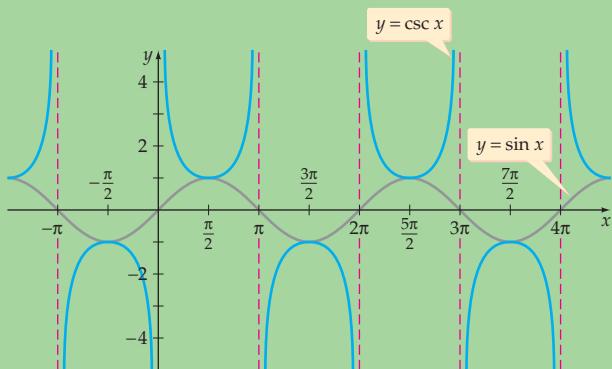


Figure 2.81

### Basic Properties

- **Domain:** All real numbers except multiples of  $\pi$
- **Range:**  $\{y \mid y \geq 1, y \leq -1\}$
- **Period:**  $2\pi$
- **Symmetry:** With respect to the origin
- **$x$ -intercepts:** None
- **Vertical asymptotes:** At multiples of  $\pi$
- **Reciprocal relationship:** If  $(x, y)$ ,  $y \neq 0$ , is a point on the graph of  $y = \sin x$ , then  $(x, 1/y)$  is a corresponding point on the graph of  $y = \csc x$ .

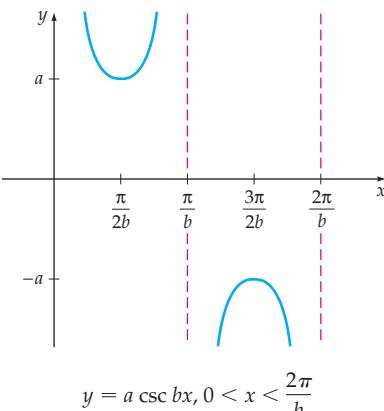


Figure 2.82

### take note

The **maximum value of a function** is its largest range value. The **minimum value of a function** is its smallest range value.

The value  $f(a)$  is called a **relative maximum of a function**  $f$ , provided  $f(a)$  is greater than or equal to the values of  $f$  near  $x = a$ . The value of  $f(b)$  is called a **relative minimum of a function**  $f$ , provided  $f(b)$  is less than or equal to the values of  $f$  near  $x = b$ .

**Figure 2.82** shows one cycle of the graph of  $y = a \csc bx$  for both  $a$  and  $b$  positive. Note from the graph the following properties of the function  $y = a \csc bx$ .

- The period is  $\frac{2\pi}{b}$ .
- The vertical asymptotes of  $y = a \csc bx$  are located at the zeros of  $y = a \sin bx$ .
- The graph passes through  $\left(\frac{\pi}{2b}, a\right)$  and  $\left(\frac{3\pi}{2b}, -a\right)$ .
- If  $a < 0$ , then the graph is reflected across the  $x$ -axis.

One procedure for graphing  $y = a \csc bx$  is to begin by graphing  $y = a \sin bx$ . For instance, in **Figure 2.83**, we have used the graph of  $y = 2 \sin 4x$  to produce the graph of  $y = 2 \csc 4x$ . Observe that

- The vertical asymptotes of the graph of the cosecant function pass through the  $x$ -intercepts of the graph of the sine function.
- The maximum values of the sine function are the relative minimum values of the cosecant function, and the minimum values of the sine function are the relative maximum values of the cosecant function.

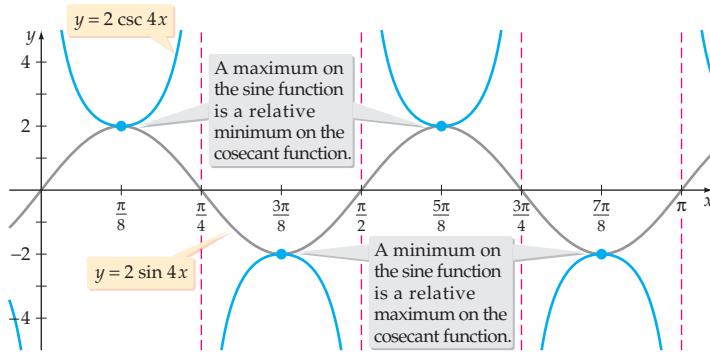


Figure 2.83

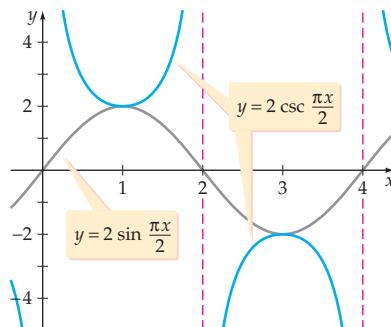
**EXAMPLE 4** **Graph  $y = a \csc bx$** 

Graph one complete period of  $y = 2 \csc \frac{\pi x}{2}$ .

**Solution**

Graph one period of  $y = 2 \sin \frac{\pi x}{2}$  and draw vertical asymptotes through the  $x$ -intercepts. Use the asymptotes as guides to draw the cosecant function.

The maximum values of  $y = 2 \sin \frac{\pi x}{2}$  are the relative minimum values of  $y = 2 \csc \frac{\pi x}{2}$ , and the minimum values of  $y = 2 \sin \frac{\pi x}{2}$  are the relative maximum values of  $y = 2 \csc \frac{\pi x}{2}$ .



**Try Exercise 40, page 190**

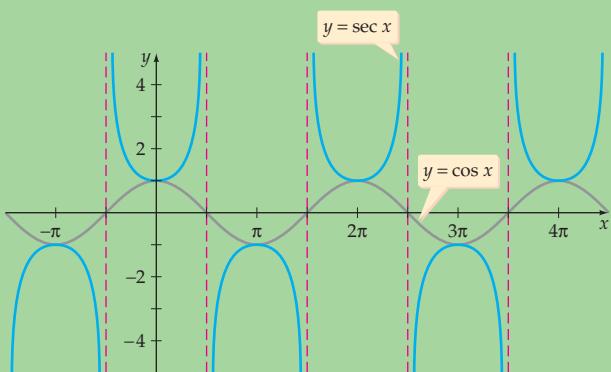
**The Graph of the Secant Function**

Because  $\sec x = \frac{1}{\cos x}$ , the value of  $\sec x$  is the reciprocal of the value of  $\cos x$ .

Therefore,  $\sec x$  is undefined when  $\cos x = 0$  or when  $x = \frac{\pi}{2} + k\pi$ ,  $k$  an integer.

The graph of  $y = \sec x$  has vertical asymptotes at  $\frac{\pi}{2} + k\pi$ . Because  $y = \sec x$  has period  $2\pi$ , the graph will be replicated along the  $x$ -axis every  $2\pi$  units. A graph of  $y = \sec x$  is shown in **Figure 2.84** on page 188.

The graph of  $y = \cos x$  is also shown in **Figure 2.84**. Note the relationships among the  $x$ -intercepts of  $y = \cos x$  and the asymptotes of  $y = \sec x$ .

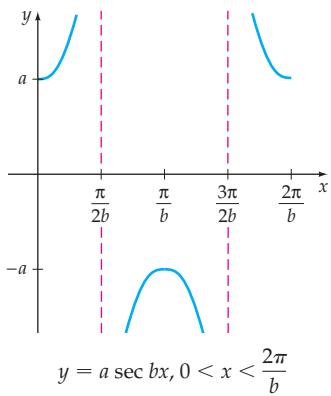
**The Graph of  $y = \sec x$** **Figure 2.84****Basic Properties**

- **Domain:** All real numbers except odd multiples of  $\frac{\pi}{2}$
- **Range:**  $\{y \mid y \geq 1, y \leq -1\}$
- **Period:**  $2\pi$
- **Symmetry:** With respect to the  $y$ -axis
- **$x$ -intercepts:** None
- **Vertical asymptotes:** At odd multiples of  $\frac{\pi}{2}$
- **Reciprocal relationship:** If  $(x, y)$ ,  $y \neq 0$ , is a point on the graph of  $y = \cos x$ , then  $(x, 1/y)$  is a corresponding point on the graph of  $y = \sec x$ .

The procedure for graphing  $y = a \sec bx$  is analogous to the procedure used to graph cosecant functions. First graph  $y = a \cos bx$  to determine its  $x$ -intercepts and its maximum and minimum points.

- The vertical asymptotes of the graph of the secant function pass through the  $x$ -intercepts of the graph of the cosine function.
- The maximum values of the cosine function are the relative minimum values of the secant function, and the minimum values of the cosine function are the relative maximum values of the secant function.

**Figure 2.85** shows one cycle of the graph of  $y = a \sec bx$  for both  $a$  and  $b$  positive. Note from the graph the following properties of the function  $y = a \sec bx$ .

**Figure 2.85**

- The period is  $\frac{2\pi}{b}$ .
- The vertical asymptotes of  $y = a \sec bx$  are located at the  $x$ -intercepts of  $y = a \cos bx$ .
- The graph passes through  $(0, a)$ ,  $\left(\frac{\pi}{b}, -a\right)$ , and  $\left(\frac{2\pi}{b}, a\right)$ .
- If  $a < 0$ , then the graph is reflected across the  $x$ -axis.

**EXAMPLE 5 » Graph  $y = a \sec bx$** 

Graph:  $y = -3 \sec \frac{x}{2}$

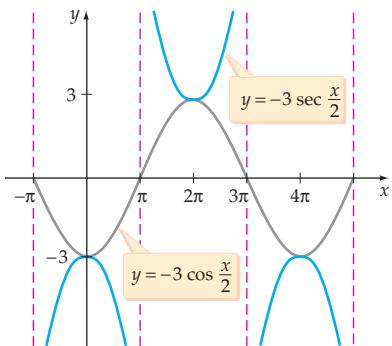


Figure 2.86

**Solution**

First sketch the graph of  $y = -3 \cos \frac{x}{2}$  and draw vertical asymptotes through the  $x$ -intercepts. Now sketch the graph of  $y = -3 \sec \frac{x}{2}$ , using the asymptotes as guides for the graph, as shown in **Figure 2.86**.

» Try Exercise 44, page 190

**Topics for Discussion**

1. Explain how you could use the graph of  $y = \sin x$  to produce the graph of  $y = \csc x$ .
2. Explain how to determine the period of  $y = \tan bx$ .
3. What is the amplitude of the function  $y = \cot \frac{x}{2}$ ? Explain.
4. Explain how to determine the location of the vertical asymptotes for the graph of  $y = 2 \tan bx$ .

## Exercise Set 2.6

1. For what values of  $x$  is  $y = \tan x$  undefined?

2. For what values of  $x$  is  $y = \cot x$  undefined?

3. For what values of  $x$  is  $y = \sec x$  undefined?

4. For what values of  $x$  is  $y = \csc x$  undefined?

17.  $y = \cot \pi x$

18.  $y = \cot \frac{\pi x}{3}$

19.  $y = 0.5 \tan \left( \frac{\pi}{5} t \right)$

20.  $y = 1.6 \cot \left( \frac{\pi}{2} t \right)$

21.  $y = 2.4 \csc \left( \frac{\pi}{4.25} t \right)$

22.  $y = 4.5 \sec \left( \frac{\pi}{2.5} t \right)$

In Exercises 5 to 22, state the period of each function.

5.  $y = \sec x$

6.  $y = \cot x$

7.  $y = \tan x$

In Exercises 23 to 42, sketch one full period of the graph of each function.

8.  $y = \csc x$

9.  $y = 2 \tan \frac{x}{2}$

10.  $y = \frac{1}{2} \cot 2x$

23.  $y = 3 \tan x$

24.  $y = \frac{1}{3} \tan x$

11.  $y = \csc 3x$

12.  $y = \csc \frac{x}{2}$

25.  $y = \frac{3}{2} \cot x$

26.  $y = 4 \cot x$

13.  $y = -\tan 3x$

14.  $y = -3 \cot \frac{2x}{3}$

27.  $y = 2 \sec x$

28.  $y = \frac{3}{4} \sec x$

15.  $y = -3 \sec \frac{x}{4}$

16.  $y = -\frac{1}{2} \csc 2x$

29.  $y = \frac{1}{2} \csc x$

30.  $y = 2 \csc x$

31.  $y = 2 \tan \frac{x}{2}$

32.  $y = -3 \tan 3x$

33.  $y = -3 \cot \frac{x}{2}$

34.  $y = \frac{1}{2} \cot 2x$

35.  $y = -2 \csc \frac{x}{3}$

36.  $y = \frac{3}{2} \csc 3x$

37.  $y = \frac{1}{2} \sec 2x$

38.  $y = -3 \sec \frac{2x}{3}$

39.  $y = -2 \sec \pi x$

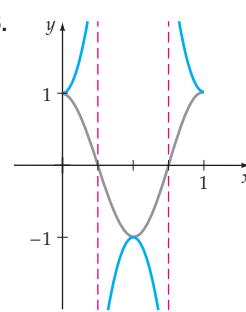
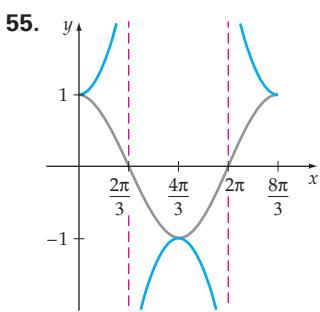
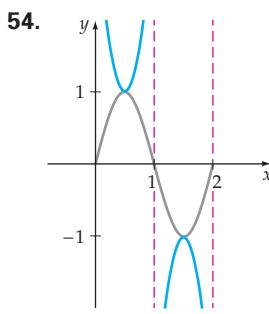
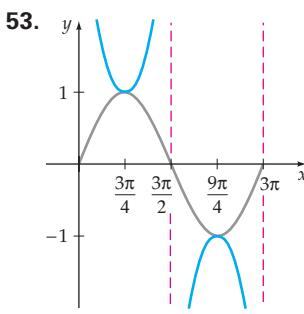
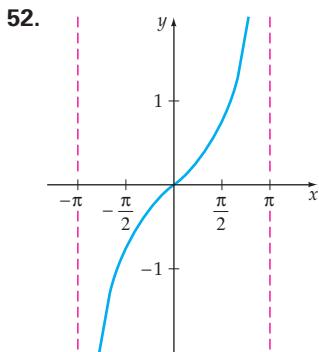
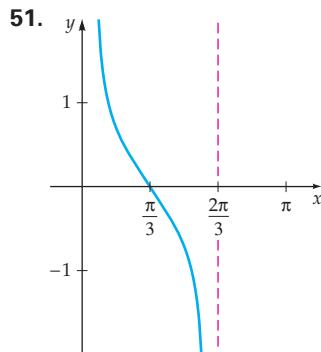
40.  $y = 3 \csc \frac{\pi x}{2}$

41.  $y = 3 \tan 2\pi x$

42.  $y = -\frac{1}{2} \cot \frac{\pi x}{2}$

43. Graph  $y = 2 \csc 3x$  from  $-2\pi$  to  $2\pi$ .44. Graph  $y = \sec \frac{x}{2}$  from  $-4\pi$  to  $4\pi$ .45. Graph  $y = 3 \sec \pi x$  from  $-2$  to  $4$ .46. Graph  $y = \csc \frac{\pi x}{2}$  from  $-4$  to  $4$ .47. Graph  $y = 2 \cot 2x$  from  $-\pi$  to  $\pi$ .48. Graph  $y = \frac{1}{2} \tan \frac{x}{2}$  from  $-4\pi$  to  $4\pi$ .49. Graph  $y = 3 \tan \pi x$  from  $-2$  to  $2$ .50. Graph  $y = \cot \frac{\pi x}{2}$  from  $-4$  to  $4$ .

**In Exercises 51 to 56, each blue graph displays one cycle of the graph of a trigonometric function. Find an equation of each blue graph.**



**In Exercises 57 to 60, graph each equation.**

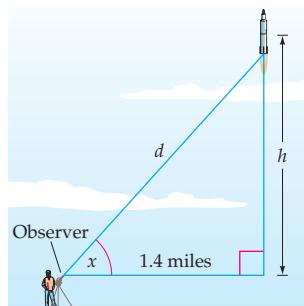
57.  $y = \tan |x|$

58.  $y = \sec |x|$

59.  $y = |\csc x|$

60.  $y = |\cot x|$

61. **ROCKET LAUNCH** An observer is 1.4 miles from the launch pad of a rocket. The rocket is launched upward as shown in the accompanying figure.



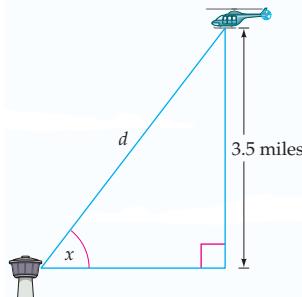
- Write the height  $h$ , in miles, of the rocket as a function of the angle of elevation  $x$ .
- Write the distance  $d$ , in miles, from the observer to the rocket as a function of angle  $x$ .
- Graph both functions in the same viewing window with  $X_{\min} = 0$ ,  $X_{\max} = \pi/2$ ,  $Y_{\min} = 0$ , and  $Y_{\max} = 20$ .

- d.  Describe the relationship between the graphs in c.

62. **AVIATION** A helicopter maintains an elevation of 3.5 miles and is flying toward a control tower. The angle of elevation from the tower to the helicopter is  $x$  radians.

a. Write the distance  $d$ , in miles, between the tower and the helicopter as a function of  $x$ .

b. Find  $d$  when  $x = 1$  and  $x = 1.2$ . Round to the nearest hundredth of a mile.



In Exercises 63 to 70, write an equation of the form  $y = \tan bx$ ,  $y = \cot bx$ ,  $y = \sec bx$ , or  $y = \csc bx$  that satisfies the given conditions.

- 63.** Tangent, period  $\frac{\pi}{3}$

**64.** Cotangent, period  $\frac{\pi}{2}$

**65.** Secant, period  $\frac{3\pi}{4}$

**66.** Cosecant, period  $\frac{5\pi}{2}$

- 67.** Cotangent, period 2      **68.** Tangent, period 0.5  
**69.** Cosecant, period 1.5      **70.** Secant, period 3

- 1.**  **A TECHNOLOGY QUESTION** A student's calculator shows the display at the right. Note that the domain values 4.7123 and 4.7124 are close together, but the range values are over 100,000 units apart. Explain how this is possible.

**2.**  **SOLUTIONS OF A TRIGONOMETRIC EQUATION** How many solutions does  $\tan \frac{1}{x} = 0$  have on the interval  $-1 \leq x \leq 1$ ? Explain.

$\tan 4.7123$       11238.43194  
 $\tan 4.7124$       -90747.26955

For more information about the study, please contact Dr. Michael J. Hwang at (310) 206-6500 or via email at [mhwang@ucla.edu](mailto:mhwang@ucla.edu).

## Section 2.7

- Translations of Trigonometric Functions
- Addition of Ordinates
- Damping Factor

# Graphing Techniques

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A12.

**PS1.** Find the amplitude and period of the graph of  $y = 2 \sin 2x$ . [2.5]

**PS2.** Find the amplitude and period of the graph of  $y = \frac{2}{3} \cos \frac{x}{3}$ . [2.5]

**PS3.** Find the amplitude and period of the graph of  $y = -4 \sin 2\pi x$ . [2.5]

**PS4.** What is the maximum value of  $f(x) = 2 \sin x$ ? [2.5]

**PS5.** What is the minimum value of  $f(x) = 3 \cos 2x$ ? [2.5]

**PS6.** Is the graph of  $f(x) = \cos x$  symmetric with respect to the origin or with respect to the  $y$ -axis? [2.5]

## Translations of Trigonometric Functions

Recall that the graph of  $y = f(x) \pm c$  is a *vertical translation* of the graph of  $y = f(x)$ . For  $c > 0$ , the graph of  $y = f(x) - c$  is the graph of  $y = f(x)$  shifted  $c$  units down; the graph of  $y = f(x) + c$  is the graph of  $y = f(x)$  shifted  $c$  units up. The graph in **Figure 2.87** is a graph of the equation  $y = 2 \sin \pi x - 3$ , which is a vertical translation of  $y = 2 \sin \pi x$  down 3 units. Note that subtracting 3 from  $y = 2 \sin \pi x$  changes neither its amplitude nor its period.

Also, the graph of  $y = f(x \pm c)$  is a *horizontal translation* of the graph of  $y = f(x)$ . For  $c > 0$ , the graph of  $y = f(x - c)$  is the graph of  $y = f(x)$  shifted  $c$  units to the right; the graph of  $y = f(x + c)$  is the graph of  $y = f(x)$  shifted  $c$  units to the left. The graph in **Figure 2.88** is a graph of the equation  $y = 2 \sin\left(x - \frac{\pi}{4}\right)$ , which is the graph of  $y = 2 \sin x$  translated  $\frac{\pi}{4}$  units to the right. Note that neither the period nor the amplitude is affected. The horizontal shift of the graph of a trigonometric function is called its **phase shift**.

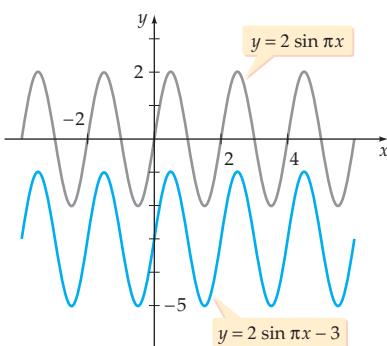


Figure 2.87

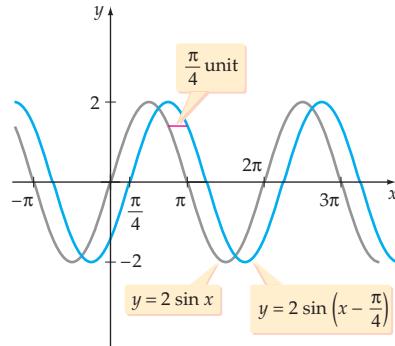


Figure 2.88

## TO REVIEW

### Translations of Graphs

See page 58.

Because one cycle of  $y = a \sin x$  is completed for  $0 \leq x \leq 2\pi$ , one cycle of the graph of  $y = a \sin(bx + c)$ , where  $b > 0$ , is completed for  $0 \leq bx + c \leq 2\pi$ . Solving this inequality for  $x$ , we have

$$\begin{aligned} 0 &\leq bx + c \leq 2\pi \\ -c &\leq bx \leq -c + 2\pi \\ -\frac{c}{b} &\leq x \leq -\frac{c}{b} + \frac{2\pi}{b} \end{aligned}$$

The number  $-\frac{c}{b}$  is the phase shift for  $y = a \sin(bx + c)$ . The graph of the equation  $y = a \sin(bx + c)$  is the graph of  $y = a \sin bx$  shifted  $-\frac{c}{b}$  units horizontally. Similar arguments apply to the remaining trigonometric functions.

### The Graphs of $\sin$ and $\cos$

The graphs of  $y = a \sin(bx + c)$  and  $y = a \cos(bx + c)$ , have

$$\text{Amplitude: } |a| \quad \text{Period: } \frac{2\pi}{|b|} \quad \text{Phase shift: } -\frac{c}{b}$$

To graph  $y = a \sin(bx + c)$ , shift the graph of  $y = a \sin bx$  horizontally  $-\frac{c}{b}$  units.

To graph  $y = a \cos(bx + c)$ , shift the graph of  $y = a \cos bx$  horizontally  $-\frac{c}{b}$  units.

**QUESTION** What is the phase shift of the graph of  $y = 3 \sin\left(\frac{1}{2}x - \frac{\pi}{6}\right)$ ?

### EXAMPLE 1 » Graph by Using a Translation

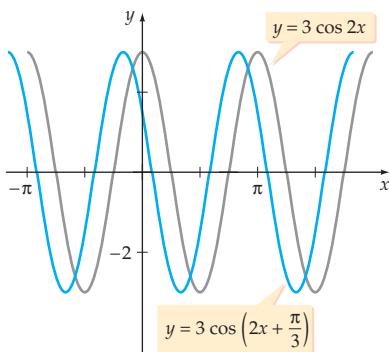


Figure 2.89

$$\text{Graph: } y = 3 \cos\left(2x + \frac{\pi}{3}\right)$$

#### Solution

The phase shift is  $-\frac{c}{b} = -\frac{\pi/3}{2} = -\frac{\pi}{6}$ . The graph of the equation

$y = 3 \cos\left(2x + \frac{\pi}{3}\right)$  is the graph of  $y = 3 \cos 2x$  shifted  $\frac{\pi}{6}$  units to the left, as shown in **Figure 2.89**.

» Try Exercise 20, page 199

**The Graphs of** **and**

The graphs of  $y = a \tan(bx + c)$  and  $y = a \cot(bx + c)$  have

$$\text{Period: } \frac{\pi}{|b|} \quad \text{Phase shift: } -\frac{c}{b}$$

To graph  $y = a \tan(bx + c)$ , shift the graph of  $y = a \tan bx$  horizontally  $-\frac{c}{b}$  units.

To graph  $y = a \cot(bx + c)$ , shift the graph of  $y = a \cot bx$  horizontally  $-\frac{c}{b}$  units.

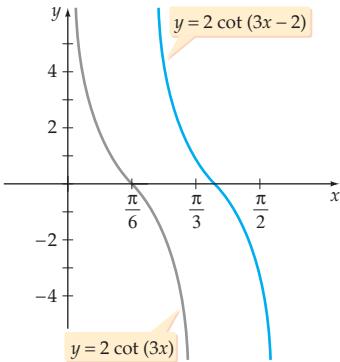


Figure 2.90

**E X A M P L E 2 »» Graph by Using a Translation**

Graph one period of the function  $y = 2 \cot(3x - 2)$ .

**Solution**

The phase shift is

$$-\frac{c}{b} = -\frac{-2}{3} = \frac{2}{3} \quad \bullet 3x - 2 = 3x + (-2)$$

The graph of  $y = 2 \cot(3x - 2)$  is the graph of  $y = 2 \cot(3x)$  shifted  $\frac{2}{3}$  unit to the right, as shown in **Figure 2.90**.

» Try Exercise 22, page 199

In Example 3 we use both a horizontal translation and a vertical translation to graph a function of the form  $y = a \sin(bx + c) + d$ .

**E X A M P L E 3 »» Graph  $y = a \sin(bx + c) + d$**

$$\text{Graph: } y = \frac{1}{2} \sin\left(x - \frac{\pi}{4}\right) - 2$$

**Solution**

The phase shift is  $-\frac{c}{b} = -\frac{-\pi/4}{1} = \frac{\pi}{4}$ . The vertical shift is 2 units down.

The graph of  $y = \frac{1}{2} \sin\left(x - \frac{\pi}{4}\right) - 2$  is the graph of  $y = \frac{1}{2} \sin x$  shifted  $\frac{\pi}{4}$  units to the right and 2 units down, as shown in **Figure 2.91**.

» Try Exercise 40, page 199

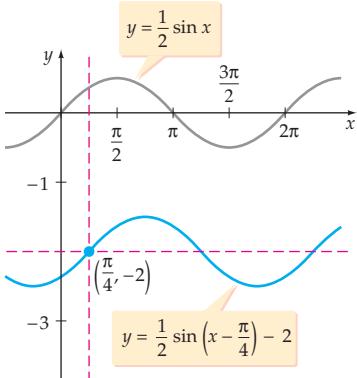
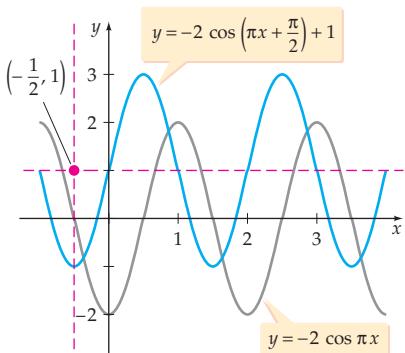


Figure 2.91

In Example 4 we use both a horizontal translation and a vertical translation to graph a function of the form  $y = a \cos(bx + c) + d$ .

**EXAMPLE 4** » Graph  $y = a \cos(bx + c) + d$ **Figure 2.92**

Graph:  $y = -2 \cos\left(\pi x + \frac{\pi}{2}\right) + 1$

**Solution**

The phase shift is  $-\frac{c}{b} = -\frac{\pi/2}{\pi} = -\frac{1}{2}$ . The vertical shift is 1 unit up. The graph of  $y = -2 \cos\left(\pi x + \frac{\pi}{2}\right) + 1$  is the graph of  $y = -2 \cos \pi x$  shifted  $\frac{1}{2}$  unit to the left and 1 unit up, as shown in **Figure 2.92**.

» Try Exercise 42, page 199

The following application involves a function of the form  $y = \cos(bx + c) + d$ .

**EXAMPLE 5** » A Mathematical Model of a Patient's Blood Pressure

The function  $bp(t) = 32 \cos\left(\frac{10\pi}{3}t - \frac{\pi}{3}\right) + 112$ ,  $0 \leq t \leq 20$ , gives the blood pressure, in millimeters of mercury (mm Hg), of a patient during a 20-second interval.

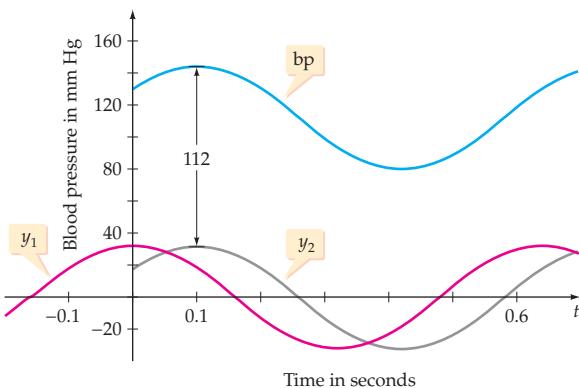
- Find the phase shift and the period of  $bp$ .
- Graph one period of  $bp$ .
- What are the patient's maximum (*systolic*) and minimum (*diastolic*) blood pressure readings during the given time interval?
- What is the patient's pulse rate in beats per minute?

**Solution**

a. Phase shift =  $-\frac{c}{b} = -\frac{\left(-\frac{\pi}{3}\right)}{\left(\frac{10\pi}{3}\right)} = 0.1$

Period =  $\frac{2\pi}{|b|} = \frac{2\pi}{\left(\frac{10\pi}{3}\right)} = 0.6$  second

- b. The graph of  $bp$  is the graph of  $y_1 = 32 \cos\left(\frac{10\pi}{3}t\right)$  shifted one tenth of a unit to the right, shown by  $y_2 = 32 \cos\left(\frac{10\pi}{3}t - \frac{\pi}{3}\right)$  in the graph at the top of the next page, and 112 units upward.



- c. The function  $y_2 = 32 \cos\left(\frac{10\pi}{3}t - \frac{\pi}{3}\right)$  has a maximum of 32 and a minimum of -32. Thus the patient's maximum blood pressure is  $32 + 112 = 144$  mm Hg, and the patient's minimum blood pressure is  $-32 + 112 = 80$  mm Hg.
- d. From a. we know that the patient has 1 heartbeat every 0.6 second. Therefore, during the given time interval, the patient has a pulse rate of

$$\left(\frac{1 \text{ heartbeat}}{0.6 \text{ second}}\right) \left(\frac{60 \text{ seconds}}{1 \text{ minute}}\right) = 100 \text{ heartbeats per minute}$$

» Try Exercise 64, page 200

The translation techniques used to graph sine and cosine functions can also be used to graph secant and cosecant functions.

### EXAMPLE 6 » Graph a Cosecant Function

Graph:  $y = 2 \csc(2x - \pi)$

#### Solution

The phase shift is  $-\frac{c}{b} = -\frac{-\pi}{2} = \frac{\pi}{2}$ . The graph of the equation

$y = 2 \csc(2x - \pi)$  is the graph of  $y = 2 \csc 2x$  shifted  $\frac{\pi}{2}$  units to the right. Sketch the graph of the equation  $y = 2 \sin 2x$  shifted  $\frac{\pi}{2}$  units to the right. Use this graph to draw the graph of  $y = 2 \csc(2x - \pi)$ , as shown in Figure 2.93.

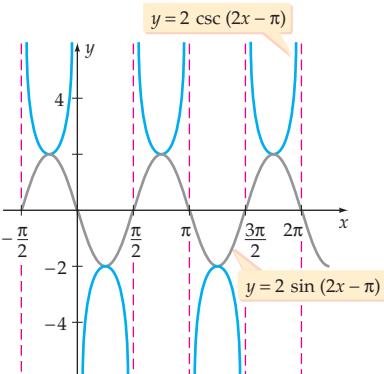


Figure 2.93

» Try Exercise 48, page 199

## ■ Addition of Ordinates

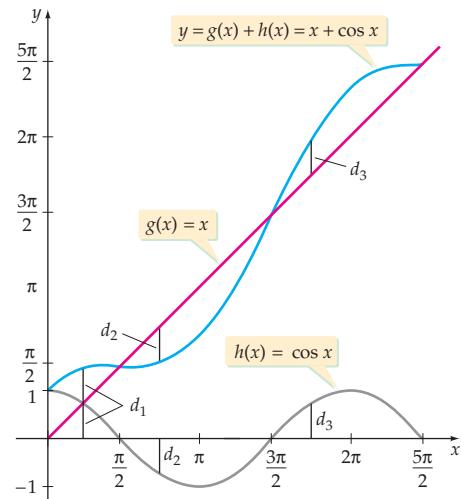
Given two functions  $g$  and  $h$ , the sum of the functions is the function  $f$  defined by  $f(x) = g(x) + h(x)$ . The graph of the sum  $f$  can be obtained by graphing  $g$  and  $h$  separately and then geometrically adding the  $y$ -coordinates of each function for a given value of  $x$ . It is convenient, when we are drawing the graph of the sum of two functions, to pick zeros of the functions.

### EXAMPLE 7 » Graph the Sum of Two Functions

Graph:  $y = x + \cos x$

#### Solution

Graph  $g(x) = x$  and  $h(x) = \cos x$  on the same coordinate grid. Then add the  $y$ -coordinates geometrically point by point. The figure below shows the results of adding, by using a ruler, the  $y$ -coordinates of the two functions for selected values of  $x$ . Notice that when  $h(x) = \cos x = 0$ , the graph of  $y$  intersects the graph of  $g(x) = x$ .



### » Try Exercise 52, page 199

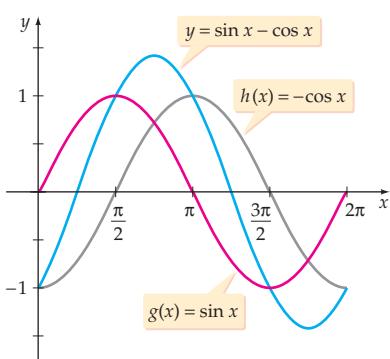
In Example 8 we graph  $y = \sin x - \cos x$  by geometrically adding the  $y$  values of selected points on the graphs of  $g(x) = \sin x$  and  $h(x) = -\cos x$ .

### EXAMPLE 8 » Graph the Difference of Two Functions

Graph  $y = \sin x - \cos x$  for  $0 \leq x \leq 2\pi$ .

#### Solution

Graph  $g(x) = \sin x$  and  $h(x) = -\cos x$  on the same coordinate grid. For selected values of  $x$ , add  $g(x)$  and  $h(x)$  geometrically. Now draw a smooth curve through the points. See **Figure 2.94**.



**Figure 2.94**

### » Try Exercise 56, page 199

### Damping Factor

The factor  $\frac{1}{4}x$  in  $f(x) = \frac{1}{4}x \cos x$  is referred to as the **damping factor**. In the next example we analyze the role of the damping factor.

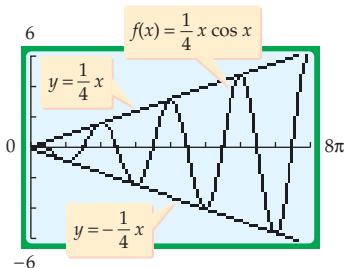
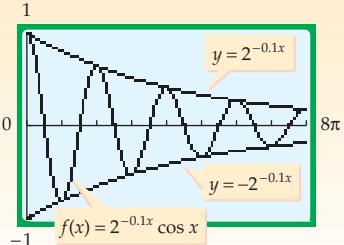


Figure 2.95

#### take note

Replacing the damping factor can make a dramatic change in the graph of a function. For instance, the graph of  $f(x) = 2^{-0.1x} \cos x$  approaches 0 as  $x$  approaches  $\infty$ .



### Solution

**Figure 2.95** shows that the graph of  $f$  intersects

- the graph of  $y = \frac{1}{4}x$  for  $x = 0, 2\pi, 4\pi, \dots$
  - the graph of  $y = -\frac{1}{4}x$  for  $x = \pi, 3\pi, 5\pi, \dots$
  - the  $x$ -axis for  $x = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \dots$
- Because  $\cos x = 1$  for  $x = 2n\pi$
  - Because  $\cos x = -1$  for  $x = (2n - 1)\pi$
  - Because  $\cos x = 0$  for  $x = \frac{2n - 1}{2}\pi$

**Figure 2.95** also shows that the graph of  $f$  lies on or between the lines

$$y = \frac{1}{4}x \quad \text{and} \quad y = -\frac{1}{4}x \quad \bullet \text{Because } |\cos x| \leq 1 \text{ for all } x$$

» Try Exercise 78, page 202



### Topics for Discussion

1. The maximum value of  $f(x) = 3 \sin x + 4$  is 7. Thus  $f$  has an amplitude of 7. Do you agree? Explain.
2. The graph of  $y = \sec x$  has a period of  $2\pi$ . What is the period of the graph of  $y = |\sec x|$ ?
3. The zeros of  $y = \sin x$  are the same as the zeros of  $y = x \sin x$ . Do you agree? Explain.
4. What is the phase shift of the graph of  $y = \tan\left(3x - \frac{\pi}{6}\right)$ ?


**Exercise Set 2.7**

**In Exercises 1 to 8, find the amplitude, phase shift, and period for the graph of each function.**

1.  $y = 2 \sin\left(x - \frac{\pi}{2}\right)$

2.  $y = -3 \sin(x + \pi)$

3.  $y = \cos\left(2x - \frac{\pi}{4}\right)$

4.  $y = \frac{3}{4} \cos\left(\frac{x}{2} + \frac{\pi}{3}\right)$

5.  $y = -4 \sin\left(\frac{2x}{3} + \frac{\pi}{6}\right)$

6.  $y = \frac{3}{2} \sin\left(\frac{x}{4} - \frac{3\pi}{4}\right)$

7.  $y = \frac{5}{4} \cos(3x - 2\pi)$

8.  $y = 6 \cos\left(\frac{x}{3} - \frac{\pi}{6}\right)$

**In Exercises 9 to 16, find the phase shift and the period for the graph of each function.**

9.  $y = 2 \tan\left(2x - \frac{\pi}{4}\right)$

10.  $y = \frac{1}{2} \tan\left(\frac{x}{2} - \pi\right)$

11.  $y = -3 \csc\left(\frac{x}{3} + \pi\right)$

12.  $y = -4 \csc\left(3x - \frac{\pi}{6}\right)$

13.  $y = 2 \sec\left(2x - \frac{\pi}{8}\right)$

14.  $y = 3 \sec\left(\frac{x}{4} - \frac{\pi}{2}\right)$

15.  $y = -3 \cot\left(\frac{x}{4} + 3\pi\right)$

16.  $y = \frac{3}{2} \cot\left(2x - \frac{\pi}{4}\right)$

**In Exercises 17 to 32, graph one full period of each function.**

17.  $y = \sin\left(x - \frac{\pi}{2}\right)$

18.  $y = \sin\left(x + \frac{\pi}{6}\right)$

19.  $y = \cos\left(\frac{x}{2} + \frac{\pi}{3}\right)$

20.  $y = \cos\left(2x - \frac{\pi}{3}\right)$

21.  $y = \tan\left(x + \frac{\pi}{4}\right)$

22.  $y = \tan(x - \pi)$

23.  $y = 2 \cot\left(\frac{x}{2} - \frac{\pi}{8}\right)$

24.  $y = \frac{3}{2} \cot\left(3x + \frac{\pi}{4}\right)$

25.  $y = \sec\left(x + \frac{\pi}{4}\right)$

26.  $y = \csc(2x + \pi)$

27.  $y = \csc\left(\frac{x}{3} - \frac{\pi}{2}\right)$

28.  $y = \sec\left(2x + \frac{\pi}{6}\right)$

29.  $y = -2 \sin\left(\frac{x}{3} - \frac{2\pi}{3}\right)$

30.  $y = -\frac{3}{2} \sin\left(2x + \frac{\pi}{4}\right)$

31.  $y = -3 \cos\left(3x + \frac{\pi}{4}\right)$

32.  $y = -4 \cos\left(\frac{3x}{2} + 2\pi\right)$

**In Exercises 33 to 50, graph each function by using translations.**

33.  $y = \sin x + 1$

34.  $y = -\sin x + 1$

35.  $y = -\cos x - 2$

36.  $y = 2 \sin x + 3$

37.  $y = \sin 2x - 2$

38.  $y = -\cos \frac{x}{2} + 2$

39.  $y = 4 \cos(\pi x - 2) + 1$

40.  $\ggg y = 2 \sin\left(\frac{\pi x}{2} + 1\right) - 2$

41.  $y = -\sin(\pi x + 1) - 2$

42.  $\ggg y = -3 \cos(2\pi x - 3) + 1$

43.  $y = \sin\left(x - \frac{\pi}{2}\right) - \frac{1}{2}$

44.  $y = -2 \cos\left(x + \frac{\pi}{3}\right) + 3$

45.  $y = \tan \frac{x}{2} - 4$

46.  $y = \cot 2x + 3$

47.  $y = \sec 2x - 2$

48.  $\ggg y = \csc \frac{x}{3} + 4$

49.  $y = \csc \frac{x}{2} - 1$

50.  $y = \sec\left(x - \frac{\pi}{2}\right) + 1$

**In Exercises 51 to 56, graph the given function by using the addition-of-ordinates method.**

51.  $y = x - \sin x$

52.  $\ggg y = \frac{x}{2} + \cos x$

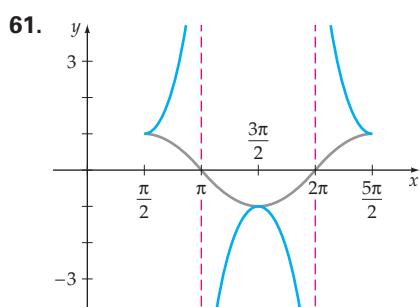
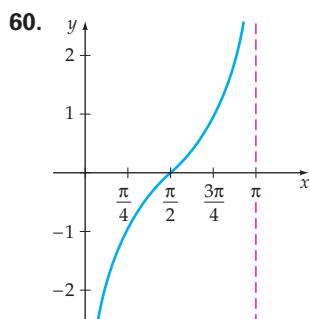
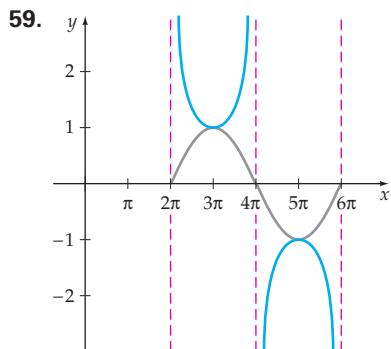
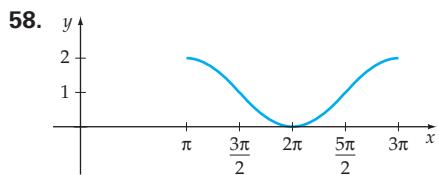
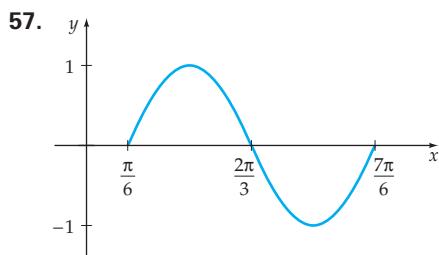
53.  $y = x + \sin 2x$

54.  $y = \frac{2x}{3} - \sin x$

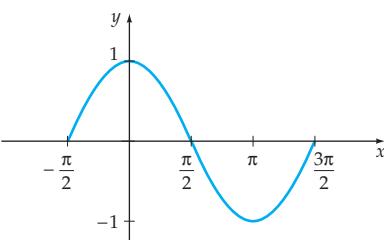
55.  $y = \sin x + \cos x$

56.  $\ggg y = \cos x - \sin x$

In Exercises 57 to 62, each blue graph displays one cycle of the graph of a trigonometric function. Find an equation of each blue graph.



62.



63. **RETAIL SALES** The manager of a major department store finds that the number of men's suits  $S$ , in hundreds, that the store sells is given by

$$S = 4.1 \cos\left(\frac{\pi}{6}t - 1.25\pi\right) + 7$$

where  $t$  is time measured in months, with  $t = 0$  representing January 1.

- Find the phase shift and the period of  $S$ .
- Graph one period of  $S$ .
- Use the graph from b. to determine in which month the store sells the most suits.

64. **RETAIL SALES** The owner of a shoe store finds that the number of pairs of shoes  $S$ , in hundreds, that the store sells can be modeled by the function

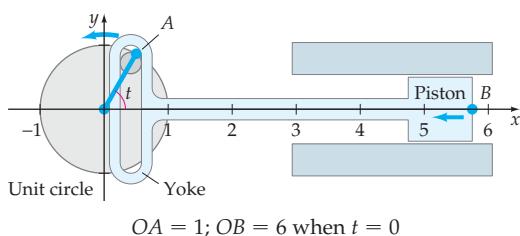
$$S = 2.7 \cos\left(\frac{\pi}{6}t - \frac{7}{12}\pi\right) + 4$$

where  $t$  is time measured in months, with  $t = 0$  representing January 1.

- Find the phase shift and the period of  $S$ .
- Graph one period of  $S$ .
- Use the graph from b. to determine in which month the store sells the most shoes.

65. **CARBON DIOXIDE LEVELS** Because of seasonal changes in vegetation, carbon dioxide ( $\text{CO}_2$ ) levels, as a product of photosynthesis, rise and fall during the year. Besides the naturally occurring  $\text{CO}_2$  from plants, additional  $\text{CO}_2$  is given off as a pollutant. A reasonable model of  $\text{CO}_2$  levels in a city for the years 1990–2006 is given by  $y = 2.3 \sin 2\pi t + 1.25t + 315$ , where  $t$  is the number of years since 1990 and  $y$  is the concentration of  $\text{CO}_2$  in parts per million (ppm). Find the difference in  $\text{CO}_2$  levels between the beginning of 1990 and the beginning of 2006.

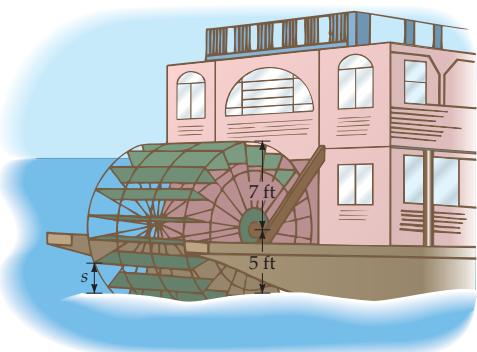
- 66. ROTATIONAL MOTION TO LINEAR MOTION** A Scotch Yoke is a mechanical device that is used to convert rotational motion into linear motion. In the accompanying figure, the starter of an engine rotates point  $A$  in a counterclockwise direction at a constant rate.



As  $A$  rotates, it also moves up and down in the yoke, which causes the piston to move to the left and to the right.

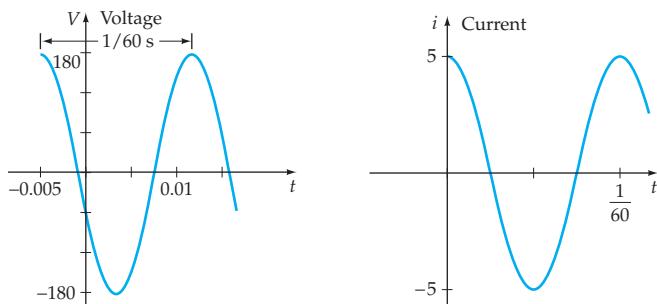
- Write the coordinates of  $A$  in terms of angle  $t$ .
- Point  $B$  is on the right end of the piston. Write the coordinates of  $B$  in terms of  $t$ .
- Describe the motion of  $B$  as  $A$  makes one complete revolution.

- 67. HEIGHT OF A PADDLE** The paddle wheel on a riverboat is shown in the accompanying figure. Write an equation for the height of a paddle relative to the water at time  $t$ . The radius of the paddle wheel is 7 feet, and the distance from the center of the paddle wheel to the water is 5 feet. Assume that the paddle wheel rotates at 5 revolutions per minute and that the paddle is at its highest point at  $t = 0$ . Graph the equation for  $0 \leq t \leq 0.20$  minute.

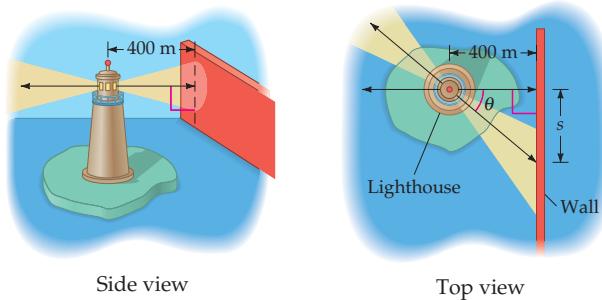


- 68. VOLTAGE AND AMPERAGE** The graphs of the voltage and amperage of an alternating household circuit are shown in the following figures, where  $t$  is measured in seconds. Note that there is a phase shift between the graph of the voltage and the graph of the current. The current is said to

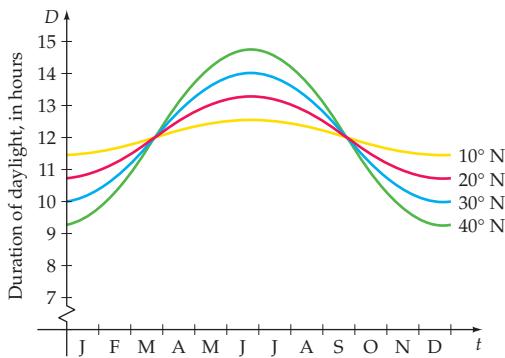
lag the voltage by 0.005 second. Write an equation for the voltage and an equation for the current.



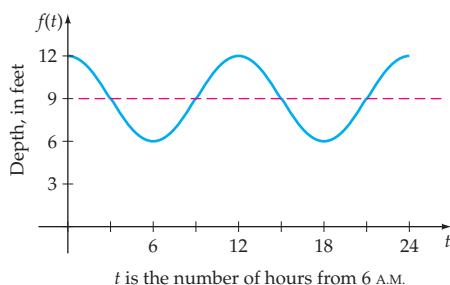
- 69. A LIGHTHOUSE BEACON** The beacon of a lighthouse 400 meters from a straight sea wall rotates at 6 revolutions per minute. Using the accompanying figures, write an equation expressing the distance  $s$ , measured in meters, in terms of time  $t$ . Assume that when  $t = 0$ , the beam is perpendicular to the sea wall. Sketch a graph of the equation for  $0 \leq t \leq 10$  seconds.



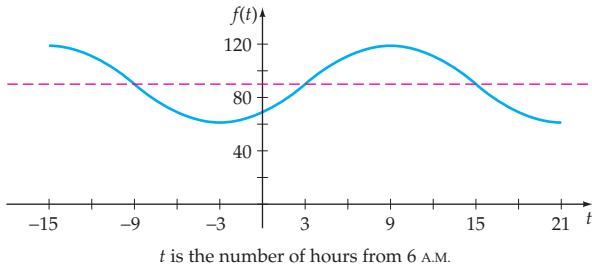
- 70. HOURS OF DAYLIGHT** The duration of daylight for a region is dependent not only on the time of year but also on the latitude of the region. The following graph gives the daylight hours for a one-year period at various latitudes. Assuming that a sine function can model these curves, write an equation for each curve.



- 71. TIDES** During a 24-hour day, the tides raise and lower the depth of water at a pier as shown in the figure below. Write an equation of the form  $f(t) = A \cos Bt + d$  that gives the depth of the water at time  $t$ , and find the depth of the water at 6 P.M.



- 72. TEMPERATURE** During a summer day, the ground temperature at a desert location was recorded and graphed as a function of time, as shown in the following figure. The graph can be approximated by  $f(t) = A \cos(bt + c) + d$ . Find the equation and approximate the temperature (to the nearest degree) at 1:00 P.M.



**In Exercises 73 to 80, use a graphing utility to graph each function.**

**73.**  $y = \sin x - \cos \frac{x}{2}$

**74.**  $y = 2 \sin 2x - \cos x$

**75.**  $y = 2 \cos x + \sin \frac{x}{2}$

**76.**  $y = -\frac{1}{2} \cos 2x + \sin \frac{x}{2}$

**77.**  $y = \frac{x}{2} \sin x$

**78.**  $y = x \cos x$

**79.**  $y = x \sin \frac{x}{2}$

**80.**  $y = \frac{x}{2} \cos \frac{x}{2}$



**BEATS** When two sound waves have approximately the same frequency, the sound waves interfere with one another and produce phenomena called *beats*, which are heard as variations in the loudness of the sound. A piano tuner can use these phenomena to tune a piano. By striking a tuning fork and then tapping the corresponding key on a piano, the piano tuner listens for beats and adjusts the tension in the string until the beats disappear. Use a graphing utility to graph the functions in Exercises 81 and 82, which are based on beats.

$$81. \ y = \sin(5\pi x) \cdot \sin\left(-\frac{\pi}{2}x\right)$$

**82.**  $y = \sin(17\pi x) \cdot \sin\left(-\frac{\pi}{2}x\right)$

- 83.** Find an equation of the sine function with amplitude 2, period  $\pi$ , and phase shift  $\frac{\pi}{3}$ .

**84.** Find an equation of the cosine function with amplitude 3, period  $3\pi$ , and phase shift  $-\frac{\pi}{4}$ .

**85.** Find an equation of the tangent function with period  $2\pi$  and phase shift  $\frac{\pi}{2}$ .

**86.** Find an equation of the cotangent function with period  $\frac{\pi}{2}$  and phase shift  $-\frac{\pi}{4}$ .



**In Exercises 89 to 92, use a graphing utility to graph each function.**

89.  $y = \frac{\sin x}{x}$

**90.**  $y = 2 + \sec \frac{x}{2}$

**91.**  $y = |x| \sin x$

**92.**  $y = |x| \cos x$

## »»» Projects

-  **PREDATOR-PREY RELATIONSHIP** Predator-prey interactions can produce cyclic population growth for both the predator population and the prey population. Consider an animal reserve where the rabbit population  $r$  is given by

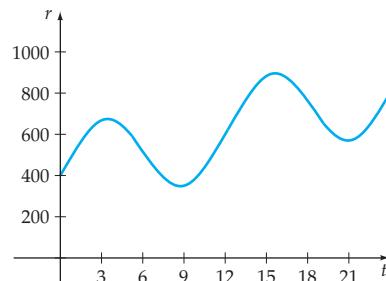
$$r(t) = 850 + 210 \sin\left(\frac{\pi}{6}t\right)$$

and the wolf population  $w$  is given by

$$w(t) = 120 + 30 \sin\left(\frac{\pi}{6}t - 2\right)$$

where  $t$  is the number of months after March 1, 2006. Graph  $r(t)$  and  $w(t)$  on the same coordinate system for  $0 \leq t \leq 24$ . Write a few sentences that explain a possible relationship between the two populations.

Write an equation that could be used to model the rabbit population shown by the graph below, where  $t$  is measured in months.



## Section 2.8

- Simple Harmonic Motion
- Damped Harmonic Motion

# Harmonic Motion—An Application of the Sine and Cosine Functions

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A13.

**PS1.** Find the reciprocal of  $\frac{2\pi}{3}$ .

**PS2.** Find the reciprocal of  $\frac{2}{5}$ .

**PS3.** Evaluate  $a \cos 2\pi t$  for  $a = 4$  and  $t = 0$ . [2.4]

**PS4.** Evaluate  $\sqrt{\frac{k}{m}}$  for  $k = 18$  and  $m = 2$ .

**PS5.** Evaluate  $4 \cos\left(\sqrt{\frac{k}{m}} t\right)$  for  $k = 16$ ,  $m = 4$ , and  $t = 2\pi$ . [2.4]

**PS6.** Write an equation of the form  $y = a \cos bx$  whose graph has an amplitude of 4 and a period of 2. [2.5]

The vibratory motion of a mass on a spring is called **harmonic motion**. In this section, we will examine two types of harmonic motion.

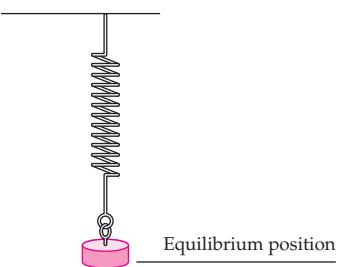


Figure 2.96

**take note**

The graph of the displacement  $y$  as a function of the time  $t$  is a cosine curve.

## Simple Harmonic Motion

We will consider a mass on a spring to illustrate vibratory motion. Assume that we have placed a mass on a spring and allowed the spring to come to rest, as shown in **Figure 2.96**. The system is said to be in *equilibrium* when the mass is at rest. The point of rest is called the *origin* of the system. We consider the distance above the equilibrium point as positive and the distance below the equilibrium point as negative.

If the mass is now lifted a distance  $a$  and released, the mass will oscillate up and down in periodic motion. If there is no friction, the motion repeats itself in a certain period of time. The number of times the mass oscillates in 1 unit of time is called the *frequency*  $f$  of the motion, and the time one oscillation takes is the *period*  $p$  of the motion. The motion is referred to as *simple harmonic motion*.

**Figure 2.97** shows the **displacement**  $y$  of the mass for one oscillation at times

$$t = 0, \frac{p}{4}, \frac{p}{2}, \frac{3p}{4}, \text{ and } p.$$

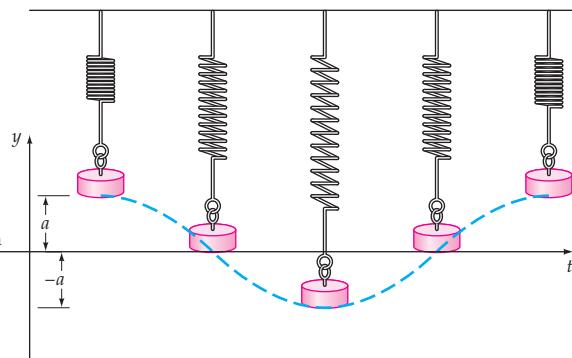


Figure 2.97

The frequency and the period are related by the formulas

$$f = \frac{1}{p} \quad \text{and} \quad p = \frac{1}{f}$$

The maximum displacement from the equilibrium position is called the *amplitude of the motion*. Vibratory motion can be quite complicated. However, the simple harmonic motion of a mass on a spring can be described by one of the following equations.

**take note**

Function (1) is used if the displacement from the origin is at a maximum at time  $t = 0$ . Function (2) is used if the displacement at time  $t = 0$  is zero.

### Simple Harmonic Motion

**Simple harmonic motion** can be modeled by one of the following functions:

$$y = a \cos 2\pi ft \quad (1) \quad \text{or} \quad y = a \sin 2\pi ft \quad (2)$$

where  $|a|$  is the amplitude (maximum displacement),  $f$  is the frequency,  $\frac{1}{f}$  is the period,  $y$  is the displacement, and  $t$  is the time.

**QUESTION** A simple harmonic motion has a frequency of 2 cycles per second. What is the period of the simple harmonic motion?

### EXAMPLE 1 » Find the Equation of Motion of a Mass on a Spring

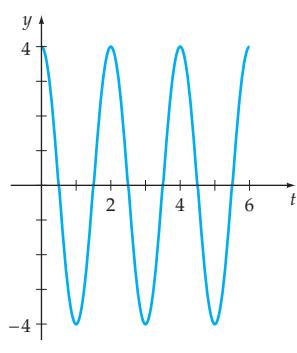


Figure 2.98

A mass on a spring has been displaced 4 centimeters above the equilibrium point and released. The mass is vibrating with a frequency of  $\frac{1}{2}$  cycle per second. Write the equation of simple harmonic motion, and graph three cycles of the displacement as a function of time.

#### Solution

Because the maximum displacement is 4 centimeters when  $t = 0$ , use  $y = a \cos 2\pi ft$ .

$$\begin{aligned} y &= a \cos 2\pi ft && \text{• Equation for simple harmonic motion} \\ &= 4 \cos 2\pi \left(\frac{1}{2}\right)t && \bullet a = 4, f = \frac{1}{2} \\ &= 4 \cos \pi t \end{aligned}$$

The equation of simple harmonic motion is  $y = 4 \cos \pi t$ . See **Figure 2.98**.

#### » Try Exercise 20, page 208

From physical laws determined by experiment, the frequency of oscillation of a mass on a spring is given by

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

where  $k$  is a **spring constant** determined by experiment and  $m$  is the mass. The simple harmonic motion of the mass on the spring (with maximum displacement at  $t = 0$ ) can then be described by

$$\begin{aligned} y &= a \cos 2\pi ft = a \cos 2\pi \left(\frac{1}{2\pi} \sqrt{\frac{k}{m}}\right)t \\ y &= a \cos \sqrt{\frac{k}{m}} t \end{aligned} \tag{3}$$

The equation of the simple harmonic motion for zero displacement at  $t = 0$  is

$$y = a \sin \sqrt{\frac{k}{m}} t \tag{4}$$

---

**ANSWER**  $p = \frac{1}{f} = \frac{1}{2}$  second per cycle

**EXAMPLE 2** » Find the Equation of Motion of a Mass on a Spring

A mass of 2 units is in equilibrium suspended from a spring with a spring constant of  $k = 18$ . The mass is pulled down 0.5 unit and released. Find the period, frequency, and amplitude of the resulting simple harmonic motion. Write the equation of the motion, and graph two cycles of the displacement as a function of time.

**Solution**

At the start of the motion, the displacement is at a maximum but in the negative direction. The resulting motion is described by Equation (3), using  $a = -0.5$ ,  $k = 18$ , and  $m = 2$ .

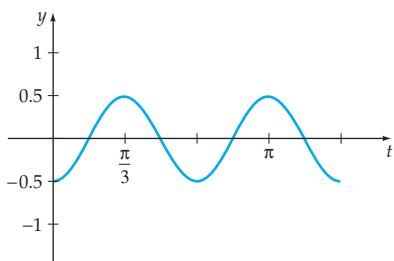


Figure 2.99

$$y = a \cos \sqrt{\frac{k}{m}} t = -0.5 \cos \sqrt{\frac{18}{2}} t \quad \bullet \text{Substitute for } a, k, \text{ and } m.$$

$$y = -0.5 \cos 3t \quad \bullet \text{Equation of motion}$$

$$\text{Period: } \frac{2\pi}{|b|} = \frac{2\pi}{3}$$

$$\text{Frequency: } \frac{1}{\text{period}} = \frac{3}{2\pi}$$

$$\text{Amplitude: } |a| = |-0.5| = 0.5$$

See Figure 2.99.

» Try Exercise 28, page 208

**Damped Harmonic Motion**

The previous examples have assumed that there is no friction within the spring and no air resistance. If we consider friction and air resistance, then the motion of the mass tends to decrease as  $t$  increases. The motion is called **damped harmonic motion** and can be modeled by an equation in the form  $f(t) = ae^{-kt} \cos wt$ .

**EXAMPLE 3** » Model Damped Harmonic Motion


A mass on a spring has been displaced 14 inches below the equilibrium point and released. The damped harmonic motion of the mass is given by

$$f(t) = -14e^{-0.4t} \cos 2t, \quad t \geq 0$$

where  $f(t)$  is measured in inches and  $t$  is the time in seconds.

- Find the values of  $t$  for which  $f(t) = 0$ .
- Use a graphing utility to determine how long it will be until the absolute value of the displacement of the mass is always less than 0.01 inch.

**take note**

The motion in Example 3 is not periodic in a strict sense because the motion does not repeat exactly but tends to diminish as  $t$  increases. However, because the motion cycles every  $\pi$  seconds, we call  $\pi$  the *pseudoperiod* of the motion. In general, the damped harmonic motion modeled by  $f(t) = ae^{-kt} \cos \omega t$  has

**pseudoperiod**  $p = \frac{2\pi}{\omega}$  and

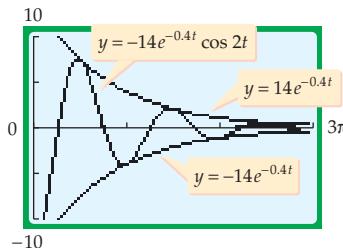
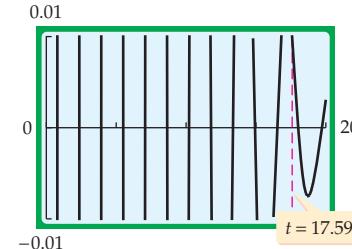
frequency  $f = \frac{\omega}{2\pi}$ .

**Solution**

- a.  $f(t) = 0$  if and only if  $\cos 2t = 0$ , and  $\cos 2t = 0$  when  $2t = \frac{\pi}{2} + n\pi$ .

Therefore,  $f(t) = 0$  when  $t = \frac{\pi}{4} + n\left(\frac{\pi}{2}\right) \approx 0.79 + n(1.57)$ . The displacement  $f(t) = 0$  first occurs when  $t \approx 0.79$  second. The graph of  $f$  in **Figure 2.100** confirms these results. **Figure 2.100** also shows that the graph of  $f$  lies on or between the graphs of  $y = -14e^{-0.4t}$  and  $y = 14e^{-0.4t}$ . Recall from Section 2.7 that  $-14e^{-0.4t}$  is the damping factor.

- b. We need to find the smallest value of  $t$  for which the absolute value of the displacement of the mass is always less than 0.01. **Figure 2.101** shows a graph of  $f$  using a viewing window of  $0 \leq t \leq 20$  and  $-0.01 \leq f(t) \leq 0.01$ . Use the TRACE or INTERSECT feature to determine that  $t \approx 17.59$  seconds.

**Figure 2.100****Figure 2.101**

» Try Exercise 34, page 209

The examples in this section involve the harmonic motion of a mass on a spring; however, there are other types of applications that involve harmonic motion. Some of these applications are illustrated in Exercises 29–32 on pages 208 and 209.

**Topics for Discussion**

1. The period of a simple harmonic motion is the same as the frequency of the motion. Do you agree? Explain.
2. In the simple harmonic motion modeled by  $y = 3 \cos 2\pi t$ , does the displacement  $y$  approach 0 as  $t$  increases without bound? Explain.
3. Explain how you know whether to use

$$y = a \cos 2\pi ft \quad \text{or} \quad y = a \sin 2\pi ft$$

to model a particular harmonic motion.



## Exercise Set 2.8

**In Exercises 1 to 8, find the amplitude, period, and frequency of the simple harmonic motion.**

1.  $y = 2 \sin 2t$

2.  $y = \frac{2}{3} \cos \frac{t}{3}$

3.  $y = 3 \cos \frac{2t}{3}$

4.  $y = 4 \sin 3t$

5.  $y = 4 \cos \pi t$

6.  $y = 2 \sin \frac{\pi t}{3}$

7.  $y = \frac{3}{4} \sin \frac{\pi t}{2}$

8.  $y = 5 \cos 2\pi t$

**In Exercises 9 to 12, write an equation for the simple harmonic motion that satisfies the given conditions. Assume that the maximum displacement occurs at  $t = 0$ . Sketch a graph of the equation.**

 9. Frequency = 1.5 cycles per second,  $a = 4$  inches

 10. Frequency = 0.8 cycle per second,  $a = 4$  centimeters

 11. Period = 1.5 seconds,  $a = \frac{3}{2}$  feet

 12. Period = 0.6 second,  $a = 1$  meter

**In Exercises 13 to 18, write an equation for the simple harmonic motion that satisfies the given conditions. Assume zero displacement at  $t = 0$ . Sketch a graph of the equation.**

 13. Amplitude 2 centimeters, period  $\pi$  seconds

 14. Amplitude 4 inches, period  $\frac{\pi}{2}$  seconds

15. Amplitude 1 inch, period 2 seconds

16. Amplitude 3 centimeters, period 1 second

17. Amplitude 2 centimeters, frequency 1 second

18. Amplitude 4 inches, frequency 4 seconds

**In Exercises 19 to 26, write an equation for the simple harmonic motion that satisfies the given conditions. Assume that the maximum displacement occurs when  $t = 0$ .**

 19. Amplitude  $\frac{1}{2}$  centimeter, frequency  $\frac{2}{\pi}$  cycle per second

20. Amplitude 3 inches, frequency  $\frac{1}{\pi}$  cycle per second

21. Amplitude 2.5 inches, frequency 0.5 cycle per second

 22. Amplitude 5 inches, frequency  $\frac{1}{8}$  cycle per second

 23. Amplitude  $\frac{1}{2}$  inch, period 3 seconds

24. Amplitude 5 centimeters, period 5 seconds

 25. Amplitude 4 inches, period  $\frac{\pi}{2}$  seconds

 26. Amplitude 2 centimeters, period  $\pi$  seconds

27. **SIMPLE HARMONIC MOTION** A mass of 32 units is in equilibrium suspended from a spring. The mass is pulled down 2 feet and released. Find the period, frequency, and amplitude of the resulting simple harmonic motion. Write an equation of the motion. Assume a spring constant of  $k = 8$ .

28. **SIMPLE HARMONIC MOTION** A mass of 27 units is in equilibrium suspended from a spring. The mass is pulled down 1.5 feet and released. Find the period, frequency, and amplitude of the resulting simple harmonic motion. Write an equation of the motion. Assume a spring constant of  $k = 3$ .

29. **SOUND MODEL** A trumpet player plays a musical note and sustains the volume of the sound for several seconds. A model of the sound wave is  $V(t) = 0.1 \sin 392\pi t$ , where  $V(t)$  is the variation in air pressure (from normal air pressure) after  $t$  seconds, measured in pounds per square inch.

a. Find the frequency and period of the sound wave. Assume the frequency is in cycles per second.

b. If the trumpet player increases the volume of the sound, how does the model need to change?

30. **TIDAL CYCLE** During a 12.5-hour period, the water level at Grays Harbor, Washington, started at mean sea level, rose to 9.3 feet above sea level, dropped to 9.3 feet below sea level, and then returned to mean sea level. Find a simple harmonic motion equation that models the height  $h$  of the tide above or below mean sea level for this 12.5-hour period.

- 31. AMUSEMENT PARK RIDE** A person riding on a circular Ferris wheel reaches a maximum height of 78 feet and a minimum height of 4 feet above ground level. When in uniform motion, the Ferris wheel makes one complete revolution every 45 seconds. Find an equation that gives the height  $h$  above ground level of a person riding the Ferris wheel as a function of the time  $t$ . Assume the Ferris wheel is in motion and the person is 4 feet above ground level at  $t = 0$  seconds.

**32. TIME IN MOTION** A clock with a minute hand that is 12 inches long is hanging on a wall. At precisely 8:00 A.M. the distance from the tip of the minute hand to the floor is 6.5 feet. Thirty minutes later, this distance is 4.5 feet. Find an equation that gives the distance  $d$ , in feet, from the floor to the tip of the minute hand as a function of the number of minutes  $t$  after 8:00 A.M.

## »»» Connecting Concepts

- 41.** Assume that a mass of  $m$  pounds on the end of a spring is oscillating in simple harmonic motion. What will be the effect on the period of the motion if the mass is increased to  $9m$ ?

**42.** A mass on the end of a spring oscillates in simple harmonic motion. What will be the effect on the frequency of the motion if the spring is replaced with a stiffer spring that has a spring constant four times that of the original spring?

**In Exercises 33 to 40, each of the equations models the damped harmonic motion of a mass on a spring.**

- a. Find the number of complete oscillations that occur during the time interval  $0 \leq t \leq 10$  seconds.

b. Use a graph to determine how long it will be (to the nearest tenth of a second) until the absolute value of the displacement of the mass is always less than 0.01.

33.  $f(t) = 4e^{-0.1t} \cos 2t$       34.  $f(t) = 12e^{-0.6t} \cos t$

35.  $f(t) = -6e^{-0.09t} \cos 2\pi t$       36.  $f(t) = -11e^{-0.4t} \cos \pi t$

37.  $f(t) = e^{-0.5t} \cos 2\pi t$       38.  $f(t) = e^{-0.2t} \cos 3\pi t$

39.  $f(t) = e^{-0.75t} \cos 2\pi t$       40.  $f(t) = e^{-t} \cos 2\pi t$

 In Exercises 43 to 46, use a graphing utility to determine whether both of the given functions model the same damped harmonic motion.

$$43. f(t) = \sqrt{2} e^{-0.2t} \sin\left(t + \frac{\pi}{4}\right)$$

$$g(t) = e^{-0.2t}(\cos t + \sin t)$$

**44.**  $f(t) = 5e^{-0.3t} \cos(t - 0.927295)$   
 $g(t) = e^{-0.2t}(3 \cos t + 4 \sin t)$

$$45. f(t) = 13e^{-0.4t} \cos(t - 1.176005)$$

$$g(t) = e^{-0.4t} (5 \cos t + 12 \sin t)$$

$$46. \quad f(t) = \sqrt{2} e^{-0.2t} \cos(t + 0.785398)$$

$$g(t) = e^{-0.2t}(\cos t - \sin t)$$

## »»» Projects

1.   THREE TYPES OF DAMPED HARMONIC MOTION

In some cases the damped harmonic motion of a mass on the end of a spring does not cycle about the equilibrium point. The following three functions illustrate different types of damped harmonic motion. Graph each function and then write a few sentences that explain the major differences among the motions.

$$f(t) = (0.5t + 1)e^{-0.5t} \quad g(t) = -2e^{-0.4t} + 5e^{-0.2t}$$

- 2.**  **LOGARITHMIC DECREMENT** If a damped harmonic motion is modeled by

$$f(t) = ae^{-\alpha t} \cos \omega t$$

then the ratio of any two consecutive relative maxima of the motion is a constant,  $\gamma$ .

- a. Determine  $\gamma$  for the damped harmonic motion modeled by

$$f(t) = -14e^{-0.4t} \cos 2t, \quad t \geq 0$$

- b. The constant  $\Delta = \frac{2\pi\alpha}{\omega}$  is called the **logarithmic decrement** of the motion. Compute  $\Delta$  for the damped harmonic motion in the equation in a. How does  $\ln \gamma$  compare with  $\Delta$ ?



## Exploring Concepts with Technology

### Sinusoidal Families



Some graphing calculators have a feature that allows you to graph a family of functions easily. For instance, entering  $Y_1=\{2,4,6\}\sin(X)$  in the  $Y=$  menu and pressing the **GRAPH** key on a TI-83/TI-83 Plus/TI-84 Plus calculator produces a graph of the three functions  $y = 2 \sin x$ ,  $y = 4 \sin x$ , and  $y = 6 \sin x$ , all displayed in the same window.

1. Use a graphing calculator to graph  $Y_1=\{2,4,6\}\sin(X)$ . Write a sentence that indicates the similarities and differences among the three graphs.
2. Use a graphing calculator to graph  $Y_1=\sin(\{\pi, 2\pi, 4\pi\}X)$ . Write a sentence that indicates the similarities and differences among the three graphs.
3. Use a graphing calculator to graph  $Y_1=\sin(X+\{\pi/4, \pi/6, \pi/12\})$ . Write a sentence that indicates the similarities and differences among the three graphs.
4. A student has used a graphing calculator to graph  $Y_1=\sin(X+\{\pi, 3\pi, 5\pi\})$  and expects to see three graphs. However, the student sees only one graph displayed in the graph window. Has the calculator displayed all three graphs? Explain.

## Chapter 2 Summary

### 2.1 Angles and Arcs

- An angle is in standard position when its initial side is along the positive  $x$ -axis and its vertex is at the origin of the coordinate axes.
- Angle  $\alpha$  is an acute angle when  $0^\circ < \alpha < 90^\circ$ ; it is an obtuse angle when  $90^\circ < \alpha < 180^\circ$ .
- $\alpha$  and  $\beta$  are complementary angles when  $\alpha + \beta = 90^\circ$ ; they are supplementary angles when  $\alpha + \beta = 180^\circ$ .
- One radian is the measure of a central angle subtended by an arc of length  $r$  on a circle of radius  $r$ .
- The length of the arc  $s$  that subtends the central angle  $\theta$  (in radians) on a circle of radius  $r$  is given by  $s = r\theta$ .
- A point moves on a circular path with radius  $r$  at a constant rate of  $\theta$  radians per unit of time  $t$ . Its linear speed is  $v = \frac{s}{t}$ , and its angular speed is  $\omega = \frac{\theta}{t}$ .

### 2.2 Right Triangle Trigonometry

- Let  $\theta$  be an acute angle of a right triangle. The six trigonometric functions of  $\theta$  are given by

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

### 2.3 Trigonometric Functions of Any Angle

- Let  $P(x, y)$  be a point, except the origin, on the terminal side of an angle  $\theta$  in standard position. The six trigonometric functions of  $\theta$  are

$$\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y}, \quad y \neq 0$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x}, \quad x \neq 0$$

$$\tan \theta = \frac{y}{x}, \quad x \neq 0 \quad \cot \theta = \frac{x}{y}, \quad y \neq 0$$

- Given  $\angle\theta$  in standard position, its reference angle  $\theta'$  is the smallest positive angle formed by the terminal side of  $\angle\theta$  and the  $x$ -axis.

### 2.4 Trigonometric Functions of Real Numbers

- The wrapping function pairs a real number with a point on the unit circle.

- Let  $W$  be the wrapping function,  $t$  be a real number, and  $W(t) = P(x, y)$ . Then the trigonometric functions of the real number  $t$  are defined as follows:

$$\sin t = y$$

$$\csc t = \frac{1}{y}, \quad y \neq 0$$

$$\cos t = x$$

$$\sec t = \frac{1}{x}, \quad x \neq 0$$

$$\tan t = \frac{y}{x}, \quad x \neq 0$$

$$\cot t = \frac{x}{y}, \quad y \neq 0$$

- $\sin t$ ,  $\csc t$ ,  $\tan t$ , and  $\cot t$  are odd functions.

- $\cos t$  and  $\sec t$  are even functions.

- $\sin t$ ,  $\cos t$ ,  $\sec t$ , and  $\csc t$  have period  $2\pi$ .

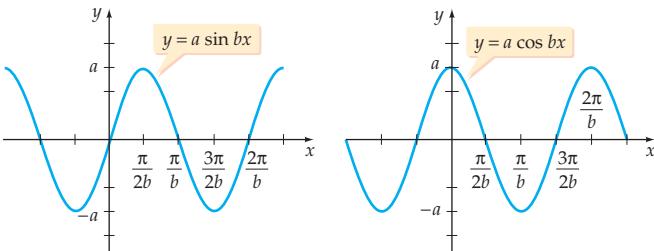
- $\tan t$  and  $\cot t$  have period  $\pi$ .

- The domain and range of each trigonometric function is given in Table 2.6 on page 160.

- An identity is an equation that is true for every number in the domain of the equation. The Fundamental Trigonometric Identities are given on page 163.

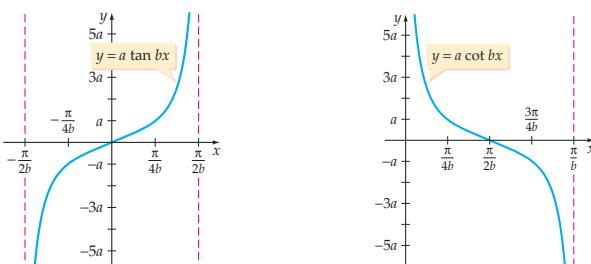
## 2.5 Graphs of the Sine and Cosine Functions

- The graphs of  $y = a \sin bx$  and  $y = a \cos bx$  both have an amplitude of  $|a|$  and a period of  $\frac{2\pi}{b}$ . The graph of each function for  $a > 0$  and  $b > 0$  is shown below.

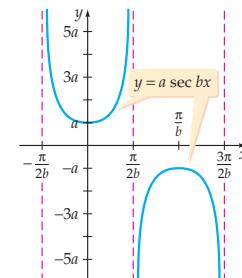
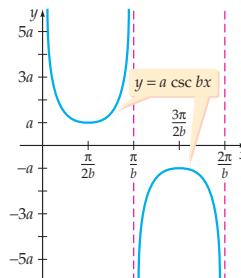


## 2.6 Graphs of the Other Trigonometric Functions

- The graphs of  $y = a \tan bx$  and  $y = a \cot bx$  both have a period of  $\frac{\pi}{b}$ . The graph of one period of each function for  $a > 0$  and  $b > 0$  is shown below.



- The graphs of  $y = a \csc bx$  and  $y = a \sec bx$  both have a period of  $\frac{2\pi}{b}$ . The graphs of one period of each function, with  $a > 0$  and  $b > 0$ , are shown below.



## 2.7 Graphing Techniques

- If  $y = f(bx + c)$ , where  $f$  is a trigonometric function, then the graph of  $y = f(bx + c)$  can be produced by shifting the graph of  $y = f(bx)$  horizontally  $-\frac{c}{b}$  units. The value  $-\frac{c}{b}$  is called the phase shift of  $y = f(bx + c)$ .
- Addition of ordinates is a method of graphing the sum of two functions by geometrically adding the values of their  $y$ -coordinates.
- The factor  $g(x)$  in  $f(x) = g(x) \cos x$  is called a damping factor. The graph of  $f$  lies on or between the graphs of the equations  $y = g(x)$  and  $y = -g(x)$ .

## 2.8 Harmonic Motion—An Application of the Sine and Cosine Functions

- The equations of simple harmonic motion are  $y = a \cos 2\pi ft$  and  $y = a \sin 2\pi ft$  where  $|a|$  is the amplitude,  $f$  is the frequency,  $y$  is the displacement, and  $t$  is the time.
- Functions of the form  $f(t) = ae^{-kt} \cos \omega t$  are used to model some forms of damped harmonic motion.



## Chapter 2 Assessing Concepts

1. True or False: In the formula  $s = r\theta$ , the measure of angle  $\theta$  must be in radians.
2. True or False:  $\sec^2 \theta + \tan^2 \theta = 1$  is an identity.
3. True or False: The measure of one radian differs depending on the radius of the circle used.
4. True or False: The graph of  $y = \sin x$  is symmetric with respect to the origin.
5. What is the measure of the reference angle for the angle  $\theta = \frac{3\pi}{4}$ ?
6. What is the point defined by  $W\left(\frac{\pi}{2}\right)$ ?
7. What is the period of the graph of  $y = \frac{1}{2} \cos \frac{3\pi}{4} x$ ?
8. Explain how to use the graph of  $y_1 = \sin \frac{x}{2}$  to produce the graph of  $y_2 = \sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$ .
9. What is the domain of the function  $y = \cot x$ ?
10. Consider the graph of  $y = \sec x$  on the interval  $[0, 2\pi]$ . What are the equations of the vertical asymptotes of the graph?



## Chapter 2 Review Exercises

1. Find the complement and supplement of the angle  $\theta$  whose measure is  $65^\circ$ .
2. Find the measure of the reference angle  $\theta'$  for the angle  $\theta$  whose measure is  $980^\circ$ .
3. Convert 2 radians to the nearest hundredth of a degree.
4. Convert  $315^\circ$  to radian measure.
5. Find the length (to the nearest hundredth of a meter) of the arc on a circle of radius 3 meters that subtends an angle of  $75^\circ$ .
6. Find the radian measure of the angle subtended by an arc of length 12 centimeters on a circle whose radius is 40 centimeters.
7. A car with a 16-inch-radius wheel is moving with a speed of 50 mph. Find the angular speed (to the nearest radian per second) of the wheel.

**In Exercises 8 to 11, let  $\theta$  be an acute angle of a right triangle and  $\csc \theta = \frac{3}{2}$ . Evaluate each function.**

8.  $\cos \theta$
9.  $\cot \theta$
10.  $\sin \theta$
11.  $\sec \theta$
12. Find the values of the six trigonometric functions of an angle in standard position with the point  $P(1, -3)$  on the terminal side of the angle.
13. Find the exact value of
  - a.  $\sec 150^\circ$
  - b.  $\tan\left(-\frac{3\pi}{4}\right)$
  - c.  $\cot(-225^\circ)$
  - d.  $\cos\left(\frac{2\pi}{3}\right)$
14.  Find the value of each of the following to the nearest ten-thousandth.
  - a.  $\cos 123^\circ$
  - b.  $\cot 4.22$
  - c.  $\sec 612^\circ$
  - d.  $\tan \frac{2\pi}{5}$
15. Given  $\cos \phi = -\frac{\sqrt{3}}{2}$ ,  $180^\circ < \phi < 270^\circ$ , find the exact value of
  - a.  $\sin \phi$
  - b.  $\tan \phi$
16. Given  $\tan \phi = -\frac{\sqrt{3}}{3}$ ,  $90^\circ < \phi < 180^\circ$ , find the exact value of
  - a.  $\sec \phi$
  - b.  $\csc \phi$
17. Given  $\sin \phi = -\frac{\sqrt{2}}{2}$ ,  $270^\circ < \phi < 360^\circ$ , find the exact value of
  - a.  $\cos \phi$
  - b.  $\cot \phi$

- 18.** Let  $W$  be the wrapping function. Evaluate

a.  $W(\pi)$     b.  $W\left(-\frac{\pi}{3}\right)$     c.  $W\left(\frac{5\pi}{4}\right)$     d.  $W(28\pi)$

- 19.** Is the function defined by  $f(x) = \sin(x) \tan(x)$  even, odd, or neither?

**In Exercises 20 and 21, use the unit circle to show that each equation is an identity.**

**20.**  $\cos(\pi + t) = -\cos t$

**21.**  $\tan(-t) = -\tan t$

**In Exercises 22 to 27, use trigonometric identities to write each expression in terms of a single trigonometric function or as a constant.**

**22.**  $1 + \frac{\sin^2 \phi}{\cos^2 \phi}$

**23.**  $\frac{\tan \phi + 1}{\cot \phi + 1}$

**24.**  $\frac{\cos^2 \phi + \sin^2 \phi}{\csc \phi}$

**25.**  $\sin^2 \phi (\tan^2 \phi + 1)$

**26.**  $1 + \frac{1}{\tan^2 \phi}$

**27.**  $\frac{\cos^2 \phi}{1 - \sin^2 \phi} - 1$

**In Exercises 28 to 33, state the amplitude (if it exists), period, and phase shift of the graph of each function.**

**28.**  $y = 3 \cos(2x - \pi)$

**29.**  $y = 2 \tan 3x$

**30.**  $y = -2 \sin\left(3x + \frac{\pi}{3}\right)$

**31.**  $y = \cos\left(2x - \frac{2\pi}{3}\right) + 2$

**32.**  $y = -4 \sec\left(4x - \frac{3\pi}{2}\right)$

**33.**  $y = 2 \csc\left(x - \frac{\pi}{4}\right) - 3$

**In Exercises 34 to 51, graph each function.**

**34.**  $y = 2 \cos \pi x$

**35.**  $y = -\sin \frac{2x}{3}$

**36.**  $y = 2 \sin \frac{3x}{2}$

**37.**  $y = \cos\left(x - \frac{\pi}{2}\right)$

**38.**  $y = \frac{1}{2} \sin\left(2x + \frac{\pi}{4}\right)$

**39.**  $y = 3 \cos 3(x - \pi)$

**40.**  $y = -\tan \frac{x}{2}$

**41.**  $y = 2 \cot 2x$

**42.**  $y = \tan\left(x - \frac{\pi}{2}\right)$

**43.**  $y = -\cot\left(2x + \frac{\pi}{4}\right)$

**44.**  $y = -2 \csc\left(2x - \frac{\pi}{3}\right)$

**45.**  $y = 3 \sec\left(x + \frac{\pi}{4}\right)$

**46.**  $y = 3 \sin 2x - 3$

**47.**  $y = 2 \cos 3x + 3$

**48.**  $y = -\cos\left(3x + \frac{\pi}{2}\right) + 2$

**49.**  $y = 3 \sin\left(4x - \frac{2\pi}{3}\right) - 3$

**50.**  $y = 2 - \sin 2x$

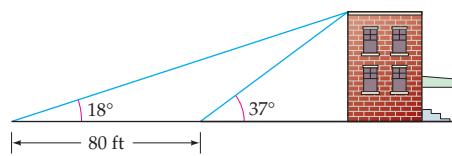
**51.**  $y = \sin x - \sqrt{3} \cos x$

- 52.** A car climbs a hill that has a constant angle of  $4.5^\circ$  for a distance of 1.14 miles. What is the car's increase in altitude?

- 53.** **HEIGHT OF A TREE** A tree casts a shadow of 8.55 feet when the angle of elevation of the sun is  $55.3^\circ$ . Find the height of the tree.

- 54.** **LINEAR SPEEDS ON A CAROUSEL** A carousel has two circular rings of horses. The inner ring has a radius of 14.5 feet and the outer ring has a radius of 21.0 feet. The carousel makes one complete revolution every 24 seconds. How much greater, in feet per second, is the linear speed of a horse in the outer ring than the linear speed of a horse in the inner ring? Round to the nearest tenth of a foot per second.

- 55.** **HEIGHT OF A BUILDING** Find the height of a building if the angle of elevation to the top of the building changes from  $18^\circ$  to  $37^\circ$  as an observer moves a distance of 80 feet toward the building.



- 56.** Find the amplitude, period, and frequency of the simple harmonic motion given by  $y = 2.5 \sin 50t$ .

- 57.** A mass of 5 kilograms is in equilibrium suspended from a spring. The mass is pulled down 0.5 foot and released. Find the period, frequency, and amplitude of the motion, assuming the mass oscillates in simple harmonic motion. Write an equation of motion. Assume  $k = 20$ .

- 58.** Graph the damped harmonic motion that is modeled by

$$f(t) = 3e^{-0.75t} \cos \pi t$$

where  $t$  is in seconds. Use the graph to determine how long (to the nearest tenth of a second) it will be until the absolute value of the displacement of the mass is always less than 0.01.

»»» Quantitative Reasoning: *Find the Periods of Trigonometric Functions and Combined Musical Sound Tracks* »»»

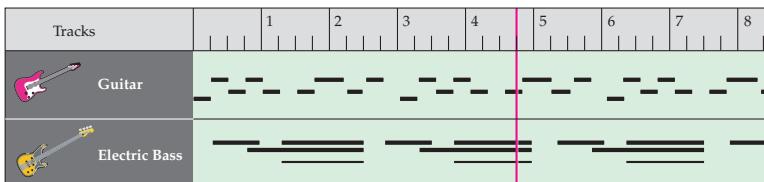
**QR1.** Let  $f$  be a function with period  $j$  and let  $g$  be a function with period  $k$ . The function  $f + g$  is a periodic function provided there exist natural numbers  $m$  and  $n$  such that  $\frac{j}{k} = \frac{m}{n}$ . Furthermore, if  $m$  and  $n$  have no common prime factors, then the period of  $f + g$  is  $n \cdot j = m \cdot k$ . For each of the following, determine the period of  $f + g$ .

a.  $f(x) = \sin x, g(x) = \cos \frac{2}{3}x$       b.  $f(x) = \cos 3\pi x, g(x) = \sin \frac{1}{2}\pi x$

c.  $f(x) = \tan 2x, g(x) = \sin 3x$       d.  $f(x) = \cot \frac{2}{3}x, g(x) = \cos \frac{3}{4}x$

e.  $f(x) = \sec \frac{4\pi}{5}x, g(x) = \cot \frac{2\pi}{3}x$       f.  $f(x) = \csc \frac{5}{2}x, g(x) = \tan \frac{1}{4}x$

**QR2.** The following figure shows a guitar sound track with a period of 3 seconds and an electric bass track with a period of 2.5 seconds. Use the procedure in Exercise QR1 to find the period of the music produced when the two sound tracks are played together.



**QR3.** A tambourine sound track has a period of 1.25 seconds and a drum sound track has a period of 2.25 seconds. Use the procedure in Exercise QR1 to find the period of the music produced when the two sound tracks are played together.

**QR4.** An alto sax sound track has a period of 6 seconds, a piano sound track has a period of 4.5 seconds, and an electric guitar sound track has a period of 27 seconds. Find the period of the music produced when the three sound tracks are played together.

## Chapter 2 Test

- Convert  $150^\circ$  to exact radian measure.
- Find the supplement of the angle whose radian measure is  $\frac{11}{12}\pi$ . Express your answer in terms of  $\pi$ .
- Find the length (to the nearest tenth of a centimeter) of an arc that subtends a central angle of  $75^\circ$  in a circle of radius 10 centimeters.
- ANGULAR SPEED** A wheel is rotating at 6 revolutions per second. Find the angular speed in radians per second.
- LINEAR SPEED** A wheel with a diameter of 16 centimeters is rotating at 10 radians per second. Find the linear speed (in centimeters per second) of a point on the edge of the wheel.
- If  $\theta$  is an acute angle of a right triangle and  $\tan \theta = \frac{3}{7}$ , find  $\sec \theta$ .

7. Use a calculator to find the value of  $\csc 67^\circ$  to the nearest ten-thousandth.
8. Find the exact value of  $\tan \frac{\pi}{6} \cos \frac{\pi}{3} - \sin \frac{\pi}{2}$ .
9. Find the exact coordinates of  $W\left(\frac{11\pi}{6}\right)$ .
10. Express  $\frac{\sec^2 t - 1}{\sec^2 t}$  in terms of a single trigonometric function.
11. State the period of  $y = -4 \tan 3x$ .
12. State the amplitude, period, and phase shift for the function  $y = -3 \cos\left(2x + \frac{\pi}{2}\right)$ .
13. State the period and phase shift for the function  $y = 2 \cot\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$ .
14. Graph one full period of  $y = 3 \cos \frac{1}{2}x$ .
15. Graph one full period of  $y = -2 \sec \frac{1}{2}x$ .
16. Write a sentence that explains how to obtain the graph of  $y = 2 \sin\left(2x - \frac{\pi}{2}\right) - 1$  from the graph of  $y = 2 \sin 2x$ .
17. Graph one full period of  $y = 2 - \sin \frac{x}{2}$ .
18. Graph one full period of  $y = \sin x - \cos 2x$ .
19. **HEIGHT OF A TREE** The angle of elevation from point  $A$  to the top of a tree is  $42.2^\circ$ . From point  $B$ , which is 5.24 meters from  $A$  and on a line through the base of the tree and  $A$ , the angle of elevation to the top of the tree is  $37.4^\circ$ . Find the height of the tree.
20. Write the equation for simple harmonic motion given that the amplitude is 13 feet, the period is 5 seconds, and the displacement is zero when  $t = 0$ .



## Cumulative Review Exercises

1. Find the distance between the points  $P(-3, 2)$  and  $Q(4, 1)$ .
2. The hypotenuse of a right triangle has a length of 1 and one of its legs has a length of  $\frac{1}{2}$ . Find the length of the other leg.
3. Find the  $x$ - and the  $y$ -intercept(s) of the graph of  $f(x) = x^2 - 9$ .
4. Determine whether  $f(x) = \frac{x}{x^2 + 1}$  is an even function or an odd function.
5. Find the inverse of  $f(x) = \frac{x}{2x - 3}$ .
6. Use interval notation to state the domain of  $f(x) = \frac{2}{x - 4}$ .
7. Solve:  $x^2 + x - 6 = 0$
8. Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = f(x - 3)$ .
9. Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = f(-x)$ .
10. Convert  $300^\circ$  to radians.
11. Convert  $\frac{5\pi}{4}$  to degrees.
12. Evaluate  $f(x) = \sin\left(x + \frac{\pi}{6}\right)$  for  $x = \frac{\pi}{3}$ .
13. Evaluate  $f(x) = \sin x + \sin \frac{\pi}{6}$  for  $x = \frac{\pi}{3}$ .
14. Find the exact value of  $\cos^2 45^\circ + \sin^2 60^\circ$ .
15. Determine the sign of  $\tan \theta$  given that  $\frac{\pi}{2} < \theta < \pi$ .
16. What is the measure of the reference angle for the angle  $\theta = 210^\circ$ ?
17. What is the measure of the reference angle for the angle  $\theta = \frac{2\pi}{3}$ ?
18. Use interval notation to state the domain of  $f(x) = \sin x$ , where  $x$  is a real number.
19. Use interval notation to state the range of  $f(x) = \cos x$ , where  $x$  is a real number.
20. If  $\theta$  is an acute angle of a right triangle and  $\tan \theta = \frac{3}{4}$ , find  $\sin \theta$ .

# 3

# Trigonometric Identities and Equations

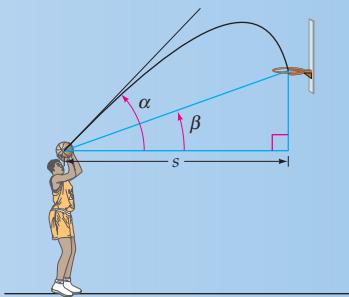


- 3.1 Verification of Trigonometric Identities**
- 3.2 Sum, Difference, and Cofunction Identities**
- 3.3 Double- and Half-Angle Identities**
- 3.4 Identities Involving the Sum of Trigonometric Functions**
- 3.5 Inverse Trigonometric Functions**
- 3.6 Trigonometric Equations**

## Trigonometric Equations and Applications

This chapter explores trigonometric identities, inverse trigonometric functions, and trigonometric equations. Understanding these concepts will provide you with a foundation for more advanced mathematical studies. You will also gain additional problem-solving skills and learn

procedures that can be used to solve a wide variety of real-life applications. For example, in **Exercise 83 on page 266**, you will determine the minimum altitude a communications satellite must attain to provide service over a desired region. In **Quantitative Reasoning Exercise 2a on page 289**, you will find the launch angle that minimizes the velocity at which a basketball must be launched to make a shot from a given distance.



## Online Study Center

For online student resources, such as section quizzes, visit this website:  
[college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

## Section 3.1

- Fundamental Trigonometric Identities
- Verification of Trigonometric Identities

# Verification of Trigonometric Identities

## ■ Fundamental Trigonometric Identities

The domain of an equation consists of all values of the variable for which every term is defined. For example, the domain of

$$\frac{\sin x \cos x}{\sin x} = \cos x \quad (1)$$

includes all real numbers  $x$  except  $x = k\pi$ , where  $k$  is an integer, because  $\sin x = 0$  for  $x = k\pi$ , and division by 0 is undefined. An **identity** is an equation that is true for all of its domain values. **Table 3.1** lists identities that were introduced earlier.

**Table 3.1 Fundamental Trigonometric Identities**

|                               |   |  |   |
|-------------------------------|---|--|---|
| <b>Reciprocal identities</b>  | $\sin x = \frac{1}{\csc x}$                 | $\cos x = \frac{1}{\sec x}$                  | $\tan x = \frac{1}{\cot x}$                 |
| <b>Ratio identities</b>       | $\tan x = \frac{\sin x}{\cos x}$            | $\cot x = \frac{\cos x}{\sin x}$             |   |
| <b>Pythagorean identities</b> | $\sin^2 x + \cos^2 x = 1$                   | $\tan^2 x + 1 = \sec^2 x$                    | $1 + \cot^2 x = \csc^2 x$                   |
| <b>Odd–even identities</b>    | $\sin(-x) = -\sin x$<br>$\cos(-x) = \cos x$ | $\tan(-x) = -\tan x$<br>$\cot(-x) = -\cot x$ | $\sec(-x) = \sec x$<br>$\csc(-x) = -\csc x$ |

## ■ Verification of Trigonometric Identities

To verify an identity, we show that one side of the identity can be rewritten in an equivalent form that is identical to the other side. There is no one method that can be used to verify every identity; however, the following guidelines should prove useful.

### Guidelines for Verifying Trigonometric Identities

- If one side of the identity is more complex than the other, then it is generally best to try first to simplify the more complex side until it becomes identical to the other side.
- Perform indicated operations such as adding fractions or squaring a binomial. Also be aware of any factorization that may help you to achieve your goal of producing the expression on the other side.
- Make use of previously established identities that enable you to rewrite one side of the identity in an equivalent form.
- Rewrite one side of the identity so that it involves only sines and/or cosines.

Rewrite one side of the identity in terms of a single trigonometric function.

- Multiplying both the numerator and the denominator of a fraction by the same factor (such as the conjugate of the denominator or the conjugate of the numerator) may get you closer to your goal.
- Keep your goal in mind. Does it involve products, quotients, sums, radicals, or powers? Knowing exactly what your goal is may provide the insight you need to verify the identity.

In Example 1 we verify an identity by rewriting the more complicated side so that it involves only sines and cosines.

### **E X A M P L E 1** »» Change to Sines and Cosines to Verify an Identity

Verify the identity  $\sin x \cot x \sec x = 1$ .

#### Solution

The left side of the identity is more complicated than the right side. We will try to verify the identity by rewriting the left side so that it involves only sines and cosines.

$$\begin{aligned}\sin x \cot x \sec x &= \sin x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} \\&= \frac{\cancel{\sin x} \cos x}{\cancel{\sin x} \cos x} \\&= 1\end{aligned}$$

- Apply the fundamental identities  $\cot x = \frac{\cos x}{\sin x}$  and  $\sec x = \frac{1}{\cos x}$ .
- Multiply the fractions to produce a single fraction.
- Simplify by dividing out the common factors.

We have rewritten the left side of the equation so that it is identical to the right side. Thus we have verified that the equation is an identity.

#### »» Try Exercise 2, page 222

**QUESTION** Is  $\cos(-x) = \cos x$  an identity?

### **E X A M P L E 2** »» Use a Pythagorean Identity to Verify an Identity

Verify the identity  $1 - 2 \sin^2 x = 2 \cos^2 x - 1$ .

**ANSWER** Yes,  $\cos(-x) = \cos x$  is one of the odd–even identities shown in **Table 3.1**.

**take note**

Each of the Pythagorean identities can be written in several different forms. For instance,

$$\sin^2 x + \cos^2 x = 1$$

also can be written as

$$\sin^2 x = 1 - \cos^2 x$$

and as

$$\cos^2 x = 1 - \sin^2 x$$

**Solution**

Rewrite the right side of the equation.

$$\begin{aligned} 2 \cos^2 x - 1 &= 2(1 - \sin^2 x) - 1 \\ &= 2 - 2 \sin^2 x - 1 \\ &= 1 - 2 \sin^2 x \end{aligned}$$

- $\cos^2 x = 1 - \sin^2 x$

- Simplify.

» Try Exercise 12, page 223

**Figure 3.1** shows the graph of  $f(x) = 1 - 2 \sin^2 x$  and the graph of  $g(x) = 2 \cos^2 x - 1$  on the same coordinate axes. The fact that the graphs appear to be identical on the interval  $[-2\pi, 2\pi]$  supports the verification in Example 2.

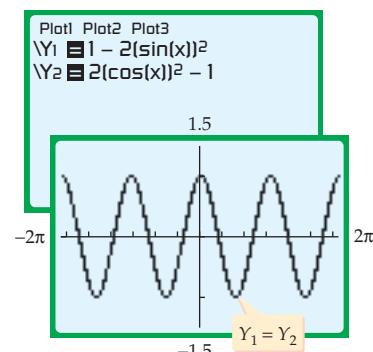


Figure 3.1

**EXAMPLE 3 » Factor to Verify an Identity**

Verify the identity  $\csc^2 x - \cos^2 x \csc^2 x = 1$ .

**Solution**

Simplify the left side of the equation.

$$\begin{aligned} \csc^2 x - \cos^2 x \csc^2 x &= \csc^2 x(1 - \cos^2 x) \\ &= \csc^2 x \sin^2 x \\ &= \frac{1}{\sin^2 x} \cdot \sin^2 x = 1 \end{aligned}$$

- Factor out  $\csc^2 x$ .
- $1 - \cos^2 x = \sin^2 x$
- $\csc^2 x = \frac{1}{\sin^2 x}$

» Try Exercise 24, page 223

In the next example we make use of the guideline that states that it may be helpful to multiply both the numerator and the denominator of a fraction by the same factor.

**EXAMPLE 4 » Multiply by a Conjugate to Verify an Identity**

$$\text{Verify the identity } \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}.$$

Continued ►

**take note**

The sum  $a + b$  and the difference  $a - b$  are called conjugates of each other.

**Solution**

Multiply the numerator and denominator of the left side of the identity by the conjugate of  $1 + \cos x$ , which is  $1 - \cos x$ .

$$\begin{aligned}\frac{\sin x}{1 + \cos x} &= \frac{\sin x}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} \\&= \frac{\sin x(1 - \cos x)}{1 - \cos^2 x} \\&= \frac{\sin x(1 - \cos x)}{\sin^2 x} = \frac{1 - \cos x}{\sin x}\end{aligned}$$

» Try Exercise 34, page 223

In Example 5 we verify an identity by first rewriting the more complicated side so that it involves only sines and cosines. After further algebraic simplification we are able to establish that the equation is an identity.

### EXAMPLE 5 » Change to Sines and Cosines to Verify an Identity

Verify the identity  $\frac{\sin x + \tan x}{1 + \cos x} = \tan x$ .

**Solution**

The left side of the identity is more complicated than the right side. We will try to verify the identity by rewriting the left side so that it involves only sines and cosines.

$$\begin{aligned}\frac{\sin x + \tan x}{1 + \cos x} &= \frac{\sin x + \frac{\sin x}{\cos x}}{1 + \cos x} \\&= \frac{\frac{\sin x \cos x}{\cos x} + \frac{\sin x}{\cos x}}{1 + \cos x}\end{aligned}$$

- Use the identity  
 $\tan x = \frac{\sin x}{\cos x}$ .

$$\begin{aligned}&= \frac{\sin x \cos x + \sin x}{\cos x} \\&= \frac{\sin x \cos x + \sin x}{\cos x} \div \frac{1 + \cos x}{1} \\&= \frac{\sin x \cos x + \sin x}{\cos x} \cdot \frac{1}{1 + \cos x} \\&= \frac{\sin x (\cos x + 1)}{\cos x} \cdot \frac{1}{1 + \cos x}\end{aligned}$$

- The common denominator for the terms in the numerator is  $\cos x$ . Rewrite the numerator so that each term has a denominator of  $\cos x$ .

- Add the terms in the numerator.

- Rewrite the complex fraction as a division.

- Invert and multiply.

- Factor.

$$\begin{aligned}
 &= \frac{\sin x (1 + \cos x)}{\cos x} \cdot \frac{1}{1 + \cos x} \quad \bullet \text{ Simplify.} \\
 &= \frac{\sin x}{\cos x} \\
 &= \tan x \quad \bullet \text{ Use the identity } \frac{\sin x}{\cos x} = \tan x.
 \end{aligned}$$

We have rewritten the left side of the equation so that it is identical to the right side. Thus we have verified that the equation is an identity.

**Try Exercise 44, page 224**

In the previous examples we verified trigonometric identities by rewriting one side of the identity in equivalent forms until that side appeared identical to the other side. A second method is to work with each side *separately* to produce a trigonometric expression that is equivalent to both sides. In the following paragraphs we use this procedure to verify the identity

$$\frac{1 + \cos x}{1 - \cos x} = (\csc x + \cot x)^2$$

Working with the left side gives us

$$\frac{1 + \cos x}{1 - \cos x} = \frac{1 + \cos x}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x} \quad (1)$$

• Multiply both numerator and denominator by the conjugate of the denominator.

$$= \frac{1 + 2 \cos x + \cos^2 x}{1 - \cos^2 x} \quad (2)$$

• Multiply and rewrite as a single fraction.

$$= \frac{1 + 2 \cos x + \cos^2 x}{\sin^2 x} \quad (3)$$

• Use the identity  $1 - \cos^2 x = \sin^2 x$ .

When you reach a point at which you are unsure how to proceed, stop and see what results can be obtained by working with the right side.

$$(\csc x + \cot x)^2 = \csc^2 x + 2 \csc x \cot x + \cot^2 x \quad (4) \quad \bullet \text{ Square the binomial.}$$

$$= \frac{1}{\sin^2 x} + 2 \left( \frac{1}{\sin x} \right) \left( \frac{\cos x}{\sin x} \right) + \frac{\cos^2 x}{\sin^2 x} \quad (5) \quad \bullet \text{ Rewrite using fundamental identities.}$$

$$= \frac{1 + 2 \cos x + \cos^2 x}{\sin^2 x} \quad (6) \quad \bullet \text{ Add the fractions.}$$

At this point we have verified the identity because we have shown that each side is equal to the same trigonometric expression. It is worth noting that if we arranged the above equations in the order (1), (2), (3), (5), and (4), we would have a verification that starts with the left side of the original identity and produces the right side.

When we verify an identity, we are not allowed to perform the same mathematical operation on both sides of the equation because this can lead to an incorrect result. For instance, consider the equation  $\tan x = -\tan x$ , which is not an identity. If we square both sides of the equation, we produce the identity  $\tan^2 x = \tan^2 x$ . We can avoid the problem of converting a non-identity into an identity and vice versa by using only mathematical operations and procedures that are *reversible*. We can see that the operation of squaring is not a *reversible* operation because it is not possible to take square roots (reverse the squaring operation) and convert  $\tan^2 x = \tan^2 x$  back into  $\tan x = -\tan x$ .

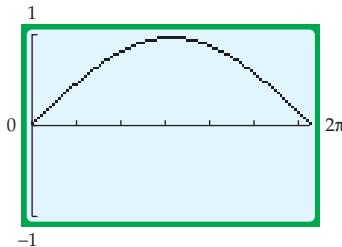
There is often more than one way to verify a trigonometric identity. Some of these approaches are shorter and considered more elegant than others. As a beginner, you should be content with finding a verification. After you have had lots of practice, you may decide to seek out different verifications and compare them to see which you think is most efficient.



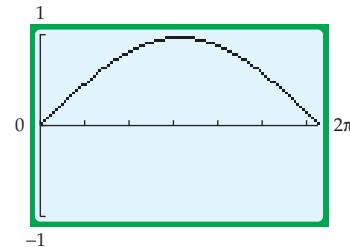
### Topics for Discussion

1. Explain why  $\tan = \frac{\sin}{\cos}$  is not an identity.
2. Is  $\cos |x| = |\cos x|$  an identity? Explain. What about  $\cos |x| = \cos x$ ? Explain.
3. The identity  $\sin^2 x + \cos^2 x = 1$  is one of the Pythagorean identities. What are the other two Pythagorean identities, and how are they derived?
4. The graph of  $y = \sin \frac{x}{2}$  for  $0 \leq x \leq 2\pi$  is shown on the left below. The graph of  $y = \sqrt{\frac{1 - \cos x}{2}}$  for  $0 \leq x \leq 2\pi$  is shown on the right below.

of  $y = \sqrt{\frac{1 - \cos x}{2}}$  for  $0 \leq x \leq 2\pi$  is shown on the right below.



$$y = \sin \frac{x}{2}$$



$$y = \sqrt{\frac{1 - \cos x}{2}}$$

The graphs appear identical, but the equation

$$\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}}$$

is not an identity. Explain.

## Exercise Set 3.1

In Exercises 1 to 56, verify each identity.

1.  $\tan x \csc x \cos x = 1$

2.  $\tan x \sec x \sin x = \tan^2 x$

3.  $\frac{4 \sin^2 x - 1}{2 \sin x + 1} = 2 \sin x - 1$

4.  $\frac{\sin^2 x - 2 \sin x + 1}{\sin x - 1} = \sin x - 1$

5.  $(\sin x - \cos x)(\sin x + \cos x) = 1 - 2 \cos^2 x$

6.  $(\tan x)(1 - \cot x) = \tan x - 1$

7.  $\frac{1}{\sin x} - \frac{1}{\cos x} = \frac{\cos x - \sin x}{\sin x \cos x}$

8.  $\frac{1}{\sin x} + \frac{3}{\cos x} = \frac{\cos x + 3 \sin x}{\sin x \cos x}$

9.  $\frac{\cos x}{1 - \sin x} = \sec x + \tan x$

10.  $\frac{\sin x}{1 - \cos x} = \csc x + \cot x$

11.  $\frac{1 - \tan^4 x}{\sec^2 x} = 1 - \tan^2 x$

12.  $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

13.  $\frac{1 + \tan^3 x}{1 + \tan x} = 1 - \tan x + \tan^2 x$

14.  $\frac{\cos x \tan x - \sin x}{\cot x} = 0$

15.  $\frac{\sin x - 2 + \frac{1}{\sin x}}{\sin x - \frac{1}{\sin x}} = \frac{\sin x - 1}{\sin x + 1}$

16.  $\frac{\sin x}{1 - \cos x} - \frac{\sin x}{1 + \cos x} = 2 \cot x$

17.  $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$

18.  $(\tan x + 1)^2 = \sec^2 x + 2 \tan x$

19.  $\frac{\cos x}{1 + \sin x} = \sec x - \tan x$

20.  $\frac{\sin x}{1 + \cos x} = \csc x - \cot x$

21.  $\csc x = \frac{\cot x + \tan x}{\sec x}$

22.  $\sec x = \frac{\cot x + \tan x}{\csc x}$

23.  $\frac{\cos x \tan x + 2 \cos x - \tan x - 2}{\tan x + 2} = \cos x - 1$

24.  $\frac{2 \sin x \cot x + \sin x - 4 \cot x - 2}{2 \cot x + 1} = \sin x - 2$

25.  $\sec x - \tan x = \frac{1 - \sin x}{\cos x}$

26.  $\cot x - \csc x = \frac{\cos x - 1}{\sin x}$

27.  $\sin^2 x - \cos^2 x = 2 \sin^2 x - 1$

28.  $\sin^2 x - \cos^2 x = 1 - 2 \cos^2 x$

29.  $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \csc^2 x \sec^2 x$

30.  $\frac{1}{\tan^2 x} - \frac{1}{\cot^2 x} = \csc^2 x - \sec^2 x$

31.  $\sec x - \cos x = \sin x \tan x$

32.  $\tan x + \cot x = \sec x \csc x$

33.  $\frac{\frac{1}{\sin x} + 1}{\frac{1}{\sin x} - 1} = \tan^2 x + 2 \tan x \sec x + \sec^2 x$

34.  $\frac{\frac{1}{\sin x} + \frac{1}{\cos x}}{\frac{1}{\sin x} - \frac{1}{\cos x}} = \frac{\cos^2 x - \sin^2 x}{1 - 2 \cos x \sin x}$

35.  $\sin^4 x - \cos^4 x = 2 \sin^2 x - 1$

36.  $\sin^6 x + \cos^6 x = \sin^4 x - \sin^2 x \cos^2 x + \cos^4 x$

37.  $\frac{1}{1 - \cos x} = \frac{1 + \cos x}{\sin^2 x}$

38.  $1 + \sin x = \frac{\cos^2 x}{1 - \sin x}$

39.  $\frac{\sin x}{1 - \sin x} - \frac{\cos x}{1 - \sin x} = \frac{1 - \cot x}{\csc x - 1}$

40.  $\frac{\tan x}{1 + \tan x} - \frac{\cot x}{1 + \tan x} = 1 - \cot x$

41.  $\frac{1}{1 + \cos x} - \frac{1}{1 - \cos x} = -2 \cot x \csc x$

42.  $\frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} = 2 \tan x \sec x$

43.  $\frac{\frac{1}{\sin x} + \csc x}{\frac{1}{\sin x} - \sin x} = \frac{2}{\cos^2 x}$



$$75. \frac{\sin x(\tan x + 1) - 2 \tan x \cos x}{\sin x - \cos x} = \tan x$$

$$76. \frac{\sin^2 x \cos x + \cos^3 x - \sin^3 x \cos x - \sin x \cos^3 x}{1 - \sin^2 x}$$

$$= \frac{\cos x}{1 + \sin x}$$

- 77.** Verify the identity  $\sin^4 x + \cos^4 x = 1 - 2 \sin^2 x \cos^2 x$  by completing the square of the left side of the identity.

**78.** Verify the identity  $\tan^4 x + \sec^4 x = 1 + 2 \tan^2 x \sec^2 x$  by completing the square of the left side of the identity.

## »»» Projects

- 1.**  **GRADING A QUIZ** Suppose that you are a teacher's assistant. You are to assist the teacher of a trigonometry class by grading a four-question quiz. Each question asks the student to find a trigonometric expression that models a given application. The teacher has prepared an answer key. These answers are shown in the next column. A student gives as answers the expressions shown in the far right column. Determine for which problems the student has given a correct response.

## Section 3.2

- Identities That Involve  $(\alpha \pm \beta)$
  - Cofunctions
  - Additional Sum and Difference Identities
  - Reduction Formulas

# Sum, Difference, and Cofunction Identities

## **PREPARE FOR THIS SECTION**

**Prepare for this section by completing the following exercises. The answers can be found on page A16.**

- PS1.** Compare  $\cos(\alpha - \beta)$  and  $\cos \alpha \cos \beta + \sin \alpha \sin \beta$  for  $\alpha = \frac{\pi}{2}$  and  $\beta = \frac{\pi}{6}$ . [2.2]

**PS2.** Compare  $\sin(\alpha + \beta)$  and  $\sin \alpha \cos \beta + \cos \alpha \sin \beta$  for  $\alpha = \frac{\pi}{2}$  and  $\beta = \frac{\pi}{3}$ . [2.2]

**PS3.** Compare  $\sin(90^\circ - \theta)$  and  $\cos \theta$  for  $\theta = 30^\circ$ ,  $\theta = 45^\circ$ , and  $\theta = 120^\circ$ . [2.2]

**PS4.** Compare  $\tan\left(\frac{\pi}{2} - \theta\right)$  and  $\cot \theta$  for  $\theta = \frac{\pi}{6}$ ,  $\theta = \frac{\pi}{4}$ , and  $\theta = \frac{4\pi}{3}$ . [2.2]

**PS5.** Compare  $\tan(\alpha - \beta)$  and  $\frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$  for  $\alpha = \frac{\pi}{3}$  and  $\beta = \frac{\pi}{6}$ . [2.2]

**PS6.** Find the value of  $\sin[(2k + 1)\pi]$ , where  $k$  is any arbitrary integer. [2.2]

### ■ Identities That Involve $(\alpha \pm \beta)$

Each identity in Section 3.1 involved only one variable. We now consider identities that involve a trigonometric function of the sum or difference of two variables.

#### Sum and Difference Identities

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

**Proof** To establish the identity for  $\cos(\alpha - \beta)$ , we make use of the unit circle shown in **Figure 3.2**. The angles  $\alpha$  and  $\beta$  are drawn in standard position, with  $OA$  and  $OB$  as the terminal sides of  $\alpha$  and  $\beta$ , respectively. The coordinates of  $A$  are  $(\cos \alpha, \sin \alpha)$ , and the coordinates of  $B$  are  $(\cos \beta, \sin \beta)$ . The angle  $(\alpha - \beta)$  is formed by the terminal sides of the angles  $\alpha$  and  $\beta$  (angle  $AOB$ ).

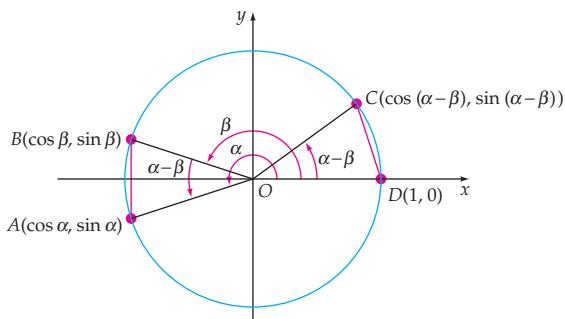


Figure 3.2

An angle equal in measure to angle  $(\alpha - \beta)$  is placed in standard position in the same figure (angle  $COD$ ). From geometry, if two central angles of a circle have the same measure, then their chords are also equal in measure. Thus the chords  $AB$  and  $CD$  are equal in length. Using the distance formula, we can calculate the lengths of the chords  $AB$  and  $CD$ .

$$d(A, B) = \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2}$$

$$d(C, D) = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$$

Because  $d(A, B) = d(C, D)$ , we have

$$\sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta)]^2}$$

Squaring each side of the equation and simplifying, we obtain

$$\begin{aligned} (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 &= [\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta)]^2 \\ \cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta &= \cos^2(\alpha - \beta) - 2 \cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta) \\ \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta &= \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) + 1 - 2 \cos(\alpha - \beta) \end{aligned}$$

Simplifying by using  $\sin^2 \theta + \cos^2 \theta = 1$ , we have

$$2 - 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta = 2 - 2 \cos(\alpha - \beta)$$

Solving for  $\cos(\alpha - \beta)$  gives us

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

To derive an identity for  $\cos(\alpha + \beta)$ , write  $\cos(\alpha + \beta)$  as  $\cos[\alpha - (-\beta)]$ .

$$\cos(\alpha + \beta) = \cos[\alpha - (-\beta)] = \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta)$$

Recall that  $\cos(-\beta) = \cos \beta$  and  $\sin(-\beta) = -\sin \beta$ . Substituting into the previous equation, we obtain the identity

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

### EXAMPLE 1 » Evaluate a Trigonometric Expression

Use an identity to find the *exact* value of  $\cos(60^\circ - 45^\circ)$ .

#### Solution

Use the identity  $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$  with  $\alpha = 60^\circ$  and  $\beta = 45^\circ$ .

$$\begin{aligned} \cos(60^\circ - 45^\circ) &= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ \\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4} \end{aligned}$$

• Substitute.

• Evaluate each factor.

• Simplify.

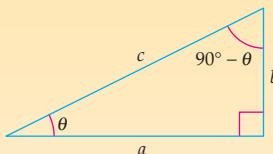
» Try Exercise 4, page 233

### ■ Cofunctions

Any pair of trigonometric functions  $f$  and  $g$  for which  $f(x) = g(90^\circ - x)$  and  $g(x) = f(90^\circ - x)$  are said to be **cofunctions**.

**take note**

To visualize the cofunction identities, consider the right triangle shown in the following figure.



If  $\theta$  is the degree measure of one of the acute angles, then the degree measure of the other acute angle is  $(90^\circ - \theta)$ . Using the definitions of the trigonometric functions gives us

$$\sin \theta = \frac{b}{c} = \cos(90^\circ - \theta)$$

$$\tan \theta = \frac{b}{a} = \cot(90^\circ - \theta)$$

$$\sec \theta = \frac{c}{a} = \csc(90^\circ - \theta)$$

These identities state that the value of a trigonometric function of  $\theta$  is equal to the cofunction of the complement of  $\theta$ .

**Cofunction Identities**

$$\sin(90^\circ - \theta) = \cos \theta \quad \cos(90^\circ - \theta) = \sin \theta$$

$$\tan(90^\circ - \theta) = \cot \theta \quad \cot(90^\circ - \theta) = \tan \theta$$

$$\sec(90^\circ - \theta) = \csc \theta \quad \csc(90^\circ - \theta) = \sec \theta$$

If  $\theta$  is in radian measure, replace  $90^\circ$  with  $\frac{\pi}{2}$ .

To verify that the sine function and the cosine function are cofunctions, we make use of the identity for  $\cos(\alpha - \beta)$ .

$$\begin{aligned}\cos(90^\circ - \beta) &= \cos 90^\circ \cos \beta + \sin 90^\circ \sin \beta \\ &= 0 \cdot \cos \beta + 1 \cdot \sin \beta\end{aligned}$$

which gives

$$\cos(90^\circ - \beta) = \sin \beta$$

Thus the sine of an angle is equal to the cosine of its complement. Using  $\cos(90^\circ - \beta) = \sin \beta$  with  $\beta = 90^\circ - \alpha$ , we have

$$\cos \alpha = \cos[90^\circ - (90^\circ - \alpha)] = \sin(90^\circ - \alpha)$$

Therefore,

$$\cos \alpha = \sin(90^\circ - \alpha)$$

We can use the ratio identities to show that the tangent and cotangent functions are cofunctions.

$$\tan(90^\circ - \theta) = \frac{\sin(90^\circ - \theta)}{\cos(90^\circ - \theta)} = \frac{\cos \theta}{\sin \theta} = \cot \theta$$

$$\cot(90^\circ - \theta) = \frac{\cos(90^\circ - \theta)}{\sin(90^\circ - \theta)} = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

The secant and cosecant functions are also cofunctions.

**EXAMPLE 2 » Write an Equivalent Expression**

Use a cofunction identity to write an equivalent expression for  $\sin 20^\circ$ .

**Solution**

The value of a given trigonometric function of  $\theta$ , measured in degrees, is equal to its cofunction of  $90^\circ - \theta$ . Thus

$$\begin{aligned}\sin 20^\circ &= \cos(90^\circ - 20^\circ) \\ &= \cos 70^\circ\end{aligned}$$

**» Try Exercise 20, page 233**

## ■ Additional Sum and Difference Identities

We can use the cofunction identities to verify the remaining sum and difference identities. To derive an identity for  $\sin(\alpha + \beta)$ , substitute  $\alpha + \beta$  for  $\theta$  in the cofunction identity  $\sin \theta = \cos(90^\circ - \theta)$ .

$$\begin{aligned}\sin \theta &= \cos(90^\circ - \theta) \\ \sin(\alpha + \beta) &= \cos[90^\circ - (\alpha + \beta)] && \bullet \text{ Replace } \theta \text{ with } \alpha + \beta. \\ &= \cos[(90^\circ - \alpha) - \beta] && \bullet \text{ Rewrite as the difference of two angles.} \\ &= \cos(90^\circ - \alpha) \cos \beta + \sin(90^\circ - \alpha) \sin \beta \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta\end{aligned}$$

Therefore,

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

We also can derive an identity for  $\sin(\alpha - \beta)$  by rewriting  $(\alpha - \beta)$  as  $[\alpha + (-\beta)]$ .

$$\begin{aligned}\sin(\alpha - \beta) &= \sin[\alpha + (-\beta)] \\ &= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta && \bullet \cos(-\beta) = \cos \beta \\ &&& \sin(-\beta) = -\sin \beta\end{aligned}$$

Thus

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

The identity for  $\tan(\alpha + \beta)$  is a result of the identity  $\tan \theta = \frac{\sin \theta}{\cos \theta}$  and the identities for  $\sin(\alpha + \beta)$  and  $\cos(\alpha + \beta)$ .

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\ &= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} && \bullet \text{ Multiply both the numerator and the denominator by } \frac{1}{\cos \alpha \cos \beta} \text{ and simplify.} \\ &= \frac{1 + \frac{\tan \alpha}{\tan \beta}}{1 - \frac{\tan \alpha}{\tan \beta}}\end{aligned}$$

Therefore,

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

The tangent function is an odd function, so  $\tan(-\theta) = -\tan \theta$ . Rewriting  $(\alpha - \beta)$  as  $[\alpha + (-\beta)]$  enables us to derive an identity for  $\tan(\alpha - \beta)$ .

$$\tan(\alpha - \beta) = \tan[\alpha + (-\beta)] = \frac{\tan \alpha + \tan(-\beta)}{1 - \tan \alpha \tan(-\beta)}$$

Therefore,

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

The sum and difference identities can be used to simplify some trigonometric expressions.

### EXAMPLE 3 » Simplify Trigonometric Expressions

Write each expression in terms of a single trigonometric function.

a.  $\sin 5x \cos 3x - \cos 5x \sin 3x$     b.  $\frac{\tan 4\alpha + \tan \alpha}{1 - \tan 4\alpha \tan \alpha}$

#### Solution

a.  $\sin 5x \cos 3x - \cos 5x \sin 3x = \sin(5x - 3x) = \sin 2x$   
 b.  $\frac{\tan 4\alpha + \tan \alpha}{1 - \tan 4\alpha \tan \alpha} = \tan(4\alpha + \alpha) = \tan 5\alpha$

» Try Exercise 26, page 233

### EXAMPLE 4 » Evaluate a Trigonometric Function

Given  $\tan \alpha = -\frac{4}{3}$  for  $\alpha$  in Quadrant II and  $\tan \beta = -\frac{5}{12}$  for  $\beta$  in Quadrant IV, find  $\sin(\alpha + \beta)$ .

#### Solution

See **Figure 3.3**. Because  $\tan \alpha = \frac{y}{x} = -\frac{4}{3}$  and the terminal side of  $\alpha$  is in Quadrant II,  $P_1(-3, 4)$  is a point on the terminal side of  $\alpha$ . Similarly,  $P_2(12, -5)$  is a point on the terminal side of  $\beta$ .

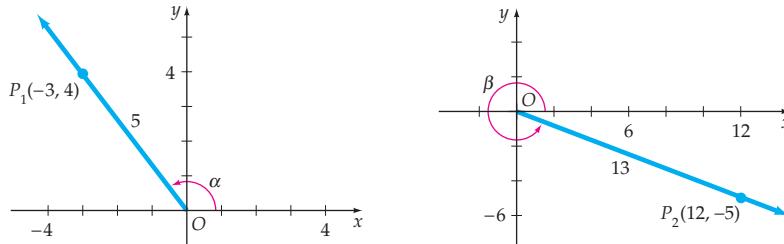


Figure 3.3

Using the Pythagorean Theorem, we find that the length of the line segment  $OP_1$  is 5 and the length of  $OP_2$  is 13.

$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \frac{4}{5} \cdot \frac{12}{13} + \frac{-3}{5} \cdot \frac{-5}{13} = \frac{48}{65} + \frac{15}{65} = \frac{63}{65}\end{aligned}$$

» Try Exercise 38, page 233

**EXAMPLE 5** » Verify an Identity

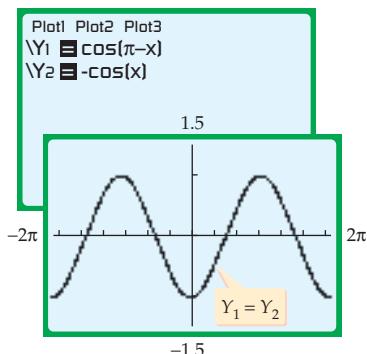
Verify the identity  $\cos(\pi - \theta) = -\cos \theta$ .

**Solution**

$$\begin{aligned}\cos(\pi - \theta) &= \cos \pi \cos \theta + \sin \pi \sin \theta \\ &= -1 \cdot \cos \theta + 0 \cdot \sin \theta \\ &= -\cos \theta\end{aligned}$$

- Use the identity for  $\cos(\alpha - \beta)$ .

» Try Exercise 50, page 234



**Figure 3.4**

**Figure 3.4** shows the graphs of  $Y_1 = \cos(\pi - \theta)$  and  $Y_2 = -\cos \theta$  on the same coordinate axes. The fact that the graphs appear to be identical supports the verification in Example 5.

**EXAMPLE 6** » Verify an Identity

$$\text{Verify the identity } \frac{\cos 4\theta}{\sin \theta} - \frac{\sin 4\theta}{\cos \theta} = \frac{\cos 5\theta}{\sin \theta \cos \theta}.$$

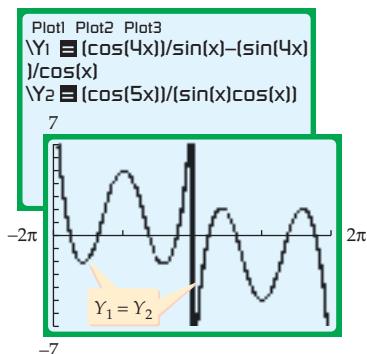
**Solution**

Subtract the fractions on the left side of the equation.

$$\begin{aligned}\frac{\cos 4\theta}{\sin \theta} - \frac{\sin 4\theta}{\cos \theta} &= \frac{\cos 4\theta \cos \theta - \sin 4\theta \sin \theta}{\sin \theta \cos \theta} \\ &= \frac{\cos(4\theta + \theta)}{\sin \theta \cos \theta} \\ &= \frac{\cos 5\theta}{\sin \theta \cos \theta}\end{aligned}$$

- Use the identity for  $\cos(\alpha + \beta)$ .

» Try Exercise 62, page 234



**Figure 3.5**

**Figure 3.5** shows the graph of  $Y_1 = \frac{\cos 4\theta}{\sin \theta} - \frac{\sin 4\theta}{\cos \theta}$  and the graph of  $Y_2 = \frac{\cos 5\theta}{\sin \theta \cos \theta}$  on the same coordinate axes. The fact that the graphs appear to be identical supports the verification in Example 6.

### ■ Reduction Formulas

The sum or difference identities can be used to write expressions such as

$$\sin(\theta + k\pi) \quad \sin(\theta + 2k\pi) \quad \text{and} \quad \cos[\theta + (2k + 1)\pi]$$

where  $k$  is an integer, as expressions involving only  $\sin \theta$  or  $\cos \theta$ . The resulting formulas are called **reduction formulas**.

#### EXAMPLE 7 » Find Reduction Formulas

Write as a function involving only  $\sin \theta$ .

$$\sin[\theta + (2k + 1)\pi], \text{ where } k \text{ is an integer}$$

#### Solution

Applying the identity  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$  yields

$$\sin[\theta + (2k + 1)\pi] = \sin \theta \cos[(2k + 1)\pi] + \cos \theta \sin[(2k + 1)\pi]$$

If  $k$  is an integer, then  $2k + 1$  is an odd integer. The cosine of any odd multiple of  $\pi$  equals  $-1$ , and the sine of any odd multiple of  $\pi$  is 0. This gives us

$$\sin[\theta + (2k + 1)\pi] = (\sin \theta)(-1) + (\cos \theta)(0) = -\sin \theta$$

Thus  $\sin[\theta + (2k + 1)\pi] = -\sin \theta$  for any integer  $k$ .

#### » Try Exercise 76, page 235

 **QUESTION** Is  $\sin(\theta + 2k\pi) = \sin \theta$  a reduction formula?



#### Topics for Discussion

1. Does  $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$  for all values of  $\alpha$  and  $\beta$ ? If not, find nonzero values of  $\alpha$  and  $\beta$  for which  $\sin(\alpha + \beta) \neq \sin \alpha + \sin \beta$ .
2. If  $k$  is an integer, then  $2k + 1$  is an odd integer. Do you agree? Explain.
3. What are the trigonometric cofunction identities? Explain.
4. Is  $\tan(\theta + k\pi) = \tan \theta$ , where  $k$  is an integer, a reduction formula? Explain.

---

 **ANSWER** Yes.  $\sin(\theta + 2k\pi) = \sin \theta \cos(2k\pi) + \cos \theta \sin(2k\pi)$   
 $= (\sin \theta)(1) + (\cos \theta)(0) = \sin \theta$ .


**Exercise Set 3.2**

In Exercises 1 to 18, find (if possible) the exact value of the expression.

1.  $\sin(45^\circ + 30^\circ)$

2.  $\sin(330^\circ + 45^\circ)$

3.  $\cos(45^\circ - 30^\circ)$

4.  $\cos(120^\circ - 45^\circ)$

5.  $\tan(45^\circ - 30^\circ)$

6.  $\tan(240^\circ - 45^\circ)$

7.  $\sin\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$

8.  $\sin\left(\frac{4\pi}{3} + \frac{\pi}{4}\right)$

9.  $\cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$

10.  $\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$

11.  $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

12.  $\tan\left(\frac{11\pi}{6} - \frac{\pi}{4}\right)$

13.  $\cos 212^\circ \cos 122^\circ + \sin 212^\circ \sin 122^\circ$

14.  $\sin 167^\circ \cos 107^\circ - \cos 167^\circ \sin 107^\circ$

15.  $\sin \frac{5\pi}{12} \cos \frac{\pi}{4} - \cos \frac{5\pi}{12} \sin \frac{\pi}{4}$

16.  $\cos \frac{\pi}{12} \cos \frac{\pi}{4} - \sin \frac{\pi}{12} \sin \frac{\pi}{4}$

17. 
$$\frac{\tan \frac{7\pi}{12} - \tan \frac{\pi}{4}}{1 + \tan \frac{7\pi}{12} \tan \frac{\pi}{4}}$$

18. 
$$\frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{3}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{3}}$$

In Exercises 19 to 24, use a cofunction identity to write an equivalent expression for the given value.

19.  $\sin 42^\circ$

20.  $\cos 80^\circ$

21.  $\tan 15^\circ$

22.  $\cot 2^\circ$

23.  $\sec 25^\circ$

24.  $\csc 84^\circ$

In Exercises 25 to 36, write each expression in terms of a single trigonometric function.

25.  $\sin 7x \cos 2x - \cos 7x \sin 2x$

26.  $\sin x \cos 3x + \cos x \sin 3x$

27.  $\cos x \cos 2x + \sin x \sin 2x$

28.  $\cos 4x \cos 2x - \sin 4x \sin 2x$

29.  $\sin 7x \cos 3x - \cos 7x \sin 3x$

30.  $\cos x \cos 5x - \sin x \sin 5x$

31.  $\cos 4x \cos(-2x) - \sin 4x \sin(-2x)$

32.  $\sin(-x) \cos 3x - \cos(-x) \sin 3x$

33.  $\sin \frac{x}{3} \cos \frac{2x}{3} + \cos \frac{x}{3} \sin \frac{2x}{3}$

34.  $\cos \frac{3x}{4} \cos \frac{x}{4} + \sin \frac{3x}{4} \sin \frac{x}{4}$

35. 
$$\frac{\tan 3x + \tan 4x}{1 - \tan 3x \tan 4x}$$

36. 
$$\frac{\tan 2x - \tan 3x}{1 + \tan 2x \tan 3x}$$

In Exercises 37 to 48, find the exact value of the given functions.

37. Given  $\tan \alpha = -\frac{4}{3}$ ,  $\alpha$  in Quadrant II, and  $\tan \beta = \frac{15}{8}$ ,  $\beta$  in Quadrant III, find

- a.  $\sin(\alpha - \beta)$       b.  $\cos(\alpha + \beta)$       c.  $\tan(\alpha - \beta)$

38. Given  $\tan \alpha = \frac{24}{7}$ ,  $\alpha$  in Quadrant I, and  $\sin \beta = -\frac{8}{17}$ ,  $\beta$  in Quadrant III, find

- a.  $\sin(\alpha + \beta)$       b.  $\cos(\alpha + \beta)$       c.  $\tan(\alpha - \beta)$

39. Given  $\sin \alpha = \frac{3}{5}$ ,  $\alpha$  in Quadrant I, and  $\cos \beta = -\frac{5}{13}$ ,  $\beta$  in Quadrant II, find

- a.  $\sin(\alpha - \beta)$       b.  $\cos(\alpha + \beta)$       c.  $\tan(\alpha - \beta)$

40. Given  $\sin \alpha = \frac{24}{25}$ ,  $\alpha$  in Quadrant II, and  $\cos \beta = -\frac{4}{5}$ ,  $\beta$  in Quadrant III, find

- a.  $\cos(\beta - \alpha)$       b.  $\sin(\alpha + \beta)$       c.  $\tan(\alpha + \beta)$

- 41.** Given  $\sin \alpha = -\frac{4}{5}$ ,  $\alpha$  in Quadrant III, and  $\cos \beta = -\frac{12}{13}$ ,  $\beta$  in Quadrant II, find

a.  $\sin(\alpha - \beta)$       b.  $\cos(\alpha + \beta)$       c.  $\tan(\alpha + \beta)$

- 42.** Given  $\sin \alpha = -\frac{7}{25}$ ,  $\alpha$  in Quadrant IV, and  $\cos \beta = \frac{8}{17}$ ,  $\beta$  in Quadrant IV, find

a.  $\sin(\alpha + \beta)$       b.  $\cos(\alpha - \beta)$       c.  $\tan(\alpha + \beta)$

- 43.** Given  $\cos \alpha = \frac{15}{17}$ ,  $\alpha$  in Quadrant I, and  $\sin \beta = -\frac{3}{5}$ ,  $\beta$  in Quadrant III, find

a.  $\sin(\alpha + \beta)$       b.  $\cos(\alpha - \beta)$       c.  $\tan(\alpha - \beta)$

- 44.** Given  $\cos \alpha = -\frac{7}{25}$ ,  $\alpha$  in Quadrant II, and  $\sin \beta = -\frac{12}{13}$ ,  $\beta$  in Quadrant IV, find

a.  $\sin(\alpha + \beta)$       b.  $\cos(\alpha + \beta)$       c.  $\tan(\alpha - \beta)$

- 45.** Given  $\cos \alpha = -\frac{3}{5}$ ,  $\alpha$  in Quadrant III, and  $\sin \beta = \frac{5}{13}$ ,  $\beta$  in Quadrant I, find

a.  $\sin(\alpha - \beta)$       b.  $\cos(\alpha + \beta)$       c.  $\tan(\alpha + \beta)$

- 46.** Given  $\cos \alpha = \frac{8}{17}$ ,  $\alpha$  in Quadrant IV, and  $\sin \beta = -\frac{24}{25}$ ,  $\beta$  in Quadrant III, find

a.  $\sin(\alpha - \beta)$       b.  $\cos(\alpha + \beta)$       c.  $\tan(\alpha + \beta)$

- 47.** Given  $\sin \alpha = \frac{3}{5}$ ,  $\alpha$  in Quadrant I, and  $\tan \beta = \frac{5}{12}$ ,  $\beta$  in Quadrant III, find

a.  $\sin(\alpha + \beta)$       b.  $\cos(\alpha - \beta)$       c.  $\tan(\alpha - \beta)$

- 48.** Given  $\tan \alpha = \frac{15}{8}$ ,  $\alpha$  in Quadrant I, and  $\tan \beta = -\frac{7}{24}$ ,  $\beta$  in Quadrant IV, find

a.  $\sin(\alpha - \beta)$       b.  $\cos(\alpha - \beta)$       c.  $\tan(\alpha + \beta)$

**In Exercises 49 to 74, verify the identity.**

**49.**  $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$

» **50.**  $\cos(\theta + \pi) = -\cos \theta$

**51.**  $\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$

**52.**  $\sin(\theta + \pi) = -\sin \theta$

**53.**  $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan \theta + 1}{1 - \tan \theta}$

**54.**  $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

**55.**  $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin \theta$

**56.**  $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos \theta$

**57.**  $\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$

**58.**  $\cot(\pi + \theta) = \cot \theta$

**59.**  $\csc(\pi - \theta) = \csc \theta$

**60.**  $\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$

**61.**  $\sin 6x \cos 2x - \cos 6x \sin 2x = 2 \sin 2x \cos 2x$

» **62.**  $\cos 5x \cos 3x + \sin 5x \sin 3x = \cos^2 x - \sin^2 x$

**63.**  $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$

**64.**  $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta$

**65.**  $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$

**66.**  $\sin(\alpha - \beta) - \sin(\alpha + \beta) = -2 \cos \alpha \sin \beta$

**67.**  $\frac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)} = \frac{\cot \alpha + \tan \beta}{1 + \cot \alpha \tan \beta}$

$$68. \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{1 + \cot \alpha \tan \beta}{1 - \cot \alpha \tan \beta}$$

$$69. \frac{\sin(x+h) - \sin x}{h} = \cos x \cdot \frac{\sin h}{h} + \sin x \cdot \frac{(\cos h - 1)}{h}$$

$$70. \frac{\cos(x+h) - \cos x}{h} = \cos x \cdot \frac{(\cos h - 1)}{h} - \sin x \cdot \frac{\sin h}{h}$$

$$71. \sin\left(\frac{\pi}{2} + \alpha - \beta\right) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$72. \cos\left(\frac{\pi}{2} + \alpha + \beta\right) = -(\sin \alpha \cos \beta + \cos \alpha \sin \beta)$$

73.  $\sin 3x = 3 \sin x - 4 \sin^3 x$

**74.**  $\cos 3x = 4 \cos^3 x - 3 \cos x$

**In Exercises 75 to 80, write the given expression as a function that involves only  $\sin \theta$ ,  $\cos \theta$ , or  $\tan \theta$ . (In Exercises 78, 79, and 80, assume  $k$  is an integer.)**

**75.**  $\cos(\theta + 3\pi)$

**76.**  $\sin(\theta + 2\pi)$

**77.**  $\tan(\theta + \pi)$

**78.**  $\cos[\theta + (2k + 1)\pi]$

**79.**  $\sin(\theta + 2k\pi)$

**80.**  $\sin(\theta - k\pi)$



**In Exercises 81 to 84, compare the graphs of each side of the equation to predict whether the equation is an identity.**

$$81. \sin\left(\frac{\pi}{2} - x\right) = \cos x$$

**82.**  $\cos(x + \pi) = -\cos x$

**83.**  $\sin 7x \cos 2x - \cos 7x \sin 2x = \sin 5x$

**84.**  $\sin 3x = 3 \sin x - 4 \sin^3 x$

**In Exercises 85 to 89, verify the identity.**

$$85. \sin(x - y) \cdot \sin(x + y) = \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$$

$$86. \sin(x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$$

$$87. \cos(x + y + z) = \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$$

$$88. \frac{\sin(x + y)}{\sin x \sin y} = \cot x + \cot y$$

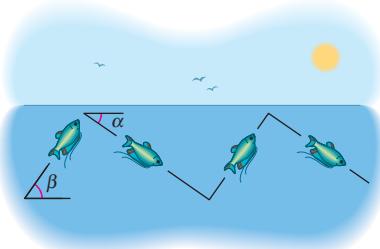
$$89. \frac{\cos(x - y)}{\cos x \sin y} = \cot y + \tan x$$

- 90. MODEL RESISTANCE** The drag (resistance) on a fish when it is swimming is two to three times the drag when it is gliding. To compensate for this, some fish swim in a saw-tooth pattern, as shown in the accompanying figure. The ratio of the amount of energy the fish expends when

swimming upward at angle  $\beta$  and then gliding down at angle  $\alpha$  to the energy it expends swimming horizontally is given by

$$E_R = \frac{k \sin \alpha + \sin \beta}{k \sin(\alpha + \beta)}$$

where  $k$  is a value such that  $2 \leq k \leq 3$ , and  $k$  depends on the assumptions we make about the amount of drag experienced by the fish. Find  $E_R$  for  $k = 2$ ,  $\alpha = 10^\circ$ , and  $\beta = 20^\circ$ .



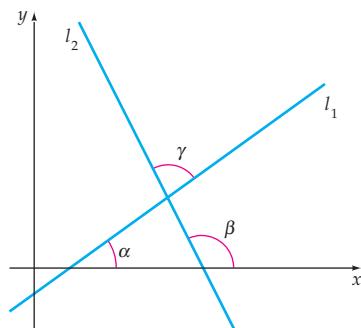
## »»» Projects

- 1. INTERSECTING LINES** In the figure shown at the right, two nonvertical lines intersect in a plane. The slope of line  $l_1$  is  $m_1$  and the slope of line  $l_2$  is  $m_2$ .

- a. Show that the tangent of the smallest positive angle  $\gamma$  between  $l_1$  and  $l_2$  is given by

$$\tan \gamma = \frac{m_2 - m_1}{1 + m_1 m_2}$$

- b. Two nonvertical lines intersect at the point  $(1, 5)$ . The measure of the smallest positive angle between the lines is  $\gamma = 60^\circ$ . The first line is given by  $y = 0.5x + 4.5$ . What is the equation (in slope-intercept form) of the second line?



## Section 3.3

- Double-Angle Identities
- Power-Reducing Identities
- Half-Angle Identities

## Double- and Half-Angle Identities

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A16.

**PS1.** Use the identity for  $\sin(\alpha + \beta)$  to rewrite  $\sin 2\alpha$ . [3.2]

**PS2.** Use the identity for  $\cos(\alpha + \beta)$  to rewrite  $\cos 2\alpha$ . [3.2]

**PS3.** Use the identity for  $\tan(\alpha + \beta)$  to rewrite  $\tan 2\alpha$ . [3.2]

**PS4.** Compare  $\tan \frac{\alpha}{2}$  and  $\frac{\sin \alpha}{1 + \cos \alpha}$  for  $\alpha = 60^\circ$ ,  $\alpha = 90^\circ$ , and  $\alpha = 120^\circ$ . [2.2]

**PS5.** Verify that  $\sin 2\alpha = 2 \sin \alpha$  is *not* an identity. [2.2]

**PS6.** Verify that  $\cos \frac{\alpha}{2} = \frac{1}{2} \cos \alpha$  is *not* an identity. [2.2]



### Integrating Technology

One way of showing that  $\sin 2x \neq 2 \sin x$  is by graphing  $y = \sin 2x$  and  $y = 2 \sin x$  and observing that the graphs are not the same.

### Double-Angle Identities

By using the sum identities, we can derive identities for  $f(2\alpha)$ , where  $f$  is a trigonometric function. These are called the *double-angle identities*. To find the sine of a double angle, substitute  $\alpha$  for  $\beta$  in the identity for  $\sin(\alpha + \beta)$ .

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

• Let  $\beta = \alpha$ .

A double-angle identity for cosine is derived in a similar manner.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$$

• Let  $\beta = \alpha$ .

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

There are two alternative forms of the double-angle identity for  $\cos 2\alpha$ . Using  $\cos^2 \alpha = 1 - \sin^2 \alpha$ , we can rewrite the identity for  $\cos 2\alpha$  as follows:

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\cos 2\alpha = (1 - \sin^2 \alpha) - \sin^2 \alpha$$

•  $\cos^2 \alpha = 1 - \sin^2 \alpha$

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

We also can rewrite  $\cos 2\alpha$  as

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - (1 - \cos^2 \alpha)$$

•  $\sin^2 \alpha = 1 - \cos^2 \alpha$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

The double-angle identity for the tangent function is derived from the identity for  $\tan(\alpha + \beta)$  with  $\beta = \alpha$ .

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha}$$

• Let  $\beta = \alpha$ .

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Here is a summary of the double-angle identities.

### Double-Angle Identities

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

The double-angle identities are often used to write a trigonometric expression in terms of a single trigonometric function.

#### EXAMPLE 1 » Simplify a Trigonometric Expression

Write  $4 \sin 5\theta \cos 5\theta$  in terms of a single trigonometric function.

##### Solution

$$\begin{aligned} 4 \sin 5\theta \cos 5\theta &= 2(2 \sin 5\theta \cos 5\theta) \\ &= 2(\sin 10\theta) = 2 \sin 10\theta \end{aligned}$$

• Use  $2 \sin \alpha \cos \alpha = \sin 2\alpha$ , with  $\alpha = 5\theta$ .

» Try Exercise 2, page 243

**QUESTION** Does  $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$ ?

In Example 2, we use given information concerning the angle  $\alpha$  to find the exact value of  $\sin 2\alpha$ .

### EXAMPLE 2 » Evaluate a Trigonometric Function

If  $\sin \alpha = \frac{4}{5}$  and  $0^\circ < \alpha < 90^\circ$ , find the exact value of  $\sin 2\alpha$ .

#### Solution

Use the identity  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ . Find  $\cos \alpha$  by substituting for  $\sin \alpha$  in  $\sin^2 \alpha + \cos^2 \alpha = 1$  and solving for  $\cos \alpha$ .

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

•  $\cos \alpha > 0$  if  $\alpha$  is in Quadrant I.

Substitute the values of  $\sin \alpha$  and  $\cos \alpha$  in the double-angle formula for  $\sin 2\alpha$ .

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \left(\frac{4}{5}\right) \left(\frac{3}{5}\right) = \frac{24}{25}$$

» Try Exercise 10, page 243

### EXAMPLE 3 » Verify an Identity

Verify the identity  $\csc 2\alpha = \frac{1}{2}(\tan \alpha + \cot \alpha)$ .

#### Solution

Work on the right-hand side of the equation.

$$\begin{aligned} \frac{1}{2}(\tan \alpha + \cot \alpha) &= \frac{1}{2} \left( \frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha} \right) \\ &= \frac{1}{2} \left( \frac{\sin^2 \alpha + \cos^2 \alpha}{\cos \alpha \sin \alpha} \right) \\ &= \frac{1}{2 \cos \alpha \sin \alpha} = \frac{1}{\sin 2\alpha} = \csc 2\alpha \end{aligned}$$

» Try Exercise 50, page 244

---

**ANSWER** Yes.  $\sin \theta \cos \theta = \frac{2 \sin \theta \cos \theta}{2} = \frac{\sin 2\theta}{2} = \frac{1}{2} \sin 2\theta$ .

## ■ Power-Reducing Identities

The double-angle identities can be used to derive the following power-reducing identities. These identities can be used to write trigonometric expressions involving even powers of sine, cosine, and tangent in terms of the first power of a cosine function.

### Calculus Connection

#### Power-Reducing Identities

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \quad \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \quad \tan^2 \alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$$

The first power-reducing identity is derived by solving the double-angle identity  $\cos 2\alpha = 1 - 2 \sin^2 \alpha$  for  $\sin^2 \alpha$ . The second identity is derived by solving the double-angle identity  $\cos 2\alpha = 2 \cos^2 \alpha - 1$  for  $\cos^2 \alpha$ . The identity for  $\tan^2 \alpha$  can be derived by using the ratio identity, as shown below.

$$\tan^2 \alpha = \frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{\frac{1 - \cos 2\alpha}{2}}{\frac{1 + \cos 2\alpha}{2}} = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$$

#### EXAMPLE 4 » Use Power-Reducing Identities

Write  $\sin^4 \alpha$  in terms of the first power of one or more cosine functions.

##### Solution

$$\sin^4 \alpha = (\sin^2 \alpha)^2$$

$$= \left( \frac{1 - \cos 2\alpha}{2} \right)^2$$

• Power-reducing identity

$$= \frac{1}{4}(1 - 2 \cos 2\alpha + \cos^2 2\alpha)$$

• Square.

$$= \frac{1}{4} \left( 1 - 2 \cos 2\alpha + \frac{1 + \cos 4\alpha}{2} \right)$$

• Power-reducing identity

$$= \frac{1}{4} \left( \frac{2 - 4 \cos 2\alpha + 1 + \cos 4\alpha}{2} \right)$$

• Simplify.

$$= \frac{1}{8}(3 - 4 \cos 2\alpha + \cos 4\alpha)$$

» Try Exercise 22, page 243

### ■ Half-Angle Identities

The following identities, called *half-angle identities*, can be derived from the power-reducing identities by replacing  $\alpha$  with  $\frac{\alpha}{2}$  and taking the square root of each side. Two additional identities are given for  $\tan \frac{\alpha}{2}$ .

#### Half-Angle Identities

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

The choice of the plus or minus sign depends on the quadrant in which  $\frac{\alpha}{2}$  lies.

In Example 5, we use a half-angle identity to find the exact value of a trigonometric function.

#### EXAMPLE 5 » Evaluate a Trigonometric Function

Find the exact value of  $\cos 105^\circ$ .

#### Solution

Because  $105^\circ = \frac{1}{2}(210^\circ)$ , we can find  $\cos 105^\circ$  by using the half-angle identity for  $\cos \frac{\alpha}{2}$  with  $\alpha = 210^\circ$ . The angle  $\frac{\alpha}{2} = 105^\circ$  lies in Quadrant II and the cosine function is negative in Quadrant II. Thus  $\cos 105^\circ < 0$ , and we must select the minus sign that precedes the radical in  $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$  to produce the correct result.

$$\cos 105^\circ = -\sqrt{\frac{1 + \cos 210^\circ}{2}}$$

$$= -\sqrt{\frac{1 + \left(-\frac{\sqrt{3}}{2}\right)}{2}}$$

$$= -\sqrt{\frac{\frac{2}{2} - \frac{\sqrt{3}}{2}}{2}}$$

- Use the formula for  $\cos \frac{\alpha}{2}$  with  $\alpha = 210^\circ$  and a minus sign in front of the radical.

- $\cos 210^\circ = -\frac{\sqrt{3}}{2}$

- Simplify the numerator of the radicand.

$$\begin{aligned}
 &= -\sqrt{\left(\frac{2-\sqrt{3}}{2}\right) \cdot \frac{1}{2}} \\
 &= -\sqrt{\frac{2-\sqrt{3}}{4}} \\
 &= -\frac{\sqrt{2-\sqrt{3}}}{2}
 \end{aligned}$$

- Definition of division
- Simplify.

» Try Exercise 28, page 243

In Example 6, we use given information concerning an angle  $\alpha$  to find the exact values of the cosine and tangent of  $\frac{\alpha}{2}$ .

### EXAMPLE 6 » Evaluate Trigonometric Functions

If  $\sin \alpha = -\frac{3}{5}$  and  $180^\circ < \alpha < 270^\circ$ , find the exact value of

- a.  $\cos \frac{\alpha}{2}$     b.  $\tan \frac{\alpha}{2}$

#### Solution

To apply the half-angle identities, we need to find  $\cos \alpha$ . We can use the identity  $\cos^2 \alpha = 1 - \sin^2 \alpha$  to find  $\cos \alpha$ , but first we need to determine the sign of  $\cos \alpha$ . Because  $180^\circ < \alpha < 270^\circ$ , we know that  $\cos \alpha < 0$ . Thus

$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25} \quad \text{and} \quad \cos \alpha = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Multiply each part of  $180^\circ < \alpha < 270^\circ$  by  $\frac{1}{2}$  to obtain  $90^\circ < \frac{\alpha}{2} < 135^\circ$ .

Therefore,  $\frac{\alpha}{2}$  lies in Quadrant II. Thus  $\cos \frac{\alpha}{2} < 0$  and  $\tan \frac{\alpha}{2} < 0$ .

- a. In the following work, we use the half-angle identity for  $\cos \frac{\alpha}{2}$  with a minus sign in front of the radical.

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{1 + \cos \alpha}{2}} = -\sqrt{\frac{1 + \left(-\frac{4}{5}\right)}{2}} = -\sqrt{\frac{1}{5} \cdot \frac{1}{2}} = -\sqrt{\frac{1}{10}} = -\frac{\sqrt{10}}{10}$$

- b. We could use the half-angle identity  $\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$ ; however, the identity  $\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$  is simpler and easier to evaluate.

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{-\frac{3}{5}}{1 + \left(-\frac{4}{5}\right)} = \left(-\frac{3}{5}\right) \div \frac{1}{5} = \left(-\frac{3}{5}\right) \cdot 5 = -3$$

» Try Exercise 38, page 243

The power-reducing identities and half-angle identities can be used to verify other identities.

### EXAMPLE 7 » Verify an Identity

Verify the identity  $2 \csc x \cos^2 \frac{x}{2} = \frac{\sin x}{1 - \cos x}$ .

#### Solution

Work on the left side of the identity.

$$\begin{aligned}
 2 \csc x \cos^2 \frac{x}{2} &= 2 \csc x \left( \frac{1 + \cos x}{2} \right) && \bullet \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} \\
 &= \frac{1 + \cos x}{\sin x} && \bullet \csc x = \frac{1}{\sin x} \\
 &= \frac{1 + \cos x}{\sin x} \cdot \frac{1 - \cos x}{1 - \cos x} && \bullet \text{Multiply the numerator and denominator by the conjugate of the numerator.} \\
 &= \frac{1 - \cos^2 x}{\sin x(1 - \cos x)} \\
 &= \frac{\sin^2 x}{\sin x(1 - \cos x)} && \bullet 1 - \cos^2 x = \sin^2 x \\
 &= \frac{\sin x}{1 - \cos x} && \bullet \text{Simplify.}
 \end{aligned}$$

» Try Exercise 68, page 244



### Topics for Discussion

1. True or false: If  $\sin \alpha = \sin \beta$ , then  $\alpha = \beta$ . Why?
2. Because

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} \quad \text{and} \quad \tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$$

are both identities, it follows that

$$\frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

is also an identity. Do you agree? Explain.

3. Is  $\sin 10x = 2 \sin 5x \cos 5x$  an identity? Explain.
4. Is  $\sin \frac{\alpha}{2} = \cos \frac{\alpha}{2}$  an identity? Explain.



## Exercise Set 3.3

In Exercises 1 to 8, write each trigonometric expression in terms of a single trigonometric function.

1.  $2 \sin 2\alpha \cos 2\alpha$

2.  $2 \sin 3\theta \cos 3\theta$

3.  $1 - 2 \sin^2 5\beta$

4.  $2 \cos^2 2\beta - 1$

5.  $\cos^2 3\alpha - \sin^2 3\alpha$

6.  $\cos^2 6\alpha - \sin^2 6\alpha$

7.  $\frac{2 \tan 3\alpha}{1 - \tan^2 3\alpha}$

8.  $\frac{2 \tan 4\theta}{1 - \tan^2 4\theta}$

In Exercises 9 to 18, find the exact values of  $\sin 2\alpha$ ,  $\cos 2\alpha$ , and  $\tan 2\alpha$  given the following information.

9.  $\cos \alpha = -\frac{4}{5} \quad 90^\circ < \alpha < 180^\circ$

10.  $\cos \alpha = \frac{24}{25} \quad 270^\circ < \alpha < 360^\circ$

11.  $\sin \alpha = \frac{8}{17} \quad 90^\circ < \alpha < 180^\circ$

12.  $\sin \alpha = -\frac{9}{41} \quad 180^\circ < \alpha < 270^\circ$

13.  $\tan \alpha = -\frac{24}{7} \quad 270^\circ < \alpha < 360^\circ$

14.  $\tan \alpha = \frac{4}{3} \quad 0^\circ < \alpha < 90^\circ$

15.  $\sin \alpha = \frac{15}{17} \quad 0^\circ < \alpha < 90^\circ$

16.  $\sin \alpha = -\frac{3}{5} \quad 180^\circ < \alpha < 270^\circ$

17.  $\cos \alpha = \frac{40}{41} \quad 270^\circ < \alpha < 360^\circ$

18.  $\cos \alpha = \frac{4}{5} \quad 270^\circ < \alpha < 360^\circ$

In Exercises 19 to 24, use the power-reducing identities to write each trigonometric expression in terms of the first power of one or more cosine functions.

19.  $6 \cos^2 x$

20.  $\sin^4 x \cos^4 x$

21.  $\cos^4 x$

22.  $\sin^2 x \cos^4 x$

23.  $\sin^4 x \cos^2 x$

24.  $\sin^6 x$

In Exercises 25 to 36, use the half-angle identities to find the exact value of each trigonometric expression.

25.  $\sin 75^\circ$

26.  $\cos 105^\circ$

27.  $\tan 67.5^\circ$

28.  $\cos 165^\circ$

29.  $\cos 157.5^\circ$

30.  $\sin 112.5^\circ$   
33.  $\sin \frac{7\pi}{8}$

34.  $\cos \frac{5\pi}{8}$

35.  $\cos \frac{5\pi}{12}$

36.  $\sin \frac{3\pi}{8}$

In Exercises 37 to 44, find the exact values of the sine, cosine, and tangent of  $\frac{\alpha}{2}$  given the following information.

37.  $\sin \alpha = \frac{5}{13} \quad 90^\circ < \alpha < 180^\circ$

38.  $\sin \alpha = -\frac{7}{25} \quad 180^\circ < \alpha < 270^\circ$

39.  $\cos \alpha = -\frac{8}{17} \quad 180^\circ < \alpha < 270^\circ$

40.  $\cos \alpha = \frac{12}{13} \quad 0^\circ < \alpha < 90^\circ$

41.  $\tan \alpha = \frac{4}{3} \quad 0^\circ < \alpha < 90^\circ$

42.  $\tan \alpha = -\frac{8}{15} \quad 90^\circ < \alpha < 180^\circ$

43.  $\cos \alpha = \frac{24}{25} \quad 270^\circ < \alpha < 360^\circ$

44.  $\sin \alpha = -\frac{9}{41} \quad 270^\circ < \alpha < 360^\circ$

In Exercises 45 to 90, verify the given identity.

45.  $\sin 3x \cos 3x = \frac{1}{2} \sin 6x$

46.  $\cos 8x = \cos^2 4x - \sin^2 4x$

47.  $\sin^2 x + \cos 2x = \cos^2 x$

48.  $\frac{\cos 2x}{\sin^2 x} = \cot^2 x - 1$

71.  $2 \sin \frac{x}{2} \cos \frac{x}{2} = \sin x$

72.  $\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x$

49.  $\frac{1 + \cos 2x}{\sin 2x} = \cot x$

50.  $\frac{1}{1 - \cos 2x} = \frac{1}{2} \csc^2 x$

73.  $\left( \cos \frac{x}{2} + \sin \frac{x}{2} \right)^2 = 1 + \sin x$

51.  $\frac{\sin 2x}{1 - \sin^2 x} = 2 \tan x$

74.  $\tan^2 \frac{x}{2} = \frac{\sec x - 1}{\sec x + 1}$

52.  $\frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = \cot 2x$

75.  $\sin^2 \frac{x}{2} \sec x = \frac{1}{2}(\sec x - 1)$

53.  $1 - \tan^2 x = \frac{\cos 2x}{\cos^2 x}$

76.  $\cos^2 \frac{x}{2} \sec x = \frac{1}{2}(\sec x + 1)$

54.  $\tan 2x = \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x}$

77.  $\cos^2 \frac{x}{2} - \cos x = \sin^2 \frac{x}{2}$

55.  $\sin 2x - \tan x = \tan x \cos 2x$

78.  $\sin^2 \frac{x}{2} + \cos x = \cos^2 \frac{x}{2}$

56.  $\sin 2x - \cot x = -\cot x \cos 2x$

79.  $\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} = -\cos x$

57.  $\cos^4 x - \sin^4 x = \cos 2x$

80.  $\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1}{2} \csc x \sin 2x$

58.  $\sin 4x = 4 \sin x \cos^3 x - 4 \cos x \sin^3 x$

81.  $\sin 2x - \cos x = (\cos x)(2 \sin x - 1)$

59.  $\cos^2 x - 2 \sin^2 x \cos^2 x - \sin^2 x + 2 \sin^4 x = \cos^2 2x$

82.  $\frac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

60.  $2 \cos^4 x - \cos^2 x - 2 \sin^2 x \cos^2 x + \sin^2 x = \cos^2 2x$

83.  $\tan 2x = \frac{2}{\cot x - \tan x}$

61.  $\cos 4x = 1 - 8 \cos^2 x + 8 \cos^4 x$

84.  $\frac{2 \cos 2x}{\sin 2x} = \cot x - \tan x$

62.  $\sin 4x = 4 \sin x \cos x - 8 \cos x \sin^3 x$

85.  $2 \tan \frac{x}{2} = \frac{\sin^2 x + 1 - \cos^2 x}{(\sin x)(1 + \cos x)}$

63.  $\cos 3x - \cos x = 4 \cos^3 x - 4 \cos x$

86.  $\frac{1}{2} \csc^2 \frac{x}{2} = \csc^2 x + \cot x \csc x$

64.  $\sin 3x + \sin x = 4 \sin x - 4 \sin^3 x$

87.  $\csc 2x = \frac{1}{2} \csc x \sec x$

65.  $\sin^3 x + \cos^3 x = (\sin x + \cos x) \left( 1 - \frac{1}{2} \sin 2x \right)$

88.  $\sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}$

66.  $\cos^3 x - \sin^3 x = (\cos x - \sin x) \left( 1 + \frac{1}{2} \sin 2x \right)$

67.  $\sin^2 \frac{x}{2} = \frac{\sec x - 1}{2 \sec x}$

68.  $\cos^2 \frac{x}{2} = \frac{\sec x + 1}{2 \sec x}$

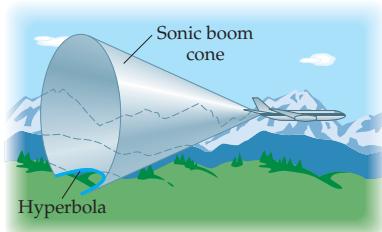
89.  $\cos \frac{x}{5} = 1 - 2 \sin^2 \frac{x}{10}$

69.  $\tan \frac{x}{2} = \csc x - \cot x$

70.  $\tan \frac{x}{2} = \frac{\tan x}{\sec x + 1}$

90.  $\sec^2 \frac{x}{2} = \frac{2}{1 + \cos x}$

- 91. MACH NUMBERS** Ernst Mach (1838–1916) was an Austrian physicist who made a study of the motion of objects at high speeds. Today we often state the speed of aircraft in terms of a *Mach number*. A **Mach number** is the speed of an object divided by the speed of sound. For example, a plane flying at the speed of sound is said to have a speed  $M$  of Mach 1. Mach 2 is twice the speed of sound. An airplane that travels faster than the speed of sound creates a sonic boom. This sonic boom emanates from the airplane in the shape of a cone.



## »»» Connecting Concepts



**In Exercises 92 to 95, compare the graphs of each side of the equation to predict whether the equation is an identity.**

**92.**  $\sin^2 x + \cos 2x = \cos^2 x$       **93.**  $\frac{\sin 2x}{1 - \sin^2 x} = 2 \tan x$

$$93. \frac{\sin 2x}{1 - \sin^2 x} = 2 \tan x$$

**94.**  $\sin \frac{x}{2} \cos \frac{x}{2} = \sin x$

**95.**  $\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2 = 1 + \sin x$

**In Exercises 96 to 98, verify the identity.**

$$96. \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \frac{1}{2} \sin 2x$$

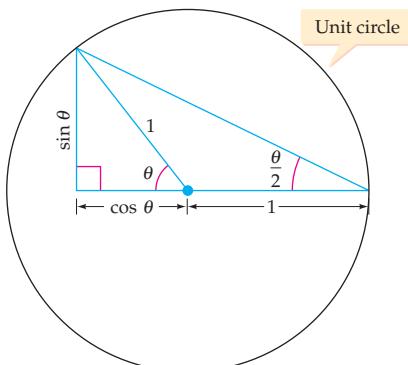
$$97. \cos^4 x = \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8}$$

$$98. \frac{\sin x - \sin 2x}{\cos x + \cos 2x} = -\tan \frac{x}{2}$$

# »»» Projects

- 1.**  **VISUAL INSIGHT** Explain how the figure at the right can be used to verify the half-angle identity

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$$



## Section 3.4

- The Product-to-Sum Identities
- The Sum-to-Product Identities
- Functions of the Form  $f(x) = a \sin x + b \cos x$

# Identities Involving the Sum of Trigonometric Functions

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A17.

**PS1.** Use sum and difference identities to rewrite  $\frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$ . [3.2]

**PS2.** Use sum and difference identities to rewrite  $\frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$ . [3.2]

**PS3.** Compare  $\sin x - \sin y$  and  $2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$  for  $x = \pi$  and  $y = \frac{\pi}{6}$ . [2.4]

**PS4.** Use a sum identity to rewrite  $\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ . [3.2]

**PS5.** Find a real number  $x$  and a real number  $y$  to verify that  $\sin x - \sin y = \sin(x - y)$  is *not* an identity. [2.4]

**PS6.** Evaluate  $\sqrt{a^2 + b^2}$  for  $a = -1$  and  $b = \sqrt{3}$ .

## The Product-to-Sum Identities

Some applications require that a product of trigonometric functions be written as a sum or difference of these functions. Other applications require that the sum or difference of trigonometric functions be represented as a product of these functions. The *product-to-sum identities* are particularly useful in these types of problems.

### Product-to-Sum Identities

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

The product-to-sum identities can be derived by using the sum or difference identities. Adding the identities for  $\sin(\alpha + \beta)$  and  $\sin(\alpha - \beta)$ , we have

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\underline{\sin(\alpha - \beta)} = \underline{\sin \alpha \cos \beta - \cos \alpha \sin \beta}$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

• Add the identities.

Solving for  $\sin \alpha \cos \beta$ , we obtain the first product-to-sum identity:

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

The identity for  $\cos \alpha \sin \beta$  is obtained when  $\sin(\alpha - \beta)$  is subtracted from  $\sin(\alpha + \beta)$ . The result is

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

In like manner, the identities for  $\cos(\alpha + \beta)$  and  $\cos(\alpha - \beta)$  are used to derive the identities for  $\cos \alpha \cos \beta$  and  $\sin \alpha \sin \beta$ .

The product-to-sum identities can be used to verify some identities.

### EXAMPLE 1 » Verify an Identity

Verify the identity  $\cos 2x \sin 5x = \frac{1}{2} (\sin 7x + \sin 3x)$ .

#### Solution

$$\begin{aligned}\cos 2x \sin 5x &= \frac{1}{2} [\sin(2x + 5x) - \sin(2x - 5x)] \\&= \frac{1}{2} [\sin 7x - \sin(-3x)] \\&= \frac{1}{2} (\sin 7x + \sin 3x)\end{aligned}$$

• Use the product-to-sum identity for  $\cos \alpha \sin \beta$ .

•  $\sin(-3x) = -\sin 3x$

» Try Exercise 36, page 252

### ■ The Sum-to-Product Identities

The *sum-to-product identities* can be derived from the product-to-sum identities.

**Sum-to-Product Identities**

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

To derive the sum-to-product identity for  $\sin x + \sin y$ , we first let  $x = \alpha + \beta$  and  $y = \alpha - \beta$ . Then

$$\begin{aligned}x + y &= \alpha + \beta + \alpha - \beta && \text{and } x - y = \alpha + \beta - (\alpha - \beta) \\x + y &= 2\alpha && x - y = 2\beta \\\alpha &= \frac{x+y}{2} && \beta = \frac{x-y}{2}\end{aligned}$$

Substituting these expressions for  $\alpha$  and  $\beta$  into the product-to-sum identity

$$\frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] = \sin \alpha \cos \beta$$

yields

$$\sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) + \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right) = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

Simplifying the left side, we have the sum-to-product identity.

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

The other three sum-to-product identities can be derived in a similar manner. The proofs of these identities are left as exercises.

**E X A M P L E 2 » Write the Difference of Trigonometric Expressions as a Product**

Write  $\sin 4\theta - \sin \theta$  as the product of two functions.

**Solution**

$$\sin 4\theta - \sin \theta = 2 \cos \frac{4\theta + \theta}{2} \sin \frac{4\theta - \theta}{2} = 2 \cos \frac{5\theta}{2} \sin \frac{3\theta}{2}$$

» Try Exercise 22, page 251

**QUESTION** Does  $\cos 4\theta + \cos 2\theta = 2 \cos 3\theta \cos \theta$ ?

**EXAMPLE 3** **Verify a Sum-to-Product Identity**

Verify the identity  $\frac{\sin 6x + \sin 2x}{\sin 6x - \sin 2x} = \tan 4x \cot 2x$ .

**Solution**

$$\begin{aligned}\frac{\sin 6x + \sin 2x}{\sin 6x - \sin 2x} &= \frac{2 \sin \frac{6x + 2x}{2} \cos \frac{6x - 2x}{2}}{2 \cos \frac{6x + 2x}{2} \sin \frac{6x - 2x}{2}} \\ &= \frac{\sin 4x \cos 2x}{\cos 4x \sin 2x} \\ &= \tan 4x \cot 2x\end{aligned}$$

» Try Exercise 44, page 252

**Functions of the Form  $f(x) = a \sin x + b \cos x$**

The function given by the equation  $f(x) = a \sin x + b \cos x$  can be written in the form  $f(x) = k \sin(x + \alpha)$ . This form of the function is useful in graphing and engineering applications because the amplitude, period, and phase shift can be readily calculated.

Let  $P(a, b)$  be a point on a coordinate plane, and let  $\alpha$  represent an angle in standard position whose terminal side contains  $P$ . See **Figure 3.6**. To rewrite  $y = a \sin x + b \cos x$ , multiply and divide the expression  $a \sin x + b \cos x$  by  $\sqrt{a^2 + b^2}$ .

$$\begin{aligned}a \sin x + b \cos x &= \frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2}}(a \sin x + b \cos x) \\ &= \sqrt{a^2 + b^2} \left( \frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x \right) \quad (1)\end{aligned}$$

From the definition of the sine and cosine of an angle in standard position, let

$$k = \sqrt{a^2 + b^2}, \quad \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \quad \text{and} \quad \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

Substituting these expressions into Equation (1) yields

$$a \sin x + b \cos x = k(\cos \alpha \sin x + \sin \alpha \cos x)$$

Now, using the identity for the sine of the sum of two angles, we have

$$a \sin x + b \cos x = k \sin(x + \alpha)$$

Thus  $a \sin x + b \cos x = k \sin(x + \alpha)$ , where  $k = \sqrt{a^2 + b^2}$  and  $\alpha$  is the angle for which  $\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$  and  $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$ .

**ANSWER** Yes.  $\cos 4\theta + \cos 2\theta = 2 \cos\left(\frac{4\theta + 2\theta}{2}\right) \cos\left(\frac{4\theta - 2\theta}{2}\right) = 2 \cos 3\theta \cos \theta$ .

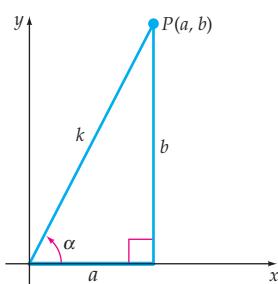


Figure 3.6

**Functions of the Form  $a \sin x + b \cos x$** 

$$a \sin x + b \cos x = k \sin(x + \alpha)$$

where  $k = \sqrt{a^2 + b^2}$ ,  $\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$ , and  $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$

**EXAMPLE 4 » Rewrite  $a \sin x + b \cos x$** 

Rewrite  $\sin x + \cos x$  in the form  $k \sin(x + \alpha)$ .

**Solution**

Comparing  $\sin x + \cos x$  to  $a \sin x + b \cos x$ ,  $a = 1$  and  $b = 1$ . Thus

$k = \sqrt{1^2 + 1^2} = \sqrt{2}$ ,  $\sin \alpha = \frac{1}{\sqrt{2}}$ , and  $\cos \alpha = \frac{1}{\sqrt{2}}$ . Thus  $\alpha = \frac{\pi}{4}$ .

$$\sin x + \cos x = k \sin(x + \alpha) = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$$

» Try Exercise 62, page 252

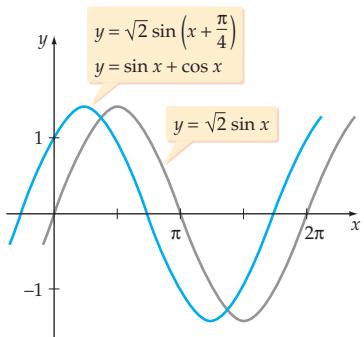


Figure 3.7

The graphs of  $y = \sin x + \cos x$  and  $y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$  are both the graph of  $y = \sqrt{2} \sin x$  shifted  $\frac{\pi}{4}$  units to the left. See **Figure 3.7**.

**EXAMPLE 5 » Graph a Function of the Form  $f(x) = a \sin x + b \cos x$** 

Graph:  $f(x) = -\sin x + \sqrt{3} \cos x$

**Solution**

First, we write  $f(x)$  as  $k \sin(x + \alpha)$ . Let  $a = -1$  and  $b = \sqrt{3}$ ; then  $k = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$ . The amplitude is 2. The point  $P(-1, \sqrt{3})$  is in the second quadrant (see **Figure 3.8**). Let  $\alpha$  be an angle in standard position with  $P$  on its terminal side. Let  $\alpha'$  be the reference angle for  $\alpha$ . Then

$$\sin \alpha' = \frac{\sqrt{3}}{2}$$

$$\alpha' = \frac{\pi}{3}$$

$$\alpha = \pi - \alpha' = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

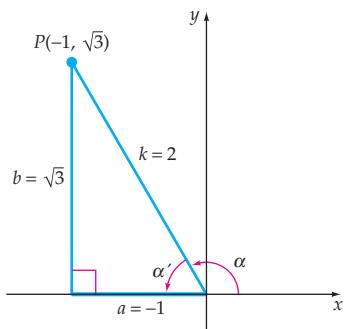


Figure 3.8

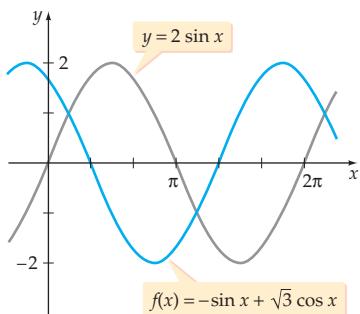


Figure 3.9

Substituting 2 for  $k$  and  $\frac{2\pi}{3}$  for  $\alpha$  in  $y = k \sin(x + \alpha)$ , we have

$$y = 2 \sin\left(x + \frac{2\pi}{3}\right)$$

The phase shift is  $-\frac{2\pi}{3}$ . The graph of  $f(x) = -\sin x + \sqrt{3} \cos x$  is the graph of  $y = 2 \sin x$  shifted  $\frac{2\pi}{3}$  units to the left. See **Figure 3.9**.

» Try Exercise 70, page 252



### Topics for Discussion

1. The exact value of  $\sin 75^\circ \cos 15^\circ$  is  $\frac{2 + \sqrt{3}}{4}$ . Do you agree? Explain.
2. Do you agree with the following work? Explain.

$$\cos 195^\circ + \cos 105^\circ = \cos(195^\circ + 105^\circ) = \cos 300^\circ = -\frac{1}{2}$$

3. The graphs of  $y_1 = \sin x + \cos x$  and  $y_2 = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$  are identical. Do you agree? Explain.
4. Explain how to find the amplitude of the graph of  $y = a \sin x + b \cos x$ .

## Exercise Set 3.4

In Exercises 1 to 8, write each expression as the sum or difference of two functions.

- |                        |                        |
|------------------------|------------------------|
| 1. $2 \sin x \cos 2x$  | 2. $2 \sin 4x \sin 2x$ |
| 3. $\cos 6x \sin 2x$   | 4. $\cos 3x \cos 5x$   |
| 5. $2 \sin 5x \cos 3x$ | 6. $2 \sin 2x \cos 6x$ |
| 7. $\sin x \sin 5x$    | 8. $\cos 3x \sin x$    |

In Exercises 9 to 16, find the exact value of each expression. Do not use a calculator.

- |  |                                    |
|--|------------------------------------|
| 9. $\cos 75^\circ \cos 15^\circ$       | 10. $\sin 105^\circ \cos 15^\circ$ |
| 11. $\cos 157.5^\circ \sin 22.5^\circ$ | 12. $\sin 195^\circ \cos 15^\circ$ |

13.  $\sin \frac{13\pi}{12} \cos \frac{\pi}{12}$

14.  $\sin \frac{11\pi}{12} \sin \frac{7\pi}{12}$

15.  $\sin \frac{\pi}{12} \cos \frac{7\pi}{12}$

16.  $\cos \frac{17\pi}{12} \sin \frac{7\pi}{12}$

In Exercises 17 to 32, write each expression as the product of two functions.

- |                                   |                                   |
|-----------------------------------|-----------------------------------|
| 17. $\sin 4\theta + \sin 2\theta$ | 18. $\cos 5\theta - \cos 3\theta$ |
| 19. $\cos 3\theta + \cos \theta$  | 20. $\sin 7\theta - \sin 3\theta$ |
| 21. $\cos 6\theta - \cos 2\theta$ | 22. $\cos 3\theta + \cos 5\theta$ |
| 23. $\cos \theta + \cos 7\theta$  | 24. $\sin 3\theta + \sin 7\theta$ |

25.  $\sin 5\theta + \sin 9\theta$

26.  $\cos 5\theta - \cos \theta$

51.  $y = \frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x$

52.  $y = \frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x$

27.  $\cos 2\theta - \cos \theta$

28.  $\sin 2\theta + \sin 6\theta$

53.  $y = \frac{1}{2} \sin x - \frac{1}{2} \cos x$

29.  $\cos \frac{\theta}{2} - \cos \theta$

30.  $\sin \frac{3\theta}{4} + \sin \frac{\theta}{2}$

54.  $y = -\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x$

31.  $\sin \frac{\theta}{2} - \sin \frac{\theta}{3}$

32.  $\cos \theta + \cos \frac{\theta}{2}$

55.  $y = -3 \sin x + 3 \cos x$

In Exercises 33 to 48, verify the identity.

33.  $2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

34.  $2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

35.  $2 \cos 3x \sin x = 2 \sin x \cos x - 8 \cos x \sin^3 x$

36.  $\sin 5x \cos 3x = \sin 4x \cos 4x + \sin x \cos x$

37.  $2 \cos 5x \cos 7x = \cos^2 6x - \sin^2 6x + 2 \cos^2 x - 1$

38.  $\sin 3x \cos x = \sin x \cos x(3 - 4 \sin^2 x)$

39.  $\sin 3x - \sin x = 2 \sin x - 4 \sin^3 x$

40.  $\cos 5x - \cos 3x = -8 \sin^2 x(2 \cos^3 x - \cos x)$

41.  $\sin 2x + \sin 4x = 2 \sin x \cos x(4 \cos^2 x - 1)$

42.  $\cos 3x + \cos x = 4 \cos^3 x - 2 \cos x$

43.  $\frac{\sin 3x - \sin x}{\cos 3x - \cos x} = -\cot 2x$

44.  $\frac{\cos 5x - \cos 3x}{\sin 5x + \sin 3x} = -\tan x$

45.  $\frac{\sin 5x + \sin 3x}{4 \sin x \cos^3 x - 4 \sin^3 x \cos x} = 2 \cos x$

46.  $\frac{\cos 4x - \cos 2x}{\sin 2x - \sin 4x} = \tan 3x$

47.  $\sin(x + y) \cos(x - y) = \sin x \cos x + \sin y \cos y$

48.  $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y$

In Exercises 49 to 58, write the given equation in the form  $y = k \sin(x + \alpha)$ , where the measure of  $\alpha$  is in degrees.

49.  $y = -\sin x - \cos x$

50.  $y = \sqrt{3} \sin x - \cos x$

53.  $y = \frac{1}{2} \sin x - \frac{1}{2} \cos x$

54.  $y = -\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x$

55.  $y = -3 \sin x + 3 \cos x$

56.  $y = \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x$

57.  $y = \pi \sin x - \pi \cos x$

58.  $y = -0.4 \sin x + 0.4 \cos x$

In Exercises 59 to 66, write the given equation in the form  $y = k \sin(x + \alpha)$ , where the measure of  $\alpha$  is in radians.

59.  $y = -\sin x + \cos x$

60.  $y = -\sqrt{3} \sin x - \cos x$

61.  $y = \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x$

62.  $y = \sin x + \sqrt{3} \cos x$

63.  $y = -10 \sin x + 10\sqrt{3} \cos x$

64.  $y = 3 \sin x - 3\sqrt{3} \cos x$

65.  $y = -5 \sin x + 5 \cos x$

66.  $y = 3 \sin x - 3 \cos x$

In Exercises 67 to 76, graph one cycle of the function. Do not use a graphing calculator.

67.  $y = -\sin x - \sqrt{3} \cos x$

68.  $y = -\sqrt{3} \sin x + \cos x$

69.  $y = 2 \sin x + 2 \cos x$

70.  $y = \sin x + \sqrt{3} \cos x$

71.  $y = -\sqrt{3} \sin x - \cos x$

72.  $y = -\sin x + \cos x$

73.  $y = -5 \sin x + 5\sqrt{3} \cos x$

74.  $y = -\sqrt{2} \sin x + \sqrt{2} \cos x$

75.  $y = 6\sqrt{3} \sin x - 6 \cos x$

76.  $y = 5\sqrt{2} \sin x - 5\sqrt{2} \cos x$

**TONES ON A TOUCH-TONE PHONE** In Exercises 77 and 78, use the following information about touch-tone phones. Every tone made on a touch-tone phone is produced by adding a pair of sounds. The following chart shows the sound fre-



$$93. \frac{\sin 2x + \sin 4x + \sin 6x}{\cos 2x + \cos 4x + \cos 6x} = \tan 4x$$

$$94. \frac{\sin 2x + \sin 6x}{\cos 6x - \cos 2x} = -\cot 2x$$

- 95.** Verify that  $\cos^2 x - \sin^2 x = \cos 2x$  by using a product-to-sum identity.

**96.** Verify that  $2 \sin x \cos x = \sin 2x$  by using a product-to-sum identity.

- 1.**  **BEATS** If two tuning forks that are close in frequency are struck at the same time, the sound we hear fluctuates between a loud tone and silence. These regular fluctuations are called **beats**. The loud periods occur when the sound waves reinforce (interfere constructively with) one another, and the silent periods occur when the waves interfere destructively with one another. If the frequencies of the tuning forks are 442 cycles per second and 440 cycles per second then the pressure produced on our eardrums by each tuning fork, respectively, is given by

$$Y_1 = \sin(2\pi \cdot 442x) \quad \text{and} \quad Y_2 = \sin(2\pi \cdot 440x)$$

where  $x$  is the time in seconds. The combined pressure produced when both tuning forks are struck simultaneously is modeled by

$$Y_3 = Y_1 + Y_2$$



- 97.** Verify that  $a \sin x + b \cos x = k \cos(x - \alpha)$ , where  
 $k = \sqrt{a^2 + b^2}$  and  $\tan \alpha = \frac{a}{b}$ .

**98.** Verify that  $a \sin cx + b \cos cx = k \sin(cx + \alpha)$ , where  
 $k = \sqrt{a^2 + b^2}$  and  $\tan \alpha = \frac{b}{a}$ .

- a. Graph Y3. Use a viewing window with  $X_{\text{min}} = 0$ ,  $X_{\text{max}} = 1$ ,  $Y_{\text{min}} = -2.1$ ,  $Y_{\text{max}} = 2.1$ . To produce a graph that better illustrates the beats produced when both tuning forks are struck simultaneously, graph Y3,  $Y_4 = 2 \cos(2\pi x)$ , and  $Y_5 = -2 \cos(2\pi x)$  together in the same window. What is the relationship between the graph of Y3 and the graphs of Y4 and Y5?
  - b. Use a sum-to-product identity to write Y3 as a product.
  - c. The rate of the beats produced by two sounds with the same intensity is the absolute value of the difference between their frequencies. Consider two tuning forks that are struck at the same time with the same force and held on a sounding board. The tuning forks have frequencies of 564 and 568 cycles per second, respectively. How many beats will be heard each second?
  - d. A piano tuner strikes a tuning fork and a key on a piano that is supposed to have the same frequency as the tuning fork. The piano tuner notices that the sound produced by the piano is lower than that produced by the tuning fork. The piano tuner also notes that the combined sound of the piano and the tuning fork has 2 beats per second. How much lower is the frequency of the piano than the frequency of the tuning fork?

## Section 3.5

- Inverse Trigonometric Functions
- Composition of Trigonometric Functions and Their Inverses
- Graphs of Inverse Trigonometric Functions

# Inverse Trigonometric Functions

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A17.

- PS1.** What is a one-to-one function? [1.3]
- PS2.** State the horizontal line test. [1.3]
- PS3.** Find  $f[g(x)]$  given that  $f(x) = 2x + 4$  and  $g(x) = \frac{1}{2}x - 2$ . [1.5]
- PS4.** If  $f$  and  $f^{-1}$  are inverse functions, then determine  $f[f^{-1}(x)]$  for any  $x$  in the domain of  $f^{-1}$ . [1.6]
- PS5.** If  $f$  and  $f^{-1}$  are inverse functions, then explain how the graph of  $f^{-1}$  is related to the graph of  $f$ . [1.6]
- PS6.** Use the horizontal line test to determine whether the graph of  $y = \sin x$ , where  $x$  is any real number, is a one-to-one function. [1.3/2.5]

### Inverse Trigonometric Functions

Because the graph of  $y = \sin x$  fails the horizontal line test, it is not the graph of a one-to-one function. Therefore, it does not have an inverse function. **Figure 3.10** shows the graph of  $y = \sin x$  on the interval  $-2\pi \leq x \leq 2\pi$  and the graph of the inverse relation  $x = \sin y$ . Note that the graph of  $x = \sin y$  does not satisfy the vertical line test and therefore is not the graph of a function.

If the domain of the function  $y = \sin x$  is restricted to  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , the graph of  $y = \sin x$  satisfies the horizontal line test and therefore the function has an inverse function. The graphs of  $y = \sin x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$  and its inverse are shown in **Figure 3.11** on the next page.

#### TO REVIEW

#### Inverse Functions

See section 1.6.

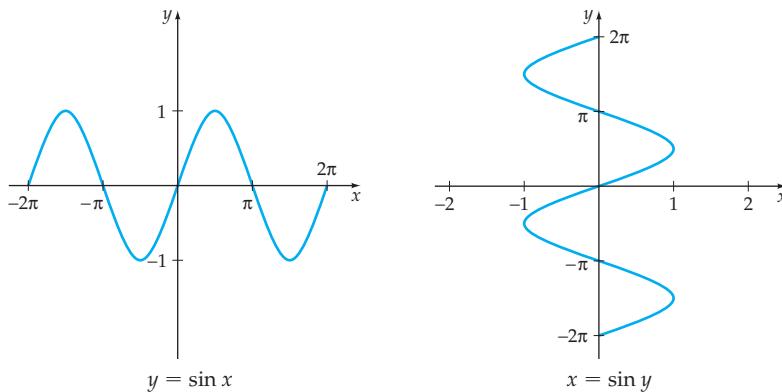


Figure 3.10

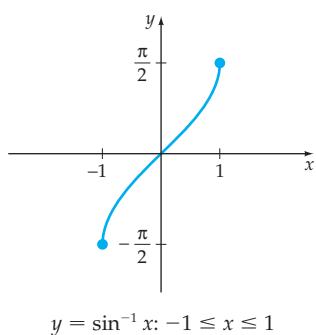
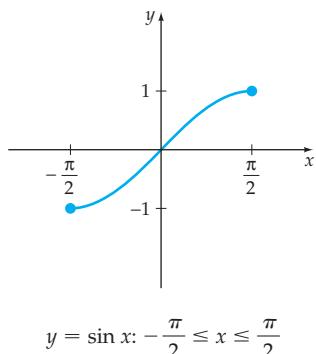


Figure 3.11

**take note**

The  $-1$  in  $\sin^{-1} x$  is not an exponent. The  $-1$  is used to denote the inverse function. To use  $-1$  as an exponent for a sine function, enclose the function in parentheses.

$$(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$$

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

To find the inverse of the function defined by  $y = \sin x$ , with  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , interchange  $x$  and  $y$ . Then solve for  $y$ .

$$\begin{array}{ll} y = \sin x & \bullet -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ x = \sin y & \bullet \text{Interchange } x \text{ and } y. \\ y = ? & \bullet \text{Solve for } y. \end{array}$$

Unfortunately, there is no algebraic solution for  $y$ . Thus we establish new notation and write

$$y = \sin^{-1} x$$

which is read “ $y$  is the inverse sine of  $x$ .” Some textbooks use the notation  $\arcsin x$  instead of  $\sin^{-1} x$ .

**Definition of**

$$y = \sin^{-1} x \text{ if and only if } x = \sin y$$

$$\text{where } -1 \leq x \leq 1 \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

It is convenient to think of the value of an inverse trigonometric function as an angle. For instance, if  $y = \sin^{-1}\left(\frac{1}{2}\right)$ , then  $y$  is the angle in the interval  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  whose sine is  $\frac{1}{2}$ . Thus  $y = \frac{\pi}{6}$ .

Because the graph of  $y = \cos x$  fails the horizontal line test, it is not the graph of a one-to-one function. Therefore, it does not have an inverse function. **Figure 3.12** shows the graph of  $y = \cos x$  on the interval  $-2\pi \leq x \leq 2\pi$  and the graph of the inverse relation  $x = \cos y$ . Note that the graph of  $x = \cos y$  does not satisfy the vertical line test and therefore is not the graph of a function.

If the domain of  $y = \cos x$  is restricted to  $0 \leq x \leq \pi$ , the graph of  $y = \cos x$  satisfies the horizontal line test and therefore is the graph of a one-to-one function. The graphs of  $y = \cos x$  for  $0 \leq x \leq \pi$  and its inverse are shown in **Figure 3.13**.

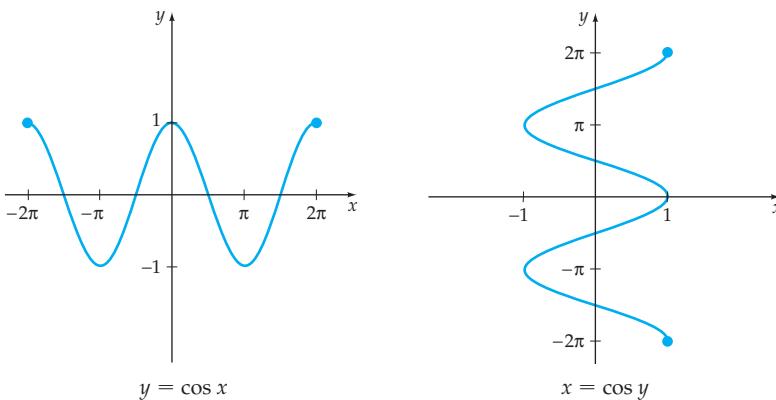
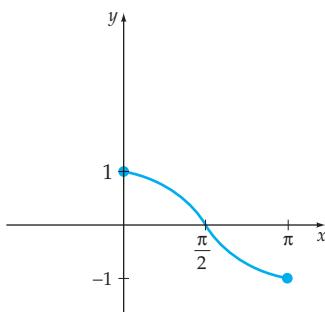
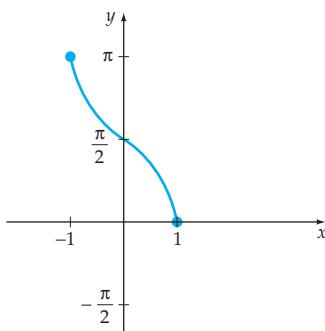


Figure 3.12



$$y = \cos x: 0 \leq x \leq \pi$$



$$y = \cos^{-1} x: -1 \leq x \leq 1$$

**Figure 3.13**

To find the inverse of the function defined by  $y = \cos x$ , with  $0 \leq x \leq \pi$ , interchange  $x$  and  $y$ . Then solve for  $y$ .

$$\begin{array}{ll} y = \cos x & \bullet 0 \leq x \leq \pi \\ x = \cos y & \bullet \text{Interchange } x \text{ and } y. \\ y = ? & \bullet \text{Solve for } y. \end{array}$$

As in the case for the inverse sine function, there is no algebraic solution for  $y$ . Thus the notation for the inverse cosine function becomes  $y = \cos^{-1} x$ . We can write the following definition of the inverse cosine function.

### Definition of

$$y = \cos^{-1} x \quad \text{if and only if } x = \cos y$$

where  $-1 \leq x \leq 1$  and  $0 \leq y \leq \pi$ .

Because the graphs of  $y = \tan x$ ,  $y = \csc x$ ,  $y = \sec x$ , and  $y = \cot x$  fail the horizontal line test, these functions are not one-to-one functions. Therefore, these functions do not have inverse functions. If the domains of all these functions are restricted in a certain way, however, the graphs satisfy the horizontal line test. Thus each of these functions has an inverse function over a restricted domain. **Table 3.2** on page 258 shows the restricted function and the inverse function for  $\tan x$ ,  $\csc x$ ,  $\sec x$ , and  $\cot x$ .

The choice of ranges for  $y = \sec^{-1} x$  and  $y = \csc^{-1} x$  is not universally accepted. For example, some calculus texts use  $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$  as the range of  $y = \sec^{-1} x$ . This definition has some advantages and some disadvantages that are explained in more advanced mathematics courses.

### EXAMPLE 1 » Evaluate Inverse Functions

Find the exact value of each inverse function.

a.  $y = \tan^{-1} \frac{\sqrt{3}}{3}$     b.  $y = \cos^{-1} \left(-\frac{\sqrt{2}}{2}\right)$

#### Solution

a. Because  $y = \tan^{-1} \frac{\sqrt{3}}{3}$ ,  $y$  is the angle whose measure is in the interval  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , and  $\tan y = \frac{\sqrt{3}}{3}$ . Therefore,  $y = \frac{\pi}{6}$ .

b. Because  $y = \cos^{-1} \left(-\frac{\sqrt{2}}{2}\right)$ ,  $y$  is the angle whose measure is in the interval  $[0, \pi]$ , and  $\cos y = -\frac{\sqrt{2}}{2}$ . Therefore,  $y = \frac{3}{4}\pi$ .

» Try Exercise 2, page 265

Table 3.2

|                   | $y = \tan x$                              | $y = \tan^{-1} x$                         | $y = \csc x$   | $y = \csc^{-1} x$                                    |
|-------------------|---|---|--|--|
| <b>Domain</b>     | $-\frac{\pi}{2} < x < \frac{\pi}{2}$      | $-\infty < x < \infty$                    | $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$ | $x \leq -1$ or $x \geq 1$                            |
| <b>Range</b>      | $-\infty < y < \infty$                    | $-\frac{\pi}{2} < y < \frac{\pi}{2}$      | $y \leq -1$ or $y \geq 1$                            | $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$ |
| <b>Asymptotes</b> | $x = -\frac{\pi}{2}, x = \frac{\pi}{2}$   | $y = -\frac{\pi}{2}, y = \frac{\pi}{2}$   | $x = 0$  | $y = 0$  |
| <b>Graph</b>      |   |   |  |  |
|                   | $y = \sec x$                              | $y = \sec^{-1} x$                         | $y = \cot x$   | $y = \cot^{-1} x$                                    |
| <b>Domain</b>     | $0 \leq x \leq \pi, x \neq \frac{\pi}{2}$ | $x \leq -1$ or $x \geq 1$                 | $0 < x < \pi$  | $-\infty < x < \infty$                               |
| <b>Range</b>      | $y \leq -1$ or $y \geq 1$                 | $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$ | $-\infty < y < \infty$                               | $0 < y < \pi$  |
| <b>Asymptotes</b> | $x = \frac{\pi}{2}$                       | $y = \frac{\pi}{2}$                       | $x = 0, x = \pi$                                     | $y = 0, y = \pi$                                     |
| <b>Graph</b>      |   |   |  |  |

A calculator may not have keys for the inverse secant, cosecant, and cotangent functions. The following procedure shows an identity for the inverse cosecant function in terms of the inverse sine function. If we need to determine  $y$ , which is the angle whose cosecant is  $x$ , we can rewrite  $y = \csc^{-1} x$  as follows:

$$y = \csc^{-1} x$$

- Domain:  $x \leq -1$  or  $x \geq 1$

- Range:  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

$$\csc y = x$$

- Definition of inverse function

$$\frac{1}{\sin y} = x$$

- Substitute  $\frac{1}{\sin y}$  for  $\csc y$ .

$$\sin y = \frac{1}{x}$$

• Solve for  $\sin y$ .

$$y = \sin^{-1} \frac{1}{x}$$

• Write using inverse notation.

$$\csc^{-1} x = \sin^{-1} \frac{1}{x}$$

• Replace  $y$  with  $\csc^{-1} x$ .

Thus  $\csc^{-1} x$  is the same as  $\sin^{-1} \frac{1}{x}$ . There is a similar identity for  $\sec^{-1} x$ .

### Identities for the Inverse Secant, Cosecant, and Cotangent Functions

If  $x \leq -1$  or  $x \geq 1$ , then

$$\csc^{-1} x = \sin^{-1} \frac{1}{x} \quad \text{and} \quad \sec^{-1} x = \cos^{-1} \frac{1}{x}$$

If  $x$  is a real number, then

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

## Composition of Trigonometric Functions and Their Inverses

Recall that a function  $f$  and its inverse  $f^{-1}$  have the property that  $f[f^{-1}(x)] = x$  for all  $x$  in the domain of  $f^{-1}$  and that  $f^{-1}[f(x)] = x$  for all  $x$  in the domain of  $f$ . Applying this property to the functions  $\sin x$ ,  $\cos x$ , and  $\tan x$  and their inverse functions produces the following theorems.

### TO REVIEW

#### Composition of Functions

See section 1.5.

### Composition of Trigonometric Functions and Their Inverses

- If  $-1 \leq x \leq 1$ , then  $\sin(\sin^{-1} x) = x$  and  $\cos(\cos^{-1} x) = x$ .
- If  $x$  is any real number, then  $\tan(\tan^{-1} x) = x$ .
- If  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , then  $\sin^{-1}(\sin x) = x$ .
- If  $0 \leq x \leq \pi$ , then  $\cos^{-1}(\cos x) = x$ .
- If  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ , then  $\tan^{-1}(\tan x) = x$ .

In the next example we make use of some of the composition theorems to evaluate trigonometric expressions.

**E X A M P L E 2 »» Evaluate the Composition of a Function and Its Inverse**

Find the exact value of each composition of functions.

- a.  $\sin(\sin^{-1} 0.357)$
- b.  $\cos^{-1}(\cos 3)$
- c.  $\tan[\tan^{-1}(-11.27)]$
- d.  $\sin(\sin^{-1} \pi)$
- e.  $\cos(\cos^{-1} 0.277)$
- f.  $\tan^{-1}\left(\tan \frac{4\pi}{3}\right)$

**Solution**

- a. Because 0.357 is in the interval  $[-1, 1]$ ,  $\sin(\sin^{-1} 0.357) = 0.357$ .
- b. Because 3 is in the interval  $[0, \pi]$ ,  $\cos^{-1}(\cos 3) = 3$ .
- c. Because  $-11.27$  is a real number,  $\tan[\tan^{-1}(-11.27)] = -11.27$ .
- d. Because  $\pi$  is not in the domain of the inverse sine function,  $\sin(\sin^{-1} \pi)$  is undefined.
- e. Because 0.277 is in the interval  $[-1, 1]$ ,  $\cos(\cos^{-1} 0.277) = 0.277$ .
- f.  $\frac{4\pi}{3}$  is not in the interval  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ; however, the reference angle for  $\theta = \frac{4\pi}{3}$  is  $\theta' = \frac{\pi}{3}$ . Thus  $\tan^{-1}\left(\tan \frac{4\pi}{3}\right) = \tan^{-1}\left(\tan \frac{\pi}{3}\right)$ . Because  $\frac{\pi}{3}$  is in the interval  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $\tan^{-1}\left(\tan \frac{\pi}{3}\right) = \frac{\pi}{3}$ . Hence  $\tan^{-1}\left(\tan \frac{4\pi}{3}\right) = \frac{\pi}{3}$ .

»» Try Exercise 28, page 265

**QUESTION** Is  $\tan^{-1}(\tan x) = x$  an identity?

It is often easy to evaluate a trigonometric expression by referring to a sketch of a right triangle that satisfies given conditions. In Example 3 we make use of this technique.

**E X A M P L E 3 »» Evaluate a Trigonometric Expression**

Find the exact value of  $\sin\left(\cos^{-1} \frac{2}{5}\right)$ .

---

**ANSWER** No.  $\tan^{-1}(\tan x) = x$  only if  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ .

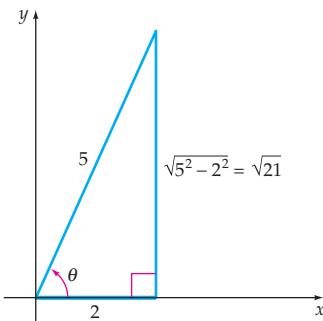


Figure 3.14

**Solution**

Let  $\theta = \cos^{-1} \frac{2}{5}$ , which implies that  $\cos \theta = \frac{2}{5}$ . Because  $\cos \theta$  is positive,  $\theta$  is a first-quadrant angle. We draw a right triangle with base 2 and hypotenuse 5 so that we can view  $\theta$ , as shown in **Figure 3.14**. The height of the triangle is  $\sqrt{5^2 - 2^2} = \sqrt{21}$ . Our goal is to find  $\sin \theta$ , which by definition is  $\frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{21}}{5}$ . Thus

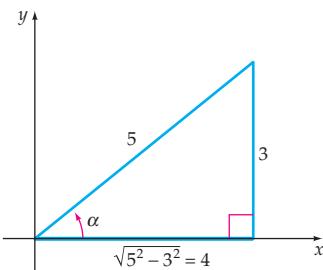
$$\sin\left(\cos^{-1} \frac{2}{5}\right) = \sin(\theta) = \frac{\sqrt{21}}{5}$$

» Try Exercise 50, page 265

In Example 4, we sketch two right triangles to evaluate the given expression.

**E X A M P L E 4 » Evaluate a Trigonometric Expression**

Find the exact value of  $\sin\left[\sin^{-1} \frac{3}{5} + \cos^{-1}\left(-\frac{5}{13}\right)\right]$ .

**Solution**

Let  $\alpha = \sin^{-1} \frac{3}{5}$ . Thus  $\sin \alpha = \frac{3}{5}$ . Let  $\beta = \cos^{-1}\left(-\frac{5}{13}\right)$ , which implies that  $\cos \beta = -\frac{5}{13}$ . Sketch angles  $\alpha$  and  $\beta$  as shown in **Figure 3.15**. We wish to evaluate

$$\begin{aligned} \sin\left[\sin^{-1} \frac{3}{5} + \cos^{-1}\left(-\frac{5}{13}\right)\right] &= \sin(\alpha + \beta) \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{aligned} \quad (1)$$

A close look at the triangles in **Figure 3.15** shows that

$$\cos \alpha = \frac{4}{5} \quad \text{and} \quad \sin \beta = \frac{12}{13}$$

Substituting in Equation (1) gives us our desired result.

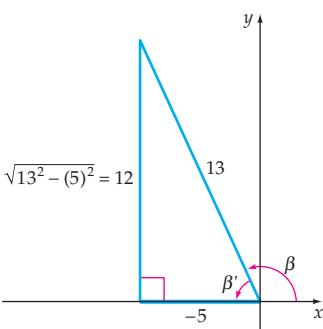


Figure 3.15

$$\begin{aligned} \sin\left[\sin^{-1} \frac{3}{5} + \cos^{-1}\left(-\frac{5}{13}\right)\right] &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{33}{65} \end{aligned}$$

» Try Exercise 56, page 266

In Example 5 we make use of the identity  $\cos(\cos^{-1} x) = x$ , where  $-1 \leq x \leq 1$ , to solve an equation.

**EXAMPLE 5** » Solve an Inverse Trigonometric Equation

Solve  $\sin^{-1} \frac{3}{5} + \cos^{-1} x = \pi$ .

**Solution**

Solve for  $\cos^{-1} x$  and then take the cosine of both sides of the equation.

$$\sin^{-1} \frac{3}{5} + \cos^{-1} x = \pi$$

$$\cos^{-1} x = \pi - \sin^{-1} \frac{3}{5}$$

$$\cos(\cos^{-1} x) = \cos\left(\pi - \sin^{-1} \frac{3}{5}\right)$$

$$x = \cos(\pi - \alpha)$$

- Let  $\alpha = \sin^{-1} \frac{3}{5}$ . Note that  $\alpha$  is the angle whose sine is  $\frac{3}{5}$ . (See Figure 3.16.)

$$= \cos \pi \cos \alpha + \sin \pi \sin \alpha$$

- Difference identity for cosine

$$= (-1) \cos \alpha + (0) \sin \alpha$$

$$= -\cos \alpha$$

$$= -\frac{4}{5}$$

- $\cos \alpha = \frac{4}{5}$  (See Figure 3.16.)

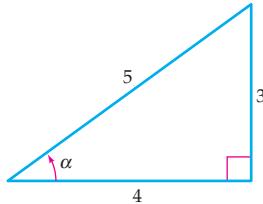


Figure 3.16

**» Try Exercise 66, page 266**
**EXAMPLE 6** » Verify a Trigonometric Identity That Involves Inverses

Verify the identity  $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ .

**Solution**

Let  $\alpha = \sin^{-1} x$  and  $\beta = \cos^{-1} x$ . These equations imply that  $\sin \alpha = x$  and  $\cos \beta = x$ . From the right triangles in Figure 3.17,

$$\cos \alpha = \sqrt{1 - x^2} \quad \text{and} \quad \sin \beta = \sqrt{1 - x^2}$$

Our goal is to show  $\sin^{-1} x + \cos^{-1} x$  equals  $\frac{\pi}{2}$ .

$$\begin{aligned} \sin^{-1} x + \cos^{-1} x &= \alpha + \beta \\ &= \cos^{-1}[\cos(\alpha + \beta)] \end{aligned}$$

- Because  $0 \leq \alpha + \beta \leq \pi$ , we can apply  $\alpha + \beta = \cos^{-1}[\cos(\alpha + \beta)]$ .

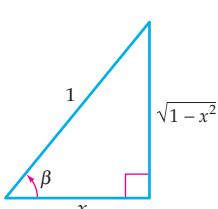
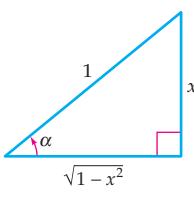


Figure 3.17

$$\begin{aligned}
 &= \cos^{-1}[\cos \alpha \cos \beta - \sin \alpha \sin \beta] \\
 &= \cos^{-1}[(\sqrt{1-x^2})(x) - (x)(\sqrt{1-x^2})] \\
 &= \cos^{-1} 0 \\
 &= \frac{\pi}{2}
 \end{aligned}$$

• Addition identity  
for cosine

» Try Exercise 72, page 266

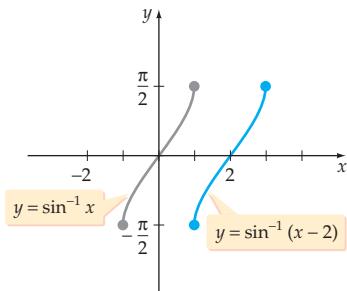


Figure 3.18

## Graphs of Inverse Trigonometric Functions

The inverse trigonometric functions can be graphed by using the procedures of stretching, shrinking, and translation that were discussed earlier in the text. For instance, the graph of  $y = \sin^{-1}(x - 2)$  is a horizontal shift 2 units to the right of the graph of  $y = \sin^{-1} x$ , as shown in **Figure 3.18**.

### EXAMPLE 7 » Graph an Inverse Function

Graph:  $y = \cos^{-1} x + 1$

#### Solution

Recall that the graph of  $y = f(x) + c$  is a vertical translation of the graph of  $f$ . Because  $c = 1$ , a positive number, the graph of  $y = \cos^{-1} x + 1$  is the graph of  $y = \cos^{-1} x$  shifted 1 unit up. See **Figure 3.19**.

» Try Exercise 76, page 266

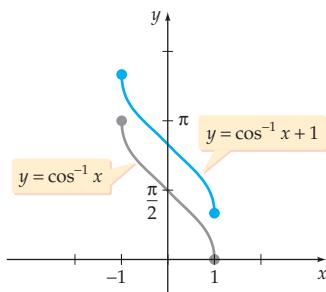


Figure 3.19

### Integrating Technology

When you use a graphing utility to draw the graph of an inverse trigonometric function, use the properties of these functions to verify the correctness of your graph. For instance, the graph of  $y = 3 \sin^{-1} 0.5x$  is shown in **Figure 3.20**. The domain of  $y = \sin^{-1} x$  is  $-1 \leq x \leq 1$ . Therefore, the domain of  $y = 3 \sin^{-1} 0.5x$  is  $-1 \leq 0.5x \leq 1$  or, multiplying the inequality by 2,  $-2 \leq x \leq 2$ . This is consistent with the graph in **Figure 3.20**.

The range of  $y = \sin^{-1} x$  is  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ . Thus the range of

$y = 3 \sin^{-1} 0.5x$  is  $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$ . This is also consistent with the graph.

Verifying some of the properties of  $y = \sin^{-1} x$  serves as a check that you have correctly entered the equation for the graph.

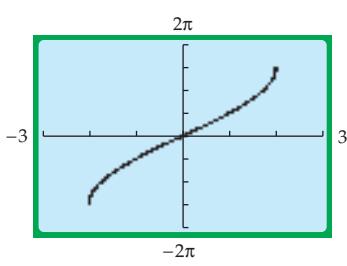


Figure 3.20

## EXAMPLE 8 » Solve an Application



A camera is placed on a deck of a pool as shown in **Figure 3.21**. A diver is 18 feet above the camera lens. The extended length of the diver is 8 feet.

- a. Show that the angle  $\theta$  subtended at the lens by the diver is

$$\theta = \tan^{-1} \frac{26}{x} - \tan^{-1} \frac{18}{x}$$

- b. For what values of  $x$  will  $\theta = 9^\circ$ ?  
c. What value of  $x$  maximizes  $\theta$ ?

## Solution

- a. From **Figure 3.21** we see that  $\alpha = \tan^{-1} \frac{26}{x}$  and  $\beta = \tan^{-1} \frac{18}{x}$ . Because

$$\theta = \alpha - \beta, \text{ we have } \theta = \tan^{-1} \frac{26}{x} - \tan^{-1} \frac{18}{x}.$$

- b. Use a graphing utility to graph  $\theta = \tan^{-1} \frac{26}{x} - \tan^{-1} \frac{18}{x}$  and  $\theta = \frac{\pi}{20} \left(9^\circ = \frac{\pi}{20} \text{ radians}\right)$ . See **Figure 3.22**. Use the “intersect” command to show that  $\theta$  is  $9^\circ$  for  $x \approx 12.22$  feet and  $x \approx 38.29$  feet.

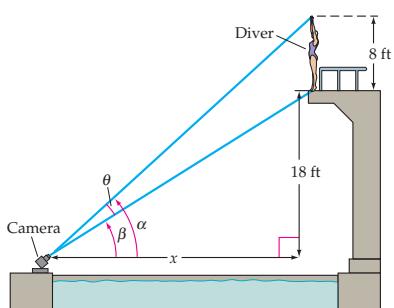


Figure 3.21

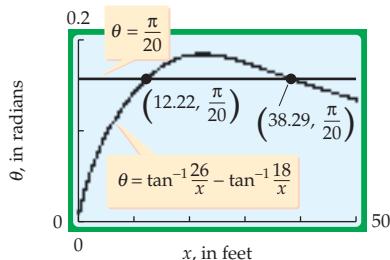


Figure 3.22

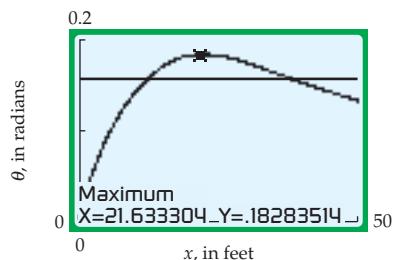


Figure 3.23

- c. Use the “maximum” command to show that the maximum value of  $\theta = \tan^{-1} \frac{26}{x} - \tan^{-1} \frac{18}{x}$  occurs when  $x \approx 21.63$  feet. See **Figure 3.23**.

## » Try Exercise 84, page 266



## Topics for Discussion

1. Is the equation

$$\tan^{-1} x = \frac{1}{\tan x}$$

true for all values of  $x$ , true for some values of  $x$ , or false for all values of  $x$ ?

2. Are there real numbers  $x$  for which the following is true? Explain.

$$\sin(\sin^{-1} x) \neq \sin^{-1}(\sin x)$$

3. Explain how to find the value of  $\sec^{-1} 3$  by using a calculator.  
 4. Explain how you can determine the range of  $y = (2 \cos^{-1} x) - 1$  using  
 a. algebra      b. a graph

## Exercise Set 3.5

In Exercises 1 to 18, find the exact radian value.

1.  $\sin^{-1} 1$



2.  $\sin^{-1} \frac{\sqrt{2}}{2}$

3.  $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

4.  $\cos^{-1}\left(-\frac{1}{2}\right)$

5.  $\tan^{-1}(-1)$

6.  $\tan^{-1} \sqrt{3}$

7.  $\cot^{-1} \frac{\sqrt{3}}{3}$

8.  $\cot^{-1} 1$

9.  $\sec^{-1} 2$

10.  $\sec^{-1} \frac{2\sqrt{3}}{3}$

11.  $\csc^{-1}(-\sqrt{2})$

12.  $\csc^{-1}(-2)$

13.  $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

14.  $\sin^{-1} \frac{1}{2}$

15.  $\cos^{-1}\left(-\frac{1}{2}\right)$

16.  $\cos^{-1} \frac{\sqrt{3}}{2}$

17.  $\tan^{-1} \frac{\sqrt{3}}{3}$

18.  $\tan^{-1} 1$



In Exercises 19 to 22, use a calculator to approximate each function accurate to four decimal places.

19. a.  $\sin^{-1}(0.8422)$

b.  $\tan^{-1}(0.2385)$

20. a.  $\cos^{-1}(-0.0356)$

b.  $\tan^{-1}(3.7555)$

21. a.  $\sec^{-1}(2.2500)$

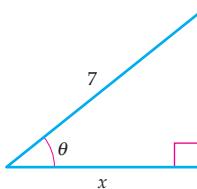
b.  $\cot^{-1}(3.4545)$

22. a.  $\csc^{-1}(1.3465)$

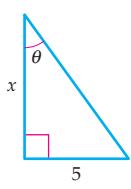
b.  $\cot^{-1}(0.1274)$

In Exercises 23 to 24, express  $\theta$  as a function of  $x$ .

23.



24.



In Exercises 25 to 58, find the exact value of the given expression. If an exact value cannot be given, give the value to the nearest ten-thousandth.

25.  $\cos\left(\cos^{-1} \frac{1}{2}\right)$

26.  $\cos(\cos^{-1} 2)$

27.  $\tan(\tan^{-1} 2)$

28.  $\tan\left(\tan^{-1} \frac{1}{2}\right)$

29.  $\sin\left(\tan^{-1} \frac{3}{4}\right)$

30.  $\cos\left(\sin^{-1} \frac{5}{13}\right)$

31.  $\tan\left(\sin^{-1} \frac{\sqrt{2}}{2}\right)$

32.  $\sin\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

33.  $\cos(\sec^{-1} 2)$

34.  $\sin^{-1}(\sin 2)$

35.  $\sin^{-1}\left(\sin \frac{\pi}{6}\right)$

36.  $\sin^{-1}\left(\sin \frac{5\pi}{6}\right)$

37.  $\cos^{-1}\left(\sin \frac{\pi}{4}\right)$

38.  $\cos^{-1}\left(\cos \frac{5\pi}{4}\right)$

39.  $\sin^{-1}\left(\tan \frac{\pi}{3}\right)$

40.  $\cos^{-1}\left(\tan \frac{2\pi}{3}\right)$

41.  $\tan^{-1}\left(\sin \frac{\pi}{6}\right)$

42.  $\cot^{-1}\left(\cos \frac{2\pi}{3}\right)$

43.  $\sin^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$

44.  $\cos^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right]$

45.  $\tan\left(\sin^{-1} \frac{1}{2}\right)$

46.  $\cot(\csc^{-1} 2)$

47.  $\sec\left(\sin^{-1} \frac{1}{4}\right)$

48.  $\csc\left(\cos^{-1} \frac{3}{4}\right)$

49.  $\cos\left(\sin^{-1} \frac{7}{25}\right)$

50.  $\tan\left(\cos^{-1} \frac{3}{5}\right)$

51.  $\cos\left(2 \sin^{-1} \frac{\sqrt{2}}{2}\right)$

52.  $\tan\left(2 \sin^{-1} \frac{\sqrt{3}}{2}\right)$

53.  $\sin\left(2 \sin^{-1} \frac{4}{5}\right)$

54.  $\cos(2 \tan^{-1} 1)$

55.  $\sin\left(\sin^{-1} \frac{2}{3} + \cos^{-1} \frac{1}{2}\right)$

56.  $\cos\left(\sin^{-1} \frac{3}{4} + \cos^{-1} \frac{5}{13}\right)$

57.  $\tan\left(\cos^{-1} \frac{1}{2} - \sin^{-1} \frac{3}{4}\right)$

58.  $\sec\left(\cos^{-1} \frac{2}{3} + \sin^{-1} \frac{2}{3}\right)$

In Exercises 59 to 68, solve the equation for  $x$  algebraically.

59.  $\sin^{-1} x = \cos^{-1} \frac{5}{13}$

60.  $\tan^{-1} x = \sin^{-1} \frac{24}{25}$

61.  $\sin^{-1}(x - 1) = \frac{\pi}{2}$

62.  $\cos^{-1}\left(x - \frac{1}{2}\right) = \frac{\pi}{3}$

63.  $\tan^{-1}\left(x + \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

64.  $\sin^{-1}(x - 2) = -\frac{\pi}{6}$

65.  $\sin^{-1} \frac{3}{5} + \cos^{-1} x = \frac{\pi}{4}$

66.  $\sin^{-1} x + \cos^{-1} \frac{4}{5} = \frac{\pi}{6}$

67.  $\sin^{-1} \frac{\sqrt{2}}{2} + \cos^{-1} x = \frac{2\pi}{3}$

68.  $\cos^{-1} x + \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{2}$

In Exercises 69 and 70, write each expression in terms of  $x$ .

69.  $\tan(\cos^{-1} x)$

70.  $\sin(\sec^{-1} x)$

In Exercises 71 to 74, verify the identity.

71.  $\sin^{-1} x + \sin^{-1}(-x) = 0$

72.  $\cos^{-1} x + \cos^{-1}(-x) = \pi$

73.  $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}, x > 0$

74.  $\sec^{-1} \frac{1}{x} + \csc^{-1} \frac{1}{x} = \frac{\pi}{2}$

In Exercises 75 to 82, use stretching, shrinking, and translation procedures to graph each equation.

75.  $y = \sin^{-1} x + 2$

76.  $y = \cos^{-1}(x - 1)$

77.  $y = \sin^{-1}(x + 1) - 2$

78.  $y = \tan^{-1}(x - 1) + 2$

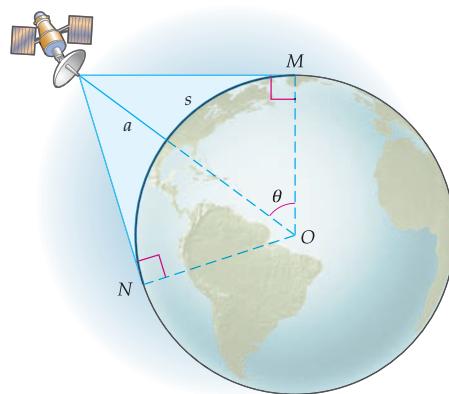
79.  $y = 2 \cos^{-1} x$

80.  $y = -2 \tan^{-1} x$

81.  $y = \tan^{-1}(x + 1) - 2$

82.  $y = \sin^{-1}(x - 2) + 1$

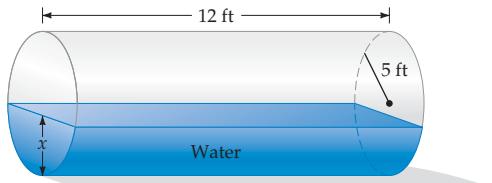
83. **SATELLITE COVERAGE** A communications satellite orbits Earth at an altitude of  $a$  miles. The beam coverage provided by the satellite is shown in the figure below.



- Find the distance  $s$  (length of arc  $MN$ ) as a function of the altitude  $a$  of the satellite. Assume the radius of Earth is 3960 miles.
- What altitude does the satellite need to attain to provide coverage over a distance of  $s = 5500$  miles? Round to the nearest 10 miles.

84. **VOLUME IN A WATER TANK** The volume  $V$  of water (measured in cubic feet) in a horizontal cylindrical tank of radius 5 feet and length 12 feet is given by

$$V(x) = 12 \left[ 25 \cos^{-1}\left(\frac{5-x}{5}\right) - (5-x)\sqrt{10x-x^2} \right]$$



where  $x$  is the depth of the water in feet.

- a. Graph  $V$  over its domain  $0 \leq x \leq 10$ .



- b.**  Write a sentence that explains why the graph of  $V$  increases more rapidly when  $x$  increases from 4.9 feet to 5 feet than it does when  $x$  increases from 0.1 foot to 0.2 foot.

**c.** If  $x = 4$  feet, find the volume (to the nearest 0.01 cubic foot) of the water in the tank.

**d.** Find the depth  $x$  (to the nearest 0.01 foot) if there are 288 cubic feet of water in the tank.

**85.** Graph  $f(x) = \cos^{-1} x$  and  $g(x) = \sin^{-1} \sqrt{1 - x^2}$  on the same coordinate axes. Does  $f(x) = g(x)$  on the interval  $[-1, 1]$ ?

In Exercises 93 to 96, verify the identity.

**93.**  $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

**94.**  $\sec(\sin^{-1} x) = \frac{\sqrt{1 - x^2}}{1 - x^2}$

**95.**  $\tan(\csc^{-1} x) = \frac{\sqrt{x^2 - 1}}{x^2 - 1}, x > 1$

**96.**  $\sin(\cot^{-1} x) = \frac{\sqrt{x^2 + 1}}{x^2 + 1}$

In Exercises 97 to 100, solve for  $y$  in terms of  $x$ .

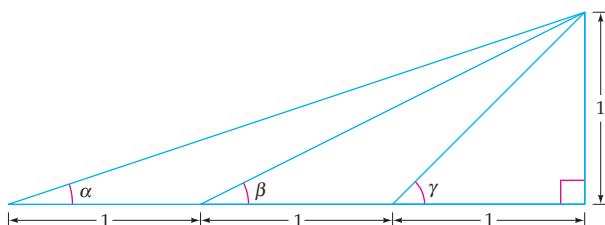
**97.**  $5x = \tan^{-1} 3y$

**98.**  $2x = \frac{1}{2} \sin^{-1} 2y$

**99.**  $x - \frac{\pi}{3} = \cos^{-1}(y - 3)$

**100.**  $x + \frac{\pi}{2} = \tan^{-1}(2y - 1)$

## 1 VISUAL INSIGHT



Explain how the figure above can be used to verify each identity.

**a.**  $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} = \frac{\pi}{4}$  [Hint: Start by using an identity to find the value of  $\tan(\alpha + \beta)$ .]

**b.**  $\alpha + \beta = \gamma$

## Section 3.6

- Solve Trigonometric Equations
- Model Sinusoidal Data

# Trigonometric Equations

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A18.

**PS1.** Use the quadratic formula to solve  $3x^2 - 5x - 4 = 0$ . [1.1]

**PS2.** Use a Pythagorean identity to write  $\sin^2 x$  as a function involving  $\cos^2 x$ . [3.1]

**PS3.** Evaluate  $\frac{\pi}{2} + 2k\pi$  for  $k = 1, 2$ , and  $3$ .

**PS4.** Factor by grouping:  $x^2 - \frac{\sqrt{3}}{2}x + x - \frac{\sqrt{3}}{2}$ .

**PS5.**  Graph the scatter plot for the following data. Use a viewing window with  $X_{\min}=0$ ,  $X_{\max}=40$ ,  $Y_{\min}=0$ , and  $Y_{\max}=100$ . [1.7]

|          |    |    |    |    |    |    |    |    |
|----------|----|----|----|----|----|----|----|----|
| <b>x</b> | 3  | 7  | 11 | 15 | 19 | 23 | 27 | 31 |
| <b>y</b> | 14 | 55 | 90 | 99 | 80 | 44 | 8  | 4  |

**PS6.** Solve  $2x^2 - 2x = 0$  by factoring. [1.1]

## Solve Trigonometric Equations

Consider the equation  $\sin x = \frac{1}{2}$ . The graph of  $y = \sin x$ , along with the line  $y = \frac{1}{2}$ , is shown in **Figure 3.24**. The  $x$  values of the intersections of the two graphs are the solutions of  $\sin x = \frac{1}{2}$ . The solutions in the interval  $0 \leq x < 2\pi$  are  $x = \frac{\pi}{6}$  and  $\frac{5\pi}{6}$ .

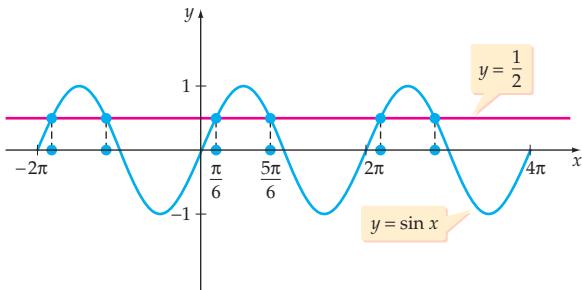


Figure 3.24

If we remove the restriction  $0 \leq x < 2\pi$ , there are many more solutions. Because the sine function is periodic with a period of  $2\pi$ , other solutions are obtained by adding  $2k\pi$ ,  $k$  an integer, to either of the previous solutions. Thus the solutions of  $\sin x = \frac{1}{2}$  are

$$x = \frac{\pi}{6} + 2k\pi, \quad k \text{ an integer}$$

$$x = \frac{5\pi}{6} + 2k\pi, \quad k \text{ an integer}$$

**QUESTION** How many solutions does the equation  $\cos x = \frac{\sqrt{3}}{2}$  have on the interval  $0 \leq x < 2\pi$ ?

Algebraic methods and trigonometric identities are used frequently to find the solutions of trigonometric equations. Algebraic methods that are often employed include solving by factoring, solving by using the quadratic formula, and squaring each side of the equation.

### EXAMPLE 1 » Solve a Trigonometric Equation by Factoring

Solve  $2 \sin^2 x \cos x - \cos x = 0$ , where  $0 \leq x < 2\pi$ .

#### ALGEBRAIC SOLUTION

$$2 \sin^2 x \cos x - \cos x = 0$$

$$\cos x(2 \sin^2 x - 1) = 0$$

$$\cos x = 0 \quad \text{or} \quad 2 \sin^2 x - 1 = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\sin^2 x = \frac{1}{2}$$

$$\sin x = \pm \frac{\sqrt{2}}{2}$$

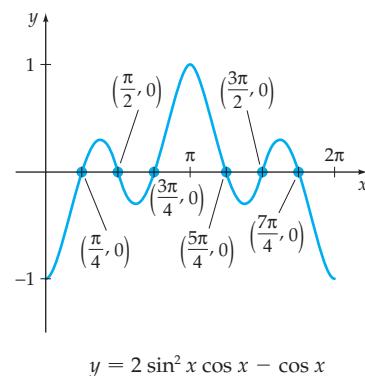
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

The solutions in the interval  $0 \leq x < 2\pi$  are

$$\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \text{ and } \frac{7\pi}{4}$$

#### VISUALIZE THE SOLUTION

The solutions are the  $x$ -coordinates of the  $x$ -intercepts of the graph of  $y = 2 \sin^2 x \cos x - \cos x$  on the interval  $[0, 2\pi]$ .



» Try Exercise 14, page 278

**ANSWER** Two

Squaring both sides of an equation may not produce an equivalent equation. Thus, when this method is used, the proposed solutions must be checked to eliminate any extraneous solutions.

### EXAMPLE 2 »» Solve a Trigonometric Equation by Squaring Each Side of the Equation

Solve  $\sin x + \cos x = 1$ , where  $0 \leq x < 2\pi$ .

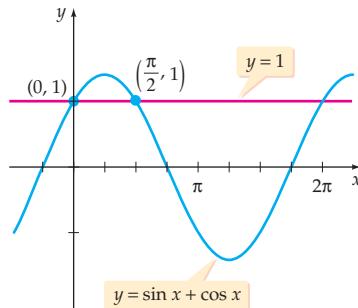
#### ALGEBRAIC SOLUTION

$$\begin{aligned} \sin x + \cos x &= 1 && \bullet \text{Solve for } \sin x. \\ \sin x &= 1 - \cos x \\ \sin^2 x &= (1 - \cos x)^2 && \bullet \text{Square each side.} \\ \sin^2 x &= 1 - 2 \cos x + \cos^2 x \\ 1 - \cos^2 x &= 1 - 2 \cos x + \cos^2 x && \bullet \sin^2 x = 1 - \cos^2 x \\ 2 \cos^2 x - 2 \cos x &= 0 \\ 2 \cos x(\cos x - 1) &= 0 && \bullet \text{Factor.} \\ 2 \cos x &= 0 \quad \text{or} \quad \cos x = 1 \\ x &= \frac{\pi}{2}, \frac{3\pi}{2} && \bullet \text{Solve each equation for } x \\ x &= 0 && \text{with } 0 \leq x < 2\pi. \end{aligned}$$

A check will show that  $0$  and  $\frac{3\pi}{2}$  are solutions but  $\frac{3\pi}{2}$  is not a solution.

#### VISUALIZE THE SOLUTION

The solutions are the  $x$ -coordinates of the points of intersection of  $y = \sin x + \cos x$  and  $y = 1$  on the interval  $[0, 2\pi]$ .



»» Try Exercise 52, page 278

### EXAMPLE 3 »» Solve a Trigonometric Equation by Using the Quadratic Formula

Solve  $3 \cos^2 x - 5 \cos x - 4 = 0$ , where  $0 \leq x < 2\pi$ .

#### ALGEBRAIC SOLUTION

The given equation is quadratic in form and cannot be factored easily. However, we can use the quadratic formula to solve for  $\cos x$ .

$$\begin{aligned} 3 \cos^2 x - 5 \cos x - 4 &= 0 && \bullet a = 3, b = -5, c = -4 \\ \cos x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-4)}}{(2)(3)} = \frac{5 \pm \sqrt{73}}{6} \end{aligned}$$

The equation  $\cos x = \frac{5 + \sqrt{73}}{6}$  does not have a solution because  $\frac{5 + \sqrt{73}}{6} > 2$ , and for any  $x$  the maximum value of  $\cos x$  is 1. Thus

#### VISUALIZE THE SOLUTION

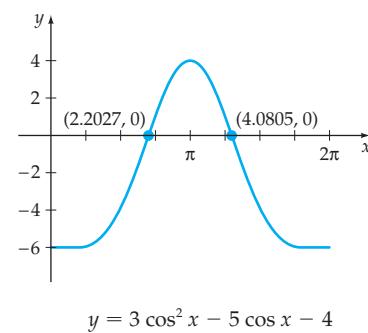
The solutions are the  $x$ -coordinates of the  $x$ -intercepts of  $y = 3 \cos^2 x - 5 \cos x - 4$  on the interval  $[0, 2\pi]$ . See the following figure.

$\cos x = \frac{5 - \sqrt{73}}{6}$ , and because  $\frac{5 - \sqrt{73}}{6}$  is a negative number (about  $-0.59$ ), the equation  $\cos x = \frac{5 - \sqrt{73}}{6}$  will have two solutions on the interval  $[0, 2\pi]$ . Thus

$$\begin{aligned}x &= \cos^{-1}\left(\frac{5 - \sqrt{73}}{6}\right) \approx 2.2027 \quad \text{or} \\x &= 2\pi - \cos^{-1}\left(\frac{5 - \sqrt{73}}{6}\right) \approx 4.0805\end{aligned}$$

To the nearest 0.0001, the solutions on  $[0, 2\pi]$  are 2.2027 and 4.0805.

» Try Exercise 56, page 278



When solving an equation that has multiple solutions, we must be sure we find all solutions of the equation for the given interval. For example, to find all solutions of  $\sin 2x = \frac{1}{2}$ , where  $0 \leq x < 2\pi$ , we first solve for  $2x$ .

$$\sin 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad 2x = \frac{5\pi}{6} + 2k\pi \quad \bullet k \text{ is an integer.}$$

Solving for  $x$ , we have  $x = \frac{\pi}{12} + k\pi$  or  $x = \frac{5\pi}{12} + k\pi$ . Substituting integers for  $k$ , we obtain

$$k = 0: \quad x = \frac{\pi}{12} \quad \text{or} \quad x = \frac{5\pi}{12}$$

$$k = 1: \quad x = \frac{13\pi}{12} \quad \text{or} \quad x = \frac{17\pi}{12}$$

$$k = 2: \quad x = \frac{25\pi}{12} \quad \text{or} \quad x = \frac{29\pi}{12}$$

Note that for  $k \geq 2$ ,  $x \geq 2\pi$  and the solutions to  $\sin 2x = \frac{1}{2}$  are not in the interval  $0 \leq x < 2\pi$ . Also if  $k < 0$ , then  $x < 0$  and no solutions in the interval  $0 \leq x < 2\pi$  are produced. Thus, for  $0 \leq x < 2\pi$ , the solutions are  $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \text{ and } \frac{17\pi}{12}$ . See

Figure 3.25

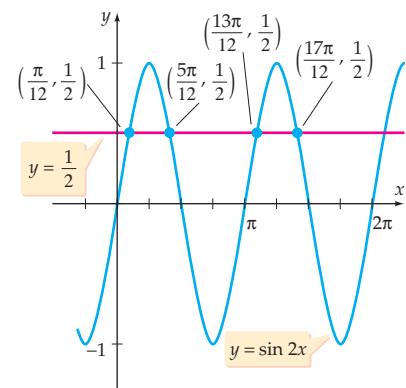


Figure 3.25

**EXAMPLE 4** » Solve a Trigonometric Equation

Solve:  $\sin 3x = 1$

**ALGEBRAIC SOLUTION**

The equation  $\sin 3x = 1$  implies

$$3x = \frac{\pi}{2} + 2k\pi, \quad k \text{ an integer}$$

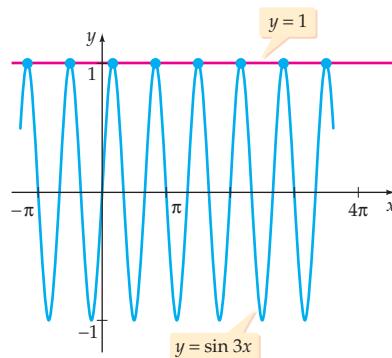
$$x = \frac{\pi}{6} + \frac{2k\pi}{3}, \quad k \text{ an integer}$$
• Divide each side by 3.

Because  $x$  is not restricted to a finite interval, the given equation has an infinite number of solutions. All of the solutions are represented by the equation

$$x = \frac{\pi}{6} + \frac{2k\pi}{3}, \quad \text{where } k \text{ is an integer}$$

**VISUALIZE THE SOLUTION**

The solutions are the  $x$ -coordinates of the points of intersection of  $y = \sin 3x$  and  $y = 1$ . The following figure shows eight of the points of intersection.



» Try Exercise 66, page 279

**EXAMPLE 5** » Solve a Trigonometric Equation

Solve  $\sin^2 2x - \frac{\sqrt{3}}{2} \sin 2x + \sin 2x - \frac{\sqrt{3}}{2} = 0$ , where  $0^\circ \leq x < 360^\circ$ .

**ALGEBRAIC SOLUTION**

Factor the left side of the equation by grouping, and then set each factor equal to zero.

$$\begin{aligned} \sin^2 2x - \frac{\sqrt{3}}{2} \sin 2x + \sin 2x - \frac{\sqrt{3}}{2} &= 0 \\ \sin 2x \left( \sin 2x - \frac{\sqrt{3}}{2} \right) + \left( \sin 2x - \frac{\sqrt{3}}{2} \right) &= 0 \\ (\sin 2x + 1) \left( \sin 2x - \frac{\sqrt{3}}{2} \right) &= 0 \\ \sin 2x + 1 &= 0 \quad \text{or} \quad \sin 2x - \frac{\sqrt{3}}{2} = 0 \end{aligned}$$

$$\sin 2x = -1 \qquad \qquad \qquad \sin 2x = \frac{\sqrt{3}}{2}$$

**VISUALIZE THE SOLUTION**

The solutions are the  $x$ -coordinates of the  $x$ -intercepts of

$$\begin{aligned} y &= \sin^2 2x - \frac{\sqrt{3}}{2} \sin 2x \\ &\quad + \sin 2x - \frac{\sqrt{3}}{2} \end{aligned}$$

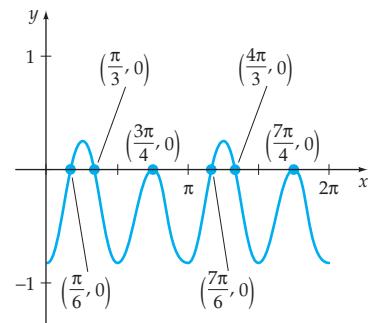
on the interval  $[0, 2\pi]$ . See the following figure.

The equation  $\sin 2x = -1$  implies that  $2x = 270^\circ + 360^\circ \cdot k$ ,  $k$  an integer. Thus  $x = 135^\circ + 180^\circ \cdot k$ . The solutions of this equation with  $0^\circ \leq x < 360^\circ$  are  $135^\circ$  and  $315^\circ$ . Similarly, the equation  $\sin 2x = \frac{\sqrt{3}}{2}$  implies

$$\begin{aligned} 2x &= 60^\circ + 360^\circ \cdot k \quad \text{or} \quad 2x = 120^\circ + 360^\circ \cdot k \\ x &= 30^\circ + 180^\circ \cdot k \quad \quad \quad x = 60^\circ + 180^\circ \cdot k \end{aligned}$$

The solutions with  $0^\circ \leq x < 360^\circ$  are  $30^\circ$ ,  $60^\circ$ ,  $210^\circ$ , and  $240^\circ$ . Combining the solutions from each equation, we have  $30^\circ$ ,  $60^\circ$ ,  $135^\circ$ ,  $210^\circ$ ,  $240^\circ$ , and  $315^\circ$  as our solutions.

» Try Exercise 84, page 279



In Example 6, algebraic methods do not provide the solutions, so we rely on a graph.

### EXAMPLE 6 » Approximate Solutions Graphically



Use a graphing utility to approximate the solutions of  $x + 3 \cos x = 0$ .

#### Solution

The solutions are the  $x$ -intercepts of  $y = x + 3 \cos x$ . See **Figure 3.26**. A close-up view of the graph of  $y = x + 3 \cos x$  shows that, to the nearest thousandth, the solutions are

$$x_1 = -1.170, \quad x_2 = 2.663, \quad \text{and} \quad x_3 = 2.938$$

» Try Exercise 86, page 279

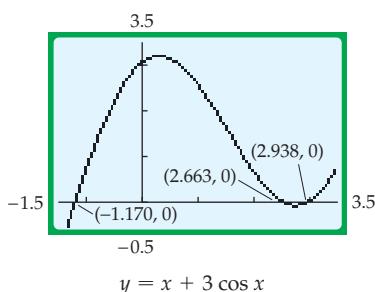


Figure 3.26

### EXAMPLE 7 » Solve a Projectile Application



A projectile is fired at an angle of inclination  $\theta$  from the horizon with an initial velocity  $v_0$ . Its range  $d$  (neglecting air resistance) is given by

$$d = \frac{(v_0)^2}{16} \sin \theta \cos \theta$$

where  $v_0$  is measured in feet per second and  $d$  is measured in feet. See **Figure 3.27**.

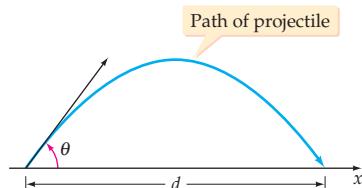


Figure 3.27

Continued ►

- If  $v_0 = 325$  feet per second, find the angles  $\theta$  (in degrees) for which the projectile will hit a target 2295 feet downrange.
- What is the maximum horizontal range for a projectile that has an initial velocity of 474 feet per second?
- Determine the angle of inclination that produces the maximum range.

### Solution

- We need to solve

$$2295 = \frac{325^2}{16} \sin \theta \cos \theta \quad (1)$$

for  $\theta$ , where  $0^\circ < \theta < 90^\circ$ .

**Method 1** The following solutions were obtained by using a graphing utility to graph  $d = 2295$  and  $d = \frac{325^2}{16} \sin \theta \cos \theta$ . See **Figure 3.28**. Thus there are two angles for which the projectile will hit the target. To the nearest thousandth of a degree, they are

$$\theta = 22.025^\circ \quad \text{and} \quad \theta = 67.975^\circ$$

It should be noted that the graph in **Figure 3.28** is *not* a graph of the path of the projectile. It is a graph of the distance  $d$  as a function of the angle  $\theta$ .

**Method 2** To solve algebraically, we proceed as follows. Multiply each side of Equation (1) by 16 and divide by  $325^2$  to produce

$$\sin \theta \cos \theta = \frac{(16)(2295)}{325^2}$$

The identity  $2 \sin \theta \cos \theta = \sin 2\theta$  gives us  $\sin \theta \cos \theta = \frac{\sin 2\theta}{2}$ .

$$\text{Hence} \quad \frac{\sin 2\theta}{2} = \frac{(16)(2295)}{325^2}$$

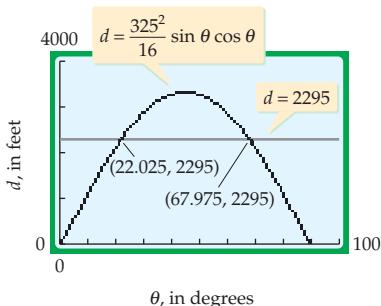
$$\sin 2\theta = 2 \cdot \frac{(16)(2295)}{325^2} \approx 0.69529$$

There are two angles in the interval  $[0^\circ, 180^\circ]$  whose sines are 0.69529.

One is  $\sin^{-1} 0.69529$ , and the other is the *reference angle* for  $\sin^{-1} 0.69529$ . Therefore,

$$\begin{aligned} 2\theta &\approx \sin^{-1} 0.69529 & \text{or} & \quad 2\theta \approx 180^\circ - \sin^{-1} 0.69529 \\ \theta &\approx \frac{1}{2} \sin^{-1} 0.69529 & \theta &\approx \frac{1}{2} (180^\circ - \sin^{-1} 0.69529) \\ \theta &\approx 22.025^\circ & \theta &\approx 67.975^\circ \end{aligned}$$

These are the same angles that we obtained using Method 1.



**Figure 3.28**

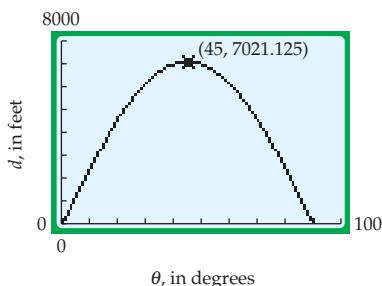


Figure 3.29

- b. Use a graphing utility to find that the graph of  $d = \frac{474^2}{16} \sin \theta \cos \theta$  has a maximum value of  $d = 7021.125$  feet. See **Figure 3.29**.

- c. In b., the maximum value is attained for  $\theta = 45^\circ$ .

» Try Exercise 92, page 279

### Model Sinusoidal Data

Data that can be closely modeled by a function of the form  $y = a \sin(bx + c) + d$  are called **sinusoidal data**. Many graphing utilities are designed to perform a **sine regression** to find the sine function that provides the best least-squares fit for sinusoidal data. For instance, the TI-83/TI-83 Plus/TI-84 Plus uses the command **SinReg** and the TI-86 uses **SinR** to perform a sine regression. The process generally works best for those sets of data for which we have a good estimate of the period.

In Example 8 we use a sine regression to model the percent of illumination of the moon as seen from Earth. See **Figure 3.30**.



Figure 3.30

### EXAMPLE 8 » Use a Sine Regression to Model Data



**Table 3.3** on the following page shows the percent of the moon illuminated, at midnight Central Standard Time, for selected days of January 2009. Find the regression function that models the percent of the moon illuminated as a function of the day number, with January 1, 2009 represented by  $x = 1$ . Use the function to estimate the percent of the moon illuminated at midnight Central Standard Time on January 21, 2009. Note: The illumination cycle of the moon has a period of about 29.53 days.

Table 3.3

| Midnight of:<br>Date in 2009<br>(CST) | Day<br>Number | % of Moon<br>Illuminated |
|---------------------------------------|---------------|--------------------------|
| Jan. 3                                | 3             | 37                       |
| Jan. 7                                | 7             | 79                       |
| Jan. 11                               | 11            | 100                      |
| Jan. 15                               | 15            | 78                       |
| Jan. 19                               | 19            | 39                       |
| Jan. 23                               | 23            | 8                        |
| Jan. 27                               | 27            | 1                        |
| Jan. 31                               | 31            | 22                       |

The percent of the moon illuminated is nearly the same regardless of one's position on Earth.

Source: The Astronomical Applications Department of the U.S. Naval Observatory.

### TO REVIEW

How to Construct a Scatter Plot  
See section 1.7.

### Solution

1. Construct a scatter plot of the data. Enter the data from **Table 3.3** into your graphing utility. See **Figure 3.31**. The sinusoidal nature of the scatter plot in **Figure 3.32** indicates that the data can be effectively modeled by a sine function.

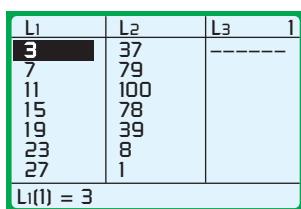


Figure 3.31

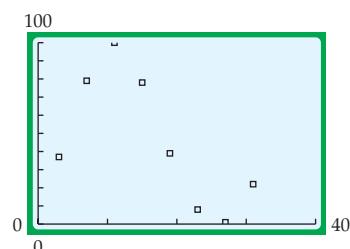


Figure 3.32

2. Find the regression equation. On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, the input shown in **Figure 3.33** produces the results in **Figure 3.34**.

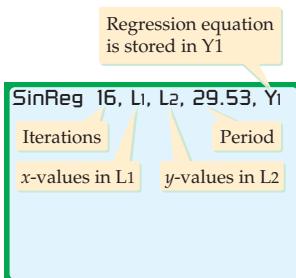


Figure 3.33

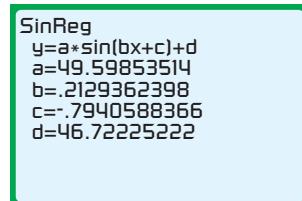


Figure 3.34

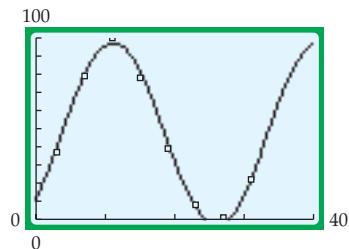


### Integrating Technology

A sine regression requires at least four data points. The iterations number 16 in **Figure 3.33** is the maximum number of times the SinReg command will iterate to find a regression equation. Any integer from 1 to 16 can be used; however, the integer 16 generally produces more accurate results than smaller integers.

The regression equation is  $y \approx 49.60 \sin(0.2129x - 0.7941) + 46.72$ .

3. **Examine the fit.** The SinReg command does not yield a correlation coefficient. However, a graph of the regression equation and the scatter plot of the data shows that the regression equation provides a good model. See **Figure 3.35**.

**Figure 3.35**

According to the following result, the percent of the moon illuminated at midnight (CST) on January 21, 2009 ( $x = 21$ ) will be about **21%**.

$$y \approx 49.60 \sin(0.2129(21) - 0.7941) + 46.72 \approx 21$$

» Try Exercise 94, page 280



### Topics for Discussion

1. A student finds that  $x = 0$  is a solution of  $\sin x = x$ . Because the function  $y = \sin x$  has a period of  $2\pi$ , the student reasons that  $\pm 2\pi, \pm 4\pi, \pm 6\pi, \dots$  are also solutions. Explain why the student is not correct.
2. How many solutions does the equation  $2 \sin\left(x - \frac{\pi}{2}\right) = 5$  have on the interval  $0 \leq x < 2\pi$ ? Explain.
3. How many solutions does the equation  $\sin \frac{1}{x} = 0$  have on the interval  $0 < x < \frac{\pi}{2}$ ? Explain.
4. On the interval  $0 \leq x < 2\pi$ , the equation  $\sin x = \frac{1}{2}$  has solutions of  $x = \frac{\pi}{6}$  and  $x = \frac{5\pi}{6}$ . How would you write the solutions of  $\sin x = \frac{1}{2}$  if the real number  $x$  were not restricted to the interval  $[0, 2\pi)$ ?


**Exercise Set 3.6**

In Exercises 1 to 22, solve each equation for solutions in the interval  $0 \leq x < 2\pi$ .

1.  $\sec x - \sqrt{2} = 0$

2.  $2 \sin x = \sqrt{3}$

3.  $\tan x - \sqrt{3} = 0$

4.  $\cos x - 1 = 0$

5.  $2 \sin x \cos x = \sqrt{2} \cos x$

6.  $2 \sin x \cos x = \sqrt{3} \sin x$

7.  $\sin^2 x - 1 = 0$

8.  $\cos^2 x - 1 = 0$

9.  $4 \sin x \cos x - 2\sqrt{3} \sin x - 2\sqrt{2} \cos x + \sqrt{6} = 0$

10.  $\sec^2 x + \sqrt{3} \sec x - \sqrt{2} \sec x - \sqrt{6} = 0$

11.  $\csc x - \sqrt{2} = 0$

12.  $3 \cot x + \sqrt{3} = 0$

13.  $2 \sin^2 x + 1 = 3 \sin x$

14.  $2 \cos^2 x + 1 = -3 \cos x$

15.  $4 \cos^2 x - 3 = 0$

16.  $2 \sin^2 x - 1 = 0$

17.  $2 \sin^3 x = \sin x$

18.  $4 \cos^3 x = 3 \cos x$

19.  $4 \sin^2 x + 2\sqrt{3} \sin x - \sqrt{3} = 2 \sin x$

20.  $\tan^2 x + \tan x - \sqrt{3} = \sqrt{3} \tan x$

21.  $\sin^4 x = \sin^2 x$

22.  $\cos^4 x = \cos^2 x$

In Exercises 23 to 60, solve each equation, where  $0^\circ \leq x < 360^\circ$ .

Round approximate solutions to the nearest tenth of a degree.

23.  $\cos x - 0.75 = 0$

24.  $\sin x + 0.432 = 0$

25.  $3 \sin x - 5 = 0$

26.  $4 \cos x - 1 = 0$

27.  $3 \sec x - 8 = 0$

28.  $4 \csc x + 9 = 0$

29.  $\cos x + 3 = 0$

30.  $\sin x - 4 = 0$

31.  $3 - 5 \sin x = 4 \sin x + 1$

32.  $4 \cos x - 5 = \cos x - 3$

33.  $\frac{1}{2} \sin x + \frac{2}{3} = \frac{3}{4} \sin x + \frac{3}{5}$

34.  $\frac{2}{5} \cos x - \frac{1}{2} = \frac{1}{3} - \frac{1}{2} \cos x$

35.  $3 \tan^2 x - 2 \tan x = 0$

36.  $4 \cot^2 x + 3 \cot x = 0$

37.  $3 \cos x + \sec x = 0$

38.  $5 \sin x - \csc x = 0$

39.  $\tan^2 x = 3 \sec^2 x - 2$

40.  $\csc^2 x - 1 = 3 \cot^2 x + 2$

41.  $2 \sin^2 x = 1 - \cos x$

42.  $\cos^2 x + 4 = 2 \sin x - 3$

43.  $3 \cos^2 x + 5 \cos x - 2 = 0$

44.  $2 \sin^2 x + 5 \sin x + 3 = 0$

45.  $2 \tan^2 x - \tan x - 10 = 0$

46.  $2 \cot^2 x - 7 \cot x + 3 = 0$

47.  $3 \sin x \cos x - \cos x = 0$

48.  $\tan x \sin x - \sin x = 0$

49.  $2 \sin x \cos x - \sin x - 2 \cos x + 1 = 0$

50.  $6 \cos x \sin x - 3 \cos x - 4 \sin x + 2 = 0$

51.  $2 \sin x - \cos x = 1$

52.  $\sin x + 2 \cos x = 1$

53.  $2 \sin x - 3 \cos x = 1$

54.  $\sqrt{3} \sin x + \cos x = 1$

55.  $3 \sin^2 x - \sin x - 1 = 0$

56.  $2 \cos^2 x - 5 \cos x - 5 = 0$

57.  $2 \cos x - 1 + 3 \sec x = 0$

58.  $3 \sin x - 5 + \csc x = 0$

59.  $\cos^2 x - 3 \sin x + 2 \sin^2 x = 0$

60.  $\sin^2 x = 2 \cos x + 3 \cos^2 x$

**In Exercises 61 to 70, find the exact solutions, in radians, of each trigonometric equation.**

61.  $\tan 2x - 1 = 0$

62.  $\sec 3x - \frac{2\sqrt{3}}{3} = 0$

63.  $\sin 5x = 1$

64.  $\cos 4x = -\frac{\sqrt{2}}{2}$

65.  $\sin 2x - \sin x = 0$

66.  $\cos 2x = -\frac{\sqrt{3}}{2}$

67.  $\sin\left(2x + \frac{\pi}{6}\right) = -\frac{1}{2}$

68.  $\cos\left(2x - \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

69.  $\sin^2 \frac{x}{2} + \cos x = 1$

70.  $\cos^2 \frac{x}{2} - \cos x = 1$

**In Exercises 71 to 84, solve each equation for exact solutions in the interval  $0 \leq x < 2\pi$ .**

71.  $\cos 2x = 1 - 3 \sin x$

72.  $\cos 2x = 2 \cos x - 1$

73.  $\sin 4x - \sin 2x = 0$

74.  $\sin 4x - \cos 2x = 0$

75.  $\tan \frac{x}{2} = \sin x$

76.  $\tan \frac{x}{2} = 1 - \cos x$

77.  $\sin 2x \cos x + \cos 2x \sin x = 0$

78.  $\cos 2x \cos x - \sin 2x \sin x = 0$

79.  $\sin x \cos 2x - \cos x \sin 2x = \frac{\sqrt{3}}{2}$

80.  $\cos 2x \cos x + \sin 2x \sin x = -1$

81.  $\sin 3x - \sin x = 0$

82.  $\cos 3x + \cos x = 0$

83.  $2 \sin x \cos x + 2 \sin x - \cos x - 1 = 0$

84.  $2 \sin x \cos x - 2\sqrt{2} \sin x - \sqrt{3} \cos x + \sqrt{6} = 0$

 **In Exercises 85 to 88, use a graphing utility to solve the equation. State each approximate solution accurate to the nearest ten-thousandth.**

85.  $\cos x = x$ , where  $0 \leq x < 2\pi$

86.  $2 \sin x = x$ , where  $0 \leq x < 2\pi$

87.  $\sin 2x = \frac{1}{x}$ , where  $-4 \leq x \leq 4$

88.  $\cos x = \frac{1}{x}$ , where  $0 \leq x \leq 5$

89.  Use a graphing utility to solve  $\cos x = x^3 - x$  by graphing each side and finding the  $x$  value of each point of intersection. Round to the nearest hundredth.

90.  Approximate the largest value of  $k$  for which the equation  $\sin x \cos x = k$  has a solution.

**PROJECTILES** Exercises 91 and 92 make use of the following. A projectile is fired at an angle of inclination  $\theta$  from the horizon with an initial velocity  $v_0$ . Its range  $d$  (neglecting air resistance) is given by

$$d = \frac{(v_0)^2 \cdot 16}{\sin \theta \cos \theta}$$

where  $v_0$  is measured in feet per second and  $d$  is measured in feet.

91.  If  $v_0 = 288$  feet per second, use a graphing utility to find the angles  $\theta$  (to the nearest hundredth of a degree) for which the projectile will hit a target 1295 feet downrange.

92.  Find the maximum horizontal range, to the nearest tenth of a foot, for a projectile that has an initial velocity of 375 feet per second. What value of  $\theta$  produces this maximum horizontal range?

93.   **SUNRISE TIME** The table below shows the sunrise time for Atlanta, Georgia, for selected days in 2008.

| Date    | Day Number | Sunrise Time |
|---------|------------|--------------|
| Jan. 1  | 1          | 7:42         |
| Feb. 1  | 32         | 7:35         |
| Mar. 1  | 61         | 7:06         |
| April 1 | 92         | 6:25         |
| May 1   | 122        | 5:48         |
| June 1  | 153        | 5:28         |
| July 1  | 183        | 5:31         |
| Aug. 1  | 214        | 5:50         |
| Sept. 1 | 245        | 6:12         |
| Oct. 1  | 275        | 6:32         |
| Nov. 1  | 306        | 6:57         |
| Dec. 1  | 336        | 7:25         |

Source: The U.S. Naval Observatory. Note: The times do not reflect daylight savings time.

- a. Find the sine regression function that models the sunrise time, in hours, as a function of the day number. Let  $x = 1$  represent January 1, 2008. Assume that the sunrise times have a period of 365.25 days.
- b. Use the regression function to estimate the sunrise time (to the nearest minute) for March 11, 2008 ( $x = 71$ ).

94.



**SUNSET TIME** The table below shows the sunset time for Omaha, Nebraska, for selected days in 2009.

| Date    | Day Number | Sunset Time |
|---------|------------|-------------|
| Jan. 1  | 1          | 17:06       |
| Feb. 1  | 32         | 17:41       |
| Mar. 1  | 60         | 18:15       |
| April 1 | 91         | 18:49       |
| May 1   | 121        | 19:22       |
| June 1  | 152        | 19:51       |
| July 1  | 182        | 20:01       |
| Aug. 1  | 213        | 19:40       |
| Sept. 1 | 244        | 18:56       |
| Oct. 1  | 274        | 18:05       |
| Nov. 1  | 305        | 17:19       |
| Dec. 1  | 335        | 16:56       |

Source: The U.S. Naval Observatory. Note: The times do not reflect daylight savings time.

- a. Find the sine regression function that models the sunset time, in hours, as a function of the day number. Let  $x = 1$  represent January 1, 2009. Assume that the sunset times have a period of 365.25 days.
- b. Use the regression function to estimate the sunset time (to the nearest minute) for March 14, 2009 ( $x = 73$ ).

95.



**PERCENT OF THE MOON ILLUMINATED** The table below shows the percent of the moon illuminated at midnight, Central Standard Time, for selected days in October and November of 2008.

| Midnight of:<br>Date in 2008 | Day<br>Number | % of Moon<br>Illuminated |
|------------------------------|---------------|--------------------------|
| Oct. 1                       | 1             | 4                        |
| Oct. 5                       | 5             | 31                       |
| Oct. 9                       | 9             | 68                       |
| Oct. 13                      | 13            | 97                       |
| Oct. 17                      | 17            | 92                       |
| Oct. 21                      | 21            | 53                       |
| Oct. 25                      | 25            | 14                       |
| Oct. 29                      | 29            | 0                        |
| Nov. 2                       | 33            | 16                       |
| Nov. 6                       | 37            | 51                       |
| Nov. 10                      | 41            | 88                       |
| Nov. 14                      | 45            | 98                       |
| Nov. 18                      | 49            | 68                       |
| Nov. 22                      | 53            | 26                       |
| Nov. 26                      | 57            | 2                        |
| Nov. 30                      | 61            | 6                        |

Source: The U.S. Naval Observatory.

- a. Find the sine regression function that models the percent of the moon illuminated as a function of the day number. Let  $x = 1$  represent October 1, 2008. Use 29.53 days for the period of the data.
- b. Use the regression function to estimate the percent of the moon illuminated (to the nearest 1 percent) at midnight Central Standard Time on October 31, 2008.



- 96.** **HOURS OF DAYLIGHT** The table below shows the hours of daylight for Dallas, Texas, for selected days in 2009.

| Date    | Day Number | Hours of Daylight (hours:minutes) |
|---------|------------|-----------------------------------|
| Jan. 1  | 1          | 10:03                             |
| Feb. 1  | 32         | 10:38                             |
| Mar. 1  | 60         | 11:30                             |
| April 1 | 91         | 12:33                             |
| May 1   | 121        | 13:29                             |
| June 1  | 152        | 14:11                             |
| July 1  | 182        | 14:16                             |
| Aug. 1  | 213        | 13:44                             |
| Sept. 1 | 244        | 12:50                             |
| Oct. 1  | 274        | 11:50                             |
| Nov. 1  | 305        | 10:51                             |
| Dec. 1  | 335        | 10:09                             |

Source: Data extracted from sunrise-sunset times given on the World Wide Web by the U.S. Naval Observatory.

- a. Find the sine regression function that models the hours of daylight as a function of the day number. Let  $x = 1$  represent January 1, 2009. Use 365.25 for the period of the data.
- b. Use the regression function to estimate the hours of daylight (stated in hours and minutes, with the minutes rounded to the nearest minute) for Dallas on May 12, 2009.



- 97. ALTITUDE OF THE SUN** The table at the top of the next column shows the altitude of the sun for Detroit, Michigan, at selected times during October 19, 2007.

- a. Find the sine regression function that models the altitude, in degrees, of the sun as a function of the time of day. Use 24.03 hours (the time from sunrise October 19 to sunrise October 20) for the period.
- b. Use the regression function to estimate the altitude of the sun (to the nearest 0.1 degree) on October 19, 2007, at 9:25.

| Time of day | Altitude (degrees) |
|-------------|--------------------|
| 6:00        | -9.9               |
| 7:00        | 1.4                |
| 8:00        | 11.5               |
| 9:00        | 21.0               |
| 10:00       | 29.0               |
| 11:00       | 34.7               |
| 12:00       | 37.5               |
| 13:00       | 36.7               |
| 14:00       | 32.6               |
| 15:00       | 25.7               |
| 16:00       | 17.0               |
| 17:00       | 7.2                |
| 18:00       | -3.6               |

Source: The U.S. Naval Observatory.



- 98. MODEL THE DAYLIGHT HOURS** For a particular day of the year  $t$ , the number of daylight hours in New Orleans can be approximated by

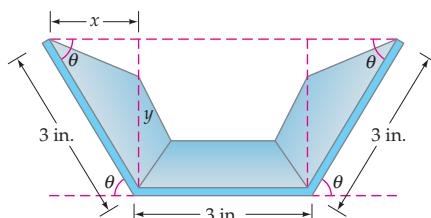
$$d(t) = 1.792 \sin\left(\frac{2\pi(t - 80)}{365}\right) + 12.145$$

where  $t$  is an integer and  $t = 1$  corresponds to January 1. According to  $d$ , how many days per year will New Orleans have at least 10.75 hours of daylight?

- 99. CROSS-SECTIONAL AREA** A rain gutter is constructed from a long sheet of aluminum that measures 9 inches in width. The aluminum is to be folded as shown by the cross section in the following diagram.

- a. Verify that the area of the cross section is given by  $A = 9 \sin \theta (\cos \theta + 1)$ , where  $0^\circ < \theta \leq 90^\circ$ .
- b. **Calculator** What values of  $\theta$ , to the nearest degree, produce a cross-sectional area of 10.5 square inches?

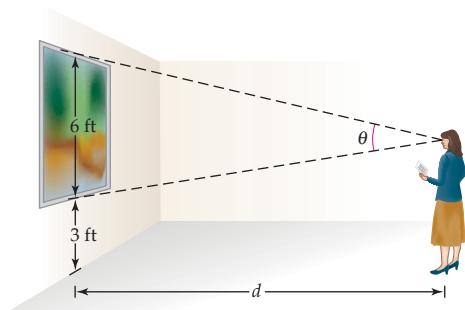
- c. **Calculator** Determine the value of  $\theta$  that produces the cross section with the maximum area.



- 100. OBSERVATION ANGLE** A person with an eye level of 5 feet 6 inches is standing in front of a painting, as shown in the following diagram. The bottom of the painting is 3 feet above floor level, and the painting is 6 feet in height. The angle  $\theta$  shown in the figure is called the *observation angle* for the painting. The person is  $d$  feet from the painting.

a. Verify that  $\theta = \tan^{-1}\left(\frac{6d}{d^2 - 8.75}\right)$ .

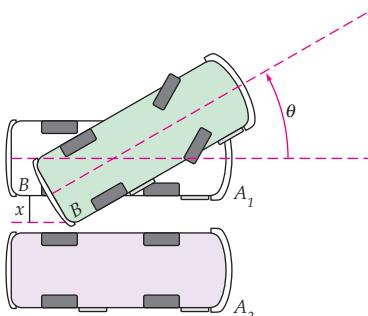
- b.**  Find the distance  $d$ , to the nearest tenth of a foot, for which  $\theta = \frac{\pi}{6}$ .



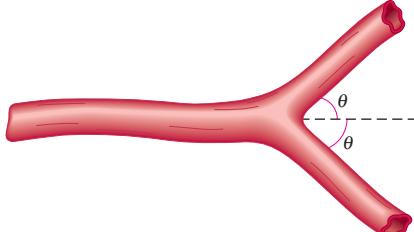
- 101. MODEL THE MOVEMENT OF A BUS** As bus  $A_1$  makes a left turn, the back  $B$  of the bus moves to the right. If bus  $A_2$  were waiting at a stoplight while  $A_1$  turned left, as shown in the following figure, there is a chance the two buses would scrape against one another. For a bus 28 feet long and 8 feet wide, the movement of the back of the bus to the right can be approximated by

$$x = \sqrt{(4 + 18 \cot \theta)^2 + 100} - (4 + 18 \cot \theta)$$

where  $\theta$  is the angle the bus driver has turned the front of the bus. Find the value of  $x$  for  $\theta = 20^\circ$  and  $\theta = 30^\circ$ . Round to the nearest hundredth of a foot.



- 102. OPTIMAL BRANCHING OF BLOOD VESSELS** It is hypothesized that the system of blood vessels in primates has evolved so that it has an optimal structure. In the case of a blood vessel splitting into two vessels, as shown in the accompanying figure, we assume that both new branches carry equal amounts of blood. A model of the angle  $\theta$  is given by the equation  $\cos \theta = 2^{(x-4)/(x+4)}$ . The value of  $x$  is such that  $1 \leq x \leq 2$  and depends on assumptions about the thickness of the blood vessels. Assuming this is an accurate model, find the range of values of the angle  $\theta$ .



**In Exercises 103 to 108, solve each equation for exact solutions in the interval  $0 \leq x \leq 2\pi$ .**

**103.**  $\sqrt{3} \sin x + \cos x = \sqrt{3}$

**104.**  $\sin x - \cos x = 1$

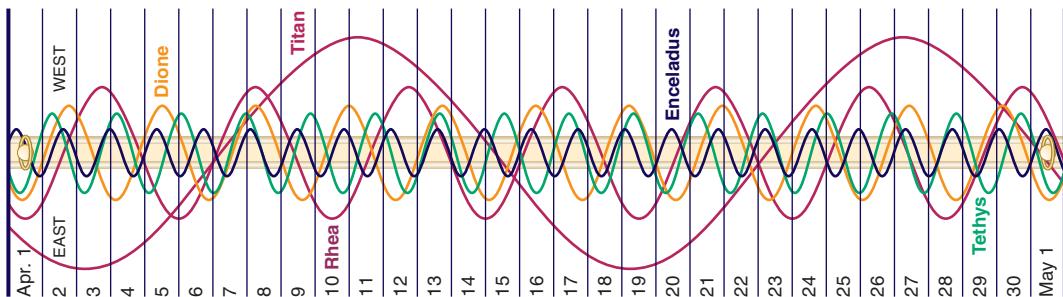
**105.**  $-\sin x + \sqrt{3} \cos x = \sqrt{3}$

**106.**  $-\sqrt{3} \sin x - \cos x = 1$

**107.**  $\cos 5x - \cos 3x = 0$

**108.**  $\cos 5x - \cos x - \sin 3x = 0$

- 1. THE MOONS OF SATURN** The accompanying figure shows the east-west displacement of five moons of Saturn. The figure shows that the period of the sine curve that models the displacement of the moon Titan is about 15.95 Earth days. The period of the sine curve that models the displacement of the moon Rhea is about 4.52 Earth days.



*Source:* "Saturn's Satellites" from *Sky & Telescope*, April 2006, p. 56. Copyright 2006 by Sky Publishing Company. Reprinted with permission.

- a. Find a sine regression function that models the displacement of the moon Titan. Use  $x = 0$  to represent the beginning of April 1, 2006, and think of "east" as a negative displacement and "west" as a positive displacement. Use 1 for the amplitude.
  - b. Work a. for the moon Rhea. Use 0.4 for the amplitude.
  - c. An astronomer viewed the moons Titan and Rhea at 11:59 P.M. on May 10, 2006 ( $x = 40$ ). Were these moons on opposite sides of Saturn or on the same (east/west) side at that time?



# Exploring Concepts with Technology

## Approximate an Inverse Trigonometric Function with Polynomials

The function  $y = \sin^{-1} x$  can be approximated by polynomials. For example, consider the following:

$$f_1(x) = x + \frac{x^3}{2 \cdot 3} \quad \text{where } -1 \leq x \leq 1$$

$$f_2(x) = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} \quad \text{where } -1 \leq x \leq 1$$

$$f_3(x) = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} \quad \text{where } -1 \leq x \leq 1$$

$$f_4(x) = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7x^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} \quad \text{where } -1 \leq x \leq 1$$

.

.

.

$$f_n(x) = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)}$$

where  $-1 \leq x \leq 1$ ,  $n! = 1 \cdot 2 \cdot 3 \cdots (n-1)n$

and  $(2n)! = 1 \cdot 2 \cdot 3 \cdots (2n-1)(2n)$



Use a graphing utility for the following exercises.

- Graph  $y = f_1(x)$ ,  $y = f_2(x)$ ,  $y = f_3(x)$ , and  $y = f_4(x)$  in the viewing window  $X_{\min} = -1$ ,  $X_{\max} = 1$ ,  $Y_{\min} = -1.5708$ ,  $Y_{\max} = 1.5708$ .
- Determine the values of  $x$  for which  $f_3(x)$  and  $\sin^{-1} x$  differ by less than 0.001. That is, determine the values of  $x$  for which  $|f_3(x) - \sin^{-1} x| < 0.001$ .
- Determine the values of  $x$  for which  $|f_4(x) - \sin^{-1} x| < 0.001$ .
- Write all seven terms of  $f_6(x)$ . Graph  $y = f_6(x)$  and  $y = \sin^{-1} x$  on the viewing window  $X_{\min} = -1$ ,  $X_{\max} = 1$ ,  $Y_{\min} = -\frac{\pi}{2}$ ,  $Y_{\max} = \frac{\pi}{2}$ .
- Write all seven terms of  $f_6(1)$ . What do you notice about the size of a term compared to that of the preceding term?
- What is the largest-degree term in  $f_{10}(x)$ ?

## Chapter 3 Summary

### 3.1 Verification of Trigonometric Identities

- Trigonometric identities are verified by using algebraic methods and previously proved identities.
- The Fundamental Trigonometric Identities are given in Table 3.1, page 217.
- Guidelines for verifying trigonometric identities are given on pages 217 and 218.

### 3.2 Sum, Difference, and Cofunction Identities

- Sum and difference identities for the cosine function

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

- Sum and difference identities for the sine function

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

- Sum and difference identities for the tangent function

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

- Cofunction identities with  $\theta$  in degrees

$$\sin(90^\circ - \theta) = \cos \theta \quad \cos(90^\circ - \theta) = \sin \theta$$

$$\tan(90^\circ - \theta) = \cot \theta \quad \cot(90^\circ - \theta) = \tan \theta$$

$$\sec(90^\circ - \theta) = \csc \theta \quad \csc(90^\circ - \theta) = \sec \theta$$

If  $\theta$  is in radian measure, replace  $90^\circ$  with  $\frac{\pi}{2}$ .

### 3.3 Double- and Half-Angle Identities

- Double-angle identities

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$= 1 - 2 \sin^2 \alpha$$

$$= 2 \cos^2 \alpha - 1$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

- Power-reducing identities

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\tan^2 \alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$$

- Half-angle identities

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

The choice of the plus or minus sign depends on the quadrant in which  $\frac{\alpha}{2}$  lies.

### 3.4 Identities Involving the Sum of Trigonometric Functions

- Product-to-sum identities

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

- Sum-to-product identities

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

- For sums of the form  $a \sin x + b \cos x$ ,

$$a \sin x + b \cos x = k \sin(x + \alpha)$$

where  $k = \sqrt{a^2 + b^2}$ ,  $\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$ , and

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}.$$

### 3.5 Inverse Trigonometric Functions

- The inverse of  $y = \sin x$  is  $y = \sin^{-1} x$ , with  $-1 \leq x \leq 1$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .
- The inverse of  $y = \cos x$  is  $y = \cos^{-1} x$ , with  $-1 \leq x \leq 1$  and  $0 \leq y \leq \pi$ .
- The inverse of  $y = \tan x$  is  $y = \tan^{-1} x$ , with  $-\infty < x < \infty$  and  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ .
- The inverse of  $y = \cot x$  is  $y = \cot^{-1} x$ , with  $-\infty < x < \infty$  and  $0 < y < \pi$ .
- The inverse of  $y = \csc x$  is  $y = \csc^{-1} x$ , with  $x \leq -1$  or  $x \geq 1$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ ,  $y \neq 0$ .
- The inverse of  $y = \sec x$  is  $y = \sec^{-1} x$ , with  $x \leq -1$  or  $x \geq 1$  and  $0 \leq y \leq \pi$ ,  $y \neq \frac{\pi}{2}$ .

### 3.6 Trigonometric Equations

- Algebraic methods and identities are used to solve trigonometric equations. Because the trigonometric functions are periodic, there may be an infinite number of solutions. If solutions cannot be found by algebraic methods, then we often use a graphing utility to find approximate solutions.



## Chapter 3 Assessing Concepts

1. True or False:  $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$  is an identity.
2. True or False: For all real numbers  $x$ ,  $\cos^{-1}(\cos x) = x$ .
3. True or False: For all real numbers  $x$ ,  $\cos(\cos^{-1} x) = x$ .
4. True or False: The graph of  $y = \sin^{-1} x$  is symmetric with respect to the origin.
5. How many solutions does the equation  $\sin 2x = 0.3$  have on the interval  $0 \leq x < 2\pi$ ?
6. What is the domain of  $f(x) = \cos^{-1} 2x$ ?
7. What is the range of  $y = \cos^{-1} x$ ?
8. Determine the exact value of  $\sin^{-1}\left(\sin \frac{7\pi}{3}\right)$ .
9. What is the amplitude of  $f(x) = \sin x + \cos x$ ?
10. Determine the exact value of  $\tan 75^\circ$ .



## Chapter 3 Review Exercises

**In Exercises 1 to 10, find the exact value.**

1.  $\cos(45^\circ + 30^\circ)$
2.  $\tan(210^\circ - 45^\circ)$
3.  $\sin\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$
4.  $\sec\left(\frac{4\pi}{3} - \frac{\pi}{4}\right)$
5.  $\sin(60^\circ - 135^\circ)$
6.  $\cos\left(\frac{5\pi}{3} - \frac{7\pi}{4}\right)$
7.  $\sin\left(22\frac{1}{2}^\circ\right)$
8.  $\cos 105^\circ$
9.  $\tan\left(67\frac{1}{2}^\circ\right)$
10.  $\sin 112.5^\circ$

**In Exercises 11 to 14, find the exact values of the given functions.**

11. Given  $\sin \alpha = \frac{1}{2}$  with  $0^\circ < \alpha < 90^\circ$ , and  $\cos \beta = \frac{1}{2}$  with  $270^\circ < \beta < 360^\circ$ , find
  - a.  $\cos(\alpha - \beta)$
  - b.  $\tan 2\alpha$
  - c.  $\sin\left(\frac{\beta}{2}\right)$
12. Given  $\sin \alpha = \frac{\sqrt{3}}{2}$  with  $90^\circ < \alpha < 180^\circ$ , and  $\cos \beta = -\frac{1}{2}$  with  $180^\circ < \beta < 270^\circ$ , find
  - a.  $\sin(\alpha + \beta)$
  - b.  $\sec 2\beta$
  - c.  $\cos\left(\frac{\alpha}{2}\right)$

13. Given  $\sin \alpha = -\frac{1}{2}$  with  $270^\circ < \alpha < 360^\circ$ , and

$\cos \beta = -\frac{\sqrt{3}}{2}$  with  $180^\circ < \beta < 270^\circ$ , find

- a.  $\sin(\alpha - \beta)$
- b.  $\tan 2\alpha$
- c.  $\cos\left(\frac{\beta}{2}\right)$

14. Given  $\sin \alpha = \frac{\sqrt{2}}{2}$  with  $0^\circ < \alpha < 90^\circ$ , and  $\cos \beta = \frac{\sqrt{3}}{2}$  with  $270^\circ < \beta < 360^\circ$ , find

- a.  $\cos(\alpha - \beta)$
- b.  $\tan 2\beta$
- c.  $\sin 2\alpha$

**In Exercises 15 to 20, write the given expression as a single trigonometric function.**

15.  $2 \sin 3x \cos 3x$
16.  $\frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$
17.  $\sin 4x \cos x - \cos 4x \sin x$
18.  $\cos^2 2\theta - \sin^2 2\theta$
19.  $\frac{\sin 2\theta}{\cos 2\theta}$
20.  $\frac{1 - \cos 2\theta}{\sin 2\theta}$

**In Exercises 21 to 24, write each expression as the product of two trigonometric functions.**

21.  $\cos 2\theta - \cos 4\theta$
22.  $\sin 3\theta - \sin 5\theta$
23.  $\sin 6\theta + \sin 2\theta$
24.  $\sin 5\theta - \sin \theta$

**In Exercises 25 to 42, verify the identity.**

25.  $\frac{1}{\sin x - 1} + \frac{1}{\sin x + 1} = -2 \tan x \sec x$

26.  $\frac{\sin x}{1 - \cos x} = \csc x + \cot x, \quad 0 < x < \frac{\pi}{2}$

27.  $\frac{1 + \sin x}{\cos^2 x} = \tan^2 x + 1 + \tan x \sec x$

28.  $\cos^2 x - \sin^2 x - \sin 2x = \frac{\cos^2 2x - \sin^2 2x}{\cos 2x + \sin 2x}$

29.  $\frac{1}{\cos x} - \cos x = \tan x \sin x$

30.  $\sin(270^\circ - \theta) - \cos(270^\circ - \theta) = \sin \theta - \cos \theta$

31.  $\sin\left(\frac{\pi}{4} - \alpha\right) = \frac{\sqrt{2}}{2}(\cos \alpha - \sin \alpha)$

32.  $\sin(180^\circ - \alpha + \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

33.  $\frac{\sin 4x - \sin 2x}{\cos 4x - \cos 2x} = -\cot 3x$

34.  $2 \sin x \sin 3x = (1 - \cos 2x)(1 + 2 \cos 2x)$

35.  $\sin x - \cos 2x = (2 \sin x - 1)(\sin x + 1)$

36.  $\cos 4x = 1 - 8 \sin^2 x + 8 \sin^4 x$

37.  $\tan 4x = \frac{4 \tan x - 4 \tan^3 x}{1 - 6 \tan^2 x + \tan^4 x}$

38.  $\frac{\sin 2x - \sin x}{\cos 2x + \cos x} = \frac{1 - \cos x}{\sin x}$

39.  $2 \cos 4x \sin 2x = 2 \sin 3x \cos 3x - 2 \sin x \cos x$

40.  $2 \sin x \sin 2x = 4 \cos x \sin^2 x$

41.  $\cos(x + y) \cos(x - y) = \cos^2 x + \cos^2 y - 1$

42.  $\cos(x + y) \sin(x - y) = \sin x \cos x - \sin y \cos y$

**In Exercises 43 to 46, evaluate each expression.**

43.  $\sec\left(\sin^{-1}\frac{12}{13}\right)$

44.  $\cos\left(\sin^{-1}\frac{3}{5}\right)$

45.  $\cos\left[\sin^{-1}\left(-\frac{3}{5}\right) + \cos^{-1}\frac{5}{13}\right]$

46.  $\cos\left(2 \sin^{-1}\frac{3}{5}\right)$

**In Exercises 47 and 48, solve each equation.**

47.  $2 \sin^{-1}(x - 1) = \frac{\pi}{3}$

48.  $\sin^{-1} x + \cos^{-1} \frac{4}{5} = \frac{\pi}{2}$

**In Exercises 49 and 50, find all solutions of each equation with  $0^\circ \leq x < 360^\circ$ .**

49.  $4 \sin^2 x + 2\sqrt{3} \sin x - 2 \sin x - \sqrt{3} = 0$

50.  $2 \sin x \cos x - \sqrt{2} \cos x - 2 \sin x + \sqrt{2} = 0$

**In Exercises 51 and 52, solve the trigonometric equation where  $x$  is in radians. Round approximate solutions to four decimal places.**

51.  $3 \cos^2 x + \sin x = 1$

52.  $\tan^2 x - 2 \tan x - 3 = 0$

**In Exercises 53 and 54, solve each equation on  $0 \leq x < 2\pi$ .**

53.  $\sin 3x \cos x - \cos 3x \sin x = \frac{1}{2}$

54.  $\cos\left(2x - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

**In Exercises 55 to 58, write the equation in the form  $y = k \sin(x + \alpha)$ , where the measure of  $\alpha$  is in radians. Graph one period of each function.**

55.  $f(x) = \sqrt{3} \sin x + \cos x$

56.  $f(x) = -2 \sin x - 2 \cos x$

57.  $f(x) = -\sin x - \sqrt{3} \cos x$

58.  $f(x) = \frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x$

**In Exercises 59 to 62, graph each function.**

59.  $f(x) = 2 \cos^{-1} x$

60.  $f(x) = \sin^{-1}(x - 1)$

61.  $f(x) = \sin^{-1} \frac{x}{2}$

62.  $f(x) = \sec^{-1} 2x$

63.



**SUNRISE TIME** The table below shows the sunrise time for Flagstaff, Arizona, for selected days in 2009.

| Date    | Day Number | Sunrise Time (hours:minutes) |
|---------|------------|------------------------------|
| Jan. 1  | 1          | 7:35                         |
| Feb. 1  | 32         | 7:25                         |
| Mar. 1  | 60         | 6:56                         |
| April 1 | 91         | 6:13                         |
| May 1   | 121        | 5:35                         |
| June 1  | 152        | 5:13                         |
| July 1  | 182        | 5:16                         |
| Aug. 1  | 213        | 5:36                         |
| Sept. 1 | 244        | 5:59                         |
| Oct. 1  | 274        | 6:22                         |
| Nov. 1  | 305        | 6:48                         |
| Dec. 1  | 335        | 7:17                         |

Source: The U.S. Naval Observatory. Note: The times are Mountain Standard Times.

- a. Find the sine regression function that models the sunrise time, in hours, as a function of the day number. Let  $x = 1$  represent January 1, 2009. Assume that the sunrise times have a period of 365.25 days.

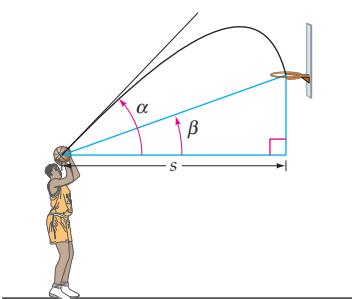
- b. Use the regression function to estimate the sunrise time (to the nearest minute) for April 14, 2009 ( $x = 104$ ).

### »»» Quantitative Reasoning: Basketball and Trigonometric Equations »»»

- QR1.** When a basketball player shoots at the basket, there are several factors that determine the horizontal distance the basketball travels. One equation for the distance  $s$ , in feet, is given by

$$s = \frac{v^2 \cos \alpha \sec \beta \sin(\alpha - \beta)}{16} \quad (1)$$

where the angles  $\alpha$  and  $\beta$  are as shown in the diagram at the left and  $v$  is the velocity in feet per second at which the ball leaves the player's hand.



- a. Graph Equation (1) for  $\beta = \frac{\pi}{18}$ ,  $v = 28$  feet per second, and  $\beta < \alpha < 1.35$ .

Use the graph to find the value of  $\alpha$  that maximizes  $s$  for the given values.

- b. Show that the value of  $\alpha$  in a. is approximately  $\frac{\pi}{4} + \frac{\beta}{2}$ .

- QR2.** Verify the identity  $\cos \alpha \sec \beta \sin(\alpha - \beta) = \cos^2 \alpha (\tan \alpha - \tan \beta)$  and then use the identity to show that Equation (1) can be written as

$$v = 4 \sec \alpha \sqrt{\frac{s}{\tan \alpha - \tan \beta}} \quad (2)$$

*Note:* Equation (2) gives the velocity at which a ball must be launched for various launch angles to reach a basket  $s$  feet away.

- a. Graph Equation (2) for  $\beta = \frac{\pi}{18}$ ,  $s = 15$  feet (the distance of a free throw), and  $\beta < \alpha < 1.35$ . Use the graph to find the value of  $\alpha$  that maximizes  $v$  for the given values.
- b. Show that the value of  $\alpha$  in part a. is approximately  $\frac{\pi}{4} + \frac{\beta}{2}$ . *Note:* The significance of this result is that the ball must be launched at an angle of  $\alpha = \frac{\pi}{4} + \frac{\beta}{2}$  to travel the required distance with the least effort.

## Chapter 3 Test

1. Verify the identity  $1 + \sin^2 x \sec^2 x = \sec^2 x$ .

2. Verify the identity

$$\frac{1}{\sec x - \tan x} - \frac{1}{\sec x + \tan x} = 2 \tan x$$

3. Verify the identity  $\cos^3 x + \cos x \sin^2 x = \cos x$ .

4. Verify the identity  $\csc x - \cot x = \frac{1 - \cos x}{\sin x}$ .

5. Find the exact value of  $\sin 195^\circ$ .

6. Given  $\sin \alpha = -\frac{3}{5}$ ,  $\alpha$  in Quadrant III, and  $\cos \beta = -\frac{\sqrt{2}}{2}$ ,  $\beta$  in Quadrant II, find  $\sin(\alpha + \beta)$ .

7. Verify the identity  $\sin\left(\theta - \frac{3\pi}{2}\right) = \cos \theta$ .

8. Write  $\cos 6x \sin 3x + \sin 6x \cos 3x$  in terms of a single trigonometric function.

9. Find the exact value of  $\cos 2\theta$  given that  $\sin \theta = \frac{4}{5}$  and  $\theta$  is in Quadrant II.

10. Verify the identity  $\tan \frac{\theta}{2} + \frac{\cos \theta}{\sin \theta} = \csc \theta$ .

11. Verify the identity  $\sin^2 2x + 4 \cos^4 x = 4 \cos^2 x$ .

12. Find the exact value of  $\sin 15^\circ \cos 75^\circ$ .

13. Write  $y = -\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x$  in the form  $y = k \sin(x + \alpha)$ , where  $\alpha$  is measured in radians.

14. Use a calculator to approximate the radian measure of  $\cos^{-1} 0.7644$  to the nearest thousandth.

15. Find the exact value of  $\sin\left(\cos^{-1} \frac{12}{13}\right)$ .

16. Graph  $y = \sin^{-1}(x + 2)$ .

17. Solve  $3 \sin x - 2 = 0$ , where  $0^\circ \leq x < 360^\circ$ . (State solutions to the nearest tenth of a degree.)

18. Solve  $\sin x \cos x - \frac{\sqrt{3}}{2} \sin x = 0$ , where  $0 \leq x < 2\pi$ .

19. Find the exact solutions of  
 $\sin 2x + \sin x - 2 \cos x - 1 = 0$ , where  $0 \leq x < 2\pi$ .

20.   **HOURS OF DAYLIGHT** The table below shows the hours of daylight for Tampa, Florida, for selected days in 2009.

| Date    | Day Number | Hours of Daylight |
|---------|------------|-------------------|
| Jan. 1  | 1          | 10:24             |
| Feb. 1  | 32         | 10:53             |
| Mar. 1  | 60         | 11:37             |
| April 1 | 91         | 12:28             |
| May 1   | 121        | 13:15             |
| June 1  | 152        | 13:48             |
| July 1  | 182        | 13:53             |
| Aug. 1  | 213        | 13:26             |
| Sept. 1 | 244        | 12:41             |
| Oct. 1  | 274        | 11:52             |
| Nov. 1  | 305        | 11:04             |
| Dec. 1  | 335        | 10:30             |

Source: The U.S. Naval Observatory

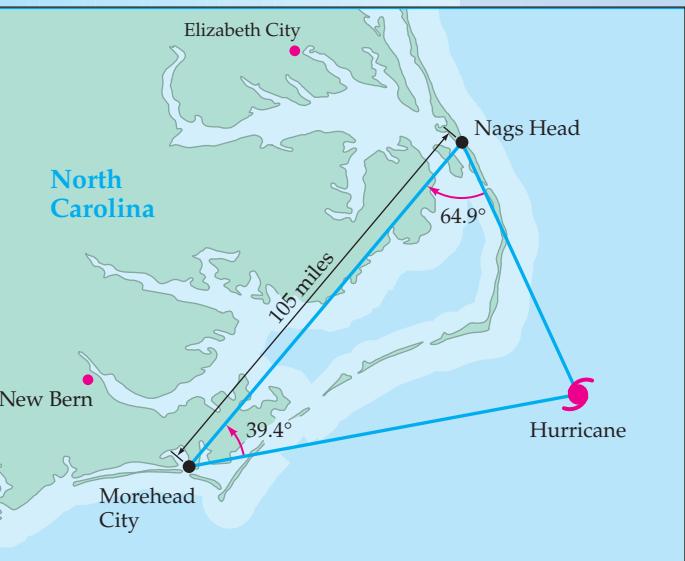
- a. Find the sine regression function that models the hours of daylight as a function of the day number. Let  $x = 1$  represent January 1, 2009. Use 365.25 for the period of the data.
- b. Use the regression function to estimate the hours of daylight (stated in hours and minutes, with the minutes rounded to the nearest minute) for Tampa on March 16, 2009 ( $x = 75$ ).

## Cumulative Review Exercises

- Solve:  $-2x + 1 < 7$
- Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = f(x + 1) + 2$ .
- Explain how to use the graph of  $y = f(x)$  to produce the graph of  $y = -f(x)$ .
- Determine whether  $f(x) = x - \sin x$  is an even function or an odd function.
- Find the inverse of  $f(x) = \frac{5x}{x - 1}$ .
- Convert  $240^\circ$  to radians.
- Convert  $\frac{5\pi}{3}$  to degrees.
- Evaluate:  $\sin \frac{\pi}{3}$
- Evaluate:  $\csc 60^\circ$
- Find  $\tan \theta$ , given that  $\theta$  is an acute angle and  $\sin \theta = \frac{2}{3}$ .

11. Determine the sign of  $\cot \theta$  given that  $\pi < \theta < \frac{3\pi}{2}$ .
12. What is the measure of the reference angle for the angle  $\theta = 310^\circ$ ?
13. What is the measure of the reference angle for the angle  $\theta = \frac{5\pi}{3}$ ?
14. Find the  $x$ - and  $y$ -coordinates of the point defined by  $W\left(\frac{\pi}{3}\right)$ , where  $W$  is the wrapping function.
15. Find the amplitude, the period, and the phase shift for the graph of  $y = 0.43 \sin\left(2x - \frac{\pi}{6}\right)$ .
16. Evaluate:  $\sin^{-1} \frac{1}{2}$
17. Use a calculator to evaluate  $\cos^{-1}(-0.8)$ . Round to the nearest thousandth.
18. Use interval notation to state the domain of  $f(x) = \cos^{-1} x$ .
19. Use interval notation to state the range of  $f(x) = \tan^{-1} x$ .
20. Find the exact solutions of  $2 \cos^2 x - 1 = -\sin x$ , where  $0 \leq x < 2\pi$ .

# 4 Applications of Trigonometry



## 4.1 The Law of Sines

## 4.2 The Law of Cosines and Area

## 4.3 Vectors

### Trigonometry and Indirect Measurement

In Chapter 2 we used trigonometric functions to find the unknown length of a side of a given *right triangle*. In this chapter we develop theorems that can be used to find the length of a side or the measure of an angle of *any triangle*, even if it is not a right triangle. These theorems are often used in the areas of navigation, surveying, and building design. Meteorologists use these theorems to estimate the distance from an approaching hurricane to cities in the projected path of the hurricane. For instance, in the diagram on the left, the distance from the hurricane to Nags Head can be determined using the Law of Sines, a theorem presented in this chapter.

See [Exercises 30 and 31 on page 299](#) for additional applications that can be solved by using the Law of Sines.

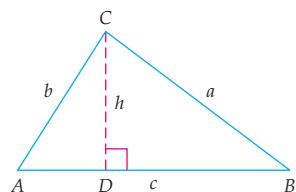


### Online Study Center

For online student resources, such as section quizzes, visit this website:  
[college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

## Section 4.1

- The Law of Sines
- The Ambiguous Case (SSA)
- Applications of the Law of Sines



**Figure 4.1**

# The Law of Sines

## The Law of Sines

Solving a triangle involves finding the lengths of all sides and the measures of all angles in the triangle. In this section and the next we develop formulas for solving an **oblique triangle**, which is a triangle that does not contain a right angle. The *Law of Sines* can be used to solve oblique triangles in which either two angles and a side or two sides and an angle opposite one of the sides are known. In **Figure 4.1**, altitude  $CD$  is drawn from  $C$ . The length of the altitude is  $h$ . Triangles  $ACD$  and  $BCD$  are right triangles.

Using the definition of the sine of an angle of a right triangle, we have from **Figure 4.1**

$$\sin B = \frac{h}{a} \quad \sin A = \frac{h}{b}$$

$$h = a \sin B \quad (1) \quad h = b \sin A \quad (2)$$

Equating the values of  $h$  in Equations (1) and (2), we obtain

$$a \sin B = b \sin A$$

Dividing each side of the equation by  $\sin A \sin B$ , we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, when an altitude is drawn to a different side, the following formulas result:

$$\frac{c}{\sin C} = \frac{b}{\sin B} \quad \text{and} \quad \frac{c}{\sin C} = \frac{a}{\sin A}$$

### take note

The Law of Sines may also be written as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

### The Law of Sines

If  $A$ ,  $B$ , and  $C$  are the measures of the angles of a triangle and  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite those angles, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

### EXAMPLE 1 » Solve a Triangle Using the Law of Sines

Solve triangle  $ABC$  if  $A = 42^\circ$ ,  $B = 63^\circ$ , and  $c = 18$  centimeters.

Continued ▶

**Solution**

Find  $C$  by using the fact that the sum of the interior angles of a triangle is  $180^\circ$ .

$$A + B + C = 180^\circ$$

$$42^\circ + 63^\circ + C = 180^\circ$$

$$C = 75^\circ$$

**take note**

We have used the rounding conventions stated on page 140 to determine the number of significant digits to be used for  $a$  and  $b$ .

Use the Law of Sines to find  $a$ .

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 42^\circ} = \frac{18}{\sin 75^\circ} \quad \bullet A = 42^\circ, c = 18, C = 75^\circ$$

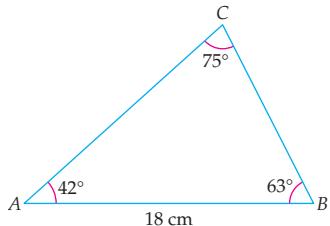
$$a = \frac{18 \sin 42^\circ}{\sin 75^\circ} \approx 12 \text{ centimeters}$$

Use the Law of Sines again, this time to find  $b$ .

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 63^\circ} = \frac{18}{\sin 75^\circ} \quad \bullet B = 63^\circ, c = 18, C = 75^\circ$$

$$b = \frac{18 \sin 63^\circ}{\sin 75^\circ} \approx 17 \text{ centimeters}$$



**Figure 4.2**

The solution is  $C = 75^\circ$ ,  $a \approx 12$  centimeters, and  $b \approx 17$  centimeters. A scale drawing can be used to see if these results are reasonable. See **Figure 4.2**.

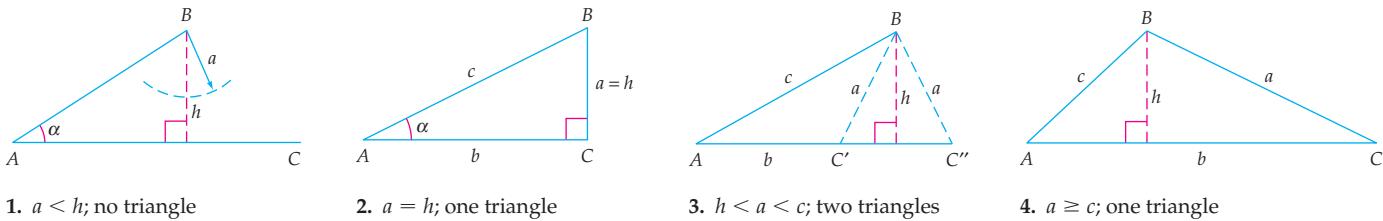
**» Try Exercise 4, page 298****■ The Ambiguous Case (SSA)**

When you are given two sides of a triangle and an angle opposite one of them, you may find that the triangle is not unique. Some information may result in two triangles, and some may result in no triangle at all. It is because of this that the case of knowing two sides and an angle opposite one of them (SSA) is called the *ambiguous case* of the Law of Sines.

Suppose that sides  $a$  and  $c$  and the nonincluded angle  $A$  of a triangle are known and we are asked to solve triangle  $ABC$ . The relationships among  $h$ , the height of the triangle,  $a$  (the side opposite  $\angle A$ ), and  $c$  determine whether there are no, one, or two triangles.

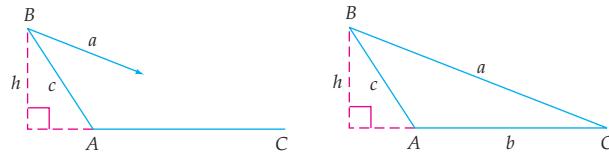
**Case 1** First consider the case in which  $\angle A$  is an acute angle (see **Figure 4.3**). There are four possible situations.

1.  $a < h$ ; there is no possible triangle.
2.  $a = h$ ; there is one triangle, a right triangle.
3.  $h < a < c$ ; there are two possible triangles.
4.  $a \geq c$ ; there is one triangle, which is not a right triangle.

**Figure 4.3**Case 1:  $A$  is an acute angle.

**Case 2** Now consider the case in which  $\angle A$  is an obtuse angle (see **Figure 4.4**). Here, there are two possible situations.

1.  $a \leq c$ ; there is no triangle.
2.  $a > c$ ; there is one triangle.

**Figure 4.4**Case 2:  $A$  is an obtuse angle.**take note**

When you solve for an angle of a triangle by using the Law of Sines, be aware that the value of the inverse sine function will give the measure of an acute angle. If the situation is the ambiguous case (SSA), you must consider a second, obtuse angle by using the supplement of the angle. You can use a scale drawing to see whether your results are reasonable.

**EXAMPLE 2 » Solve a Triangle Using the Law of Sines (SSA)**

- a. Find  $A$ , given triangle  $ABC$  with  $B = 32^\circ$ ,  $a = 42$ , and  $b = 30$ .
- b. Find  $C$ , given triangle  $ABC$  with  $A = 57^\circ$ ,  $a = 15$  feet, and  $c = 20$  feet.

**Solution**

$$\text{a. } \frac{b}{\sin B} = \frac{a}{\sin A}$$

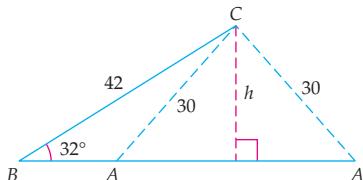
$$\frac{30}{\sin 32^\circ} = \frac{42}{\sin A}$$

$$\sin A = \frac{42 \sin 32^\circ}{30} \approx 0.7419$$

$$A \approx 48^\circ \text{ or } 132^\circ$$

- The two angles with measure between  $0^\circ$  and  $180^\circ$  that have a sine of 0.7419 are approximately  $48^\circ$  and  $132^\circ$ .

To check that  $A \approx 132^\circ$  is a valid result, add  $132^\circ$  to the measure of the given angle  $B$  ( $32^\circ$ ). Because  $132^\circ + 32^\circ < 180^\circ$ , we know that  $A \approx 132^\circ$  is a valid result. Thus angle  $A \approx 48^\circ$  or  $A \approx 132^\circ$  ( $\angle BAC$  in **Figure 4.5**).

**Figure 4.5**

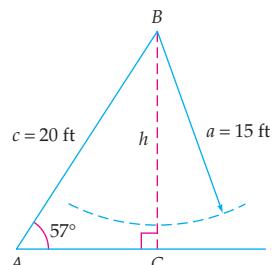


Figure 4.6

b.  $\frac{a}{\sin A} = \frac{c}{\sin C}$

$$\frac{15}{\sin 57^\circ} = \frac{20}{\sin C}$$

$$\sin C = \frac{20 \sin 57^\circ}{15} \approx 1.1182$$

Because 1.1182 is not in the range of the sine function, there is no solution of the equation. Thus **there is no triangle for these values of  $A$ ,  $a$ , and  $c$ .** See **Figure 4.6.**

» Try Exercise 20, page 298

## Applications of the Law of Sines

### EXAMPLE 3 » Solve an Application Using the Law of Sines

A radio antenna 85 feet high is located on top of an office building. At a distance  $AD$  from the base of the building, the angle of elevation to the top of the antenna is  $26^\circ$ , and the angle of elevation to the bottom of the antenna is  $16^\circ$ . Find the height of the building.

#### Solution

Sketch a diagram. See **Figure 4.7.** Find the measure of angle  $B$  and the measure of angle  $\beta$ .

$$B = 90^\circ - 26^\circ = 64^\circ$$

$$\beta = 26^\circ - 16^\circ = 10^\circ$$

Because we know the length  $BC$  and the measure of  $\beta$ , we can use triangle  $ABC$  and the Law of Sines to find length  $AC$ .

$$\frac{BC}{\sin \beta} = \frac{AC}{\sin B}$$

$$\frac{85}{\sin 10^\circ} = \frac{AC}{\sin 64^\circ}$$

$$AC = \frac{85 \sin 64^\circ}{\sin 10^\circ}$$

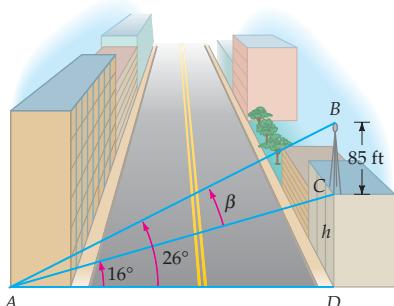


Figure 4.7

#### take note

In Example 3 we rounded the height of the building to two significant digits to comply with the rounding conventions given on page 140.

Having found  $AC$ , we can now find the height of the building.

$$\sin 16^\circ = \frac{h}{AC}$$

$$h = AC \sin 16^\circ$$

$$= \frac{85 \sin 64^\circ}{\sin 10^\circ} \sin 16^\circ \approx 121 \text{ feet}$$

• Substitute for  $AC$ .

The height of the building, to two significant digits, is 120 feet.

» Try Exercise 30, page 299

In navigation and surveying problems, there are two commonly used methods for specifying direction. The angular direction in which a craft is pointed is called the **heading**. Heading is expressed in terms of an angle measured clockwise from north. **Figure 4.8** shows a heading of  $65^\circ$  and a heading of  $285^\circ$ .

The angular direction used to locate one object in relation to another object is called the **bearing**. Bearing is expressed in terms of the acute angle formed by a north-south line and the line of direction. **Figure 4.9** shows a bearing of  $N38^\circ W$  and a bearing of  $S15^\circ E$ .

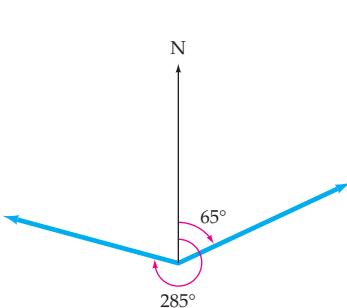


Figure 4.8

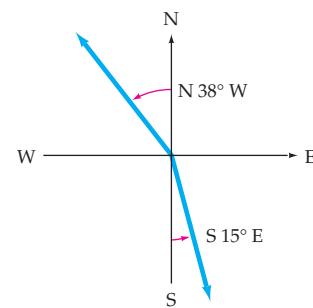


Figure 4.9

**QUESTION** Can a bearing of  $N50^\circ E$  be written as  $N310^\circ W$ ?

### EXAMPLE 4 » Solve an Application

A ship with a heading of  $330^\circ$  first sighted a lighthouse at a bearing of  $N65^\circ E$ . After traveling 8.5 miles, the ship observed the lighthouse at a bearing of  $S50^\circ E$ . Find the distance from the ship to the lighthouse when the first sighting was made.

#### Solution

Use the given information to draw a diagram. See **Figure 4.10** which shows that the measure of  $\angle CAB = 65^\circ + 30^\circ = 95^\circ$ , the measure of  $\angle BCA = 50^\circ - 30^\circ = 20^\circ$ , and  $B = 180^\circ - 95^\circ - 20^\circ = 65^\circ$ . Use triangle ABC and the Law of Sines to find  $c$ .

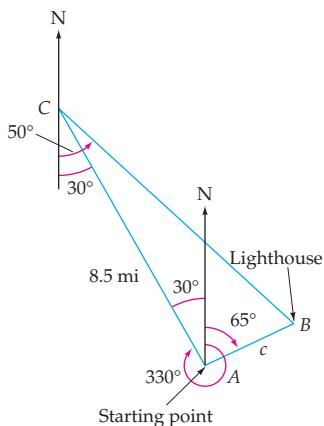


Figure 4.10

$$\begin{aligned} \frac{b}{\sin B} &= \frac{c}{\sin C} \\ \frac{8.5}{\sin 65^\circ} &= \frac{c}{\sin 20^\circ} \quad \cdot b = 8.5, B = 65^\circ, C = 20^\circ \\ c &= \frac{8.5 \sin 20^\circ}{\sin 65^\circ} \approx 3.2 \end{aligned}$$

The lighthouse was 3.2 miles (to two significant digits) from the ship when the first sighting was made.

» Try Exercise 40, page 300

**ANSWER** No. A bearing is always expressed using an acute angle.



### Topics for Discussion

- Is it possible to solve a triangle if the only given information consists of the measures of the three angles of the triangle? Explain.
- Explain why it is not possible (in general) to use the Law of Sines to solve a triangle for which we are given only the lengths of all the sides.
- Draw a triangle with dimensions  $A = 30^\circ$ ,  $c = 3$  inches, and  $a = 2.5$  inches. Is your answer unique? That is, can more than one triangle with the given dimensions be drawn?

## Exercise Set 4.1

In Exercises 1 to 44, round answers according to the rounding conventions on page 140.

In Exercises 1 to 14, solve the triangles.

1.  $A = 42^\circ$ ,  $B = 61^\circ$ ,  $a = 12$

2.  $B = 25^\circ$ ,  $C = 125^\circ$ ,  $b = 5.0$

3.  $A = 110^\circ$ ,  $C = 32^\circ$ ,  $b = 12$

4.  $B = 28^\circ$ ,  $C = 78^\circ$ ,  $c = 44$

5.  $A = 132^\circ$ ,  $a = 22$ ,  $b = 16$

6.  $B = 82.0^\circ$ ,  $b = 6.0$ ,  $c = 3.0$

7.  $A = 22.5^\circ$ ,  $B = 112.4^\circ$ ,  $a = 16.3$

8.  $A = 21.5^\circ$ ,  $B = 104.2^\circ$ ,  $c = 57.4$

9.  $A = 82.0^\circ$ ,  $B = 65.4^\circ$ ,  $b = 36.5$

10.  $B = 54.8^\circ$ ,  $C = 72.6^\circ$ ,  $a = 14.4$

11.  $A = 33.8^\circ$ ,  $C = 98.5^\circ$ ,  $c = 102$

12.  $B = 36.9^\circ$ ,  $C = 69.2^\circ$ ,  $a = 166$

13.  $C = 114.2^\circ$ ,  $c = 87.2$ ,  $b = 12.1$

14.  $A = 54.32^\circ$ ,  $a = 24.42$ ,  $c = 16.92$

In Exercises 15 to 28, solve the triangles that exist.

15.  $A = 37^\circ$ ,  $c = 40$ ,  $a = 28$

16.  $B = 32^\circ$ ,  $c = 14$ ,  $b = 9.0$

17.  $C = 65^\circ$ ,  $b = 10$ ,  $c = 8.0$

18.  $A = 42^\circ$ ,  $a = 12$ ,  $c = 18$

19.  $A = 30^\circ$ ,  $a = 1.0$ ,  $b = 2.4$

20.  $B = 22.6^\circ$ ,  $b = 5.55$ ,  $a = 13.8$

21.  $A = 14.8^\circ$ ,  $c = 6.35$ ,  $a = 4.80$

22.  $C = 37.9^\circ$ ,  $b = 3.50$ ,  $c = 2.84$

23.  $C = 47.2^\circ$ ,  $a = 8.25$ ,  $c = 5.80$

24.  $B = 52.7^\circ$ ,  $b = 12.3$ ,  $c = 16.3$

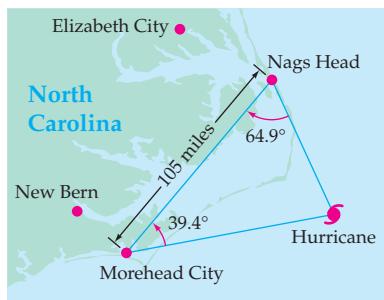
25.  $B = 117.32^\circ$ ,  $b = 67.25$ ,  $a = 15.05$

26.  $A = 49.22^\circ$ ,  $a = 16.92$ ,  $c = 24.62$

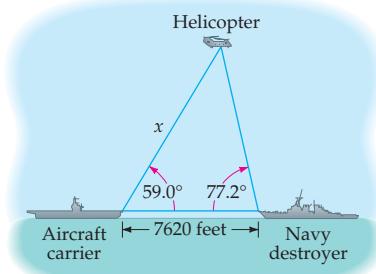
27.  $A = 20.5^\circ$ ,  $a = 10.3$ ,  $c = 14.1$

28.  $B = 41.2^\circ$ ,  $a = 31.5$ ,  $b = 21.6$

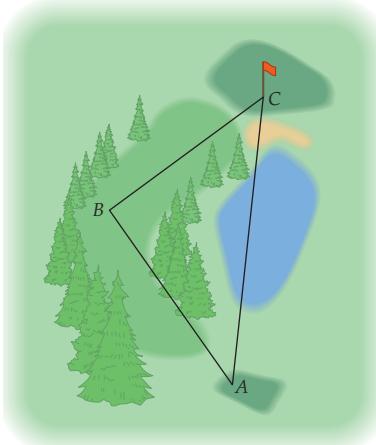
29. **HURRICANE WATCH** A satellite weather map shows a hurricane off the coast of North Carolina. Use the information in the map to find the distance from the hurricane to Nags Head.



- 30. NAVAL MANEUVERS** The distance between an aircraft carrier and a Navy destroyer is 7620 feet. The angle of elevation from the destroyer to a helicopter is  $77.2^\circ$ , and the angle of elevation from the aircraft carrier to the helicopter is  $59.0^\circ$ . The helicopter is in the same vertical plane as the two ships, as shown in the following figure. Use this data to determine the distance  $x$  from the helicopter to the aircraft carrier.



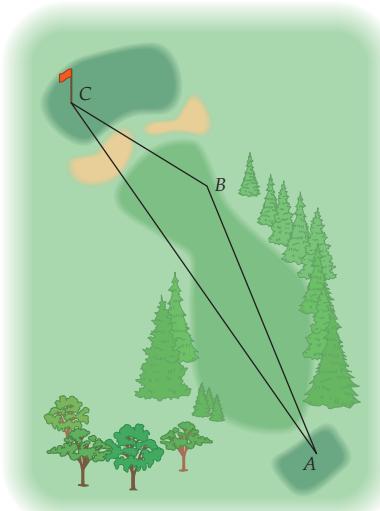
- 31. CHOOSING A GOLF STRATEGY** The following diagram shows two ways to play a golf hole. One is to hit the ball down the fairway on your first shot and then hit an approach shot to the green on your second shot. A second way is to hit directly toward the pin. Due to the water hazard, this is a more risky strategy. The distance  $AB$  is 165 yards,  $BC$  is 155 yards, and angle  $A = 42.0^\circ$ . Find the distance  $AC$  from the tee directly to the pin. Assume that angle  $B$  is an obtuse angle.



- 32. DRIVING DISTANCE** A golfer drives a golf ball from the tee at point  $A$  to point  $B$ , as shown in the following diagram.

The distance  $AC$  from the tee directly to the pin is 365 yards. Angle  $A$  measures  $11.2^\circ$  and angle  $C$  measures  $22.9^\circ$ .

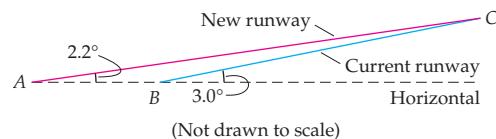
- Find the distance  $AB$  that the golfer drove the ball.
- Find the distance  $BC$  from the present position of the ball to the pin.



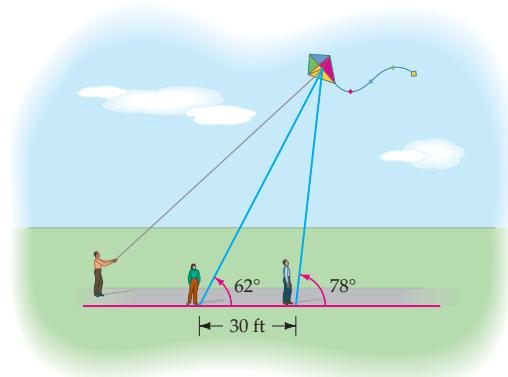
- 33. DISTANCE TO A HOT AIR BALLOON** The angle of elevation to a balloon from one observer is  $67^\circ$ , and the angle of elevation from another observer, 220 feet away, is  $31^\circ$ . If the balloon is in the same vertical plane as the two observers and between them, find the distance of the balloon from the first observer.

- 34. HEIGHT OF A SPACE SHUTTLE** Use the Law of Sines to solve Example 6 on page 141. Compare this method with the method used in Example 6. Which method do you prefer? Explain.

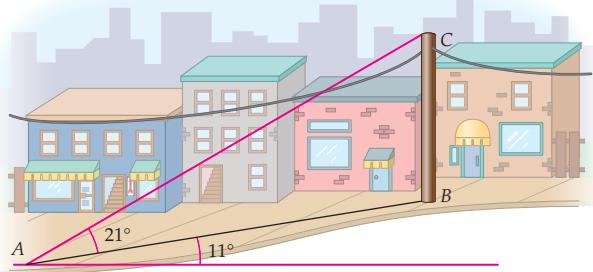
- 35. RUNWAY REPLACEMENT** An airport runway is 3550 feet long and has an incline of  $3.0^\circ$ . The airport planning committee plans to replace this runway with a new runway, as shown in the following figure. The new runway will be inclined at an angle of  $2.2^\circ$ . What will be the length of the new runway?



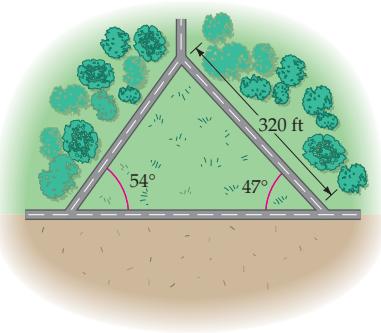
- 36. HEIGHT OF A KITE** Two observers, in the same vertical plane as a kite and at a distance of 30 feet apart, observe the kite at angles of  $62^\circ$  and  $78^\circ$ , as shown in the following diagram. Find the height of the kite.



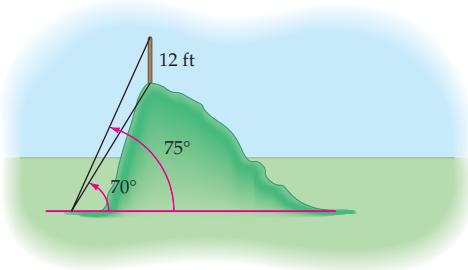
- 37. LENGTH OF A GUY WIRE** A telephone pole 35 feet high is situated on an  $11^\circ$  slope from the horizontal. The measure of angle  $CAB$  is  $21^\circ$ . Find the length of the guy wire  $AC$ .



- 38. DIMENSIONS OF A PLOT OF LAND** Three roads intersect in such a way as to form a triangular piece of land. See the accompanying figure. Find the lengths of the other two sides of the land.



- 39. HEIGHT OF A HILL** A surveying team determines the height of a hill by placing a 12-foot pole at the top of the hill and measuring the angles of elevation to the bottom and the top of the pole. They find the angles of elevation to be as shown in the following figure. Find the height of the hill.



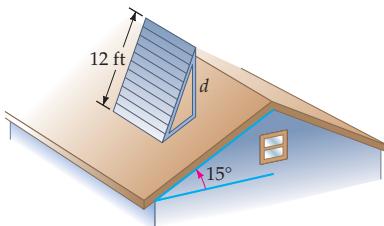
- 40. DISTANCE TO A FIRE** Two fire lookouts are located on mountains 20 miles apart. Lookout  $B$  is at a bearing of  $S65^\circ E$  from lookout  $A$ . A fire was sighted at a bearing of  $N50^\circ E$  from  $A$  and at a bearing of  $N8^\circ E$  from  $B$ . Find the distance of the fire from lookout  $A$ .

- 41. DISTANCE TO A LIGHTHOUSE** A navigator on a ship sights a lighthouse at a bearing of  $N36^\circ E$ . After traveling 8.0 miles at a heading of  $332^\circ$ , the ship sights the lighthouse at a bearing of  $S82^\circ E$ . How far is the ship from the lighthouse at the second sighting?

- 42. MINIMUM DISTANCE** The navigator on a ship traveling due east at 8 mph sights a lighthouse at a bearing of  $S55^\circ E$ . One hour later the lighthouse is sighted at a bearing of  $S25^\circ W$ . Find the closest the ship came to the lighthouse.

- 43. DISTANCE BETWEEN AIRPORTS** An airplane flew 450 miles at a bearing of  $N65^\circ E$  from airport  $A$  to airport  $B$ . The plane then flew at a bearing of  $S38^\circ E$  to airport  $C$ . Find the distance from  $A$  to  $C$  if the bearing from airport  $A$  to airport  $C$  is  $S60^\circ E$ .

- 44. LENGTH OF A BRACE** A 12-foot solar panel is to be installed on a roof with a  $15^\circ$  pitch. Find the length of the vertical brace  $d$  if the panel must be installed to make a  $40^\circ$  angle with the horizontal.



- 45. DISTANCES BETWEEN HOUSES** House  $B$  is located at a bearing of  $N67^\circ E$  from house  $A$ . House  $C$  is 300 meters from house  $A$  at a bearing of  $S68^\circ E$ . House  $B$  is located at a bearing of  $N11^\circ W$  from house  $C$ . Find the distance from house  $A$  to house  $B$ .

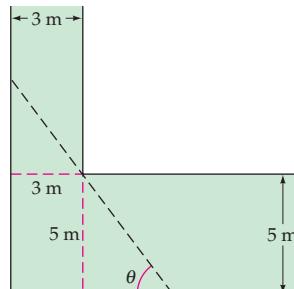
**46.** Show that for any triangle  $ABC$ ,  $\frac{a - b}{b} = \frac{\sin A - \sin B}{\sin B}$ .

**47.** Show that for any triangle  $ABC$ ,  $\frac{a + b}{b} = \frac{\sin A + \sin B}{\sin B}$ .

**48.** Show that for any triangle  $ABC$ ,  $\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}$ .

**49.**  **MAXIMUM LENGTH OF A ROD** The longest rod that can be carried horizontally around a corner from a hall 3 meters wide into one that is 5 meters wide is the

minimum of the length  $L$  of the black dashed line shown in the figure below.



Use similar triangles to show that the length  $L$  is a function of the angle  $\theta$ , given by

$$L(\theta) = \frac{5}{\sin \theta} + \frac{3}{\cos \theta}$$

Graph  $L$  and estimate the minimum value of  $L$ . Round to the nearest hundredth of a meter.

- 1. FERMAT'S PRINCIPLE AND SNELL'S LAW** State Fermat's Principle and Snell's Law. The refractive index of the glass in a ring is found to be 1.82. The refractive index of a particular diamond in a ring is 2.38. Use Snell's Law to explain what this means in terms of the reflective properties of the glass and the diamond.

## Section 4.2

- The Law of Cosines
- Area of a Triangle
- Heron's Formula

# The Law of Cosines and Area

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A20.

**PS1.** Evaluate  $\sqrt{a^2 + b^2 - 2ab \cos C}$  for  $a = 10.0$ ,  $b = 15.0$ , and  $C = 110.0^\circ$ . Round your result to the nearest tenth. [2.3]

**PS2.** Find the area of a triangle with a base of 6 inches and a height of 8.5 inches.

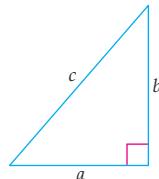
**PS3.** Solve  $c^2 = a^2 + b^2 - 2ab \cos C$  for  $C$ . [3.5]

**PS4.** The **semiperimeter** of a triangle is defined as one-half the perimeter of the triangle. Find the semiperimeter of a triangle with sides of 6 meters, 9 meters, and 10 meters.

**PS5.** Evaluate  $\sqrt{s(s - a)(s - b)(s - c)}$  for  $a = 3$ ,  $b = 4$ ,  $c = 5$ , and

$$s = \frac{a + b + c}{2}.$$

**PS6.** State a relationship between the lengths  $a$ ,  $b$ , and  $c$  in the triangle shown at the right. [1.2]



### The Law of Cosines

The *Law of Cosines* can be used to solve triangles in which two sides and the included angle (SAS) are known or in which three sides (SSS) are known. Consider the triangle in **Figure 4.11**. The height  $BD$  is drawn from  $B$  perpendicular to the  $x$ -axis. The triangle  $BDA$  is a right triangle, and the coordinates of  $B$  are  $(a \cos C, a \sin C)$ . The coordinates of  $A$  are  $(b, 0)$ . Using the distance formula, we can find the distance  $c$ .

$$\begin{aligned}c &= \sqrt{(a \cos C - b)^2 + (a \sin C - 0)^2} \\c^2 &= a^2 \cos^2 C - 2ab \cos C + b^2 + a^2 \sin^2 C \\c^2 &= a^2(\cos^2 C + \sin^2 C) + b^2 - 2ab \cos C \\c^2 &= a^2 + b^2 - 2ab \cos C\end{aligned}$$

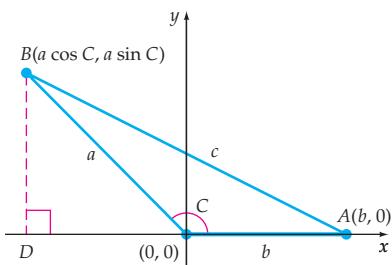


Figure 4.11

### The Law of Cosines

If  $A$ ,  $B$ , and  $C$  are the measures of the angles of a triangle and  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite these angles, then

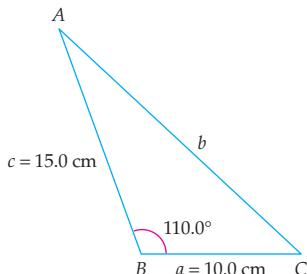
$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

**EXAMPLE 1** » Use the Law of Cosines (SAS)

In triangle  $ABC$ ,  $B = 110.0^\circ$ ,  $a = 10.0$  centimeters, and  $c = 15.0$  centimeters. See **Figure 4.12**. Find  $b$ .



**Figure 4.12**

### Solution

The Law of Cosines can be used because two sides and the included angle are known.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ &= 10.0^2 + 15.0^2 - 2(10.0)(15.0) \cos 110.0^\circ \\ b &= \sqrt{10.0^2 + 15.0^2 - 2(10.0)(15.0) \cos 110.0^\circ} \\ b &\approx 20.7 \text{ centimeters} \end{aligned}$$

» Try Exercise 12, page 308

In the next example we know the length of each side, but we do not know the measure of any of the angles.

**EXAMPLE 2** » Use the Law of Cosines (SSS)

In triangle  $ABC$ ,  $a = 32$  feet,  $b = 20$  feet, and  $c = 40$  feet. Find  $B$ . This is the SSS case.

### Solution

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

• Solve for  $\cos B$ .

$$= \frac{32^2 + 40^2 - 20^2}{2(32)(40)}$$

• Substitute for  $a$ ,  $b$ , and  $c$ .

$$B = \cos^{-1}\left(\frac{32^2 + 40^2 - 20^2}{2(32)(40)}\right)$$

• Solve for angle  $B$ .

$$B \approx 30^\circ$$

• To the nearest degree

» Try Exercise 18, page 308

**EXAMPLE 3** » Solve an Application Using the Law of Cosines

A boat sailed 3.0 miles at a heading of  $78^\circ$  and then turned to a heading of  $138^\circ$  and sailed another 4.3 miles. Find the distance and the bearing of the boat from the starting point.

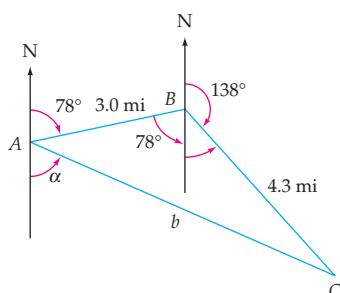


Figure 4.13

**Solution**

Sketch a diagram (see **Figure 4.13**). First find the measure of angle  $B$  in triangle  $ABC$ .

$$B = 78^\circ + (180^\circ - 138^\circ) = 120^\circ$$

Use the Law of Cosines first to find  $b$  and then to find  $A$ .

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ &= 4.3^2 + 3.0^2 - 2(4.3)(3.0) \cos 120^\circ \end{aligned} \quad \bullet \text{ Substitute for } a, c, \text{ and } B.$$

$$b = \sqrt{4.3^2 + 3.0^2 - 2(4.3)(3.0) \cos 120^\circ}$$

$$b \approx 6.4 \text{ miles}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \bullet \text{ Solve the Law of Cosines for } \cos A.$$

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}\left(\frac{6.4^2 + 3.0^2 - 4.3^2}{(2)(6.4)(3.0)}\right) \approx 35^\circ$$

The bearing of the present position of the boat from the starting point  $A$  can be determined by calculating the measure of angle  $\alpha$  in **Figure 4.13**.

$$\alpha \approx 180^\circ - (78^\circ + 35^\circ) = 67^\circ$$

**The distance is approximately 6.4 miles, and the bearing (to the nearest degree) is S67°E.**

**take note**

The measure of angle  $A$  in Example 3 can also be determined by using the Law of Sines.

**» Try Exercise 52, page 310**

There are five different cases that we may encounter when solving an oblique triangle. Each case is listed below under the law that can be used to solve the triangle.

**Choosing Between the Law of Sines and the Law of Cosines**

*Apply the Law of Sines to solve an oblique triangle for each of the following cases.*

**ASA** The measures of two angles of the triangle and the length of the included side are known.

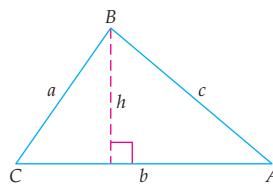
**AAS** The measures of two angles of the triangle and the length of a side opposite one of these angles are known.

**SSA** The lengths of two sides of the triangle and the measure of an angle opposite one of these sides are known. This case is called the ambiguous case. It may yield one solution, two solutions, or no solution.

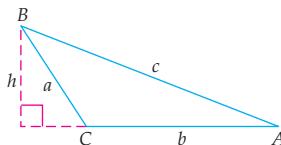
*Apply the Law of Cosines to solve an oblique triangle for each of the following cases.*

**SSS** The lengths of all three sides of the triangle are known. After finding the measure of an angle, you can complete your solution by using the Law of Sines.

**SAS** The lengths of two sides of the triangle and the measure of the included angle are known. After finding the measure of the third side, you can complete your solution by using the Law of Sines.



Acute triangle



Obtuse triangle

**Figure 4.14**

**QUESTION** In triangle  $ABC$ ,  $A = 40^\circ$ ,  $C = 60^\circ$ , and  $b = 114$ . Should you use the Law of Sines or the Law of Cosines to solve this triangle?

### Area of a Triangle

The formula  $A = \frac{1}{2}bh$  can be used to find the area of a triangle when the base and height are given. In this section we will find the areas of triangles when the height is not given. We will use  $K$  for the area of a triangle because the letter  $A$  is often used to represent the measure of an angle.

Consider the areas of the acute and obtuse triangles in **Figure 4.14**.

$$\text{Height of each triangle: } h = c \sin A$$

$$\text{Area of each triangle: } K = \frac{1}{2}bh$$

$$K = \frac{1}{2}bc \sin A \quad \bullet \text{ Substitute for } h.$$

Thus we have established the following theorem.

#### take note

Because each formula requires two sides and the included angle, it is necessary to learn only one formula.

#### Area of a Triangle

The area  $K$  of triangle  $ABC$  is one-half the product of the lengths of any two sides and the sine of the included angle. Thus

$$K = \frac{1}{2}bc \sin A \quad K = \frac{1}{2}ab \sin C \quad K = \frac{1}{2}ac \sin B$$

### EXAMPLE 4 » Find the Area of a Triangle

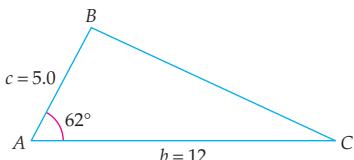
Given angle  $A = 62^\circ$ ,  $b = 12$  meters, and  $c = 5.0$  meters, find the area of triangle  $ABC$ .

#### Solution

In **Figure 4.15**, two sides and the included angle of the triangle are given. Using the formula for area, we have

$$K = \frac{1}{2}bc \sin A = \frac{1}{2}(12)(5.0)(\sin 62^\circ) \approx 26 \text{ square meters}$$

» Try Exercise 30, page 309

**Figure 4.15**

**ANSWER** Because the measure of two angles and the length of the included side are given, the triangle can be solved by using the Law of Sines.

When two angles and an included side are given, the Law of Sines is used to derive a formula for the area of a triangle. First, solve for  $c$  in the Law of Sines.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$c = \frac{b \sin C}{\sin B}$$

Substitute for  $c$  in the formula  $K = \frac{1}{2}bc \sin A$ .

$$K = \frac{1}{2}bc \sin A = \frac{1}{2}b\left(\frac{b \sin C}{\sin B}\right) \sin A$$

$$K = \frac{b^2 \sin C \sin A}{2 \sin B}$$

In like manner, the following two alternative formulas can be derived for the area of a triangle.

$$K = \frac{a^2 \sin B \sin C}{2 \sin A} \quad \text{and} \quad K = \frac{c^2 \sin A \sin B}{2 \sin C}$$

### EXAMPLE 5 » Find the Area of a Triangle

Given  $A = 32^\circ$ ,  $C = 77^\circ$ , and  $a = 14$  inches, find the area of triangle  $ABC$ .

#### Solution

To use the above area formula, we need to know two angles and the included side. Therefore, we need to determine the measure of angle  $B$ .

$$B = 180^\circ - 32^\circ - 77^\circ = 71^\circ$$

Thus

$$K = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{14^2 \sin 71^\circ \sin 77^\circ}{2 \sin 32^\circ} \approx 170 \text{ square inches}$$

» Try Exercise 32, page 309

#### Math Matters

Recent findings indicate that Heron's formula for finding the area of a triangle was first discovered by Archimedes. However, the formula is called Heron's formula in honor of the geometer Heron of Alexandria (A.D. 50), who gave an ingenious proof of the theorem in his work *Metrica*. Because Heron of Alexandria was also known as Hero, some texts refer to Heron's formula as Hero's formula.

### ■ Heron's Formula

The Law of Cosines can be used to derive *Heron's formula* for the area of a triangle in which three sides of the triangle are given.

#### Heron's Formula for Finding the Area of a Triangle

If  $a$ ,  $b$ , and  $c$  are the lengths of the sides of a triangle, then the area  $K$  of the triangle is

$$K = \sqrt{s(s - a)(s - b)(s - c)}, \quad \text{where } s = \frac{1}{2}(a + b + c)$$

Because  $s$  is one-half the perimeter of the triangle, it is called the **semiperimeter**.

### EXAMPLE 6 » Find an Area by Heron's Formula

Find, to two significant digits, the area of the triangle with  $a = 7.0$  meters,  $b = 15$  meters, and  $c = 12$  meters.

#### Solution

Calculate the semiperimeter  $s$ .

$$s = \frac{a + b + c}{2} = \frac{7.0 + 15 + 12}{2} = 17$$

Use Heron's formula.

$$\begin{aligned} K &= \sqrt{s(s - a)(s - b)(s - c)} \\ &= \sqrt{17(17 - 7.0)(17 - 15)(17 - 12)} \\ &= \sqrt{1700} \approx 41 \text{ square meters} \end{aligned}$$

» Try Exercise 40, page 309

### EXAMPLE 7 » Use Heron's Formula to Solve an Application

 The original portion of the Luxor Hotel in Las Vegas has the shape of a square pyramid. Each face of the pyramid is an isosceles triangle with a base of 646 feet and sides of length 576 feet. Assuming that the glass on the exterior of the Luxor Hotel costs \$35 per square foot, determine the cost of the glass, to the nearest \$10,000, for one of the triangular faces of the hotel.

#### Solution

The lengths (in feet) of the sides of a triangular face are  $a = 646$ ,  $b = 576$ , and  $c = 576$ .

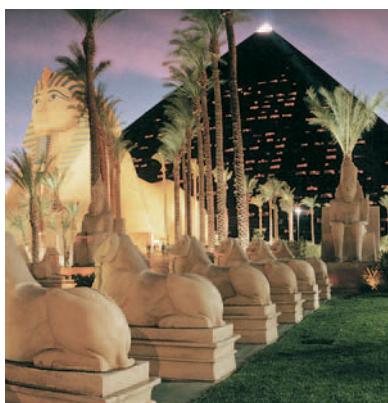
$$s = \frac{a + b + c}{2} = \frac{646 + 576 + 576}{2} = 899 \text{ feet}$$

$$\begin{aligned} K &= \sqrt{s(s - a)(s - b)(s - c)} \\ &= \sqrt{899(899 - 646)(899 - 576)(899 - 576)} \\ &= \sqrt{23,729,318,063} \\ &\approx 154,043 \text{ square feet} \end{aligned}$$

The cost  $C$  of the glass is the product of the cost per square foot and the area.

$$C \approx 35 \cdot 154,043 = 5,391,505$$

The approximate cost of the glass for one face of the Luxor Hotel is \$5,390,000.



The pyramid portion of the Luxor Hotel in Las Vegas, Nevada

» Try Exercise 60, page 310



### Topics for Discussion

1. Explain why there is no triangle that has sides of lengths  $a = 2$  inches,  $b = 11$  inches, and  $c = 3$  inches.
2. The Pythagorean Theorem is a special case of the Law of Cosines. Explain.
3. To solve a triangle in which the lengths of the three sides are given (SSS), a mathematics professor recommends the following procedure.
  - (i) Use the Law of Cosines to find the measure of the largest angle.
  - (ii) Use the Law of Sines to find the measure of a second angle.
  - (iii) Find the measure of the third angle by using the formula  
$$A + B + C = 180^\circ$$

Explain why this procedure is easier than using the Law of Cosines to find the measure of all three angles.

4. Explain why the Law of Cosines cannot be used to solve a triangle in which you are given the measures of two angles and the length of the included side (ASA).

## Exercise Set 4.2

**In Exercises 1 to 52, round answers according to the rounding conventions on page 140.**

**In Exercises 1 to 14, find the third side of the triangle.**

1.  $a = 12, b = 18, C = 44^\circ$

2.  $b = 30, c = 24, A = 120^\circ$

3.  $a = 120, c = 180, B = 56^\circ$

4.  $a = 400, b = 620, C = 116^\circ$

5.  $b = 60, c = 84, A = 13^\circ$

6.  $a = 122, c = 144, B = 48^\circ$

7.  $a = 9.0, b = 7.0, C = 72^\circ$

8.  $b = 12, c = 22, A = 55^\circ$

9.  $a = 4.6, b = 7.2, C = 124^\circ$

10.  $b = 12.3, c = 14.5, A = 6.5^\circ$

11.  $a = 25.9, c = 33.4, B = 84.0^\circ$

12.  $a = 14.2, b = 9.30, C = 9.20^\circ$

13.  $a = 122, c = 55.9, B = 44.2^\circ$

14.  $b = 444.8, c = 389.6, A = 78.44^\circ$

**In Exercises 15 to 24, given three sides of a triangle, find the specified angle.**

15.  $a = 25, b = 32, c = 40$ ; find  $A$ .

16.  $a = 60, b = 88, c = 120$ ; find  $B$ .

17.  $a = 8.0, b = 9.0, c = 12$ ; find  $C$ .

18.  $a = 108, b = 132, c = 160$ ; find  $A$ .

19.  $a = 80.0, b = 92.0, c = 124$ ; find  $B$ .

20.  $a = 166, b = 124, c = 139$ ; find  $B$ .

21.  $a = 1025, b = 625.0, c = 1420$ ; find  $C$ .

22.  $a = 4.7, b = 3.2, c = 5.9$ ; find  $A$ .

23.  $a = 32.5, b = 40.1, c = 29.6$ ; find  $B$ .

24.  $a = 112.4, b = 96.80, c = 129.2$ ; find  $C$ .



12.  $a = 14.2, b = 9.30, C = 9.20^\circ$

In Exercises 25 to 28, solve the triangle.

25.  $A = 39.4^\circ, b = 15.5, c = 17.2$

26.  $C = 98.4^\circ, a = 141, b = 92.3$

27.  $a = 83.6, b = 144, c = 98.1$

28.  $a = 25.4, b = 36.3, c = 38.2$

In Exercises 29 to 40, find the area of the given triangle. Round each area to the same number of significant digits given for each of the given sides.

29.  $A = 105^\circ, b = 12, c = 24$

30.  $B = 127^\circ, a = 32, c = 25$

31.  $A = 42^\circ, B = 76^\circ, c = 12$

32.  $B = 102^\circ, C = 27^\circ, a = 8.5$

33.  $a = 16, b = 12, c = 14$

34.  $a = 32, b = 24, c = 36$

35.  $B = 54.3^\circ, a = 22.4, b = 26.9$

36.  $C = 18.2^\circ, b = 13.4, a = 9.84$

37.  $A = 116^\circ, B = 34^\circ, c = 8.5$

38.  $B = 42.8^\circ, C = 76.3^\circ, c = 17.9$

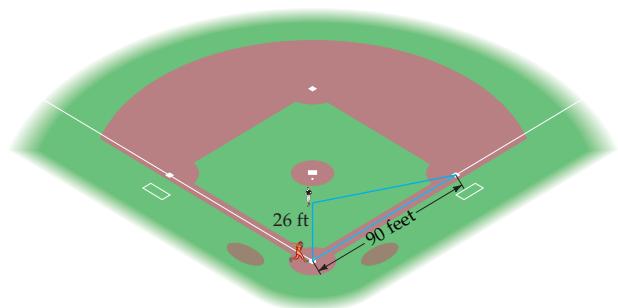
39.  $a = 3.6, b = 4.2, c = 4.8$

40.  $a = 10.2, b = 13.3, c = 15.4$

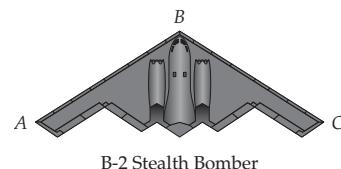
41. **DISTANCE BETWEEN AIRPORTS** A plane leaves airport  $A$  and travels 560 miles to airport  $B$  at a bearing of N32°E. The plane leaves airport  $B$  and travels to airport  $C$  320 miles away at a bearing of S72°E. Find the distance from airport  $A$  to airport  $C$ .

42. **LENGTH OF A STREET** A developer owns a triangular lot at the intersection of two streets. The streets meet at an angle of  $72^\circ$ , and the lot has 300 feet of frontage along one street and 416 feet of frontage along the other street. Find the length of the third side of the lot.

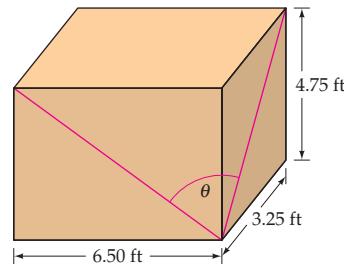
43. **BASEBALL** In a baseball game, a batter hits a ground ball 26 feet in the direction of the pitcher's mound. See the figure at the top of the next column. The pitcher runs forward and reaches for the ball. At that moment, how far is the ball from first base? (Note: A baseball infield is a square that measures 90 feet on each side.)



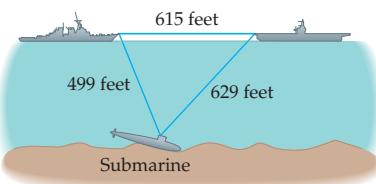
44.  **B-2 BOMBER** The leading edge of each wing of the B-2 Stealth Bomber measures 105.6 feet in length. The angle between the wing's leading edges ( $\angle ABC$ ) is  $109.05^\circ$ . What is the wing span (the distance from  $A$  to  $C$ ) of the B-2 Bomber?



45. **ANGLE BETWEEN THE DIAGONALS OF A BOX** The rectangular box in the figure measures 6.50 feet by 3.25 feet by 4.75 feet. Find the measure of the angle  $\theta$  that is formed by the union of the diagonal shown on the front of the box and the diagonal shown on the right side of the box.

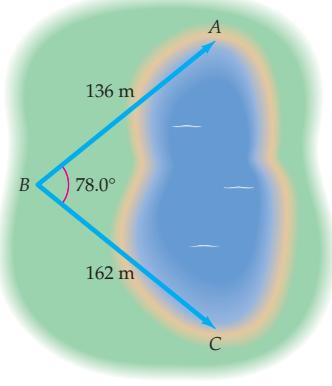


46. **SUBMARINE RESCUE MISSION** Use the distances shown in the following figure to determine the depth of the submarine below the surface of the water. Assume that the line segment between the surface ships is directly above the submarine.



- 47. DISTANCE BETWEEN SHIPS** Two ships left a port at the same time. One ship traveled at a speed of 18 mph at a heading of  $318^\circ$ . The other ship traveled at a speed of 22 mph at a heading of  $198^\circ$ . Find the distance between the two ships after 10 hours of travel.

- 48. DISTANCE ACROSS A LAKE** Find the distance across the lake shown in the figure.



- 49. GEOMETRY** A regular hexagon is inscribed in a circle with a radius of exactly 40 centimeters. Find the exact length of one side of the hexagon.

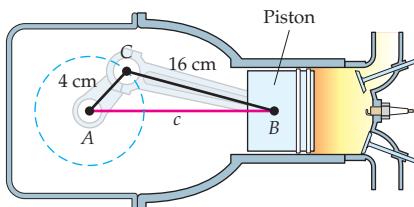
- 50. ANGLE BETWEEN BOUNDARIES OF A LOT** A triangular city lot has sides of 224 feet, 182 feet, and 165 feet. Find the angle between the longer two sides of the lot.

- 51. DISTANCE TO A PLANE** A plane traveling at 180 mph passes 400 feet directly over an observer. The plane is traveling along a straight path with an angle of elevation of  $14^\circ$ . Find the distance of the plane from the observer 10 seconds after the plane has passed directly overhead.

- 52. DISTANCE BETWEEN SHIPS** A ship leaves a port at a speed of 16 mph at a heading of  $32^\circ$ . One hour later another ship leaves the port at a speed of 22 mph at a heading of  $254^\circ$ . Find the distance between the ships 4 hours after the first ship leaves the port.

- 53. DISTANCE AND BEARING FROM A STARTING POINT** A plane flew 181 miles at a heading of  $108.5^\circ$  and then turned to a heading of  $124.6^\circ$  and flew another 225 miles. Find the distance and the bearing of the plane from the starting point.

- 54. ENGINE DESIGN** An engine has a 16-centimeter connecting rod that is attached to a rotating crank with a 4-centimeter radius. See the following figure.



- Use the Law of Cosines to find an equation that relates  $c$  and  $A$ .
- Use the quadratic formula to solve the equation in **a.** for  $c$ .
- Use the equation from **b.** to find  $c$  when  $A = 55^\circ$ . Round to the nearest centimeter.
- Use the Law of Sines to find  $c$  when  $A = 55^\circ$ . Round to the nearest centimeter. How does this result compare with the result in **c.**?

- 55. AREA OF A TRIANGULAR LOT** Find the area of a triangular piece of land that is bounded by sides of 236 meters, 620 meters, and 814 meters. Round to the nearest hundred square meters.

- 56. GEOMETRY** Find the exact area of a parallelogram with sides of exactly 8 feet and 12 feet. The shorter diagonal is exactly 10 feet.

- 57. GEOMETRY** Find the exact area of a square inscribed in a circle with a radius of exactly 9 inches.

- 58. GEOMETRY** Find the exact area of a regular hexagon inscribed in a circle with a radius of exactly 24 centimeters.

- 59. COST OF A LOT** A commercial piece of real estate is priced at \$2.20 per square foot. Find, to the nearest \$1000, the cost of a triangular lot measuring 212 feet by 185 feet by 240 feet.

- 60. COST OF A LOT** An industrial piece of real estate is priced at \$4.15 per square foot. Find, to the nearest \$1000, the cost of a triangular lot measuring 324 feet by 516 feet by 412 feet.

- 61. AREA OF A PASTURE** Find the number of acres in a pasture whose shape is a triangle measuring 800 feet by 1020 feet by 680 feet. Round to the nearest hundredth of an acre. (An acre is 43,560 square feet.)

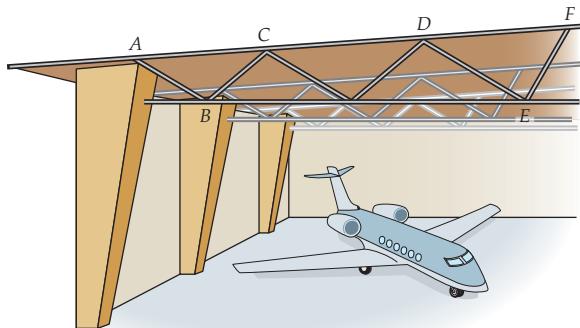
- 62. AREA OF A HOUSING TRACT** Find the number of acres in a housing tract whose shape is a triangle measuring 420 yards by 540 yards by 500 yards. Round to the nearest hundredth of an acre. (An acre is 4840 square yards.)

The identity at the right is one of *Mollweide's formulas*. It applies to any triangle  $ABC$ .

$$\frac{a - b}{c} = \frac{\sin\left(\frac{A - B}{2}\right)}{\cos\left(\frac{C}{2}\right)}$$

The formula is intriguing because it contains the dimensions of all angles and all sides of triangle  $ABC$ . The formula can be used to check whether a triangle has been solved correctly. Substitute the dimensions of a given triangle in the formula and compare the value of the left side of the formula with the value of the right side. If the two results are not reasonably close, then you know that at least one dimension is incorrect. The results generally will not be identical because each dimension is an approximation. In Exercises 63 and 64, you can assume that a triangle has an incorrect dimension if the value of the left side and the value of the right side of the above formula differ by more than 0.02.

- 63.**  **CHECK DIMENSIONS OF TRUSSES** The following diagram shows some of the steel trusses in an airplane hangar.



## »»» Connecting Concepts

- 65.** Find the measure of the angle formed by the sides  $P_1P_2$  and  $P_1P_3$  of a triangle with vertices at  $P_1(-2, 4)$ ,  $P_2(2, 1)$ , and  $P_3(4, -3)$ .

**66.** Given a triangle  $ABC$ , prove that for any triangle  $ABC$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

**67.** Use the Law of Cosines to show that for any triangle  $ABC$

$$\cos A = \frac{(b + c - a)(b + c + a)}{2bc} - 1$$

**68.** Prove that  $K = xy \sin A$  for a parallelogram, where  $x$  and  $y$  are the lengths of adjacent sides,  $A$  is the measure of the angle between side  $x$  and side  $y$ , and  $K$  is the area of the parallelogram.

An architect has determined the following dimensions for  $\triangle ABC$  and  $\triangle DEF$ :

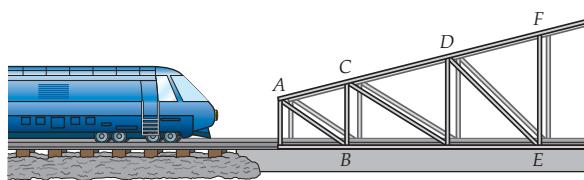
$$\triangle ABC: A = 53.5^\circ, B = 86.5^\circ, C = 40.0^\circ$$

$a = 13.0$  feet,  $b = 16.1$  feet,  $c = 10.4$  feet

$$\triangle DEF: D = 52.1^\circ, E = 59.9^\circ, F = 68.0^\circ$$

Use Mollweide's formula to determine if either  $\triangle ABC$  or  $\triangle DEF$  has an incorrect dimension.

- 64.**  **CHECK DIMENSIONS OF TRUSSES** The following diagram shows some of the steel trusses in a railroad bridge.



A structural engineer has determined the following dimensions for  $\triangle ABC$  and  $\triangle DEF$ :

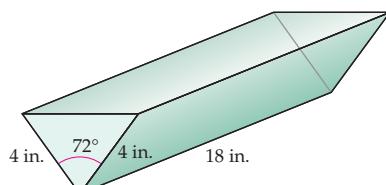
$$\triangle ABC: A = 34.1^\circ, B = 66.2^\circ, C = 79.7^\circ$$

$$\triangle DEF: D = 45.0^\circ, E = 56.2^\circ, F = 78.8^\circ$$

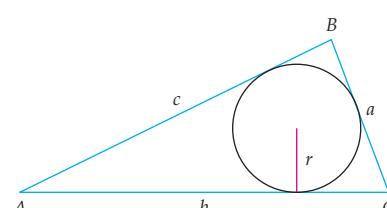
$$d = 13.6 \text{ feet}, e = 16.0 \text{ feet}, f = 18.9 \text{ feet}$$

Use Mollweide's formula to determine if either  $\triangle ABC$  or  $\triangle DEF$  has an incorrect dimension.

- 69.** Find the volume of the triangular prism shown in the figure.



- 70.** Show that the area of the circumscribed triangle in the following figure is  $K = rs$ , where  $s = \frac{a + b + c}{2}$ .



- 1. VISUAL INSIGHT** Give the reason that justifies each step in the following proof of the Law of Cosines.

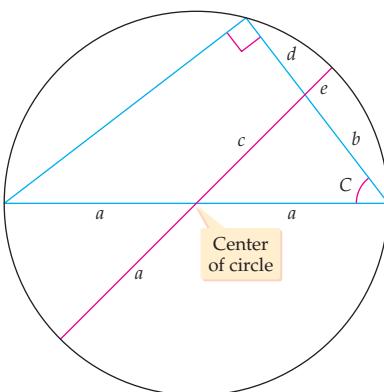
$$bd = (a + c)e$$

$$bd = (a + c)(a - c)$$

$$b(2a \cos C - b) = (a + c)(a - c)$$

$$2ab \cos C - b^2 = a^2 - c^2$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



## Section 4.3

- Vectors
  - Unit Vectors
  - Applications of Vectors
  - Dot Product
  - Scalar Projection
  - Parallel and Perpendicular Vectors
  - Work: An Application of the Dot Product

# Vectors

## PREPARE FOR THIS SECTION

**Prepare for this section by completing the following exercises. The answers can be found on page A20.**

**PS1.** Evaluate:  $\sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$

**PS2.** Use a calculator to evaluate  $10 \cos 228^\circ$ . Round to the nearest thousandth. [2.3]

**PS3.** Solve  $\tan \alpha = \left| \frac{-\sqrt{3}}{3} \right|$  for  $\alpha$ , where  $\alpha$  is an acute angle measured in degrees. [3.5]

**PS4.** Solve  $\cos \alpha = \frac{-17}{\sqrt{338}}$  for  $\alpha$ , where  $\alpha$  is an obtuse angle measured in degrees. Round to the nearest tenth of a degree. [3.5]

**PS5.** Rationalize the denominator of  $\frac{1}{\sqrt{5}}$ .

**PS6.** Rationalize the denominator of  $\frac{28}{\sqrt{68}}$ .

## Vectors

In scientific applications some measurements, such as area, mass, distance, speed, and time, are completely described by a real number and a unit. Examples include 30 square feet (area), 25 meters/second (speed), and 5 hours (time). These measurements are called **scalar quantities**, and the number used to indicate the magnitude of the measurement is called a **scalar**. Two other examples of scalar quantities are volume and temperature.

For other quantities, in addition to the numerical and unit description, it is also necessary to include a *direction* to describe the quantity completely. For

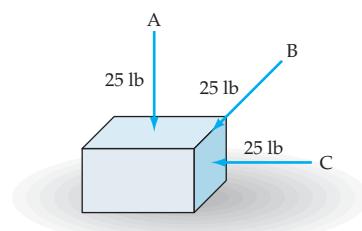


Figure 4.16

example, applying a force of 25 pounds at various angles to a small metal box will influence how the box moves. In **Figure 4.16**, applying the 25-pound force straight down (A) will not move the box to the left. However, applying the 25-pound force parallel to the floor (C) will move the box along the floor.

**Vector quantities** have a *magnitude* (numerical and unit description) and a *direction*. Force is a vector quantity. Velocity is another. Velocity includes the speed (magnitude) and a direction. A velocity of 40 mph east is different from a velocity of 40 mph north. Displacement is another vector quantity; it consists of distance (a scalar) moved in a certain direction. For example, we might speak of a displacement of 13 centimeters at an angle of  $15^\circ$  from the positive  $x$ -axis.

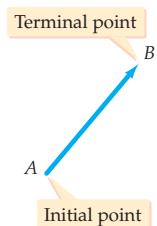


Figure 4.17

### Definition of a Vector

A **vector** is a directed line segment. The length of the line segment is the **magnitude of the vector**, and the direction of the vector is measured by an angle.

The point  $A$  for the vector in **Figure 4.17** is called the **initial point** (or tail) of the vector, and the point  $B$  is the **terminal point** (or head) of the vector. An arrow over the letters ( $\vec{AB}$ ), an arrow over a single letter ( $\vec{V}$ ), or boldface type ( $\mathbf{AB}$  or  $\mathbf{V}$ ) is used to denote a vector. The magnitude of the vector is the length of the line segment and is denoted by  $\|\vec{AB}\|$ ,  $\|\vec{V}\|$ ,  $\|\mathbf{AB}\|$ , or  $\|\mathbf{V}\|$ .

**Equivalent vectors** have the same magnitude and the same direction. The vectors in **Figure 4.18** are equivalent. They have the same magnitude and direction.

Multiplying a vector by a positive real number (other than 1) changes the magnitude of the vector but not its direction. If  $v$  is any vector, then  $2v$  is the vector that has the same direction as  $v$  but is twice the magnitude of  $v$ . The multiplication of 2 and  $v$  is called the **scalar multiplication** of the vector  $v$  and the scalar 2. Multiplying a vector by a negative number  $a$  reverses the direction of the vector and multiplies the magnitude of the vector by  $|a|$ . See **Figure 4.19**.

The sum of two vectors, called the **resultant vector** or the **resultant**, is the single equivalent vector that will have the same effect as the application of those two vectors. For example, a displacement of 40 meters along the positive  $x$ -axis and then 30 meters in the positive  $y$  direction is equivalent to a vector of magnitude 50 meters at an angle of approximately  $37^\circ$  to the positive  $x$ -axis. See **Figure 4.20**.

Vectors can be added graphically by using the *triangle method* or the *parallelogram method*. In the triangle method, shown in **Figure 4.21**, the initial point of  $V$  is placed at the terminal point of  $U$ . The vector connecting the initial point of  $U$  with the terminal point of  $V$  is the sum  $U + V$ .

The parallelogram method of adding two vectors graphically places the initial point of the two vectors  $U$  and  $V$  together, as in **Figure 4.22**. Complete the parallelogram so that  $U$  and  $V$  are sides of the parallelogram. The diagonal beginning at the initial point of the two vectors is  $U + V$ .

To find the difference between two vectors, first rewrite the expression as  $V - U = V + (-U)$ . The difference is shown geometrically in **Figure 4.23**.

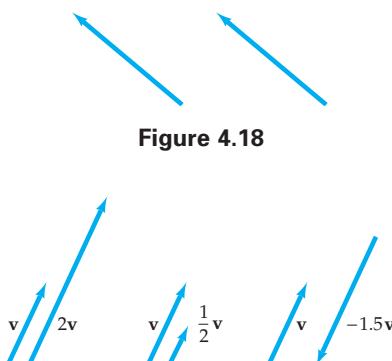


Figure 4.19

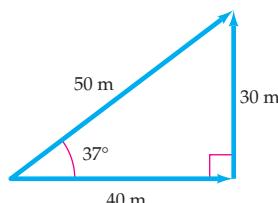


Figure 4.20

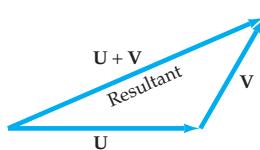


Figure 4.21

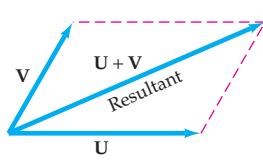


Figure 4.22

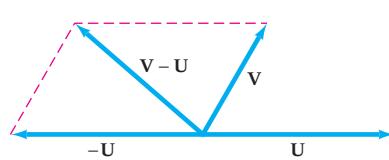


Figure 4.23

By introducing a coordinate plane, it is possible to develop an analytic approach to vectors. Recall from our discussion of equivalent vectors that a vector can be moved in the plane as long as *the magnitude and direction* are not changed.

With this in mind, consider  $\mathbf{AB}$ , whose initial point is  $A(2, -1)$  and whose terminal point is  $B(-3, 4)$ . If this vector is moved so that the initial point is at the origin  $O$ , the terminal point becomes  $P(-5, 5)$ , as shown in **Figure 4.24**. The vector  $\mathbf{OP}$  is equivalent to the vector  $\mathbf{AB}$ .

In **Figure 4.25**, let  $P_1(x_1, y_1)$  be the initial point of a vector and  $P_2(x_2, y_2)$  its terminal point. Then an equivalent vector  $\mathbf{OP}$  has its initial point at the origin and its terminal point at  $P(a, b)$ , where  $a = x_2 - x_1$  and  $b = y_2 - y_1$ . The vector  $\mathbf{OP}$  can be denoted by  $\mathbf{v} = \langle a, b \rangle$ ;  $a$  and  $b$  are called the **components** of the vector.

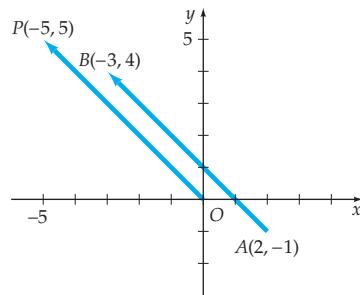


Figure 4.24

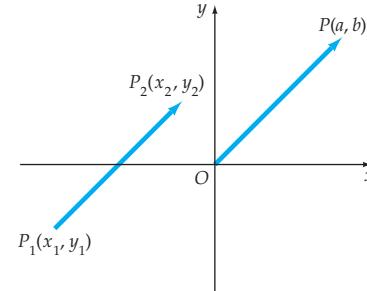


Figure 4.25

### EXAMPLE 1 » Find the Components of a Vector

Find the components of the vector  $\mathbf{AB}$  whose initial point is  $A(2, -1)$  and whose terminal point is  $B(-2, 6)$ . Determine a vector  $\mathbf{v}$  that is equivalent to  $\mathbf{AB}$  and has its initial point at the origin.

#### ALGEBRAIC SOLUTION

The components of  $\mathbf{AB}$  are  $\langle a, b \rangle$ , where

$$a = x_2 - x_1 = -2 - 2 = -4 \quad \text{and} \quad b = y_2 - y_1 = 6 - (-1) = 7$$

Thus  $\mathbf{v} = \langle -4, 7 \rangle$ .

#### VISUALIZE THE SOLUTION

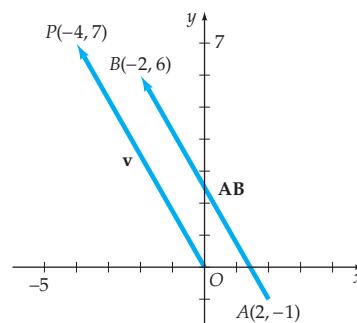


Figure 4.26

### » Try Exercise 10, page 325

The magnitude and direction of a vector can be found from its components. For instance, the terminal point of vector  $\mathbf{v}$  sketched in **Figure 4.26** is the ordered pair  $(-4, 7)$ . Applying the Pythagorean Theorem, we find

$$\|\mathbf{v}\| = \sqrt{(-4)^2 + 7^2} = \sqrt{16 + 49} = \sqrt{65}$$

Let  $\theta$  be the angle made by the positive  $x$ -axis and  $\mathbf{v}$ . Let  $\alpha$  be the reference angle for  $\theta$ . Then

$$\tan \alpha = \left| \frac{b}{a} \right| = \left| \frac{7}{-4} \right| = \frac{7}{4}$$

$$\alpha = \tan^{-1} \frac{7}{4} \approx 60^\circ$$

$$\theta = 180^\circ - 60^\circ = 120^\circ$$

•  $\alpha$  is the reference angle.

•  $\theta$  is the angle made by the vector and the positive  $x$ -axis.

The magnitude of  $\mathbf{v}$  is  $\sqrt{65}$ , and its direction is  $120^\circ$  as measured from the positive  $x$ -axis. The angle between a vector and the positive  $x$ -axis is called the **direction angle** of the vector. Because  $\mathbf{AB}$  in **Figure 4.26** is equivalent to  $\mathbf{v}$ ,  $\|\mathbf{AB}\| = \sqrt{65}$  and the direction angle of  $\mathbf{AB}$  is also  $120^\circ$ .

Expressing vectors in terms of components provides a convenient method for performing operations on vectors.

### Definitions of Fundamental Vector Operations

If  $\mathbf{v} = \langle a, b \rangle$  and  $\mathbf{w} = \langle c, d \rangle$  are two vectors and  $k$  is a real number, then

1.  $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$
2.  $\mathbf{v} + \mathbf{w} = \langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$
3.  $k\mathbf{v} = k\langle a, b \rangle = \langle ka, kb \rangle$

In terms of components, the **zero vector**  $\mathbf{0} = \langle 0, 0 \rangle$ . The **additive inverse of a vector**  $\mathbf{v} = \langle a, b \rangle$  is given by  $-\mathbf{v} = \langle -a, -b \rangle$ .

### EXAMPLE 2 » Perform Operations on Vectors

Given  $\mathbf{v} = \langle -2, 3 \rangle$  and  $\mathbf{w} = \langle 4, -1 \rangle$ , find

- a.  $\|\mathbf{w}\|$     b.  $\mathbf{v} + \mathbf{w}$     c.  $-3\mathbf{v}$     d.  $2\mathbf{v} - 3\mathbf{w}$

#### Solution

- |  |  |
|--|--|
| <p>a. <math>\ \mathbf{w}\  = \sqrt{4^2 + (-1)^2} = \sqrt{17}</math></p> <p>b. <math>\mathbf{v} + \mathbf{w} = \langle -2, 3 \rangle + \langle 4, -1 \rangle</math><br/> <math>= \langle -2 + 4, 3 + (-1) \rangle</math><br/> <math>= \langle 2, 2 \rangle</math></p> | <p>c. <math>-3\mathbf{v} = -3\langle -2, 3 \rangle = \langle 6, -9 \rangle</math></p> <p>d. <math>2\mathbf{v} - 3\mathbf{w} = 2\langle -2, 3 \rangle - 3\langle 4, -1 \rangle</math><br/> <math>= \langle -4, 6 \rangle - \langle 12, -3 \rangle</math><br/> <math>= \langle -16, 9 \rangle</math></p> |
|--|--|

» Try Exercise 24, page 325

### Unit Vectors

A **unit vector** is a vector whose magnitude is 1. For example, the vector  $\mathbf{v} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$  is a unit vector because

$$\|\mathbf{v}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

Given any nonzero vector  $\mathbf{v}$ , we can obtain a unit vector in the direction of  $\mathbf{v}$  by dividing each component of  $\mathbf{v}$  by the magnitude of  $\mathbf{v}$ ,  $\|\mathbf{v}\|$ .

### EXAMPLE 3 » Find a Unit Vector

Find a unit vector  $\mathbf{u}$  in the direction of  $\mathbf{v} = \langle -4, 2 \rangle$ .

#### Solution

Find the magnitude of  $\mathbf{v}$ .

$$\|\mathbf{v}\| = \sqrt{(-4)^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$$

Divide each component of  $\mathbf{v}$  by  $\|\mathbf{v}\|$ .

$$\mathbf{u} = \left\langle \frac{-4}{2\sqrt{5}}, \frac{2}{2\sqrt{5}} \right\rangle = \left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = \left\langle \frac{-2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right\rangle.$$

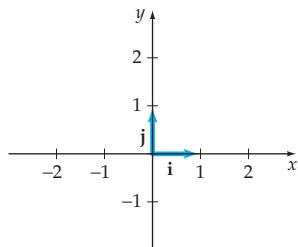
A unit vector in the direction of  $\mathbf{v}$  is  $\mathbf{u}$ .

### » Try Exercise 12, page 325

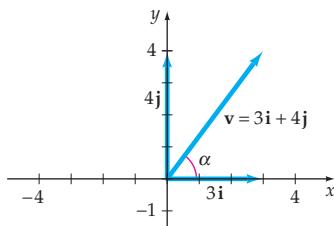
Two unit vectors, one parallel to the  $x$ -axis and one parallel to the  $y$ -axis, are of special importance. See **Figure 4.27**.

#### Definitions of Unit Vectors $\mathbf{i}$ and $\mathbf{j}$

$$\mathbf{i} = \langle 1, 0 \rangle \quad \mathbf{j} = \langle 0, 1 \rangle$$



**Figure 4.27**



**Figure 4.28**

The vector  $\mathbf{v} = \langle 3, 4 \rangle$  can be written in terms of the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$  as shown in **Figure 4.28**.

$$\begin{aligned} \langle 3, 4 \rangle &= \langle 3, 0 \rangle + \langle 0, 4 \rangle && \bullet \text{Definition of vector addition} \\ &= 3\langle 1, 0 \rangle + 4\langle 0, 1 \rangle && \bullet \text{Definition of scalar multiplication of a vector} \\ &= 3\mathbf{i} + 4\mathbf{j} && \bullet \text{Definition of } \mathbf{i} \text{ and } \mathbf{j} \end{aligned}$$

By means of scalar multiplication and addition of vectors, any vector can be expressed in terms of the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ . Let  $\mathbf{v} = \langle a_1, a_2 \rangle$ . Then

$$\mathbf{v} = \langle a_1, a_2 \rangle = a_1\langle 1, 0 \rangle + a_2\langle 0, 1 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

This gives the following result.

#### Representation of a Vector in Terms of $\mathbf{i}$ and $\mathbf{j}$

If  $\mathbf{v}$  is a vector and  $\mathbf{v} = \langle a_1, a_2 \rangle$ , then  $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$ .

The definitions for addition and scalar multiplication of vectors can be restated in terms of  $\mathbf{i}$  and  $\mathbf{j}$ . If  $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$  and  $\mathbf{w} = b_1\mathbf{i} + b_2\mathbf{j}$ , then

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (a_1\mathbf{i} + a_2\mathbf{j}) + (b_1\mathbf{i} + b_2\mathbf{j}) = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} \\ k\mathbf{v} &= k(a_1\mathbf{i} + a_2\mathbf{j}) = ka_1\mathbf{i} + ka_2\mathbf{j} \end{aligned}$$

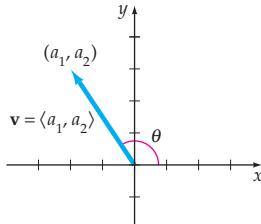
**EXAMPLE 4** » Operate on Vectors Written in Terms of  $\mathbf{i}$  and  $\mathbf{j}$ 

Given  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$  and  $\mathbf{w} = 5\mathbf{i} + 3\mathbf{j}$ , find  $3\mathbf{v} - 2\mathbf{w}$ .

**Solution**

$$\begin{aligned}3\mathbf{v} - 2\mathbf{w} &= 3(3\mathbf{i} - 4\mathbf{j}) - 2(5\mathbf{i} + 3\mathbf{j}) \\&= (9\mathbf{i} - 12\mathbf{j}) - (10\mathbf{i} + 6\mathbf{j}) \\&= (9 - 10)\mathbf{i} + (-12 - 6)\mathbf{j} \\&= -\mathbf{i} - 18\mathbf{j}\end{aligned}$$

## » Try Exercise 30, page 325



**Figure 4.29**

The components  $a_1$  and  $a_2$  of the vector  $\mathbf{v} = \langle a_1, a_2 \rangle$  can be expressed in terms of the magnitude of  $\mathbf{v}$  and the direction angle of  $\mathbf{v}$  (the angle that  $\mathbf{v}$  makes with the positive  $x$ -axis). Consider the vector  $\mathbf{v}$  in **Figure 4.29**. Then

$$\|\mathbf{v}\| = \sqrt{(a_1)^2 + (a_2)^2}$$

From the definitions of sine and cosine, we have

$$\cos \theta = \frac{a_1}{\|\mathbf{v}\|} \quad \text{and} \quad \sin \theta = \frac{a_2}{\|\mathbf{v}\|}$$

Rewriting the last two equations, we find that the components of  $\mathbf{v}$  are

$$a_1 = \|\mathbf{v}\| \cos \theta \quad \text{and} \quad a_2 = \|\mathbf{v}\| \sin \theta$$

**Definitions of Horizontal and Vertical Components of a Vector**

Let  $\mathbf{v} = \langle a_1, a_2 \rangle$ , where  $\mathbf{v} \neq \mathbf{0}$ , the zero vector. Then

$$a_1 = \|\mathbf{v}\| \cos \theta \quad \text{and} \quad a_2 = \|\mathbf{v}\| \sin \theta$$

where  $\theta$  is the angle between the positive  $x$ -axis and  $\mathbf{v}$ .

The **horizontal component** of  $\mathbf{v}$  is  $\|\mathbf{v}\| \cos \theta$ . The **vertical component** of  $\mathbf{v}$  is  $\|\mathbf{v}\| \sin \theta$ .

?

**QUESTION** Is  $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$  a unit vector?

Any nonzero vector can be written in terms of its horizontal and vertical components. Let  $\mathbf{v} = a_1 \mathbf{i} + a_2 \mathbf{j}$ . Then

$$\begin{aligned}\mathbf{v} &= a_1 \mathbf{i} + a_2 \mathbf{j} \\&= (\|\mathbf{v}\| \cos \theta) \mathbf{i} + (\|\mathbf{v}\| \sin \theta) \mathbf{j} \\&= \|\mathbf{v}\|(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})\end{aligned}$$

where  $\|\mathbf{v}\|$  is the magnitude of  $\mathbf{v}$  and the vector  $\cos \theta \mathbf{i} + \sin \theta \mathbf{j}$  is a unit vector. The last equation shows that any vector  $\mathbf{v}$  can be written as the product of its magnitude and a unit vector in the direction of  $\mathbf{v}$ .

?

**ANSWER** Yes, because  $\|\cos \theta \mathbf{i} + \sin \theta \mathbf{j}\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1$ .

**EXAMPLE 5** » Find the Horizontal and Vertical Components of a Vector

Find, to the nearest tenth, the horizontal and vertical components of a vector  $\mathbf{v}$  of magnitude 10 meters with direction angle  $228^\circ$ . Write the vector in the form  $a_1\mathbf{i} + a_2\mathbf{j}$ .

**Solution**

$$a_1 = 10 \cos 228^\circ \approx -6.7$$

$$a_2 = 10 \sin 228^\circ \approx -7.4$$

The approximate horizontal and vertical components are  $-6.7$  and  $-7.4$ , respectively.

$$\mathbf{v} \approx -6.7\mathbf{i} - 7.4\mathbf{j}$$

## » Try Exercise 40, page 325

**take note**

Ground speed is the magnitude of the resultant of the plane's velocity vector and the wind velocity vector.

**Applications of Vectors**

Vectors are used to solve applied problems in which forces are acting simultaneously on an object. For instance, an airplane flying in a wind is being acted upon by the force of its engine,  $\mathbf{F}_1$ , and the force of the wind,  $\mathbf{F}_2$ . The combined effect of these two forces is given by the resultant  $\mathbf{F}_1 + \mathbf{F}_2$ . In the following example, the **airspeed** of the plane is the speed at which the plane would be moving if there were no wind. The **actual velocity** of the plane is the plane's velocity with respect to the ground. The magnitude of the plane's actual velocity is called its **ground speed**.

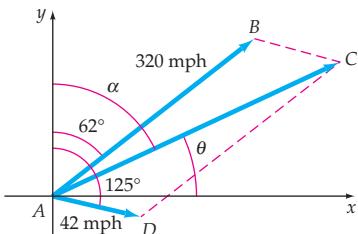
**EXAMPLE 6** » Solve an Application Involving Airspeed


Figure 4.30

An airplane is traveling with an airspeed of 320 mph and a heading of  $62^\circ$ . A wind of 42 mph is blowing at a heading of  $125^\circ$ . Find the ground speed and the course of the airplane.

**Solution**

Sketch a diagram similar to **Figure 4.30** showing the relevant vectors.  $\mathbf{AB}$  represents the heading and the airspeed,  $\mathbf{AD}$  represents the wind velocity, and  $\mathbf{AC}$  represents the course and the ground speed. By vector addition,  $\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$ . From the figure,

$$\mathbf{AB} = 320(\cos 28^\circ\mathbf{i} + \sin 28^\circ\mathbf{j})$$

$$\mathbf{AD} = 42[\cos(-35^\circ)\mathbf{i} + \sin(-35^\circ)\mathbf{j}]$$

$$\begin{aligned}\mathbf{AC} &= 320(\cos 28^\circ\mathbf{i} + \sin 28^\circ\mathbf{j}) + 42[\cos(-35^\circ)\mathbf{i} + \sin(-35^\circ)\mathbf{j}] \\ &\approx (282.5\mathbf{i} + 150.2\mathbf{j}) + (34.4\mathbf{i} - 24.1\mathbf{j}) \\ &= 316.9\mathbf{i} + 126.1\mathbf{j}\end{aligned}$$

$\mathbf{AC}$  is the course of the plane. The ground speed is  $\|\mathbf{AC}\|$ . The heading is  $\alpha = 90^\circ - \theta$ .

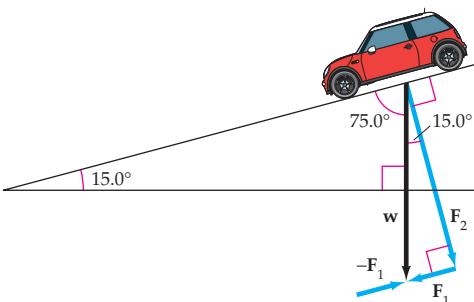
$$\|\mathbf{AC}\| = \sqrt{(316.9)^2 + (126.1)^2} \approx 340$$

$$\alpha = 90^\circ - \theta = 90^\circ - \tan^{-1}\left(\frac{126.1}{316.9}\right) \approx 68^\circ$$

The ground speed is approximately 340 mph at a heading of  $68^\circ$ .

## » Try Exercise 44, page 325

We can add two vectors to produce a resultant vector, and we can also find two vectors whose sum is a given vector. The process of finding two vectors whose sum is a given vector is called **resolving the vector**. Many applied problems can be analyzed by resolving a vector into two vectors that are *perpendicular* to each other. For instance, **Figure 4.31** shows a car on a ramp. The force of gravity on the car is shown by the downward vector  $w$ , which has been resolved into the vectors  $F_1$  and  $F_2$ . The vector  $F_1$  is parallel to the ramp and represents the force pushing the car down the ramp. The vector  $F_2$  is perpendicular to the ramp and represents the force the car exerts against the ramp. These two perpendicular forces  $F_1$  and  $F_2$  are **vector components** of  $w$ . That is,  $w = F_1 + F_2$ . The force needed to keep the car from rolling down the ramp is  $-F_1$ . The angle formed by  $w$  and  $F_2$  is complementary to the  $75.0^\circ$  angle. Thus the angle formed by  $w$  and  $F_2$  has a measure of  $15.0^\circ$ , which is the same as the angular measure of the incline of the ramp.



**Figure 4.31**

### EXAMPLE 7 » Solve an Application Involving Force

The car shown in **Figure 4.31** has a weight of 2855 pounds.

- Find the magnitude of the force needed to keep the car from rolling down the ramp. Round to the nearest pound.
- Find the magnitude of the force the car exerts against the ramp. Round to the nearest pound.

#### Solution

- The force needed to keep the car from rolling down the ramp is represented by  $-F_1$  in **Figure 4.31**. The magnitude of  $-F_1$  equals the magnitude of  $F_1$ . The triangle formed by  $w$ ,  $F_1$ , and  $F_2$  is a right triangle in which  $\sin 15.0^\circ = \frac{\|F_1\|}{\|w\|}$ . Use this equation to find  $\|F_1\|$ .

$$\sin 15.0^\circ = \frac{\|F_1\|}{\|w\|}$$

$$\sin 15.0^\circ = \frac{\|F_1\|}{2855} \quad \bullet \|w\| = 2855$$

$$\begin{aligned} \|F_1\| &= 2855 \sin 15.0^\circ \\ &\approx 739 \end{aligned} \quad \bullet \text{Solve for } \|F_1\|.$$

Approximately 739 pounds of force is needed to keep the car from rolling down the ramp.

- b. The vector  $\mathbf{F}_2$  in **Figure 4.31** is the force the car exerts against the ramp. The triangle formed by  $\mathbf{w}$ ,  $\mathbf{F}_1$ , and  $\mathbf{F}_2$  is a right triangle in which  $\cos 15.0^\circ = \frac{\|\mathbf{F}_2\|}{\|\mathbf{w}\|}$ . Use this equation to find  $\|\mathbf{F}_2\|$ .

$$\cos 15.0^\circ = \frac{\|\mathbf{F}_2\|}{\|\mathbf{w}\|}$$

$$\cos 15.0^\circ = \frac{\|\mathbf{F}_2\|}{2855}$$

$$\begin{aligned} \|\mathbf{F}_2\| &= 2855 \cos 15.0^\circ && \bullet \text{ Solve for } \|\mathbf{F}_2\|. \\ &\approx 2758 \end{aligned}$$

The magnitude of the force the car exerts against the ramp is approximately 2758 pounds.

» Try Exercise 48, Page 326

### Dot Product

We have considered the product of a real number (scalar) and a vector. We now turn our attention to the product of two vectors. Finding the *dot product* of two vectors is one way to multiply a vector by a vector. The dot product of two vectors is a real number and *not* a vector. The dot product is also called the *inner product* or the *scalar product*. This product is useful in engineering and physics.

#### Definition of Dot Product

Given  $\mathbf{v} = \langle a, b \rangle$  and  $\mathbf{w} = \langle c, d \rangle$ , the **dot product** of  $\mathbf{v}$  and  $\mathbf{w}$  is given by

$$\mathbf{v} \cdot \mathbf{w} = ac + bd$$

#### EXAMPLE 8 » Find the Dot Product of Two Vectors

Find the dot product of  $\mathbf{v} = \langle 6, -2 \rangle$  and  $\mathbf{w} = \langle -3, 4 \rangle$ .

#### Solution

$$\mathbf{v} \cdot \mathbf{w} = 6(-3) + (-2)4 = -18 - 8 = -26$$

» Try Exercise 60, page 326

**QUESTION** Is the dot product of two vectors a vector or a real number?

If the vectors in Example 8 were given in terms of the vectors  $\mathbf{i}$  and  $\mathbf{j}$ , then  $\mathbf{v} = 6\mathbf{i} - 2\mathbf{j}$  and  $\mathbf{w} = -3\mathbf{i} + 4\mathbf{j}$ . In this case,

$$\mathbf{v} \cdot \mathbf{w} = (6\mathbf{i} - 2\mathbf{j}) \cdot (-3\mathbf{i} + 4\mathbf{j}) = 6(-3) + (-2)4 = -26$$

---

**ANSWER** A real number

### Properties of the Dot Product

In the following properties,  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors and  $a$  is a scalar.

1.  $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
2.  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3.  $a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v})$
4.  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5.  $\mathbf{0} \cdot \mathbf{v} = 0$
6.  $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$
7.  $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$

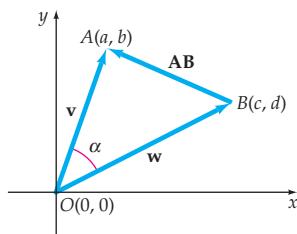
The proofs of these properties follow from the definition of dot product. Here is the proof of the fourth property. Let  $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ .

$$\mathbf{v} \cdot \mathbf{v} = (a\mathbf{i} + b\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j}) = a^2 + b^2 = \|\mathbf{v}\|^2$$

Rewriting the fourth property of the dot product yields an alternative way of expressing the magnitude of a vector.

### Magnitude of a Vector in Terms of the Dot Product

If  $\mathbf{v} = \langle a, b \rangle$ , then  $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ .



**Figure 4.32**

The Law of Cosines can be used to derive an alternative formula for the dot product. Consider the vectors  $\mathbf{v} = \langle a, b \rangle$  and  $\mathbf{w} = \langle c, d \rangle$  as shown in **Figure 4.32**. Using the Law of Cosines for triangle  $OAB$ , we have

$$\|\mathbf{AB}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\alpha$$

By the distance formula,  $\|\mathbf{AB}\|^2 = (a - c)^2 + (b - d)^2$ ,  $\|\mathbf{v}\|^2 = a^2 + b^2$ , and  $\|\mathbf{w}\|^2 = c^2 + d^2$ . Thus

$$\begin{aligned} (a - c)^2 + (b - d)^2 &= (a^2 + b^2) + (c^2 + d^2) - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\alpha \\ a^2 - 2ac + c^2 + b^2 - 2bd + d^2 &= a^2 + b^2 + c^2 + d^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\alpha \\ -2ac - 2bd &= -2\|\mathbf{v}\|\|\mathbf{w}\|\cos\alpha \\ ac + bd &= \|\mathbf{v}\|\|\mathbf{w}\|\cos\alpha \\ \mathbf{v} \cdot \mathbf{w} &= \|\mathbf{v}\|\|\mathbf{w}\|\cos\alpha \end{aligned} \quad \bullet \mathbf{v} \cdot \mathbf{w} = ac + bd$$

### Alternative Formula for the Dot Product

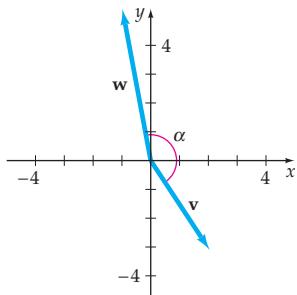
If  $\mathbf{v}$  and  $\mathbf{w}$  are two nonzero vectors and  $\alpha$  is the smallest nonnegative angle between  $\mathbf{v}$  and  $\mathbf{w}$ , then  $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\alpha$ .

Solving the alternative formula for the dot product for  $\cos\alpha$ , we have a formula for the cosine of the angle between two vectors.

### Angle Between Two Vectors

If  $\mathbf{v}$  and  $\mathbf{w}$  are two nonzero vectors and  $\alpha$  is the smallest nonnegative angle between  $\mathbf{v}$  and  $\mathbf{w}$ , then  $\cos \alpha = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$  and  $\alpha = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}\right)$ .

### EXAMPLE 9 » Find the Angle Between Two Vectors



**Figure 4.33**

Find, to the nearest tenth of a degree, the measure of the smallest nonnegative angle between the vectors  $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$  and  $\mathbf{w} = -\mathbf{i} + 5\mathbf{j}$ , as shown in **Figure 4.33**.

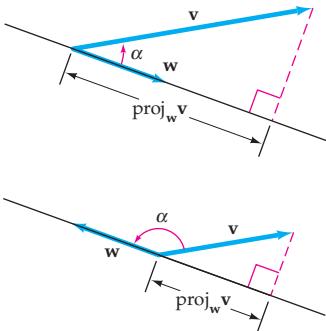
#### Solution

Use the equation for the angle between two vectors.

$$\begin{aligned}\cos \alpha &= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{(2\mathbf{i} - 3\mathbf{j}) \cdot (-\mathbf{i} + 5\mathbf{j})}{(\sqrt{2^2 + (-3)^2})(\sqrt{(-1)^2 + 5^2})} \\ &= \frac{-2 - 15}{\sqrt{13} \sqrt{26}} = \frac{-17}{\sqrt{338}} \\ \alpha &= \cos^{-1}\left(\frac{-17}{\sqrt{338}}\right) \approx 157.6^\circ\end{aligned}$$

The measure of the smallest nonnegative angle between the two vectors is approximately  $157.6^\circ$ .

### » Try Exercise 70, page 326



**Figure 4.34**

### Scalar Projection

Let  $\mathbf{v} = \langle a_1, a_2 \rangle$  and  $\mathbf{w} = \langle b_1, b_2 \rangle$  be two nonzero vectors, and let  $\alpha$  be the angle between the vectors. Two possible configurations, one for which  $\alpha$  is an acute angle and one for which  $\alpha$  is an obtuse angle, are shown in **Figure 4.34**. In each case, a right triangle is formed by drawing a line segment from the terminal point of  $\mathbf{v}$  to a line through  $\mathbf{w}$ .

#### Definition of the Scalar Projection of $\mathbf{v}$ onto $\mathbf{w}$

If  $\mathbf{v}$  and  $\mathbf{w}$  are two nonzero vectors and  $\alpha$  is the angle between  $\mathbf{v}$  and  $\mathbf{w}$ , then the scalar projection of  $\mathbf{v}$  onto  $\mathbf{w}$ ,  $\text{proj}_{\mathbf{w}} \mathbf{v}$ , is given by

$$\text{proj}_{\mathbf{w}} \mathbf{v} = \|\mathbf{v}\| \cos \alpha$$

To derive an alternate formula for  $\text{proj}_{\mathbf{w}} \mathbf{v}$ , consider the dot product  $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \alpha$ . Solving for  $\|\mathbf{v}\| \cos \alpha$ , which is  $\text{proj}_{\mathbf{w}} \mathbf{v}$ , we have

$$\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}$$

When the angle  $\alpha$  between the two vectors is an acute angle,  $\text{proj}_{\mathbf{w}} \mathbf{v}$  is positive. When  $\alpha$  is an obtuse angle,  $\text{proj}_{\mathbf{w}} \mathbf{v}$  is negative.

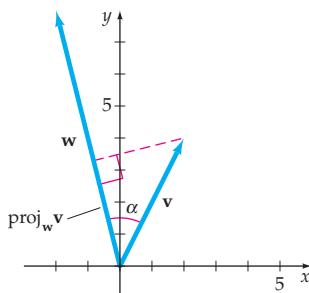


Figure 4.35

**EXAMPLE 10** » Find the Projection of  $\mathbf{v}$  onto  $\mathbf{w}$ 

Given  $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$  and  $\mathbf{w} = -2\mathbf{i} + 8\mathbf{j}$  as shown in **Figure 4.35**, find  $\text{proj}_{\mathbf{w}}\mathbf{v}$ .

**Solution**

Use the equation  $\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}$ .

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{(2\mathbf{i} + 4\mathbf{j}) \cdot (-2\mathbf{i} + 8\mathbf{j})}{\sqrt{(-2)^2 + 8^2}} = \frac{28}{\sqrt{68}} = \frac{14\sqrt{17}}{17} \approx 3.4$$

» Try Exercise 72, page 326

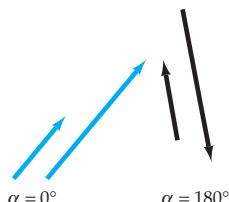


Figure 4.36

**Parallel and Perpendicular Vectors**

Two vectors are *parallel* when the angle  $\alpha$  between the vectors is  $0^\circ$  or  $180^\circ$ , as shown in **Figure 4.36**. When the angle  $\alpha$  is  $0^\circ$ , the vectors point in the same direction; the vectors point in opposite directions when  $\alpha$  is  $180^\circ$ .

Let  $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ , let  $c$  be a real number, and let  $\mathbf{w} = c\mathbf{v}$ . Because  $\mathbf{w}$  is a constant multiple of  $\mathbf{v}$ ,  $\mathbf{w}$  and  $\mathbf{v}$  are parallel vectors. When  $c > 0$ , the vectors point in the same direction. When  $c < 0$ , the vectors point in opposite directions.

Two vectors are *perpendicular* when the angle between the vectors is  $90^\circ$ . See **Figure 4.37**. Perpendicular vectors are referred to as **orthogonal vectors**. If  $\mathbf{v}$  and  $\mathbf{w}$  are two nonzero orthogonal vectors, then from the formula for the angle between two vectors and the fact that  $\cos \alpha = 0$ , we have

$$0 = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

If a fraction equals zero, the numerator must be zero. Thus, for orthogonal vectors  $\mathbf{v}$  and  $\mathbf{w}$ ,  $\mathbf{v} \cdot \mathbf{w} = 0$ . This gives the following result.

**Condition for Perpendicular Vectors**

Two nonzero vectors  $\mathbf{v}$  and  $\mathbf{w}$  are orthogonal if and only if  $\mathbf{v} \cdot \mathbf{w} = 0$ .

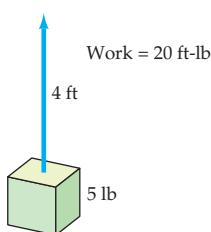
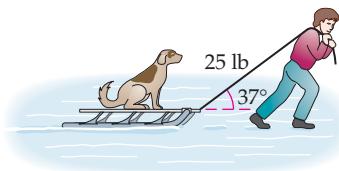


Figure 4.38

**Work: An Application of the Dot Product**

When a 5-pound force is used to lift a box from the ground a distance of 4 feet, *work* is done. The amount of *work* is the product of the force on the box and the distance the box is moved. In this case the work is 20 foot-pounds. When the box is lifted, the force and the displacement vector (the direction in which and the distance the box was moved) are in the same direction. (See **Figure 4.38**.)

Now consider a sled being pulled by a child along the ground by a rope attached to the sled, as shown in **Figure 4.39**. The force vector (along the rope) is *not* in the same direction as the displacement vector (parallel to the ground). In this case the dot product is used to determine the work done by the force.



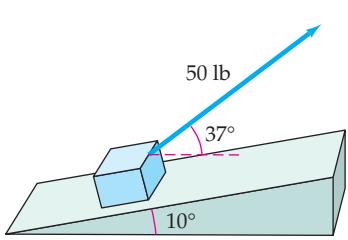
**Figure 4.39**

### Definition of Work

The work  $W$  done by a force  $\mathbf{F}$  applied along a displacement  $\mathbf{s}$  is

$$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha$$

where  $\alpha$  is the angle between  $\mathbf{F}$  and  $\mathbf{s}$ .



**Figure 4.40**

In the case of the child pulling the sled a horizontal distance of 7 feet, the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{s} \\ &= \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha && \bullet \alpha \text{ is the angle between } \mathbf{F} \text{ and } \mathbf{s}. \\ &= (25)(7) \cos 37^\circ \approx 140 \text{ foot-pounds} \end{aligned}$$

### EXAMPLE 11 » Solve a Work Problem

A force of 50 pounds on a rope is used to drag a box up a ramp that is inclined  $10^\circ$ . If the rope makes an angle of  $37^\circ$  with the ground, find the work done in moving the box 15 feet along the ramp. See **Figure 4.40**.

### Solution

In the formula  $W = \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha$ ,  $\alpha$  is the angle between the force and the displacement. Thus  $\alpha = 37^\circ - 10^\circ = 27^\circ$ . The work done is

$$W = \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha = 50 \cdot 15 \cdot \cos 27^\circ \approx 670 \text{ foot-pounds}$$

» Try Exercise 80, page 326



### Topics for Discussion

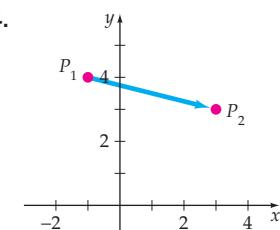
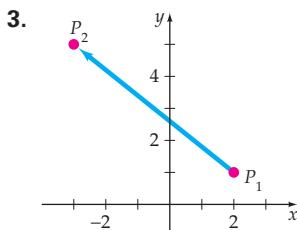
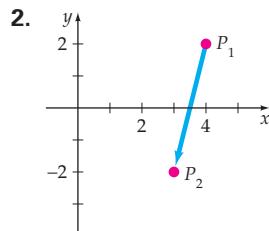
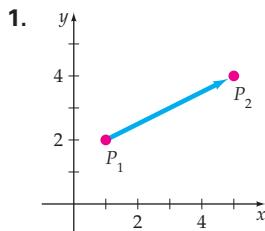
1. Is the dot product of two vectors a vector or a scalar? Explain.
2. Is the projection of  $\mathbf{v}$  onto  $\mathbf{w}$  a vector or a scalar? Explain.
3. Is the nonzero vector  $\langle a, b \rangle$  perpendicular to the vector  $\langle -b, a \rangle$ ? Explain.
4. Consider the nonzero vector  $\mathbf{u} = \langle a, b \rangle$  and the vector

$$\mathbf{v} = \left\langle \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right\rangle$$

- a. Are the vectors parallel? Explain.
- b. Which one of the vectors is a unit vector?
- c. Which vector has the larger magnitude? Explain.

## Exercise Set 4.3

In Exercises 1 to 10, find the components of the vector with the initial point  $P_1$  and terminal point  $P_2$ . Use these components to write a vector that is equivalent to  $\overrightarrow{P_1 P_2}$ .



5.  $P_1(-3, 0); P_2(4, -1)$

6.  $P_1(5, -1); P_2(3, 1)$

7.  $P_1(4, 2); P_2(-3, -3)$

8.  $P_1(0, -3); P_2(0, 4)$

9.  $P_1(2, -5); P_2(2, 3)$

10.  $P_1(3, -2); P_2(3, 0)$

In Exercises 11 to 18, find the magnitude and direction of each vector. Find the unit vector in the direction of the given vector.

11.  $\mathbf{v} = \langle -3, 4 \rangle$

12.  $\mathbf{v} = \langle 6, 10 \rangle$

13.  $\mathbf{v} = \langle 20, -40 \rangle$

14.  $\mathbf{v} = \langle -50, 30 \rangle$

15.  $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

16.  $\mathbf{v} = -5\mathbf{i} + 6\mathbf{j}$

17.  $\mathbf{v} = 42\mathbf{i} - 18\mathbf{j}$

18.  $\mathbf{v} = -22\mathbf{i} - 32\mathbf{j}$

In Exercises 19 to 27, perform the indicated operations, where  $\mathbf{u} = \langle -2, 4 \rangle$  and  $\mathbf{v} = \langle -3, -2 \rangle$ .

19.  $3\mathbf{u}$

20.  $-4\mathbf{v}$

21.  $2\mathbf{u} - \mathbf{v}$

22.  $4\mathbf{v} - 2\mathbf{u}$

23.  $\frac{2}{3}\mathbf{u} + \frac{1}{6}\mathbf{v}$

24.  $\frac{3}{4}\mathbf{u} - 2\mathbf{v}$

25.  $\|\mathbf{u}\|$

26.  $\|\mathbf{v} + 2\mathbf{u}\|$

27.  $\|3\mathbf{u} - 4\mathbf{v}\|$

In Exercises 28 to 36, perform the indicated operations, where  $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$  and  $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$ .

28.  $-2\mathbf{u}$

29.  $4\mathbf{v}$

30.  $3\mathbf{u} + 2\mathbf{v}$

31.  $6\mathbf{u} + 2\mathbf{v}$

32.  $\frac{1}{2}\mathbf{u} - \frac{3}{4}\mathbf{v}$

33.  $\frac{2}{3}\mathbf{v} + \frac{3}{4}\mathbf{u}$

34.  $\|\mathbf{v}\|$

35.  $\|\mathbf{u} - 2\mathbf{v}\|$

36.  $\|2\mathbf{v} + 3\mathbf{u}\|$

In Exercises 37 to 40, find the horizontal and vertical components of each vector. Round to the nearest tenth. Write an equivalent vector in the form  $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$ .

37. Magnitude = 5, direction angle =  $27^\circ$

38. Magnitude = 4, direction angle =  $127^\circ$

39. Magnitude = 4, direction angle =  $\frac{\pi}{4}$

40. Magnitude = 2, direction angle =  $\frac{8\pi}{7}$

41. **GROUND SPEED OF A PLANE** A plane is flying at an air-speed of 340 mph at a heading of  $124^\circ$ . A wind of 45 mph is blowing from the west. Find the ground speed of the plane.

42. **HEADING OF A BOAT** A person who can row 2.6 mph in still water wants to row due east across a river. The river is flowing from the north at a rate of 0.8 mph. Determine the heading of the boat required for the boat to travel due east across the river.

43. **GROUND SPEED AND COURSE OF A PLANE** A pilot is flying at a heading of  $96^\circ$  at 225 mph. A 50-mph wind is blowing from the southwest at a heading of  $37^\circ$ . Find the ground speed and course of the plane.

44. **COURSE OF A BOAT** The captain of a boat is steering at a heading of  $327^\circ$  at 18 mph. The current is flowing at 4 mph at a heading of  $60^\circ$ . Find the course (to the nearest degree) of the boat.

45. **MAGNITUDE OF A FORCE** Find the magnitude of the force necessary to keep a 3000-pound car from sliding down a ramp inclined at an angle of  $5.6^\circ$ . Round to the nearest pound.

- 46. ANGLE OF A RAMP** A 120-pound force keeps an 800-pound object from sliding down an inclined ramp. Find the angle of the ramp. Round to the nearest tenth of a degree.

- 47. MAGNITUDE OF A FORCE** A 345-pound box is placed on a ramp that is inclined  $22.4^\circ$ .

- Find the magnitude of the force needed to keep the box from sliding down the ramp. Ignore the effects of friction. Round to the nearest pound.
- Find the magnitude of the force the box exerts against the ramp. Round to the nearest pound.

- 48. MAGNITUDE OF A FORCE** A motorcycle that weighs 811 pounds is placed on a ramp that is inclined  $31.8^\circ$ .

- Find the magnitude of the force needed to keep the motorcycle from rolling down the ramp. Round to the nearest pound.
- Find the magnitude of the force the motorcycle exerts against the ramp. Round to the nearest pound.

The forces  $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \dots, \mathbf{F}_n$  acting on an object are in **equilibrium** provided the resultant of all the forces is the zero vector:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n = \mathbf{0}$$

In Exercises 49 to 53, determine whether the given forces are in equilibrium. If the forces are not in equilibrium, determine an additional force that would bring the forces into equilibrium.

49.  $\mathbf{F}_1 = \langle 18.2, 13.1 \rangle, \mathbf{F}_2 = \langle -12.4, 3.8 \rangle, \mathbf{F}_3 = \langle -5.8, -16.9 \rangle$

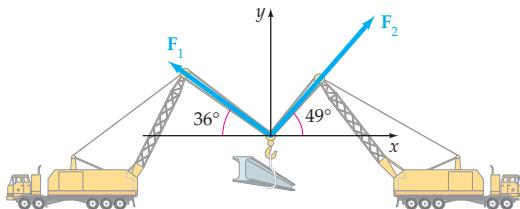
50.  $\mathbf{F}_1 = \langle -4.6, 5.3 \rangle, \mathbf{F}_2 = \langle 6.2, 4.9 \rangle, \mathbf{F}_3 = \langle -1.6, -10.2 \rangle$

51.  $\mathbf{F}_1 = 155\mathbf{i} - 257\mathbf{j}, \mathbf{F}_2 = -124\mathbf{i} + 149\mathbf{j}, \mathbf{F}_3 = -31\mathbf{i} + 98\mathbf{j}$

52.  $\mathbf{F}_1 = 23.5\mathbf{i} + 18.9\mathbf{j}, \mathbf{F}_2 = -18.7\mathbf{i} + 2.5\mathbf{j}, \mathbf{F}_3 = -5.6\mathbf{i} - 15.6\mathbf{j}$

53.  $\mathbf{F}_1 = 189.3\mathbf{i} + 235.7\mathbf{j}, \mathbf{F}_2 = 45.8\mathbf{i} - 205.6\mathbf{j}, \mathbf{F}_3 = -175.2\mathbf{i} - 37.7\mathbf{j}, \mathbf{F}_4 = -59.9\mathbf{i} + 7.6\mathbf{j}$

54. The cranes in the following figure are holding a 9450-pound steel girder in midair. The forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$ , along with the force  $\mathbf{F}_3 = \langle 0, -9450 \rangle$ , are in equilibrium as defined above Exercise 49. Find the magnitude of  $\mathbf{F}_2$ , given that  $\|\mathbf{F}_1\| = 6223$  pounds. Round to the nearest 10 pounds.



In Exercises 55 to 62, find the dot product of the vectors.

55.  $\mathbf{v} = \langle 3, -2 \rangle; \mathbf{w} = \langle 1, 3 \rangle$

56.  $\mathbf{v} = \langle 2, 4 \rangle; \mathbf{w} = \langle 0, 2 \rangle$

57.  $\mathbf{v} = \langle 4, 1 \rangle; \mathbf{w} = \langle -1, 4 \rangle$

58.  $\mathbf{v} = \langle 2, -3 \rangle; \mathbf{w} = \langle 3, 2 \rangle$

59.  $\mathbf{v} = \mathbf{i} + 2\mathbf{j}; \mathbf{w} = -\mathbf{i} + \mathbf{j}$

60.  $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}; \mathbf{w} = 4\mathbf{i} - 2\mathbf{j}$

61.  $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j}; \mathbf{w} = -2\mathbf{i} - 3\mathbf{j}$

62.  $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j}; \mathbf{w} = -2\mathbf{i} - 4\mathbf{j}$

In Exercises 63 to 70, find the measure of the angle between the two vectors. State which pairs of vectors are orthogonal. Round approximate measures to the nearest tenth of a degree.

63.  $\mathbf{v} = \langle 2, -1 \rangle; \mathbf{w} = \langle 3, 4 \rangle$

64.  $\mathbf{v} = \langle 1, -5 \rangle; \mathbf{w} = \langle -2, 3 \rangle$

65.  $\mathbf{v} = \langle 0, 3 \rangle; \mathbf{w} = \langle 2, 2 \rangle$

66.  $\mathbf{v} = \langle -1, 7 \rangle; \mathbf{w} = \langle 3, -2 \rangle$

67.  $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j}; \mathbf{w} = 2\mathbf{i} + 5\mathbf{j}$

68.  $\mathbf{v} = 8\mathbf{i} + \mathbf{j}; \mathbf{w} = -\mathbf{i} + 8\mathbf{j}$

69.  $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}; \mathbf{w} = -5\mathbf{i} - 2\mathbf{j}$

70.  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}; \mathbf{w} = 6\mathbf{i} - 12\mathbf{j}$

In Exercises 71 to 78, find  $\text{proj}_{\mathbf{w}} \mathbf{v}$ .

71.  $\mathbf{v} = \langle 6, 7 \rangle; \mathbf{w} = \langle 3, 4 \rangle$

72.  $\mathbf{v} = \langle -7, 5 \rangle; \mathbf{w} = \langle -4, 1 \rangle$

73.  $\mathbf{v} = \langle -3, 4 \rangle; \mathbf{w} = \langle 2, 5 \rangle$

74.  $\mathbf{v} = \langle 2, 4 \rangle; \mathbf{w} = \langle -1, 5 \rangle$

75.  $\mathbf{v} = 2\mathbf{i} + \mathbf{j}; \mathbf{w} = 6\mathbf{i} + 3\mathbf{j}$

76.  $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}; \mathbf{w} = -5\mathbf{i} - 2\mathbf{j}$

77.  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}; \mathbf{w} = -6\mathbf{i} + 12\mathbf{j}$

78.  $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j}; \mathbf{w} = -4\mathbf{i} - 2\mathbf{j}$

79. **WORK** A 150-pound box is dragged 15 feet along a level floor. Find the work done if a force of 75 pounds with a direction angle of  $32^\circ$  is used. Round to the nearest foot-pound.

80. **WORK** A 100-pound force is pulling a sled loaded with bricks that weighs 400 pounds. The force is at an angle of  $42^\circ$  with the displacement. Find the work done in moving the sled 25 feet.

81. **WORK** A rope is being used to pull a box up a ramp that is inclined at  $15^\circ$ . The rope exerts a force of 75 pounds on the box, and it makes an angle of  $30^\circ$  with the plane of the ramp. Find the work done in moving the box 12 feet.

82. **WORK** A dock worker exerts a force on a box sliding down the ramp of a truck. The ramp makes an angle of  $48^\circ$  with the road, and the worker exerts a 50-pound force parallel to the road. Find the work done in sliding the box 6 feet.

## »»» Connecting Concepts

- 83.** For  $\mathbf{u} = \langle -1, 1 \rangle$ ,  $\mathbf{v} = \langle 2, 3 \rangle$ , and  $\mathbf{w} = \langle 5, 5 \rangle$ , find the sum of the three vectors geometrically by using the triangle method of adding vectors.

**84.** For  $\mathbf{u} = \langle 1, 2 \rangle$ ,  $\mathbf{v} = \langle 3, -2 \rangle$ , and  $\mathbf{w} = \langle -1, 4 \rangle$ , find  $\mathbf{u} + \mathbf{v} - \mathbf{w}$  geometrically by using the triangle method of adding vectors.

**85.** Find a vector that has initial point  $(3, -1)$  and is equivalent to  $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ .

**86.** Find a vector that has initial point  $(-2, 4)$  and is equivalent to  $\mathbf{v} = \langle -1, 3 \rangle$ .

**87.**  If  $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$  and  $\mathbf{w} = 5\mathbf{i} + 2\mathbf{j}$  have the same initial point, is  $\mathbf{v}$  perpendicular to  $\mathbf{w}$ ? Why or why not?

**88.**  If  $\mathbf{v} = \langle 5, 6 \rangle$  and  $\mathbf{w} = \langle 6, 5 \rangle$  have the same initial point, is  $\mathbf{v}$  perpendicular to  $\mathbf{w}$ ? Why or why not?

**89.** Let  $\mathbf{v} = \langle -2, 7 \rangle$ . Find a vector perpendicular to  $\mathbf{v}$ .

**90.** Let  $\mathbf{w} = 4\mathbf{i} + \mathbf{j}$ . Find a vector perpendicular to  $\mathbf{w}$ .

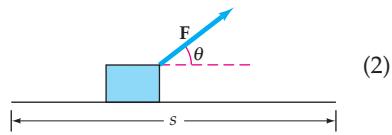
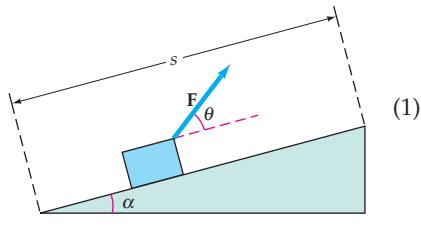
$$(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})?$$

- 92.** Prove that  $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ .

- 93.** Prove that  $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w}$ .

- 94.** Show that the dot product of two nonzero vectors is positive if the angle between the vectors is an acute angle and negative if the angle between the two vectors is an obtuse angle.

- 95. COMPARISON OF WORK DONE** Consider the following two situations. (1) A rope is being used to pull a box up a ramp inclined at an angle  $\alpha$ . The rope exerts a force  $F$  on the box, and the rope makes an angle  $\theta$  with the ramp. The box is pulled  $s$  feet. (2) A rope is being used to pull the same box along a level floor. The rope exerts the same force  $F$  on the box. The box is pulled the same  $s$  feet. In which case is more work done?



## »»» Projects

- 1. SAME DIRECTION OR OPPOSITE DIRECTIONS** Let  $\mathbf{v} = cw$ , where  $c$  is a nonzero real number and  $\mathbf{w}$  is a nonzero vector. Show that  $\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \pm 1$  and that the result is 1 when  $c > 0$  and  $-1$  when  $c < 0$ .

- ## 2. THE LAW OF COSINES AND VECTORS

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\mathbf{v} \cdot \mathbf{w}$$

- 3. PROJECTION RELATIONSHIPS** Determine the relationship between the nonzero vectors  $\mathbf{v}$  and  $\mathbf{w}$  if

- a.**  $\text{proj}_w v = 0$



## Exploring Concepts with Technology

### Optimal Branching of Arteries

The physiologist Jean Louis Poiseuille (1799–1869) developed several laws concerning the flow of blood. One of his laws states that the resistance  $R$  of a blood vessel of length  $l$  and radius  $r$  is given by

$$R = k \frac{l}{r^4} \quad (1)$$

The number  $k$  is a variation constant that depends on the viscosity of the blood.

**Figure 4.41** shows a large artery with radius  $r_1$  and a smaller artery with radius  $r_2$ . The branching angle between the arteries is  $\theta$ . Make use of Poiseuille's Law, Equation (1), to show that the resistance  $R$  of the blood along the path  $P_1P_2P_3$  is

$$R = k \left( \frac{a - b \cot \theta}{(r_1)^4} + \frac{b \csc \theta}{(r_2)^4} \right) \quad (2)$$

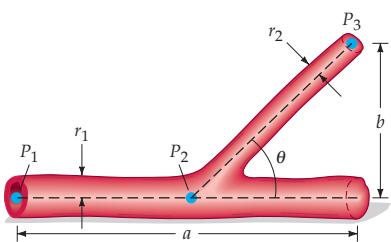


Figure 4.41

Use a graphing utility to graph  $R$  with  $k = 0.0563$ ,  $a = 8$  centimeters,  $b = 4$  centimeters,  $r_1 = 0.4$  centimeter, and  $r_2 = \frac{3}{4}r_1 = 0.3$  centimeter. Then estimate (to the nearest degree) the angle  $\theta$  that minimizes  $R$ . By using calculus, it can be demonstrated that  $R$  is minimized when

$$\cos \theta = \left( \frac{r_2}{r_1} \right)^4 \quad (3)$$

This equation is remarkable because it is much simpler than Equation (2) and because it does not involve the distance  $a$  or  $b$ . Solve Equation (3) for  $\theta$ , with  $r_2 = \frac{3}{4}r_1$ . How does this value of  $\theta$  compare with the value of  $\theta$  you obtained by graphing?

## Chapter 4 Summary

### 4.1 The Law of Sines

- The following Law of Sines is used to solve triangles when two angles and a side are given or when two sides and an angle opposite one of them are given.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

### 4.2 The Law of Cosines and Area

- The Law of Cosines,  $a^2 = b^2 + c^2 - 2bc \cos A$ , is used to solve general triangles when two sides and the included angle or three sides of the triangle are given.

- The area  $K$  of triangle  $ABC$  is

$$K = \frac{1}{2}bc \sin A = \frac{b^2 \sin C \sin A}{2 \sin B}$$

- The area of a triangle for which three sides are given (Heron's formula) is

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c)$$

### 4.3 Vectors

- A vector is a directed line segment. The length of the line segment is the magnitude of the vector, and the direction of

the vector is measured by an angle. The angle between a vector and the positive  $x$ -axis is called the direction angle of the vector. Two vectors are equivalent if they have the same magnitude and the same direction. The resultant of two or more vectors is the sum of the vectors.

- Vectors can be added by the parallelogram method, the triangle method, or addition of the  $x$ - and  $y$ -components.

- If  $\mathbf{v} = \langle a, b \rangle$  and  $k$  is a real number, then  $k\mathbf{v} = \langle ka, kb \rangle$ .
- The dot product of  $\mathbf{v} = \langle a, b \rangle$  and  $\mathbf{w} = \langle c, d \rangle$  is given by  

$$\mathbf{v} \cdot \mathbf{w} = ac + bd$$
- If  $\mathbf{v}$  and  $\mathbf{w}$  are two nonzero vectors and  $\alpha$  is the smallest nonnegative angle between  $\mathbf{v}$  and  $\mathbf{w}$ , then  $\cos \alpha = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$ .

## Chapter 4 Assessing Concepts

1. What is an oblique triangle?
2. In triangle  $ABC$ ,  $a = 4.5$ ,  $b = 6.2$ , and  $C = 107^\circ$ . Which law, the Law of Sines or the Law of Cosines, can be used to find  $c$ ?
3. Which of the following cases, ASA, AAS, SSA, SSS, or SAS is known as the ambiguous case of the Law of Sines?
4. In Heron's formula, what does the variable  $s$  represent?
5. Is the dot product of two vectors a vector or a scalar?
6. Let  $\mathbf{v}$  and  $\mathbf{w}$  be nonzero vectors. Is  $\text{proj}_{\mathbf{w}} \mathbf{v}$  a vector or a scalar?
7. True or false: The vector  $\left\langle \frac{12}{13}, -\frac{5}{13} \right\rangle$  is a unit vector.
8. True or false:  $\mathbf{i} \cdot \mathbf{j} = 0$ .
9. True or false: The Law of Sines can be used to solve any triangle, given two angles and any side.
10. True or false: If two nonzero vectors are orthogonal, then their dot product is 0.

## Chapter 4 Review Exercises

**In Exercises 1 to 10, solve each triangle.**

1.  $A = 37^\circ$ ,  $b = 14$ ,  $C = 92^\circ$
2.  $B = 77.4^\circ$ ,  $c = 11.8$ ,  $C = 94.0^\circ$
3.  $a = 12$ ,  $b = 15$ ,  $c = 20$
4.  $a = 24$ ,  $b = 32$ ,  $c = 28$
5.  $a = 18$ ,  $b = 22$ ,  $C = 35^\circ$
6.  $b = 102$ ,  $c = 150$ ,  $A = 82^\circ$
7.  $A = 105^\circ$ ,  $a = 8$ ,  $c = 10$
8.  $C = 55^\circ$ ,  $c = 80$ ,  $b = 110$
9.  $A = 55^\circ$ ,  $B = 80^\circ$ ,  $c = 25$
10.  $B = 25^\circ$ ,  $C = 40^\circ$ ,  $c = 40$

**In Exercises 11 to 18, find the area of each triangle. Round each area accurate to two significant digits.**

11.  $a = 24$ ,  $b = 30$ ,  $c = 36$
12.  $a = 9.0$ ,  $b = 7.0$ ,  $c = 12$
13.  $a = 60$ ,  $b = 44$ ,  $C = 44^\circ$
14.  $b = 8.0$ ,  $c = 12$ ,  $A = 75^\circ$
15.  $b = 50$ ,  $c = 75$ ,  $C = 15^\circ$
16.  $b = 18$ ,  $a = 25$ ,  $A = 68^\circ$
17.  $A = 110^\circ$ ,  $a = 32$ ,  $b = 15$
18.  $A = 45^\circ$ ,  $c = 22$ ,  $b = 18$

**In Exercises 19 and 20, find the components of each vector with the given initial and terminal points. Write an equivalent vector in terms of its components.**

19.  $P_1(-2, 4); P_2(3, 7)$
20.  $P_1(-4, 0); P_2(-3, 6)$

**In Exercises 21 to 24, find the magnitude and direction angle of each vector.**

21.  $\mathbf{v} = \langle -4, 2 \rangle$
22.  $\mathbf{v} = \langle 6, -3 \rangle$
23.  $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$
24.  $\mathbf{u} = -4\mathbf{i} - 7\mathbf{j}$

In Exercises 25 to 28, find a unit vector in the direction of the given vector.

25.  $\mathbf{w} = \langle -8, 5 \rangle$

26.  $\mathbf{w} = \langle 7, -12 \rangle$

27.  $\mathbf{v} = 5\mathbf{i} + \mathbf{j}$

28.  $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}$

In Exercises 29 and 30, perform the indicated operation, where  $\mathbf{u} = \langle 3, 2 \rangle$  and  $\mathbf{v} = \langle -4, -1 \rangle$ .

29.  $\mathbf{v} - \mathbf{u}$

30.  $2\mathbf{u} - 3\mathbf{v}$

In Exercises 31 and 32, perform the indicated operation, where  $\mathbf{u} = 10\mathbf{i} + 6\mathbf{j}$  and  $\mathbf{v} = 8\mathbf{i} - 5\mathbf{j}$ .

31.  $-\mathbf{u} + \frac{1}{2}\mathbf{v}$

32.  $\frac{2}{3}\mathbf{v} - \frac{3}{4}\mathbf{u}$

**33. GROUND SPEED OF A PLANE** A plane is flying at an air-speed of 400 mph at a heading of  $204^\circ$ . A wind of 45 mph is blowing from the east. Find the ground speed of the plane.

**34. ANGLE OF A RAMP** A 40-pound force keeps a 320-pound object from sliding down an inclined ramp. Find the angle of the ramp.

In Exercises 35 to 38, find the dot product of the vectors.

35.  $\mathbf{u} = \langle 3, 7 \rangle; \mathbf{v} = \langle -1, 3 \rangle$

36.  $\mathbf{v} = \langle -8, 5 \rangle; \mathbf{u} = \langle 2, -1 \rangle$

37.  $\mathbf{v} = -4\mathbf{i} - \mathbf{j}; \mathbf{u} = 2\mathbf{i} + \mathbf{j}$

38.  $\mathbf{u} = -3\mathbf{i} + 7\mathbf{j}; \mathbf{v} = -2\mathbf{i} + 2\mathbf{j}$

In Exercises 39 to 42, find the angle between the vectors. Round to the nearest degree.

39.  $\mathbf{u} = \langle 7, -4 \rangle; \mathbf{v} = \langle 2, 3 \rangle$

40.  $\mathbf{v} = \langle -5, 2 \rangle; \mathbf{u} = \langle 2, -4 \rangle$

41.  $\mathbf{v} = 6\mathbf{i} - 11\mathbf{j}; \mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$

42.  $\mathbf{u} = \mathbf{i} - 5\mathbf{j}; \mathbf{v} = \mathbf{i} + 5\mathbf{j}$

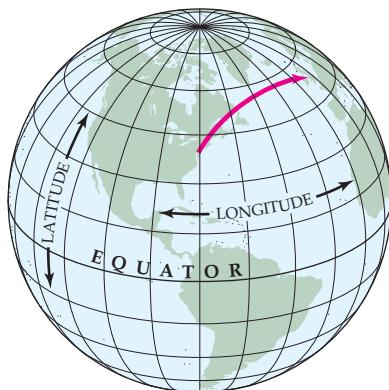
In Exercises 43 and 44, find  $\text{proj}_{\mathbf{w}} \mathbf{v}$ .

43.  $\mathbf{v} = \langle -2, 5 \rangle; \mathbf{w} = \langle 5, 4 \rangle$

44.  $\mathbf{v} = 4\mathbf{i} - 7\mathbf{j}; \mathbf{w} = -2\mathbf{i} - 5\mathbf{j}$

**45. WORK** A 120-pound box is dragged 14 feet along a level floor. Find the work done if a force of 60 pounds with a direction angle of  $38^\circ$  is used. Round to the nearest foot-pound.

## »»» Quantitative Reasoning: *Trigonometry and Great Circle Routes* »»»



Great circle route from New York's John F. Kennedy airport to London's Heathrow airport.

A **great circle** is a circle on a sphere's surface whose center is the same as the center of the sphere. A great circle is a path on the sphere with the smallest curvature, and hence an arc of a great circle is the shortest path between two given points on the sphere. The distance between any two points on a sphere is called the **great circle distance** for the given points.

Great circle routes are often used by pilots when winds are not a significant factor. A formula that is used to estimate the great circle distance  $d$ , measured in radians, between two airports is given by

$$d = \cos^{-1}[\cos(\text{lat1}) \cos(\text{lat2}) \cos(\text{lon1} - \text{lon2}) + \sin(\text{lat1}) \sin(\text{lat2})] \quad (1)$$

where  $\text{lat1}$  is the latitude of airport 1,  $\text{lon1}$  is the longitude of airport 1,  $\text{lat2}$  is the latitude of airport 2, and  $\text{lon2}$  is the longitude of airport 2.

For example, the coordinates of New York's John F. Kennedy airport (JFK) are  $40^\circ 39' \text{ N}$  and  $73^\circ 47' \text{ W}$ . The coordinates of London's Heathrow airport (LHR) are  $51^\circ 29' \text{ N}$  and  $0^\circ 27' \text{ W}$ . Converting each of these coordinates to radians gives

$$(\text{lat1}, \text{lon1}) \approx (0.709476, -1.287756)$$

$$(\text{lat2}, \text{lon2}) \approx (0.898554, -0.007854)$$

Note: North latitudes and east longitudes are represented by positive radian values, whereas south latitudes and west longitudes are represented by negative radian values. Substitute the radian values into Formula (1) to produce  $d \approx 0.869496$  radian. Because the earth is nearly spherical with a radius of about 3960 miles, we can convert  $d$  to miles by multiplying by 3960.

$$d \approx 0.869496 \times 3960 \approx 3443$$

Rounded to the nearest 10 miles, the great circle distance from JFK to LHR is approximately 3440 miles.

To fly a great circle route, the pilot must adjust his or her heading several times during the flight. The following formulas can be used to find the *initial heading* ( $h_1$ ) a pilot should use to fly a great circle route from airport 1 to airport 2.

If  $\sin(\text{lon}_2 - \text{lon}_1) > 0$ , then

$$h_1 = \cos^{-1} \left[ \frac{\sin(\text{lat}_2) - \sin(\text{lat}_1) \cos(d)}{\sin(d) \cos(\text{lat}_1)} \right] \quad (2)$$

Otherwise,

$$h_1 = 2\pi - \cos^{-1} \left[ \frac{\sin(\text{lat}_2) - \sin(\text{lat}_1) \cos(d)}{\sin(d) \cos(\text{lat}_1)} \right] \quad (3)$$

In Formulas (2) and (3), each of the values  $\text{lat}_1$ ,  $\text{lat}_2$ , and  $d$  must be in radians. Also, these formulas are not valid if airport 1 is located at the North or South Pole.

To find the initial heading required to fly the great circle route from JFK to LHR, we first note that

$$\sin(\text{lon}_2 - \text{lon}_1) = \sin[-0.007854 - (-1.287756)] \approx 0.957988$$

Because  $\sin(\text{lon}_2 - \text{lon}_1) > 0$ , we use Formula (2) to find the initial heading. Substitute 0.709476 for  $\text{lat}_1$ , 0.898554 for  $\text{lat}_2$ , and 0.869496 for  $d$  to produce  $h_1 \approx 0.896040$  radian. Converting to degrees gives  $51^\circ$  as the initial heading from JFK to LHR, to the nearest degree.

- QR1.** Find the great circle distance, in miles, between Orlando International airport (KMCO) (latitude = 0.496187 radian, longitude =  $-1.419110$  radians) and Los Angeles International airport (LAX) (latitude = 0.592409 radian, longitude =  $-2.066611$  radians). Use 3960 miles as the radius of Earth. Round to the nearest 10 miles.
- QR2.** Find the initial heading needed to fly the great circle route from KMCO to LAX. Round to the nearest degree.
- QR3.** Find the great circle distance, in miles, between JFK and Denver International airport (KDEN) (latitude = 0.695717 radian, longitude =  $-1.826892$  radians). Use 3960 miles as the radius of Earth. Round to the nearest 10 miles.
- QR4.** Find the initial heading required to fly the great circle route from KDEN to JFK. Round to the nearest degree.

 Chapter 4 Test

1. Solve triangle  $ABC$  if  $A = 70^\circ$ ,  $C = 16^\circ$ , and  $c = 14$ .
2. Find  $B$  in triangle  $ABC$  if  $A = 140^\circ$ ,  $b = 13$ , and  $a = 45$ .
3. In triangle  $ABC$ ,  $C = 42^\circ$ ,  $a = 20$ , and  $b = 12$ . Find side  $c$ .
4. In triangle  $ABC$ ,  $a = 32$ ,  $b = 24$ , and  $c = 18$ . Find angle  $B$ .

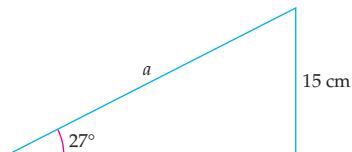
**In Exercises 5 to 7, round your answers to two significant digits.**

5. Given angle  $C = 110^\circ$ , side  $a = 7.0$ , and side  $b = 12$ , find the area of triangle  $ABC$ .
6. Given angle  $B = 42^\circ$ , angle  $C = 75^\circ$ , and side  $b = 12$ , find the area of triangle  $ABC$ .
7. Given side  $a = 17$ , side  $b = 55$ , and side  $c = 42$ , find the area of triangle  $ABC$ .
8. Given  $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$ , find  $\|\mathbf{v}\|$ .
9. A vector has a magnitude of 12 and direction  $220^\circ$ . Write an equivalent vector in the form  $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$ . Round  $a_1$  and  $a_2$  to four significant digits.

10. Find  $3\mathbf{u} - 5\mathbf{v}$  given the vectors  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$  and  $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$ .
11. Find the dot product of  $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$  and  $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}$ .
12. Find the smallest positive angle, to the nearest degree, between the vectors  $\mathbf{u} = \langle 3, 5 \rangle$  and  $\mathbf{v} = \langle -6, 2 \rangle$ .
13. One ship leaves a port at 1:00 P.M. traveling at 12 mph at a heading of  $65^\circ$ . At 2:00 P.M. another ship leaves the port traveling at 18 mph at a heading of  $142^\circ$ . Find the distance between the ships at 3:00 P.M.
14. Two fire lookouts are located 12 miles apart. Lookout  $A$  is at a bearing of  $N32^\circ W$  from lookout  $B$ . A fire was sighted at a bearing of  $S82^\circ E$  from  $A$  and  $N72^\circ E$  from  $B$ . Find the distance of the fire from lookout  $B$ .
15. A triangular commercial piece of real estate is priced at \$8.50 per square foot. Find the cost, to the nearest \$100, of the lot, which measures 112 feet by 165 feet by 140 feet.

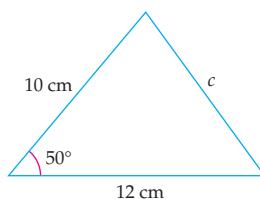
 Cumulative Review Exercises

1. Find the distance between  $P_1(-3, 4)$  and  $P_2(4, -1)$ .
2. Given  $f(x) = \cos x$  and  $g(x) = \sin x$ , find  $(f + g)(x)$ .
3. Given  $f(x) = \sec x$  and  $g(x) = \cos x$ , find  $(f \circ g)(x)$ .
4. Given  $f(x) = \frac{1}{2}x - 3$ , find  $f^{-1}(x)$ .
5. How is the graph of  $F(x) = f(x - 2) + 3$  related to the graph of  $y = f(x)$ ?
6. For the right triangle shown at the right, find  $a$ .



7. Graph  $y = 3 \sin \pi x$ .

8. Graph  $y = \frac{1}{4} \tan 2x$ .
9. Graph  $y = 2 \sin(\pi x) + 1$ .
10. Find the amplitude, period, and phase shift of the graph of  $y = 3 \sin\left(\frac{1}{3}x - \frac{\pi}{2}\right)$ .
11. Find the amplitude, period, and phase shift of the graph of  $y = \sin x + \cos x$ .
12. Find  $c$  for the triangle at the right.



13. Verify the identity  $\frac{1}{\cos x} - \cos x = \sin x \tan x$ .

14. Evaluate  $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ .

15. Evaluate  $\tan\left(\cos^{-1}\left(\frac{12}{13}\right)\right)$ .

16. Solve  $\sin x \tan x - \frac{1}{2} \tan x = 0$  for  $0 \leq x < 2\pi$ .

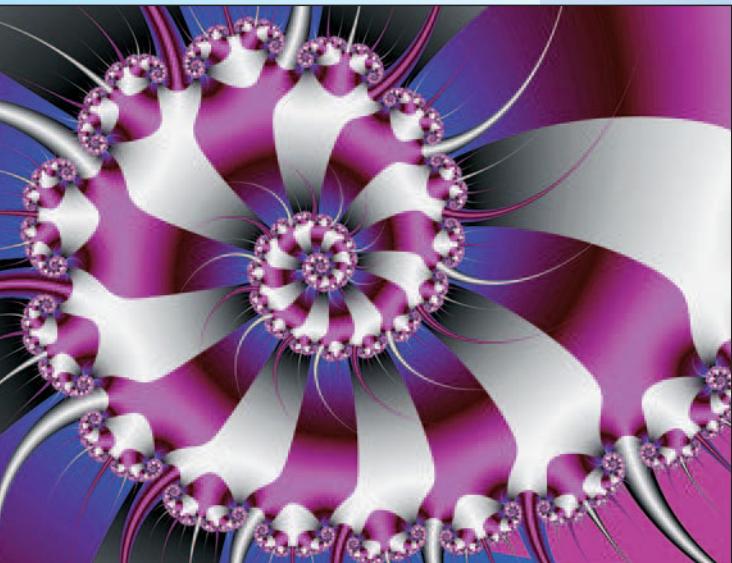
17. Find the magnitude and the positive direction angle for the vector  $\langle 4, -3 \rangle$ . Round the angle to the nearest tenth of a degree.

18. Find the angle between the vectors  $\mathbf{v} = \langle 1, 2 \rangle$  and  $\mathbf{w} = \langle -2, 3 \rangle$ . Round to the nearest tenth of a degree.

19. **HEADING OF A BOAT** A person who can row at 3 mph in still water wants to row due west across a river. This river is flowing north to south at a rate of 1 mph. Determine the heading of the boat that is required to travel due west across the river.

20. **GROUND SPEED AND COURSE OF A PLANE** An airplane is traveling with an airspeed of 515 mph at a heading of  $54.0^\circ$ . A wind of 150 mph is blowing at a heading of  $120.0^\circ$ . Find the ground speed and the course of the plane.

# 5 Complex Numbers



A fractal image produced on a computer. As you zoom in on a region, additional details of the fractal are displayed.



Fractal images can be produced using complex numbers and simple algebraic procedures.

## 5.1 Complex Numbers

## 5.2 Trigonometric Form of Complex Numbers

## 5.3 De Moivre's Theorem

### Fractals

A fractal is a geometric figure that reveals greater detail and complexity as it is magnified over and over. Traditional Euclidean figures, such as a circle, a parabola, and a hyperbola, appear simpler as they are magnified. For instance, if you use a computer to repeatedly zoom in on a region of one of these figures, the figure starts to appear more and more like a straight line. However, as you repeatedly zoom in on a region of a fractal, more and more complex details of the figure are displayed.

The word *fractal* was first used by the mathematician Benoit B. Mandelbrot (1924– ). Earlier mathematicians such as Cantor, Lebesgue, Julia, Koch, Hausdorff, Peano, Bolzano, and Sierpinski developed many concepts regarding fractals; however, these ideas were largely neglected until Mandelbrot publicized *The Fractal Geometry of Nature* in 1977.

Many fractals are generated using complex numbers and a feedback process. The process starts by substituting a complex number into a complex function to produce an output. The output is substituted back into the function and the process is repeated. The **Exploring Concepts with Technology** on page 356 and the **Quantitative Reasoning feature** on page 359 explain the details of how a fractal called the *Mandelbrot set* is generated and how you can use a graphing calculator to produce its graph.



### Online Study Center

For online student resources, such as section quizzes, visit this website:  
[college.hmco.com/info/aufmannCAT6e](http://college.hmco.com/info/aufmannCAT6e)

## Section 5.1

- Introduction to Complex Numbers
- Addition and Subtraction of Complex Numbers
- Multiplication of Complex Numbers
- Division of Complex Numbers
- Powers of  $i$

### Math Matters

It may seem strange to just invent new numbers, but that is how mathematics evolves. For instance, negative numbers were not an accepted part of mathematics until well into the thirteenth century. In fact, these numbers often were referred to as “fictitious numbers.”

In the seventeenth century, René Descartes called square roots of negative numbers “imaginary numbers,” an unfortunate choice of words, and started using the letter  $i$  to denote these numbers. These numbers were subjected to the same skepticism as negative numbers.

It is important to understand that these numbers are not *imaginary* in the dictionary sense of the word. This misleading word is similar to the situation of negative numbers being called *fictitious*.

If you think of a number line, then the numbers to the right of zero are positive numbers and the numbers to the left of zero are negative numbers. One way to think of an imaginary number is to visualize it as *up* or *down* from zero. See the Project on page 342 for more information on this topic.

# Complex Numbers

## Introduction to Complex Numbers

Recall that  $\sqrt{9} = 3$  because  $3^2 = 9$ . Now consider the expression  $\sqrt{-9}$ . To find  $\sqrt{-9}$ , we need to find a number  $c$  such that  $c^2 = -9$ . However, the square of any real number  $c$  (except zero) is a *positive* number. Consequently, we must expand our concept of number to include numbers whose squares are negative numbers.

Around the seventeenth century, a new number, called an *imaginary number*, was defined so that a negative number would have a square root. The letter  $i$  was chosen to represent the number whose square is  $-1$ .

### Definition of $i$

The **imaginary unit**, designated by the letter  $i$ , is the number such that  $i^2 = -1$ .

The principal square root of a negative number is defined in terms of  $i$ .

### Definition of an Imaginary Number

If  $a$  is a positive real number, then  $\sqrt{-a} = i\sqrt{a}$ . The number  $i\sqrt{a}$  is called an **imaginary number**.

#### Example

$$\begin{array}{ll} \sqrt{-36} = i\sqrt{36} = 6i & \sqrt{-18} = i\sqrt{18} = 3i\sqrt{2} \\ \sqrt{-23} = i\sqrt{23} & \sqrt{-1} = i\sqrt{1} = i \end{array}$$

It is customary to write  $i$  in front of a radical sign, as we did for  $i\sqrt{23}$ , to avoid confusing  $\sqrt{a}i$  with  $\sqrt{ai}$ .

### Definition of a Complex Number

A **complex number** is a number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$ . The number  $a$  is the **real part** of  $a + bi$ , and  $b$  is the **imaginary part**.

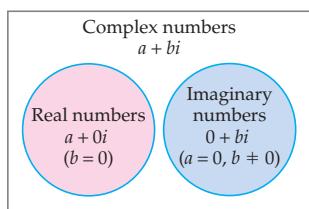
#### Example

|           |   |
|-----------|---|
| $-3 + 5i$ | • Real part: $-3$ ; imaginary part: $5$ |
| $2 - 6i$  | • Real part: $2$ ; imaginary part: $-6$ |
| $5$       | • Real part: $5$ ; imaginary part: $0$  |
| $7i$      | • Real part: $0$ ; imaginary part: $7$  |

Note from these examples that a real number is a complex number whose imaginary part is zero, and an imaginary number is a complex number whose real part is zero, and whose imaginary part is not zero.

**Math Matters**

The imaginary unit  $i$  is important in the field of electrical engineering. However, because the letter  $i$  is used by engineers as the symbol for electric current, these engineers use  $j$  for the complex unit.



**QUESTION** What are the real part and imaginary part of  $3 - 5i$ ?

Note from the diagram at the left that the set of real numbers is a subset of the complex numbers, and the set of imaginary numbers is a separate subset of the complex numbers. The set of real numbers and the set of imaginary numbers are disjoint sets.

Example 1 illustrates how to write a complex number in the **standard form**  $a + bi$ .

**EXAMPLE 1 » Write a Complex Number in Standard Form**

Write  $7 + \sqrt{-45}$  in the form  $a + bi$ .

**Solution**

$$\begin{aligned} 7 + \sqrt{-45} &= 7 + i\sqrt{45} \\ &= 7 + i\sqrt{9 \cdot 5} \\ &= 7 + 3i\sqrt{5} \end{aligned}$$

» Try Exercise 8, page 340

**■ Addition and Subtraction of Complex Numbers**

All the standard arithmetic operations that are applied to real numbers can be applied to complex numbers.

**Definition of Addition and Subtraction of Complex Numbers**

If  $a + bi$  and  $c + di$  are complex numbers, then

$$\text{Addition} \quad (a + bi) + (c + di) = (a + c) + (b + d)i$$

$$\text{Subtraction} \quad (a + bi) - (c + di) = (a - c) + (b - d)i$$

Basically, these definitions say that to add two complex numbers, add the real parts and add the imaginary parts. To subtract two complex numbers, subtract the real parts and subtract the imaginary parts.

**EXAMPLE 2 » Add or Subtract Complex Numbers**

Simplify.

a.  $(7 - 2i) + (-2 + 4i)$     b.  $(-9 + 4i) - (2 - 6i)$

**Solution**

a.  $(7 - 2i) + (-2 + 4i) = (7 + (-2)) + (-2 + 4)i = 5 + 2i$

b.  $(-9 + 4i) - (2 - 6i) = (-9 - 2) + (4 - (-6))i = -11 + 10i$

» Try Exercise 18, page 340

**ANSWER** Real part: 3; imaginary part:  $-5$

## Multiplication of Complex Numbers

When multiplying complex numbers, the term  $i^2$  is frequently a part of the product. Recall that  $i^2 = -1$ . Therefore,

$$\begin{aligned}3i(5i) &= 15i^2 = 15(-1) = \textcolor{blue}{-15} \\-2i(6i) &= -12i^2 = -12(-1) = \textcolor{blue}{12} \\4i(3 - 2i) &= 12i - 8i^2 = 12i - 8(-1) = \textcolor{blue}{8 + 12i}\end{aligned}$$

### take note

Recall that the definition of the product of radical expressions requires that the radicand be a positive number. Therefore, when multiplying expressions containing negative radicands, we first must rewrite the expression using  $i$  and a positive radicand.

When multiplying square roots of negative numbers, first rewrite the radical expressions using  $i$ . For instance,

$$\begin{aligned}\sqrt{-6} \cdot \sqrt{-24} &= i\sqrt{6} \cdot i\sqrt{24} & \bullet \sqrt{-6} = i\sqrt{6}, \sqrt{-24} = i\sqrt{24} \\&= i^2\sqrt{144} = -1 \cdot 12 \\&= \textcolor{blue}{-12}\end{aligned}$$

Note from this example that it would have been incorrect to multiply the radicands of the two radical expressions. To illustrate:

$$\sqrt{-6} \cdot \sqrt{-24} \neq \sqrt{(-6)(-24)}$$

**QUESTION** What is the product of  $\sqrt{-2}$  and  $\sqrt{-8}$ ?

To multiply two complex numbers, we use the following definition.

### Definition of Multiplication of Complex Numbers

If  $a + bi$  and  $c + di$  are complex numbers, then

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Because every complex number can be written as a sum of two terms, it is natural to perform multiplication on complex numbers in a manner consistent with the operation defined on binomials and the definition  $i^2 = -1$ . By using this analogy, you can multiply complex numbers without memorizing the definition.

### EXAMPLE 3 » Multiply Complex Numbers

Simplify. **a.**  $(3 - 4i)(2 + 5i)$     **b.**  $(2 + \sqrt{-3})(4 - 5\sqrt{-3})$

#### Solution

$$\begin{aligned}\textbf{a. } (3 - 4i)(2 + 5i) &= 6 + 15i - 8i - 20i^2 && \bullet \text{ Replace } i^2 \text{ by } -1. \\&= 6 + 15i - 8i - 20(-1) && \bullet \text{ Simplify.} \\&= 6 + 15i - 8i + 20 \\&= \textcolor{blue}{26 + 7i}\end{aligned}$$

Continued ►

**ANSWER**  $\sqrt{-2} \cdot \sqrt{-8} = i\sqrt{2} \cdot i\sqrt{8} = i^2\sqrt{16} = -1 \cdot 4 = -4$



## Integrating Technology

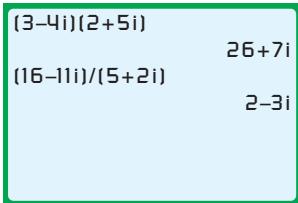
Some graphing calculators can be used to perform operations on complex numbers. Here are some typical screens for a TI-83/TI-83 Plus/TI-84 Plus graphing calculator.

Press **[MODE]**. Use the down arrow key to highlight  $a + bi$ .



Press **[ENTER]** **[2nd]** **[QUIT]**.

The following screen shows two examples of computations on complex numbers. To enter an  $i$ , use **[2nd]** **[i]**, which is located above the decimal point key.



$$\begin{aligned} \text{b. } (2 + \sqrt{-3})(4 - 5\sqrt{-3}) &= (2 + i\sqrt{3})(4 - 5i\sqrt{3}) \\ &= 8 - 10i\sqrt{3} + 4i\sqrt{3} - 5i^2(3) \\ &= 8 - 10i\sqrt{3} + 4i\sqrt{3} - 5(-1)(3) \\ &= 8 - 10i\sqrt{3} + 4i\sqrt{3} + 15 = 23 - 6i\sqrt{3} \end{aligned}$$

» Try Exercise 34, page 340

## Division of Complex Numbers

Recall that the number  $\frac{3}{\sqrt{2}}$  is not in simplest form because there is a radical expression in the denominator. Similarly,  $\frac{3}{i}$  is not in simplest form because  $i = \sqrt{-1}$ . To write this expression in simplest form, multiply the numerator and denominator by  $i$ .

$$\frac{3}{i} \cdot \frac{i}{i} = \frac{3i}{i^2} = \frac{3i}{-1} = -3i$$

Here is another example.

$$\frac{3 - 6i}{2i} = \frac{3 - 6i}{2i} \cdot \frac{i}{i} = \frac{3i - 6i^2}{2i^2} = \frac{3i - 6(-1)}{2(-1)} = \frac{3i + 6}{-2} = -3 - \frac{3}{2}i$$

Recall that to simplify the quotient  $\frac{2 + \sqrt{3}}{5 + 2\sqrt{3}}$ , we multiply the numerator and denominator by the conjugate of  $5 + 2\sqrt{3}$ , which is  $5 - 2\sqrt{3}$ . In a similar manner, to find the quotient of two complex numbers, we multiply the numerator and denominator by the conjugate of the denominator.

The complex numbers  $a + bi$  and  $a - bi$  are called **complex conjugates** or **conjugates** of each other. The conjugate of the complex number  $z$  is denoted by  $\bar{z}$ . For instance,

$$\overline{2 + 5i} = 2 - 5i \quad \text{and} \quad \overline{3 - 4i} = 3 + 4i$$

Consider the product of a complex number and its conjugate. For instance,

$$\begin{aligned} (2 + 5i)(2 - 5i) &= 4 - 10i + 10i - 25i^2 \\ &= 4 - 25(-1) = 4 + 25 \\ &= 29 \end{aligned}$$

Note that the product is a *real* number. This is always true.

### Product of Complex Conjugates

The product of a complex number and its conjugate is a real number. That is,  $(a + bi)(a - bi) = a^2 + b^2$ .

#### Example

$$(5 + 3i)(5 - 3i) = 5^2 + 3^2 = 25 + 9 = 34$$

The next example shows how the quotient of two complex numbers is determined by using conjugates.

### EXAMPLE 4 » Divide Complex Numbers

Simplify:  $\frac{16 - 11i}{5 + 2i}$

#### Solution

$$\begin{aligned}\frac{16 - 11i}{5 + 2i} &= \frac{16 - 11i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i} \\&= \frac{80 - 32i - 55i + 22i^2}{5^2 + 2^2} \\&= \frac{80 - 32i - 55i + 22(-1)}{25 + 4} \\&= \frac{80 - 87i - 22}{29} \\&= \frac{58 - 87i}{29} \\&= \frac{29(2 - 3i)}{29} = 2 - 3i\end{aligned}$$

- Multiply numerator and denominator by the conjugate of the denominator.

» Try Exercise 48, page 341

### ■ Powers of $i$

The following powers of  $i$  illustrate a pattern:

$$\begin{array}{ll} i^1 = i & i^5 = i^4 \cdot i = 1 \cdot i = i \\ i^2 = -1 & i^6 = i^4 \cdot i^2 = 1(-1) = -1 \\ i^3 = i^2 \cdot i = (-1)i = -i & i^7 = i^4 \cdot i^3 = 1(-i) = -i \\ i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 & i^8 = (i^4)^2 = 1^2 = 1 \end{array}$$

Because  $i^4 = 1$ ,  $(i^4)^n = 1^n = 1$  for any integer  $n$ . Thus it is possible to evaluate powers of  $i$  by factoring out powers of  $i^4$ , as shown in the following:

$$i^{27} = (i^4)^6 \cdot i^3 = 1^6 \cdot i^3 = 1 \cdot (-i) = -i$$

The following theorem can also be used to evaluate powers of  $i$ .

#### Powers of $i$

If  $n$  is a positive integer, then  $i^n = i^r$ , where  $r$  is the remainder of the division of  $n$  by 4.

**EXAMPLE 5** » Evaluate a Power of  $i$ Evaluate:  $i^{153}$ **Solution**Use the powers of  $i$  theorem.

$$i^{153} = i^1 = i$$

• Remainder of  $153 \div 4$  is 1.

» Try Exercise 60, page 341

**Topics for Discussion**

1. What is an imaginary number? What is a complex number?
2. How are the real numbers related to the complex numbers?
3. Is zero a complex number?
4. What is the conjugate of a complex number?
5. If  $a$  and  $b$  are real numbers and  $ab = 0$ , then  $a = 0$  or  $b = 0$ . Is the same true for complex numbers? That is, if  $u$  and  $v$  are complex numbers and  $uv = 0$ , must one of the complex numbers be zero?

**Exercise Set 5.1**

In Exercises 1 to 10, write the complex number in standard form.

1.  $\sqrt{-81}$

2.  $\sqrt{-64}$

3.  $\sqrt{-98}$

4.  $\sqrt{-27}$

5.  $\sqrt{16} + \sqrt{-81}$

6.  $\sqrt{25} + \sqrt{-9}$

7.  $5 + \sqrt{-49}$

» 8.  $6 - \sqrt{-1}$

9.  $8 - \sqrt{-18}$

10.  $11 + \sqrt{-48}$

19.  $8i - (2 - 8i)$

21.  $5i \cdot 8i$

23.  $\sqrt{-50} \cdot \sqrt{-2}$

25.  $3(2 + 5i) - 2(3 - 2i)$

27.  $(4 + 2i)(3 - 4i)$

29.  $(-3 - 4i)(2 + 7i)$

31.  $(4 - 5i)(4 + 5i)$

20.  $3 - (4 - 5i)$

22.  $(-3i)(2i)$

24.  $\sqrt{-12} \cdot \sqrt{-27}$

26.  $3i(2 + 5i) + 2i(3 - 4i)$

28.  $(6 + 5i)(2 - 5i)$

30.  $(-5 - i)(2 + 3i)$

In Exercises 11 to 36, simplify and write the complex number in standard form.

11.  $(5 + 2i) + (6 - 7i)$

12.  $(4 - 8i) + (5 + 3i)$

32.  $(3 + 7i)(3 - 7i)$

33.  $(3 + \sqrt{-4})(2 - \sqrt{-9})$

13.  $(-2 - 4i) - (5 - 8i)$

14.  $(3 - 5i) - (8 - 2i)$

» 34.  $(5 + 2\sqrt{-16})(1 - \sqrt{-25})$

15.  $(1 - 3i) + (7 - 2i)$

16.  $(2 - 6i) + (4 - 7i)$

35.  $(3 + 2\sqrt{-18})(2 + 2\sqrt{-50})$

17.  $(-3 - 5i) - (7 - 5i)$

» 18.  $(5 - 3i) - (2 + 9i)$

36.  $(5 - 3\sqrt{-48})(2 - 4\sqrt{-27})$



- 82.** It is possible to find the square root of a complex number.  
 Verify that  $\sqrt{i} = \frac{\sqrt{2}}{2}(1 + i)$  by showing that

$$\left[ \frac{\sqrt{2}}{2}(1+i) \right]^2 = i.$$

- 83.** Simplify  $i + i^2 + i^3 + i^4 + \dots + i^{28}$ .

- 84.** Simplify  $i + i^2 + i^3 + i^4 + \dots + i^{100}$ .

**ARGAND DIAGRAM** Just as we can graph a real number on a real number line, we can graph a complex number. This is accomplished by using one number line for the real part of the complex number and one number line for the imaginary part of the complex number. These two number lines are drawn perpendicular to each other and pass through their respective origins, as shown below.

The result is called the *complex plane* or an *Argand diagram*, after Jean-Robert Argand (1768–1822), an accountant and amateur mathematician. Although he is given credit for this representation of complex numbers, Caspar Wessel (1745–1818) actually conceived the idea before Argand.

To graph the complex number  $3 + 4i$ , start at 3 on the real axis. Now move 4 units up (for positive numbers move up; for negative numbers move down) and place a dot at that point, as shown in the diagram. The graphs of several other complex numbers are also shown.

In Exercises 1 to 8, graph the complex number.

- 1.**  $2 + 5i$       **2.**  $4 - 3i$       **3.**  $-2 + 6i$       **4.**  $-3 - 5i$

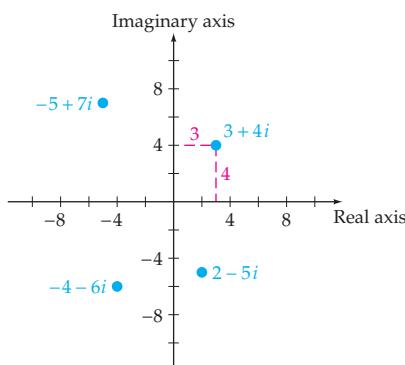
**5.**  $4$       **6.**  $-2i$       **7.**  $3i$       **8.**  $-5$

The absolute value of a complex number is given by  $|a + bi| = \sqrt{a^2 + b^2}$ . In Exercises 9 to 12, find the absolute value of the complex number.

- 9.**  $2 + 5i$       **10.**  $4 - 3i$       **11.**  $-2 + 6i$       **12.**  $-3 - 5i$

**13.** The additive inverse of  $a + bi$  is  $-a - bi$ . Show that the absolute value of a complex number and the absolute value of its additive inverse are equal.

**14.** A *real* number and its additive inverse are the same distance from zero but on opposite sides of zero on a real number line. Describe the relationship between the graphs of a complex number and its additive inverse.



## Section 5.2

- Trigonometric Form of a Complex Number
- The Product and Quotient of Complex Numbers Written in Trigonometric Form

# Trigonometric Form of Complex Numbers

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A22.

**PS1.** Simplify:  $(1 + i)(2 + i)$  [5.1]

**PS2.** Simplify:  $\frac{2+i}{3-i}$  [5.1]

**PS3.** What is the conjugate of  $2 + 3i$ ? [5.1]

**PS4.** What is the conjugate of  $3 - 5i$ ? [5.1]

**PS5.** Use the quadratic formula to find the solutions of  $x^2 + x = -1$ . [1.1/5.1]

**PS6.** Solve:  $x^2 + 9 = 0$  [1.1/5.1]

## ■ Trigonometric Form of a Complex Number

Real numbers are graphed as points on a number line. Complex numbers can be graphed in a coordinate plane called the **complex plane**. The horizontal axis of the complex plane is called the **real axis**; the vertical axis is called the **imaginary axis**.

A complex number written in the form  $z = a + bi$  is written in **standard form** or **rectangular form**. The graph of  $a + bi$  is associated with the point  $P(a, b)$  in the complex plane. **Figure 5.1** shows the graphs of several complex numbers.

The length of the line segment from the origin to the point  $(-3, 4)$  in the complex plane is the *absolute value* of  $z = -3 + 4i$ . See **Figure 5.2**. From the Pythagorean Theorem, the absolute value of  $z = -3 + 4i$  is

$$\sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$

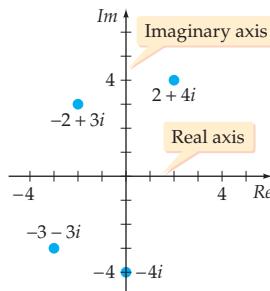


Figure 5.1

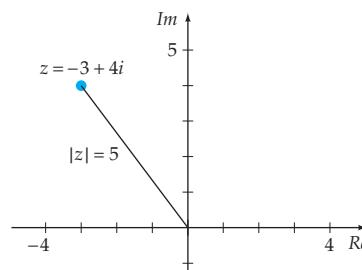


Figure 5.2

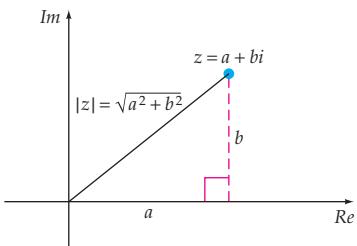


Figure 5.3

**Definition of the Absolute Value of a Complex Number**

The absolute value of the complex number  $z = a + bi$ , denoted by  $|z|$ , is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

Thus  $|z|$  is the distance from the origin to  $z$  (see **Figure 5.3**).

**QUESTION** The conjugate of  $a + bi$  is  $a - bi$ . Does  $|a + bi| = |a - bi|$ ?

A complex number  $z = a + bi$  can be written in terms of trigonometric functions. Consider the complex number graphed in **Figure 5.4**. We can write  $a$  and  $b$  in terms of the sine and the cosine.

$$\begin{aligned}\cos \theta &= \frac{a}{r} & \sin \theta &= \frac{b}{r} \\ a &= r \cos \theta & b &= r \sin \theta\end{aligned}$$

where  $r = |z| = \sqrt{a^2 + b^2}$ . Substituting for  $a$  and  $b$  in  $z = a + bi$ , we obtain

$$z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)$$

The expression  $z = r(\cos \theta + i \sin \theta)$  is known as the **trigonometric form** of a complex number. The trigonometric form of a complex number is also called the **polar form** of the complex number. The notation  $\cos \theta + i \sin \theta$  is often abbreviated as  $\text{cis } \theta$  using the *c* from  $\cos \theta$ , the imaginary unit  $i$ , and the *s* from  $\sin \theta$ .

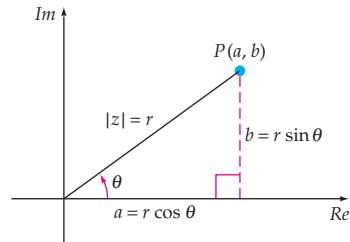


Figure 5.4

**Trigonometric Form of a Complex Number**

The complex number  $z = a + bi$  can be written in trigonometric form as

$$z = r(\cos \theta + i \sin \theta) = r \text{ cis } \theta$$

where  $a = r \cos \theta$ ,  $b = r \sin \theta$ ,  $r = \sqrt{a^2 + b^2}$ , and  $\tan \theta = \frac{b}{a}$ .

In this text we will often write the trigonometric form of a complex number in its abbreviated form  $z = r \text{ cis } \theta$ . The value of  $r$  is called the **modulus** of the complex number  $z$ , and the angle  $\theta$  is called the **argument** of the complex number  $z$ . The modulus  $r$  and the argument  $\theta$  of a complex number  $z = a + bi$  are given by

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \cos \theta = \frac{a}{r}, \quad \sin \theta = \frac{b}{r}$$

---

**ANSWER** Yes, because  $\sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2}$ .

We also can write  $\alpha = \tan^{-1} \left| \frac{b}{a} \right|$ , where  $\alpha$  is the reference angle for  $\theta$ . Due to the periodic nature of the sine and cosine functions, the trigonometric form of a complex number is not unique. Because  $\cos \theta = \cos(\theta + 2k\pi)$  and  $\sin \theta = \sin(\theta + 2k\pi)$ , where  $k$  is an integer, the complex numbers  $r \operatorname{cis} \theta$  and  $r \operatorname{cis}(\theta + 2k\pi)$  are equal. For example,  $2 \operatorname{cis} \frac{\pi}{6} = 2 \operatorname{cis} \left( \frac{\pi}{6} + 2\pi \right)$ .

### EXAMPLE 1 » Write a Complex Number in Trigonometric Form

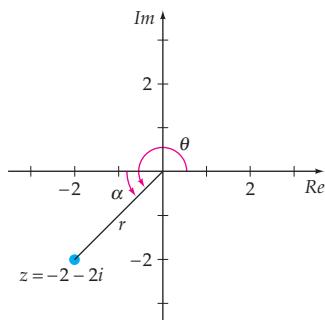
Write  $z = -2 - 2i$  in trigonometric form.

#### Solution

Find the modulus and the argument of  $z$ . Then substitute these values in the trigonometric form of  $z$ .

$$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

To determine  $\theta$ , we first determine  $\alpha$ . See **Figure 5.5**.



**Figure 5.5**

$$\begin{aligned} \alpha &= \tan^{-1} \left| \frac{b}{a} \right| && \bullet \alpha \text{ is the reference angle of angle } \theta. \\ \alpha &= \tan^{-1} \left| \frac{-2}{-2} \right| = \tan^{-1} 1 = 45^\circ && \bullet a = -2 \text{ and } b = -2 \\ \theta &= 180^\circ + 45^\circ = 225^\circ && \bullet \text{Because } z \text{ is in the third quadrant,} \\ &&& 180^\circ < \theta < 270^\circ. \end{aligned}$$

The trigonometric form is

$$z = r \operatorname{cis} \theta = 2\sqrt{2} \operatorname{cis} 225^\circ \quad \bullet r = 2\sqrt{2}, \theta = 225^\circ$$

### » Try Exercise 12, page 349

### EXAMPLE 2 » Write a Complex Number in Standard Form

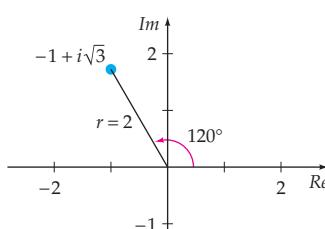
Write  $z = 2 \operatorname{cis} 120^\circ$  in standard form.

#### Solution

Write  $z$  in the form  $r(\cos \theta + i \sin \theta)$  and then evaluate  $\cos \theta$  and  $\sin \theta$ . See **Figure 5.6**.

$$z = 2 \operatorname{cis} 120^\circ = 2(\cos 120^\circ + i \sin 120^\circ) = 2 \left( -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = -1 + i\sqrt{3}$$

### » Try Exercise 30, page 349



**Figure 5.6**

## The Product and Quotient of Complex Numbers Written in Trigonometric Form

Let  $z_1$  and  $z_2$  be two complex numbers written in trigonometric form. The product of  $z_1$  and  $z_2$  can be found by using trigonometric identities. If  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ , then

$$\begin{aligned} z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2(\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\ &= r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

• Identities for  $\cos(\theta_1 + \theta_2)$   
and  $\sin(\theta_1 + \theta_2)$

Thus, using cis notation, we have  $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ .

The above result is called the **Product Property of Complex Numbers**. It states that the modulus for the product of two complex numbers in trigonometric form is the product of the moduli of the two numbers, and the argument of the product is the sum of the arguments of the two numbers.

### EXAMPLE 3 » Find the Product of Two Complex Numbers

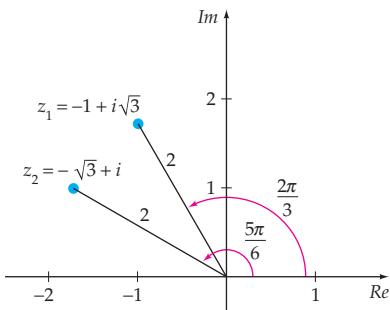


Figure 5.7

Find the product of  $z_1 = -1 + i\sqrt{3}$  and  $z_2 = -\sqrt{3} + i$  by using the trigonometric forms of the complex numbers. Write the answer in standard form.

#### Solution

Write each complex number in trigonometric form. Then use the Product Property of Complex Numbers. See **Figure 5.7**.

$$z_1 = -1 + i\sqrt{3} = 2 \operatorname{cis} \frac{2\pi}{3}$$

•  $r_1 = 2$ ,  $\theta_1 = \frac{2\pi}{3}$

$$z_2 = -\sqrt{3} + i = 2 \operatorname{cis} \frac{5\pi}{6}$$

•  $r_2 = 2$ ,  $\theta_2 = \frac{5\pi}{6}$

$$\begin{aligned} z_1 z_2 &= 2 \operatorname{cis} \frac{2\pi}{3} \cdot 2 \operatorname{cis} \frac{5\pi}{6} \\ &= 4 \operatorname{cis} \left( \frac{2\pi}{3} + \frac{5\pi}{6} \right) \\ &= 4 \operatorname{cis} \frac{3\pi}{2} \\ &= 4 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \\ &= 4(0 - i) = -4i \end{aligned}$$

- The Product Property of Complex Numbers
- Simplify.

» Try Exercise 56, page 349

Similarly, the quotient of  $z_1$  and  $z_2$  can be found by using trigonometric identities. If  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ , then

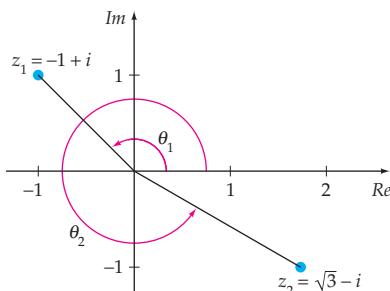
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \\ &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{r_2(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \\ &= \frac{r_1(\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - i^2 \sin \theta_1 \sin \theta_2)}{r_2(\cos^2 \theta_2 - i^2 \sin^2 \theta_2)} \\ &= \frac{r_1[(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)]}{r_2(\cos^2 \theta_2 + \sin^2 \theta_2)} \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \end{aligned}$$

**Identities for  $\cos(\theta_1 - \theta_2)$ ,  $\sin(\theta_1 - \theta_2)$ , and  $\cos^2 \theta_2 + \sin^2 \theta_2$**

Thus, using cis notation, we have  $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$ .

The above result is called the **Quotient Property of Complex Numbers**. It states that the modulus for the quotient of two complex numbers in trigonometric form is the quotient of the moduli of the two numbers, and the argument of the quotient is the difference of the arguments of the two numbers.

### EXAMPLE 4 » Find the Quotient of Two Complex Numbers



**Figure 5.8**

Find the quotient of  $z_1 = -1 + i$  and  $z_2 = \sqrt{3} - i$  by using the trigonometric forms of the complex numbers. Write the answer in standard form. Round approximate constants to the nearest thousandth.

#### Solution

Write the numbers in trigonometric form. Then use the Quotient Property of Complex Numbers. See **Figure 5.8**.

$$\begin{aligned} z_1 &= -1 + i = \sqrt{2} \operatorname{cis} 135^\circ & \bullet r_1 = \sqrt{2}, \theta_1 = 135^\circ \\ z_2 &= \sqrt{3} - i = 2 \operatorname{cis} 330^\circ & \bullet r_2 = 2, \theta_2 = 330^\circ \\ \frac{z_1}{z_2} &= \frac{-1 + i}{\sqrt{3} - i} = \frac{\sqrt{2} \operatorname{cis} 135^\circ}{2 \operatorname{cis} 330^\circ} & \bullet \text{The Quotient Property of Complex Numbers} \\ &= \frac{\sqrt{2}}{2} \operatorname{cis}(-195^\circ) & \bullet \text{Simplify.} \\ &= \frac{\sqrt{2}}{2} [\cos(-195^\circ) + i \sin(-195^\circ)] & \bullet \cos(-x) = \cos x, \\ &= \frac{\sqrt{2}}{2} (\cos 195^\circ - i \sin 195^\circ) & \sin(-x) = -\sin x \\ &\approx -0.683 + 0.183i \end{aligned}$$

» Try Exercise 60, page 349

Here is a summary of the product and quotient theorems.

### Product and Quotient Properties of Complex Numbers Written in Trigonometric Form

#### *Product Property of Complex Numbers*

Let  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ . Then

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

• Using **cis** notation

#### *Quotient Property of Complex Numbers*

Let  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ . Then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

• Using **cis** notation



### Topics for Discussion

1. Describe an algebraic procedure that can be used to verify that the absolute value of  $a + bi$  is equal to the absolute value of  $-a + bi$ .
2. Describe the graph of all complex numbers with an absolute value of 5.
3. Describe two different methods that can be used to simplify  $\frac{4}{i}$ .
4. The complex numbers  $z_1$  and  $z_2$  both have an absolute value of 1. What is the absolute value of the product  $z_1 z_2$ ? Explain.
5. Explain how to use the Product Property of Complex Numbers to prove that the product of a complex number and its conjugate is a real number.


**Exercise Set 5.2**

**In Exercises 1 to 8, graph each complex number. Find the absolute value of each complex number.**

1.  $z = -2 - 2i$

2.  $z = 4 - 4i$

3.  $z = \sqrt{3} - i$

4.  $z = 1 + i\sqrt{3}$

5.  $z = -2i$

6.  $z = -5$

7.  $z = 3 - 5i$

8.  $z = -5 - 4i$

**In Exercises 9 to 20, write each complex number in trigonometric form.**

9.  $z = 1 - i$

10.  $z = -4 - 4i$

11.  $z = \sqrt{3} - i$

12.  $z = 1 + i\sqrt{3}$

13.  $z = 3i$

14.  $z = -2i$

15.  $z = -5$

16.  $z = 3$

17.  $z = -8 + 8i\sqrt{3}$

18.  $z = -2\sqrt{2} + 2i\sqrt{2}$

19.  $z = -2 - 2i\sqrt{3}$

20.  $z = \sqrt{2} - i\sqrt{2}$

**In Exercises 21 to 38, write each complex number in standard form.**

21.  $z = 2(\cos 45^\circ + i \sin 45^\circ)$

22.  $z = 3(\cos 240^\circ + i \sin 240^\circ)$

23.  $z = (\cos 315^\circ + i \sin 315^\circ)$

24.  $z = 5(\cos 120^\circ + i \sin 120^\circ)$

25.  $z = 6 \text{ cis } 135^\circ$

26.  $z = \text{cis } 315^\circ$

27.  $z = 8 \text{ cis } 0^\circ$

28.  $z = 5 \text{ cis } 90^\circ$

29.  $z = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

30.  $z = 4\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$

31.  $z = 3\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$     32.  $z = 5(\cos \pi + i \sin \pi)$

33.  $z = 8 \text{ cis } \frac{3\pi}{4}$

34.  $z = 9 \text{ cis } \frac{4\pi}{3}$

35.  $z = 9 \text{ cis } \frac{11\pi}{6}$

36.  $z = \text{cis } \frac{3\pi}{2}$

37.  $z = 2 \text{ cis } 2$

38.  $z = 5 \text{ cis } 4$

**In Exercises 39 to 46, multiply the complex numbers. Write the answer in trigonometric form.**

39.  $2 \text{ cis } 30^\circ \cdot 3 \text{ cis } 225^\circ$

40.  $4 \text{ cis } 120^\circ \cdot 6 \text{ cis } 315^\circ$

41.  $3(\cos 122^\circ + i \sin 122^\circ) \cdot 4(\cos 213^\circ + i \sin 213^\circ)$

42.  $8(\cos 88^\circ + i \sin 88^\circ) \cdot 12(\cos 112^\circ + i \sin 112^\circ)$

43.  $5\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \cdot 2\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)$

44.  $5 \text{ cis } \frac{11\pi}{12} \cdot 3 \text{ cis } \frac{4\pi}{3}$

45.  $4 \text{ cis } 2.4 \cdot 6 \text{ cis } 4.1$

46.  $7 \text{ cis } 0.88 \cdot 5 \text{ cis } 1.32$

**In Exercises 47 to 54, divide the complex numbers. Write the answer in standard form. Round approximate constants to the nearest thousandth.**

47.  $\frac{32 \text{ cis } 30^\circ}{4 \text{ cis } 150^\circ}$

48.  $\frac{15 \text{ cis } 240^\circ}{3 \text{ cis } 135^\circ}$

49.  $\frac{27(\cos 315^\circ + i \sin 315^\circ)}{9(\cos 225^\circ + i \sin 225^\circ)}$

50.  $\frac{9(\cos 25^\circ + i \sin 25^\circ)}{3(\cos 175^\circ + i \sin 175^\circ)}$

51.  $\frac{12 \text{ cis } \frac{2\pi}{3}}{4 \text{ cis } \frac{11\pi}{6}}$

52.  $\frac{10 \text{ cis } \frac{\pi}{3}}{5 \text{ cis } \frac{\pi}{4}}$

53.  $\frac{25(\cos 3.5 + i \sin 3.5)}{5(\cos 1.5 + i \sin 1.5)}$

54.  $\frac{18(\cos 0.56 + i \sin 0.56)}{6(\cos 1.22 + i \sin 1.22)}$

**In Exercises 55 to 62, perform the indicated operation in trigonometric form. Write the solution in standard form. Round approximate constants to the nearest ten-thousandth.**

55.  $(1 - i\sqrt{3})(1 + i)$

56.  $(\sqrt{3} - i)(1 + i\sqrt{3})$

57.  $(3 - 3i)(1 + i)$

58.  $(2 + 2i)(\sqrt{3} - i)$

59.  $\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}$

60.  $\frac{1 + i}{1 - i}$

61.  $\frac{\sqrt{2} - i\sqrt{2}}{1 + i}$

62.  $\frac{1 + i\sqrt{3}}{4 - 4i}$

**In Exercises 63 to 68, perform the indicated operation in trigonometric form. Write the solution in standard form.**

**63.**  $(\sqrt{3} - i)(2 + 2i)(2 - 2i\sqrt{3})$

**64.**  $(1 - i)(1 + i\sqrt{3})(\sqrt{3} - i)$

**65.**  $\frac{\sqrt{3} + i\sqrt{3}}{(1 - i\sqrt{3})(2 - 2i)}$

**66.** 
$$\frac{(2 - 2i\sqrt{3})(1 - i\sqrt{3})}{4\sqrt{3} + 4i}$$

$$67 \quad (1 - 3i)(2 + 3i)(4 + 5i)$$

**68.** 
$$\frac{(2 - 5i)(1 - 6i)}{3 + 4i}$$

**In Exercises 69 and 70, the notation  $\bar{z}$  is used to represent the conjugate of the complex number  $z$ .**

**69.** Use the trigonometric forms of  $z$  and  $\bar{z}$  to find  $z \cdot \bar{z}$ .

**70.** Use the trigonometric forms of  $z$  and  $\bar{z}$  to find  $\frac{z}{\bar{z}}$ .

- 1. A GEOMETRIC INTERPRETATION** Multiplying a real number by a number greater than 1 increases the magnitude of that number. For example, multiplying 2 by 3 triples the magnitude of 2. Explain the effect of multiplying a real

number by  $i$ , the imaginary unit. Now multiply a complex number by  $i$  and note the effect. The use of a complex plane may be helpful in your explanation.

## Section 5.3

- De Moivre's Theorem
  - De Moivre's Theorem for Finding Roots

# De Moivre's Theorem

## **PREPARE FOR THIS SECTION**

**Prepare for this section by completing the following exercises. The answers can be found on page A22.**

**PS1.** Find  $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2$ . [5.1]

**PS2.** Find the real root of  $x^3 - 8 = 0$ .

**PS3.** Find the real root of  $x^5 - 243 = 0$ .

**PS4.** Write  $2 + 2i$  in trigonometric form. [5.2]

**PS5.** Write  $2(\cos 150^\circ + i \sin 150^\circ)$  in standard form. [5.2]

**PS6.** Find the absolute value of  $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ . [5.2]

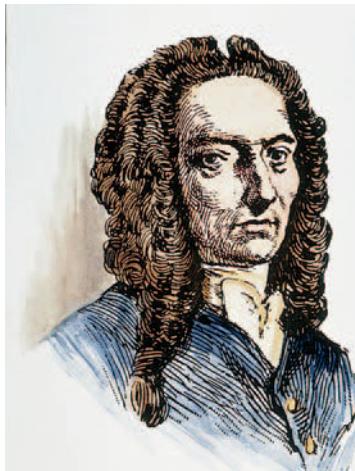
## ■ De Moivre's Theorem

De Moivre's Theorem is a procedure for finding powers and roots of complex numbers when the complex numbers are expressed in trigonometric form. This theorem can be illustrated by repeated multiplication of a complex number.

Let  $z = r \operatorname{cis} \theta$ . Then  $z^2$  can be written as

$$z \cdot z = r \operatorname{cis} \theta \cdot r \operatorname{cis} \theta$$

$$z^2 = r^2 \operatorname{cis} 2\theta$$

**Math Matters**

Abraham de Moivre (1667–1754) was a French mathematician who fled to England during the expulsion of the Huguenots in 1685.

De Moivre made important contributions to probability theory and analytic geometry. In 1718 he published *The Doctrine of Chance*, in which he developed the theory of annuities, mortality statistics, and the concept of statistical independence. In 1730 de Moivre stated the theorem shown above Example 1, which we now call De Moivre's Theorem. This theorem is significant because it provides a connection between trigonometry and mathematical analysis.

Although de Moivre was well respected in the mathematical community and was elected to the Royal Society of England, he never was able to secure a university teaching position. His income came mainly from tutoring, and he died in poverty.

De Moivre is often remembered for predicting the day of his own death. At one point in his life, he noticed that he was sleeping a few minutes longer each night. Thus he calculated that he would die when he needed 24 hours of sleep. As it turned out, his calculation was correct.

The product  $z^2 \cdot z$  is

$$z^2 \cdot z = r^2 \operatorname{cis} 2\theta \cdot r \operatorname{cis} \theta$$

$$z^3 = r^3 \operatorname{cis} 3\theta$$

If we continue this process, the results suggest a formula known as De Moivre's Theorem for the  $n$ th power of a complex number.

**De Moivre's Theorem**

If  $z = r \operatorname{cis} \theta$  and  $n$  is a positive integer, then

$$z^n = r^n \operatorname{cis} n\theta$$

**EXAMPLE 1** » **Find the Power of a Complex Number**

Find  $(2 \operatorname{cis} 30^\circ)^5$ . Write the answer in standard form.

**Solution**

By De Moivre's Theorem,

$$\begin{aligned}(2 \operatorname{cis} 30^\circ)^5 &= 2^5 \operatorname{cis}(5 \cdot 30^\circ) \\&= 2^5 [\cos(5 \cdot 30^\circ) + i \sin(5 \cdot 30^\circ)] \\&= 32(\cos 150^\circ + i \sin 150^\circ) \\&= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -16\sqrt{3} + 16i\end{aligned}$$

» Try Exercise 6, page 354

**EXAMPLE 2** » **Use De Moivre's Theorem**

Find  $(1 + i)^8$  using De Moivre's Theorem. Write the answer in standard form.

**Solution**

Convert  $1 + i$  to trigonometric form and then use De Moivre's Theorem.

$$\begin{aligned}(1 + i)^8 &= (\sqrt{2} \operatorname{cis} 45^\circ)^8 = (\sqrt{2})^8 \operatorname{cis} 8(45^\circ) = 16 \operatorname{cis} 360^\circ \\&= 16(\cos 360^\circ + i \sin 360^\circ) = 16(1 + 0i) = 16\end{aligned}$$

» Try Exercise 16, page 354

**QUESTION** Is  $(1 + i)^4$  a real number?

**ANSWER** Yes.  $(1 + i)^4 = (\sqrt{2} \operatorname{cis} 45^\circ)^4 = (\sqrt{2})^4 \operatorname{cis} 180^\circ = -4$ .

## ■ De Moivre's Theorem for Finding Roots

De Moivre's Theorem can be used to find the  $n$ th roots of any number.

### De Moivre's Theorem for Finding Roots

If  $z = r \operatorname{cis} \theta$  is a complex number, then there exist  $n$  distinct  $n$ th roots of  $z$  given by

$$w_k = r^{1/n} \operatorname{cis} \frac{\theta + 360^\circ k}{n} \quad \text{for } k = 0, 1, 2, \dots, n-1, \text{ and } n \geq 1$$

### EXAMPLE 3 » Find Cube Roots by De Moivre's Theorem

Find the three cube roots of 27.

#### ALGEBRAIC SOLUTION

Write 27 in trigonometric form:  $27 = 27 \operatorname{cis} 0^\circ$ . Then, from De Moivre's Theorem for finding roots, the cube roots of 27 are

$$w_k = 27^{1/3} \operatorname{cis} \frac{0^\circ + 360^\circ k}{3} \quad \text{for } k = 0, 1, 2$$

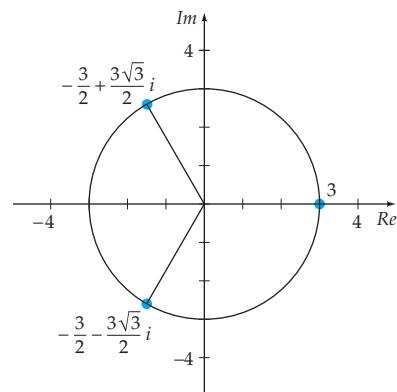
Substitute for  $k$  to find the three cube roots of 27.

$$\begin{aligned} w_0 &= 27^{1/3} \operatorname{cis} 0^\circ & \bullet k = 0; \frac{0^\circ + 360^\circ(0)}{3} = 0^\circ \\ &= 3(\cos 0^\circ + i \sin 0^\circ) \\ &= 3 \\ w_1 &= 27^{1/3} \operatorname{cis} 120^\circ & \bullet k = 1; \frac{0^\circ + 360^\circ(1)}{3} = 120^\circ \\ &= 3(\cos 120^\circ + i \sin 120^\circ) \\ &= -\frac{3}{2} + \frac{3\sqrt{3}}{2}i \\ w_2 &= 27^{1/3} \operatorname{cis} 240^\circ & \bullet k = 2; \frac{0^\circ + 360^\circ(2)}{3} = 240^\circ \\ &= 3(\cos 240^\circ + i \sin 240^\circ) \\ &= -\frac{3}{2} - \frac{3\sqrt{3}}{2}i \end{aligned}$$

For  $k = 3$ ,  $\frac{0^\circ + 1080^\circ}{3} = 360^\circ$ . The angles start repeating; thus there are

only three cube roots of 27. The three cube roots are graphed in **Figure 5.9**.

#### VISUALIZE THE SOLUTION



**Figure 5.9**

Note that the arguments of the three cube roots of 27 are  $0^\circ, 120^\circ$ , and  $240^\circ$  and that  $|w_0| = |w_1| = |w_2| = 3$ . In geometric terms, this means that the three cube roots of 27 are equally spaced on a circle centered at the origin with a radius of 3.

**EXAMPLE 4** » Find the Fifth Roots of a Complex Number

Find the fifth roots of  $z = 1 + i\sqrt{3}$ .

**ALGEBRAIC SOLUTION**

Write  $z$  in trigonometric form:  $z = r \operatorname{cis} \theta$ .

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$z = 2 \operatorname{cis} 60^\circ$$

$$\bullet \theta = \tan^{-1} \frac{\sqrt{3}}{1} = 60^\circ$$

From De Moivre's Theorem, the modulus of each root is  $\sqrt[5]{2}$ , and the arguments are determined by  $\frac{60^\circ + 360^\circ k}{5}$ ,  $k = 0, 1, 2, 3, 4$ .

$$w_k = \sqrt[5]{2} \operatorname{cis} \frac{60^\circ + 360^\circ k}{5} \quad \bullet k = 0, 1, 2, 3, 4$$

Substitute for  $k$  to find the five fifth roots of  $z$ .

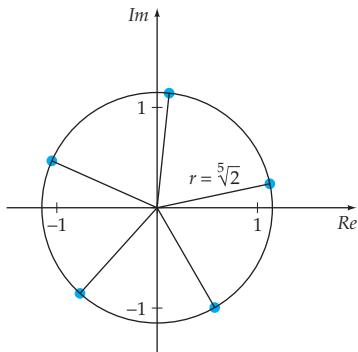
$$w_0 = \sqrt[5]{2} \operatorname{cis} 12^\circ \quad \bullet k = 0; \frac{60^\circ + 360^\circ(0)}{5} = 12^\circ$$

$$w_1 = \sqrt[5]{2} \operatorname{cis} 84^\circ \quad \bullet k = 1; \frac{60^\circ + 360^\circ(1)}{5} = 84^\circ$$

$$w_2 = \sqrt[5]{2} \operatorname{cis} 156^\circ \quad \bullet k = 2; \frac{60^\circ + 360^\circ(2)}{5} = 156^\circ$$

$$w_3 = \sqrt[5]{2} \operatorname{cis} 228^\circ \quad \bullet k = 3; \frac{60^\circ + 360^\circ(3)}{5} = 228^\circ$$

$$w_4 = \sqrt[5]{2} \operatorname{cis} 300^\circ \quad \bullet k = 4; \frac{60^\circ + 360^\circ(4)}{5} = 300^\circ$$

**VISUALIZE THE SOLUTION**

**Figure 5.10**

The five fifth roots of  $1 + i\sqrt{3}$  are graphed in **Figure 5.10**. Note that the roots are equally spaced on a circle with center  $(0, 0)$  and a radius of  $\sqrt[5]{2} \approx 1.15$ .

**Try Exercise 30, page 355**

Keep the following properties in mind as you compute the  $n$  distinct  $n$ th roots of the complex number  $z$ .

**Properties of the  $n$ th Roots of  $z$** **Geometric Property**

All  $n$ th roots of  $z$  are equally spaced on a circle with center  $(0, 0)$  and a radius of  $|z|^{1/n}$ .

**Absolute Value Properties**

1. If  $|z| = 1$ , then each  $n$ th root of  $z$  has an absolute value of 1.
2. If  $|z| > 1$ , then each  $n$ th root of  $z$  has an absolute value of  $|z|^{1/n}$ , where  $|z|^{1/n}$  is greater than 1 but less than  $|z|$ .
3. If  $|z| < 1$ , then each  $n$ th root of  $z$  has an absolute value of  $|z|^{1/n}$ , where  $|z|^{1/n}$  is less than 1 but greater than  $|z|$ .

**Integrating Technology**

A web applet is available to show the  $n$ th roots of a complex number. This applet, *Nth Roots*, can be found on our website at [college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

**Argument Property**

Given that the argument of  $z$  is  $\theta$ , then the argument of  $w_0$  is  $\frac{\theta}{n}$  and the arguments of the remaining  $n$ th roots can be determined by adding multiples of  $\frac{360^\circ}{n}$  (or  $\frac{2\pi}{n}$  if you are using radians) to  $\frac{\theta}{n}$ .

- QUESTION** Are all fourth roots of 1 equally spaced on a circle with center  $(0, 0)$  and radius 1?

**Topics for Discussion**

- How many solutions are there for  $z = (a + bi)^8$ ? How many solutions are there for  $z^8 = a + bi$ ? Explain.
- To solve  $z^4 = 16$ , a student first observes that  $z = 2$  is one solution. The other solutions are equally spaced on a circle with center  $(0, 0)$  and radius 2, so the student reasons that  $z = 2i$ ,  $z = -2$ , and  $z = -2i$  are the other three solutions. Do you agree? Explain.
- If  $|z| = 1$ , then the  $n$  solutions of  $w^n = z$  all have an absolute value of 1. Do you agree? Explain.
- If  $z$  is a solution of  $z^2 = c + di$ , then the conjugate of  $z$  is also a solution. Do you agree? Explain.

- 
- ANSWER** Yes.

## Exercise Set 5.3

In Exercises 1 to 16, find the indicated power. Write the answer in standard form.

1.  $[2(\cos 30^\circ + i \sin 30^\circ)]^8$

2.  $(\cos 240^\circ + i \sin 240^\circ)^{12}$

3.  $[2(\cos 240^\circ + i \sin 240^\circ)]^5$

4.  $[2(\cos 45^\circ + i \sin 45^\circ)]^{10}$

5.  $(2 \operatorname{cis} 225^\circ)^5$

6.  $(2 \operatorname{cis} 330^\circ)^4$

7.  $\left(2 \operatorname{cis} \frac{2\pi}{3}\right)^6$

8.  $\left(4 \operatorname{cis} \frac{5\pi}{6}\right)^3$

9.  $(1 - i)^{10}$

10.  $(1 + i\sqrt{3})^8$

11.  $(1 + i)^4$

12.  $(2 - 2i\sqrt{3})^3$

13.  $(2 + 2i)^7$

15.  $\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^6$

14.  $(2\sqrt{3} - 2i)^5$

16.  $\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^{12}$

In Exercises 17 to 30, find all of the indicated roots. Write all answers in standard form. Round approximate constants to the nearest thousandth.

17. The two square roots of 9

18. The two square roots of 16

19. The six sixth roots of 64

20. The five fifth roots of 32

- 21.** The five fifth roots of  $-1$

**22.** The four fourth roots of  $-16$

**23.** The three cube roots of  $1$

**24.** The three cube roots of  $i$

**25.** The four fourth roots of  $1 + i$

**26.** The five fifth roots of  $-1 + i$

**27.** The three cube roots of  $2 - 2i\sqrt{3}$

**28.** The three cube roots of  $-2 + 2i\sqrt{3}$

**29.** The two square roots of  $-16 + 16i\sqrt{3}$

**30.** The two square roots of  $-1 + i\sqrt{3}$

**In Exercises 31 to 42, find all roots of the equation. Write the answers in trigonometric form.**

**31.**  $x^3 + 8 = 0$

**32.**  $x^5 - 32 = 0$

**33.**  $x^4 + i = 0$

**34.**  $x^3 - 2i = 0$

**35.**  $x^3 - 27 = 0$

**36.**  $x^5 + 32i = 0$

**37.**  $x^4 + 81 = 0$

**38.**  $x^3 - 64i = 0$

**39.**  $x^4 - (1 - i\sqrt{3}) = 0$

**40.**  $x^3 + (2\sqrt{3} - 2i) = 0$

**41.**  $x^3 + (1 + i\sqrt{3}) = 0$

**42.**  $x^6 - (4 - 4i) = 0$

## »»» Connecting Concepts »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»»

- 43.** Show that the conjugate of  $z = r(\cos \theta + i \sin \theta)$  is equal to  $\bar{z} = r(\cos \theta - i \sin \theta)$ .

**44.** Show that if  $z = r(\cos \theta + i \sin \theta)$ , then

$$z^{-1} = r^{-1}(\cos \theta - i \sin \theta)$$

**45.** Show that if  $z = r(\cos \theta + i \sin \theta)$ , then

$$z^{-2} = r^{-2}(\cos 2\theta - i \sin 2\theta)$$

**Note that Exercises 44 and 45 suggest that if  $z = r(\cos \theta + i \sin \theta)$ , then, for positive integers  $n$ ,**

$$z^{-n} = r^{-n}(\cos n\theta - i \sin n\theta)$$

**47. SUM OF THE  $n$ TH ROOTS OF 1** Make a conjecture about the sum of the  $n$ th roots of 1 for any natural number  $n \geq 2$ . (Hint: Experiment by finding the sum of the two square roots of 1, the sum of the three cube roots of 1, the sum of the four fourth roots of 1, the sum of the five fifth roots of 1, and the sum of the six sixth roots of 1.)

**48. PRODUCT OF THE  $n$ TH ROOTS OF 1** Make a conjecture about the product of the  $n$ th roots of 1 for any natural number  $n \geq 2$ . (Hint: Experiment by finding the product of the two square roots of 1, the product of the three cube roots of 1, the product of the four fourth roots of 1, the product of the five fifth roots of 1, and the product of the six sixth roots of 1.)

- VERIFY IDENTITIES** Raise  $(\cos \theta + i \sin \theta)$  to the second power by using De Moivre's Theorem. Now square  $(\cos \theta + i \sin \theta)$  as a binomial. Equate the real and imaginary parts of the two complex numbers and show that
    - $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
    - $\sin 2\theta = 2 \sin \theta \cos \theta$
  - DISCOVER IDENTITIES** Raise  $(\cos \theta + i \sin \theta)$  to the fourth power by using De Moivre's Theorem. Now find the fourth power of the binomial  $(\cos \theta + i \sin \theta)$  by multiplying. Equate the real and imaginary parts of the two complex numbers.
    - What identity have you discovered for  $\cos 4\theta$ ?
    - What identity have you discovered for  $\sin 4\theta$ ?



## Exploring Concepts with Technology

### The Mandelbrot Iteration Procedure

The following procedure is called the **Mandelbrot iteration procedure**.

#### Mandelbrot Iteration Procedure

Choose a complex number  $z_0$ .

1. Square  $z_0$  and add the result to  $z_0$ .
2. Square the preceding result and add it to  $z_0$ .
3. Repeat step 2.

Applying the Mandelbrot iteration procedure to a complex number  $z_0$  generates a sequence of numbers  $z_1, z_2, z_3, \dots$ . The numbers  $z_1, z_2, z_3, \dots$  are called **iterates**. In this case, the iterates are generated using the complex function  $f(z) = z^2 + z_0$ . The number  $z_0$  is referred to as the *seed* or *initial value*. If you change the seed, a different sequence of iterates is generated. Some seeds generate iterates that grow without bound; some seeds generate iterates that approach a constant; and some seeds generate iterates that are cyclic. For instance:

- Let  $z_0 = 1$ . Then

$$z_1 = 1^2 + 1 = 2, z_2 = 2^2 + 1 = 5, z_3 = 5^2 + 1 = 26, z_4 = 26^2 + 1 = 677$$

In this case, the iterates grow larger and larger.

- Let  $z_0 = -1$ . Then

$$z_1 = (-1)^2 + (-1) = 0, z_2 = 0^2 + (-1) = -1, z_3 = (-1)^2 + (-1) = 0$$

In this case, the iterates cycle:  $0, -1, 0, -1, 0, \dots$

- Let  $z_0 = 0.25$ . Then

$$\begin{aligned} z_1 &= 0.25^2 + 0.25 = 0.3125, z_2 = 0.3125^2 + 0.25 = 0.34765625, \\ z_3 &= 0.34765625^2 + 0.25 \approx 0.3708648682 \end{aligned}$$

In this case, subsequent iterates can be calculated easily using a graphing calculator and the method shown below.

Store the seed in Z. Press ENTER.

**0.25→Z**

**.25** =  $Z_0$

Store  $Z^2$  plus the seed in Z. Press  
ENTER to produce the next iterate.

**Z<sup>2</sup>+0.25→Z**

**.3125** =  $Z_1$

**.34765625** =  $Z_2$

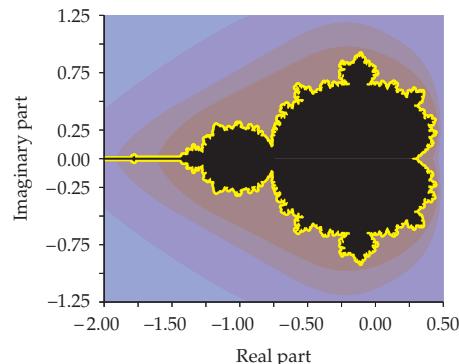
**.3708648682** =  $Z_3$

**.3875407504** =  $Z_4$

1. Use a graphing calculator to continue the above Mandelbrot iteration procedure, with  $z_0 = 0.25$ , to find  $z_5, z_{10}, z_{100}$ , and  $z_{200}$ . It can be shown that the sequence of iterates is approaching a constant. What constant do you think the sequence of iterates is approaching?

2. Let  $z_0 = i$ . Use the Mandelbrot iteration procedure to find  $z_1, z_2, z_3$ , and  $z_4$ . What happens as the iteration procedure progresses?

The black region in **Figure 5.11** is called the **Mandelbrot set**. The Mandelbrot set consists of all complex numbers for which the absolute value of each of their Mandelbrot iteration procedure iterates  $z_1, z_2, z_3, \dots$  is less than or equal to 2.



**Figure 5.11**

The Mandelbrot set has been called the most complex object in the realm of mathematics. There are many Internet sites that provide additional information on the Mandelbrot set. Visit some of these sites and use the computer programs that are provided to graph the Mandelbrot set and zoom in on different regions of the graph to see some of its complex structure.

## Chapter 5 Summary

### 5.1 Complex Numbers

- The number  $i$ , called the *imaginary unit*, is the number such that  $i^2 = -1$ .
- If  $a$  is positive real number, then  $\sqrt{-a} = i\sqrt{a}$ . The number  $i\sqrt{a}$  is called an *imaginary number*.
- A complex number is a number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$ . The number  $a$  is the *real part* of  $a + bi$ , and  $b$  is the *imaginary part*.
- The complex number  $a + bi$  and  $a - bi$  are called *complex conjugates* or *conjugates* of each other.

### Operations on Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}$$

- Multiply the numerator and denominator by the conjugate of the denominator.**

### 5.2 Trigonometric Form of Complex Numbers

- The complex number  $z = a + bi$  can be written in trigonometric form as

$$z = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$$

where  $a = r \cos \theta$ ,  $b = r \sin \theta$ ,  $r = \sqrt{a^2 + b^2}$ , and  $\tan \theta = \frac{b}{a}$

- If  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ , then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

### 5.3 De Moivre's Theorem

#### • De Moivre's Theorem

If  $z = r \operatorname{cis} \theta$  and  $n$  is a positive integer, then

$$z^n = r^n \operatorname{cis} n\theta$$

- If  $z = r \operatorname{cis} \theta$ , then the  $n$  distinct roots of  $z$  are given by

$$w_k = r^{1/n} \operatorname{cis} \frac{\theta + 360^\circ k}{n} \text{ for } k = 0, 1, 2, \dots, n - 1$$



## Chapter 5 Assessing Concepts

1. True or false: The real number 7 is also a complex number.
2. True or false: The product of a complex number  $z$  and its conjugate  $\bar{z}$  is a real number.
3. True or false:  $z = \cos 45^\circ + i \sin 45^\circ$  is a square root of  $i$ .
4. True or false:  $\sqrt{-4} \cdot \sqrt{-9} = 6$
5. If  $|z| = 1$ , describe the geometric relationship among the graphs of the fourth roots of  $z$ .
6. How many solutions exist for the equation  $z = (1 + i)^5$ ?
7. How many solutions exist for the equation  $z^5 = (1 + i)$ ?
8. Does  $|a + bi| = \sqrt{a^2 + (bi)^2}$ ?
9. What is the conjugate of  $-3 - 5i$ ?
10. What is the modulus of  $2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$ ?



## Chapter 5 Review Exercises

In Exercises 1 to 4, write the complex number in standard form and give its conjugate.

1.  $3 - \sqrt{-64}$
2.  $\sqrt{-4} + 6$
3.  $-2 + \sqrt{-5}$
4.  $-5 - \sqrt{-27}$

In Exercises 5 to 20, simplify and write the complex number in standard form.

5.  $(\sqrt{-4})(\sqrt{-4})$
6.  $(-\sqrt{-27})(\sqrt{-3})$
7.  $(3 + 7i) + (2 - 5i)$
8.  $(3 - 4i) + (-6 + 8i)$
9.  $(6 - 8i) - (9 - 11i)$
10.  $(-3 - 5i) - (2 + 10i)$
11.  $(5 + 3i)(2 - 5i)$
12.  $(-2 - 3i)(-4 + 7i)$
13.  $\frac{-2i}{3 - 4i}$
14.  $\frac{4 + i}{7 - 2i}$
15.  $i(2i) - (1 + i)^2$
16.  $(2 - i)^3$
17.  $(3 + \sqrt{-4}) - (-3 - \sqrt{-16})$
18.  $(-2 + \sqrt{-9}) + (-3 - \sqrt{-81})$
19.  $(2 - \sqrt{-3})(2 + \sqrt{-3})$
20.  $(3 - \sqrt{-5})(2 + \sqrt{-5})$

In Exercises 21 to 24, simplify and write each complex number as  $i$ ,  $-i$ , 1, or  $-1$ .

21.  $i^{27}$
22.  $i^{105}$
23.  $\frac{i}{i^{17}}$
24.  $i^{62}$

In Exercises 25 to 28, find the absolute value of each complex number.

25.  $| -8i |$
26.  $| 2 - 3i |$
27.  $| -4 + 5i |$
28.  $| -1 - i |$

In Exercises 29 to 32, write the complex number in trigonometric form.

29.  $z = 2 - 2i$
30.  $z = -\sqrt{3} + i$
31.  $z = -3 + 2i$
32.  $z = 4 - i$

In Exercises 33 to 36, write the complex number in standard form. Round approximate constants to the nearest thousandth.

33.  $z = 5(\cos 315^\circ + i \sin 315^\circ)$
34.  $z = 6\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$
35.  $z = 2(\cos 2 + i \sin 2)$
36.  $z = 3(\cos 115^\circ + i \sin 115^\circ)$

**In Exercises 37 to 42, multiply the complex numbers. Write the answer in standard form.**

37.  $3(\cos 225^\circ + i \sin 225^\circ) \cdot 10(\cos 45^\circ + i \sin 45^\circ)$

38.  $5(\cos 162^\circ + i \sin 162^\circ) \cdot 2(\cos 63^\circ + i \sin 63^\circ)$

39.  $3(\cos 12^\circ + i \sin 12^\circ) \cdot 4(\cos 126^\circ + i \sin 126^\circ)$

40.  $(\cos 23^\circ + i \sin 23^\circ) \cdot 4(\cos 233^\circ + i \sin 233^\circ)$

41.  $3(\cos 1.8 + i \sin 1.8) \cdot 5(\cos 2.5 + i \sin 2.5)$

42.  $6(\cos 3.1 + i \sin 3.1) \cdot 5(\cos 4.3 + i \sin 4.3)$

**In Exercises 43 to 48, divide the complex numbers. Write the answer in trigonometric form.**

43.  $\frac{6(\cos 50^\circ + i \sin 50^\circ)}{2(\cos 150^\circ + i \sin 150^\circ)}$

44.  $\frac{30(\cos 165^\circ + i \sin 165^\circ)}{10(\cos 55^\circ + i \sin 55^\circ)}$

45.  $\frac{40(\cos 66^\circ + i \sin 66^\circ)}{8(\cos 125^\circ + i \sin 125^\circ)}$

46.  $\frac{2(\cos 150^\circ + i \sin 150^\circ)}{\sqrt{2}(\cos 200^\circ + i \sin 200^\circ)}$

47.  $\frac{10(\cos 3.7 + i \sin 3.7)}{6(\cos 1.8 + i \sin 1.8)}$

48.  $\frac{4(\cos 1.2 + i \sin 1.2)}{8(\cos 5.2 + i \sin 5.2)}$

**In Exercises 49 to 54, find the indicated power. Write the answer in standard form.**

49.  $[3(\cos 45^\circ + i \sin 45^\circ)]^5$

50.  $\left[ \cos\left(\frac{11\pi}{8}\right) + i \sin\left(\frac{11\pi}{8}\right) \right]^8$

51.  $(1 - i\sqrt{3})^7$

52.  $(-2 - 2i)^{10}$

53.  $(\sqrt{2} - i\sqrt{2})^5$

54.  $(3 - 4i)^5$

**In Exercises 55 to 60, find the indicated roots. Write the answer in trigonometric form.**

55. The three cube roots of  $27i$

56. The four fourth roots of  $8i$

57. The four fourth roots of  $256$

58. The five fifth roots of  $-16\sqrt{2} - 16\sqrt{2}i$

59. The four fourth roots of  $81$

60. The three cube roots of  $-125$

## »»» Quantitative Reasoning: Graphing the Mandelbrot Set »»»

```
PROGRAM: MANBROT
Disp " THIS PROGRAM "
Disp " GRAPHS THE TOP "
Disp " HALF OF THE "
Disp " MANDELBROT SET "
Disp " "
Disp " PRESS ENTER "
Disp " TO START "
Pause
24→I
AxesOff:ClrDraw:FnOff
-2→Xmin:1.03→Xmax
0→Ymin:2→Ymax
For(A,-2,.4,.032)
For(B,0,√(4-A²),.032)
Pt-On(A,B)
(A+B)i→S
S→Z
For (C,0,I)
(Z²+S)→Z
If abs(Z)>2
Then
Pt-Off(A,B)
I→C
End:End:End
StorePic 2
```



The T1-83/T1-83 Plus/T1-84 Plus program at the left graphs the top half of the Mandelbrot set. The bottom half of the Mandelbrot set is a reflection across the  $x$ -axis of the top half. See **Figure 5.11** on page 357. The program uses the Mandelbrot iteration procedure, presented in the Exploring Concepts with Technology on page 356, to generate the iterates  $z_1, z_2, z_3, \dots$ . In theory, a complex number  $z_0$  is defined to be an element of the Mandelbrot set if and only if the absolute value of each of its iterates is less than or equal to 2. In reality, a calculator or computer program cannot check an infinite number of iterates, but a fairly accurate graph can be produced by checking the first 24 iterates. If the program finds that the absolute values of the first 24 iterates of  $z_0$  are all less than or equal to 2, then the program assumes that  $z_0$  is an element of the Mandelbrot set and plots a point at  $z_0$  in the complex plane. If the program finds an iterate whose absolute value is greater than 2, then the program does not plot a point at  $z_0$ . After the graph is produced, it is stored in memory, location **Pic 2**. You can recall the graph by pressing **RecallPic 2**. The **RecallPic** instruction is found in the **DRAW STO** menu.

**QR1.** Use the **MANBROT** program to graph the top half of the Mandelbrot set. The program requires about 25 minutes to complete the graph. The program step  $24 \rightarrow I$  sets the number of iterates to check. You can make the program run faster by using a natural number less than 24, but the graph will be less accurate. The graph will be slightly more accurate if you use a natural number larger than 24, but the program will run slower.

**QR2.** Examine the graph from Exercise QR1 or **Figure 5.11** to determine which of the following complex numbers are elements of the Mandelbrot set. The **WINDOW** settings for the program are  $X_{\min} = -2$ ,  $X_{\max} = 1.03$ ,  $Y_{\min} = 0$ , and  $Y_{\max} = 2$ .

$$-0.25 + 0.25i, -1 + 0.1i, -0.75 + 0.75i, 0.1 + 0.2i$$

**QR3.** Examine the iterates of  $-2$  to verify that  $-2$  is an element of the Mandelbrot set.

**QR4.** What is the first iterate of  $2i$ ? Is  $2i$  an element of the Mandelbrot set?

**QR5.** Which step in the program applies the Mandelbrot iteration procedure?

**QR6.** Search the Internet for additional information on the Mandelbrot set. Describe two properties of the Mandelbrot set that you find the most interesting.

## Chapter 5 Test

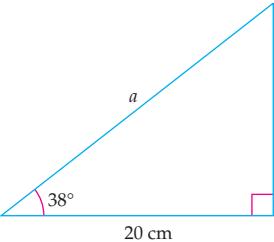
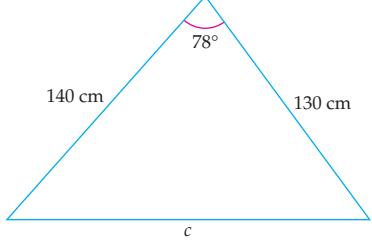
1. Write  $6 + \sqrt{-9}$  in the form  $a + bi$ .
2. Simplify:  $\sqrt{-18}$
3. Simplify  $(3 + \sqrt{-4}) + (7 - \sqrt{-9})$ . Write the answer in standard form.
4. Simplify  $(-1 + \sqrt{-25}) - (8 - \sqrt{-16})$ . Write the answer in standard form.
5. Simplify:  $(\sqrt{-12})(\sqrt{-3})$
6. Simplify:  $i^{263}$
7. Simplify:  $(3 + 7i) - (-2 - 9i)$
8. Simplify:  $(-6 - 9i)(4 + 3i)$
9. Simplify:  $(3 - 5i)(-3 + 5i)$
10. Simplify:  $\frac{4 - 5i}{i}$
11. Simplify:  $\frac{2 - 7i}{4 + 3i}$
12. Simplify:  $\frac{6 + 2i}{1 - i}$
13. Find the absolute value of  $3 - 5i$ .
14. Write  $3 - 3i$  in trigonometric form.
15. Write  $-6i$  in trigonometric form.
16. Write  $4(\cos 120^\circ + i \sin 120^\circ)$  in standard form.
17. Write  $5(\cos 225^\circ + i \sin 225^\circ)$  in standard form.
18. Simplify  $3(\cos 28^\circ + i \sin 28^\circ) \cdot 4(\cos 17^\circ + i \sin 17^\circ)$ . Write the answer in standard form.
19. Simplify  $5(\cos 115^\circ + i \sin 115^\circ) \cdot 4(\cos 10^\circ + i \sin 10^\circ)$ . Write the answer in standard form.

20. Simplify  $\frac{24(\cos 258^\circ + i \sin 258^\circ)}{6(\cos 78^\circ + i \sin 78^\circ)}$ . Write the answer in standard form.
21. Simplify  $\frac{18(\cos 50^\circ + i \sin 50^\circ)}{3(\cos 140^\circ + i \sin 140^\circ)}$ . Write the answer in standard form.
22. Simplify  $(2 - 2i\sqrt{3})^{12}$ . Write the answer in standard form.

**In Exercises 23 to 25, write the indicated roots and solutions in trigonometric form.**

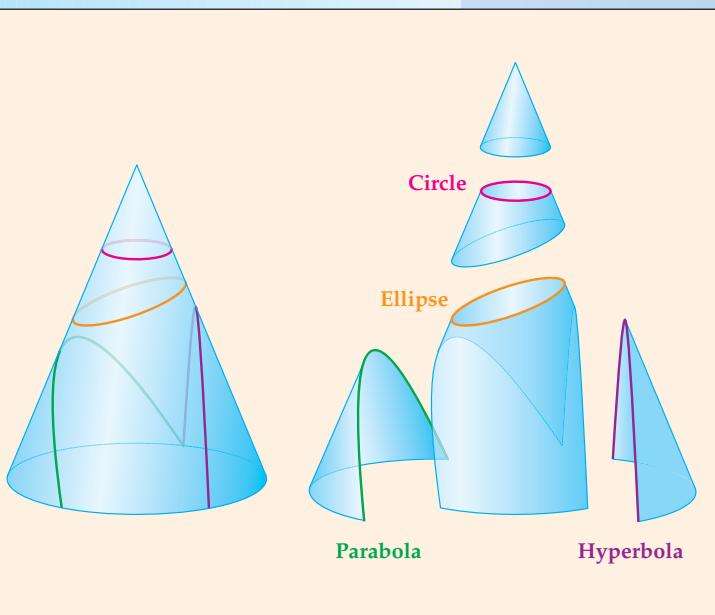
23. Find the six sixth roots of 64.
24. Find the three cube roots of  $-1 + i\sqrt{3}$ .
25. Find the five solutions of  $z^5 + 32 = 0$ .

## Cumulative Review Exercises

1. Solve  $x^2 - x - 6 \leq 0$ . Write the answer using interval notation.
2. What is the domain of  $f(x) = \frac{x^2}{x^2 - 4}$ ?
3. Find  $c$  in the domain of  $f(x) = \frac{x}{x + 1}$  such that  $f(c) = 2$ .
4. Given  $f(x) = \sin 3x$  and  $g(x) = \frac{x^2 - 1}{3}$ , find  $(f \circ g)(x)$ .
5. Given  $f(x) = \frac{x}{x - 1}$ , find  $f^{-1}(3)$ .
6. Convert  $\frac{3\pi}{2}$  radians to degrees.
7. Find the length of the hypotenuse  $a$  for the right triangle shown at the right.
- 
8. If  $t$  is any real number, what are the values of  $a$  and  $b$  for the inequality  $a \leq \sin t \leq b$ ?
9. Graph  $y = 3 \sin \pi x$ .
10. Graph  $y = \frac{1}{2} \tan \frac{\pi x}{4}$ .
11. Verify the identity  $\frac{\sin x}{1 + \cos x} = \csc x - \cot x$ .
12. Express  $\sin 2x \cos 3x - \sin 3x \cos 2x$  in terms of the sine function.
13. Given  $\sin \alpha = \frac{4}{5}$  in Quadrant I and  $\cos \beta = \frac{12}{13}$  in Quadrant IV, find  $\cos(\alpha + \beta)$ .
14. Find the exact value of  $\sin \left[ \sin^{-1} \left( \frac{3}{5} \right) + \cos^{-1} \left( -\frac{5}{13} \right) \right]$ .
15. Solve  $\sin 2x = \sqrt{3} \sin x$  for  $0 \leq x < 2\pi$ .
16. For the triangle at the right, find the length of side  $c$ . Round to the nearest ten.
- 
17. Find the angle between the vectors  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$  and  $\mathbf{w} = 5\mathbf{i} - 3\mathbf{j}$ . Round to the nearest tenth of a degree.
18. **WORK** A force of 100 pounds on a rope is used to drag a box up a ramp that is inclined  $15^\circ$ . If the rope makes an angle of  $30^\circ$  with the ground, find the work done in moving the box 15 feet along the ramp. Round to the nearest foot-pound.
19. Write the complex number  $2 + 2i$  in trigonometric form.
20. Find the three cube roots of  $-27$ . Write the roots in standard form.

# 6

# Topics in Analytic Geometry



## 6.1 Parabolas

## 6.2 Ellipses

## 6.3 Hyperbolas

## 6.4 Rotation of Axes

## 6.5 Introduction to Polar Coordinates

## 6.6 Polar Equations of the Conics

## 6.7 Parametric Equations

### Conic Sections and Their Applications

The study of the geometric figures called *conic sections* is one of the topics of this chapter. Each of these figures is formed by the intersection of a plane and a cone.<sup>1</sup>

The ancient Greeks were the first to study the conic sections. Their study was motivated by the many

new and interesting mathematical concepts they were able to discover, without regard to finding or producing practical applications. The ancient Greeks would be surprised to learn that their study of the conic sections helped produce a body of knowledge with many practical applications in several diverse fields, including astronomy, architecture, engineering, and satellite communications. **Exercise 55 on page 386** illustrates a medical application of conic sections, and **Exercise 47 on page 373** illustrates an application of conic sections in the design of a ski with a parabolic sidecut.



### Online Study Center

For online student resources, such as section quizzes, visit this website:  
[college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

<sup>1</sup>Only one branch of the hyperbola in the figure is displayed; however, all hyperbolas have two branches that are formed by the intersection of a plane and a double-napped cone as shown in Figure 6.1 on page 363.

## Section 6.1

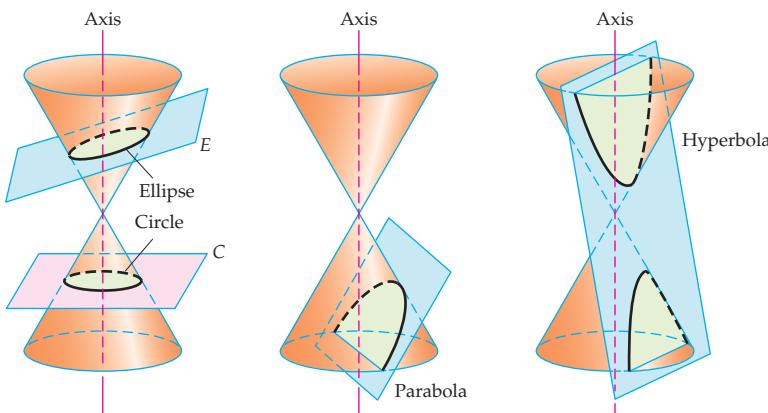
- **Parabolas with Vertex at  $(0, 0)$**
- **Parabolas with Vertex at  $(h, k)$**
- **Applications**

### take note

If the intersection of a plane and a cone is a point, a line, or two intersecting lines, then the intersection is called a *degenerate conic section*.

# Parabolas

The graph of a circle, an ellipse, a parabola, or a hyperbola can be formed by the intersection of a plane and a cone. Hence these figures are referred to as *conic sections*. See **Figure 6.1**.



**Figure 6.1**

Cones intersected by planes

A plane perpendicular to the axis of the cone intersects the cone in a circle (plane  $C$ ). The plane  $E$ , tilted so that it is not perpendicular to the axis, intersects the cone in an ellipse. When the plane is parallel to a line on the surface of the cone, the plane intersects the cone in a parabola. When the plane intersects both portions of the cone, a hyperbola is formed.

### Math Matters

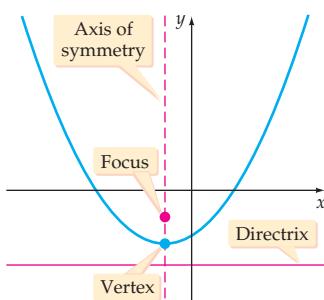
Appollonius (262–200 B.C.) wrote an eight-volume treatise entitled *On Conic Sections* in which he derived the formulas for all the conic sections. He was the first to use the words *parabola*, *ellipse*, and *hyperbola*.

### Parabolas with Vertex at $(0, 0)$

In addition to the geometric description of a conic section just given, a conic section can be defined as a set of points. This method uses specified conditions about a curve to determine which points in the coordinate system are points of the graph. For example, a parabola can be defined by the following set of points.

#### Definition of a Parabola

A **parabola** is the set of points in a plane that are equidistant from a fixed line (the **directrix**) and a fixed point (the **focus**) not on the directrix.



**Figure 6.2**

The line that passes through the focus and is perpendicular to the directrix is called the **axis of symmetry** of the parabola. The midpoint of the line segment between the focus and directrix on the axis of symmetry is the **vertex** of the parabola, as shown in **Figure 6.2**.

Using this definition of a parabola, we can determine an equation of a parabola. Suppose that the coordinates of the vertex of a parabola are  $V(0, 0)$  and the axis of symmetry is the  $y$ -axis. The equation of the directrix is  $y = -p$ ,  $p > 0$ . The focus lies on the axis of symmetry and is the same distance from the vertex as the vertex is from the directrix. Thus the coordinates of the focus are  $F(0, p)$ , as shown in **Figure 6.3**.

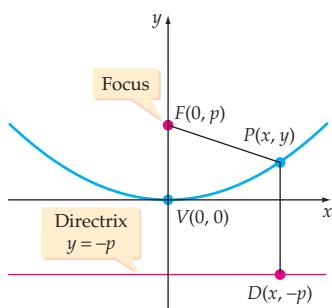


Figure 6.3

**TO REVIEW**  
**Axis of Symmetry**  
 See page 54.

**take note**

The tests for  $y$ -axis and  $x$ -axis symmetry can be used to verify these statements and provide connections to earlier topics on symmetry.

Let  $P(x, y)$  be any point  $P$  on the parabola. Then, using the distance formula and the fact that the distance between any point  $P$  on the parabola and the focus is equal to the distance from the point  $P$  to the directrix, we can write the equation

$$d(P, F) = d(P, D)$$

By the distance formula,

$$\sqrt{(x - 0)^2 + (y - p)^2} = y + p$$

Now, squaring each side and simplifying, we get

$$\begin{aligned} (\sqrt{(x - 0)^2 + (y - p)^2})^2 &= (y + p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 &= 4py \end{aligned}$$

This is the standard form of the equation of a parabola with vertex at the origin and the  $y$ -axis as its axis of symmetry. The standard form of the equation of a parabola with vertex at the origin and the  $x$ -axis as its axis of symmetry is derived in a similar manner.

### Standard Forms of the Equation of a Parabola with Vertex at the Origin

#### Axis of Symmetry Is the $y$ -Axis

The standard form of the equation of a parabola with vertex  $(0, 0)$  and the  $y$ -axis as its axis of symmetry is

$$x^2 = 4py$$

The focus is  $(0, p)$ , and the equation of the directrix is  $y = -p$ . If  $p > 0$ , the graph of the parabola opens up. See **Figure 6.4a**. If  $p < 0$ , the graph of the parabola opens down. See **Figure 6.4b**.

#### Axis of Symmetry Is the $x$ -Axis

The standard form of the equation of a parabola with vertex  $(0, 0)$  and the  $x$ -axis as its axis of symmetry is

$$y^2 = 4px$$

The focus is  $(p, 0)$ , and the equation of the directrix is  $x = -p$ . If  $p > 0$ , the graph of the parabola opens to the right. See **Figure 6.4c**. If  $p < 0$ , the graph of the parabola opens to the left. See **Figure 6.4d**.

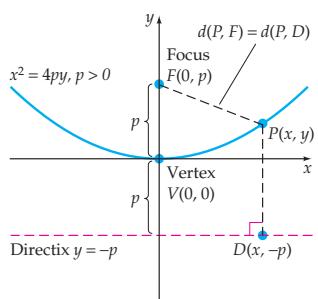
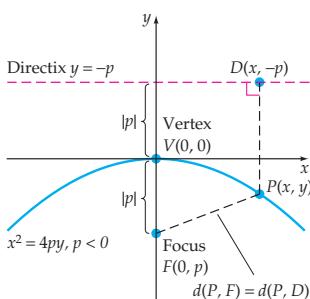
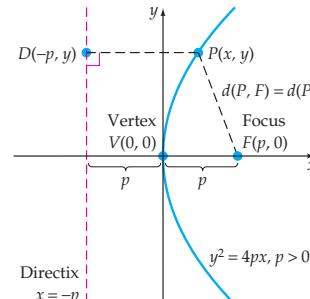
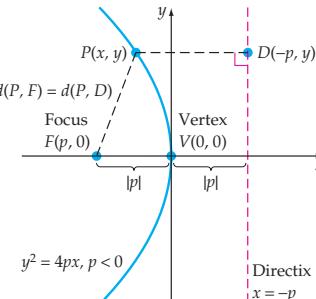
a. The graph of  $x^2 = 4py$  with  $p > 0$ b. The graph of  $x^2 = 4py$  with  $p < 0$ c. The graph of  $y^2 = 4px$  with  $p > 0$ d. The graph of  $y^2 = 4px$  with  $p < 0$ 

Figure 6.4

**QUESTION** Does the graph of  $y^2 = -4x$  open up, down, to the left, or to the right?

### EXAMPLE 1 » Find the Focus and Directrix of a Parabola

Find the focus and directrix of the parabola given by the equation

$$y = -\frac{1}{2}x^2.$$

#### Solution

Because the  $x$  term is squared, the standard form of the equation is  $x^2 = 4py$ .

$$y = -\frac{1}{2}x^2$$

$$x^2 = -2y$$

- Write the given equation in standard form.

Comparing this equation with  $x^2 = 4py$  gives

$$4p = -2$$

$$p = -\frac{1}{2}$$

Because  $p$  is negative, the parabola opens down, and the focus is below the vertex  $(0, 0)$ , as shown in **Figure 6.5**. The coordinates of the focus are  $\left(0, -\frac{1}{2}\right)$ . The equation of the directrix is  $y = \frac{1}{2}$ .

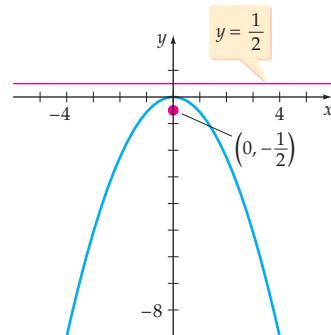


Figure 6.5

» Try Exercise 6, page 371

### EXAMPLE 2 » Find the Equation of a Parabola in Standard Form

Find the equation in standard form of the parabola with vertex at the origin and focus at  $(-2, 0)$ .

#### Solution

Because the vertex is  $(0, 0)$  and the focus is at  $(-2, 0)$ ,  $p = -2$ . The graph of the parabola opens toward the focus, so in this case the parabola opens to

Continued ►

**ANSWER** The graph opens to the left.

the left. The equation in standard form of the parabola that opens to the left is  $y^2 = 4px$ . Substitute  $-2$  for  $p$  in this equation and simplify.

$$y^2 = 4(-2)x = -8x$$

The equation of the parabola is  $y^2 = -8x$ .

» Try Exercise 30, page 371

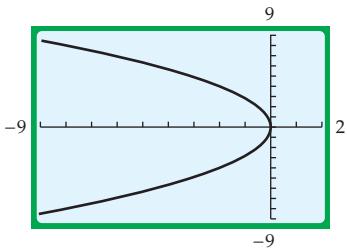


Figure 6.6



### Integrating Technology

The graph of  $y^2 = -8x$  is shown in **Figure 6.6**. Note that the graph is not the graph of a function. To graph  $y^2 = -8x$  with a graphing utility, we first solve for  $y$  to produce  $y = \pm\sqrt{-8x}$ . From this equation we can see that for any  $x < 0$ , there are two values of  $y$ . For example, when  $x = -2$ ,

$$y = \pm\sqrt{(-8)(-2)} = \pm\sqrt{16} = \pm 4$$

The graph of  $y^2 = -8x$  in **Figure 6.6** was constructed by graphing both  $Y_1 = \sqrt{-8x}$  and  $Y_2 = -\sqrt{-8x}$  in the same window.

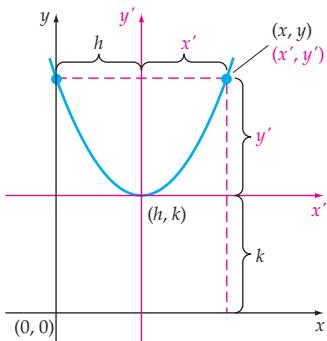


Figure 6.7

### Parabolas with Vertex at $(h, k)$

The equation of a parabola with a vertical or horizontal axis of symmetry and with vertex at a point  $(h, k)$  can be found by using the translations discussed previously. Consider a coordinate system with coordinate axes labeled  $x'$  and  $y'$  placed so that its origin is at  $(h, k)$  of the  $xy$ -coordinate system.

The relationship between an ordered pair in the  $x'y'$ -coordinate system and in the  $xy$ -coordinate system is given by the **transformation equations**

$$\begin{aligned} x' &= x - h \\ y' &= y - k \end{aligned} \tag{1}$$

Now consider a parabola with vertex at  $(h, k)$ , as shown in **Figure 6.7**. Create a new coordinate system with axes labeled  $x'$  and  $y'$  and with its origin at  $(h, k)$ . The equation of a parabola in the  $x'y'$ -coordinate system is

$$(x')^2 = 4py' \tag{2}$$

Using the transformation Equations (1), we can substitute the expressions for  $x'$  and  $y'$  into Equation (2). The standard form of the equation of a parabola with vertex  $(h, k)$  and a vertical axis of symmetry is

$$(x - h)^2 = 4p(y - k)$$

Similarly, we can derive the standard form of the equation of a parabola with vertex  $(h, k)$  and a horizontal axis of symmetry.

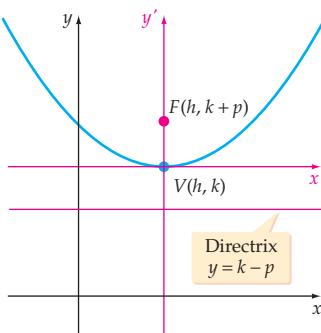


Figure 6.8

### Standard Forms of the Equation of a Parabola with Vertex at $(h, k)$

#### Vertical Axis of Symmetry

The standard form of the equation of a parabola with vertex  $(h, k)$  and a vertical axis of symmetry is

$$(x - h)^2 = 4p(y - k)$$

The focus is  $(h, k + p)$ , and the equation of the directrix is  $y = k - p$ . If  $p > 0$ , the parabola opens up. See **Figure 6.8**. If  $p < 0$ , the parabola opens down.

#### Horizontal Axis of Symmetry

The standard form of the equation of a parabola with vertex  $(h, k)$  and a horizontal axis of symmetry is

$$(y - k)^2 = 4p(x - h)$$

The focus is  $(h + p, k)$ , and the equation of the directrix is  $x = h - p$ . If  $p > 0$ , the parabola opens to the right. If  $p < 0$ , the parabola opens to the left.

In Example 3 we complete the square to find the standard form of a parabola, and then use the standard form to determine the vertex, focus, and directrix of the parabola.

### EXAMPLE 3 » Find the Focus and Directrix of a Parabola

Find the equation of the directrix and the coordinates of the vertex and focus of the parabola given by the equation  $3x + 2y^2 + 8y - 4 = 0$ .

#### Solution

Rewrite the equation so that the  $y$  terms are on one side of the equation, and then complete the square on  $y$ .

$$3x + 2y^2 + 8y - 4 = 0$$

$$2y^2 + 8y = -3x + 4$$

$$2(y^2 + 4y) = -3x + 4$$

$$2(y^2 + 4y + 4) = -3x + 4 + 8$$

$$2(y + 2)^2 = -3(x - 4)$$

$$(y + 2)^2 = -\frac{3}{2}(x - 4)$$

- Complete the square. Note that  $2 \cdot 4 = 8$  is added to each side.

- Simplify and factor.

- Write the equation in standard form.

Comparing this equation to  $(y - k)^2 = 4p(x - h)$ , we have a parabola that opens to the left with vertex  $(4, -2)$  and  $4p = -\frac{3}{2}$ . Thus  $p = -\frac{3}{8}$ .

The coordinates of the focus are

$$\left(4 + \left(-\frac{3}{8}\right), -2\right) = \left(\frac{29}{8}, -2\right)$$

**TO REVIEW**  
Completing the Square  
See page 26.

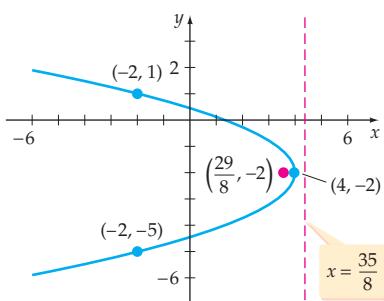


Figure 6.9

The equation of the directrix is

$$x = 4 - \left(-\frac{3}{8}\right) = \frac{35}{8}$$

Choosing some values for  $y$  and finding the corresponding values for  $x$ , we plot a few points. Because the line  $y = -2$  is the axis of symmetry, for each point on one side of the axis of symmetry there is a corresponding point on the other side. Two points are  $(-2, 1)$  and  $(-2, -5)$ . See **Figure 6.9**.

» Try Exercise 22, page 371

#### EXAMPLE 4 » Find the Equation in Standard Form of a Parabola

Find the equation in standard form of the parabola with directrix  $x = -1$  and focus  $(3, 2)$ .

##### Solution

The vertex is the midpoint of the line segment joining the focus  $(3, 2)$  and the point  $(-1, 2)$  on the directrix.

$$(h, k) = \left(\frac{-1 + 3}{2}, \frac{2 + 2}{2}\right) = (1, 2)$$

The standard form of the equation is  $(y - k)^2 = 4p(x - h)$ . The distance from the vertex to the focus is 2. Thus  $4p = 4(2) = 8$ , and the equation of the parabola in standard form is  $(y - 2)^2 = 8(x - 1)$ . See **Figure 6.10**.

» Try Exercise 32, page 371

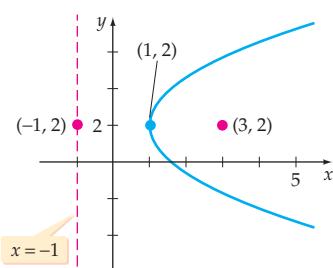


Figure 6.10

#### Applications

A principle of physics states that when light is reflected from a point  $P$  on a surface, the angle of incidence (that of the incoming ray) equals the angle of reflection (that of the outgoing ray). See **Figure 6.11**. This principle applied to parabolas has some useful consequences.

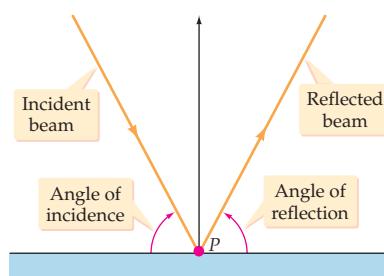


Figure 6.11

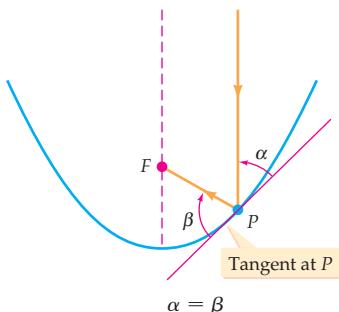


Figure 6.12

### Reflective Property of a Parabola

The line tangent to a parabola at a point  $P$  makes equal angles with the line through  $P$  and parallel to the axis of symmetry and the line through  $P$  and the focus of the parabola (see **Figure 6.12**).

A cross section of the reflecting mirror of a telescope has the shape of a parabola. The incoming parallel rays of light are reflected from the surface of the mirror to the eyepiece. See **Figure 6.13**.

Flashlights and car headlights also make use of this reflective property. The light bulb is positioned at the focus of the parabolic reflector, which causes the reflected light to be reflected outward in parallel rays. See **Figure 6.14**.

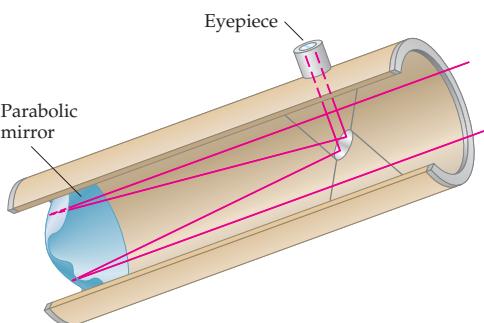


Figure 6.13

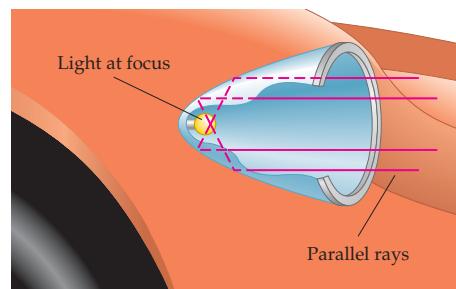


Figure 6.14

When a parabola is revolved about its axis, it produces a three-dimensional surface called a **paraboloid**. The **focus of a paraboloid** is the same as the focus of the parabola that was revolved to generate the paraboloid. The **vertex of a paraboloid** is the same as the vertex of the parabola that was revolved to generate the paraboloid. In Example 5 we find the focus of a satellite dish that has the shape of a paraboloid.

### EXAMPLE 5 >> Find the Focus of a Satellite Dish

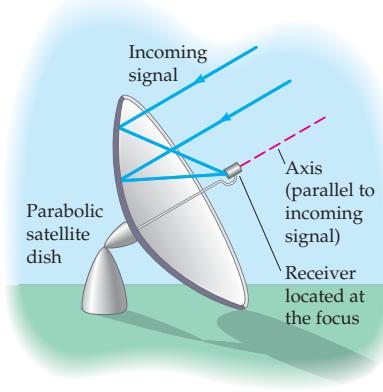


Figure 6.15

A satellite dish has the shape of a paraboloid. The signals that it receives are reflected to a receiver that is located at the focus of the paraboloid. If the dish is 8 feet across at its opening and 1.25 feet deep at its center, determine the location of its focus.

#### Solution

**Figure 6.15** shows that a cross section of the paraboloid along its axis of symmetry is a parabola. **Figure 6.16** shows this cross section placed in a rectangular coordinate system with the vertex of the parabola at  $(0, 0)$  and the axis of symmetry of the parabola on the  $y$ -axis. The parabola has an equation of the form

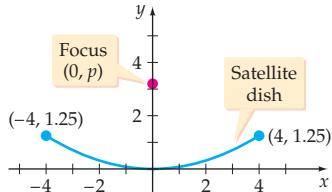
$$4py = x^2$$

Because the parabola contains the point  $(4, 1.25)$ , this equation is satisfied by the substitutions  $x = 4$  and  $y = 1.25$ . Thus we have

$$4p(1.25) = 4^2$$

$$5p = 16$$

$$p = \frac{16}{5}$$



**Figure 6.16**

The focus of the satellite dish is on the axis of symmetry of the dish, and it is  $3\frac{1}{5}$  feet above the vertex of the dish. See **Figure 6.16**.

» Try Exercise 40, page 372



### Topics for Discussion

1. Do the graphs of the parabola given by  $y = x^2$  and the vertical line given by  $x = 10,000$  intersect? Explain.
2. A student claims that the focus of the parabola given by  $y = 8x^2$  is at  $(0, 2)$  because  $4p = 8$  implies that  $p = 2$ . Explain the error in the student's reasoning.
3. The vertex of a parabola is always halfway between its focus and its directrix. Do you agree? Explain.
4. A tutor claims that the graph of  $(x - h)^2 = 4p(y - k)$  has a  $y$ -intercept of  $\left(0, \frac{h^2}{4p} + k\right)$ . Explain why the tutor is correct.

## Exercise Set 6.1

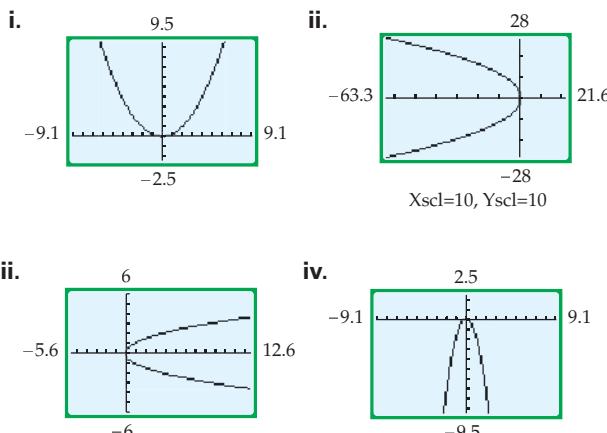
1. Examine the following four equations and the graphs labeled **i**, **ii**, **iii**, and **iv**. Determine which graph is the graph of each equation.

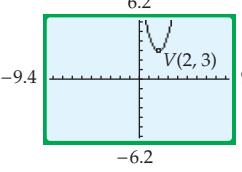
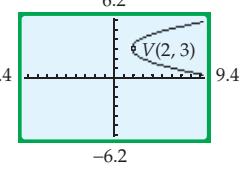
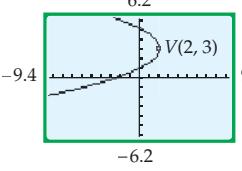
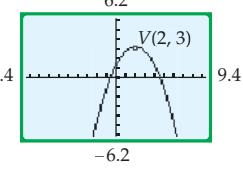
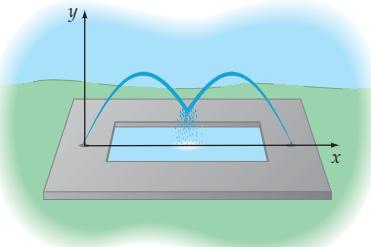
a.  $y^2 = x$

b.  $x^2 = 4y$

c.  $x^2 = -\frac{1}{2}y$

d.  $y^2 = -12x$



2. Examine the following four equations and the graphs labeled **i**, **ii**, **iii**, and **iv**. Determine which graph is the graph of each equation.
- a.  $(y - 3)^2 = x - 2$       b.  $(y - 3)^2 = -2(x - 2)$   
 c.  $\frac{1}{2}(y - 3) = (x - 2)^2$       d.  $-2(y - 3) = (x - 2)^2$
- i. 
- ii. 
- iii. 
- iv. 
- In Exercises 3 to 28, find the vertex, focus, and directrix of the parabola given by each equation. Sketch the graph.
3.  $x^2 = -4y$       4.  $2y^2 = x$   
 5.  $y^2 = \frac{1}{3}x$       6.  $x^2 = -\frac{1}{4}y$   
 7.  $(x - 2)^2 = 8(y + 3)$       8.  $(y + 1)^2 = 6(x - 1)$   
 9.  $(y + 4)^2 = -4(x - 2)$       10.  $(x - 3)^2 = -(y + 2)$   
 11.  $(y - 1)^2 = 2x + 8$       12.  $(x + 2)^2 = 3y - 6$   
 13.  $(2x - 4)^2 = 8y - 16$       14.  $(3x + 6)^2 = 18y - 36$   
 15.  $x^2 + 8x - y + 6 = 0$       16.  $x^2 - 6x + y + 10 = 0$   
 17.  $x + y^2 - 3y + 4 = 0$       18.  $x - y^2 - 4y + 9 = 0$   
 19.  $2x - y^2 - 6y + 1 = 0$       20.  $3x + y^2 + 8y + 4 = 0$   
 21.  $x^2 + 3x + 3y - 1 = 0$       22.  $x^2 + 5x - 4y - 1 = 0$   
 23.  $2x^2 - 8x - 4y + 3 = 0$       24.  $6x - 3y^2 - 12y + 4 = 0$   
 25.  $2x + 4y^2 + 8y - 5 = 0$       26.  $4x^2 - 12x + 12y + 7 = 0$   
 27.  $3x^2 - 6x - 9y + 4 = 0$       28.  $2x - 3y^2 + 9y + 5 = 0$
29. Find the equation in standard form of the parabola with vertex at the origin and focus  $(0, -4)$ .
30. Find the equation in standard form of the parabola with vertex at the origin and focus  $(5, 0)$ .
31. Find the equation in standard form of the parabola with vertex at  $(-1, 2)$  and focus  $(-1, 3)$ .
32. Find the equation in standard form of the parabola with vertex at  $(2, -3)$  and focus  $(0, -3)$ .
33. Find the equation in standard form of the parabola with focus  $(3, -3)$  and directrix  $y = -5$ .
34. Find the equation in standard form of the parabola with focus  $(-2, 4)$  and directrix  $x = 4$ .
35. Find the equation in standard form of the parabola that has vertex  $(-4, 1)$ , has its axis of symmetry parallel to the  $y$ -axis, and passes through the point  $(-2, 2)$ .
36. Find the equation in standard form of the parabola that has vertex  $(3, -5)$ , has its axis of symmetry parallel to the  $x$ -axis, and passes through the point  $(4, 3)$ .
37. **STRUCTURAL DEFECTS** Ultrasound is used as a nondestructive method of determining whether a support beam for a structure has an internal fracture. In one scanning procedure, if the resulting image is a parabola, engineers know that there is a structural defect. Suppose that a scan produced an image whose equation is
- $$x = -0.325y^2 + 13y + 120$$
- Determine the vertex and focus of the graph of this parabola.
38. **FOUNTAIN DESIGN** A fountain in a shopping mall has two parabolic arcs of water intersecting as shown below. The equation of one parabola is  $y = -0.25x^2 + 2x$  and the equation of the second parabola is
- $$y = -0.25x^2 + 4.5x - 16.25$$
- How high above the base of the fountain do the parabolas intersect? All dimensions are in feet.
- 

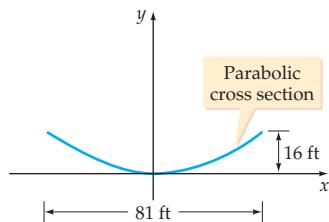
- 39. SATELLITE DISH** A satellite dish has the shape of a paraboloid. The signals that it receives are reflected to a receiver that is located at the focus of the paraboloid. If the dish is 8 feet across at its opening and 1 foot deep at its vertex, determine the location (distance above the vertex of the dish) of its focus.

- 40. RADIO TELESCOPES** The antenna of a radio telescope is a paraboloid measuring 81 feet across with a depth of 16 feet. Determine, to the nearest tenth of a foot, the distance from the vertex to the focus of this antenna.

- 41. CAPTURING SOUND** During televised football games, a parabolic microphone is used to capture sounds. The shield of the microphone is a paraboloid with a diameter of 18.75 inches and a depth of 3.66 inches. To pick up the sounds, a microphone is placed at the focus of the paraboloid. How far (to the nearest tenth of an inch) from the vertex of the paraboloid should the microphone be placed?



- 42. THE LOVELL TELESCOPE** The Lovell Telescope is a radio telescope located at the Jodrell Bank Observatory in Cheshire, England. The dish of the telescope has the shape of a paraboloid with a diameter of 250 feet. The distance from the vertex of the dish to its focus is 75 feet.



- a. Find an equation of a cross section of the paraboloid that passes through the vertex of the paraboloid. Assume that the dish has its vertex at  $(0, 0)$  and a vertical axis of symmetry.

- b. Find the depth of the dish. Round to the nearest foot.

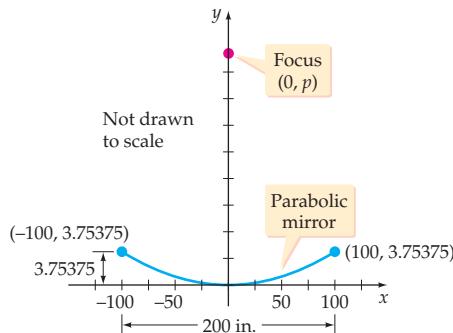
- 43.** The surface area of a paraboloid with radius  $r$  and depth  $d$  is given by  $S = \frac{\pi r}{6d^2}[(r^2 + 4d^2)^{3/2} - r^3]$ .

Approximate (to the nearest 100 square feet) the surface area of

- a. the radio telescope in Exercise 40.

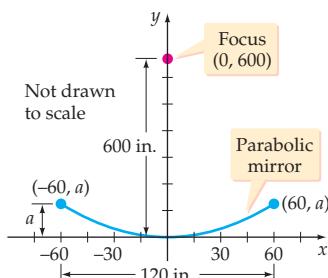
- b. the Lovell Telescope (see Exercise 42).

- 44. THE HALE TELESCOPE** The parabolic mirror in the Hale telescope at the Palomar Observatory in southern California has a diameter of 200 inches and a concave depth of 3.75375 inches. Determine the location of its focus (to the nearest inch).



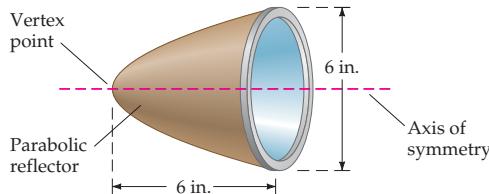
Cross Section of the Mirror in the Hale Telescope

- 45. THE LICK TELESCOPE** The parabolic mirror in the Lick telescope at the Lick Observatory on Mount Hamilton has a diameter of 120 inches and a focal length of 600 inches. (Note: The **focal length** of a parabolic mirror is the distance from the vertex of the mirror to the mirror's focus.) In the construction of the mirror, workers ground the mirror as shown in the following diagram. Determine the dimension  $a$ , which is the concave depth of the mirror.

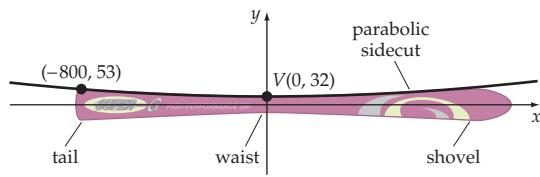


Cross Section of the Mirror in the Lick Telescope

- 46. HEADLIGHT DESIGN** A light source is to be placed on the axis of symmetry of the parabolic reflector shown in the figure below. How far to the right of the vertex point should the light source be located if the designer wishes the reflected light rays to form a beam of parallel rays?



- 47. Ski Design** Many contemporary skis have parabolic sidecuts that allow a skier to carve tighter turns than are possible with traditional skis. In the following diagram,  $x$  is the directed distance to the right or left of the  $y$ -axis (with  $x$  and  $y$  measured in millimeters). The  $x$ -axis is on the center horizontal axis of the ski. The vertex of the parabolic sidecut is  $V(0, 32)$ .

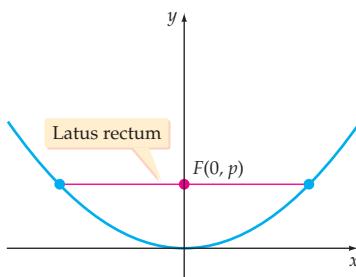


- a. Find the equation in standard form of the parabolic sidecut.

b. How wide is the ski at its shovel (the widest point near the front of the ski), where  $x = 900$ ? Round to the nearest millimeter.

 The only information we have about a particular parabola is that  $(2, 3)$  and  $(-2, 3)$  are points on the parabola. Explain why it is not possible to find the equation of this particular parabola using just this information.

In Exercises 49 to 51, use the following definition of latus rectum: The line segment that has endpoints on a parabola, passes through the focus of the parabola, and is perpendicular to the axis of symmetry is called the *latus rectum* of the parabola.



49. Find the length of the latus rectum for the parabola  $x^2 = 4y$ .

50. Find the length of the latus rectum for the parabola  $y^2 = -8x$ .

51. Find the length of the latus rectum for any parabola in terms of  $|p|$ , the distance from the vertex of the parabola to its focus.

The result of Exercise 51 can be stated as the following theorem: Two points on a parabola will be  $2|p|$  units on each side of the axis of symmetry on the line through the focus and perpendicular to that axis.

52. Use the theorem to sketch a graph of the parabola given by the equation  $(x - 3)^2 = 2(y + 1)$ .

53. Use the theorem to sketch a graph of the parabola given by the equation  $(y + 4)^2 = -(x - 1)$ .

54. By using the definition of a parabola, find the equation in standard form of the parabola with  $V(0, 0)$ ,  $F(-c, 0)$ , and directrix  $x = c$ .

55. Sketch a graph of  $4(y - 2) = |x| - 1$ .

56. Find the equation of the directrix of the parabola with vertex at the origin and focus at the point  $(1, 1)$ .

57. Find the equation of the parabola with vertex at the origin and focus at the point  $(1, 1)$ . (*Hint:* You will need the answer to Exercise 56 and the definition of a parabola.)

## »»» Projects

1.  **3-D OPTICAL ILLUSION** In the photo below, a strawberry appears to float above a black disc.



Mirage by InnovaToys

*Photo source: <http://innovatoys.com/c/OPT>*

If you try to touch the strawberry, you will discover that the strawberry is not located where it appears. Search the Internet to discover how parabolic mirrors are used to create this illusion. Write a few sentences, along with a diagram that explains how the illusion is accomplished.

## Section 6.2

- Ellipses with Center at  $(0, 0)$
- Ellipses with Center at  $(h, k)$
- Eccentricity of an Ellipse
- Applications

## Ellipses

### PREPARE FOR THIS SECTION

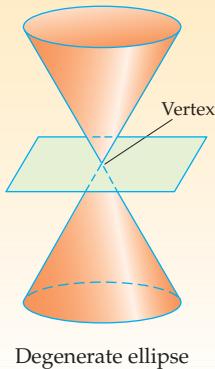
Prepare for this section by completing the following exercises. The answers can be found on page A25.

- PS1.** Find the midpoint and the length of the line segment between  $P_1(5, 1)$  and  $P_2(-1, 5)$ . [1.2]
- PS2.** Solve:  $x^2 + 6x - 16 = 0$  [1.1]
- PS3.** Solve:  $x^2 - 2x = 2$  [1.1]
- PS4.** Complete the square of  $x^2 - 8x$  and write the result as the square of a binomial. [1.2]
- PS5.** Solve  $(x - 2)^2 + y^2 = 4$  for  $y$ . [1.2]
- PS6.** Graph:  $(x - 2)^2 + (y + 3)^2 = 16$  [1.2]

An ellipse is another of the conic sections formed when a plane intersects a right circular cone. If  $\beta$  is the angle at which the plane intersects the axis of the cone and

**take note**

If a plane intersects a cone at the vertex of the cone so that the resulting figure is a point, the point is called a *degenerate ellipse*. See the accompanying figure.



Degenerate ellipse

$\alpha$  is the angle shown in **Figure 6.17**, an ellipse is formed when  $\alpha < \beta < 90^\circ$ . If  $\beta = 90^\circ$ , then a circle is formed.

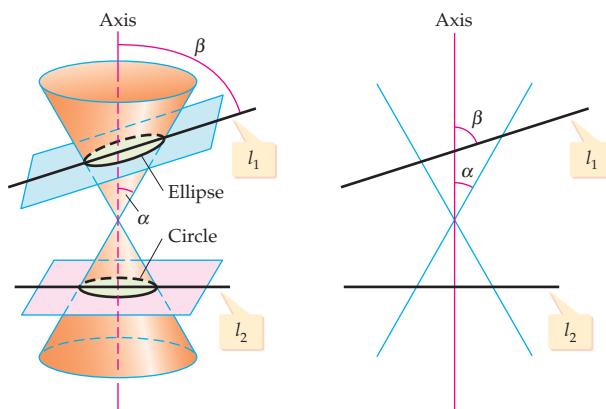


Figure 6.17

As was the case for a parabola, there is a definition for an ellipse in terms of a certain set of points in the plane.

**Definition of an Ellipse**

An **ellipse** is the set of all points in the plane the sum of whose distances from two fixed points (**foci**) is a positive constant.

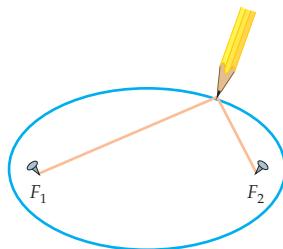


Figure 6.18

Equipped only with a piece of string and two tacks, we can use this definition to draw an ellipse (see **Figure 6.18**). Tack the ends of the string to the foci, and trace a curve with a pencil held tight against the string. The resulting curve is an ellipse. The positive constant mentioned in the definition of an ellipse is the length of the string.

**Ellipses with Center at  $(0, 0)$** 

The graph of an ellipse has two axes of symmetry (see **Figure 6.19**). The longer axis is called the **major axis**. The foci of the ellipse are on the major axis. The shorter axis is called the **minor axis**. It is customary to denote the length of the major axis by  $2a$  and the length of the minor axis by  $2b$ . The **center** of the ellipse is the midpoint of the major axis. The endpoints of the major axis are the **vertices** (plural of *vertex*) of the ellipse.

A **semimajor axis** of an ellipse is a line segment that connects the center point of the ellipse with a vertex. Its length is half the length of the major axis. A **semiminor axis** of an ellipse is a line segment that lies on the minor axis and connects the center point with a point on the ellipse. Its length is half the length of the minor axis.

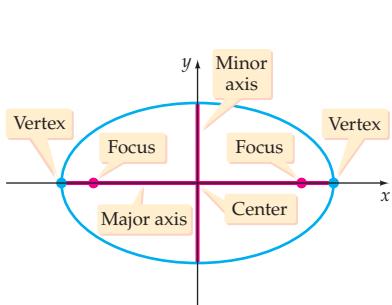


Figure 6.19

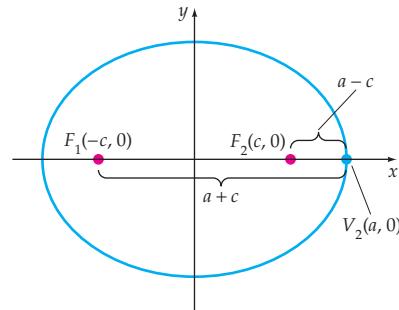


Figure 6.20

Consider the point  $V_2(a, 0)$ , which is one vertex of an ellipse, and the points  $F_2(c, 0)$  and  $F_1(-c, 0)$ , which are the foci of the ellipse shown in **Figure 6.20**. The distance from  $V_2$  to  $F_1$  is  $a + c$ . Similarly, the distance from  $V_2$  to  $F_2$  is  $a - c$ . From the definition of an ellipse, the sum of the distances from any point on the ellipse to the foci is a positive constant. By adding the expressions  $a + c$  and  $a - c$ , we have

$$(a + c) + (a - c) = 2a$$

Thus the positive constant referred to in the definition of an ellipse is  $2a$ , the length of the major axis.

Now let  $P(x, y)$  be any point on the ellipse (see **Figure 6.21**). By using the definition of an ellipse, we have

$$\begin{aligned} d(P, F_1) + d(P, F_2) &= 2a \\ \sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} &= 2a \end{aligned}$$

Subtract the second radical from each side of the equation, and then square each side.

$$\begin{aligned} [\sqrt{(x + c)^2 + y^2}]^2 &= [2a - \sqrt{(x - c)^2 + y^2}]^2 \\ (x + c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2 \\ x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 \\ 4cx - 4a^2 &= -4a\sqrt{(x - c)^2 + y^2} \\ [-cx + a^2]^2 &= [a\sqrt{(x - c)^2 + y^2}]^2 \end{aligned}$$

• Divide by  $-4$ , and then square each side.  
 • Rewrite with  $x$  and  $y$  terms on the left side.  
 • Factor.  
 • Let  $b^2 = a^2 - c^2$ .  
 • Divide each side by  $-a^2b^2$ . The result is an equation of an ellipse with center at  $(0, 0)$ .

$$\begin{aligned} c^2x^2 - 2cxa^2 + a^4 &= a^2x^2 - 2cxa^2 + a^2c^2 + a^2y^2 \\ -a^2x^2 + c^2x^2 - a^2y^2 &= -a^4 + a^2c^2 \\ -(a^2 - c^2)x^2 - a^2y^2 &= -a^2(a^2 - c^2) \\ -b^2x^2 - a^2y^2 &= -a^2b^2 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \end{aligned}$$

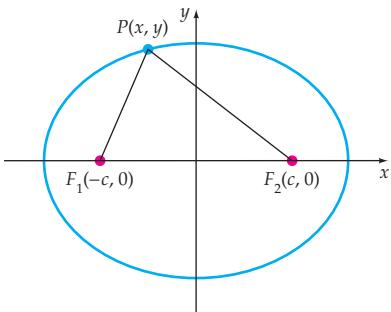


Figure 6.21

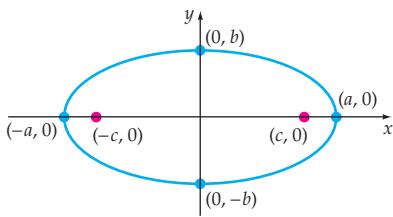
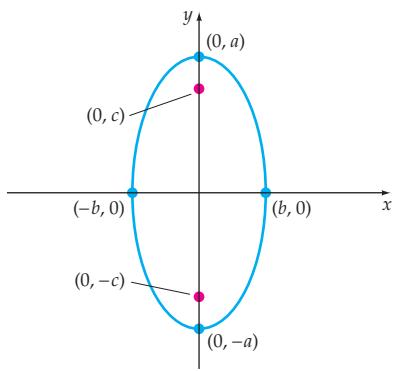
a. Major axis on  $x$ -axisb. Major axis on  $y$ -axis

Figure 6.22

### Standard Forms of the Equation of an Ellipse with Center at the Origin

#### Major Axis on the $x$ -Axis

The standard form of the equation of an ellipse with center at the origin and major axis on the  $x$ -axis (see Figure 6.22a) is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$$

The length of the major axis is  $2a$ . The length of the minor axis is  $2b$ . The coordinates of the vertices are  $(a, 0)$  and  $(-a, 0)$ , and the coordinates of the foci are  $(c, 0)$  and  $(-c, 0)$ , where  $c^2 = a^2 - b^2$ .

#### Major Axis on the $y$ -Axis

The standard form of the equation of an ellipse with center at the origin and major axis on the  $y$ -axis (see Figure 6.22b) is given by

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b$$

The length of the major axis is  $2a$ . The length of the minor axis is  $2b$ . The coordinates of the vertices are  $(0, a)$  and  $(0, -a)$ , and the coordinates of the foci are  $(0, c)$  and  $(0, -c)$ , where  $c^2 = a^2 - b^2$ .

**QUESTION** For the graph of  $\frac{x^2}{16} + \frac{y^2}{25} = 1$ , is the major axis on the  $x$ -axis or the  $y$ -axis?

### EXAMPLE 1 » Find the Vertices and Foci of an Ellipse

Find the vertices and foci of the ellipse given by the equation  $\frac{x^2}{25} + \frac{y^2}{49} = 1$ .

Sketch the graph.

#### Solution

Because the  $y^2$  term has the larger denominator, the major axis is on the  $y$ -axis.

$$\begin{aligned} a^2 &= 49 & b^2 &= 25 & c^2 &= a^2 - b^2 \\ a &= 7 & b &= 5 & &= 49 - 25 = 24 \\ & & & & c &= \sqrt{24} = 2\sqrt{6} \end{aligned}$$

Continued ►

**ANSWER** Because  $25 > 16$ , the major axis is on the  $y$ -axis.

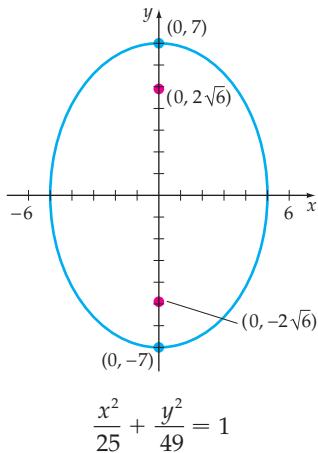


Figure 6.23

The vertices are  $(0, 7)$  and  $(0, -7)$ . The foci are  $(0, 2\sqrt{6})$  and  $(0, -2\sqrt{6})$ . See Figure 6.23.

» Try Exercise 22, page 385

An ellipse with foci  $(3, 0)$  and  $(-3, 0)$  and major axis of length 10 is shown in Figure 6.24. To find the equation of the ellipse in standard form, we must find  $a^2$  and  $b^2$ . Because the foci are on the major axis, the major axis is on the  $x$ -axis. The length of the major axis is  $2a$ . Thus  $2a = 10$ . Solving for  $a$ , we have  $a = 5$  and  $a^2 = 25$ .

Because the foci are  $(3, 0)$  and  $(-3, 0)$  and the center of the ellipse is the midpoint between the two foci, the distance from the center of the ellipse to a focus is 3. Therefore,  $c = 3$ . To find  $b^2$ , use the equation

$$\begin{aligned}c^2 &= a^2 - b^2 \\9 &= 25 - b^2 \\b^2 &= 16\end{aligned}$$

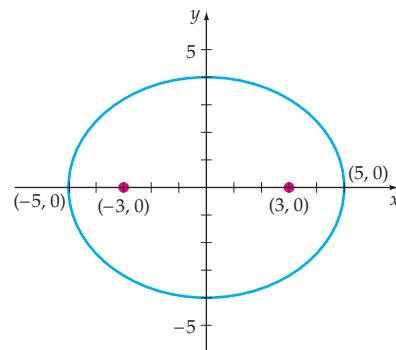


Figure 6.24

The equation of the ellipse in standard form is  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ .

### Ellipses with Center at $(h, k)$

The equation of an ellipse with center at  $(h, k)$  and with a horizontal or vertical major axis can be found by using a translation of coordinates. On a coordinate system with axes labeled  $x'$  and  $y'$ , the standard form of the equation of an ellipse with center at the origin of the  $x'y'$ -coordinate system is

$$\frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} = 1$$

Now place the origin of the  $x'y'$ -coordinate system at  $(h, k)$  in an  $xy$ -coordinate system. See Figure 6.25.

The relationship between an ordered pair in the  $x'y'$ -coordinate system and one in the  $xy$ -coordinate system is given by the transformation equations

$$\begin{aligned}x' &= x - h \\y' &= y - k\end{aligned}$$

Substitute the expressions for  $x'$  and  $y'$  into the equation of an ellipse. The equation of the ellipse with center at  $(h, k)$  is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

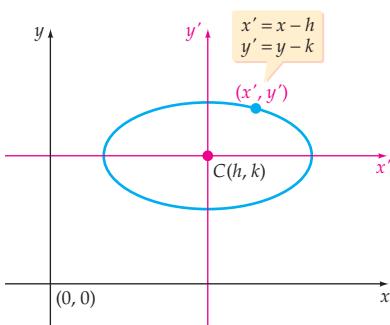
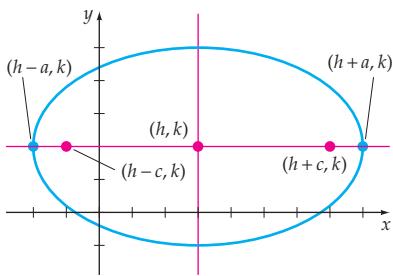
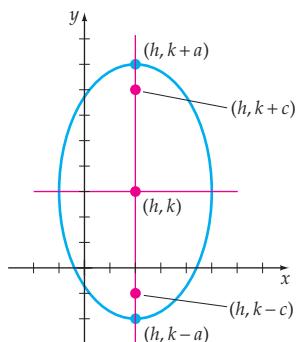


Figure 6.25



a. Major axis parallel to x-axis



b. Major axis parallel to y-axis

Figure 6.26

### Standard Forms of the Equation of an Ellipse with Center at $(h, k)$

#### Major Axis Parallel to the $x$ -Axis

The standard form of the equation of an ellipse with center at  $(h, k)$  and major axis parallel to the  $x$ -axis (see **Figure 6.26a**) is given by

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \quad a > b$$

The length of the major axis is  $2a$ . The length of the minor axis is  $2b$ . The coordinates of the vertices are  $(h + a, k)$  and  $(h - a, k)$ , and the coordinates of the foci are  $(h + c, k)$  and  $(h - c, k)$ , where  $c^2 = a^2 - b^2$ .

#### Major Axis Parallel to the $y$ -Axis

The standard form of the equation of an ellipse with center at  $(h, k)$  and major axis parallel to the  $y$ -axis (see **Figure 6.26b**) is given by

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1, \quad a > b$$

The length of the major axis is  $2a$ . The length of the minor axis is  $2b$ . The coordinates of the vertices are  $(h, k + a)$  and  $(h, k - a)$ , and the coordinates of the foci are  $(h, k + c)$  and  $(h, k - c)$ , where  $c^2 = a^2 - b^2$ .

### EXAMPLE 2 Find the Center, Vertices, and Foci of an Ellipse

Find the center, vertices, and foci of the ellipse  $4x^2 + 9y^2 - 8x + 36y + 4 = 0$ . Sketch the graph.

#### Solution

Write the equation of the ellipse in standard form by completing the square.

$$4x^2 + 9y^2 - 8x + 36y + 4 = 0$$

$$4x^2 - 8x + 9y^2 + 36y = -4$$

$$4(x^2 - 2x) + 9(y^2 + 4y) = -4$$

$$4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) = -4 + 4 + 36$$

$$4(x - 1)^2 + 9(y + 2)^2 = 36$$

$$\frac{(x - 1)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

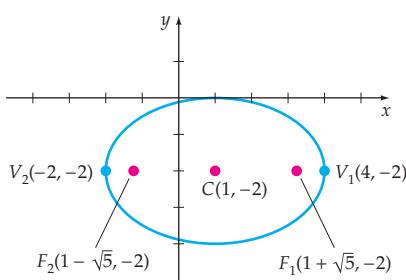
• Rearrange terms.

• Factor.

• Complete the square.

• Factor.

• Divide each side by 36.



$$\frac{(x - 1)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

Figure 6.27

From the equation of the ellipse in standard form, the **coordinates of the center of the ellipse are  $(1, -2)$** . Because the larger denominator is 9, the major axis is parallel to the  $x$ -axis and  $a^2 = 9$ . Thus  $a = 3$ . The vertices are  $(4, -2)$  and  $(-2, -2)$ .

**Continued ▶**

To find the coordinates of the foci, we find  $c$ .

$$\begin{aligned}c^2 &= a^2 - b^2 = 9 - 4 = 5 \\c &= \sqrt{5}\end{aligned}$$

The foci are  $(1 + \sqrt{5}, -2)$  and  $(1 - \sqrt{5}, -2)$ . See **Figure 6.27**.

» Try Exercise 28, page 385



### Integrating Technology

A graphing utility can be used to graph an ellipse. For instance, consider the equation  $4x^2 + 9y^2 - 8x + 36y + 4 = 0$  from Example 2. Rewrite the equation as

$$9y^2 + 36y + (4x^2 - 8x + 4) = 0$$

In this form, the equation is a quadratic equation in terms of the variable  $y$  with

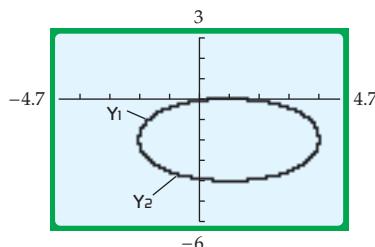
$$A = 9, B = 36, \text{ and } C = 4x^2 - 8x + 4$$

Apply the quadratic formula to produce

$$y = \frac{-36 \pm \sqrt{1296 - 36(4x^2 - 8x + 4)}}{18}$$

The graph of  $Y_1 = \frac{-36 + \sqrt{1296 - 36(4x^2 - 8x + 4)}}{18}$  is the part of the ellipse on or above the line  $y = -2$  (see **Figure 6.28**).

The graph of  $Y_2 = \frac{-36 - \sqrt{1296 - 36(4x^2 - 8x + 4)}}{18}$  is the part of the ellipse on or below the line  $y = -2$ , as shown in **Figure 6.28**.



**Figure 6.28**

One advantage of this graphing procedure is that it does not require us to write the given equation in standard form. A disadvantage of the graphing procedure is that it does not indicate where the foci of the ellipse are located.

TO REVIEW  
Quadratic Formula  
See page 5.

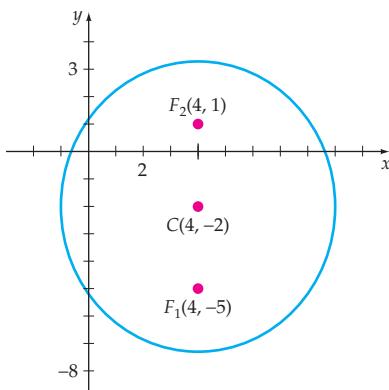


Figure 6.29

**EXAMPLE 3** » Find the Equation of an Ellipse

Find the standard form of the equation of the ellipse with center at  $(4, -2)$ , foci  $F_2(4, 1)$  and  $F_1(4, -5)$ , and minor axis of length 10, as shown in **Figure 6.29**.

**Solution**

Because the foci are on the major axis, the major axis is parallel to the  $y$ -axis. The distance from the center of the ellipse to a focus is  $c$ . The distance between the center  $(4, -2)$  and the focus  $(4, 1)$  is 3. Therefore,  $c = 3$ .

The length of the minor axis is  $2b$ . Thus  $2b = 10$  and  $b = 5$ .

To find  $a^2$ , use the equation  $c^2 = a^2 - b^2$ .

$$9 = a^2 - 25$$

$$a^2 = 34$$

Thus the equation in standard form is

$$\frac{(x - 4)^2}{25} + \frac{(y + 2)^2}{34} = 1$$

» Try Exercise 44, page 386

**take note**

Eccentric literally means “out of the center.” Eccentricity is a measure of how much an ellipse is unlike a set of points the same distance from the center. The higher the eccentricity, the more unlike a circle the ellipse is, and therefore the longer and thinner it is. A circle is also a conic section. Its standard form is given on page 25.

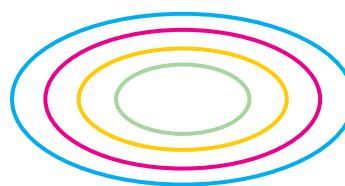
**Eccentricity of an Ellipse**

The graph of an ellipse can be very long and thin, or it can be much like a circle. The **eccentricity** of an ellipse is a measure of its “roundness.”

**Eccentricity ( $e$ ) of an Ellipse**

The eccentricity  $e$  of an ellipse is the ratio of  $c$  to  $a$ , where  $c$  is the distance from the center to a focus and  $a$  is one-half the length of the major axis. (See **Figure 6.30**.) That is,

$$e = \frac{c}{a}$$



Eccentricity = 0.87

Figure 6.30

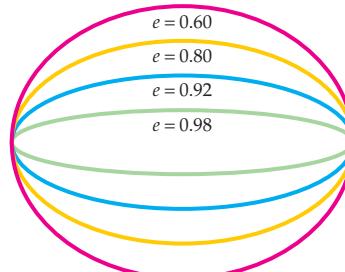


Figure 6.31

Because  $c < a$ , for an ellipse,  $0 < e < 1$ . When  $e \approx 0$ , the graph is almost a circle. When  $e \approx 1$ , the graph is long and thin. See **Figure 6.31**.

**EXAMPLE 4** » Find the Eccentricity of an Ellipse

Find the eccentricity of the ellipse given by  $8x^2 + 9y^2 = 18$ .

**Solution**

First, write the equation of the ellipse in standard form. Divide each side of the equation by 18.

$$\begin{aligned}\frac{8x^2}{18} + \frac{9y^2}{18} &= 1 \\ \frac{4x^2}{9} + \frac{y^2}{2} &= 1 \\ \frac{x^2}{9/4} + \frac{y^2}{2} &= 1 \quad \cdot \frac{4}{9} = \frac{1}{9/4}\end{aligned}$$

The last step is necessary because the standard form of the equation has coefficients of 1 in the numerator. Thus

$$a^2 = \frac{9}{4} \quad \text{and} \quad a = \frac{3}{2}$$

Use the equation  $c^2 = a^2 - b^2$  to find  $c$ .

$$c^2 = \frac{9}{4} - 2 = \frac{1}{4} \quad \text{and} \quad c = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Now find the eccentricity.

$$e = \frac{c}{a} = \frac{1/2}{3/2} = \frac{1}{3}$$

The eccentricity of the ellipse is  $\frac{1}{3}$ .

**Table 6.1**

| Planet  | Eccentricity |
|---------|--------------|
| Mercury | 0.206        |
| Venus   | 0.007        |
| Earth   | 0.017        |
| Mars    | 0.093        |
| Jupiter | 0.049        |
| Saturn  | 0.051        |
| Uranus  | 0.046        |
| Neptune | 0.005        |

## » Try Exercise 50, page 386

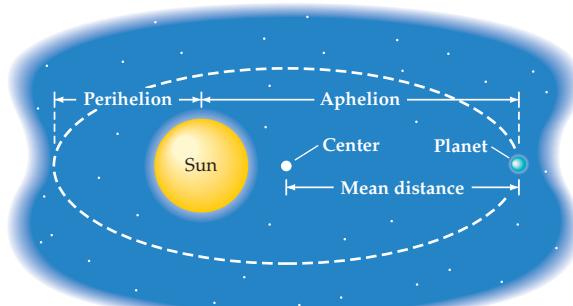
**Applications**

 The planets travel around the sun in elliptical orbits. The sun is located at a focus of the orbit. The eccentricities of the orbits for the planets in our solar system are given in **Table 6.1**.

 **QUESTION** Which planet has the most nearly circular orbit?

 **ANSWER** Neptune has the smallest eccentricity, so it is the planet with the most nearly circular orbit.

The terms *perihelion* and *aphelion* are used to denote the position of a planet in its orbit around the sun. The perihelion is the point nearest the sun; the aphelion is the point farthest from the sun. See **Figure 6.32**. The length of the semimajor axis of a planet's elliptical orbit is called the *mean distance* of the planet from the sun.



**Figure 6.32**

### EXAMPLE 5 » Determine an Equation for the Orbit of Earth



Earth has a mean distance of 93 million miles and a perihelion distance of 91.5 million miles. Find an equation for Earth's orbit.

#### Solution

A mean distance of 93 million miles implies that the length of the semi-major axis of the orbit is  $a = 93$  million miles. Earth's aphelion distance is the length of the major axis less the length of the perihelion distance. Thus

$$\text{Aphelion distance} = 2(93) - 91.5 = 94.5 \text{ million miles}$$

The distance  $c$  from the sun to the center of Earth's orbit is

$$c = \text{aphelion distance} - 93 = 94.5 - 93 = 1.5 \text{ million miles}$$

The length  $b$  of the semiminor axis of the orbit is

$$b = \sqrt{a^2 - c^2} = \sqrt{93^2 - 1.5^2} = \sqrt{8646.75}$$

An equation of Earth's orbit is

$$\frac{x^2}{93^2} + \frac{y^2}{8646.75} = 1$$

» Try Exercise 58, page 387

Sound waves, although different from light waves, have a similar reflective property. When sound is reflected from a point  $P$  on a surface, the angle of incidence equals the angle of reflection. Applying this principle to a room with an elliptical ceiling results in what are called whispering galleries. These galleries are based on the following theorem.

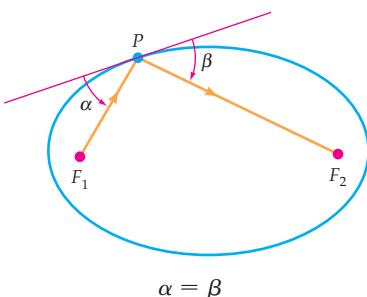


Figure 6.33

**Reflective Property of an Ellipse**

The lines from the foci to a point on an ellipse make equal angles with the tangent line at that point. See **Figure 6.33**.

The reflective property of an ellipse can be used to show that sound waves, or light waves, that emanate from one focus of an ellipse will be reflected to the other focus.

The Statuary Hall in the Capitol Building in Washington, D.C., is a whispering gallery. A person standing at one focus of the elliptical ceiling can whisper and be heard by a person standing at the other focus. John Quincy Adams, while a member of the House of Representatives, was aware of this acoustical phenomenon. He situated his desk at a focus of the elliptical ceiling, which allowed him to eavesdrop on the conversations of his political adversaries, who were located near the other focus.

**EXAMPLE 6 »» Locate the Foci of a Whispering Gallery**

A room 88 feet long is constructed to be a whispering gallery. The room has an elliptical ceiling, as shown in **Figure 6.34**. If the maximum height of the ceiling is 22 feet, determine where the foci are located.

**Solution**

The length  $a$  of the semimajor axis of the elliptical ceiling is 44 feet. The height  $b$  of the semiminor axis is 22 feet. Thus

$$\begin{aligned}c^2 &= a^2 - b^2 \\c^2 &= 44^2 - 22^2 \\c &= \sqrt{44^2 - 22^2} \approx 38.1 \text{ feet}\end{aligned}$$

The foci are located about 38.1 feet from the center of the elliptical ceiling, along its major axis.

**»» Try Exercise 60, page 387**

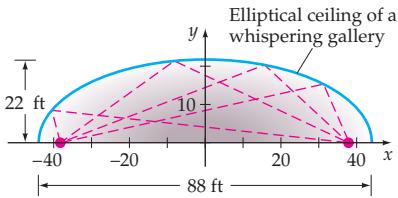


Figure 6.34

**Topics for Discussion**

1. In every ellipse, the length of the semimajor axis  $a$  is greater than the length of the semiminor axis  $b$  and greater than the distance  $c$  from a focus to the center of the ellipse. Do you agree? Explain.
2. How many vertices does an ellipse have?
3. Every ellipse has two  $y$ -intercepts. Do you agree? Explain.
4. Explain why the eccentricity of every ellipse is a number between 0 and 1.


**Exercise Set 6.2**

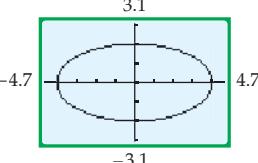
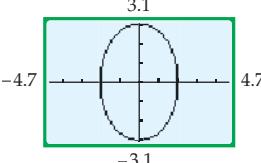
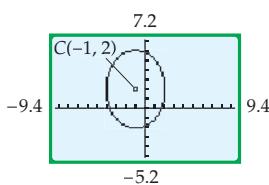
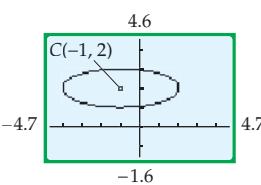
1. Examine the following four equations and the graphs labeled **i**, **ii**, **iii**, and **iv**. Determine which graph is the graph of each equation.

a.  $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{1} = 1$

b.  $\frac{x^2}{16} + \frac{y^2}{4} = 1$

c.  $\frac{x^2}{4} + \frac{y^2}{9} = 1$

d.  $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{16} = 1$

**i.****ii.****iii.****iv.**

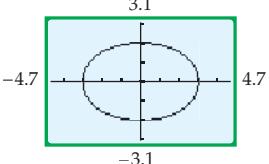
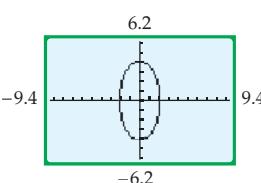
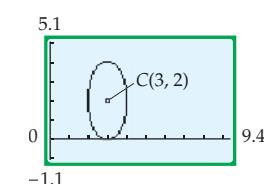
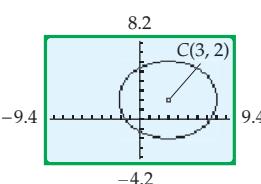
2. Examine the following four equations and the graphs labeled **i**, **ii**, **iii**, and **iv**. Determine which graph is the graph of each equation.

a.  $\frac{(x-3)^2}{1} + \frac{(y-2)^2}{4} = 1$

b.  $\frac{x^2}{9} + \frac{y^2}{4} = 1$

c.  $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$

d.  $\frac{x^2}{4} + \frac{y^2}{16} = 1$

**i.****ii.****iii.****iv.**

**In Exercises 3 to 34, find the center, vertices, and foci of the ellipse given by each equation. Sketch the graph.**

3.  $\frac{x^2}{16} + \frac{y^2}{25} = 1$

4.  $\frac{x^2}{49} + \frac{y^2}{36} = 1$

5.  $\frac{x^2}{9} + \frac{y^2}{4} = 1$

6.  $\frac{x^2}{64} + \frac{y^2}{25} = 1$

7.  $\frac{x^2}{9} + \frac{y^2}{7} = 1$

8.  $\frac{x^2}{5} + \frac{y^2}{4} = 1$

9.  $\frac{4x^2}{9} + \frac{y^2}{16} = 1$

10.  $\frac{x^2}{9} + \frac{9y^2}{16} = 1$

11.  $\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$

12.  $\frac{(x+3)^2}{9} + \frac{(y+1)^2}{16} = 1$

13.  $\frac{(x+2)^2}{9} + \frac{y^2}{25} = 1$

14.  $\frac{x^2}{25} + \frac{(y-2)^2}{81} = 1$

15.  $\frac{(x-1)^2}{21} + \frac{(y-3)^2}{4} = 1$

16.  $\frac{(x+5)^2}{9} + \frac{(y-3)^2}{7} = 1$

17.  $\frac{9(x-1)^2}{16} + \frac{(y+1)^2}{9} = 1$

18.  $\frac{(x+6)^2}{25} + \frac{25y^2}{144} = 1$

19.  $3x^2 + 4y^2 = 12$

20.  $5x^2 + 4y^2 = 20$

21.  $25x^2 + 16y^2 = 400$

22.  $25x^2 + 12y^2 = 300$

23.  $64x^2 + 25y^2 = 400$

24.  $9x^2 + 64y^2 = 144$

25.  $4x^2 + y^2 - 24x - 8y + 48 = 0$

26.  $x^2 + 9y^2 + 6x - 36y + 36 = 0$

27.  $5x^2 + 9y^2 - 20x + 54y + 56 = 0$

28.  $9x^2 + 16y^2 + 36x - 16y - 104 = 0$

29.  $16x^2 + 9y^2 - 64x - 80 = 0$

30.  $16x^2 + 9y^2 + 36y - 108 = 0$

31.  $25x^2 + 16y^2 + 50x - 32y - 359 = 0$

32.  $16x^2 + 9y^2 - 64x - 54y + 1 = 0$

33.  $8x^2 + 25y^2 - 48x + 50y + 47 = 0$

» 50. Foci at  $(0, -3)$  and  $(0, 3)$ , eccentricity  $\frac{1}{4}$

34.  $4x^2 + 9y^2 + 24x + 18y + 44 = 0$

51. Eccentricity  $\frac{2}{5}$ , foci at  $(-1, 3)$  and  $(3, 3)$

**In Exercises 35 to 46, find the equation in standard form of each ellipse, given the information provided.**

35. Center  $(0, 0)$ , major axis of length 10, foci at  $(4, 0)$  and  $(-4, 0)$

36. Center  $(0, 0)$ , minor axis of length 6, foci at  $(0, 4)$  and  $(0, -4)$

37. Vertices  $(6, 0)$ ,  $(-6, 0)$ ; ellipse passes through  $(0, -4)$  and  $(0, 4)$

38. Vertices  $(7, 0)$ ,  $(-7, 0)$ ; ellipse passes through  $(0, 5)$  and  $(0, -5)$

39. Major axis of length 12 on the  $x$ -axis, center at  $(0, 0)$ ; ellipse passes through  $(2, -3)$

40. Major axis of length 8, center at  $(0, 0)$ ; ellipse passes through  $(-2, 2)$

41. Center  $(-2, 4)$ , vertices  $(-6, 4)$  and  $(2, 4)$ , foci at  $(-5, 4)$  and  $(1, 4)$

42. Center  $(0, 3)$ , minor axis of length 4, foci at  $(0, 0)$  and  $(0, 6)$

43. Center  $(2, 4)$ , major axis parallel to the  $y$ -axis and of length 10; ellipse passes through the point  $(3, 3)$

» 44. Center  $(-4, 1)$ , minor axis parallel to the  $y$ -axis and of length 8; ellipse passes through the point  $(0, 4)$

45. Vertices  $(5, 6)$  and  $(5, -4)$ , foci at  $(5, 4)$  and  $(5, -2)$

46. Vertices  $(-7, -1)$  and  $(5, -1)$ , foci at  $(-5, -1)$  and  $(3, -1)$

**In Exercises 47 to 54, use the eccentricity of each ellipse to find its equation in standard form.**

47. Eccentricity  $\frac{2}{5}$ , major axis on the  $x$ -axis and of length 10, center at  $(0, 0)$

48. Eccentricity  $\frac{3}{4}$ , foci at  $(9, 0)$  and  $(-9, 0)$

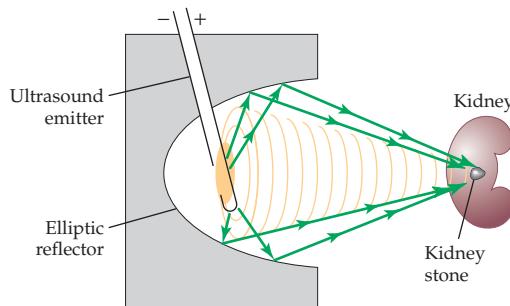
49. Foci at  $(0, -4)$  and  $(0, 4)$ , eccentricity  $\frac{2}{3}$

52. Eccentricity  $\frac{1}{4}$ , foci at  $(-2, 4)$  and  $(-2, -2)$

53. Eccentricity  $\frac{2}{3}$ , major axis of length 24 on the  $y$ -axis, center at  $(0, 0)$

54. Eccentricity  $\frac{3}{5}$ , major axis of length 15 on the  $x$ -axis, center at  $(0, 0)$

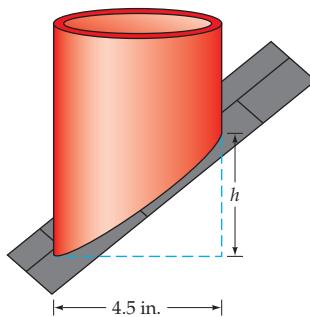
55.  MEDICINE A *lithotripter* is an instrument used to remove a kidney stone in a patient without having to do surgery. A high-frequency sound wave is emitted from a source that is located at the focus of an ellipse. The patient is placed so that the kidney stone is located at the other focus of the ellipse. If the equation of the ellipse is  $\frac{(x - 11)^2}{484} + \frac{y^2}{64} = 1$  ( $x$  and  $y$  are measured in centimeters), where, to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone?



56. CONSTRUCTION A circular vent pipe is placed on a roof that has a slope of  $\frac{4}{5}$ , as shown in the figure at the right.

a. Use the slope to find the value of  $h$ .

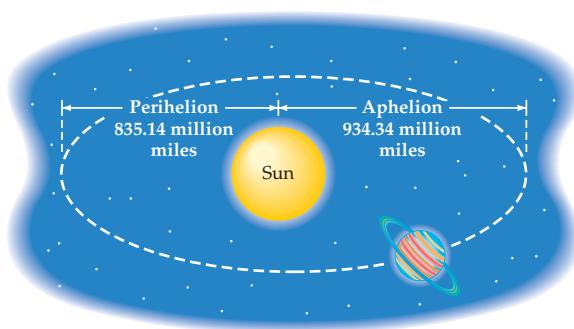
b. The intersection of the vent pipe and the roof



is an ellipse. To the nearest hundredth of an inch, what are the lengths of the major and minor axes?

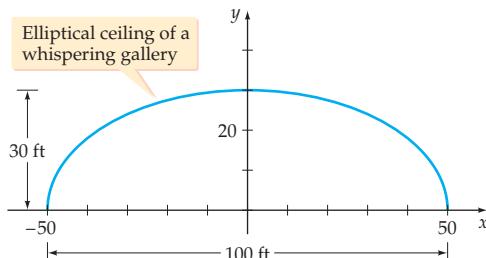
- c. Find an equation of the ellipse that should be cut from the roof so that the pipe will fit.

- 57. THE ORBIT OF SATURN** The distance from Saturn to the sun at Saturn's aphelion is 934.34 million miles, and the distance from Saturn to the sun at its perihelion is 835.14 million miles. Find an equation for the orbit of Saturn.



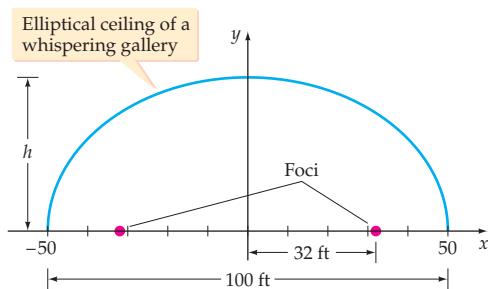
- 58. THE ORBIT OF VENUS** Venus has a mean distance from the sun of 67.08 million miles, and the distance from Venus to the sun at its aphelion is 67.58 million miles. Find an equation for the orbit of Venus.

- 59. WHISPERING GALLERY** An architect wishes to design a large room that will be a whispering gallery. The ceiling of the room has a cross section that is an ellipse, as shown in the following figure.



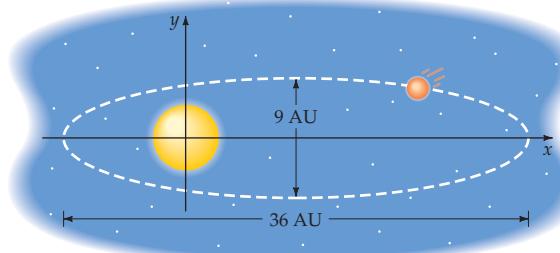
How far to the right and left of center are the foci located?

- 60. WHISPERING GALLERY** An architect wishes to design a large room 100 feet long that will be a whispering gallery. The ceiling of the room has a cross section that is an ellipse, as shown in the following figure.

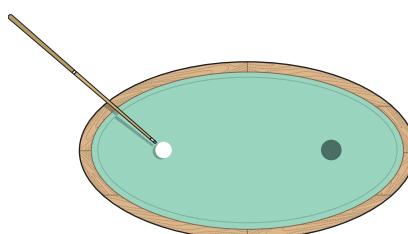


If the foci are to be located 32 feet to the right and the left of center, find the height  $h$  of the elliptical ceiling (to the nearest tenth of a foot).

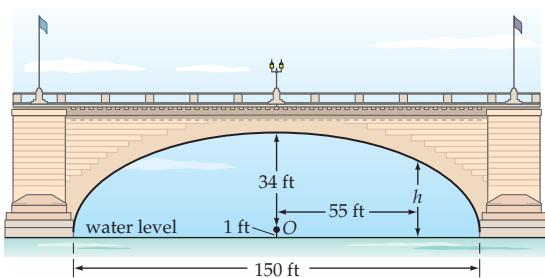
- 61. HALLEY'S COMET** Find the equation of the path of Halley's comet in astronomical units by letting one focus (the sun) be at the origin and letting the other focus be on the positive  $x$ -axis. The length of the major axis of the orbit of Halley's comet is approximately 36 astronomical units (36 AU), and the length of the minor axis is 9 AU ( $1 \text{ AU} = 92,960,000 \text{ miles}$ ).



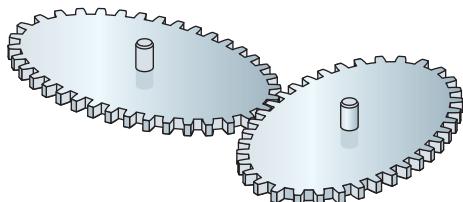
- 62. ELLIPTICAL POOL TABLE** A pool table in the shape of an ellipse has only one pocket, which is located at a focus of the ellipse. A cue ball is placed at the other focus of the ellipse. Striking the cue ball firmly in any direction causes it to go into the pocket (assuming no side or back spin is introduced to the motion of the cue ball). Explain why this happens.



- 63. BRIDGE CLEARANCE** During the 1960s, the London Bridge was dismantled and shipped to Lake Havasu City, Arizona, where it was reconstructed. Use the following diagram of an arch of the bridge to determine the vertical distance  $h$  from the water to the arch 55 feet to the right of point  $O$ , which is the center of the semielliptical arch. The distance from  $O$  to the water is 1 foot. Round to the nearest foot.



- 64. ELLIPTICAL GEARS** The figure below shows two elliptical gears. Search the Internet for an animation that demonstrates the motion of an elliptical gear that is driven by another elliptical gear rotating at a constant speed. Explain what happens to the gear on the left as the gear on the right rotates once at a constant angular speed.



Source: [http://www.stirlingsouth.com/richard/develop/engines\\_under\\_development.htm](http://www.stirlingsouth.com/richard/develop/engines_under_development.htm)

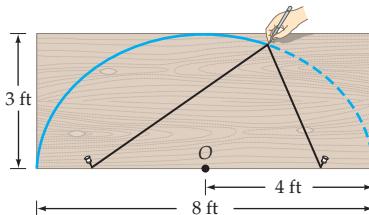
70. Explain why the graph of  $4x^2 + 9y^2 - 16x - 2 = 0$  is or is not an ellipse. Sketch the graph of this equation.

In Exercises 71 to 74, find the equation in standard form of each ellipse by using the definition of an ellipse.

71. Find the equation of the ellipse with foci at  $(-3, 0)$  and  $(3, 0)$  that passes through the point  $\left(3, \frac{9}{2}\right)$ .

- 65. CONSTRUCTION** A carpenter needs to cut a semielliptical form from a 3-foot by 8-foot sheet of plywood, as shown in the following diagram.

- a. Where should the carpenter place the push pins?
  - b. How long is the string that connects the push pins?



- 66.**  **ORBIT OF MARS** Mars travels around the sun in an elliptical orbit with the sun located at a focus of the orbit. The orbit has a major axis of 3.04 AU and a minor axis of 2.99 AU. (1 AU is 1 astronomical unit, or approximately 92,960,000 miles, the average distance of Earth from the sun.) Estimate, to the nearest million miles, the perimeter of the orbit of Mars. [Note: The approximate perimeter of an ellipse with semimajor axis  $a$  and semiminor axis  $b$  is  $P = \pi \sqrt{2(a^2 + b^2)}$ .]



 In Exercises 67 to 69, use the quadratic formula to solve for  $y$  in terms of  $x$ . Then use a graphing utility to graph each equation.

**67.**  $16x^2 + 9y^2 + 36y - 108 = 0$

**68.**  $8x^2 + 25y^2 = 48x + 50y + 47 \equiv 0$

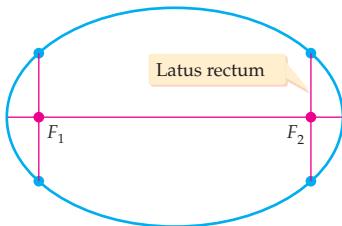
**69.**  $4x^2 + 9y^2 + 24x + 18y + 44 = 0$

- 72.** Find the equation of the ellipse with foci at  $(0, 4)$  and  $(0, -4)$  that passes through the point  $\left(\frac{9}{5}, 4\right)$ .

**73.** Find the equation of the ellipse with foci at  $(-1, 2)$  and  $(3, 2)$  that passes through the point  $(3, 5)$ .

**74.** Find the equation of the ellipse with foci at  $(-1, 1)$  and  $(-1, 7)$  that passes through the point  $\left(\frac{3}{4}, 1\right)$ .

In Exercises 75 and 76, find the latus rectum of the given ellipse. A line segment with endpoints on the ellipse that is perpendicular to the major axis and passes through a focus is a *latus rectum* of the ellipse.



- 1.**  **KEPLER'S LAWS** The German astronomer Johannes Kepler (1571–1630) derived three laws that describe how the planets orbit the sun. Write an essay that includes biographical information about Kepler and a statement of Kepler's Laws. In addition, use Kepler's Laws to answer the following questions.

  - Where is a planet located in its orbit around the sun when it achieves its greatest velocity?
  - What is the period of Mars if it has a mean distance from the sun of 1.52 astronomical units? (*Hint:* Use Earth as a reference with a period of 1 year and a mean distance from the sun of 1 astronomical unit.)

**2.**  **NEPTUNE** The position of the planet Neptune was discovered by using celestial mechanics and mathematics. Write an essay that tells how, when, and by whom Neptune was discovered.

**3.**  **GRAPH THE COLOSSEUM** Some of the Colosseum scenes in the movie *Gladiator* (Universal Studios, 2000) were computer-generated.



- 75.** Find the length of a latus rectum of the ellipse given by

$$\frac{(x - 1)^2}{9} + \frac{(y + 1)^2}{16} = 1$$

- 76.** Find the length of a latus rectum of the ellipse given by

$$9x^2 + 16y^2 - 36x + 96y + 36 = 0$$

77. Show that for any ellipse, the length of a latus rectum is  $\frac{2b^2}{a}$ .

- 78.** Use the definition of an ellipse to find the equation of an ellipse with center at  $(0, 0)$  and foci at  $(0, c)$  and  $(0, -c)$ .

- a. You can create a simple but accurate scale image of the exterior of the Colosseum by using a computer and the mathematics software program *Maple*. Open a new *Maple* worksheet and enter the following two commands.

```
with(plots);
```

```

plots[implicitplot3d]((x^2)/(307.5^2)+(y^2)/(255^2)=
1, x= -310..310, y= -260..260, z=0..157,
scaling=CONSTRAINED, style=PATCHNOGRID,
axes=FRAMED);

```

Execute each of the commands by placing the cursor in a command and pressing the **ENTER** key. After execution of the second command, a three-dimensional graph will appear. Click and drag on the graph to rotate the image.

- b.** A graphing calculator can also be used to generate a simple “graph” of the exterior of the Colosseum. Here is a procedure for the TI-83/TI-83 Plus/TI-84 Plus graphing calculator.

Enter the following in the WINDOW menu.

Xmin=-4.7 Xmax=4.7 Xscl=1 Ymin=-4  
Ymax=9 Yscl=1

Enter the following equations in the Y= menu.

$$Y_1 = \sqrt{1 - X^2} \quad Y_2 = Y_1 + 4 \quad Y_3 = -Y_1 \quad Y_4 = Y_3 + 4$$

Press: QUIT (2nd MODE)  
Enter: EXIT (Mode) ENTER



 Explain why this “Colosseum graph” appears to be constructed with ellipses even though the functions entered in the Y= menu are the equations of semicircles.

## Section 6.3

- Hyperbolas with Center at  $(0, 0)$
- Hyperbolas with Center at  $(h, k)$
- Eccentricity of a Hyperbola
- Applications

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A27.

**PS1.** Find the midpoint and the length of the line segment between  $P_1(4, -3)$  and  $P_2(-2, 1)$ . [1.2]

**PS2.** Solve:  $(x - 1)(x + 3) = 5$  [1.1]

**PS3.** Simplify:  $\frac{4}{\sqrt{8}}$

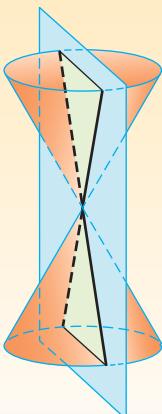
**PS4.** Complete the square of  $4x^2 + 24x$  and write the result as the square of a binomial. [1.2]

**PS5.** Solve  $\frac{x^2}{4} - \frac{y^2}{9} = 1$  for  $y$ . [1.2]

**PS6.** Graph:  $\frac{(x - 2)^2}{16} + \frac{(y + 3)^2}{9} = 1$  [6.2]

#### take note

If a plane intersects a cone along the axis of the cone, the resulting curve is two intersecting straight lines. This is the degenerate form of a hyperbola. See the accompanying figure.



Degenerate hyperbola

A hyperbola is a conic section formed when a plane intersects a right circular cone at a certain angle. If  $\beta$  is the angle at which the plane intersects the axis of the cone and  $\alpha$  is the angle shown in **Figure 6.35**, a hyperbola is formed when  $0^\circ < \beta < \alpha$  or when the plane is parallel to the axis of the cone.

As with the other conic sections, there is a definition of a hyperbola in terms of a certain set of points in the plane.

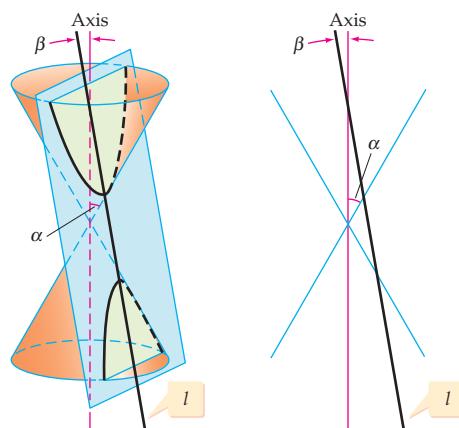


Figure 6.35

### Definition of a Hyperbola

A **hyperbola** is the set of all points in the plane the difference between whose distances from two fixed points (**foci**) is a positive constant.

This definition differs from that of an ellipse in that the ellipse was defined in terms of the *sum* of two distances, whereas the hyperbola is defined in terms of the *difference* of two distances.

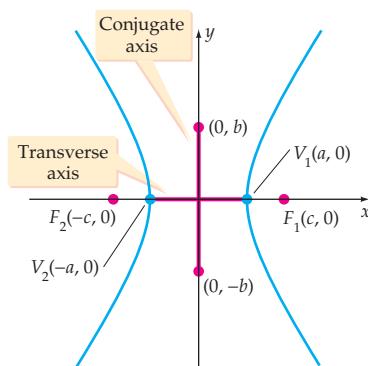


Figure 6.36

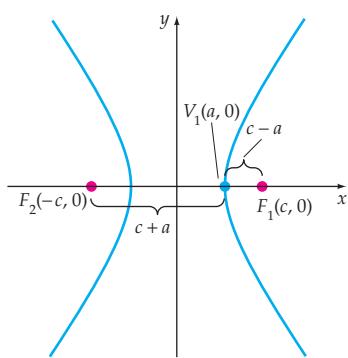


Figure 6.37

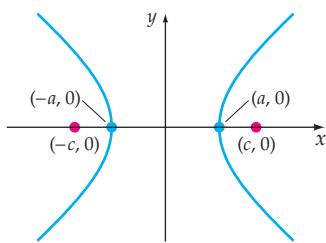
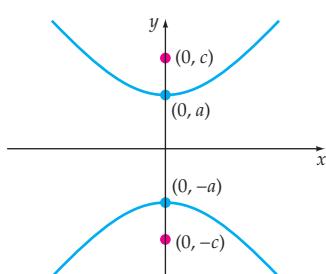
a. Transverse axis on the  $x$ -axisb. Transverse axis on the  $y$ -axis

Figure 6.38

## Hyperbolas with Center at $(0, 0)$

The **transverse axis** of a hyperbola shown in **Figure 6.36** is the line segment joining the intercepts. The midpoint of the transverse axis is called the **center** of the hyperbola. The **conjugate axis** is a line segment that passes through the center of the hyperbola and is perpendicular to the transverse axis.

The length of the transverse axis is customarily represented as  $2a$ , and the distance between the two foci is represented as  $2c$ . The length of the conjugate axis is represented as  $2b$ .

The **vertices** of a hyperbola are the points where the hyperbola intersects the transverse axis.

To determine the positive constant stated in the definition of a hyperbola, consider the point  $V_1(a, 0)$ , which is one vertex of a hyperbola, and the points  $F_1(c, 0)$  and  $F_2(-c, 0)$ , which are the foci of the hyperbola (see **Figure 6.37**). The difference between the distance from  $V_1(a, 0)$  to  $F_1(c, 0)$ ,  $c - a$ , and the distance from  $V_1(a, 0)$  to  $F_2(-c, 0)$ ,  $c + a$ , must be a constant. By subtracting these distances, we find

$$|(c - a) - (c + a)| = |-2a| = 2a$$

Thus the constant is  $2a$ , and it is the length of the transverse axis. The absolute value is used to ensure that the distance is a positive number.

### Standard Forms of the Equation of a Hyperbola with Center at the Origin

#### Transverse Axis on the $x$ -Axis

The standard form of the equation of a hyperbola with center at the origin and transverse axis on the  $x$ -axis (see **Figure 6.38a**) is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The coordinates of the vertices are  $(a, 0)$  and  $(-a, 0)$ , and the coordinates of the foci are  $(c, 0)$  and  $(-c, 0)$ , where  $c^2 = a^2 + b^2$ .

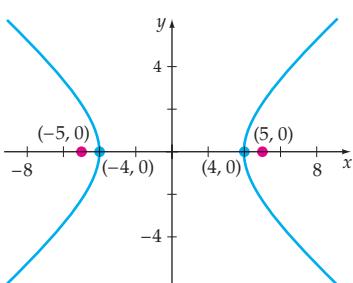
#### Transverse Axis on the $y$ -Axis

The standard form of the equation of a hyperbola with center at the origin and transverse axis on the  $y$ -axis (see **Figure 6.38b**) is given by

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

The coordinates of the vertices are  $(0, a)$  and  $(0, -a)$ , and the coordinates of the foci are  $(0, c)$  and  $(0, -c)$ , where  $c^2 = a^2 + b^2$ .

By looking at the equations, it is possible to determine the location of the transverse axis by finding which term in the equation is positive. When the  $x^2$  term is positive, the transverse axis is on the  $x$ -axis. When the  $y^2$  term is positive, the transverse axis is on the  $y$ -axis.



$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Figure 6.39

**QUESTION** For the graph of  $\frac{y^2}{9} - \frac{x^2}{4} = 1$ , is the transverse axis on the  $x$ -axis or the  $y$ -axis?

Consider the hyperbola given by the equation  $\frac{x^2}{16} - \frac{y^2}{9} = 1$ . Because the  $x^2$  term is positive, the transverse axis is on the  $x$ -axis,  $a^2 = 16$ , and thus  $a = 4$ . The vertices are  $(4, 0)$  and  $(-4, 0)$ . To find the foci, we determine  $c$ .

$$\begin{aligned}c^2 &= a^2 + b^2 = 16 + 9 = 25 \\c &= \sqrt{25} = 5\end{aligned}$$

The foci are  $(5, 0)$  and  $(-5, 0)$ . The graph is shown in **Figure 6.39**.

Each hyperbola has two asymptotes that pass through the center of the hyperbola. The asymptotes of the hyperbola are a useful guide to sketching the graph of the hyperbola.

### Asymptotes of a Hyperbola with Center at the Origin

The **asymptotes** of the hyperbola defined by  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  are given by the equations  $y = \frac{b}{a}x$  and  $y = -\frac{b}{a}x$  (see **Figure 6.40a**).

The asymptotes of the hyperbola defined by  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$  are given by the equations  $y = \frac{a}{b}x$  and  $y = -\frac{a}{b}x$  (see **Figure 6.40b**).

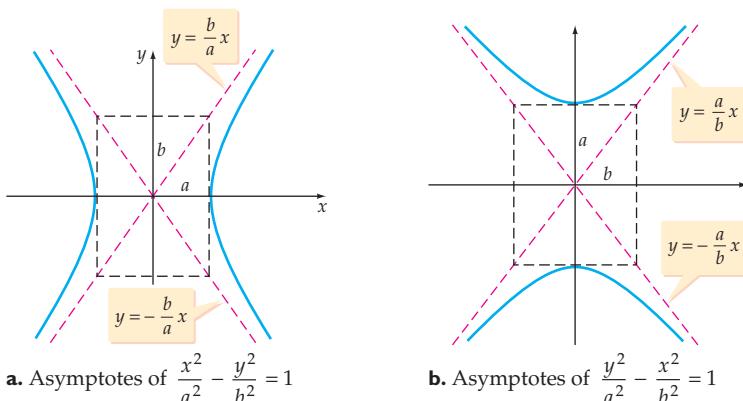


Figure 6.40

One method for remembering the equations of the asymptotes is to write the equation of a hyperbola in standard form, then replace 1 by 0 and solve for  $y$ .

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{so} \quad y^2 = \frac{b^2}{a^2}x^2, \text{ or } y = \pm \frac{b}{a}x$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0 \quad \text{so} \quad y^2 = \frac{a^2}{b^2}x^2, \text{ or } y = \pm \frac{a}{b}x$$

**ANSWER** Because the  $y^2$ -term is positive, the transverse axis is on the  $y$ -axis.

**EXAMPLE 1** » **Find the Vertices, Foci, and Asymptotes of a Hyperbola**

Find the vertices, foci, and asymptotes of the hyperbola given by the equation  $\frac{y^2}{9} - \frac{x^2}{4} = 1$ . Sketch the graph.

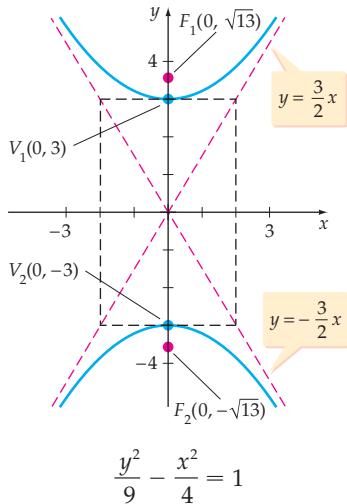


Figure 6.41

**Solution**

Because the  $y^2$  term is positive, the transverse axis is on the  $y$ -axis. We know that  $a^2 = 9$ ; thus  $a = 3$ . The vertices are  $V_1(0, 3)$  and  $V_2(0, -3)$ .

$$c^2 = a^2 + b^2 = 9 + 4$$

$$c = \sqrt{13}$$

The foci are  $F_1(0, \sqrt{13})$  and  $F_2(0, -\sqrt{13})$ .

Because  $a = 3$  and  $b = 2$  ( $b^2 = 4$ ), the equations of the asymptotes are  $y = \frac{3}{2}x$  and  $y = -\frac{3}{2}x$ .

To sketch the graph, we draw a rectangle that has its center at the origin and has dimensions equal to the lengths of the transverse and conjugate axes. The asymptotes are extensions of the diagonals of the rectangle. See **Figure 6.41**.

» Try Exercise 6, page 399

### Hyperbolas with Center at $(h, k)$

Using a translation of coordinates similar to that used for ellipses, we can write the equation of a hyperbola with center at the point  $(h, k)$ . Given coordinate axes labeled  $x'$  and  $y'$ , an equation of a hyperbola with center at the origin is

$$\frac{(x')^2}{a^2} - \frac{(y')^2}{b^2} = 1 \quad (1)$$

Now place the origin of this coordinate system at the point  $(h, k)$  of the  $xy$ -coordinate system, as shown in **Figure 6.42**. The relationship between an ordered pair in the  $x'y'$ -coordinate system and one in the  $xy$ -coordinate system is given by the transformation equations

$$\begin{aligned} x' &= x - h \\ y' &= y - k \end{aligned}$$

Substitute the expressions for  $x'$  and  $y'$  into Equation (1). The equation of a hyperbola with center at  $(h, k)$  is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

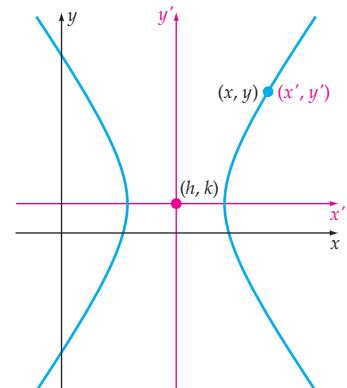


Figure 6.42

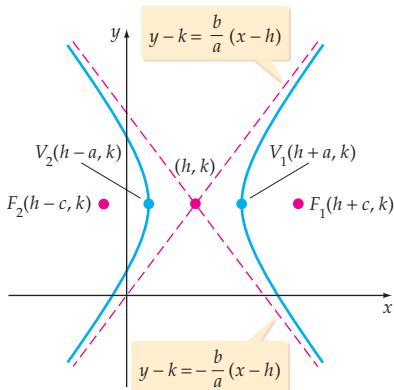
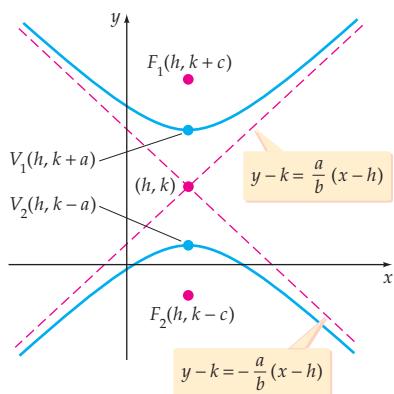
a. Transverse axis parallel to the  $x$ -axisb. Transverse axis parallel to the  $y$ -axis

Figure 6.43

### Standard Forms of the Equation of a Hyperbola with Center at $(h, k)$

#### Transverse Axis Parallel to the $x$ -Axis

The standard form of the equation of a hyperbola with center at  $(h, k)$  and transverse axis parallel to the  $x$ -axis (see **Figure 6.43a**) is given by

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

The coordinates of the vertices are  $V_1(h + a, k)$  and  $V_2(h - a, k)$ . The coordinates of the foci are  $F_1(h + c, k)$  and  $F_2(h - c, k)$ , where  $c^2 = a^2 + b^2$ .

The equations of the asymptotes are  $y - k = \pm \frac{b}{a}(x - h)$ .

#### Transverse Axis Parallel to the $y$ -Axis

The standard form of the equation of a hyperbola with center at  $(h, k)$  and transverse axis parallel to the  $y$ -axis (see **Figure 6.43b**) is given by

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

The coordinates of the vertices are  $V_1(h, k + a)$  and  $V_2(h, k - a)$ . The coordinates of the foci are  $F_1(h, k + c)$  and  $F_2(h, k - c)$ , where  $c^2 = a^2 + b^2$ .

The equations of the asymptotes are  $y - k = \pm \frac{a}{b}(x - h)$ .

### EXAMPLE 2 » Find the Center, Vertices, Foci, and Asymptotes of a Hyperbola

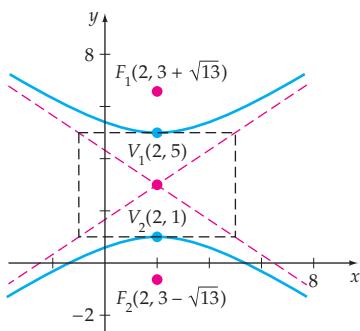
Find the center, vertices, foci, and asymptotes of the hyperbola given by the equation  $4x^2 - 9y^2 - 16x + 54y - 29 = 0$ . Sketch the graph.

#### Solution

Write the equation of the hyperbola in standard form by completing the square.

$$\begin{aligned}
 4x^2 - 9y^2 - 16x + 54y - 29 &= 0 \\
 4x^2 - 16x - 9y^2 + 54y &= 29 \\
 4(x^2 - 4x) - 9(y^2 - 6y) &= 29 \\
 4(x^2 - 4x + 4) - 9(y^2 - 6y + 9) &= 29 + 16 - 81 \\
 4(x - 2)^2 - 9(y - 3)^2 &= -36 \\
 \frac{(y - 3)^2}{4} - \frac{(x - 2)^2}{9} &= 1
 \end{aligned}$$

- Rearrange terms.
- Factor.
- Complete the square.
- Factor.
- Divide each side by  $-36$ .



$$\frac{(y - 3)^2}{4} - \frac{(x - 2)^2}{9} = 1$$

Figure 6.44

The coordinates of the center are  $(2, 3)$ . Because the term containing  $(y - 3)^2$  is positive, the transverse axis is parallel to the  $y$ -axis. We know that  $a^2 = 4$ ; thus  $a = 2$ . The vertices are  $(2, 5)$  and  $(2, 1)$ . See Figure 6.44. To find the coordinates of the foci, we find  $c$ .

$$c^2 = a^2 + b^2 = 4 + 9$$

$$c = \sqrt{13}$$

The foci are  $(2, 3 + \sqrt{13})$  and  $(2, 3 - \sqrt{13})$ . We know that  $b^2 = 9$ ; thus  $b = 3$ . The equations of the asymptotes are  $y - 3 = \pm\left(\frac{2}{3}\right)(x - 2)$ , which simplifies to

$$y = \frac{2}{3}x + \frac{5}{3} \quad \text{and} \quad y = -\frac{2}{3}x + \frac{13}{3}$$

» Try Exercise 28, page 399



### Integrating Technology

A graphing utility can be used to graph a hyperbola. For instance, consider the equation  $4x^2 - 9y^2 - 16x + 54y - 29 = 0$  from Example 2. Rewrite the equation as

$$-9y^2 + 54y + (4x^2 - 16x - 29) = 0$$

In this form, the equation is a quadratic equation in terms of the variable  $y$  with

$$A = -9, B = 54, \text{ and } C = 4x^2 - 16x - 29$$

Apply the quadratic formula to produce

$$y = \frac{-54 \pm \sqrt{2916 + 36(4x^2 - 16x - 29)}}{-18}$$

The graph of  $Y_1 = \frac{-54 + \sqrt{2916 + 36(4x^2 - 16x - 29)}}{-18}$  is the upper branch of the hyperbola (see Figure 6.45).

The graph of  $Y_2 = \frac{-54 - \sqrt{2916 + 36(4x^2 - 16x - 29)}}{-18}$  is the lower branch of the hyperbola, as shown in Figure 6.45.

One advantage of this graphing procedure is that it does not require us to write the given equation in standard form. A disadvantage of the graphing procedure is that it does not indicate where the foci of the hyperbola are located.

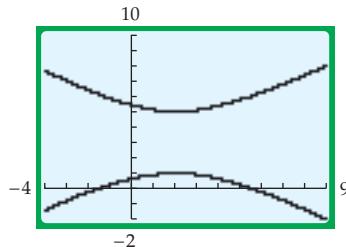


Figure 6.45

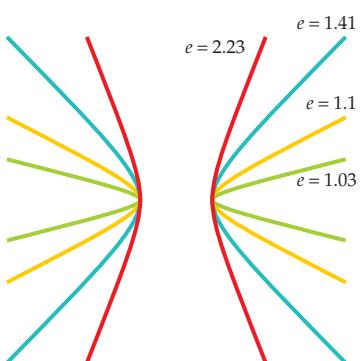


Figure 6.46

## Eccentricity of a Hyperbola

The graph of a hyperbola can be very wide or very narrow. The **eccentricity** of a hyperbola is a measure of its “wideness.”

### Definition of the Eccentricity ( $e$ ) of a Hyperbola

The eccentricity  $e$  of a hyperbola is the ratio of  $c$  to  $a$ , where  $c$  is the distance from the center to a focus and  $a$  is half the length of the transverse axis.

$$e = \frac{c}{a}$$

For a hyperbola,  $c > a$  and therefore  $e > 1$ . As the eccentricity of the hyperbola increases, the graph becomes wider and wider, as shown in **Figure 6.46**.

### EXAMPLE 3 » Find the Equation of a Hyperbola Given Its Eccentricity

Find the standard form of the equation of the hyperbola that has eccentricity  $\frac{3}{2}$ , center at the origin, and a focus at  $(6, 0)$ .

#### Solution

Because the focus is located at  $(6, 0)$  and the center is at the origin,  $c = 6$ . An extension of the transverse axis contains the foci, so the transverse axis is on the  $x$ -axis.

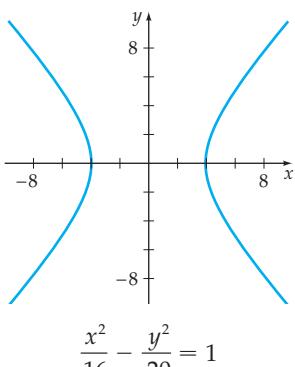


Figure 6.47

$$\begin{aligned} e &= \frac{3}{2} = \frac{c}{a} \\ \frac{3}{2} &= \frac{6}{a} && \text{• Substitute } 6 \text{ for } c. \\ a &= 4 && \text{• Solve for } a. \end{aligned}$$

To find  $b^2$ , use the equation  $c^2 = a^2 + b^2$  and the values for  $c$  and  $a$ .

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 36 &= 16 + b^2 \\ b^2 &= 20 \end{aligned}$$

The equation of the hyperbola is  $\frac{x^2}{16} - \frac{y^2}{20} = 1$ . See **Figure 6.47**.

» Try Exercise 50, page 400

## Applications

**Orbits of Comets** In Section 6.2 we noted that the orbits of the planets are elliptical. Some comets have elliptical orbits also, the most notable being Halley's comet, whose eccentricity is 0.97. See **Figure 6.48**.

**Math Matters**

**Caroline Herschel** (1750–1848) became interested in mathematics and astronomy after her brother William discovered the planet Uranus. She was the first woman to receive credit for the discovery of a comet. In fact, between 1786 and 1797, she discovered eight comets. In 1828 she completed a catalog of over 2000 nebulae, a feat for which the Royal Astronomical Society of England presented her with its prestigious gold medal.

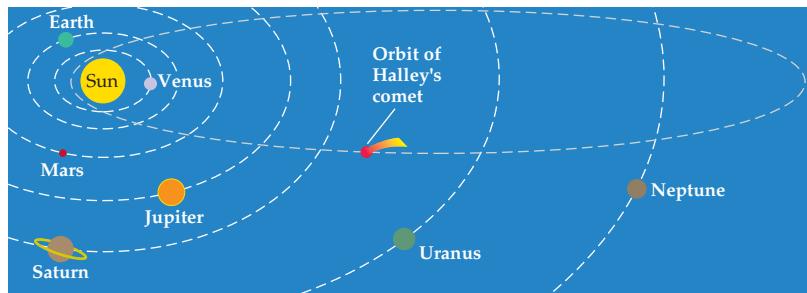


Figure 6.48

Other comets have hyperbolic orbits with the sun at a focus. These comets pass by the sun only once. The velocity of a comet determines whether its orbit is elliptical or hyperbolic.

**Hyperbolas as an Aid to Navigation** Consider two radio transmitters,  $T_1$  and  $T_2$ , placed some distance apart. A ship with electronic equipment measures the difference between the times it takes signals from the transmitters to reach the ship. Because the difference between the times is proportional to the difference between the distances of the ship from the transmitters, the ship must be located on the hyperbola with foci at the two transmitters.

Using a third transmitter,  $T_3$ , we can find a second hyperbola with foci  $T_2$  and  $T_3$ . The ship lies on the intersection of the two hyperbolas, as shown in **Figure 6.49**.

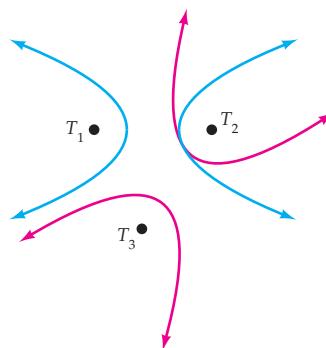


Figure 6.49

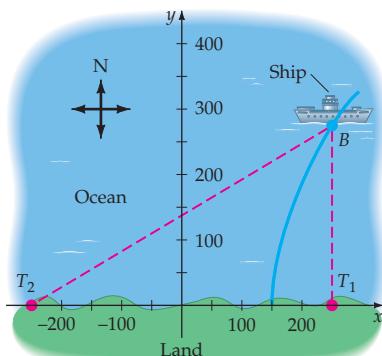


Figure 6.50

**EXAMPLE 4** » **Determine the Position of a Ship**

Two radio transmitters are positioned along a coastline, 500 miles apart. See **Figure 6.50**. Using a LORAN (LOng RAne Navigation) system, a ship determines that a radio signal from transmitter  $T_1$  reaches the ship 1600 microseconds before it receives a simultaneous signal from transmitter  $T_2$ .

- Find an equation of a hyperbola (with foci located at  $T_1$  and  $T_2$ ) on which the ship lies. See **Figure 6.50**. (Assume the radio signals travel at 0.186 mile per microsecond.)
- If the ship is directly north of transmitter  $T_1$ , determine how far (to the nearest mile) the ship is from the transmitter.

Continued ▶

**Solution**

- a. The ship lies on a hyperbola at point  $B$ , with foci at  $T_1$  and  $T_2$ . The difference of the distances  $d(T_2, B)$  and  $d(T_1, B)$  is given by

$$\begin{aligned}\text{Distance} &= \text{rate} \times \text{time} \\ &= 0.186 \text{ mile/microsecond} \times 1600 \text{ microseconds} \\ &= 297.6 \text{ mile}\end{aligned}$$

This indicates that the ship is located on a hyperbola with transverse axis of length 297.6 miles. Thus  $2a = 297.6$  miles and  $a = 148.8$  miles.

**Figure 6.50** shows that the foci are located at  $(250, 0)$  and  $(-250, 0)$ . Thus  $c = 250$  miles, and

$$b = \sqrt{c^2 - a^2} = \sqrt{250^2 - 148.8^2} \approx 200.9 \text{ miles}$$

The ship is located on the hyperbola given by

$$\frac{x^2}{148.8^2} - \frac{y^2}{200.9^2} = 1$$

- b. If the ship is directly north of  $T_1$ , then  $x = 250$ , and the distance from the ship to the transmitter  $T_1$  is  $y$ , where

$$\begin{aligned}-\frac{y^2}{200.9^2} &= 1 - \frac{250^2}{148.8^2} \\ y &= \frac{200.9}{148.8} \sqrt{250^2 - 148.8^2} \approx 271 \text{ miles}\end{aligned}$$

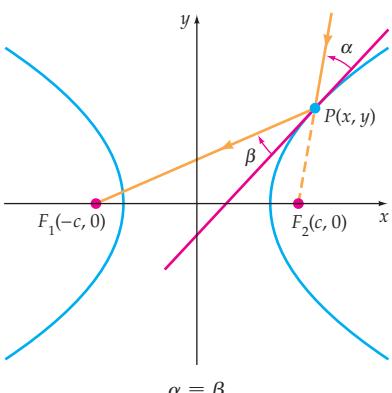
The ship is about 271 miles north of transmitter  $T_1$ .

**» Try Exercise 56, page 400**

Hyperbolas also have a reflective property that makes them useful in many applications.

**Reflective Property of a Hyperbola**

The lines from the foci to a point on a hyperbola make equal angles with the tangent line at that point. See **Figure 6.51**.



**Figure 6.51**


**Topics for Discussion**

1. In every hyperbola, the distance  $c$  from a focus to the center of the hyperbola is greater than half the length of the transverse axis. Do you agree? Explain.
2. How many vertices does a hyperbola have?
3. Explain why the eccentricity of every hyperbola is a number greater than 1.
4. Is the conjugate axis of a hyperbola perpendicular to the transverse axis of the hyperbola?


**Exercise Set 6.3**

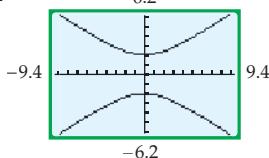
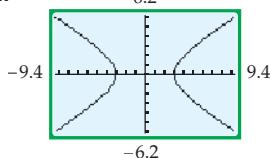
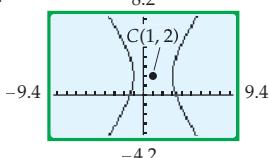
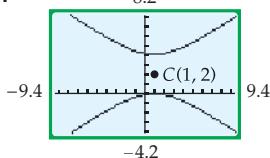
1. Examine the following four equations and the graphs labeled **i**, **ii**, **iii**, and **iv**. Determine which graph is the graph of each equation.

**a.**  $\frac{(x - 1)^2}{9} - \frac{(y - 2)^2}{16} = 1$

**b.**  $\frac{x^2}{9} - \frac{y^2}{4} = 1$

**c.**  $\frac{y^2}{4} - \frac{x^2}{9} = 1$

**d.**  $\frac{(y - 2)^2}{4} - \frac{(x - 1)^2}{9} = 1$

**i.****ii.****iii.****iv.**

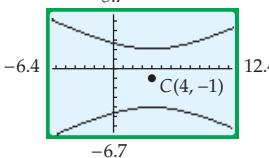
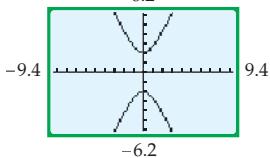
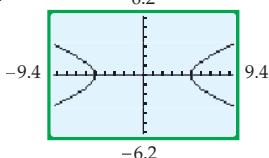
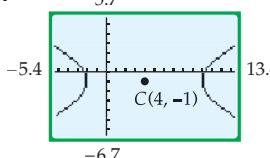
2. Examine the following four equations and the graphs labeled **i**, **ii**, **iii**, and **iv**. Determine which graph is the graph of each equation.

**a.**  $\frac{x^2}{25} - \frac{y^2}{4} = 1$

**b.**  $\frac{(y + 1)^2}{9} - \frac{(x - 4)^2}{36} = 1$

**c.**  $\frac{(x - 4)^2}{36} - \frac{(y + 1)^2}{9} = 1$

**d.**  $\frac{y^2}{4} - x^2 = 1$

**i.****ii.****iii.****iv.**

In Exercises 3 to 28, find the center, vertices, foci, and asymptotes for the hyperbola given by each equation. Graph each equation.

**3.**  $\frac{x^2}{16} - \frac{y^2}{25} = 1$

**4.**  $\frac{x^2}{16} - \frac{y^2}{9} = 1$

**5.**  $\frac{y^2}{4} - \frac{x^2}{25} = 1$

**6.**  $\frac{y^2}{25} - \frac{x^2}{36} = 1$

**7.**  $\frac{x^2}{7} - \frac{y^2}{9} = 1$

**8.**  $\frac{x^2}{5} - \frac{y^2}{4} = 1$

**9.**  $\frac{4x^2}{9} - \frac{y^2}{16} = 1$

**10.**  $\frac{x^2}{9} - \frac{9y^2}{16} = 1$

**11.**  $\frac{(x - 3)^2}{16} - \frac{(y + 4)^2}{9} = 1$

**12.**  $\frac{(x + 3)^2}{25} - \frac{y^2}{4} = 1$

**13.**  $\frac{(y + 2)^2}{4} - \frac{(x - 1)^2}{16} = 1$

**14.**  $\frac{(y - 2)^2}{36} - \frac{(x + 1)^2}{49} = 1$

**15.**  $\frac{(x + 2)^2}{9} - \frac{y^2}{25} = 1$

**16.**  $\frac{x^2}{25} - \frac{(y - 2)^2}{81} = 1$

**17.**  $\frac{9(x - 1)^2}{16} - \frac{(y + 1)^2}{9} = 1$

**18.**  $\frac{(x + 6)^2}{25} - \frac{25y^2}{144} = 1$

**19.**  $x^2 - y^2 = 9$

**20.**  $4x^2 - y^2 = 16$

**21.**  $16y^2 - 9x^2 = 144$

**22.**  $9y^2 - 25x^2 = 225$

**23.**  $9y^2 - 36x^2 = 4$

**24.**  $16x^2 - 25y^2 = 9$

**25.**  $x^2 - y^2 - 6x + 8y - 3 = 0$

**26.**  $4x^2 - 25y^2 + 16x + 50y - 109 = 0$

**27.**  $9x^2 - 4y^2 + 36x - 8y + 68 = 0$

**28.**  $16x^2 - 9y^2 - 32x - 54y + 79 = 0$



**In Exercises 29 to 34, use the quadratic formula to solve for  $y$  in terms of  $x$ . Then use a graphing utility to graph each equation.**

29.  $4x^2 - y^2 + 32x + 6y + 39 = 0$

30.  $x^2 - 16y^2 + 8x - 64y + 16 = 0$

31.  $9x^2 - 16y^2 - 36x - 64y + 116 = 0$

32.  $2x^2 - 9y^2 + 12x - 18y + 18 = 0$

33.  $4x^2 - 9y^2 + 8x - 18y - 6 = 0$

34.  $2x^2 - 9y^2 - 8x + 36y - 46 = 0$

**In Exercises 35 to 48, find the equation in standard form of the hyperbola that satisfies the stated conditions.**

35. Vertices  $(3, 0)$  and  $(-3, 0)$ , foci  $(4, 0)$  and  $(-4, 0)$

36. Vertices  $(0, 2)$  and  $(0, -2)$ , foci  $(0, 3)$  and  $(0, -3)$

37. Foci  $(0, 5)$  and  $(0, -5)$ , asymptotes  $y = 2x$  and  $y = -2x$

38. Foci  $(4, 0)$  and  $(-4, 0)$ , asymptotes  $y = x$  and  $y = -x$

39. Vertices  $(0, 3)$  and  $(0, -3)$ , passing through  $(2, 4)$

40. Vertices  $(5, 0)$  and  $(-5, 0)$ , passing through  $(-1, 3)$

41. Asymptotes  $y = \frac{1}{2}x$  and  $y = -\frac{1}{2}x$ , vertices  $(0, 4)$  and  $(0, -4)$

42. Asymptotes  $y = \frac{2}{3}x$  and  $y = -\frac{2}{3}x$ , vertices  $(6, 0)$  and  $(-6, 0)$

43. Vertices  $(6, 3)$  and  $(2, 3)$ , foci  $(7, 3)$  and  $(1, 3)$

44. Vertices  $(-1, 5)$  and  $(-1, -1)$ , foci  $(-1, 7)$  and  $(-1, -3)$

45. Foci  $(1, -2)$  and  $(7, -2)$ , slope of an asymptote  $\frac{5}{4}$

46. Foci  $(-3, -6)$  and  $(-3, -2)$ , slope of an asymptote 1

47. Passing through  $(9, 4)$ , slope of an asymptote  $\frac{1}{2}$ , center  $(7, 2)$ , transverse axis parallel to the  $y$ -axis

48. Passing through  $(6, 1)$ , slope of an asymptote 2, center  $(3, 3)$ , transverse axis parallel to the  $x$ -axis

**In Exercises 49 to 54, use the eccentricity to find the equation in standard form of each hyperbola.**

49. Vertices  $(1, 6)$  and  $(1, 8)$ , eccentricity 2

50. Vertices  $(2, 3)$  and  $(-2, 3)$ , eccentricity  $\frac{5}{2}$

51. Eccentricity 2, foci  $(4, 0)$  and  $(-4, 0)$

52. Eccentricity  $\frac{4}{3}$ , foci  $(0, 6)$  and  $(0, -6)$

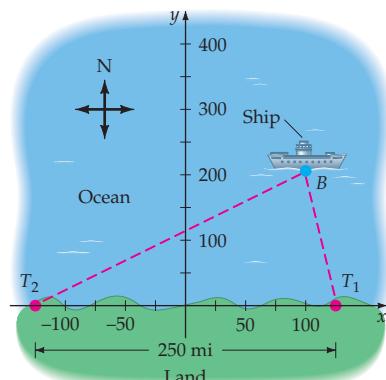
53. Center  $(4, 1)$ , conjugate axis of length 4, eccentricity  $\frac{4}{3}$   
*(Hint: There are two answers.)*

54. Center  $(-3, -3)$ , conjugate axis of length 6, eccentricity 2  
*(Hint: There are two answers.)*

55. **LORAN** Two radio transmitters are positioned along the coast, 250 miles apart. A signal is sent simultaneously from each transmitter. The signal from transmitter  $T_2$  is received by a ship's LORAN 500 microseconds after the ship receives the signal from  $T_1$ . The radio signals travel at 0.186 mile per microsecond.

- a. Find an equation of a hyperbola, with foci at  $T_1$  and  $T_2$ , on which the ship is located.

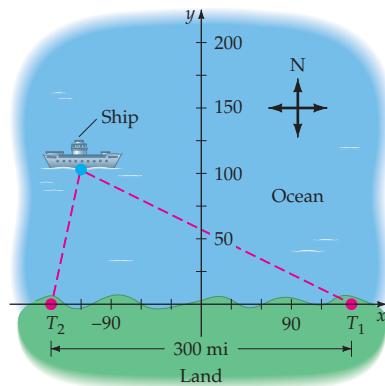
- b. If the ship is 100 miles east of the  $y$ -axis, determine its distance from the coastline (to the nearest mile).



56. **LORAN** Two radio transmitters are positioned along the coast, 300 miles apart. A signal is sent simultaneously from each transmitter. The signal from transmitter  $T_1$  is

received by a ship's LORAN 800 microseconds after the ship receives the signal from  $T_2$ . The radio signals travel at 0.186 mile per microsecond.

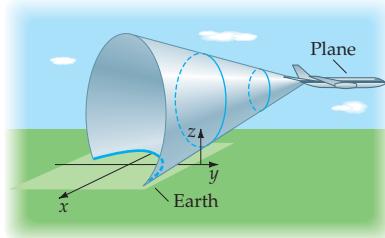
- Find an equation of a hyperbola, with foci at  $T_1$  and  $T_2$ , on which the ship is located.
- If the ship continues to travel so that the difference of 800 microseconds is maintained, determine the point at which the ship will reach the coastline.



- 57. SONIC BOOMS** When a plane exceeds the speed of sound, a sonic boom is produced by the wake of the sound waves. For a plane flying at 10,000 feet, the circular wave front can be given by

$$y^2 = x^2 + (z - 10,000)^2$$

where  $z$  is the height of the wave front above Earth. See the diagram below. Note that the  $xy$ -plane is Earth's surface, which is approximately flat over small distances.

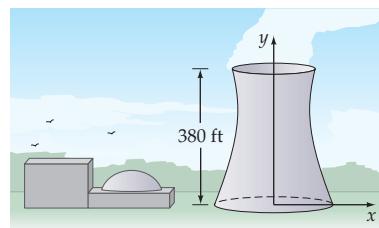


Find and name the equation formed when the wave front hits Earth.

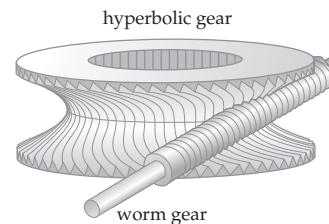
- 58. COOLING TOWER** A vertical cross section of a cooling tower is a portion of a hyperbola, as shown in the following diagram. The standard form of the equation of the hyperbolic cross section is

$$\frac{x^2}{80^2} - \frac{(y - 220)^2}{180^2} = 1, 0 \leq y \leq 380$$

where  $x$  and  $y$  are measured in feet. The horizontal cross sections of the tower are circles.

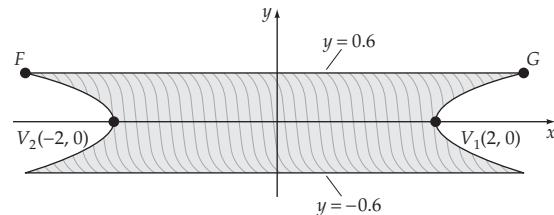


- Find the radius of the top and the base of the tower. Round to the nearest foot.
  - What is the smallest radius of a horizontal cross section?
- 59. HYPERBOLIC GEAR** The following diagram shows a cylindrical worm gear driving a hyperbolic gear.



Source: <http://www.zakgear.com/Wormoid.html>

A center vertical cross section of the hyperbolic gear is shown below. The dimensions given are in inches.

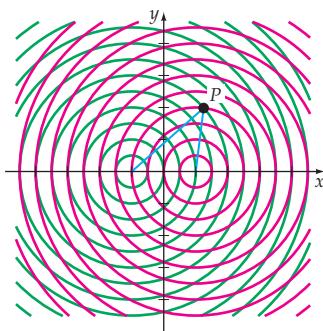


The vertical cross section of a hyperbolic gear

The eccentricity of the hyperbolic cross section is  $\frac{\sqrt{17}}{4}$ .

- What is the equation in standard form of the hyperbolic cross section?
- Find the length of diameter  $FG$ . Round to the nearest hundredth of an inch.

- 60. WATER WAVES** If two pebbles are dropped into a pond at different places  $F_1(-2, 0)$  and  $F_2(2, 0)$ , circular waves are propagated with  $F_1$  as the center of one set of circular waves (in green) and  $F_2$  as the center of the other set of circular waves (in red).



Let  $P$  be a point at which the waves intersect. In the diagram above,  $|F_1P - F_2P| = 2$ .

- a. What curve is generated by connecting all points  $P$  for which  $|F_1P - F_2P| = 2$ ?

- b.** What is the equation of the curve in a.?

**In Exercises 61 to 68, identify the graph of each equation as a parabola, an ellipse, or a hyperbola. Graph each equation.**

- 61.**  $4x^2 + 9y^2 - 16x - 36y + 16 = 0$

**62.**  $2x^2 + 3y - 8x + 2 = 0$

**63.**  $5x - 4y^2 + 24y - 11 = 0$

**64.**  $9x^2 - 25y^2 - 18x + 50y = 0$

**65.**  $x^2 + 2y - 8x = 0$

**66.**  $9x^2 + 16y^2 + 36x - 64y - 44 = 0$

**67.**  $25x^2 + 9y^2 - 50x - 72y - 56 = 0$

**68.**  $(x - 3)^2 + (y - 4)^2 = (x + 1)^2$

**In Exercises 69 to 72, use the definition of a hyperbola to find the equation of the hyperbola in standard form.**

- 69.** Foci  $(2, 0)$  and  $(-2, 0)$ ; passes through the point  $(2, 3)$

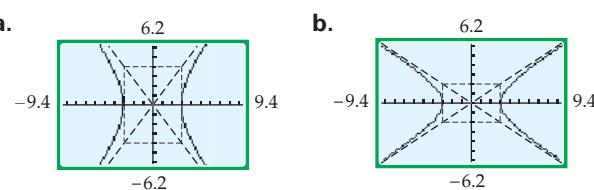
**70.** Foci  $(0, 3)$  and  $(0, -3)$ ; passes through the point  $\left(\frac{5}{2}, 3\right)$

**71.** Foci  $(0, 4)$  and  $(0, -4)$ ; passes through the point  $\left(\frac{7}{3}, 4\right)$

**72.** Foci  $(5, 0)$  and  $(-5, 0)$ ; passes through the point  $\left(5, \frac{9}{4}\right)$

- 73.** Sketch a graph of  $\frac{x|x|}{16} - \frac{y|y|}{9} = 1$ .

- 74. ECCENTRICITY** Which of the following hyperbolas has the larger eccentricity?



- 1. a.**  **A HYPERBOLIC PARABOLOID** A *hyperbolic paraboloid* is a three-dimensional figure. Make a drawing of a hyperbolic paraboloid. Explain the relationship that exists between the equations of the parabolic cross sections and the relationship that exists between the equations of the hyperbolic cross sections.

- b.  **A HYPERBOLOID OF ONE SHEET** Make a sketch of a *hyperboloid of one sheet*. Explain the different cross sections of the hyperboloid of one sheet. Do some research on nuclear power plants, and explain why nuclear cooling towers are designed in the shape of hyperboloids of one sheet.

## Section 6.4

- The Rotation Theorem for Conics
- The Conic Identification Theorem
- Use a Graphing Utility to Graph Second-Degree Equations in Two Variables

# Rotation of Axes

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A28.

**PS1.** Expand  $\cos(\alpha + \beta)$ . [3.2]

**PS2.** Expand  $\sin(\alpha + \beta)$ . [3.2]

**PS3.** Solve  $\cot 2\alpha = \frac{\sqrt{3}}{3}$  for  $0 < \alpha < \frac{\pi}{2}$ . [3.6]

**PS4.** If  $\sin \alpha = \frac{1}{2}$  and  $\cos \alpha = -\frac{\sqrt{3}}{2}$ , with  $0^\circ \leq \alpha < 360^\circ$ , find  $\alpha$ . [3.6]

**PS5.** Identify the graph of  $4x^2 - 6y^2 + 9x + 16y - 8 = 0$ . [6.3]

**PS6.** Graph:  $4x - y^2 - 2y + 3 = 0$  [6.1]

### The Rotation Theorem for Conics

The equation of a conic with axes parallel to the coordinate axes can be written in a general form.

#### General Equation of a Conic with Axes Parallel to Coordinate Axes

The **general equation of a conic** with axes parallel to the coordinate axes and not both  $A$  and  $C$  equal to zero is

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

The graph of the equation is a parabola when  $AC = 0$ , an ellipse when  $AC > 0$ , and a hyperbola when  $AC < 0$ .

The terms  $Dx$ ,  $Ey$ , and  $F$  determine the translation of the conic from the origin. The general equation of a conic is a *second-degree equation* in two variables. A more general second-degree equation can be written that contains a  $Bxy$  term.

#### take note

Some choices of the constants  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ , and  $F$  may result in a degenerate conic or an equation that has no solutions.

#### General Second-Degree Equation in Two Variables

The **general second-degree equation in two variables** is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The  $Bxy$  term ( $B \neq 0$ ) determines a rotation of the conic so that its axes are no longer parallel to the coordinate axes.

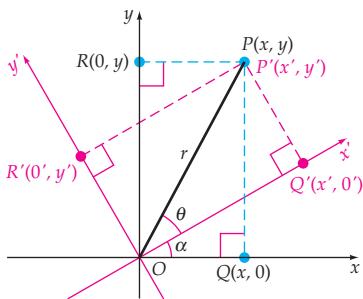


Figure 6.52

A **rotation of axes** is a rotation of the  $x$ - and  $y$ -axes about the origin to another position denoted by  $x'$  and  $y'$ . We denote the measure of the **angle of rotation** by  $\alpha$ .

Let  $P$  be some point in the plane, and let  $r$  represent the distance of  $P$  from the origin. The coordinates of  $P$  relative to the  $xy$ -coordinate system and the  $x'y'$ -coordinate system are  $P(x, y)$  and  $P(x', y')$ , respectively.

Let  $Q(x, 0)$  and  $R(0, y)$  be the projections of  $P$  onto the  $x$ - and the  $y$ -axis and let  $Q'(x', 0')$  and  $R'(0', y')$  be the projections of  $P$  onto the  $x'$ - and the  $y'$ -axis. (See **Figure 6.52**.) The angle between the  $x'$ -axis and  $OP$  is denoted by  $\theta$ . We can express the coordinates of  $P$  in each coordinate system in terms of  $\alpha$  and  $\theta$ .

$$\begin{aligned} x &= r \cos(\theta + \alpha) & x' &= r \cos \theta \\ y &= r \sin(\theta + \alpha) & y' &= r \sin \theta \end{aligned}$$

Applying the addition formulas for  $\cos(\theta + \alpha)$  and  $\sin(\theta + \alpha)$ , we get

$$\begin{aligned} x &= r \cos(\theta + \alpha) = r \cos \theta \cos \alpha - r \sin \theta \sin \alpha \\ y &= r \sin(\theta + \alpha) = r \sin \theta \cos \alpha + r \cos \theta \sin \alpha \end{aligned}$$

Now, substituting  $x'$  for  $r \cos \theta$  and  $y'$  for  $r \sin \theta$  into these equations yields

$$\begin{aligned} x &= x' \cos \alpha - y' \sin \alpha & x' &= r \cos \theta, y' = r \sin \theta \\ y &= y' \cos \alpha + x' \sin \alpha \end{aligned}$$

This proves the equations labeled (1) of the following theorem.

### Rotation-of-Axes Formulas

Suppose that an  $xy$ -coordinate system and an  $x'y'$ -coordinate system have the same origin and that  $\alpha$  is the angle between the positive  $x$ -axis and the positive  $x'$ -axis. If the coordinates of a point  $P$  are  $(x, y)$  in one system and  $(x', y')$  in the rotated system, then

$$\left. \begin{aligned} x &= x' \cos \alpha - y' \sin \alpha \\ y &= y' \cos \alpha + x' \sin \alpha \end{aligned} \right\} \quad (1) \qquad \left. \begin{aligned} x' &= x \cos \alpha + y \sin \alpha \\ y' &= y \cos \alpha - x \sin \alpha \end{aligned} \right\} \quad (2)$$

The derivations of the formulas for  $x'$  and  $y'$  are left as an exercise.

As we have noted, the appearance of the  $Bxy$  ( $B \neq 0$ ) term in the general second-degree equation indicates that the graph of the conic has been rotated. The angle through which the axes have been rotated can be determined from the following theorem.

### Rotation Theorem for Conics

Let  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ ,  $B \neq 0$ , be the equation of a conic in an  $xy$ -coordinate system, and let  $\alpha$  be an angle of rotation such that

$$\cot 2\alpha = \frac{A - C}{B}, \quad 0^\circ < 2\alpha < 180^\circ \quad (3)$$

Then the equation of the conic in the rotated coordinate system will be

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

where  $0^\circ < 2\alpha < 180^\circ$  and

$$A' = A \cos^2 \alpha + B \cos \alpha \sin \alpha + C \sin^2 \alpha \quad (4)$$

$$C' = A \sin^2 \alpha - B \cos \alpha \sin \alpha + C \cos^2 \alpha \quad (5)$$

$$D' = D \cos \alpha + E \sin \alpha \quad (6)$$

$$E' = -D \sin \alpha + E \cos \alpha \quad (7)$$

$$F' = F \quad (8)$$

Equation (3) has an infinite number of solutions. Any rotation through an angle  $\alpha$  that is a solution of  $\cot 2\alpha = \frac{A - C}{B}$  will eliminate the  $xy$  term. However, the equations of the conic in the rotated coordinate systems may differ, depending on the value of  $\alpha$  that is used to produce these transformed equations. To avoid any confusion, we will always choose  $\alpha$  to be the acute angle that satisfies Equation (3).

**QUESTION** For  $x^2 - 4xy + 9y^2 + 6x - 8y - 20 = 0$ , what is the value of  $\cot 2\alpha$ ?

### EXAMPLE 1 » Use the Rotation Theorem to Sketch a Conic

Sketch the graph of  $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$ .

#### Solution

We are given

$$A = 7, \quad B = -6\sqrt{3}, \quad C = 13, \quad D = 0, \quad E = 0, \quad \text{and} \quad F = -16$$

The angle of rotation  $\alpha$  can be determined by solving

$$\cot 2\alpha = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{-6}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

This gives us  $2\alpha = 60^\circ$ , or  $\alpha = 30^\circ$ . Because  $\alpha = 30^\circ$ , we have

$$\sin \alpha = \frac{1}{2} \quad \text{and} \quad \cos \alpha = \frac{\sqrt{3}}{2}$$

We determine the coefficients  $A'$ ,  $C'$ ,  $D'$ ,  $E'$ , and  $F'$  by using Equations (4) to (8).

$$A' = 7\left(\frac{\sqrt{3}}{2}\right)^2 + (-6\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 13\left(\frac{1}{2}\right)^2 = 4$$

$$C' = 7\left(\frac{1}{2}\right)^2 - (-6\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 13\left(\frac{\sqrt{3}}{2}\right)^2 = 16$$

Continued ►

**ANSWER**  $\cot 2\alpha = \frac{1 - 9}{-4} = 2$

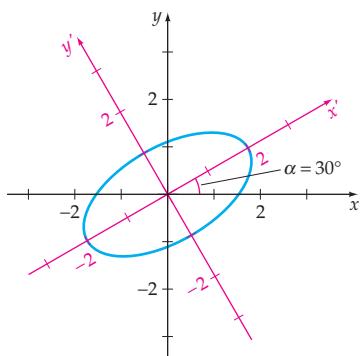


Figure 6.53

$$D' = 0\left(\frac{\sqrt{3}}{2}\right)^2 + 0\left(\frac{1}{2}\right) = 0$$

$$E' = -0\left(\frac{1}{2}\right) + 0\left(\frac{\sqrt{3}}{2}\right) = 0$$

$$F' = F = -16$$

The equation of the conic in the  $x'y'$ -plane is  $4(x')^2 + 16(y')^2 - 16 = 0$  or

$$\frac{(x')^2}{2^2} + \frac{(y')^2}{1^2} = 1$$

This is the equation of an ellipse that is centered at the origin of an  $x'y'$ -coordinate system. The ellipse has a semimajor axis  $a = 2$  and a semiminor axis  $b = 1$ . See **Figure 6.53**.

» Try Exercise 10, page 410

In Example 1, the angle of rotation  $\alpha$  was  $30^\circ$ , which is a special angle. In the next example, we demonstrate a technique that is often used when the angle of rotation is not a special angle.

### E X A M P L E 2 » Use the Rotation Theorem to Sketch a Conic

Sketch the graph of  $32x^2 - 48xy + 18y^2 - 15x - 20y = 0$ .

#### Solution

We are given

$$A = 32, \quad B = -48, \quad C = 18, \quad D = -15, \quad E = -20, \quad \text{and} \quad F = 0$$

Therefore,

$$\cot 2\alpha = \frac{A - C}{B} = \frac{32 - 18}{-48} = -\frac{7}{24}$$

**Figure 6.54** shows an angle  $2\alpha$  for which  $\cot 2\alpha = -\frac{7}{24}$ . From **Figure 6.54** we conclude that  $\cos 2\alpha = -\frac{7}{25}$ . The half-angle identities can be used to determine  $\sin \alpha$  and  $\cos \alpha$ .

$$\sin \alpha = \sqrt{\frac{1 - (-7/25)}{2}} = \frac{4}{5} \quad \text{and} \quad \cos \alpha = \sqrt{\frac{1 + (-7/25)}{2}} = \frac{3}{5}$$

A calculator can be used to determine that  $\alpha \approx 53.1^\circ$ .

Equations (4) to (8) give us

$$A' = 32\left(\frac{3}{5}\right)^2 + (-48)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + 18\left(\frac{4}{5}\right)^2 = 0$$

$$C' = 32\left(\frac{4}{5}\right)^2 - (-48)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + 18\left(\frac{3}{5}\right)^2 = 50$$

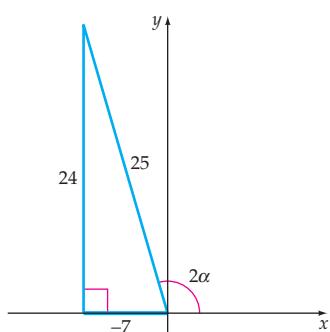
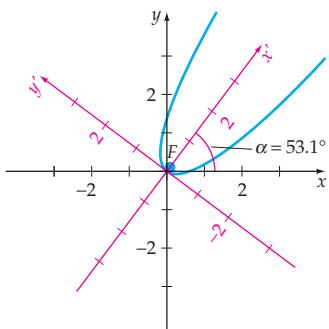


Figure 6.54



$$32x^2 - 48xy + 18y^2 - 15x - 20y = 0$$

**Figure 6.55**

$$D' = (-15)\left(\frac{3}{5}\right) + (-20)\left(\frac{4}{5}\right) = -25$$

$$E' = -(-15)\left(\frac{4}{5}\right) + (-20)\left(\frac{3}{5}\right) = 0$$

$$F' = F = 0$$

The equation of the conic in the  $x'y'$ -plane is  $50(y')^2 - 25x' = 0$ , or

$$(y')^2 = \frac{1}{2}x'$$

This is the equation of a parabola. Because  $4p = \frac{1}{2}$ , we know  $p = \frac{1}{8}$ , and the focus of the parabola is at  $\left(\frac{1}{8}, 0\right)$  on the  $x'$ -axis. See **Figure 6.55**.

» Try Exercise 20, page 410

### The Conic Identification Theorem

The following theorem provides us with a procedure that can be used to identify the type of conic that will be produced by graphing an equation that is in the form of the general second-degree equation in two variables.

#### Conic Identification Theorem

The graph of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is either a conic or a degenerate conic. If the graph is a conic, then the graph can be identified by its *discriminant*  $B^2 - 4AC$ . The graph is

- an ellipse or a circle, provided  $B^2 - 4AC < 0$ .
- a parabola, provided  $B^2 - 4AC = 0$ .
- a hyperbola, provided  $B^2 - 4AC > 0$ .

### EXAMPLE 3 » Identify Conic Sections

Each of the following equations has a graph that is a nondegenerate conic. Compute  $B^2 - 4AC$  to identify the type of conic given by each equation.

- a.  $2x^2 - 4xy + 2y^2 - 6x - 10 = 0$
- b.  $-2xy + 11 = 0$
- c.  $3x^2 + 5xy + 4y^2 - 8x + 10y + 6 = 0$
- d.  $xy - 3y^2 + 2 = 0$

Continued ►

**Solution**

- Because  $B^2 - 4AC = (-4)^2 - 4(2)(2) = 0$ , the graph is a parabola.
- Because  $B^2 - 4AC = (-2)^2 - 4(0)(0) > 0$ , the graph is a hyperbola.
- Because  $B^2 - 4AC = 5^2 - 4(3)(4) < 0$ , the graph is an ellipse or a circle.
- Because  $B^2 - 4AC = 1^2 - 4(0)(-3) > 0$ , the graph is a hyperbola.

» Try Exercise 32, page 410

### ■ Use a Graphing Utility to Graph Second-Degree Equations in Two Variables



#### Integrating Technology

To graph a general second-degree equation in two variables with a graphing utility, we first must solve the general equation for  $y$ . Consider the general second-degree equation in two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (9)$$

where  $A, B, C, D, E$ , and  $F$  are real constants, and  $C \neq 0$ . To solve Equation (9) for  $y$ , we first rewrite the equation as

$$Cy^2 + (Bx + E)y + (Ax^2 + Dx + F) = 0 \quad (10)$$

Applying the quadratic formula to Equation (10) yields

$$y = \frac{-(Bx + E) \pm \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C} \quad (11)$$

Thus the graph of Equation (9) can be constructed by graphing both

$$y_1 = \frac{-(Bx + E) + \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C} \quad (12)$$

and

$$y_2 = \frac{-(Bx + E) - \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C} \quad (13)$$

on the same grid.



#### Integrating Technology

A TI-83/TI-83 Plus/TI-84 Plus graphing calculator program is available to graph a rotated conic section by entering its coefficients. This program, ROTATE, can be found on our website at college.hmco.com/info/aufmannCAT

### E X A M P L E 4 » Use a Graphing Utility to Graph a Conic



Use a graphing utility to graph each conic.

- $7x^2 + 6xy + 2.5y^2 - 14x + 4y + 9 = 0$
- $x^2 + 5xy + 3y^2 - 25x - 84y + 375 = 0$
- $3x^2 - 6xy + 3y^2 - 15x - 12y - 8 = 0$

### Solution

Enter  $y_1$  (Equation 12) and  $y_2$  (Equation 13) into the function editing menu of a graphing utility.

- a. Store the following constants in place of the indicated variables.

$$A = 7, \quad B = 6, \quad C = 2.5, \quad D = -14, \quad E = 4, \quad \text{and} \quad F = 9$$

Graph  $y_1$  and  $y_2$  on the same screen. The union of the two graphs is an ellipse. See **Figure 6.56**.

- b. Store the following constants in place of the indicated variables.

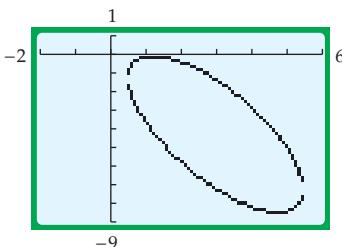
$$A = 1, \quad B = 5, \quad C = 3, \quad D = -25, \quad E = -84, \quad \text{and} \quad F = 375$$

Graph  $y_1$  and  $y_2$  on the same screen. The union of the two graphs is a hyperbola. See **Figure 6.57**.

- c. Store the following constants in place of the indicated variables.

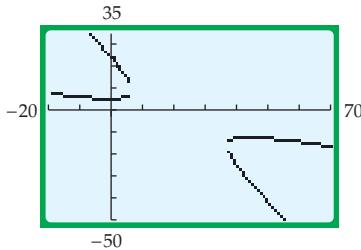
$$A = 3, \quad B = -6, \quad C = 3, \quad D = -15, \quad E = -12, \quad \text{and} \quad F = -8$$

Graph  $y_1$  and  $y_2$  on the same screen. The union of the two graphs is a parabola. See **Figure 6.58**.



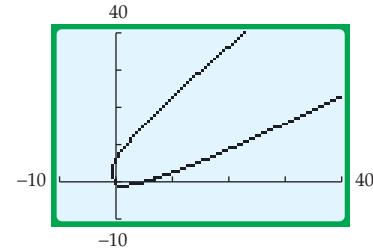
$$7x^2 + 6xy + 2.5y^2 - 14x + 4y + 9 = 0$$

**Figure 6.56**



$$x^2 + 5xy + 3y^2 - 25x - 84y + 375 = 0$$

**Figure 6.57**



$$3x^2 - 6xy + 3y^2 - 15x - 12y - 8 = 0$$

**Figure 6.58**

» Try Exercise 26, page 410



### Topics for Discussion

- Two students disagree about the graph of  $x^2 + y^2 = -4$ . One student states that the graph of the equation is an ellipse. The other student claims that the equation does not have a graph. Which student is correct? Explain.
- The graph of  $4x^2 - y^2 = 0$  consists of two intersecting lines. What are the equations of the lines?
- The graph of  $xy = 12$  is a hyperbola. What are the equations of the asymptotes of the hyperbola?
- Explain why the graph of the ellipse in **Figure 6.56** shows a gap at the left side and the right side of the ellipse.
- Explain why any conic that is given by a general second-degree equation in which  $D = 0$  and  $E = 0$  will have a quadratic equation in the rotated coordinate system with  $D' = 0$  and  $E' = 0$ .

 **Exercise Set 6.4**

**In Exercises 1 to 8, find the acute angle of rotation  $\alpha$  that eliminates the  $xy$  term. State approximate solutions to the nearest  $0.1^\circ$ .**

1.  $xy = 3$
2.  $5x^2 - 3xy - 5y^2 - 1 = 0$
3.  $9x^2 - 24xy + 16y^2 - 320x - 240y = 0$
4.  $x^2 + 4xy + 4y^2 - 6x - 5 = 0$
5.  $5x^2 - 6\sqrt{3}xy - 11y^2 + 4x - 3y + 2 = 0$
6.  $5x^2 + 4xy + 8y^2 - 6x + 3y - 12 = 0$
7.  $2x^2 + xy + y^2 - 4 = 0$
8.  $-2x^2 + \sqrt{3}xy - 3y^2 + 2x + 6y + 36 = 0$

**In Exercises 9 to 20, find the acute angle of rotation  $\alpha$  that eliminates the  $xy$  term. Then find an equation in  $x'$  and  $y'$ -coordinates. Graph the equation.**

9.  $xy = 4$
10.  $xy = -10$
11.  $6x^2 - 6xy + 14y^2 - 45 = 0$
12.  $11x^2 - 10\sqrt{3}xy + y^2 - 20 = 0$
13.  $x^2 + 4xy - 2y^2 - 1 = 0$
14.  $9x^2 - 24xy + 16y^2 + 100 = 0$
15.  $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y + 16 = 0$
16.  $x^2 + 2xy + y^2 + 2\sqrt{2}x - 2\sqrt{2}y = 0$
17.  $9x^2 - 24xy + 16y^2 - 40x - 30y + 100 = 0$
18.  $24x^2 + 16\sqrt{3}xy + 8y^2 - x + \sqrt{3}y - 8 = 0$
19.  $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$
20.  $x^2 + 4xy + 4y^2 - 2\sqrt{5}x + \sqrt{5}y = 0$

 **In Exercises 21 to 26, use a graphing utility to graph each equation.**

21.  $6x^2 - xy + 2y^2 + 4x - 12y + 7 = 0$

22.  $5x^2 - 2xy + 10y^2 - 6x - 9y - 20 = 0$
23.  $x^2 - 6xy + y^2 - 2x - 5y + 4 = 0$
24.  $2x^2 - 10xy + 3y^2 - x - 8y - 7 = 0$
25.  $3x^2 - 6xy + 3y^2 + 10x - 8y - 2 = 0$
26.  $2x^2 - 8xy + 8y^2 + 20x - 24y - 3 = 0$
27. Find the equations of the asymptotes, relative to an  $xy$ -coordinate system, for the hyperbola defined by the equation in Exercise 13. Assume that the  $xy$ -coordinate system has the same origin as the  $x'y'$ -coordinate system.
28. Find the coordinates of the foci and the equation of the directrix, relative to an  $xy$ -coordinate system, for the parabola defined by the equation in Exercise 16. Assume that the  $xy$ -coordinate system has the same origin as the  $x'y'$ -coordinate system.
29. Find the coordinates of the foci, relative to an  $xy$ -coordinate system, for the ellipse defined by the equation in Exercise 11. Assume that the  $xy$ -coordinate system has the same origin as the  $x'y'$ -coordinate system.
30.  $xy = 4$
31.  $x^2 + xy - y^2 - 40 = 0$
32.  $11x^2 - 10\sqrt{3}xy + y^2 - 20 = 0$
33.  $3x^2 + 2\sqrt{3}xy + y^2 - 3x + 2y + 20 = 0$
34.  $9x^2 - 24xy + 16y^2 + 8x - 12y - 20 = 0$
35.  $4x^2 - 4xy + y^2 - 12y + 20 = 0$
36.  $5x^2 + 4xy + 8y^2 - 6x + 3y - 12 = 0$
37.  $5x^2 - 6\sqrt{3}xy - 11y^2 + 4x - 3y + 2 = 0$
38.  $6x^2 - 6xy + 14y^2 - 14x + 12y - 60 = 0$
39.  $6x^2 + 2\sqrt{3}xy + 5y^2 - 3x + 2y - 20 = 0$
40.  $5x^2 - 2\sqrt{3}xy + 3y^2 - x + y - 12 = 0$

- 41.** By using the rotation-of-axes equations, show that for every choice of  $\alpha$ , the equation  $x^2 + y^2 = r^2$  becomes  $(x')^2 + (y')^2 = r^2$ .

**42.** The vertices of a hyperbola are  $(1, 1)$  and  $(-1, -1)$ . The foci are  $(\sqrt{2}, \sqrt{2})$  and  $(-\sqrt{2}, -\sqrt{2})$ . Find an equation of the hyperbola.

**43.** The vertices on the major axis of an ellipse are the points  $(2, 4)$  and  $(-2, -4)$ . The foci are the points  $(\sqrt{2}, 2\sqrt{2})$  and  $(-\sqrt{2}, -2\sqrt{2})$ . Find an equation of the ellipse.

**44.** The vertex of a parabola is the origin, and the focus is the point  $(1, 3)$ . Find an equation of the parabola.

## 45. AN INVARIANT THEOREM Let

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

be an equation of a conic in an  $xy$ -coordinate system. Let the equation of the conic in the rotated  $x'y'$ -coordinate system be

$$A'(x')^2 + B'x'y' + C'(y')^2 + D'x' + E'y' + F' = 0$$

Show that

$$A' + C' = A + C$$

- #### 46. AN INVARIANT THEOREM Let

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

be an equation of a conic in an  $xy$ -coordinate system. Let the equation of the conic in the rotated  $x'y'$ -coordinate system be

$$A'(x')^2 + B'x'y' + C'(y')^2 + D'x' + E'y' + F' = 0$$

Show that

$$(B')^2 - 4A'C' = B^2 - 4AC$$

- 47.** Use the result of Exercise 46 to verify the Conic Identification Theorem, stated on page 407.

**48.** Derive the equations labeled (2), page 404, of the rotation-of-axes formulas.

- 1. USE THE INVARIANT THEOREMS** The results of Exercises 45 and 46 illustrate that when the rotation theorem is used to transform equations of the form

$$Ax^2 + Bxy + Cy^2 - F = 0$$

to the form

$$A'(x')^2 + C'(\gamma')^2 - F = 0$$

the following relationships hold:

$$A + C = A' + C' \quad \text{and} \quad B^2 - 4AC = (B')^2 - 4A'C'$$

Use these equations to transform

$$10x^2 + 24xy + 17y^2 - 26 = 0$$

to the form

$$A'(x')^2 + C'(\gamma')^2 - 26 = 0$$

without applying the rotation theorem.

## Section 6.5

- The Polar Coordinate System
- Graphs of Equations in a Polar Coordinate System
- Transformations Between Rectangular and Polar Coordinates
- Write Polar Coordinate Equations as Rectangular Equations and Rectangular Coordinate Equations as Polar Equations

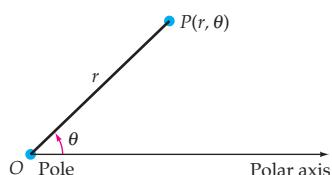


Figure 6.59

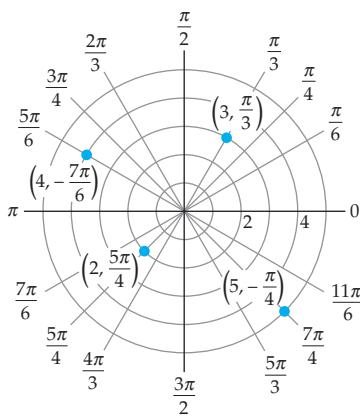


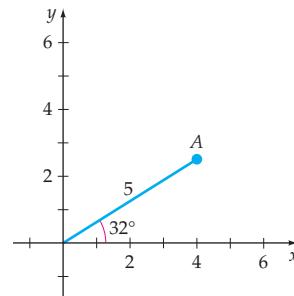
Figure 6.60

# Introduction to Polar Coordinates

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A29.

- PS1. Is  $\sin x$  an even or odd function? [2.5]
- PS2. Is  $\cos x$  an even or odd function? [2.5]
- PS3. Solve  $\tan \alpha = -\sqrt{3}$  for  $0 < \alpha < 2\pi$ . [3.6]
- PS4. If  $\sin \alpha = -\frac{\sqrt{3}}{2}$  and  $\cos \alpha = -\frac{1}{2}$ , with  $0^\circ \leq \alpha < 360^\circ$ , find  $\alpha$ . [3.6]
- PS5. Write  $(r \cos \theta)^2 + (r \sin \theta)^2$  in simplest form. [3.1]
- PS6. For the graph at the right, find the coordinates of point A. Round each coordinate to the nearest tenth. [2.2]



## The Polar Coordinate System

Until now, we have used a *rectangular coordinate system* to locate a point in the coordinate plane. An alternative method is to use a *polar coordinate system*, wherein a point is located by giving a distance from a fixed point and an angle from some fixed direction.

A **polar coordinate system** is formed by drawing a horizontal ray. The ray is called the **polar axis**, and the initial point of the ray is called the **pole**. A point  $P(r, \theta)$  in the plane is located by specifying a distance  $r$  from the pole and an angle  $\theta$  measured from the polar axis to the line segment  $OP$ . The angle can be measured in degrees or radians. See **Figure 6.59**.

The coordinates of the pole are  $(0, \theta)$ , where  $\theta$  is an arbitrary angle. Positive angles are measured counterclockwise from the polar axis. Negative angles are measured clockwise from the axis. Positive values of  $r$  are measured along the ray that makes an angle of  $\theta$  from the polar axis. Negative values of  $r$  are measured along the ray that makes an angle of  $\theta + 180^\circ$  from the polar axis. See **Figures 6.60** and **6.61**.

In a rectangular coordinate system, there is a one-to-one correspondence between the points in the plane and the ordered pairs  $(x, y)$ . This is not true for a polar coordinate system. For polar coordinates, the relationship is one-to-many. Infinitely many ordered-pair descriptions correspond to each point  $P(r, \theta)$  in a polar coordinate system.

For example, consider a point whose coordinates are  $P(3, 45^\circ)$ . Because there are  $360^\circ$  in one complete revolution around a circle, the point  $P$  also could be written as  $(3, 405^\circ)$ , as  $(3, 765^\circ)$ , as  $(3, 1125^\circ)$ , and generally as  $(3, 45^\circ + n \cdot 360^\circ)$ , where  $n$  is an integer. It is also possible to describe the point  $P(3, 45^\circ)$  by  $(-3, 225^\circ)$ ,  $(-3, -135^\circ)$ , and  $(3, -315^\circ)$ , to name just a few options.

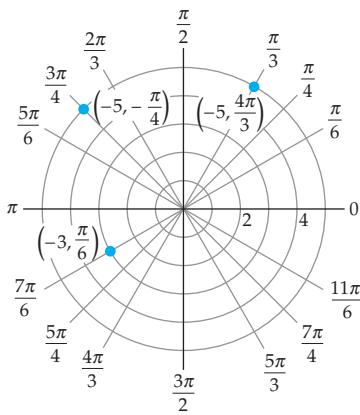


Figure 6.61

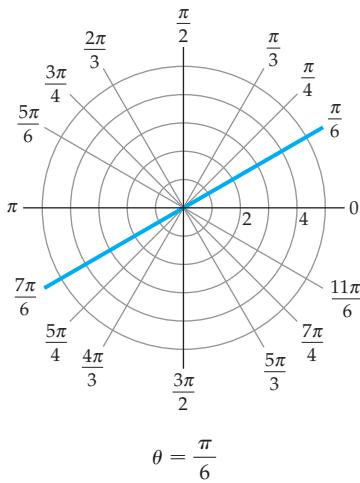


Figure 6.62

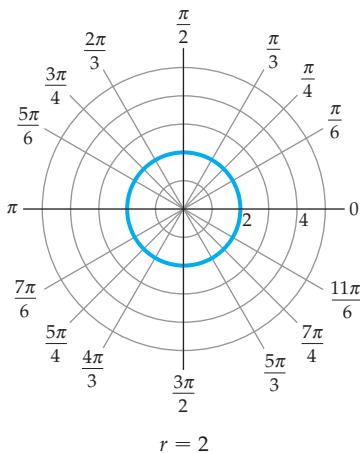


Figure 6.64

The relationship between an ordered pair and a point is not one-to-many. That is, given an ordered pair  $(r, \theta)$ , there is exactly one point in the plane that corresponds to that ordered pair.

### Graphs of Equations in a Polar Coordinate System

A **polar equation** is an equation in  $r$  and  $\theta$ . A **solution** to a polar equation is an ordered pair  $(r, \theta)$  that satisfies the equation. The **graph** of a polar equation is the set of all points whose ordered pairs are solutions of the equation.

The graph of the polar equation  $\theta = \frac{\pi}{6}$  is a line. Because  $\theta$  is independent of  $r$ ,  $\theta$  is  $\frac{\pi}{6}$  radian from the polar axis for all values of  $r$ . The graph is a line that makes an angle of  $\frac{\pi}{6}$  radian ( $30^\circ$ ) from the polar axis. See **Figure 6.62**.

#### Polar Equations of a Line

The graph of  $\theta = \alpha$  is a line through the pole at an angle of  $\alpha$  from the polar axis. See **Figure 6.63a**.

The graph of  $r \sin \theta = a$  is a horizontal line passing through the point  $\left(a, \frac{\pi}{2}\right)$ . See **Figure 6.63b**.

The graph of  $r \cos \theta = a$  is a vertical line passing through the point  $(a, 0)$ . See **Figure 6.63c**.

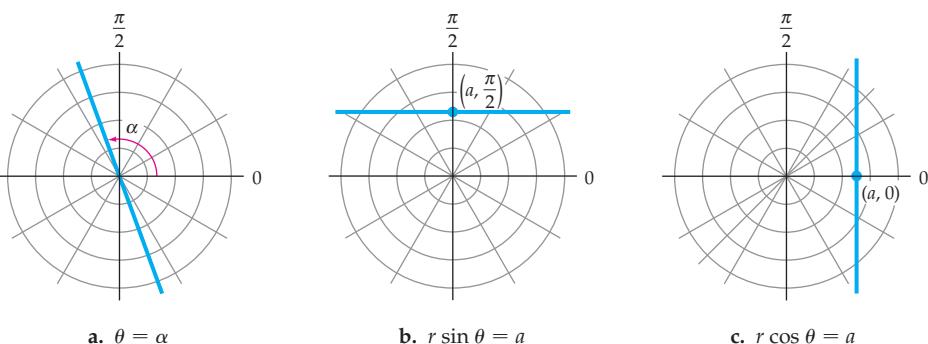
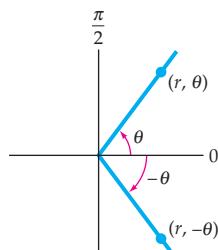


Figure 6.63

**Figure 6.64** is the graph of the polar equation  $r = 2$ . Because  $r$  is independent of  $\theta$ ,  $r$  is 2 units from the pole for all values of  $\theta$ . The graph is a circle of radius 2 with center at the pole.

#### The Graph of $r = a$

The graph of  $r = a$  is a circle with center at the pole and radius  $a$ .



Symmetry with respect to  
the line  $\theta = 0$

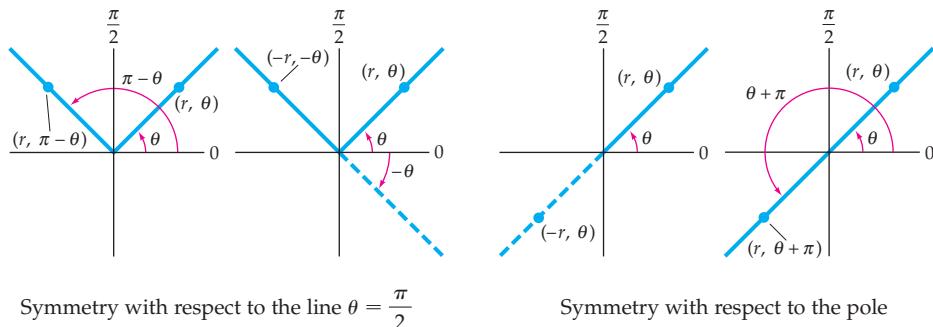
**Figure 6.65**

Suppose that whenever the ordered pair  $(r, \theta)$  lies on the graph of a polar equation,  $(r, -\theta)$  also lies on the graph. From **Figure 6.65**, the graph will have symmetry with respect to the line  $\theta = 0$ . Thus one test for symmetry is to replace  $\theta$  by  $-\theta$  in the polar equation. If the resulting equation is equivalent to the original equation, the graph is symmetric with respect to the line  $\theta = 0$ .

**Table 6.2** shows the types of symmetry and their associated tests. For each type, if the recommended substitution results in an equivalent equation, the graph will have the indicated symmetry. **Figure 6.66** illustrates the tests for symmetry with respect to the line  $\theta = \frac{\pi}{2}$  and for symmetry with respect to the pole.

**Table 6.2 Tests for Symmetry**

| Substitution                               | Symmetry with respect to          |
|--|-----------------------------------|
| $-\theta$ for $\theta$                     | The line $\theta = 0$             |
| $\pi - \theta$ for $\theta$ , $-r$ for $r$ | The line $\theta = 0$             |
| $\pi - \theta$ for $\theta$                | The line $\theta = \frac{\pi}{2}$ |
| $-\theta$ for $\theta$ , $-r$ for $r$      | The line $\theta = \frac{\pi}{2}$ |
| $-r$ for $r$                               | The pole                          |
| $\pi + \theta$ for $\theta$                | The pole                          |



**Figure 6.66**

The graph of a polar equation may have a type of symmetry even though a test for that symmetry fails. For example, as we will see later, the graph of  $r = \sin 2\theta$  is symmetric with respect to the line  $\theta = 0$ . However, using the symmetry test of substituting  $-\theta$  for  $\theta$ , we have

$$\sin 2(-\theta) = -\sin 2\theta = -r \neq r$$

Thus this test fails to show symmetry with respect to the line  $\theta = 0$ . The symmetry test of substituting  $\pi - \theta$  for  $\theta$  and  $-r$  for  $r$  establishes symmetry with respect to the line  $\theta = 0$ .

**EXAMPLE 1** » Graph a Polar Equation

Show that the graph of  $r = 4 \cos \theta$  is symmetric with respect to the line  $\theta = 0$ . Graph the equation.

**Solution**

Test for symmetry with respect to the line  $\theta = 0$ . Replace  $\theta$  by  $-\theta$ .

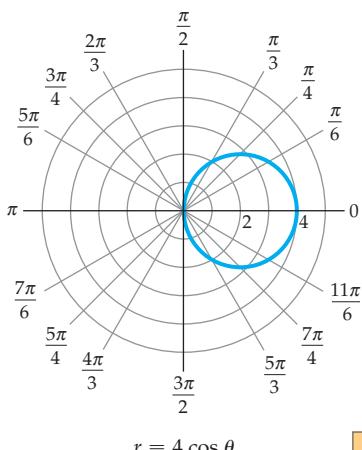
$$r = 4 \cos(-\theta) = 4 \cos \theta \quad \bullet \cos(-\theta) = \cos \theta$$

Because replacing  $\theta$  by  $-\theta$  results in the original equation  $r = 4 \cos \theta$ , the graph is symmetric with respect to the line  $\theta = 0$ .

To graph the equation, begin choosing various values of  $\theta$  and finding the corresponding values of  $r$ . However, before doing so, consider two further observations that will reduce the number of  $\theta$ -values you must choose.

First, because the cosine function is a periodic function with period  $2\pi$ , it is only necessary to choose  $\theta$ -values between 0 and  $2\pi$  ( $0^\circ$  and  $360^\circ$ ). Second, when  $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ ,  $\cos \theta$  is negative, which means that any  $\theta$  between these values will produce a negative  $r$ . Thus the point will be in the first or fourth quadrant. That is, we need consider only angles  $\theta$  in the first or fourth quadrants. However, because the graph is symmetric with respect to the line  $\theta = 0$ , it is only necessary to choose values of  $\theta$  between 0 and  $\frac{\pi}{2}$ .

**By symmetry**



**Figure 6.67**

|                            |     |                 |                 |                 |                 |                  |                  |                  |                  |
|----------------------------|-----|-----------------|-----------------|-----------------|-----------------|------------------|------------------|------------------|------------------|
| <b><math>\theta</math></b> | 0   | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $-\frac{\pi}{6}$ | $-\frac{\pi}{4}$ | $-\frac{\pi}{3}$ | $-\frac{\pi}{2}$ |
| <b><math>r</math></b>      | 4.0 | 3.5             | 2.8             | 2.0             | 0.0             | 3.5              | 2.8              | 2.0              | 0.0              |

The graph of  $r = 4 \cos \theta$  is a circle with center at  $(2, 0)$ . See **Figure 6.67**.

» Try Exercise 14, page 423

**Polar Equations of a Circle**

The graph of the equation  $r = a$  is a circle with center at the pole and radius  $a$ . See **Figure 6.68a**.

The graph of the equation  $r = a \cos \theta$  is a circle that is symmetric with respect to the line  $\theta = 0$ . See **Figure 6.68b**.

The graph of  $r = a \sin \theta$  is a circle that is symmetric with respect to the line  $\theta = \frac{\pi}{2}$ . See **Figure 6.68c**.

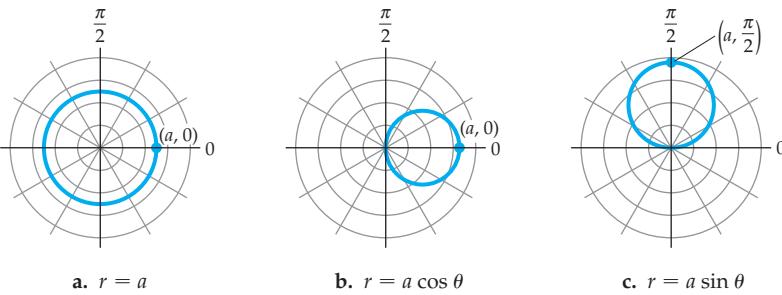


Figure 6.68

Just as there are specifically named curves in an  $xy$ -coordinate system (such as parabola and ellipse), there are named curves in an  $r\theta$ -coordinate system. Two of the many types are the *limaçon* and the *rose curve*.

### Polar Equations of a Limaçon

The graph of the equation  $r = a + b \cos \theta$  is a **limaçon** that is symmetric with respect to the line  $\theta = 0$ .

The graph of the equation  $r = a + b \sin \theta$  is a limaçon that is symmetric with respect to the line  $\theta = \frac{\pi}{2}$ .

In the special case where  $|a| = |b|$ , the graph is called a **cardioid**.

The graph of  $r = a + b \cos \theta$  is shown in **Figure 6.69** for various values of  $a$  and  $b$ .

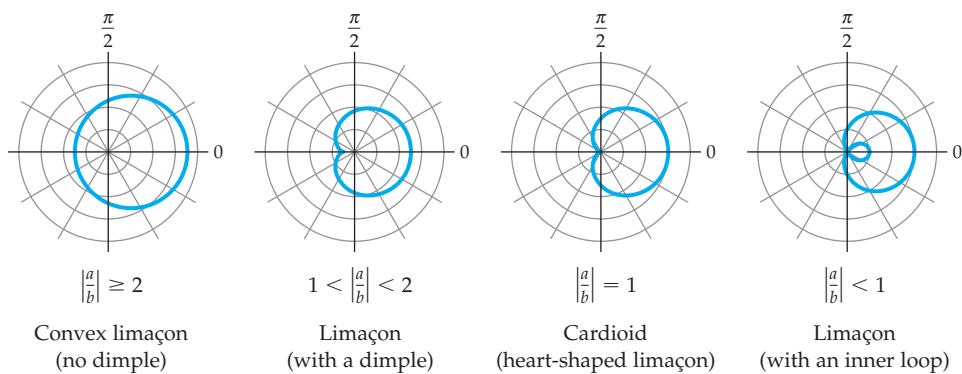


Figure 6.69

**EXAMPLE 2** Sketch the Graph of a Limaçon

Sketch the graph of  $r = 2 - 2 \sin \theta$ .

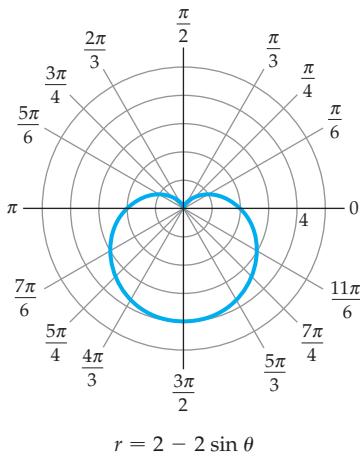
**Solution**

From the general equation of a limaçon  $r = a + b \sin \theta$  with  $|a| = |b|$  ( $|2| = |-2|$ ), the graph of  $r = 2 - 2 \sin \theta$  is a cardioid that is symmetric with respect to the line  $\theta = \frac{\pi}{2}$ .

Because we know that the graph is heart-shaped, we can sketch the graph by finding  $r$  for a few values of  $\theta$ . When  $\theta = 0$ ,  $r = 2$ . When  $\theta = \frac{\pi}{2}$ ,  $r = 0$ . When  $\theta = \pi$ ,  $r = 2$ . When  $\theta = \frac{3\pi}{2}$ ,  $r = 4$ . Sketching a heart-shaped curve through the four points

$$(2, 0), \left(0, \frac{\pi}{2}\right), (2, \pi), \text{ and } \left(4, \frac{3\pi}{2}\right)$$

produces the cardioid in **Figure 6.70**.

**Try Exercise 20, page 424**

**Figure 6.70**

Example 3 shows how to use a graphing utility to construct a polar graph.

**EXAMPLE 3** Use a Graphing Utility to Graph a Polar Equation

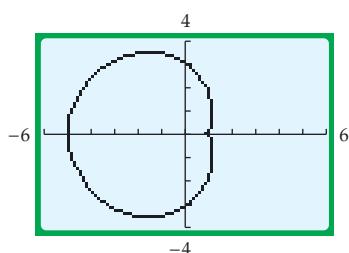
Use a graphing utility to graph  $r = 3 - 2 \cos \theta$ .

**Solution**

From the general equation of a limaçon  $r = a + b \cos \theta$ , with  $a = 3$  and  $b = -2$ , we know that the graph will be a limaçon with a dimple. The graph will be symmetric with respect to the line  $\theta = 0$ .

Use polar mode with angle measure in radians. Enter the equation  $r = 3 - 2 \cos \theta$  in the polar function editing menu. The graph in **Figure 6.71** was produced with a TI-84 by using the following window:

|                            |                    |                    |
|----------------------------|--------------------|--------------------|
| $\theta_{\min}=0$          | $X_{\min}=-6$      | $Y_{\min}=-4$      |
| $\theta_{\max}=2\pi$       | $X_{\max}=6$       | $Y_{\max}=4$       |
| $\theta_{\text{step}}=0.1$ | $X_{\text{scl}}=1$ | $Y_{\text{scl}}=1$ |



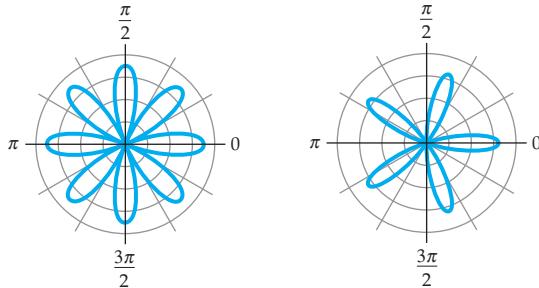
**Figure 6.71**

**Try Exercise 26, page 424**

When using a graphing utility in polar mode, choose the value of  $\theta_{\text{step}}$  carefully. If  $\theta_{\text{step}}$  is set too small, the graphing utility may require an excessively long period of time to complete the graph. If  $\theta_{\text{step}}$  is set too large, the resulting graph may give only a very rough approximation of the actual graph.

### Polar Equations of Rose Curves

The graphs of the equations  $r = a \cos n\theta$  and  $r = a \sin n\theta$  are **rose curves**. When  $n$  is an even number, the number of petals is  $2n$ . See **Figure 6.72a**. When  $n$  is an odd number, the number of petals is  $n$ . See **Figure 6.72b**.



- a.  $r = a \cos 4\theta$   
 $n = 4$  is even,  $2n = 8$  petals      b.  $r = a \cos 5\theta$   
 $n = 5$  is odd, 5 petals

**Figure 6.72**

- QUESTION** How many petals are in the graph of  
 a.  $r = 4 \cos 3\theta$ ?      b.  $r = 5 \sin 2\theta$ ?

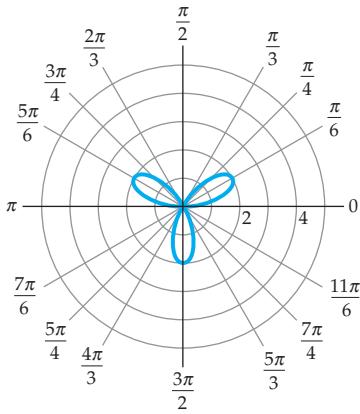
### EXAMPLE 4 » Sketch the Graph of a Rose Curve

Sketch the graph of  $r = 2 \sin 3\theta$ .

#### Solution

From the general equation of a rose curve  $r = a \sin n\theta$ , with  $a = 2$  and  $n = 3$ , the graph of  $r = 2 \sin 3\theta$  is a rose curve that is symmetric with respect to the line  $\theta = \frac{\pi}{2}$ . Because  $n$  is an odd number ( $n = 3$ ), there will be three petals in the graph.

Choose some values for  $\theta$  and find the corresponding values of  $r$ . Use symmetry to sketch the graph. See **Figure 6.73**.



$$r = 2 \sin 3\theta$$

**Figure 6.73**

|          |     |                  |                 |                   |                 |                   |                 |
|----------|-----|------------------|-----------------|-------------------|-----------------|-------------------|-----------------|
| $\theta$ | 0   | $\frac{\pi}{18}$ | $\frac{\pi}{6}$ | $\frac{5\pi}{18}$ | $\frac{\pi}{3}$ | $\frac{7\pi}{18}$ | $\frac{\pi}{2}$ |
| $r$      | 0.0 | 1.0              | 2.0             | 1.0               | 0.0             | -1.0              | -2.0            |

### » Try Exercise 16, page 423

- ANSWER** a. Because 3 is an odd number, there are three petals in the graph.  
 b. Because 2 is an even number, there are  $2(2) = 4$  petals in the graph.

**EXAMPLE 5** » Use a Graphing Utility to Graph a Rose Curve


Use a graphing utility to graph  $r = 4 \cos 2\theta$ .

**Solution**

From the general equation of a rose curve  $r = a \cos n\theta$ , with  $a = 4$  and  $n = 2$ , we know that the graph will be a rose curve with  $2n = 4$  petals. The very tip of each petal will be  $a = 4$  units away from the pole. Our symmetry tests also indicate that the graph is symmetric with respect to the line  $\theta = 0$ , the line  $\theta = \frac{\pi}{2}$ , and the pole.

Use polar mode with angle measure in radians. Enter the equation  $r = 4 \cos 2\theta$  in the polar function editing menu. The graph in **Figure 6.74** was produced with a TI-84 by using the following window:

|                            |                     |                     |
|----------------------------|---------------------|---------------------|
| $\theta_{\text{min}}=0$    | $X_{\text{min}}=-6$ | $Y_{\text{min}}=-4$ |
| $\theta_{\text{max}}=2\pi$ | $X_{\text{max}}=6$  | $Y_{\text{max}}=4$  |
| $\theta_{\text{step}}=0.1$ | $X_{\text{scl}}=1$  | $Y_{\text{scl}}=1$  |

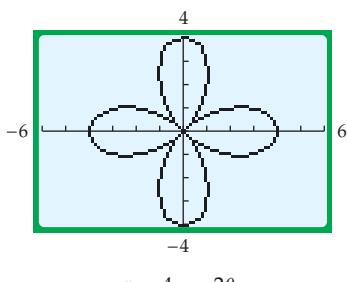


Figure 6.74

» Try Exercise 32, page 424

## Transformations Between Rectangular and Polar Coordinates

A transformation between coordinate systems is a set of equations that relate the coordinates of a point in one system with the coordinates of the point in a second system. By superimposing a rectangular coordinate system on a polar system, we can derive the set of transformation equations.

Construct a polar coordinate system and a rectangular system such that the pole coincides with the origin and the polar axis coincides with the positive  $x$ -axis. Let a point  $P$  have coordinates  $(x, y)$  in one system and  $(r, \theta)$  in the other ( $r > 0$ ).

From the definitions of  $\sin \theta$  and  $\cos \theta$ , we have

$$\frac{x}{r} = \cos \theta \quad \text{or} \quad x = r \cos \theta$$

$$\frac{y}{r} = \sin \theta \quad \text{or} \quad y = r \sin \theta$$

It can be shown that these equations are also true when  $r < 0$ .

Thus, given the point  $(r, \theta)$  in a polar coordinate system (see **Figure 6.75**), the coordinates of the point in the  $xy$ -coordinate system are given by

$$x = r \cos \theta \quad y = r \sin \theta$$

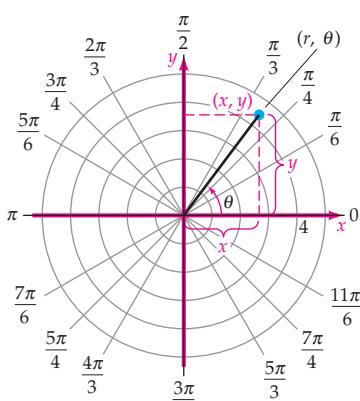


Figure 6.75

For example, to find the point in the  $xy$ -coordinate system that corresponds to the point  $\left(4, \frac{2\pi}{3}\right)$  in the  $r\theta$ -coordinate system, substitute 4 for  $r$  and  $\frac{2\pi}{3}$  for  $\theta$  into the equations and simplify.

$$x = 4 \cos \frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$$

$$y = 4 \sin \frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

The point  $\left(4, \frac{2\pi}{3}\right)$  in the  $r\theta$ -coordinate system is  $(-2, 2\sqrt{3})$  in the  $xy$ -coordinate system.

To find the polar coordinates of a given point in the  $xy$ -coordinate system, use the Pythagorean Theorem and the definition of the tangent function. Let  $P(x, y)$  be a point in the plane, and let  $r$  be the distance from the origin to the point  $P$ . Then  $r = \sqrt{x^2 + y^2}$ .

From the definition of the tangent function of an angle in a right triangle,

$$\tan \theta = \frac{y}{x}$$

Thus  $\theta$  is the angle whose tangent is  $\frac{y}{x}$ . The quadrant for  $\theta$  depends on the sign of  $x$  and the sign of  $y$ .

The equations of transformation between a polar and a rectangular coordinate system are summarized as follows:

### Transformations Between Polar and Rectangular Coordinates

Given the point  $(r, \theta)$  in the polar coordinate system, the transformation equations to change from polar to rectangular coordinates are

$$x = r \cos \theta \quad y = r \sin \theta$$

Given the point  $(x, y)$  in the rectangular coordinate system, the transformation equations to change from rectangular to polar coordinates are

$$r = \sqrt{x^2 + y^2} \quad \tan \theta = \frac{y}{x}, \quad x \neq 0$$

where  $r \geq 0$ ,  $0 \leq \theta < 2\pi$ , and  $\theta$  is chosen so that the point lies in the appropriate quadrant. If  $x = 0$ , then  $\theta = \frac{\pi}{2}$  or  $\theta = \frac{3\pi}{2}$ .

**EXAMPLE 6** » Transform from Polar to Rectangular Coordinates

Find the rectangular coordinates of the points whose polar coordinates are:

**a.**  $\left(6, \frac{3\pi}{4}\right)$

**b.**  $(-4, 30^\circ)$

**Solution**

Use the transformation equations  $x = r \cos \theta$  and  $y = r \sin \theta$ .

**a.**  $x = 6 \cos \frac{3\pi}{4} = -3\sqrt{2}$        $y = 6 \sin \frac{3\pi}{4} = 3\sqrt{2}$

The rectangular coordinates of  $\left(6, \frac{3\pi}{4}\right)$  are  $(-3\sqrt{2}, 3\sqrt{2})$ .

**b.**  $x = -4 \cos 30^\circ = -2\sqrt{3}$        $y = -4 \sin 30^\circ = -2$

The rectangular coordinates of  $(-4, 30^\circ)$  are  $(-2\sqrt{3}, -2)$ .

» Try Exercise 44, page 424

**EXAMPLE 7** » Transform from Rectangular to Polar Coordinates

Find the polar coordinates of the point whose rectangular coordinates are  $(-2, -2\sqrt{3})$ .

**Solution**

Use the transformation equations  $r = \sqrt{x^2 + y^2}$  and  $\tan \theta = \frac{y}{x}$ .

$$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\tan \theta = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$$

From this and the fact that  $(-2, -2\sqrt{3})$  lies in the third quadrant,  $\theta = \frac{4\pi}{3}$ .

The polar coordinates of  $(-2, -2\sqrt{3})$  are  $\left(4, \frac{4\pi}{3}\right)$ .

» Try Exercise 48, page 424

**■ Write Polar Coordinate Equations as Rectangular Equations and Rectangular Coordinate Equations as Polar Equations**

Using the transformation equations, it is possible to write a polar coordinate equation in rectangular form or a rectangular coordinate equation in polar form.

**EXAMPLE 8** Write a Polar Coordinate Equation in Rectangular Form

Find a rectangular form of the equation  $r^2 \cos 2\theta = 3$ .

**Solution**

$$\begin{aligned} r^2 \cos 2\theta &= 3 \\ r^2(1 - 2 \sin^2 \theta) &= 3 && \bullet \cos 2\theta = 1 - 2 \sin^2 \theta \\ r^2 - 2r^2 \sin^2 \theta &= 3 \\ r^2 - 2(r \sin \theta)^2 &= 3 && \bullet r^2 = x^2 + y^2; \sin \theta = \frac{y}{r} \\ x^2 + y^2 - 2y^2 &= 3 \\ x^2 - y^2 &= 3 \end{aligned}$$

A rectangular form of  $r^2 \cos 2\theta = 3$  is  $x^2 - y^2 = 3$ .

» Try Exercise 58, page 424

Sometimes the procedures of squaring each side of a polar equation or multiplying each side of a polar equation by  $r$  can be used to write the polar equation in rectangular form. In Example 9 we multiply each side of a polar equation by  $r$  to convert the equation from polar to rectangular form.

**EXAMPLE 9** Write a Polar Coordinate Equation in Rectangular Form

Find a rectangular form of the equation  $r = 8 \cos \theta$ .

**Solution**

$$\begin{aligned} r &= 8 \cos \theta \\ r^2 &= 8r \cos \theta && \bullet \text{Multiply each side by } r. \\ x^2 + y^2 &= 8x && \bullet \text{Use the equations } r^2 = x^2 + y^2 \text{ and } x = r \cos \theta. \\ (x^2 - 8x) + y^2 &= 0 && \bullet \text{Subtract } 8x \text{ from each side.} \\ (x^2 - 8x + 16) + y^2 &= 16 && \bullet \text{Complete the square in } x. \\ (x - 4)^2 + y^2 &= 4^2 && \bullet \text{Write in standard form.} \end{aligned}$$

A rectangular form of  $r = 8 \cos \theta$  is  $(x - 4)^2 + y^2 = 4^2$ . The graph of each of these equations is a circle with center  $(4, 0)$  and radius 4.

» Try Exercise 50, page 424

**take note**

Squaring each side of the transformation equation  $r = \sqrt{x^2 + y^2}$  produces  $r^2 = x^2 + y^2$ . This equation is often used to write polar coordinate equations in rectangular form.

In Example 10, we use the transformation equations  $x = r \cos \theta$  and  $y = r \sin \theta$  to convert a rectangular coordinate equation to polar form.

**EXAMPLE 10**  Write a Rectangular Coordinate Equation in Polar Form

Find a polar form of the equation  $x^2 + y^2 - 2x = 3$ .

**Solution**

$$\begin{aligned} x^2 + y^2 - 2x &= 3 \\ (r \cos \theta)^2 + (r \sin \theta)^2 - 2r \cos \theta &= 3 \\ r^2(\cos^2 \theta + \sin^2 \theta) - 2r \cos \theta &= 3 \\ r^2 - 2r \cos \theta &= 3 \end{aligned}$$

- Use the transformation equations  $x = r \cos \theta$  and  $y = r \sin \theta$ .

- Simplify.

A polar form of  $x^2 + y^2 - 2x = 3$  is  $r^2 - 2r \cos \theta = 3$ .

 Try Exercise 70, page 424


**Topics for Discussion**

1. In what quadrant is the point  $(-2, 150^\circ)$  located?
2. To explain why the graph of  $\theta = \frac{\pi}{6}$  is a line, a tutor rewrites the equation in the form  $\theta = \frac{\pi}{6} + 0 \cdot r$ . In this form, regardless of the value of  $r$ ,  $\theta = \frac{\pi}{6}$ . Use an analogous approach to explain why the graph of  $r = a$  is a circle.
3. Two students use a graphing calculator to graph the polar equation  $r = 2 \sin \theta$ . One graph appears to be a circle, and the other graph appears to be an ellipse. Both graphs are correct. Explain.
4. Is the graph of  $r^2 = 6 \cos 2\theta$  a rose curve with four petals? Explain your answer.


**Exercise Set 6.5**

In Exercises 1 to 8, plot the point on a polar coordinate system.

- |                                      |                                     |                                     |                          |                               |
|--------------------------------------|-------------------------------------|-------------------------------------|--------------------------|-------------------------------|
| 1. $(2, 60^\circ)$                   | 2. $(3, -90^\circ)$                 | 3. $(1, 315^\circ)$                 | 9. $r = 3$               | 10. $r = 5$                   |
| 4. $(2, 400^\circ)$                  | 5. $\left(-2, \frac{\pi}{4}\right)$ | 6. $\left(4, \frac{7\pi}{6}\right)$ | 11. $\theta = 2$         | 12. $\theta = -\frac{\pi}{3}$ |
| 7. $\left(-3, \frac{5\pi}{3}\right)$ | 8. $(-3, \pi)$                      |                                     | 13. $r = 6 \cos \theta$  | 14. $r = 4 \sin \theta$       |
|                                      |                                     |                                     | 15. $r = 4 \cos 2\theta$ | 16. $r = 5 \cos 3\theta$      |

In Exercises 9 to 24, sketch the graph of each polar equation.

17.  $r = 2 \sin 5\theta$

18.  $r = 3 \cos 5\theta$

55.  $\theta = \frac{\pi}{6}$

56.  $r \cos \theta = -4$

19.  $r = 2 - 3 \sin \theta$

20.  $r = 2 - 2 \cos \theta$

57.  $r = \tan \theta$

58.  $r = \cot \theta$

21.  $r = 4 + 3 \sin \theta$

22.  $r = 2 + 4 \sin \theta$

59.  $r = \frac{2}{1 + \cos \theta}$

60.  $r = \frac{2}{1 - \sin \theta}$

23.  $r = 2[1 + 1.5 \sin(-\theta)]$

24.  $r = 4(1 - \sin \theta)$

61.  $r(\sin \theta - 2 \cos \theta) = 6$

62.  $r(2 \cos \theta + \sin \theta) = 3$



In Exercises 25 to 40, use a graphing utility to graph each equation.

25.  $r = 3 + 3 \cos \theta$

26.  $r = 4 - 4 \sin \theta$

In Exercises 63 to 74, find a polar form of each of the equations.

27.  $r = 4 \cos 3\theta$

28.  $r = 2 \sin 4\theta$

63.  $y = 2$

64.  $x = -4$

29.  $r = 3 \sec \theta$

30.  $r = 4 \csc \theta$

65.  $y = \sqrt{3}x$

66.  $y = x^2$

31.  $r = -5 \csc \theta$

32.  $r = -4 \sec \theta$

67.  $x = 3$

68.  $xy = 4$

33.  $r = 4 \sin(3.5\theta)$

34.  $r = 6 \cos(2.25\theta)$

69.  $x^2 + y^2 = 4$

70.  $2x - 3y = 6$

35.  $r = \theta, 0 \leq \theta \leq 6\pi$

36.  $r = -\theta, 0 \leq \theta \leq 6\pi$

71.  $x^2 = 8y$

72.  $y^2 = 4y$

37.  $r = 2^\theta, 0 \leq \theta \leq 2\pi$

38.  $r = \frac{1}{\theta}, 0 \leq \theta \leq 4\pi$

73.  $x^2 - y^2 = 25$

74.  $x^2 + 4y^2 = 16$

39.  $r = \frac{6 \cos 7\theta + 2 \cos 3\theta}{\cos \theta}$

40.  $r = \frac{4 \cos 3\theta + \cos 5\theta}{\cos \theta}$

In Exercises 41 to 48, transform the given coordinates to the indicated ordered pair. Round approximate angle measures to the nearest tenth of a degree.

41.  $(1, -\sqrt{3})$  to  $(r, \theta)$

42.  $(-2\sqrt{3}, 2)$  to  $(r, \theta)$

43.  $\left(-3, \frac{2\pi}{3}\right)$  to  $(x, y)$

44.  $\left(2, -\frac{\pi}{3}\right)$  to  $(x, y)$

45.  $\left(0, -\frac{\pi}{2}\right)$  to  $(x, y)$

46.  $\left(3, \frac{5\pi}{6}\right)$  to  $(x, y)$

47.  $(3, 4)$  to  $(r, \theta)$

48.  $(12, -5)$  to  $(r, \theta)$

In Exercises 49 to 62, find a rectangular form of each of the equations.

49.  $r = 3 \cos \theta$

50.  $r = 2 \sin \theta$

51.  $r = 3 \sec \theta$

52.  $r = 4 \csc \theta$

53.  $r = 4$

54.  $\theta = \frac{\pi}{4}$



In Exercises 75 to 82, use a graphing utility to graph each equation.

75.  $r = 3 \cos\left(\theta + \frac{\pi}{4}\right)$

76.  $r = 2 \sin\left(\theta - \frac{\pi}{6}\right)$

77.  $r = 2 \sin\left(2\theta - \frac{\pi}{3}\right)$

78.  $r = 3 \cos\left(2\theta + \frac{\pi}{4}\right)$

79.  $r = 2 + 2 \sin\left(\theta - \frac{\pi}{6}\right)$

80.  $r = 3 - 2 \cos\left(\theta + \frac{\pi}{3}\right)$

81.  $r = 1 + 3 \cos\left(\theta + \frac{\pi}{3}\right)$

82.  $r = 2 - 4 \sin\left(\theta - \frac{\pi}{4}\right)$

- 83.** Explain why the graph of  $r^2 = \cos^2 \theta$  and the graph of  $r = \cos \theta$  are not the same.

**84.** Explain why the graph of  $r = \cos 2\theta$  and the graph of  $r = 2 \cos^2 \theta - 1$  are identical.

 In Exercises 85 to 92, use a graphing utility to graph each equation.

- 85.**  $r^2 = 4 \cos 2\theta$  (lemniscate)

**86.**  $r^2 = -2 \sin 2\theta$  (lemniscate)

**87.**  $r = 2(1 + \sec \theta)$  (conchoid)

**88.**  $r = 2 \cos 2\theta \sec \theta$  (strophoid)

**89.**  $r\theta = 2$  (spiral)

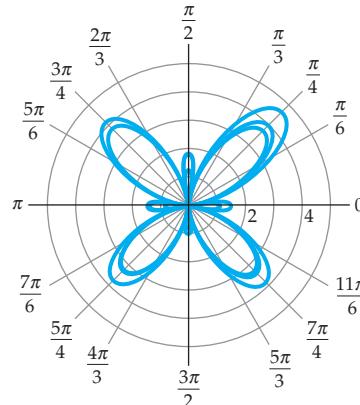
**90.**  $r = 2 \sin \theta \cos^2 2\theta$  (bifolium)

- 92.**  $r = \ln$

**93.**  The graph of

$$r = 1.5^{\sin \theta} - 2.5 \cos 4\theta + \sin^7 \frac{\theta}{15}$$

is a *butterfly curve* similar to the one shown below.



Use a graphing utility to graph the butterfly curve for

- a.**  $0 \leq \theta \leq 5\pi$       **b.**  $0 \leq \theta \leq 20\pi$

For additional information on butterfly curves, read "The Butterfly Curve" by Temple H. Fay, *The American Mathematical Monthly*, vol. 96, no. 5 (May 1989), p. 442.

- 1. A POLAR DISTANCE FORMULA** Let  $P_1(r_1, \theta_1)$  and  $P_2(r_2, \theta_2)$  be two distinct points in the  $r\theta$ -plane.

a. Verify that the distance  $d$  between the points is

$$d = \sqrt{(r_1)^2 + (r_2)^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

b. Use the above formula to find the distance (to the nearest hundredth) between  $(3, 60^\circ)$  and  $(5, 170^\circ)$ .

c.  Does the formula

$$d = \sqrt{(r_1)^2 + (r_2)^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

also produce the correct distance between  $P_1$  and  $P_2$ ? Explain.

**2. ANOTHER POLAR FORM FOR A CIRCLE**

a. Verify that the graph of the polar equation  $r = a \sin \theta + b \cos \theta$  is a circle. Assume that  $a$  and  $b$  are not both 0.

b. What are the center (in rectangular coordinates) and the radius of the circle?

## Section 6.6

- Polar Equations of the Conics
- Graph a Conic Given in Polar Form
- Write the Polar Equation of a Conic

# Polar Equations of the Conics

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A31.

**PS1.** Find the eccentricity of the graph of  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ . [6.2]

**PS2.** What is the equation of the directrix of the graph of  $y^2 = 4x$ ? [6.1]

**PS3.** Solve  $y = 2(1 + yx)$  for  $y$ . [1.1]

**PS4.** For the function  $f(x) = \frac{3}{1 + \sin x}$ , what is the smallest positive value of  $x$  that must be excluded from the domain of  $f$ ? [1.3/2.2]

**PS5.** Let  $e$  be the eccentricity of a hyperbola. Which of the following statements is true:  $e = 0$ ,  $0 < e < 1$ ,  $e = 1$ , or  $e > 1$ ? [6.3]

**PS6.** Write  $\frac{4 \sec x}{2 \sec x - 1}$  in terms of  $\cos x$ . [3.1]

## Polar Equations of the Conics

The definition of a parabola was given in terms of a point (the focus) and a line (the directrix). The definitions of both an ellipse and a hyperbola were given in terms of two points (the foci). It is possible to define each conic in terms of a point and a line.

### Calculus Connection

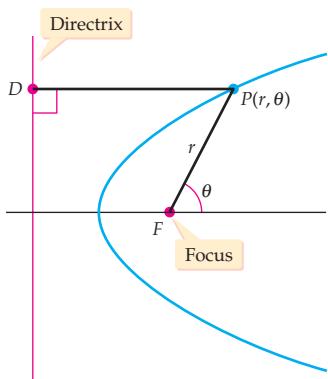


Figure 6.76

### Focus-Directrix Definitions of the Conics

Let  $F$  be a fixed point and  $D$  a fixed line in a plane. Consider the set of all points  $P$  such that  $\frac{d(P, F)}{d(P, D)} = e$ , where  $e$  is a constant. The graph is a parabola for  $e = 1$ , an ellipse for  $0 < e < 1$ , and a hyperbola for  $e > 1$ . See Figure 6.76.

The fixed point is a focus of the conic, and the fixed line is a directrix. The constant  $e$  is the eccentricity of the conic. Using this definition, we can derive the polar equations of the conics.

### Standard Forms of the Polar Equations of the Conics

Let the pole be a focus of a conic section of eccentricity  $e$ , with directrix  $d$  units from the focus. Then the equation of the conic is given by one of the following:

$$r = \frac{ed}{1 + e \cos \theta} \quad (1)$$

Vertical directrix to the right of the pole

$$r = \frac{ed}{1 - e \cos \theta} \quad (2)$$

Vertical directrix to the left of the pole

$$r = \frac{ed}{1 + e \sin \theta} \quad (3)$$

Horizontal directrix above the pole

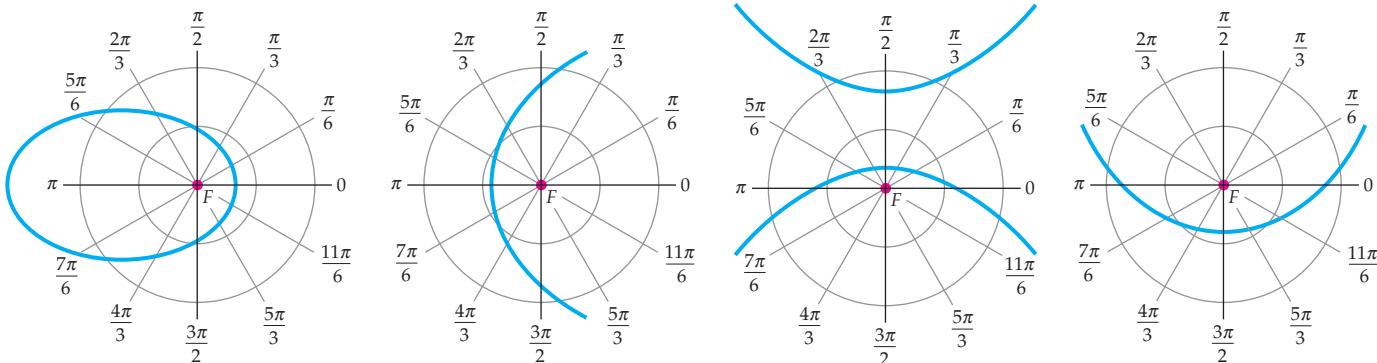
$$r = \frac{ed}{1 - e \sin \theta} \quad (4)$$

Horizontal directrix below the pole

When the equation involves  $\cos \theta$ , the line  $\theta = 0$  is an axis of symmetry.

When the equation involves  $\sin \theta$ , the line  $\theta = \frac{\pi}{2}$  is an axis of symmetry.

Graphs of examples are shown in **Figure 6.77**.



$$r = \frac{ed}{1 + e \cos \theta} \quad (1)$$

$$r = \frac{ed}{1 - e \cos \theta} \quad (2)$$

$$r = \frac{ed}{1 + e \sin \theta} \quad (3)$$

$$r = \frac{ed}{1 - e \sin \theta} \quad (4)$$

**Figure 6.77**

**QUESTION** Is the graph of  $r = \frac{4}{5 - 3 \sin \theta}$  a parabola, an ellipse, or a hyperbola?

---

**ANSWER** The eccentricity is  $\frac{3}{5}$ , which is less than 1. The graph is an ellipse.

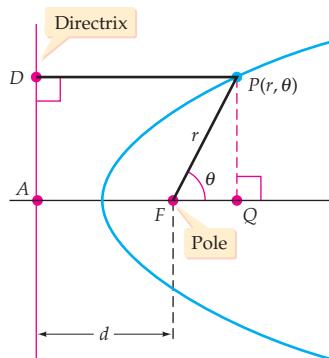


Figure 6.78

We will derive Equation (2). Let  $P(r, \theta)$  be any point on a conic section. Then, by definition,

$$\frac{d(P, F)}{d(P, D)} = e \quad \text{or} \quad d(P, F) = e \cdot d(P, D)$$

From **Figure 6.78**,  $d(P, F) = r$  and  $d(P, D) = d(A, Q)$ . But note that

$$d(A, Q) = d(A, F) + d(F, Q) = d + r \cos \theta$$

Thus

$$r = e(d + r \cos \theta) \quad \bullet \text{ } d(P, F) = e \cdot d(P, D)$$

$$= ed + er \cos \theta$$

$$r - er \cos \theta = ed$$

• Subtract  $er \cos \theta$  from each side.

$$r = \frac{ed}{1 - e \cos \theta}$$

• Solve for  $r$ .

The remaining equations can be derived in a similar manner.

## Graph a Conic Given in Polar Form

### EXAMPLE 1 » Sketch the Graph of a Hyperbola Given in Polar Form

Describe and sketch the graph of  $r = \frac{8}{2 - 3 \sin \theta}$ .

#### Solution

Write the equation in standard form by dividing the numerator and denominator by 2, the constant term in the denominator.

$$r = \frac{4}{1 - \frac{3}{2} \sin \theta}$$

Because  $e$  is the coefficient of  $\sin \theta$  and  $e = \frac{3}{2} > 1$ , the graph is a hyperbola with a focus at the pole. Because the equation contains the expression  $\sin \theta$ , the transverse axis is on the line  $\theta = \frac{\pi}{2}$ .

To find the vertices, choose  $\theta$  equal to  $\frac{\pi}{2}$  and  $\frac{3\pi}{2}$ . The corresponding values of  $r$  are  $-8$  and  $\frac{8}{5}$ . The vertices are  $(-8, \frac{\pi}{2})$  and  $(\frac{8}{5}, \frac{3\pi}{2})$ . By choosing  $\theta$  equal to  $0$  and  $\pi$ , we can determine the points  $(4, 0)$  and  $(4, \pi)$  on

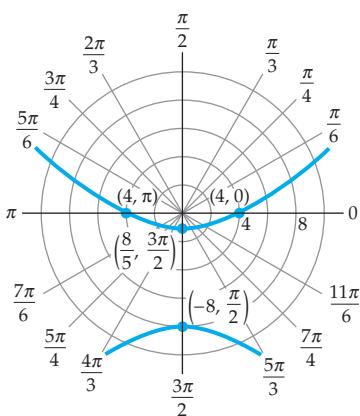


Figure 6.79

the upper branch of the hyperbola. The lower branch can be determined by symmetry.

Plot some points  $(r, \theta)$  for additional values of  $\theta$  and corresponding values of  $r$ . See **Figure 6.79**.

» Try Exercise 2, page 431

### EXAMPLE 2 » Sketch the Graph of an Ellipse Given in Polar Form

Describe and sketch the graph of  $r = \frac{4}{2 + \cos \theta}$ .

#### Solution

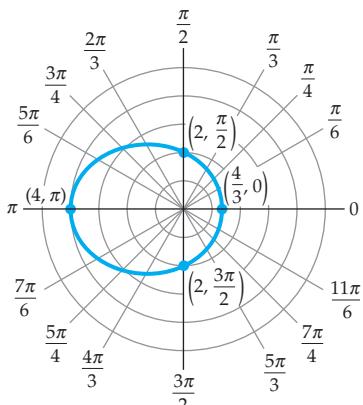
Write the equation in standard form by dividing the numerator and denominator by 2, which is the constant term in the denominator.

$$r = \frac{2}{1 + \frac{1}{2} \cos \theta}$$

Thus  $e = \frac{1}{2}$  and the graph is an ellipse with a focus at the pole. Because the equation contains the expression  $\cos \theta$ , the major axis is on the polar axis.

To find the vertices, choose  $\theta$  equal to 0 and  $\pi$ . The corresponding values for  $r$  are  $\frac{4}{3}$  and 4. The vertices on the major axis are  $\left(\frac{4}{3}, 0\right)$  and  $(4, \pi)$ . Plot some points  $(r, \theta)$  for additional values of  $\theta$  and the corresponding values of  $r$ . Two possible points are  $\left(2, \frac{\pi}{2}\right)$  and  $\left(2, \frac{3\pi}{2}\right)$ . See the graph of the ellipse in **Figure 6.80**.

» Try Exercise 4, page 431



**Figure 6.80**

### ■ Write the Polar Equation of a Conic

### EXAMPLE 3 » Find the Equation of a Conic in Polar Form

Find the polar equation of the parabola, shown in **Figure 6.81** on page 430, with vertex at  $\left(2, \frac{\pi}{2}\right)$  and focus at the pole.

Continued ▶

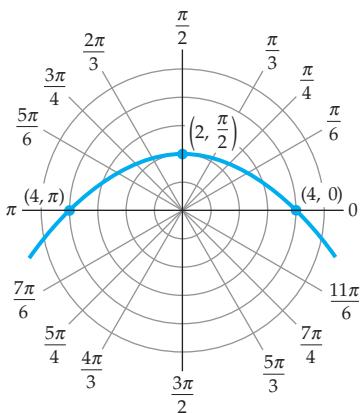


Figure 6.81

**Solution**

Because the vertex is on the line  $\theta = \frac{\pi}{2}$  and the focus is at the pole, the axis of symmetry is the line  $\theta = \frac{\pi}{2}$ . Thus the equation of the parabola must involve  $\sin \theta$ . The parabola has a horizontal directrix above the pole, so the equation has the form

$$r = \frac{ed}{1 + e \sin \theta}$$

The distance from the vertex to the focus is 2, so the distance from the focus to the directrix is 4. Because the graph of the equation is a parabola, the eccentricity is 1. The equation is

$$r = \frac{(1)(4)}{1 + (1) \sin \theta} \quad \bullet e = 1, d = 4$$

$$r = \frac{4}{1 + \sin \theta}$$

» Try Exercise 24, page 431

**QUESTION** In Example 3, why is there no point on the parabola that corresponds to  $\theta = \frac{3\pi}{2}$ ?

**Topics for Discussion**

- Is the graph of  $r = \frac{12}{2 + \cos \theta}$  a parabola? Explain.
- The graph of  $r = \frac{2 \sec \theta}{2 \sec \theta + 1}$  is an ellipse except for the fact that it has two holes. Where are the holes located?
- Are there two different ellipses that have a focus at the pole and a vertex at  $\left(1, \frac{\pi}{2}\right)$ ? Explain.
- Does the parabola given by  $r = \frac{6}{1 + \sin \theta}$  have a horizontal axis of symmetry or a vertical axis of symmetry? Explain.

---

**ANSWER** When  $\theta = \frac{3\pi}{2}$ ,  $\sin \theta = -1$ . Thus  $1 + \sin \theta = 0$ , and  $r = \frac{4}{1 + \sin \theta}$  is undefined.



## Exercise Set 6.6

In Exercises 1 to 14, describe and sketch the graph of each equation.

1.  $r = \frac{12}{3 - 6 \cos \theta}$

2.  $r = \frac{8}{2 - 4 \cos \theta}$

3.  $r = \frac{8}{4 + 3 \sin \theta}$

4.  $r = \frac{6}{3 + 2 \cos \theta}$

5.  $r = \frac{9}{3 - 3 \sin \theta}$

6.  $r = \frac{5}{2 - 2 \sin \theta}$

7.  $r = \frac{10}{5 + 6 \cos \theta}$

8.  $r = \frac{8}{2 + 4 \cos \theta}$

9.  $r = \frac{4 \sec \theta}{2 \sec \theta - 1}$

10.  $r = \frac{3 \sec \theta}{2 \sec \theta + 2}$

11.  $r = \frac{12 \csc \theta}{6 \csc \theta - 2}$

12.  $r = \frac{3 \csc \theta}{2 \csc \theta + 2}$

13.  $r = \frac{3}{\cos \theta - 1}$

14.  $r = \frac{2}{\sin \theta + 2}$

In Exercises 15 to 20, find a rectangular equation for each graph in Exercises 1 to 6.

In Exercises 21 to 28, find a polar equation of the conic with focus at the pole and the given eccentricity and directrix.

21.  $e = 2, r \cos \theta = -1$

22.  $e = \frac{3}{2}, r \sin \theta = 1$

23.  $e = 1, r \sin \theta = 2$

24.  $e = 1, r \cos \theta = -2$

25.  $e = \frac{2}{3}, r \sin \theta = -4$

26.  $e = \frac{1}{2}, r \cos \theta = 2$

27.  $e = \frac{3}{2}, r = 2 \sec \theta$

28.  $e = \frac{3}{4}, r = 2 \csc \theta$

29. Find the polar equation of the parabola with a focus at the pole and vertex  $(2, \pi)$ .

30. Find the polar equation of the ellipse with a focus at the pole, vertex at  $(4, 0)$ , and eccentricity  $\frac{1}{2}$ .

31. Find the polar equation of the hyperbola with a focus at the pole, vertex at  $(1, \frac{3\pi}{2})$ , and eccentricity 2.

32. Find the polar equation of the ellipse with a focus at the pole, vertex at  $(2, \frac{3\pi}{2})$ , and eccentricity  $\frac{2}{3}$ .



In Exercises 33 to 40, use a graphing utility to graph each equation. Write a sentence that explains how to obtain the graph from the graph of  $r$  as given in the exercise listed to the right of each equation.

33.  $r = \frac{12}{3 - 6 \cos(\theta - \frac{\pi}{6})}$  (Compare with Exercise 1.)

34.  $r = \frac{8}{2 - 4 \cos(\theta - \frac{\pi}{2})}$  (Compare with Exercise 2.)

35.  $r = \frac{8}{4 + 3 \sin(\theta - \pi)}$  (Compare with Exercise 3.)

36.  $r = \frac{6}{3 + 2 \cos(\theta - \frac{\pi}{3})}$  (Compare with Exercise 4.)

37.  $r = \frac{9}{3 - 3 \sin(\theta + \frac{\pi}{6})}$  (Compare with Exercise 5.)

38.  $r = \frac{5}{2 - 2 \sin(\theta + \frac{\pi}{2})}$  (Compare with Exercise 6.)

39.  $r = \frac{10}{5 + 6 \cos(\theta + \pi)}$  (Compare with Exercise 7.)

40.  $r = \frac{8}{2 + 4 \cos(\theta + \frac{\pi}{3})}$  (Compare with Exercise 8.)



**In Exercises 41 to 46, use a graphing utility to graph each equation.**

**41.**  $r = \frac{3}{3 - \sec \theta}$

**42.**  $r = \frac{5}{4 - 2 \csc \theta}$

**43.**  $r = \frac{3}{1 + 2 \csc \theta}$

**44.**  $r = \frac{4}{1 + 3 \sec \theta}$

**45.**  $r = 4 \sin \sqrt{2}\theta$ ,  $0 \leq \theta \leq 12\pi$

**46.**  $r = 4 \cos \sqrt{3}\theta$ ,  $0 \leq \theta \leq 8\pi$

**47.** Let  $P(r, \theta)$  satisfy the equation  $r = \frac{ed}{1 - e \cos \theta}$ . Show that  $\frac{d(P, F)}{d(P, D)} = e$ .

**48.** Show that the equation of a conic with a focus at the pole and directrix  $r \sin \theta = d$  is given by  $r = \frac{ed}{1 + e \sin \theta}$ .

## »»» Projects

- 1. POLAR EQUATION OF A LINE** Verify that the polar equation of a line that is  $d$  units from the pole is given by

$$r = \frac{d}{\cos(\theta - \theta_0)}$$

where  $\theta_p$  is the angle from the polar axis to a line segment that passes through the pole and is perpendicular to the line.

- 2. POLAR EQUATION OF A CIRCLE THAT PASSES THROUGH THE POLE** Verify that the polar equation of a circle with center  $(a, \theta_0)$  that passes through the pole is given by

$$r = 2a \cos(\theta - \theta_c)$$

where  $a$  is the length of the radius and  $\theta_c$  is the angle from the polar axis to the radius that connects the center of the circle with the pole.

## Section 6.7

- Parametric Equations
  - Eliminate the Parameter of a Pair of Parametric Equations
  - Time as a Parameter
  - The Brachistochrone Problem
  - Parametric Equations and Projectile Motion

# Parametric Equations

## PREPARE FOR THIS SECTION

**Prepare for this section by completing the following exercises. The answers can be found on page A31.**

- PS1.** Complete the square of  $y^2 + 3y$  and write the result as the square of a binomial. [1.1]

**PS2.** If  $x = 2t + 1$  and  $y = x^2$ , write  $y$  in terms of  $t$ .

**PS3.** Identify the graph of  $\left(\frac{x-2}{3^2}\right)^2 + \left(\frac{y-3}{2^2}\right)^2 = 1$ . [6.2]

**PS4.** Let  $x = \sin t$  and  $y = \cos t$ . What is the value of  $x^2 + y^2$ ? [3.1]

**PS5.** If  $x = t + 1$  and  $y = 4 - t$ , what is the ordered pair  $(x, y)$  that corresponds to  $t = 0$ ?

**PS6.** What are the domain and range of  $f(t) = 3 \cos 2t$ ? Write the answers using interval notation. [2.5]

## ■ Parametric Equations

The graph of a function is a graph for which no vertical line can intersect the graph more than once. For a graph that is not the graph of a function (an ellipse or a hyperbola, for example), it is frequently useful to describe the graph by *parametric equations*.

### Calculus Connection

#### Curve and Parametric Equations

Let  $t$  be a number in an interval  $I$ . A **curve** is a set of ordered pairs  $(x, y)$ , where

$$x = f(t), \quad y = g(t) \quad \text{for } t \in I$$

The variable  $t$  is called a **parameter**, and the equations  $x = f(t)$  and  $y = g(t)$  are **parametric equations**.

For instance,

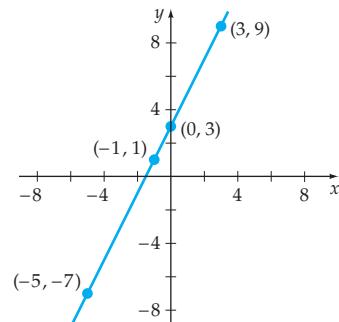
$$x = 2t - 1, \quad y = 4t + 1 \quad \text{for } t \in (-\infty, \infty)$$

is an example of a pair of parametric equations. By choosing arbitrary values of  $t$ , ordered pairs  $(x, y)$  can be created, as shown in the table below.

| <b><math>t</math></b> | <b><math>x = 2t - 1</math></b> | <b><math>y = 4t + 1</math></b> | <b><math>(x, y)</math></b> |
|-----------------------|--------------------------------|--------------------------------|----------------------------|
| -2                    | -5                             | -7                             | (-5, -7)                   |
| 0                     | -1                             | 1                              | (-1, 1)                    |
| $\frac{1}{2}$         | 0                              | 3                              | (0, 3)                     |
| 2                     | 3                              | 9                              | (3, 9)                     |

By plotting the points and drawing a curve through the points, a graph of the parametric equations is produced. See **Figure 6.82**.

**QUESTION** If  $x = t^2 + 1$  and  $y = 3 - t$ , what ordered pair corresponds to  $t = -3$ ?



**Figure 6.82**

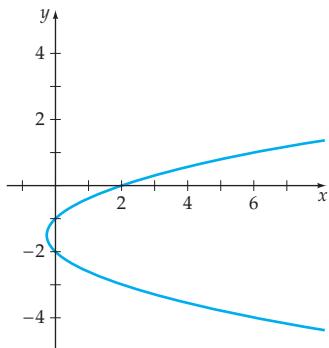
**EXAMPLE 1 » Sketch the Graph of a Curve Given in Parametric Form**

Sketch the graph of the curve given by the parametric equations

$$x = t^2 + t, \quad y = t - 1 \quad \text{for } t \in \mathbb{R}$$

**Solution**

Begin by making a table of values of  $t$  and the corresponding values of  $x$  and  $y$ . Five values of  $t$  were arbitrarily chosen for the table that follows. Many more values might be necessary to determine an accurate graph.



$$\begin{aligned} x &= t^2 + t \\ y &= t - 1 \end{aligned}$$

**Figure 6.83**

| $t$ | $x = t^2 + t$ | $y = t - 1$ | $(x, y)$ |
|-----|---------------|-------------|----------|
| -2  | 2             | -3          | (2, -3)  |
| -1  | 0             | -2          | (0, -2)  |
| 0   | 0             | -1          | (0, -1)  |
| 1   | 2             | 0           | (2, 0)   |
| 2   | 6             | 1           | (6, 1)   |

Graph the ordered pairs  $(x, y)$  and then draw a smooth curve through the points. See **Figure 6.83**.

» Try Exercise 6, page 440

**■ Eliminate the Parameter of a Pair of Parametric Equations**

It may not be clear from Example 1 and the corresponding graph that the curve is a parabola. By **eliminating the parameter**, we can write one equation in  $x$  and  $y$  that is equivalent to the two parametric equations.

To eliminate the parameter, solve  $y = t - 1$  for  $t$ .

$$y = t - 1 \quad \text{or} \quad t = y + 1$$

Substitute  $y + 1$  for  $t$  in  $x = t^2 + t$  and then simplify.

$$x = (y + 1)^2 + (y + 1)$$

$$x = y^2 + 3y + 2$$

• **The equation of a parabola**

Complete the square and write the equation in standard form.

$$\left(x + \frac{1}{4}\right) = \left(y + \frac{3}{2}\right)^2$$

• **This is the equation of a parabola vertex at  $\left(-\frac{1}{4}, -\frac{3}{2}\right)$ .**

**EXAMPLE 2 »» Eliminate the Parameter and Sketch the Graph of a Curve**

Eliminate the parameter and sketch the curve of the parametric equations

$$x = \sin t, \quad y = \cos t \quad \text{for } 0 \leq t \leq 2\pi$$

**Solution**

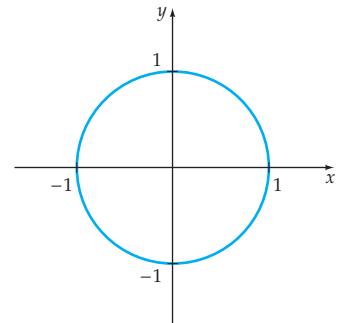
The process of eliminating the parameter sometimes involves trigonometric identities. To eliminate the parameter for the equations, square each side of each equation and then add.

$$\begin{aligned} x^2 &= \sin^2 t \\ y^2 &= \cos^2 t \\ x^2 + y^2 &= \sin^2 t + \cos^2 t \end{aligned}$$

Thus, using the trigonometric identity  $\sin^2 t + \cos^2 t = 1$ , we get

$$x^2 + y^2 = 1$$

This is the equation of a circle with center  $(0, 0)$  and radius equal to 1. See **Figure 6.84**.



**Figure 6.84**

**»» Try Exercise 12, page 440**

A parametric representation of a curve is not unique. That is, it is possible that a curve may be given by many different pairs of parametric equations. We will demonstrate this by using the equation of a line and providing two different parametric representations of the line.

Consider a line with slope  $m$  passing through the point  $(x_1, y_1)$ . By the point-slope formula, the equation of the line is

$$y - y_1 = m(x - x_1)$$

Let  $t = x - x_1$ . Then  $y - y_1 = mt$ . A parametric representation is

$$x = x_1 + t, \quad y = y_1 + mt \quad \text{for } t \text{ a real number} \quad (1)$$

Let  $x - x_1 = \cot t$ . Then  $y - y_1 = m \cot t$ . A parametric representation is

$$x = x_1 + \cot t, \quad y = y_1 + m \cot t \quad \text{for } 0 < t < \pi \quad (2)$$

It can be verified that Equations (1) and (2) represent the original line.

Example 3 illustrates that the domain of the parameter  $t$  can be used to determine the domain and range of the curve defined by the parametric equations.

**EXAMPLE 3** Sketch the Graph of a Curve Given by Parametric Equations

Eliminate the parameter and sketch the graph of the curve that is given by the parametric equations

$$x = 2 + 3 \cos t, \quad y = 3 + 2 \sin t \quad \text{for } 0 \leq t \leq \pi$$

**Solution**

Solve each equation for its trigonometric function.

$$\frac{x - 2}{3} = \cos t \quad \frac{y - 3}{2} = \sin t \quad (3)$$

Using the trigonometric identity  $\cos^2 t + \sin^2 t = 1$ , we have

$$\begin{aligned} \cos^2 t + \sin^2 t &= \left(\frac{x - 2}{3}\right)^2 + \left(\frac{y - 3}{2}\right)^2 = 1 \\ \frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{4} &= 1 \end{aligned}$$

This is the equation of an ellipse with center at  $(2, 3)$  and major axis parallel to the  $x$ -axis. However, because  $0 \leq t \leq \pi$ , it follows that  $-1 \leq \cos t \leq 1$  and  $0 \leq \sin t \leq 1$ . Therefore, we have

$$-1 \leq \frac{x - 2}{3} \leq 1 \quad 0 \leq \frac{y - 3}{2} \leq 1 \quad \bullet \text{ Using Equations (3)}$$

Solving these inequalities for  $x$  and  $y$  yields

$$-1 \leq x \leq 5 \quad \text{and} \quad 3 \leq y \leq 5$$

Because the values of  $y$  are between 3 and 5, the graph of the parametric equations is only the top half of the ellipse. See **Figure 6.85**.

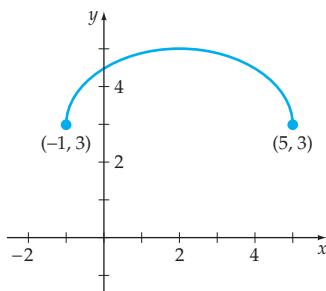


Figure 6.85

Try Exercise 14, page 440

**Time as a Parameter**

Parametric equations are often used to show the movement of a point or object as it travels along a curve. For instance, consider the parametric equations

$$x = t^2, \quad y = t + 1 \quad \text{for } -2 \leq t \leq 3$$

For any given value of  $t$  in the interval  $[-2, 3]$ , we can evaluate  $x = t^2$  and  $y = t + 1$  to determine a point  $(x, y)$  on the curve defined by the equations. See the following table and **Figure 6.86**.

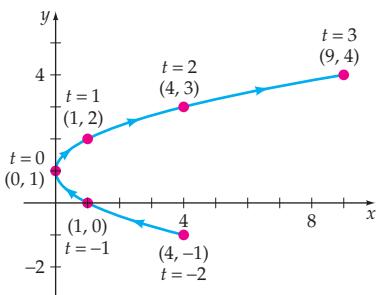


Figure 6.86

| $t$ | $x = t^2$ | $y = t + 1$ | $(x, y)$ |
|-----|-----------|-------------|----------|
| -2  | 4         | -1          | (4, -1)  |
| -1  | 1         | 0           | (1, 0)   |
| 0   | 0         | 1           | (0, 1)   |
| 1   | 1         | 2           | (1, 2)   |
| 2   | 4         | 3           | (4, 3)   |
| 3   | 9         | 4           | (9, 4)   |

As  $t$  takes on values from  $-2$  to  $3$ , the curve defined by the parametric equations is traced in a particular direction. The direction in which the curve is traced—by increasing values of the parameter  $t$ —is referred to as its **orientation**. In **Figure 6.86** the arrowheads show the orientation of the curve.

In Example 4 we let the parameter  $t$  represent time. Then the parametric equations  $x = f(t)$  and  $y = g(t)$  indicate the  $x$ - and  $y$ -coordinates of a moving point as a function of  $t$ .

#### EXAMPLE 4 » Describe the Motion of a Point

A point moves in a plane such that its position  $P(x, y)$  at time  $t$  is

$$x = \sin t, \quad y = \cos t \quad \text{for } 0 \leq t \leq 2\pi$$

Describe the motion of the point.

#### Solution

In Example 2 we determined that the graph of  $x = \sin t, y = \cos t$  for  $0 \leq t \leq 2\pi$  is a circle with center  $(0, 0)$  and radius 1. When  $t = 0$ , the point  $P$  is at  $(\sin 0, \cos 0) = (0, 1)$ . When  $t = \frac{\pi}{2}$ ,  $P$  is at  $(1, 0)$ ; when  $t = \pi$ ,  $P$  is at  $(0, -1)$ ; when  $t = \frac{3\pi}{2}$ ,  $P$  is at  $(-1, 0)$ ; and when  $t = 2\pi$ ,  $P$  is back to its starting position  $(0, 1)$ . Thus  $P$  starts at the point  $(0, 1)$  and rotates clockwise around the circle with center  $(0, 0)$  and radius 1 as the time  $t$  increases from 0 to  $2\pi$ . See **Figure 6.87**.

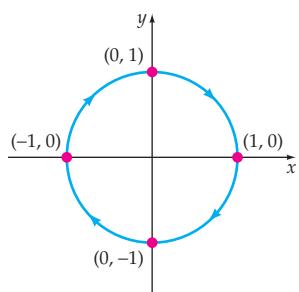


Figure 6.87

#### » Try Exercise 22, page 440

#### ■ The Brachistochrone Problem

One famous problem, involving a bead traveling down a frictionless wire, was posed in 1696 by the mathematician Johann Bernoulli. The problem was to determine the shape of a wire a bead could slide down such that the distance between two points was traveled in the shortest time. Problems that involve “shortest time” are called *brachistochrone problems*. They are very important in physics and form the basis for much of the classical theory of light propagation.

The answer to Bernoulli’s problem is an arc of an inverted cycloid. See **Figure 6.88**. A **cycloid** is formed by letting a circle of radius  $a$  roll on a straight line without slipping. See **Figure 6.89**. The curve traced by a point on the circumference of the circle is a cycloid. To find an equation for this curve, begin by placing

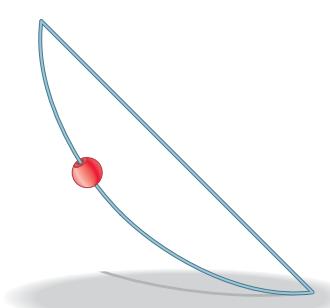


Figure 6.88

a circle tangent to the  $x$ -axis with a point  $P$  on the circle and at the origin of a rectangular coordinate system.

Roll the circle along the  $x$ -axis. After the radius of the circle has rotated through an angle  $\theta$ , the coordinates of the point  $P(x, y)$  can be given by

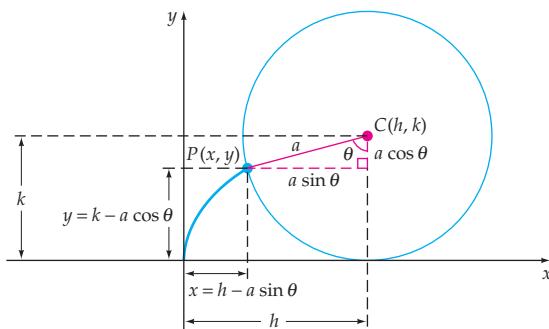
$$x = h - a \sin \theta, \quad y = k - a \cos \theta \quad (4)$$

where  $C(h, k)$  is the current center of the circle.

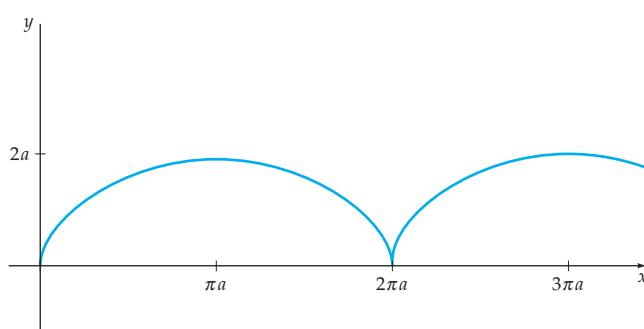
Because the radius of the circle is  $a$ ,  $k = a$ . See **Figure 6.89**. Because the circle rolls without slipping, the arc length subtended by  $\theta$  equals  $h$ . Thus  $h = a\theta$ . Substituting for  $h$  and  $k$  in Equations (4), we have, after factoring,

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta) \quad \text{for } \theta \geq 0$$

See **Figure 6.90**.



**Figure 6.89**



**Figure 6.90**

A cycloid

### EXAMPLE 5 » Graph a Cycloid



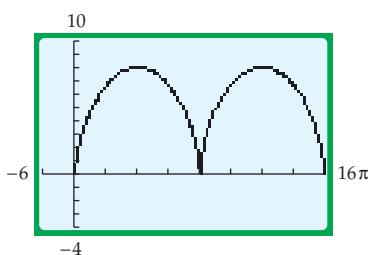
Use a graphing utility to graph the cycloid given by

$$x = 4(\theta - \sin \theta), \quad y = 4(1 - \cos \theta) \quad \text{for } 0 \leq \theta \leq 4\pi$$

#### Solution

Although  $\theta$  is the parameter in the above equations, many graphing utilities, such as the TI-83/TI-83 Plus/TI-84 Plus, use  $T$  as the parameter for parametric equations. Thus to graph the equations for  $0 \leq \theta \leq 4\pi$ , we use  $T_{\min} = 0$  and  $T_{\max} = 4\pi$ , as shown below. Use radian mode and parametric mode to produce the graph in **Figure 6.91**.

|                       |                       |                    |
|-----------------------|-----------------------|--------------------|
| $T_{\min}=0$          | $X_{\min}=-6$         | $Y_{\min}=-4$      |
| $T_{\max}=4\pi$       | $X_{\max}=16\pi$      | $Y_{\max}=10$      |
| $T_{\text{step}}=0.5$ | $X_{\text{scl}}=2\pi$ | $Y_{\text{scl}}=1$ |



$$x = 4(\theta - \sin \theta)$$

$$y = 4(1 - \cos \theta)$$

**Figure 6.91**

» Try Exercise 30, page 441

## Parametric Equations and Projectile Motion

The path of a projectile (assume air resistance is negligible) that is launched at an angle  $\theta$  from the horizon with an initial velocity of  $v_0$  feet per second is given by the parametric equations

$$x = (v_0 \cos \theta)t, \quad y = -16t^2 + (v_0 \sin \theta)t$$

where  $t$  is the time in seconds since the projectile was launched.

### EXAMPLE 6 Graph the Path of a Projectile



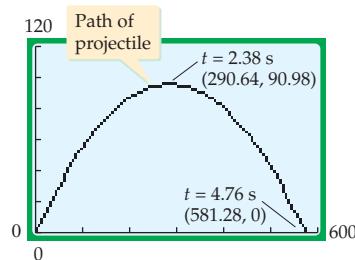
Use a graphing utility to graph the path of a projectile that is launched at an angle of  $\theta = 32^\circ$  with an initial velocity of 144 feet per second. Use the graph to determine (to the nearest foot) the maximum height of the projectile and the range of the projectile. Assume the ground is level.

#### Solution

Use degree mode and parametric mode. Graph the parametric equations

$$x = (144 \cos 32^\circ)t, \quad y = -16t^2 + (144 \sin 32^\circ)t \quad \text{for } 0 \leq t \leq 5$$

to produce the graph in **Figure 6.92**. Use the TRACE feature to determine that the maximum height of 91 feet is attained when  $t \approx 2.38$  seconds and that the projectile strikes the ground about 581 feet downrange when  $t \approx 4.76$  seconds.



$$\begin{aligned} x &= (144 \cos 32^\circ)t \\ y &= -16t^2 + (144 \sin 32^\circ)t \end{aligned}$$

**Figure 6.92**

In **Figure 6.92**, the angle of launch does not appear to be  $32^\circ$  because 1 foot on the  $x$ -axis is smaller than 1 foot on the  $y$ -axis.

### Try Exercise 34, page 441



#### Topics for Discussion

- It is always possible to eliminate the parameter of a pair of parametric equations. Do you agree? Explain.
- The line  $y = 3x + 5$  has more than one parametric representation. Do you agree? Explain.

3. Parametric equations are used only to graph functions. Do you agree? Explain.
4. Every function  $y = f(x)$  can be written in parametric form by letting  $x = t$  and  $y = f(t)$ . Do you agree? Explain.

## Exercise Set 6.7

**In Exercises 1 to 10, graph the parametric equations by plotting several points.**

1.  $x = 2t, y = -t$ , for  $t \in R$
2.  $x = -3t, y = 6t$ , for  $t \in R$
3.  $x = -t, y = t^2 - 1$ , for  $t \in R$
4.  $x = 2t, y = 2t^2 - t + 1$ , for  $t \in R$
5.  $x = t^2, y = t^3$ , for  $t \in R$

6.  $x = t^2 + 1, y = t^2 - 1$ , for  $t \in R$

7.  $x = 2 \cos t, y = 3 \sin t$ , for  $0 \leq t < 2\pi$
8.  $x = 1 - \sin t, y = 1 + \cos t$ , for  $0 \leq t < 2\pi$
9.  $x = 2^t, y = 2^{t+1}$ , for  $t \in R$

10.  $x = t - 1, y = \sqrt{t}$ , for  $0 \leq t \leq 9$

**In Exercises 11 to 18, eliminate the parameter and graph the equation.**

11.  $x = \sec t, y = \tan t$ , for  $-\frac{\pi}{2} < t < \frac{\pi}{2}$

12.  $x = 3 + 2 \cos t, y = -1 - 3 \sin t$ , for  $0 \leq t < 2\pi$

13.  $x = 2 - t^2, y = 3 + 2t^2$ , for  $t \in R$

14.  $x = 1 + t^2, y = 2 - t^2$ , for  $t \in R$

15.  $x = \cos^3 t, y = \sin^3 t$ , for  $0 \leq t < 2\pi$

16.  $x = 3 \sin t, y = \cos t$ , for  $0 \leq t \leq \frac{\pi}{2}$

17.  $x = \sqrt{t+1}, y = t$ , for  $t \geq -1$

18.  $x = \sqrt{t}, y = 2t - 1$ , for  $t \geq 0$

**In Exercises 19 to 24, the parameter  $t$  represents time and the parametric equations  $x = f(t)$  and  $y = g(t)$  indicate the  $x$ - and  $y$ -coordinates of a moving point  $P$  as a function of  $t$ . Describe the motion of the point as  $t$  increases.**

19.  $x = 2 + 3 \cos t, y = 3 + 2 \sin t$ , for  $0 \leq t \leq \pi$

20.  $x = \sin t, y = -\cos t$ , for  $0 \leq t \leq \frac{3\pi}{2}$

21.  $x = 2t - 1, y = t + 1$ , for  $0 \leq t \leq 3$

22.  $x = t + 1, y = \sqrt{t}$ , for  $0 \leq t \leq 4$

23.  $x = \tan\left(\frac{\pi}{4} - t\right), y = \sec\left(\frac{\pi}{4} - t\right)$ , for  $0 \leq t \leq \frac{\pi}{2}$

24.  $x = 1 - t, y = t^2$ , for  $0 \leq t \leq 2$

25. Eliminate the parameter for the curves

$$C_1: x = 2 + t^2, \quad y = 1 - 2t^2$$

$$\text{and} \quad C_2: x = 2 + t, \quad y = 1 - 2t$$

and then discuss the differences between their graphs.

26. Eliminate the parameter for the curves

$$C_1: x = \sec^2 t, \quad y = \tan^2 t$$

$$\text{and} \quad C_2: x = 1 + t^2, \quad y = t^2$$

for  $0 \leq t < \frac{\pi}{2}$ , and then discuss the differences between their graphs.

27. Sketch the graph of

$$x = \sin t, \quad y = \csc t \quad \text{for } 0 < t \leq \frac{\pi}{2}$$

Sketch another graph for the same pair of equations but choose the domain of  $t$  to be  $\pi < t \leq \frac{3\pi}{2}$ .

28. Discuss the differences between the graphs of

$$C_1: x = \cos t, y = \cos^2 t$$

$$\text{and} \quad C_2: x = \sin t, y = \sin^2 t$$

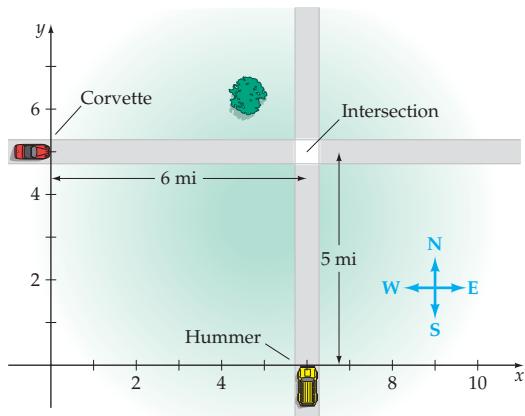
for  $0 \leq t \leq \pi$ .

29. Use a graphing utility to graph the cycloid  $x = 2(t - \sin t)$ ,  $y = 2(1 - \cos t)$  for  $0 \leq t < 2\pi$ .

30. Use a graphing utility to graph the cycloid  $x = 3(t - \sin t)$ ,  $y = 3(1 - \cos t)$  for  $0 \leq t \leq 12\pi$ .

In Exercises 31 and 32, the parameter  $t$  represents time and the parametric equations  $x = f(t)$  and  $y = g(t)$  indicate the  $x$ - and  $y$ -coordinates of a moving object as a function of  $t$ .

31. **SIMULATE UNIFORM MOTION** A Corvette is traveling east at 65 miles per hour. A Hummer is traveling north at 60 miles per hour. Both vehicles are heading toward the same intersection. At time  $t = 0$  hours, the Corvette is 6 miles from the intersection and the Hummer is 5 miles from the intersection.

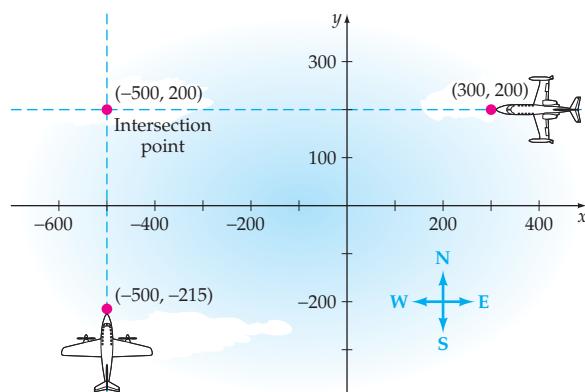


- a. The location of the Corvette is described by the equations  $x = 65t$ ,  $y = 5$ , for  $t \geq 0$ . Find a pair of parametric equations that describes the location of the Hummer at the time  $t$ , where  $t$  is measured in hours.
- b. Graph the parametric equations of the vehicles to produce a simulation. In the MODE menu, select SIMUL so that the motion of both vehicles will be shown simultaneously. Use the following window settings.

**Tmin=0      Xmin=-0.6      Ymin=0  
Tmax=0.11    Xmax=10        Ymax=7  
Tstep=0.0005   Xscl=1        Yscl=1**

Which vehicle was the first to reach the intersection?

32. **SIMULATE UNIFORM MOTION** A Learjet is flying west at 420 miles per hour. A twin engine Piper Seneca is flying north at 235 miles per hour. Both planes are flying at the same altitude. At time  $t = 0$  hours, the Learjet is 800 miles from the intersection point of the flight paths and the Piper Seneca is 415 miles from the intersection point.



- a. The location of the Piper Seneca is described by  $x = -500 + 235t$ ,  $y = -215$ ,  $t \geq 0$ . Find a pair of parametric equations that describes the location of the Learjet at time  $t$ , where  $t$  is measured in hours.
- b. Graph the parametric equations of the planes to produce a simulation. In the MODE menu, select SIMUL so that the motion of both planes will be shown simultaneously. Use the following window settings.

**Tmin=0      Xmin=-600      Ymin=-330  
Tmax=2        Xmax=400        Ymax=330  
Tstep=0.005    Xscl=100       Yscl=100**

Which plane was the first to reach the intersection point?

In Exercises 33 to 36, graph the path of the projectile that is launched at an angle of  $\theta$  with the horizon with an initial velocity of  $v_0$ . In each exercise, use the graph to determine the maximum height and the range of the projectile (to the nearest foot). Also state the time  $t$  at which the projectile reaches its maximum height and the time it hits the ground. Assume that the ground is level and the only force acting on the projectile is gravity.

33.  $\theta = 55^\circ$ ,  $v_0 = 210$  feet per second

34.  $\theta = 35^\circ$ ,  $v_0 = 195$  feet per second

35.  $\theta = 42^\circ$ ,  $v_0 = 315$  feet per second

36.  $\theta = 52^\circ$ ,  $v_0 = 315$  feet per second



 Parametric equations of the form  $x = a \sin \alpha t$ ,  $y = b \cos \beta t$ , for  $t \geq 0$ , are encountered in electrical circuit theory. The graphs of these equations are called *Lissajous figures*. In Exercises 37 to 40, use  $T_{\text{step}} = 0.05$  and  $0 \leq t \leq 2\pi$ .

- 37.** Graph:  $x = 5 \sin 2t$ ,  $y = 5 \cos t$

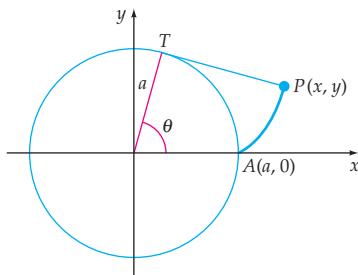
- 41.** Let  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  be two distinct points in the plane, and consider the line  $L$  passing through those points. Choose a point  $P(x, y)$  on the line  $L$ . Show that

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

Use this result to demonstrate that  $x = (x_2 - x_1)t + x_1$ ,  $y = (y_2 - y_1)t + y_1$  is a parametric representation of the line through the two points.

42. Show that  $x = h + a \sin t$ ,  $y = k + b \cos t$ , for  $a > 0$ ,  $b > 0$ , and  $0 \leq t < 2\pi$ , are parametric equations for an ellipse with center at  $(h, k)$ .

43. Suppose a string, held taut, is unwound from the circumference of a circle of radius  $a$ . The path traced by the end of the string is called the *involute* of a circle. Find parametric equations for the involute of a circle.

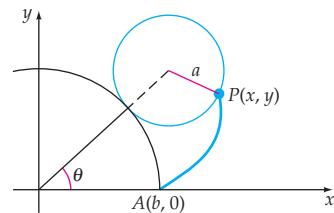


- 38.** Graph:  $x = 5 \sin 3t$ ,  $y = 5 \cos 2t$

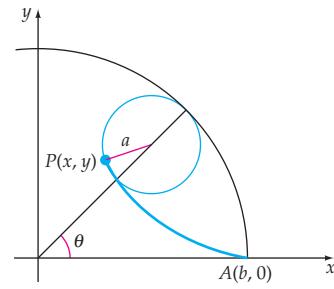
- 39.** Graph:  $x = 4 \sin 2t$ ,  $y = 4 \cos 3t$

- 40.** Graph:  $x = 5 \sin 10t$ ,  $y = 5 \cos 9t$

- 44.** A circle of radius  $a$  rolls without slipping on the outside of a circle of radius  $b > a$ . Find the parametric equations of a point  $P$  on the smaller circle. The curve is called an *epicycloid*.



- 45.** A circle of radius  $a$  rolls without slipping on the inside of a circle of radius  $b > a$ . Find the parametric equations of a point  $P$  on the smaller circle. The curve is called a *hypocycloid*.



## 1. PARAMETRIC EQUATIONS IN AN XYZ-COORDINATE SYSTEM

- a. Graph the three-dimensional curve given by

$$x = 3 \cos t, \quad y = 3 \sin t, \quad z = 0.5t$$

- b.** Graph the three-dimensional curve given by

$$x = 3 \cos t, \quad y = 6 \sin t, \quad z = 0.5t$$

- c. What is the main difference between these curves?

- d. What name is given to curves of this type?



## Exploring Concepts with Technology

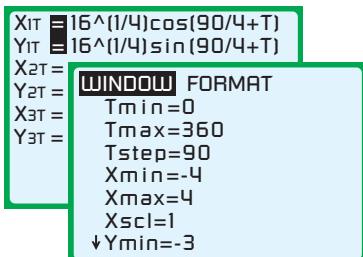
### Using a Graphing Calculator to Find the $n$ th Roots of $z$

In Chapter 5 we used De Moivre's Theorem to find the  $n$ th roots of a number. The parametric feature of a graphing calculator can also be used to find and display the  $n$ th roots of  $z = r(\cos \theta + i \sin \theta)$ . Here is the procedure for a TI-83/TI-83 Plus/TI-84 Plus graphing calculator. Put the calculator in parametric and degree mode. See **Figure 6.93**. To find the  $n$ th roots of  $z = r(\cos \theta + i \sin \theta)$ , enter in the **Y=** menu

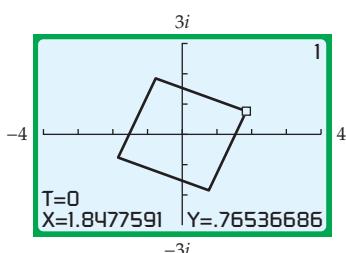
$$X_{IT}=r^{(1/n)}\cos(\theta/n+T) \quad \text{and} \quad Y_{IT}=r^{(1/n)}\sin(\theta/n+T)$$

In the **WINDOW** menu, set  $T_{min}=0$ ,  $T_{max}=360$ , and  $T_{step}=360/n$ . Set  $X_{min}$ ,  $X_{max}$ ,  $Y_{min}$ , and  $Y_{max}$  to appropriate values that will allow the roots to be seen in the graph window. Press **GRAPH** to display a polygon. The  $x$ - and  $y$ -coordinates of each vertex of the polygon represent a root of  $z$  in the rectangular form  $x + yi$ . Here is a specific example that illustrates this procedure.

**Figure 6.93**



**Figure 6.94**



The vertices of the quadrilateral represent the fourth roots of  $16i$  in the complex plane.

**Figure 6.95**

*Example* Find the fourth roots of  $z = 16i$ .

In trigonometric form,  $z = 16(\cos 90^\circ + i \sin 90^\circ)$ . Thus, in this example,  $r = 16$ ,  $\theta = 90^\circ$ , and  $n = 4$ . In the **Y=** menu, enter

$$X_{IT}=16^{(1/4)}\cos(90/4+T) \quad \text{and} \quad Y_{IT}=16^{(1/4)}\sin(90/4+T)$$

In the **WINDOW** menu, set

|                  |              |              |
|------------------|--------------|--------------|
| $T_{min}=0$      | $X_{min}=-4$ | $Y_{min}=-3$ |
| $T_{max}=360$    | $X_{max}=4$  | $Y_{max}=3$  |
| $T_{step}=360/4$ | $X_{scl}=1$  | $Y_{scl}=1$  |

See **Figure 6.94**. Press **GRAPH** to produce the quadrilateral in **Figure 6.95**. Use **TRACE** and the arrow key **►** to move to each of the vertices of the quadrilateral. **Figure 6.95** shows that one of the roots of  $z = 16i$  is  $1.8477591 + 0.76536686i$ . Continue to press the arrow key **►** to find the other three roots, which are

$$-0.7653669 + 1.847759i, -1.847759 - 0.7653669i, \text{ and } 0.76536686 - 1.847759i$$

Use a graphing calculator to estimate, in rectangular form, each of the following.

1. The cube roots of  $-27$
2. The fifth roots of  $32i$
3. The fourth roots of  $\sqrt{8} + \sqrt{8}i$
4. The sixth roots of  $-64i$

## Chapter 6 Summary

### 6.1 Parabolas

- The equations of a parabola with vertex at  $(h, k)$  and axis of symmetry parallel to a coordinate axis are given by

$$(x - h)^2 = 4p(y - k); \text{ focus } (h, k + p); \text{ directrix } y = k - p$$

$$(y - k)^2 = 4p(x - h); \text{ focus } (h + p, k); \text{ directrix } x = h - p$$

### 6.2 Ellipses

- The equations of an ellipse with center at  $(h, k)$  and major axis parallel to a coordinate axis are given by

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1; \text{ foci } (h \pm c, k); \text{ vertices } (h \pm a, k)$$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1; \text{ foci } (h, k \pm c); \text{ vertices } (h, k \pm a)$$

For each equation,  $a > b$  and  $c^2 = a^2 - b^2$ .

- The eccentricity  $e$  of an ellipse is given by  $e = \frac{c}{a}$ .

### 6.3 Hyperbolas

- The equations of a hyperbola with center at  $(h, k)$  and transverse axis parallel to a coordinate axis are given by

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1; \text{ foci } (h \pm c, k); \text{ vertices } (h \pm a, k)$$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1; \text{ foci } (h, k \pm c); \text{ vertices } (h, k \pm a)$$

For each equation,  $c^2 = a^2 + b^2$ .

- The eccentricity  $e$  of a hyperbola is given by  $e = \frac{c}{a}$ .

### 6.4 Rotation of Axes

- The rotation-of-axes formulas are

$$\begin{cases} x = x' \cos \alpha - y' \sin \alpha \\ y = y' \cos \alpha + x' \sin \alpha \end{cases} \quad \begin{cases} x' = x \cos \alpha + y \sin \alpha \\ y' = y \cos \alpha - x \sin \alpha \end{cases}$$

- To eliminate the  $xy$  term from the general quadratic equation, rotate the coordinate axes through an angle  $\alpha$ , where

$$\cot 2\alpha = \frac{A - C}{B}, \quad B \neq 0, \quad 0^\circ < 2\alpha < 180^\circ$$

- The graph of  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  is either a conic or a degenerate conic. If the graph is a conic,

then the graph can be identified by its *discriminant*  $B^2 - 4AC$ . The graph is

an ellipse or a circle, provided  $B^2 - 4AC < 0$ .

a parabola, provided  $B^2 - 4AC = 0$ .

a hyperbola, provided  $B^2 - 4AC > 0$ .

- The graph of  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  can be constructed by using a graphing utility to graph both

$$y_1 = \frac{-(Bx + E) + \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}$$

and

$$y_2 = \frac{-(Bx + E) - \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}$$

### 6.5 Introduction to Polar Coordinates

- A polar coordinate system is formed by drawing a horizontal ray (*polar axis*). The *pole* is the origin of a polar coordinate system.
- A point is specified by coordinates  $(r, \theta)$ , where  $r$  is a directed distance from the pole and  $\theta$  is an angle measured from the polar axis.

The transformation equations between a polar coordinate system and a rectangular coordinate system are

$$\text{Polar to rectangular: } x = r \cos \theta \quad y = r \sin \theta$$

$$\text{Rectangular to polar: } r = \sqrt{x^2 + y^2} \quad \tan \theta = \frac{y}{x}, x \neq 0$$

- The polar equations of:

lines are on page 413.

circles are on page 415.

limaçons are on page 416.

rose curves are on page 418.

### 6.6 Polar Equations of the Conics

- The polar equations of the conics are given by

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

where  $e$  is the eccentricity and  $d$  is the distance of the directrix from the focus.

When

$0 < e < 1$ , the graph is an ellipse.

$e = 1$ , the graph is a parabola.

$e > 1$ , the graph is a hyperbola.

### 6.7 Parametric Equations

- Let  $t$  be a number in an interval  $I$ . A *curve* is a set of ordered pairs  $(x, y)$ , where

$$x = f(t), \quad y = g(t) \quad \text{for } t \in I$$

The variable  $t$  is called a *parameter*, and the pair of equations are *parametric equations*.

- To *eliminate the parameter* is to find an equation in  $x$  and  $y$  that has the same graph as the given parametric equations.

- Parametric equations can be used to show the movement of a point as it travels along a curve.
- A cycloid is traced by a point on a circle as the circle rolls on a line. Parametric equations of a cycloid are given on page 438.
- The path of a projectile (assume air resistance is negligible) that is launched at an angle  $\theta$  from the horizon with an initial velocity  $v_0$  feet per second is given by

$$x = (v_0 \cos \theta)t, \quad y = -16t^2 + (v_0 \sin \theta)t$$

where  $t$  is the time in seconds since the projectile was launched.

## Chapter 6 Assessing Concepts

In Exercises 1 to 12, match an equation to each description. A letter may be used more than once. Some letters may not be needed.

- Equation, in rectangular form, of an ellipse with major axis of length 10 and minor axis of length 6 \_\_\_\_\_
- Equation, in rectangular form, of a hyperbola with center  $(4, 3)$  \_\_\_\_\_
- Equation, in rectangular form, of a parabola with vertex  $(2, -3)$  \_\_\_\_\_
- Equation, in rectangular form, of an ellipse with eccentricity  $\frac{1}{2}$  \_\_\_\_\_
- Equation, in rectangular form, of a hyperbola with eccentricity of  $\frac{\sqrt{7}}{2}$  \_\_\_\_\_
- Equation, in rectangular form, of a parabola with focus  $(3, 2)$  \_\_\_\_\_
- Equation, in rectangular form, of a parabola with vertex  $(2, 3)$  \_\_\_\_\_
- Polar equation of an ellipse with major axis on the polar axis \_\_\_\_\_
- Polar equation of a hyperbola with transverse axis on the polar axis \_\_\_\_\_
- Polar equation of a parabola with axis of symmetry on the polar axis \_\_\_\_\_
- Parametric equations of an ellipse with major axis parallel to the  $x$ -axis \_\_\_\_\_
- Parametric equations of an ellipse with center  $(2, 3)$  \_\_\_\_\_

a.  $3x^2 - 4y^2 = 12$

b.  $\frac{(x - 4)^2}{4^2} - \frac{(y - 3)^2}{5^2} = 1$

c.  $3x^2 + 4y^2 = 12$

d.  $\frac{x^2}{25} + \frac{y^2}{9} = 1$

e.  $-x + 2y^2 + 12y + 20 = 0$

f.  $-8x + y^2 - 4y + 12 = 0$

g.  $3x + y^2 - 6y + 3 = 0$

h.  $r = \frac{5}{\frac{3}{5} + 2 \cos \theta}$

i.  $r = \frac{6}{2 + \cos \theta}$

j.  $r = \frac{1}{\frac{2}{3} \cos \theta - \frac{2}{3}}$

k.  $x = 2 + 4 \cos t, y = 3 + 2 \sin t$ , for  $0 \leq t < 2\pi$

l.  $x = -1 + 3 \sin t, y = 3 + 8 \cos t$ , for  $0 \leq t < 2\pi$



## Chapter 6 Review Exercises

**In Exercises 1 to 12, if the equation is that of an ellipse or a hyperbola, find the center, vertices, and foci. For hyperbolas, find the equations of the asymptotes. If the equation is that of a parabola, find the vertex, the focus, and the equation of the directrix. Graph each equation.**

1.  $x^2 - y^2 = 4$

2.  $y^2 = 16x$

3.  $x^2 + 4y^2 - 6x + 8y - 3 = 0$

4.  $3x^2 - 4y^2 + 12x - 24y - 36 = 0$

5.  $3x - 4y^2 + 8y + 2 = 0$

6.  $3x + 2y^2 - 4y - 7 = 0$

7.  $9x^2 + 4y^2 + 36x - 8y + 4 = 0$

8.  $11x^2 - 25y^2 - 44x - 50y - 256 = 0$

9.  $4x^2 - 9y^2 - 8x + 12y - 144 = 0$

10.  $9x^2 + 16y^2 + 36x - 16y - 104 = 0$

11.  $4x^2 + 28x + 32y + 81 = 0$

12.  $x^2 - 6x - 9y + 27 = 0$

**In Exercises 13 to 20, find the equation of the conic that satisfies the given conditions.**

13. Ellipse with vertices at  $(7, 3)$  and  $(-3, 3)$ ; length of minor axis is 8.

14. Hyperbola with vertices at  $(4, 1)$  and  $(-2, 1)$  and eccentricity  $\frac{4}{3}$

15. Hyperbola with foci  $(-5, 2)$  and  $(1, 2)$ ; length of transverse axis is 4.

16. Parabola with focus  $(2, -3)$  and directrix  $x = 6$

17. Parabola with vertex  $(0, -2)$  and passing through the point  $(3, 4)$

18. Ellipse with eccentricity  $\frac{2}{3}$  and foci  $(-4, -1)$  and  $(0, -1)$

19. Hyperbola with vertices  $(\pm 6, 0)$  and asymptotes whose equations are  $y = \pm \frac{1}{9}x$ .

20. Parabola passing through the points  $(1, 0)$ ,  $(2, 1)$ , and  $(0, 1)$  with axis of symmetry parallel to the  $y$ -axis.

**In Exercises 21 to 24, write the equation without an  $xy$  term. Name the graph of the equation.**

21.  $11x^2 - 6xy + 19y^2 - 40 = 0$

22.  $3x^2 + 6xy + 3y^2 - 4x + 5y - 12 = 0$

23.  $x^2 + 2\sqrt{3}xy + 3y^2 + 8\sqrt{3}x - 8y + 32 = 0$

24.  $xy - x - y - 1 = 0$

**In Exercises 25 to 34, graph each polar equation.**

25.  $r = 4 \cos 3\theta$

26.  $r = 1 + \cos \theta$

27.  $r = 2(1 - 2 \sin \theta)$

28.  $r = 4 \sin 4\theta$

29.  $r = 5 \sin \theta$

30.  $r = 3 \sec \theta$

31.  $r = 4 \csc \theta$

32.  $r = 4 \cos \theta$

33.  $r = 3 + 2 \cos \theta$

34.  $r = 4 + 2 \sin \theta$

**In Exercises 35 to 38, find a polar form of each equation.**

35.  $y^2 = 16x$

36.  $x^2 + y^2 + 4x + 3y = 0$

37.  $3x - 2y = 6$

38.  $xy = 4$

**In Exercises 39 to 42, find a rectangular form of each equation.**

39.  $r = \frac{4}{1 - \cos \theta}$

40.  $r = 3 \cos \theta - 4 \sin \theta$

41.  $r^2 = \cos 2\theta$

42.  $\theta = 1$

**In Exercises 43 to 46, graph the conic given by each polar equation.**

43.  $r = \frac{4}{3 - 6 \sin \theta}$

44.  $r = \frac{2}{1 + \cos \theta}$

45.  $r = \frac{2}{2 - \cos \theta}$

46.  $r = \frac{6}{4 + 3 \sin \theta}$

In Exercises 47 to 53, eliminate the parameter and graph the curve given by the parametric equations.

47.  $x = 4t - 2, y = 3t + 1$ , for  $t \in \mathbb{R}$

48.  $x = 1 - t^2, y = 3 - 2t^2$ , for  $t \in \mathbb{R}$

49.  $x = 4 \sin t, y = 3 \cos t$ , for  $0 \leq t < 2\pi$

50.  $x = \sec t, y = 4 \tan t$ , for  $-\frac{\pi}{2} < t < \frac{\pi}{2}$

51.  $x = \frac{1}{t}, y = -\frac{2}{t}$ , for  $t > 0$

52.  $x = 1 + \cos t, y = 2 - \sin t$ , for  $0 \leq t < 2\pi$

53.  $x = \sqrt{t}, y = 2^{-t}$ , for  $t \geq 0$

54. Use a graphing utility to graph the cycloid given by  

$$x = 3(t - \sin t), \quad y = 3(1 - \cos t)$$
  
 for  $0 \leq t \leq 18\pi$ .

55. Use a graphing utility to graph the conic given by  

$$x^2 + 4xy + 2y^2 - 2x + 5y + 1 = 0$$

56. Use a graphing utility to graph

$$r = \frac{6}{3 + \sin\left(\theta + \frac{\pi}{4}\right)}$$

57. **PHYSICS** The path of a projectile (assume air resistance is negligible) that is launched at an angle  $\theta$  from the horizon with an initial velocity of  $v_0$  feet per second is given by the parametric equations

$$x = (v_0 \cos \theta)t, \quad y = -16t^2 + (v_0 \sin \theta)t$$

where  $t$  is the time in seconds since the projectile was launched. Use a graphing utility to graph the path of a projectile that is launched at an angle of  $33^\circ$  with an initial velocity of 245 feet per second. Use the graph to determine the maximum height of the projectile to the nearest foot.

## »»» Quantitative Reasoning: *The Mathematics of a Rotary Engine* »»»

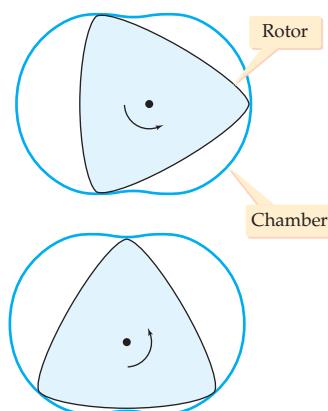


Figure 6.96

During the 1950s, Felix Wankel (1902–1988) developed a rotary engine. The engine has several remarkable properties.

- A triangular rotor<sup>2</sup> (see Figure 6.96) rotates inside a chamber in a manner such that all three vertices of the rotor maintain constant contact with the chamber.
- As the rotor rotates, the centroid (center) of the rotor moves on a circular path.
- Every time the rotor makes one complete revolution, the centroid makes three complete revolutions around its circular path.

The graph of the parametric equations

$$x = 3 \cos T + 0.5 \cos(3T), \quad y = 3 \sin T + 0.5 \sin(3T), \text{ for } 0 \leq T \leq 2\pi \quad (1)$$

where  $T$  is the counterclockwise angle of rotation as measured from the positive  $x$ -axis, yields the curve shown in Figure 6.97 on page 448. The curve has the same shape as the chamber of a rotary engine.

The following TI-83/TI-83 Plus/TI-84 Plus program **WANKEL** simulates the motion of the rotor in a rotary engine.

<sup>2</sup>The actual rotor used in a rotary engine is a modified equilateral triangle with curved sides, as shown in Figure 6.96.

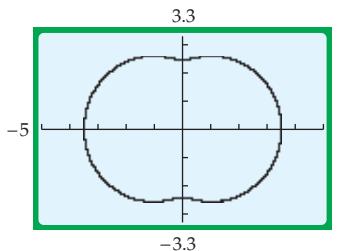


Figure 6.97

```

PROGRAM:WANKEL
ClrDraw:Radian:FnOff
-5→Xmin:5→Xmax:1→Xscl
-3.3→Ymin:3.3→Ymax:1→Yscl
For(T,0,2π/3,π/40)
Line(3cos(T)+.5cos(3T),
  3sin(T)+.5sin(3T),
  3cos(T+2π/3)+.5cos(3T),
  3sin(T+2π/3)+.5sin(3T))
Line(3cos(T+2π/3)+.5cos(3T),
  3sin(T+2π/3)+.5sin(3T),
  3cos(T+4π/3)+.5cos(3T),
  3sin(T+4π/3)+.5sin(3T))
Line(3cos(T+4π/3)+.5cos(3T),
  3sin(T+4π/3)+.5sin(3T),
  3cos(T+2π)+.5cos(3T),
  3sin(T+2π)+.5sin(3T))
Pt-On(.5cos(3T),.5sin(3T))
End

```

- QR1.** Run the **WANKEL** program. How does the path that is traced out by the vertices of the rotor compare with the graph in **Figure 6.97**?

Felix Wankel designed the shape of the chamber of his rotary engine using trial and error. He did not know that mathematicians had made a study of this figure many years before by graphing the path that is traced out by a point on the radius of a circle that rolls around another circle. For example, **Figure 6.98** shows a circle rolling around a larger circle. Point *B*, the midpoint of radius *AC* of the small circle, generates curve *E* as the small circle is rolled around the larger circle. Curve *E* is called an *epitrochoid*.

The following TI-83/TI-83 Plus/TI-84 Plus program **EPIT** simulates the process of generating an epitrochoid by rolling a small circle around a larger circle and plotting the trace of the midpoint of a radius of the smaller circle (point *B* in **Figure 6.98**).

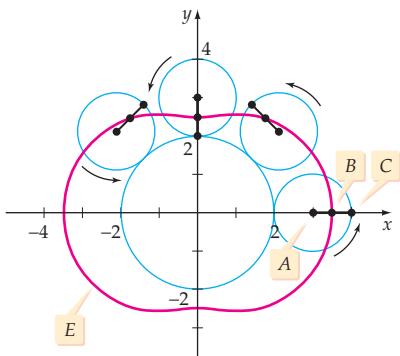


Figure 6.98

```

PROGRAM:EPIT
Radian:FnOff
ClrDraw
-1→Xmin:6→Xmax
1→Xscl
-.33→Ymin:4.3→Ymax
1→Yscl
Circle(0,0,2)
For(K,0,π/2,π/30)
Circle(3cos(K),3sin(K),1)
Circle(3cos(K)+.5cos(3K),
  3sin(K)+.5sin(3K),.09)
End

```

- QR2.** Run the **EPIT** program. How does the path that is traced out by the midpoint of the radius of the small circle compare with the graph in **Figure 6.98**, in Quadrant I?

- QR3.** Construct a two-dimensional model of a rotary engine. Demonstrate to your classmates how the rotor revolves inside the epitrochoid chamber. One method for constructing the model is to use thick cardboard from which you cut out an epitrochoid. To generate the epitrochoid, use a computer to print a large graph of the parametric equations given in (1).

Cut out an equilateral triangle from the cardboard to form the rotor. What is the length, to the nearest tenth of a unit, of each side of the triangle? (*Hint:* The length of each side of the triangle equals the distance between

the points on the epitrochoid (1) at  $T = 0$  and  $T = \frac{2\pi}{3}$ .)

For additional information on rotary engines and epitrochoids, visit website <http://auto.howstuffworks.com/rotary-engine.htm> and <http://www.math.dartmouth.edu/~dlittle/java/SpiroGraph/>.

## Chapter 6 Test

1. Find the vertex, focus, and directrix of the parabola given by the equation  $y = \frac{1}{8}x^2$ .
2. Graph:  $\frac{x^2}{16} + \frac{y^2}{1} = 1$
3. Find the vertices and foci of the ellipse given by the equation  $25x^2 - 150x + 9y^2 + 18y + 9 = 0$ .
4. Find the equation in standard form of the ellipse with center  $(0, -3)$ , foci  $(-6, -3)$  and  $(6, -3)$ , and minor axis of length 6.
5. Graph:  $\frac{y^2}{25} - \frac{x^2}{16} = 1$
6. Find the vertices, foci, and asymptotes of the hyperbola given by the equation  $\frac{x^2}{36} - \frac{y^2}{64} = 1$ .
7. Graph:  $16y^2 + 32y - 4x^2 - 24x = 84$
8. For the equation  $x^2 - 4xy - 5y^2 + 3x - 5y - 20 = 0$ , determine what acute angle of rotation (to the nearest  $0.01^\circ$ ) would eliminate the  $xy$  term.
9. Determine whether the graph of the following equation is the graph of a parabola, an ellipse, or a hyperbola.  

$$8x^2 + 5xy + 2y^2 - 10x + 5y + 4 = 0$$
10.  $P(1, -\sqrt{3})$  are the coordinates of a point in an  $xy$ -coordinate system. Find the polar coordinates of  $P$ .
11. Graph:  $r = 4 \cos \theta$
12. Graph:  $r = 3(1 - \sin \theta)$
13. Graph:  $r = 2 \sin 4\theta$
14. Find the rectangular coordinates of the point whose polar coordinates are  $\left(5, \frac{7\pi}{3}\right)$ .
15. Find the rectangular form of  $r = r \cos \theta = 4$ .
16. Write  $r = \frac{4}{1 + \sin \theta}$  as an equation in rectangular coordinates.
17. Eliminate the parameter and graph the curve given by the parametric equations  $x = t - 3$ ,  $y = 2t^2$ .
18. Eliminate the parameter and graph the curve given by the parametric equations  $x = 4 \sin \theta$ ,  $y = \cos \theta + 2$ , where  $0 \leq \theta < 2\pi$ .
19. Use a graphing utility to graph the cycloid given by  

$$x = 2(t - \sin t), \quad y = 2(1 - \cos t)$$
for  $0 \leq t \leq 12\pi$ .
20. The path of a projectile that is launched at an angle of  $30^\circ$  from the horizon with an initial velocity of 128 feet per second is given by  

$$x = (128 \cos 30^\circ)t, \quad y = -16t^2 + (128 \sin 30^\circ)t$$
where  $t$  is the time in seconds after the projectile is launched. Use a graphing utility to determine how far (to the nearest foot) the projectile will travel downrange if the ground is level.



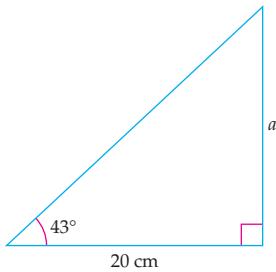
## Cumulative Review Exercises

1. Solve:  $x^2 + 4x + 6 = 0$
2. Is the graph of  $f(x) = x^3 - 4x$  symmetric with respect to the  $x$ -axis, the  $y$ -axis, or the origin, or does it exhibit none of these symmetries?
3. Given  $f(x) = \sin x$  and  $g(x) = 3x - 2$ , find  $(g \circ f)(x)$ .
4. Convert  $240^\circ$  to radians.

5. An electric cart has 10-inch-radius wheels. What is the linear speed in miles per hour of this cart when the wheels are rotating at 3 radians per second? Round to the nearest mile per hour.

6. Given  $\sin t = -\frac{\sqrt{3}}{2}$ ,  $\frac{3\pi}{2} < t < 2\pi$ , find  $\tan t$ .

7. Find the measure of  $a$  for the right triangle shown at the right. Round to the nearest centimeter.



8. What are the amplitude and period of  $y = \frac{1}{2} \cos\left(\frac{\pi x}{3}\right)$ ?
9. What is the period of  $y = 2 \tan\left(\frac{\pi x}{3}\right)$ ?
10. Verify the identity  $\frac{\sin x}{1 - \cos x} = \csc x + \cot x$ .

11. Given  $\sin \alpha = \frac{3}{5}$  in Quadrant II and  $\cos \beta = -\frac{5}{13}$  in Quadrant III, find  $\sin(\alpha + \beta)$ .
12. Find the exact value of  $\sin\left(\cos^{-1} \frac{1}{5}\right)$ .
13. Solve:  $\cos^{-1} x = \sin^{-1} \frac{12}{13}$
14. To find the distance across a ravine, a surveying team locates points  $A$  and  $B$  on one side of the ravine and point  $C$  on the other side of the ravine. The distance between  $A$  and  $B$  is 155 feet. The measure of angle  $CAB$  is  $71^\circ$ , and the measure of angle  $CBA$  is  $80^\circ$ . Find the distance across the ravine. Round to the nearest foot.
15. The lengths of the sides of a triangular piece of wood are 2.5 feet, 4 feet, and 3.6 feet. Find the angle between the two longer sides of the triangle. Round to the nearest degree.
16. The magnitude of vector  $v$  is 30 and the direction angle is  $145^\circ$ . Write the vector in the form  $v = ai + bj$ , where  $a$  and  $b$  are rounded to the nearest tenth.
17. Are the vectors  $v = 3\mathbf{i} + 2\mathbf{j}$  and  $w = 5\mathbf{i} - 7\mathbf{j}$  orthogonal?
18. Write the complex number  $z = -2 + 2i\sqrt{3}$  in polar form.
19. Sketch the graph of  $r = 3 \sin 2\theta$ .
20. Write  $x$  in terms of  $y$  by eliminating the parameter from the parametric equations  $x = 2t - 1$ ,  $y = 4t^2 + 1$ .

## 7

# Exponential and Logarithmic Functions

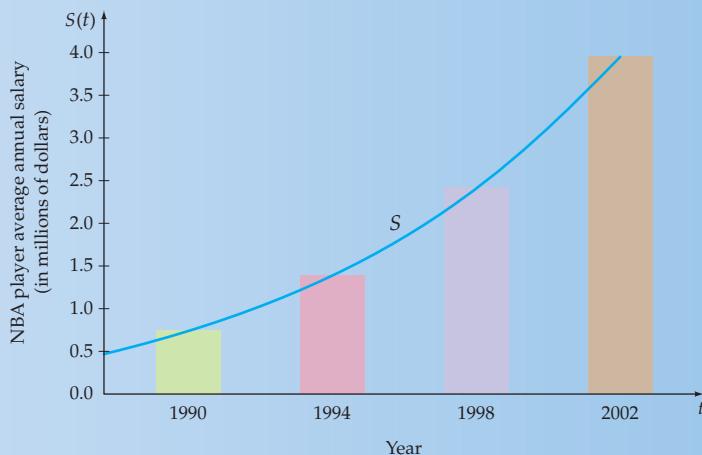


- 7.1 Exponential Functions and Their Applications**
- 7.2 Logarithmic Functions and Their Applications**
- 7.3 Properties of Logarithms and Logarithmic Scales**
- 7.4 Exponential and Logarithmic Equations**
- 7.5 Exponential Growth and Decay**
- 7.6 Modeling Data with Exponential and Logarithmic Functions**

## Modeling Data with an Exponential Function

The average annual salary of a basketball player in the National Basketball Association (NBA) has increased from \$12,000 a year in 1957 to over \$4.5 million a year in 2004.

The following bar graph shows the average annual NBA salary for selected years from 1990 to 2002.



Source: [www.sportsfansofamerica.com](http://www.sportsfansofamerica.com) and the NBA Players Association

The function  $S(t) = 0.7739(1.1485)^t$  closely models the average annual salary of an NBA player for the years 1990 ( $t = 0$ ) to 2002 ( $t = 12$ ). This function was determined by using the data from the bar graph and exponential regression, which is one of the topics of Section 7.6. See **Exercise 23, page 531** for another application in which an exponential function is used to model data.



### Online Study Center

For online student resources, such as section quizzes, visit this website:  
[college.hmco.com/info/aufmannCAT](http://college.hmco.com/info/aufmannCAT)

## Section 7.1

- Exponential Functions
- Graphs of Exponential Functions
- The Natural Exponential Function

# Exponential Functions and Their Applications

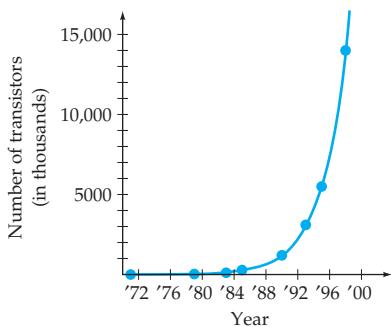
## ■ Exponential Functions

 In 1965, Gordon Moore, one of the cofounders of Intel Corporation, observed that the maximum number of transistors that could be placed on a microprocessor seemed to be doubling every 18 to 24 months. This observation is known as Moore's Law. **Table 7.1** below shows how the maximum number of transistors on various Intel processors has changed over time. (Source: Intel Museum home page.)

The curve that approximately passes through the points is a mathematical model of the data. See **Figure 7.1**. The model is based on an *exponential* function.

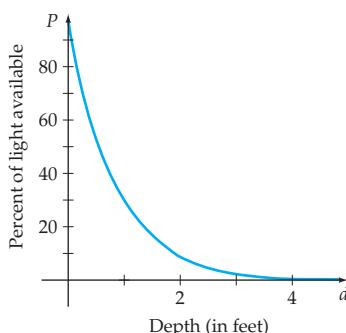
**Table 7.1**

| Year  | 1971 | 1979 | 1983 | 1985 | 1990 | 1993 | 1995 | 1998   | 2000   | 2004    |
|---|------|------|------|------|------|------|------|--------|--------|---------|
| Number of transistors per microprocessor (in thousands) | 2.3  | 31   | 110  | 280  | 1200 | 3100 | 5500 | 14,000 | 42,000 | 592,000 |



**Figure 7.1**  
Moore's Law

When light enters water, the intensity of the light decreases with the depth of the water. The graph in **Figure 7.2** shows a model, for Lake Michigan, of the decrease in the percentage of available light as the depth of the water increases.



**Figure 7.2**

This model is also based on an exponential function.

## Definition of an Exponential Function

The **exponential function with base  $b$**  is defined by

$$f(x) = b^x$$

where  $b > 0$ ,  $b \neq 1$ , and  $x$  is a real number.

The base  $b$  of  $f(x) = b^x$  is required to be positive. If the base were a negative number, the value of the function would be a complex number for some values of  $x$ . For instance, if  $b = -4$  and  $x = \frac{1}{2}$ , then  $f\left(\frac{1}{2}\right) = (-4)^{1/2} = 2i$ . To avoid complex number values of a function, the base of any exponential function must be a positive number. Also,  $b$  is defined such that  $b \neq 1$  because  $f(x) = 1^x = 1$  is a constant function.

In the following examples we evaluate  $f(x) = 2^x$  at  $x = 3$  and  $x = -2$ .

$$f(3) = 2^3 = 8 \quad f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

To evaluate the exponential function  $f(x) = 2^x$  at an irrational number such as  $x = \sqrt{2}$ , we use a rational approximation of  $\sqrt{2}$ , such as 1.4142, and a calculator to obtain an approximation of the function. For instance, if  $f(x) = 2^x$ , then  $f(\sqrt{2}) = 2^{\sqrt{2}} \approx 2^{1.4142} \approx 2.6651$ .

### EXAMPLE 1 » Evaluate an Exponential Function

Evaluate  $f(x) = 3^x$  at  $x = 2$ ,  $x = -4$ , and  $x = \pi$ .

#### Solution

$$f(2) = 3^2 = 9$$

$$f(-4) = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$$

$$f(\pi) = 3^\pi \approx 3^{3.1415927} \approx 31.54428 \quad \bullet \text{ Evaluate with the aid of a calculator.}$$

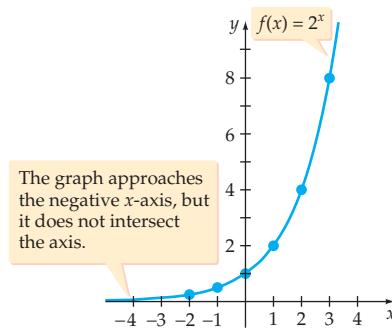
» Try Exercise 2, page 461

### Graphs of Exponential Functions

The graph of  $f(x) = 2^x$  is shown in **Figure 7.3**. The coordinates of some of the points on the curve are given in **Table 7.2**.

**Table 7.2**

| $x$ | $y = f(x) = 2^x$               | $(x, y)$                       |
|-----|--------------------------------|--------------------------------|
| -2  | $f(-2) = 2^{-2} = \frac{1}{4}$ | $\left(-2, \frac{1}{4}\right)$ |
| -1  | $f(-1) = 2^{-1} = \frac{1}{2}$ | $\left(-1, \frac{1}{2}\right)$ |
| 0   | $f(0) = 2^0 = 1$               | $(0, 1)$                       |
| 1   | $f(1) = 2^1 = 2$               | $(1, 2)$                       |
| 2   | $f(2) = 2^2 = 4$               | $(2, 4)$                       |
| 3   | $f(3) = 2^3 = 8$               | $(3, 8)$                       |

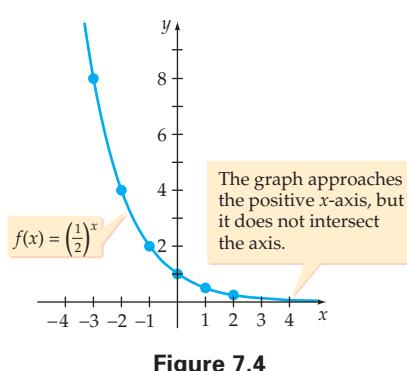


**Figure 7.3**

Note the following properties of the graph of the exponential function  $f(x) = 2^x$ .

- The  $y$ -intercept is  $(0, 1)$ .
- The graph passes through  $(1, 2)$ .
- As  $x$  decreases without bound (that is, as  $x \rightarrow -\infty$ ),  $f(x) \rightarrow 0$ .
- The graph is a smooth, continuous increasing curve.

Now consider the graph of an exponential function for which the base is between 0 and 1. The graph of  $f(x) = \left(\frac{1}{2}\right)^x$  is shown in **Figure 7.4**. The coordinates of some of the points on the curve are given in **Table 7.3**.



**Figure 7.4**

**Table 7.3**

| $x$ | $y = f(x) = \left(\frac{1}{2}\right)^x$           | $(x, y)$                      |
|-----|---|-------------------------------|
| -3  | $f(-3) = \left(\frac{1}{2}\right)^{-3} = 8$       | $(-3, 8)$                     |
| -2  | $f(-2) = \left(\frac{1}{2}\right)^{-2} = 4$       | $(-2, 4)$                     |
| -1  | $f(-1) = \left(\frac{1}{2}\right)^{-1} = 2$       | $(-1, 2)$                     |
| 0   | $f(0) = \left(\frac{1}{2}\right)^0 = 1$           | $(0, 1)$                      |
| 1   | $f(1) = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$ | $\left(1, \frac{1}{2}\right)$ |
| 2   | $f(2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ | $\left(2, \frac{1}{4}\right)$ |

Note the following properties of the graph of  $f(x) = \left(\frac{1}{2}\right)^x$ .

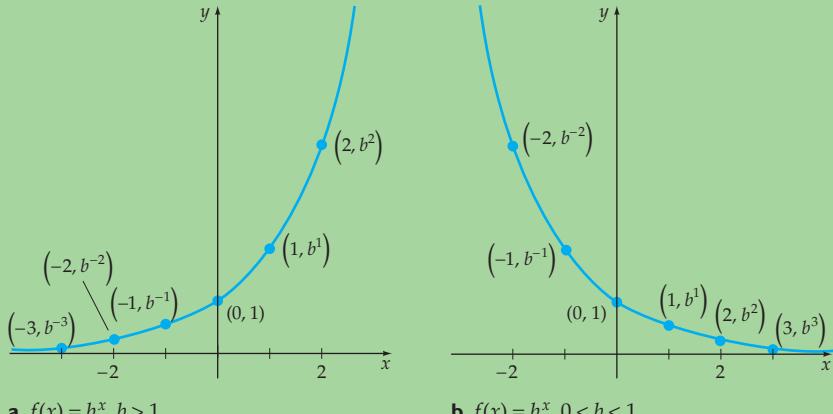
- The  $y$ -intercept is  $(0, 1)$ .
- The graph passes through  $\left(1, \frac{1}{2}\right)$ .
- As  $x$  increases without bound (that is, as  $x \rightarrow \infty$ ),  $f(x) \rightarrow 0$ .
- The graph is a smooth, continuous decreasing curve.

The basic properties of exponential functions are provided in the following summary.

### Properties of

For positive real numbers  $b$ ,  $b \neq 1$ , the exponential function defined by  $f(x) = b^x$  has the following properties:

1. The function  $f$  is a one-to-one function. It has the set of real numbers as its domain and the set of positive real numbers as its range.
2. The graph of  $f$  is a smooth, continuous curve with a  $y$ -intercept of  $(0, 1)$ , and the graph passes through  $(1, b)$ .
3. If  $b > 1$ ,  $f$  is an increasing function and the graph of  $f$  is asymptotic to the negative  $x$ -axis. [As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$ , and as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0$ .] See **Figure 7.5a**.
4. If  $0 < b < 1$ ,  $f$  is a decreasing function and the graph of  $f$  is asymptotic to the positive  $x$ -axis. [As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$ , and as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0$ .] See **Figure 7.5b**.



**Figure 7.5**

**QUESTION** What is the  $x$ -intercept of the graph of  $f(x) = \left(\frac{1}{3}\right)^x$ ?

### EXAMPLE 2 » Graph an Exponential Function

Graph:  $g(x) = \left(\frac{3}{4}\right)^x$

Continued ►

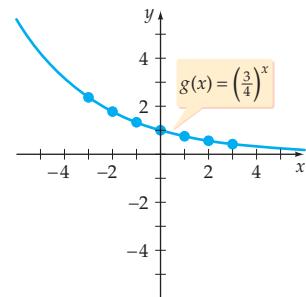
**ANSWER** The graph does not have an  $x$ -intercept. As  $x$  increases without bound, the graph approaches the  $x$ -axis, but it does not intersect the  $x$ -axis.

**Solution**

Because the base  $\frac{3}{4}$  is less than 1, we know that the graph of  $g$  is a decreasing function that is asymptotic to the positive  $x$ -axis. The  $y$ -intercept of the graph is the point  $(0, 1)$ , and the graph also passes through  $\left(1, \frac{3}{4}\right)$ . Plot a few additional points (see **Table 7.4**), and then draw a smooth curve through the points as in **Figure 7.6**.

**Table 7.4**

| $x$ | $y = g(x) = \left(\frac{3}{4}\right)^x$         | $(x, y)$                         |
|-----|---|----------------------------------|
| -3  | $\left(\frac{3}{4}\right)^{-3} = \frac{64}{27}$ | $\left(-3, \frac{64}{27}\right)$ |
| -2  | $\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}$  | $\left(-2, \frac{16}{9}\right)$  |
| -1  | $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$   | $\left(-1, \frac{4}{3}\right)$   |
| 2   | $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$     | $\left(2, \frac{9}{16}\right)$   |
| 3   | $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$    | $\left(3, \frac{27}{64}\right)$  |

**Figure 7.6**

» Try Exercise 22, page 462

Consider the functions  $F(x) = 2^x - 3$  and  $G(x) = 2^{x-3}$ . You can construct the graphs of these functions by plotting points; however, it is easier to construct their graphs by using translations of the graph of  $f(x) = 2^x$ , as shown in Example 3.

**EXAMPLE 3 » Use a Translation to Produce a Graph**

- Explain how to use the graph of  $f(x) = 2^x$  to produce the graph of  $F(x) = 2^x - 3$ .
- Explain how to use the graph of  $f(x) = 2^x$  to produce the graph of  $G(x) = 2^{x-3}$ .

**Solution**

- $F(x) = 2^x - 3 = f(x) - 3$ . The graph of  $F$  is a vertical translation of  $f$  down 3 units, as shown in **Figure 7.7**.
- $G(x) = 2^{x-3} = f(x-3)$ . The graph of  $G$  is a horizontal translation of  $f$  to the right 3 units, as shown in **Figure 7.8**.

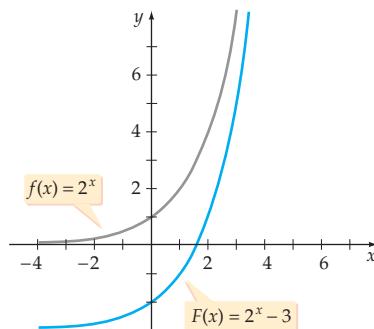


Figure 7.7

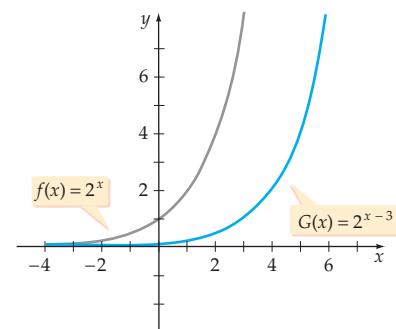


Figure 7.8

» Try Exercise 28, page 462

The graphs of some functions can be constructed by stretching, compressing, or reflecting the graph of an exponential function.

**EXAMPLE 4** » **Use Stretching or Reflecting Procedures to Produce a Graph**

- Explain how to use the graph of  $f(x) = 2^x$  to produce the graph of  $M(x) = 2(2^x)$ .
- Explain how to use the graph of  $f(x) = 2^x$  to produce the graph of  $N(x) = 2^{-x}$ .

**Solution**

- $M(x) = 2(2^x) = 2f(x)$ . The graph of  $M$  is a vertical stretching of  $f$  away from the  $x$ -axis by a factor of 2, as shown in **Figure 7.9**. (Note: If  $(x, y)$  is a point on the graph of  $f(x) = 2^x$ , then  $(x, 2y)$  is a point on the graph of  $M$ .)
- $N(x) = 2^{-x} = f(-x)$ . The graph of  $N$  is the graph of  $f$  reflected across the  $y$ -axis, as shown in **Figure 7.10**. (Note: If  $(x, y)$  is a point on the graph of  $f(x) = 2^x$ , then  $(-x, y)$  is a point on the graph of  $N$ .)

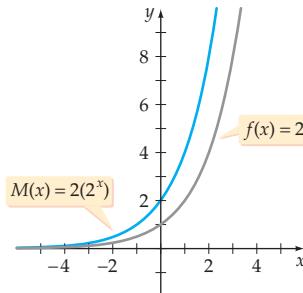


Figure 7.9

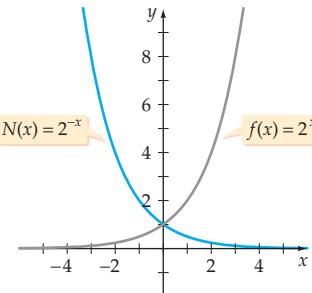


Figure 7.10

» Try Exercise 30, page 462

**Math Matters**

Leonhard Euler (1707–1783)

Some mathematicians consider Euler to be the greatest mathematician of all time. He certainly was the most prolific writer of mathematics of all time. He made substantial contributions in the areas of number theory, geometry, calculus, differential equations, differential geometry, topology, complex variables, and analysis, to name but a few. Euler was the first to introduce many of the mathematical notations that we use today. For instance, he introduced the symbol  $i$  for the square root of  $-1$ , the symbol  $\pi$  for pi, the functional notation  $f(x)$ , and the letter  $e$  for the base of the natural exponential function. Euler's computational skills were truly amazing. The mathematician François Arago remarked, "Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind."

**The Natural Exponential Function**

The irrational number  $\pi$  is often used in applications that involve circles. Another irrational number, denoted by the letter  $e$ , is useful in many applications that involve growth or decay.

**Definition of  $e$** 

The **number  $e$**  is defined as the number that

$$\left(1 + \frac{1}{n}\right)^n$$

approaches as  $n$  increases without bound.

The letter  $e$  was chosen in honor of the Swiss mathematician Leonhard Euler. He was able to compute the value of  $e$  to several decimal places by evaluating  $\left(1 + \frac{1}{n}\right)^n$  for large values of  $n$ , as shown in **Table 7.5**.

**Table 7.5**

| Value of $n$ | Value of $\left(1 + \frac{1}{n}\right)^n$ |
|--------------|---|
| 1            | 2   |
| 10           | 2.59374246                                |
| 100          | 2.704813829                               |
| 1000         | 2.716923932                               |
| 10,000       | 2.718145927                               |
| 100,000      | 2.718268237                               |
| 1,000,000    | 2.718280469                               |
| 10,000,000   | 2.718281693                               |

The value of  $e$  accurate to eight decimal places is 2.71828183.

**Definition of the Natural Exponential Function**

For all real numbers  $x$ , the function defined by

$$f(x) = e^x$$

is called the **natural exponential function**.

**Calculus Connection**

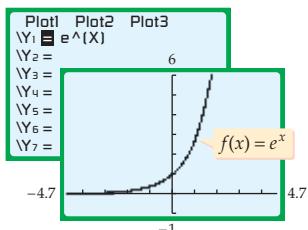
A calculator can be used to evaluate  $e^x$  for specific values of  $x$ . For instance,

$$e^2 \approx 7.389056, \quad e^{3.5} \approx 33.115452, \quad \text{and} \quad e^{-1.4} \approx 0.246597$$

On a TI-83/TI-83 Plus/TI-84 Plus calculator the  $e^x$  function is located above the **LN** key.

**Integrating Technology**

The graph of  $f(x) = e^x$  below was produced on a TI-83/TI-83 Plus/ TI-84 Plus graphing calculator by entering  $e^x$  in the Y= menu.

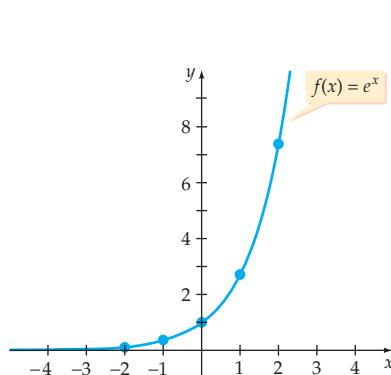
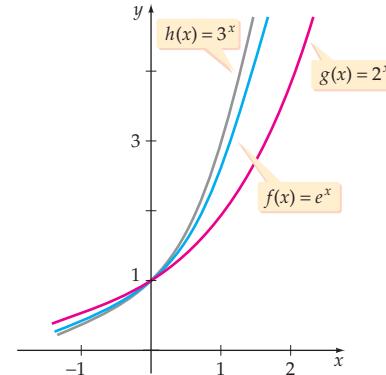


To graph  $f(x) = e^x$ , use a calculator to find the range values for a few domain values. The range values in **Table 7.6** have been rounded to the nearest tenth.

**Table 7.6**

| $x$          | -2  | -1  | 0   | 1   | 2   |
|--------------|-----|-----|-----|-----|-----|
| $f(x) = e^x$ | 0.1 | 0.4 | 1.0 | 2.7 | 7.4 |

Plot the points given in **Table 7.6**, and then connect the points with a smooth curve. Because  $e > 1$ , we know that the graph is an increasing function. To the far left, the graph will approach the  $x$ -axis. The  $y$ -intercept is  $(0, 1)$ . See **Figure 7.11**. Note in **Figure 7.12** how the graph of  $f(x) = e^x$  compares with the graphs of  $g(x) = 2^x$  and  $h(x) = 3^x$ . You may have anticipated that the graph of  $f(x) = e^x$  would lie between the two other graphs because  $e$  is between 2 and 3.

**Figure 7.11****Figure 7.12**

Many applications can be modeled effectively by functions that involve an exponential function. For instance, in Example 5 we make use of a function that involves an exponential function to model the temperature of a cup of coffee.

**EXAMPLE 5** **Use a Mathematical Model**

A cup of coffee is heated to 160°F and placed in a room that maintains a temperature of 70°F. The temperature  $T$  of the coffee, in degrees Fahrenheit, after  $t$  minutes is given by

$$T = 70 + 90e^{-0.0485t}$$

- Find the temperature of the coffee, to the nearest degree, 20 minutes after it is placed in the room.
- Use a graphing utility to determine when the temperature of the coffee will reach 90°F.

**take note**

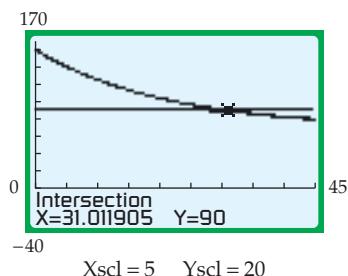
In Example 5b, we use a graphing utility to solve the equation  $90 = 70 + 90e^{-0.0485t}$ . Analytic methods of solving this type of equation without the use of a graphing utility will be developed in Section 7.4.

**Solution**

a.  $T = 70 + 90e^{-0.0485t}$   
 $= 70 + 90e^{-0.0485 \cdot (20)}$  • Substitute 20 for  $t$ .  
 $\approx 70 + 34.1$   
 $\approx 104.1$

After 20 minutes the temperature of the coffee is about 104°F.

- b. Graph  $T = 70 + 90e^{-0.0485t}$  and  $T = 90$ . See the following figure.



The graphs intersect at about (31.01, 90). It takes the coffee about 31 minutes to cool to 90°F.

» Try Exercise 48, page 463

**EXAMPLE 6 » Use a Mathematical Model**

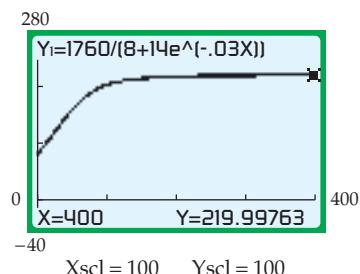
The weekly revenue  $R$ , in dollars, from the sale of a product varies with time according to the function

$$R(x) = \frac{1760}{8 + 14e^{-0.03x}}$$

where  $x$  is the number of weeks that have passed since the product was put on the market. What will the weekly revenue approach as time goes by?

**Solution**

**Method 1** Use a graphing utility to graph  $R(x)$ , and use the TRACE feature to see what happens to the revenue as the time increases. The graph on the right shows that as the weeks go by, the weekly revenue will increase and approach \$220.00 per week.



**Method 2** Write the revenue function in the following form.

$$R(x) = \frac{1760}{8 + \frac{14}{e^{0.03x}}} \quad \bullet 14e^{-0.03x} = \frac{14}{e^{0.03x}}$$

As  $x$  increases without bound,  $e^{0.03x}$  increases without bound, and the fraction  $\frac{14}{e^{0.03x}}$  approaches 0. Therefore, as  $x \rightarrow \infty$ ,  $R(x) \rightarrow \frac{1760}{8 + 0} = 220$ . Both methods indicate that as the number of weeks increases, the revenue approaches \$220 per week.

» Try Exercise 54, page 464



### Topics for Discussion

- Explain how to use the graph of  $f(x) = 2^x$  to produce the graph of  $g(x) = 2^{(x-3)} + 4$ .
- At what point does the function  $g(x) = e^{-x^2/2}$  take on its maximum value?
- Without using a graphing utility, determine whether the revenue function  $R(t) = 10 + e^{-0.05t}$  is an increasing function or a decreasing function.
- Discuss the properties of the graph of  $f(x) = b^x$  when  $b > 1$ .
- What is the base of the natural exponential function? How is it calculated? What is its approximate value?



## Exercise Set 7.1

In Exercises 1 to 8, evaluate the exponential function for the given  $x$ -values.

- $f(x) = 3^x$ ;  $x = 0$  and  $x = 4$
- $f(x) = 5^x$ ;  $x = 3$  and  $x = -2$
- $g(x) = 10^x$ ;  $x = -2$  and  $x = 3$
- $g(x) = 4^x$ ;  $x = 0$  and  $x = -1$
- $h(x) = \left(\frac{3}{2}\right)^x$ ;  $x = 2$  and  $x = -3$
- $h(x) = \left(\frac{2}{5}\right)^x$ ;  $x = -1$  and  $x = 3$

7.  $j(x) = \left(\frac{1}{2}\right)^x$ ;  $x = -2$  and  $x = 4$

8.  $j(x) = \left(\frac{1}{4}\right)^x$ ;  $x = -1$  and  $x = 5$

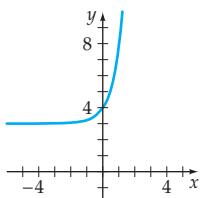
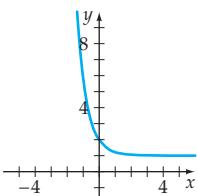
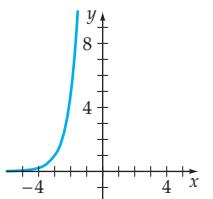
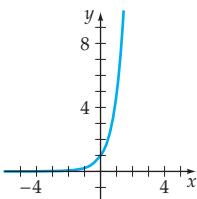


In Exercises 9 to 14, use a calculator to evaluate the exponential function for the given  $x$ -value. Round to the nearest hundredth.

- |                                   |                                |
|-----------------------------------|--------------------------------|
| 9. $f(x) = 2^x$ , $x = 3.2$       | 10. $f(x) = 3^x$ , $x = -1.5$  |
| 11. $g(x) = e^x$ , $x = 2.2$      | 12. $g(x) = e^x$ , $x = -1.3$  |
| 13. $h(x) = 5^x$ , $x = \sqrt{2}$ | 14. $h(x) = 0.5^x$ , $x = \pi$ |

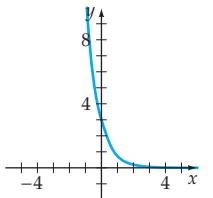
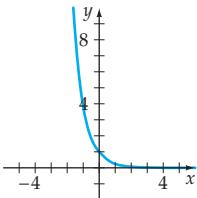
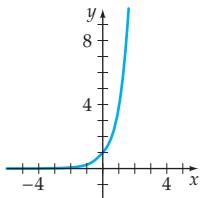
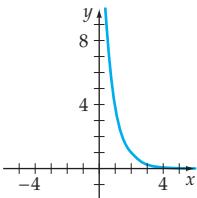
15. Examine the following four functions and the graphs labeled **a**, **b**, **c**, and **d**. For each graph, determine which function has been graphed.

$$\begin{array}{ll} f(x) = 5^x & g(x) = 1 + 5^{-x} \\ h(x) = 5^{x+3} & k(x) = 5^x + 3 \end{array}$$

**a.****b.****c.****d.**

16. Examine the following four functions and the graphs labeled **a**, **b**, **c**, and **d**. For each graph, determine which function has been graphed.

$$\begin{array}{ll} f(x) = \left(\frac{1}{4}\right)^x & g(x) = \left(\frac{1}{4}\right)^{-x} \\ h(x) = \left(\frac{1}{4}\right)^{x-2} & k(x) = 3\left(\frac{1}{4}\right)^x \end{array}$$

**a.****b.****c.****d.**

In Exercises 17 to 24, sketch the graph of each function.

17.  $f(x) = 3^x$

18.  $f(x) = 4^x$

19.  $f(x) = 10^x$

20.  $f(x) = 6^x$

21.  $f(x) = \left(\frac{3}{2}\right)^x$

» 22.  $f(x) = \left(\frac{5}{2}\right)^x$

23.  $f(x) = \left(\frac{1}{3}\right)^x$

24.  $f(x) = \left(\frac{2}{3}\right)^x$

In Exercises 25 to 38, explain how to use the graph of the first function  $f$  to produce the graph of the second function  $F$ .

25.  $f(x) = 3^x, F(x) = 3^x + 2$

26.  $f(x) = 4^x, F(x) = 4^x - 3$

27.  $f(x) = 10^x, F(x) = 10^{x-2}$

» 28.  $f(x) = 6^x, F(x) = 6^{x+5}$

29.  $f(x) = \left(\frac{3}{2}\right)^x, F(x) = \left(\frac{3}{2}\right)^{-x}$

» 30.  $f(x) = \left(\frac{5}{2}\right)^x, F(x) = -\left[\left(\frac{5}{2}\right)^x\right]$

31.  $f(x) = \left(\frac{1}{3}\right)^x, F(x) = 2\left[\left(\frac{1}{3}\right)^x\right]$

32.  $f(x) = \left(\frac{2}{3}\right)^x, F(x) = \frac{1}{2}\left[\left(\frac{2}{3}\right)^x\right]$

33.  $f(x) = e^x, F(x) = e^{-x} + 2$

34.  $f(x) = e^x, F(x) = e^{x-3} + 1$

35.  $f(x) = 2^x, F(x) = -(2^{x-4})$

36.  $f(x) = 2^x, F(x) = -(2^{-x})$

37.  $f(x) = 0.5^x, F(x) = 3 + 0.5^{-x}$

38.  $f(x) = 0.5^x, F(x) = 3(0.5^{x+2}) - 1$



In Exercises 39 to 46, use a graphing utility to graph each function. If the function has a horizontal asymptote, state the equation of the horizontal asymptote.

39.  $f(x) = \frac{3^x + 3^{-x}}{2}$

40.  $f(x) = 4 \cdot 3^{-x^2}$

41.  $f(x) = \frac{e^x - e^{-x}}{2}$

42.  $f(x) = \frac{e^x + e^{-x}}{2}$

43.  $f(x) = -e^{(x-4)}$

44.  $f(x) = 0.5e^{-x}$

45.  $f(x) = \frac{10}{1 + 0.4e^{-0.5x}}, x \geq 0$

46.  $f(x) = \frac{10}{1 + 1.5e^{-0.5x}}, x \geq 0$

47.   **INTERNET CONNECTIONS** Data from Forrester Research suggest that the number of broadband [cable and digital subscriber line (DSL)] connections to the Internet can be modeled by  $f(x) = 1.353(1.9025)^x$ , where  $x$  is the number of years after January 1, 1998, and  $f(x)$  is the number of connections in millions.

- How many broadband Internet connections, to the nearest million, does this model predict will exist on January 1, 2007?
- According to the model, in what year will the number of broadband connections first reach 1 billion? [Hint: Use the intersect feature of a graphing utility to determine the  $x$ -coordinate of the point of intersection of the graphs of  $f(x)$  and  $y = 1000$ .]

48.  **MEDICATION IN BLOODSTREAM** The function  $A(t) = 200e^{-0.014t}$  gives the amount of medication, in milligrams, in a patient's bloodstream  $t$  minutes after the medication has been injected into the patient's bloodstream.

- Find the amount of medication, to the nearest milligram, in the patient's bloodstream after 45 minutes.
- Use a graphing utility to determine how long it will take, to the nearest minute, for the amount of medication in the patient's bloodstream to reach 50 milligrams.

49. **DEMAND FOR A PRODUCT** The demand  $d$  for a specific product, in items per month, is given by

$$d(p) = 25 + 880e^{-0.18p}$$

where  $p$  is the price, in dollars, of the product.

- What will be the monthly demand, to the nearest unit, when the price of the product is \$8 and when the price is \$18?
- What will happen to the demand as the price increases without bound?

50. **SALES** The monthly income  $I$ , in dollars, from a new product is given by

$$I(t) = 24,000 - 22,000e^{-0.005t}$$

where  $t$  is the time, in months, since the product was first put on the market.

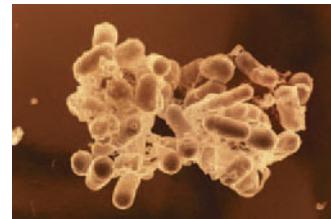
- What was the monthly income after the 10th month and after the 100th month?

- What will the monthly income from the product approach as the time increases without bound?

51.

**E. COLI INFEC-****TION**

*Escherichia coli* (*E. coli*) is a bacterium that can reproduce at an exponential rate. The *E. coli* reproduce by dividing. A small number of *E. coli* bacteria in the large intestine of a human can trigger a serious infection within a few hours. Consider a particular *E. coli* infection that starts with 100 *E. coli* bacteria. Each bacterium splits into two parts every half hour. Assuming none of the bacteria die, the size of the *E. coli* population after  $t$  hours is given by  $P(t) = 100 \cdot 2^{2t}$ , where  $0 \leq t \leq 16$ .



- Find  $P(3)$  and  $P(6)$ .

- Use a graphing utility to find the time, to the nearest tenth of an hour, it takes for the *E. coli* population to number 1 billion.

52.



**RADIATION** Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate  $x$  millimeters of lead shielding is given by  $I(x) = 100e^{-1.5x}$ .

- What percentage of radiation, to the nearest tenth of a percent, will penetrate a lead shield that is 1 millimeter thick?
- How many millimeters of lead shielding are required so that less than 0.05% of the radiation penetrates the shielding? Round to the nearest millimeter.

53.



**PHOTOCROMATIC EYEGLASS LENSES** Photochromatic eyeglass lenses contain molecules of silver chloride or silver halide. These molecules are transparent in the absence of UV rays. UV rays are normally absent in artificial lighting. However, when the lenses are exposed to UV rays, as in direct sunlight, the molecules take on a new molecular structure, which causes the lenses to darken. The number of molecules affected varies with the intensity of the UV rays. The intensity of UV rays is measured using a scale called the UV index. On this scale, a value near 0 indicates a low UV intensity and a value near 10 indicates a high UV intensity.

For the photochromatic lenses shown below, the function  $P(x) = (0.9)^x$  models the transparency  $P$  of the lenses as a function of the UV index  $x$ .



UV index, 0  
lens transparency, 100%



UV index, 5  
lens transparency, 59.0%



UV index, 9  
lens transparency, 38.7%

- Find the transparency of these lenses, to the nearest tenth of a percent, when they are exposed to light rays with a UV index of 3.5.
- What is the UV index of light rays that cause these photochromatic lenses to have a transparency of 45%? Round to the nearest tenth.

- 54.** **FISH POPULATION** The number of bass in a lake is given by

$$P(t) = \frac{3600}{1 + 7e^{-0.05t}}$$

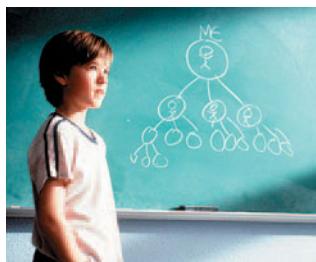
where  $t$  is the number of months that have passed since the lake was stocked with bass.



- How many bass were in the lake immediately after it was stocked?
- How many bass were in the lake 1 year after the lake was stocked? Round to the nearest bass.
- What will happen to the bass population as  $t$  increases without bound?

- 55.** **THE PAY IT FORWARD MODEL**

In the movie *Pay It Forward*, Trevor McKinney, played by Haley Joel Osment, is given a school assignment to “think of an idea to change the world—and then put it into action.” In response to this assignment, Trevor develops a *pay it forward* project. In this project, anyone who benefits from another person’s good deed must do a good deed for three additional people. Each of these three people is then obligated to do a good deed for another three people, and so on.

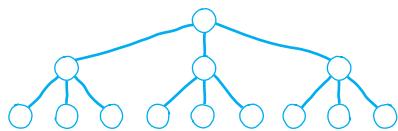


The following diagram shows the number of people who have been a beneficiary of a good deed after 1 round and after 2 rounds of this project.

Three beneficiaries after one round.



A total of 12 beneficiaries after two rounds ( $3 + 9 = 12$ ).



A mathematical model for the number of pay-it-forward beneficiaries after  $n$  rounds is given by  $B(n) = \frac{3^{n+1} - 3}{2}$ .

Use this model to determine

- the number of beneficiaries after 5 rounds and after 10 rounds. Assume that no person is a beneficiary of more than one good deed.
- how many rounds are required to produce at least 2 million beneficiaries.

- 56.** **INTENSITY OF LIGHT**

The percent  $I(x)$  of the original intensity of light striking the surface of a lake that is available  $x$  feet below the surface of the lake is given by  $I(x) = 100e^{-0.95x}$ .

- What percentage of the light, to the nearest tenth of a percent, is available 2 feet below the surface of the lake?
- At what depth, to the nearest hundredth of a foot, is the intensity of the light one-half the intensity at the surface?

- 57.**  **A TEMPERATURE MODEL** A cup of coffee is heated to 180°F and placed in a room that maintains a temperature of 65°F. The temperature of the coffee after  $t$  minutes is given by  $T(t) = 65 + 115e^{-0.042t}$ .

- a. Find the temperature, to the nearest degree, of the coffee 10 minutes after it is placed in the room.
  - b. Use a graphing utility to determine when, to the nearest tenth of a minute, the temperature of the coffee will reach  $100^{\circ}\text{F}$ .

- 58.**  **A TEMPERATURE MODEL** Soup that is at a temperature of  $170^{\circ}\text{F}$  is poured into a bowl in a room that maintains a constant temperature. The temperature of the soup decreases according to the model given by  $T(t) = 75 + 95e^{-0.12t}$ , where  $t$  is time in minutes after the soup is poured.

- a. What is the temperature, to the nearest tenth of a degree, of the soup after 2 minutes?
  - b. A certain customer prefers soup at a temperature of  $110^{\circ}\text{F}$ . How many minutes, to the nearest 0.1 minute, after the soup is poured does the soup reach that temperature?
  - c. What is the temperature of the room?

### »»» Connecting Concepts »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»»

- 60.** Verify that the hyperbolic cosine function  $\cosh(x) = \frac{e^x + e^{-x}}{2}$  is an even function.

- 61.** Verify that the hyperbolic sine function  $\sinh(x) = \frac{e^x - e^{-x}}{2}$  is an odd function.

- 62.** Graph  $g(x) = 10^x$ , and then sketch the graph of  $g$  reflected across the line given by  $y = x$ .

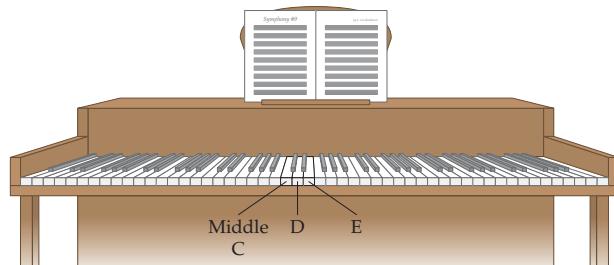
- 63.** Graph  $f(x) = e^x$ , and then sketch the graph of  $f$  reflected across the line given by  $y = x$ .

**In Exercises 64 to 67, determine the domain of the given function. Write the domain using interval notation.**

**64.**  $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

**65.**  $f(x) = \frac{e^{|x|}}{1 + e^x}$

- 59.**  **MUSICAL SCALES** Starting on the left side of a standard 88-key piano, the frequency, in vibrations per second, of the  $n$ th note is given by  $f(n) = (27.5)2^{(n-1)/12}$ .



- a. Using this formula, determine the frequency, to the nearest hundredth of a vibration per second, of middle C, key number 40 on an 88-key piano.
  - b. Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)? Explain.

**66.**  $f(x) = \sqrt{1 - e^x}$

**67.**  $f(x) = \sqrt{e^x - e^{-x}}$

- 68.** Let  $h(x) = 2^{(x^2+4)}$ . Express  $h$  as the composition of an exponential function  $f$  and a polynomial function  $g$ .

- 69.** Let  $h(x) = e^{(2x-5)}$ . Express  $h$  as the composition of an exponential function  $f$  and a polynomial function  $g$ .

- 70.** Explain why the graph of

$$f(x) = \frac{e^x + e^{-x}}{2}$$

can be produced by plotting the average height of  $g(x) = e^x$  and  $h(x) = e^{-x}$  for each value of  $x$ .

## »»» Projects

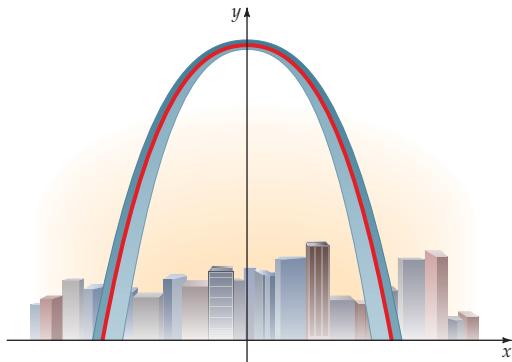


### THE SAINT LOUIS GATEWAY ARCH

The Gateway Arch in Saint Louis was designed in the shape of an inverted **catenary**, as shown by the red curve in the drawing below. The Gateway Arch is one of the largest optical illusions ever created. As you look at the arch (and its basic shape defined by the catenary curve), it appears to be much taller than it is wide. However, this is not the case. The height of the catenary is given by

$$h(x) = 693.8597 - 68.7672 \left( \frac{e^{0.0100333x} + e^{-0.0100333x}}{2} \right)$$

where  $x$  and  $h(x)$  are measured in feet and  $x = 0$  represents the position at ground level that is directly below the highest point of the catenary.



- Use a graphing utility to graph  $h(x)$ .
- Use your graph to find the height of the catenary for  $x = 0, 100, 200$ , and  $299$  feet. Round each result to the nearest tenth of a foot.
- What is the width of the catenary at ground level and what is the maximum height of the catenary? Round each result to the nearest tenth of a foot.
- By how much does the maximum height of the catenary exceed its width at ground level? Round to the nearest tenth of a foot.



### AN EXPONENTIAL REWARD

According to legend, when Sissa Ben Dahir of India invented the game of chess, King Shirham was so impressed with the game that he summoned the game's inventor and offered him the reward of his choosing. The inventor pointed to the chessboard and requested, for his reward, one grain of wheat on the first square, two grains of wheat on the second square, four grains on the third square, eight grains on the fourth square, and so on for all 64 squares on the chessboard. The king considered this a very modest reward and said he would grant the inventor's wish. The following table shows how many grains of wheat are on each of the first six squares and the total number of grains of wheat needed to cover squares 1 to  $n$  for  $n \leq 6$ .

| Square number, $n$ | Number of grains of wheat on square $n$ | Total number of grains of wheat on squares 1 through $n$ |
|--------------------|---|--|
| 1                  | 1                                       | 1  |
| 2                  | 2                                       | 3  |
| 3                  | 4                                       | 7  |
| 4                  | 8                                       | 15   |
| 5                  | 16                                      | 31   |
| 6                  | 32                                      | 63   |

- If all 64 squares of the chessboard are piled with wheat as requested by Sissa Ben Dahir, how many grains of wheat are on the board?
- A grain of wheat weighs approximately 0.000008 kilogram. Find the total weight of the wheat requested by Sissa Ben Dahir.
- In a recent year, a total of  $6.5 \times 10^8$  metric tons of wheat were produced in the world. At this level, how many years, to the nearest year, of wheat production would be required to fill the request of Sissa Ben Dahir? One metric ton equals 1000 kilograms.

## Section 7.2

- Logarithmic Functions
- Graphs of Logarithmic Functions
- Domains of Logarithmic Functions
- Common and Natural Logarithms

# Logarithmic Functions and Their Applications

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A37.

**PS1.** If  $2^x = 16$ , determine the value of  $x$ . [7.1]

**PS2.** If  $3^{-x} = \frac{1}{27}$ , determine the value of  $x$ . [7.1]

**PS3.** If  $x^4 = 625$ , determine the value of  $x$ . [7.1]

**PS4.** Find the inverse of  $f(x) = \frac{2x}{x+3}$ . [1.6]

**PS5.** State the domain of  $g(x) = \sqrt{x-2}$ . [1.3]

**PS6.** If the range of  $h(x)$  is the set of all positive real numbers, then what is the domain of  $h^{-1}(x)$ ? [1.6]

## Logarithmic Functions

Every exponential function of the form  $g(x) = b^x$  is a one-to-one function and therefore has an inverse function. Sometimes we can determine the inverse of a function represented by an equation by interchanging the variables of its equation and then solving for the dependent variable. If we attempt to use this procedure for  $g(x) = b^x$ , we obtain

$$\begin{aligned} g(x) &= b^x \\ y &= b^x \\ x &= b^y \end{aligned} \quad \bullet \text{ Interchange the variables.}$$

None of our previous methods can be used to solve the equation  $x = b^y$  for the exponent  $y$ . Thus we need to develop a new procedure. One method would be to merely write

$y =$  the power of  $b$  that produces  $x$

Although this would work, it is not very concise. We need a compact notation to represent “ $y$  is the power of  $b$  that produces  $x$ .” This more compact notation is given in the following definition.

### Definition of a Logarithm and a Logarithmic Function

If  $x > 0$  and  $b$  is a positive constant ( $b \neq 1$ ), then

$$y = \log_b x \quad \text{if and only if} \quad b^y = x$$

The notation  $\log_b x$  is read “the **logarithm** (or log) base  $b$  of  $x$ .” The function defined by  $f(x) = \log_b x$  is a **logarithmic function** with base  $b$ . This function is the inverse of the exponential function  $g(x) = b^x$ .

### Math Matters



Logarithms were developed by John Napier (1550–1617) as a means of simplifying the calculations of astronomers. One of his ideas was to devise a method by which the product of two numbers could be determined by performing an addition.

It is essential to remember that  $f(x) = \log_b x$  is the inverse function of  $g(x) = b^x$ . Because these functions are inverses and because functions that are inverses have the property that  $f(g(x)) = x$  and  $g(f(x)) = x$ , we have the following important relationships.

### Composition of Logarithmic and Exponential Functions

Let  $g(x) = b^x$  and  $f(x) = \log_b x$  ( $x > 0, b > 0, b \neq 1$ ). Then

$$g(f(x)) = b^{\log_b x} = x \quad \text{and} \quad f(g(x)) = \log_b b^x = x$$

#### *take note*

The notation  $\log_b x$  replaces the phrase “the power of  $b$  that produces  $x$ .” For instance, “3 is the power of 2 that produces 8” is abbreviated  $3 = \log_2 8$ . In your work with logarithms, remember that a logarithm is an exponent.

As an example of these relationships, let  $g(x) = 2^x$  and  $f(x) = \log_2 x$ . Then

$$2^{\log_2 x} = x \quad \text{and} \quad \log_2 2^x = x$$

The equations

$$y = \log_b x \quad \text{and} \quad b^y = x$$

are different ways of expressing the same concept.

### Definition of Exponential Form and Logarithmic Form

The **exponential form** of  $y = \log_b x$  is  $b^y = x$ .

The **logarithmic form** of  $b^y = x$  is  $y = \log_b x$ .

These concepts are illustrated in the next two examples.

### EXAMPLE 1 »» Change from Logarithmic to Exponential Form

Write each equation in its exponential form.

- a.  $3 = \log_2 8$     b.  $2 = \log_{10}(x + 5)$     c.  $\log_e x = 4$     d.  $\log_b b^3 = 3$

#### Solution

Use the definition  $y = \log_b x$  if and only if  $b^y = x$ .

**Logarithms are exponents.**

a.  $3 = \log_2 8$     if and only if  $\boxed{2^3 = 8}$

Base

- b.  $2 = \log_{10}(x + 5)$  if and only if  $10^2 = x + 5$ .
- c.  $\log_e x = 4$  if and only if  $e^4 = x$ .
- d.  $\log_b b^3 = 3$  if and only if  $b^3 = b^3$ .

»» Try Exercise 4, page 476

## EXAMPLE 2 »» Change from Exponential to Logarithmic Form

Write each equation in its logarithmic form.

- a.**  $3^2 = 9$     **b.**  $5^3 = x$     **c.**  $a^b = c$     **d.**  $b^{\log_b 5} = 5$

## Solution

The logarithmic form of  $b^y = x$  is  $y = \log_b x$ .

- Exponent**

a.  $3^2 = 9$  if and only if  $2 = \log_3 9$

**Base**

b.  $5^3 = x$  if and only if  $3 = \log_5 x$ .

c.  $a^b = c$  if and only if  $b = \log_a c$ .

d.  $b^{\log_b 5} = 5$  if and only if  $\log_b 5 = \log_b 5$ .

» Try Exercise 14, page 476

The definition of a logarithm and the definition of an inverse function can be used to establish many properties of logarithms. For instance:

- $\log_b b = 1$  because  $b = b^1$ .
  - $\log_b 1 = 0$  because  $1 = b^0$ .
  - $\log_b(b^x) = x$  because  $b^x = b^x$ .
  - $b^{\log_b x} = x$  because  $f(x) = \log_b x$  and  $g(x) = b^x$  are inverse functions. Thus  $g[f(x)] = x$ .

We will refer to the preceding properties as the *basic logarithmic properties*.

## Basic Logarithmic Properties

- $$1. \log_b b = 1 \quad 2. \log_b 1 = 0 \quad 3. \log_b(b^x) = x \quad 4. b^{\log_b x} = x$$

### EXAMPLE 3 » Apply the Basic Logarithmic Properties

Evaluate each of the following logarithms.

- a.**  $\log_8 1$     **b.**  $\log_5 5$     **c.**  $\log_2(2^4)$     **d.**  $3^{\log_3 7}$

## Solution

- a. By Property 2,  $\log_8 1 = 0$ .
  - b. By Property 1,  $\log_5 5 = 1$ .
  - c. By Property 3,  $\log_2(2^4) = 4$ .
  - d. By Property 4,  $3^{\log_3 7} = 7$ .

» Try Exercise 32, page 476

Some logarithms can be evaluated just by remembering that a logarithm is an exponent. For instance,  $\log_5 25$  equals 2 because the base 5 raised to the second power equals 25.

- $\log_{10} 100 = 2$  because  $10^2 = 100$ .
- $\log_4 64 = 3$  because  $4^3 = 64$ .
- $\log_7 \frac{1}{49} = -2$  because  $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$ .

**QUESTION** What is the value of  $\log_5 625$ ?

### Graphs of Logarithmic Functions

Because  $f(x) = \log_b x$  is the inverse function of  $g(x) = b^x$ , the graph of  $f$  is a reflection of the graph of  $g$  across the line given by  $y = x$ . The graph of  $g(x) = 2^x$  is shown in **Figure 7.13**. **Table 7.7** below shows some of the ordered pairs of the graph of  $g$ .

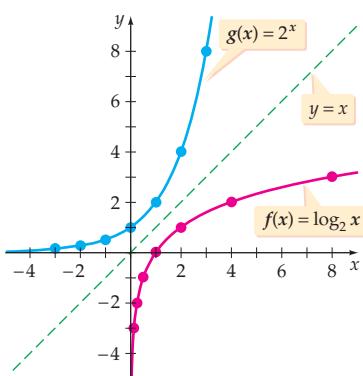


Figure 7.13

Table 7.7

| x            | -3            | -2            | -1            | 0 | 1 | 2 | 3 |
|--------------|---------------|---------------|---------------|---|---|---|---|
| $g(x) = 2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |

The graph of the inverse of  $g$ , which is  $f(x) = \log_2 x$ , is also shown in **Figure 7.13**. Some of the ordered pairs of  $f$  are shown in **Table 7.8**. Note that if  $(x, y)$  is a point on the graph of  $g$ , then  $(y, x)$  is a point on the graph of  $f$ . Also notice that the graph of  $f$  is a reflection of the graph of  $g$  across the line given by  $y = x$ .

Table 7.8

| x                 | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|-------------------|---------------|---------------|---------------|---|---|---|---|
| $f(x) = \log_2 x$ | -3            | -2            | -1            | 0 | 1 | 2 | 3 |

The graph of a logarithmic function can be drawn by first rewriting the function in its exponential form. This procedure is illustrated in Example 4.

---

**ANSWER**  $\log_5 625 = 4$  because  $5^4 = 625$ .

**EXAMPLE 4** » Graph a Logarithmic Function

Graph  $f(x) = \log_3 x$ .

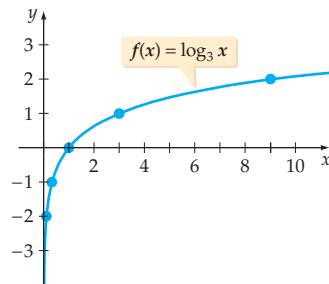
**Solution**

To graph  $f(x) = \log_3 x$ , consider the equivalent exponential equation  $x = 3^y$ . Because this equation is solved for  $x$ , choose values of  $y$  and calculate the corresponding values of  $x$ , as shown in **Table 7.9**.

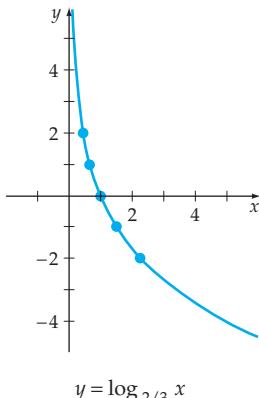
**Table 7.9**

|           |               |               |   |   |   |
|-----------|---------------|---------------|---|---|---|
| $x = 3^y$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 |
| $y$       | -2            | -1            | 0 | 1 | 2 |

Now plot the ordered pairs and connect the points with a smooth curve, as shown in **Figure 7.14**.



**Figure 7.14**

**» Try Exercise 44, page 476**


**Figure 7.15**

We can use a similar procedure to draw the graph of a logarithmic function with a fractional base. For instance, consider  $y = \log_{2/3} x$ . Rewriting this in exponential form gives us  $\left(\frac{2}{3}\right)^y = x$ . Choose values of  $y$  and calculate the corresponding  $x$  values. See **Table 7.10**. Plot the points corresponding to the ordered pairs  $(x, y)$ , and then draw a smooth curve through the points, as shown in **Figure 7.15**.

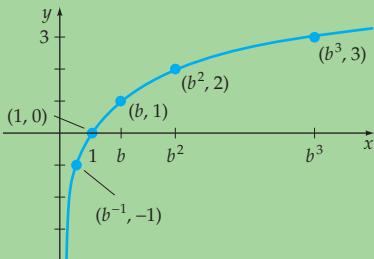
**Table 7.10**

|                                  |   |   |                                  |  |  |
|----------------------------------|---|---|----------------------------------|--|--|
| $x = \left(\frac{2}{3}\right)^y$ | $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$ | $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$ | $\left(\frac{2}{3}\right)^0 = 1$ | $\left(\frac{2}{3}\right)^1 = \frac{2}{3}$ | $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$ |
| $y$                              | -2  | -1  | 0                                | 1  | 2  |

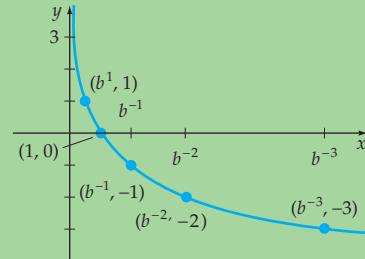
### Properties of

For all positive real numbers  $b$ ,  $b \neq 1$ , the function  $f(x) = \log_b x$  has the following properties:

1. The domain of  $f$  consists of the set of positive real numbers, and its range consists of the set of all real numbers.
2. The graph of  $f$  has an  $x$ -intercept of  $(1, 0)$  and passes through  $(b, 1)$ .
3. If  $b > 1$ ,  $f$  is an increasing function and its graph is asymptotic to the negative  $y$ -axis. [As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$ , and as  $x \rightarrow 0$  from the right,  $f(x) \rightarrow -\infty$ .] See **Figure 7.16a**.
4. If  $0 < b < 1$ ,  $f$  is a decreasing function and its graph is asymptotic to the positive  $y$ -axis. [As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$ , and as  $x \rightarrow 0$  from the right,  $f(x) \rightarrow \infty$ .] See **Figure 7.16b**.



a.  $f(x) = \log_b x$ ,  $b > 1$



b.  $f(x) = \log_b x$ ,  $0 < b < 1$

**Figure 7.16**

### Domains of Logarithmic Functions

The function  $f(x) = \log_b x$  has as its domain the set of positive real numbers. The function  $f(x) = \log_b(g(x))$  has as its domain the set of all  $x$  for which  $g(x) > 0$ . To determine the domain of a function such as  $f(x) = \log_b(g(x))$ , we must determine the values of  $x$  that make  $g(x)$  positive. This process is illustrated in Example 5.

#### EXAMPLE 5 » Find the Domain of a Logarithmic Function

Find the domain of each of the following logarithmic functions.

a.  $f(x) = \log_6(x - 3)$     b.  $F(x) = \log_2|x + 2|$     c.  $R(x) = \log_5\left(\frac{x}{8 - x}\right)$

#### Solution

- a. Solving  $(x - 3) > 0$  for  $x$  gives us  $x > 3$ . The domain of  $f$  consists of all real numbers greater than 3. In interval notation, **the domain is  $(3, \infty)$** .
- b. The solution set of  $|x + 2| > 0$  consists of all real numbers  $x$  except  $x = -2$ . The domain of  $F$  consists of all real numbers  $x \neq -2$ . In interval notation, **the domain is  $(-\infty, -2) \cup (-2, \infty)$** .

- c. Solving  $\left(\frac{x}{8-x}\right) > 0$  yields the set of all real numbers  $x$  between 0 and 8.

The domain of  $R$  is all real numbers  $x$  such that  $0 < x < 8$ . In interval notation, the domain is  $(0, 8)$ .

» Try Exercise 52, page 476

Some logarithmic functions can be graphed by using horizontal and/or vertical translations of a previously drawn graph.

**E X A M P L E 6** » Use Translations to Graph Logarithmic Functions

Graph. a.  $f(x) = \log_4(x + 3)$     b.  $f(x) = \log_4 x + 3$

**Solution**

- a. The graph of  $f(x) = \log_4(x + 3)$  can be obtained by shifting the graph of  $g(x) = \log_4 x$  to the left 3 units. See **Figure 7.17**. Note that the domain of  $f$  consists of all real numbers  $x$  greater than  $-3$  because  $x + 3 > 0$  for  $x > -3$ . The graph of  $f$  is asymptotic to the vertical line  $x = -3$ .
- b. The graph of  $f(x) = \log_4 x + 3$  can be obtained by shifting the graph of  $g(x) = \log_4 x$  upward 3 units. See **Figure 7.18**.

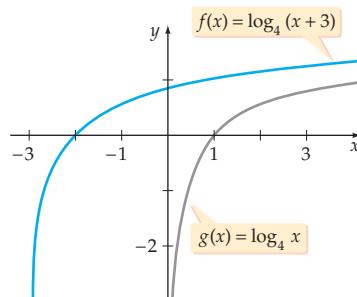


Figure 7.17

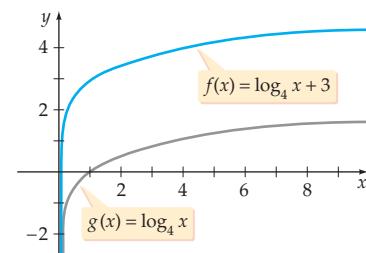


Figure 7.18

» Try Exercise 66, page 476

**Common and Natural Logarithms**

Two of the most frequently used logarithmic functions are *common logarithms*, which have base 10, and *natural logarithms*, which have base  $e$  (the base of the natural exponential function).

**Definition of Common and Natural Logarithms**

The function defined by  $f(x) = \log_{10} x$  is called the **common logarithmic function**. It is customarily written without stating the base as  $f(x) = \log x$ .

The function defined by  $f(x) = \log_e x$  is called the **natural logarithmic function**. It is customarily written as  $f(x) = \ln x$ .

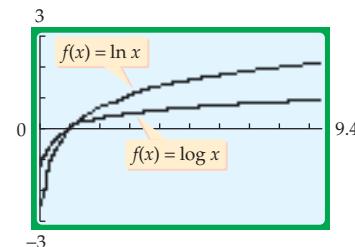
Most scientific or graphing calculators have a **LOG** key for evaluating common logarithms and an **LN** key to evaluate natural logarithms. For instance, using a graphing calculator,

$$\log 24 \approx 1.3802112 \quad \text{and} \quad \ln 81 \approx 4.3944492$$

The graphs of  $f(x) = \log x$  and  $f(x) = \ln x$  can be drawn using the same techniques we used to draw the graphs in the preceding examples. However, these graphs also can be produced with a graphing calculator by entering  $\log x$  and  $\ln x$  into the  $Y=$  menu. See **Figure 7.19** and **Figure 7.20**.



**Figure 7.19**



**Figure 7.20**

Observe that each graph passes through  $(1, 0)$ . Also note that as  $x \rightarrow 0$  from the right, the functional values  $f(x) \rightarrow -\infty$ . Thus the  $y$ -axis is a vertical asymptote for each of the graphs. The domain of both  $f(x) = \log x$  and  $f(x) = \ln x$  is the set of positive real numbers. Each of these functions has a range consisting of the set of real numbers.

Many applications can be modeled by logarithmic functions.

### EXAMPLE 7 » Applied Physiology



In the study *The Pace of Life*, M. H. Bornstein and H. G. Bornstein (*Nature*, Vol. 259, pp. 557–558, 1976) reported that as the population of a city increases, the average walking speed of a pedestrian also increases. An approximate relation between the average pedestrian walking speed  $s$ , in miles per hour, and the population  $x$ , in thousands, of a city is given by the function

$$s(x) = 0.37 \ln x + 0.05$$

- Determine the average walking speed, to the nearest tenth of a mile per hour, in San Francisco, which has a population of 780,000, and in Round Rock, Texas, which has a population of 62,000.
- Estimate the population of a city for which the average pedestrian walking speed is 3.1 miles per hour. Round to the nearest hundred-thousand.

**Math Matters**

Although logarithms were originally developed to assist with computations, logarithmic functions have a much broader use today. They are often used in such disciplines as geology, acoustics, chemistry, physics, and economics, to name a few.

**Solution**

- a. The population of San Francisco, in thousands, is 780.

$$s(x) = 0.37 \ln x + 0.05$$

$$s(780) = 0.37 \ln 780 + 0.05 \quad \bullet \text{ Substitute 780 for } x.$$

$$\approx 2.5$$

**• Use a calculator to evaluate.**

The average walking speed in San Francisco is about 2.5 miles per hour.

The population of Round Rock, in thousands, is 62.

$$s(x) = 0.37 \ln x + 0.05$$

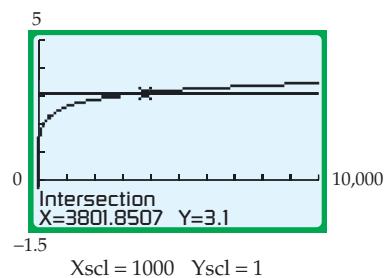
$$s(62) = 0.37 \ln 62 + 0.05 \quad \bullet \text{ Substitute 62 for } x.$$

$$\approx 1.6$$

**• Use a calculator to evaluate.**

The average walking speed in Round Rock is about 1.6 miles per hour.

- b. Graph  $s(x) = 0.37 \ln x + 0.05$  and  $s = 3.1$  in the same viewing window.



The  $x$  value of the intersection point represents the population in thousands. The function indicates that a city with an average pedestrian walking speed of 3.1 miles per hour should have a population of about 3,800,000.

» Try Exercise 86, page 477

**Topics for Discussion**

1. If  $m > n$ , must  $\log_b m > \log_b n$ ?
2. For what values of  $x$  is  $\ln x > \log x$ ?
3. What is the domain of  $f(x) = \log(x^2 + 1)$ ? Explain why the graph of  $f$  does not have a vertical asymptote.
4. The subtraction  $3 - 5$  does not have an answer if we require that the answer be positive. Keep this idea in mind as you work the rest of this discussion exercise.

Press the **MODE** key of a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, and choose “Real” from the menu. Now use the calculator to evaluate  $\log(-2)$ . What output is given by the calculator? Press the **MODE** key, and choose “ $a + bi$ ” from the menu. Now use the calculator to evaluate  $\log(-2)$ . What output is given by the calculator? Explain why the output is different for these two evaluations.

 **Exercise Set 7.2**

**In Exercises 1 to 12, write each equation in its exponential form.**

1.  $1 = \log 10$

2.  $4 = \log 10,000$

3.  $2 = \log_8 64$

4.  $3 = \log_4 64$

5.  $0 = \log_7 x$

6.  $-4 = \log_3 \frac{1}{81}$

7.  $\ln x = 4$

8.  $\ln e^2 = 2$

9.  $\ln 1 = 0$

10.  $\ln x = -3$

11.  $2 = \log(3x + 1)$

12.  $\frac{1}{3} = \ln\left(\frac{x+1}{x^2}\right)$

**In Exercises 13 to 24, write each equation in its logarithmic form. Assume  $y > 0$  and  $b > 0$ .**

13.  $3^2 = 9$

14.  $5^3 = 125$

15.  $4^{-2} = \frac{1}{16}$

16.  $10^0 = 1$

17.  $b^x = y$

18.  $2^x = y$

19.  $y = e^x$

20.  $5^1 = 5$

21.  $100 = 10^2$

22.  $2^{-4} = \frac{1}{16}$

23.  $e^2 = x + 5$

24.  $3^x = 47$

**In Exercises 25 to 42, evaluate each logarithm. Do not use a calculator.**

25.  $\log_4 16$

26.  $\log_{3/2} \frac{8}{27}$

27.  $\log_3 \frac{1}{243}$

28.  $\log_b 1$

29.  $\ln e^3$

30.  $\log_b b$

31.  $\log \frac{1}{100}$

32.  $\log 1,000,000$

33.  $\log_{0.5} 16$

34.  $\log_{0.3} \frac{100}{9}$

35.  $4 \log 1000$

36.  $\log_5 125^2$

37.  $2 \log_7 2401$

38.  $3 \log_{11} 161,051$

39.  $\log_3 \sqrt[5]{9}$

40.  $\log_6 \sqrt[3]{36}$

41.  $5 \log_{13} \sqrt[3]{169}$

42.  $2 \log_7 \sqrt[7]{343}$

**In Exercises 43 to 50, graph each function by using its exponential form.**

43.  $f(x) = \log_4 x$

44.  $f(x) = \log_6 x$

45.  $f(x) = \log_{12} x$

46.  $f(x) = \log_8 x$

47.  $f(x) = \log_{1/2} x$

48.  $f(x) = \log_{1/4} x$

49.  $f(x) = \log_{5/2} x$

50.  $f(x) = \log_{7/3} x$

**In Exercises 51 to 64, find the domain of the function. Write the domain using interval notation.**

51.  $f(x) = \log_5(x - 3)$

52.  $k(x) = \log_4(5 - x)$

53.  $k(x) = \log_{2/3}(11 - x)$

54.  $H(x) = \log_{1/4}(x^2 + 1)$

55.  $P(x) = \ln(x^2 - 4)$

56.  $J(x) = \ln\left(\frac{x-3}{x}\right)$

57.  $h(x) = \ln\left(\frac{x^2}{x-4}\right)$

58.  $R(x) = \ln(x^4 - x^2)$

59.  $N(x) = \log_2(x^3 - x)$

60.  $s(x) = \log_7(x^2 + 7x + 10)$

61.  $g(x) = \log \sqrt{2x - 11}$

62.  $m(x) = \log |4x - 8|$

63.  $t(x) = 2 \ln(3x - 7)$

64.  $v(x) = \ln(x - 4)^2$

**In Exercises 65 to 72, use translations of the graphs in Exercises 43 to 50 to produce the graph of the given function.**

65.  $f(x) = \log_4(x - 3)$

66.  $f(x) = \log_6(x + 3)$

67.  $f(x) = \log_{12} x + 2$

68.  $f(x) = \log_8 x - 4$

69.  $f(x) = 3 + \log_{1/2} x$

70.  $f(x) = 2 + \log_{1/4} x$

71.  $f(x) = 1 + \log_{5/2}(x - 4)$

72.  $f(x) = \log_{7/3}(x - 3) - 1$

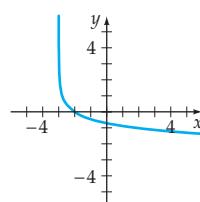
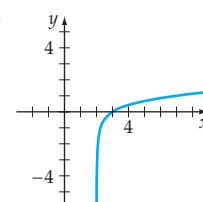
**73.** Examine the following four functions and the graphs labeled **a**, **b**, **c**, and **d**. For each graph, determine which function has been graphed.

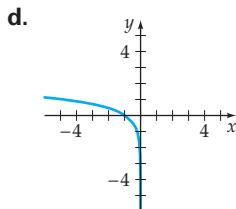
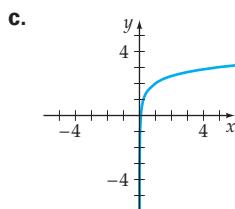
$f(x) = \log_5(x - 2)$

$g(x) = 2 + \log_5 x$

$h(x) = \log_5(-x)$

$k(x) = -\log_5(x + 3)$

**a.****b.**



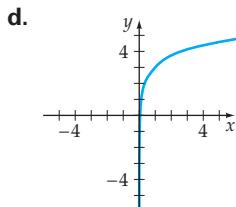
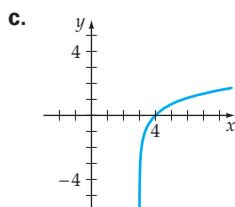
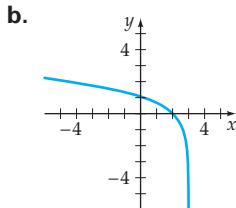
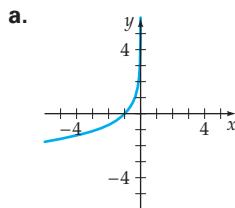
74. Examine the following four functions and the graphs labeled **a**, **b**, **c**, and **d**. For each graph, determine which function has been graphed.

$$f(x) = \ln x + 3$$

$$g(x) = \ln(x - 3)$$

$$h(x) = \ln(3 - x)$$

$$k(x) = -\ln(-x)$$



In Exercises 75 to 84, use a graphing utility to graph the function.

75.  $f(x) = -2 \ln x$

76.  $f(x) = -\log x$

77.  $f(x) = |\ln x|$

78.  $f(x) = \ln|x|$

79.  $f(x) = \log \sqrt[3]{x}$

80.  $f(x) = \ln \sqrt{x}$

81.  $f(x) = \log(x + 10)$

82.  $f(x) = \ln(x + 3)$

83.  $f(x) = 3 \log|2x + 10|$

84.  $f(x) = \frac{1}{2} \ln|x - 4|$

85. **MONEY MARKET RATES** The function

$$r(t) = 0.69607 + 0.60781 \ln t$$

gives the annual interest rate  $r$ , as a percent, a bank will pay on its money market accounts, where  $t$  is the term (the time the money is invested) in months.

- a. What interest rate, to the nearest tenth of a percent, will the bank pay on a money market account with a term of 9 months?
- b. What is the minimum number of complete months during which a person must invest to receive an interest rate of at least 3%?

- 86. AVERAGE TYPING SPEED** The following function models the average typing speed  $S$ , in words per minute, of a student who has been typing for  $t$  months.

$$S(t) = 5 + 29 \ln(t + 1), \quad 0 \leq t \leq 16$$

- a. What was the student's average typing speed, to the nearest word per minute, when the student first started to type? What was the student's average typing speed, to the nearest word per minute, after 3 months?

- b. Use a graph of  $S$  to determine how long, to the nearest tenth of a month, it will take the student to achieve an average typing speed of 65 words per minute.

87. **ADVERTISING COSTS AND SALES** The function

$$N(x) = 2750 + 180 \ln\left(\frac{x}{1000} + 1\right)$$

models the relationship between the dollar amount  $x$  spent on advertising a product and the number of units  $N$  that a company can sell.

- a. Find the number of units that will be sold with advertising expenditures of \$20,000, \$40,000, and \$60,000.
- b. How many units will be sold if the company does not pay to advertise the product?



**MEDICINE** In anesthesiology it is necessary to accurately estimate the body surface area of a patient. One formula for estimating body surface area (BSA) was developed by Edith Boyd (University of Minnesota Press, 1935). Her formula for the BSA (in square meters) of a patient of height  $H$  (in centimeters) and weight  $W$  (in grams) is



$$\text{BSA} = 0.0003207 \cdot H^{0.3} \cdot W^{0.7285 - 0.0188 \log W}$$

In Exercises 88 and 89, use Boyd's formula to estimate the body surface area of a patient with the given weight and height. Round to the nearest hundredth of a square meter.

88.  $W = 110$  pounds (49,895.2 grams);  $H = 5$  feet 4 inches (162.56 centimeters)

89.  $W = 180$  pounds (81,646.6 grams);  $H = 6$  feet 1 inch (185.42 centimeters)



**ASTRONOMY** Astronomers measure the apparent brightness of a star by a unit called the **apparent magnitude**. This unit was created in the second century B.C. when the Greek astronomer Hipparchus classified the relative brightness of several stars. In his list he assigned the number 1 to the stars that appeared to be the brightest (Sirius, Vega, and Deneb). They are

first-magnitude stars. Hipparchus assigned the number 2 to all the stars in the Big Dipper. They are second-magnitude stars. The following table shows the relationship between a star's brightness relative to a first-magnitude star and the star's apparent magnitude. Notice from the table that a first-magnitude star appears to be about 2.51 times as bright as a second-magnitude star.

| Brightness, $x$<br>relative to a<br>first-magnitude<br>star | Apparent<br>magnitude<br>$M(x)$ |
|---|---------------------------------|
| 1   | 1                               |
| $\frac{1}{2.51}$  | 2                               |
| $\frac{1}{6.31} \approx \frac{1}{2.51^2}$                   | 3                               |
| $\frac{1}{15.85} \approx \frac{1}{2.51^3}$                  | 4                               |
| $\frac{1}{39.82} \approx \frac{1}{2.51^4}$                  | 5                               |
| $\frac{1}{100} \approx \frac{1}{2.51^5}$                    | 6                               |

The following logarithmic function gives the apparent magnitude  $M(x)$  of a star as a function of its brightness  $x$ .

$$M(x) = -2.51 \log x + 1, \quad 0 < x \leq 1$$

- a. Use  $M(x)$  to find the apparent magnitude of a star that is  $\frac{1}{10}$  as bright as a first-magnitude star. Round to the nearest hundredth.

b. Find the approximate apparent magnitude of a star that is  $\frac{1}{400}$  as bright as a first-magnitude star. Round to the nearest hundredth.

- 93.**  Use a graphing utility to graph  $f(x) = \frac{e^x - e^{-x}}{2}$  and  $g(x) = \ln(x + \sqrt{x^2 + 1})$  on the same screen. Use a square viewing window. What appears to be the relationship between  $f$  and  $g$ ?

**94.**  Use a graphing utility to graph  $f(x) = \frac{e^x + e^{-x}}{2}$  for  $x \geq 0$  and  $g(x) = \ln(x + \sqrt{x^2 - 1})$  for  $x \geq 1$  on the same screen. Use a square viewing window. What appears to be the relationship between  $f$  and  $g$ ?

- 95.** The functions  $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$  and  $g(x) = \frac{1}{2} \ln \frac{1+x}{1-x}$  are inverse functions. The domain of  $f$  is the set of all real numbers. The domain of  $g$  is  $\{x \mid -1 < x < 1\}$ . Use this information to determine the range of  $f$  and the range of  $g$ .

- 96.** Use a graph of  $f(x) = \frac{2}{e^x + e^{-x}}$  to determine the domain and range of  $f$ .

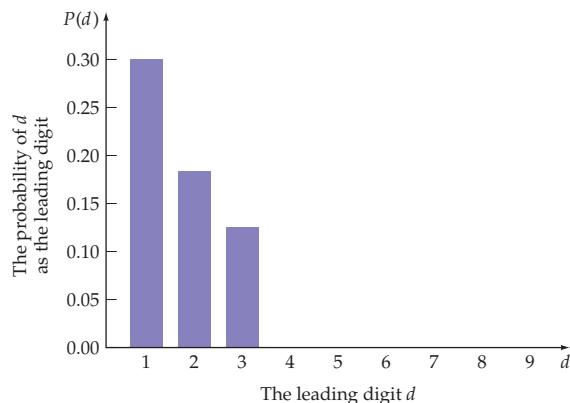
-  **BENFORD'S LAW** The authors of this text know some interesting details about your finances. For instance, of the last 150 checks you have written, about 30% are for amounts that start with the number 1. Also, you have written about 3 times as many checks for amounts that start with the number 2 as you have for amounts that start with the number 7.

We are sure of these results because of a mathematical formula known as **Benford's Law**. This law was first discovered by the mathematician Simon Newcomb in 1881 and then rediscovered by the physicist Frank Benford in 1938. Benford's Law states that the probability  $P$  that the first digit of a number selected from a wide range of numbers is  $d$  is given by

$$P(d) = \log\left(1 + \frac{1}{d}\right)$$

- a. Use Benford's Law to complete the table below and the bar graph at the top of the next column.

| <b>d</b> | $P(d) = \log\left(1 + \frac{1}{d}\right)$ |
|----------|---|
| 1        | 0.301                                     |
| 2        | 0.176                                     |
| 3        | 0.125                                     |
| 4        |   |
| 5        |   |
| 6        |   |
| 7        |   |
| 8        |   |
| 9        |   |



Benford's Law applies to many sets of data with a wide range. For instance, it applies to the populations of the cities in the U.S., the numbers of dollars in the savings accounts at your local bank, and the number of miles driven during a month by each person in a state.

- b. Use the table in a. to find the probability that in a U.S. city selected at random, the number of telephones in that city will be a number starting with 6.
  - c. Use the table in a. to estimate how many times as many purchases you have made for dollar amounts that start with a 1 than for dollar amounts that start with a 9.
  - d.  Explain why Benford's Law would not apply to the set of all the ages, in years, of students at a local high school.

**AN APPLICATION OF BENFORD'S LAW** Benford's Law has been used to identify fraudulent accountants. In most cases these accountants are unaware of Benford's Law and have replaced valid numbers with numbers selected at random. Their numbers do not conform to Benford's Law. Hence an audit is warranted.

## Section 7.3

- Properties of Logarithms
- Change-of-Base Formula
- Logarithmic Scales

# Properties of Logarithms and Logarithmic Scales

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A38.



In Exercises PS1 to PS6, use a calculator to compare the values of the given expressions.

**PS1.**  $\log 3 + \log 2; \log 6$  [7.2]

**PS2.**  $\ln 8 - \ln 3; \ln\left(\frac{8}{3}\right)$  [7.2]

**PS3.**  $3 \log 4; \log(4^3)$  [7.2]

**PS4.**  $2 \ln 5; \ln(5^2)$  [7.2]

**PS5.**  $\ln 5; \frac{\log 5}{\log e}$  [7.2]

**PS6.**  $\log 8; \frac{\ln 8}{\ln 10}$  [7.2]

## Properties of Logarithms

In Section 7.2 we introduced the following basic properties of logarithms.

$$\log_b b = 1 \quad \text{and} \quad \log_b 1 = 0$$

Also, because exponential functions and logarithmic functions are inverses of each other, we observed the relationships

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b x} = x$$

We can use the properties of exponents to establish the following additional logarithmic properties.

### take note

Pay close attention to these properties. Note that

$\log_b(MN) \neq \log_b M \cdot \log_b N$   
and

$$\log_b \frac{M}{N} \neq \frac{\log_b M}{\log_b N}$$

Also,

$$\log_b(M + N) \neq \log_b M + \log_b N$$

In fact, the expression  $\log_b(M + N)$  cannot be expanded at all.

### Properties of Logarithms

In the following properties,  $b$ ,  $M$ , and  $N$  are positive real numbers ( $b \neq 1$ ).

#### Product property

$$\log_b(MN) = \log_b M + \log_b N$$

#### Quotient property

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

#### Power property

$$\log_b(M^p) = p \log_b M$$

#### Logarithm-of-each-side property

$$M = N \quad \text{implies} \quad \log_b M = \log_b N$$

#### One-to-one property

$$\log_b M = \log_b N \quad \text{implies} \quad M = N$$

Here is a proof of the product property.

**Proof:** Let  $r = \log_b M$  and  $s = \log_b N$ . These equations can be written in exponential form as

$$M = b^r \quad \text{and} \quad N = b^s$$

Now consider the product  $MN$ .

$$MN = b^r b^s$$

$$MN = b^{r+s}$$

$$\log_b MN = r + s$$

$$\log_b MN = \log_b M + \log_b N$$

- Substitute for  $M$  and  $N$ .

- Product property of exponents

- Write in logarithmic form.

- Substitute for  $r$  and  $s$ .

The last equation is our desired result. ◆

The quotient property and the power property can be proved in a similar manner. See Exercises 87 and 88 on page 492.

The properties of logarithms are often used to rewrite logarithmic expressions in an equivalent form. The process of using the product or quotient rules to rewrite a single logarithm as the sum or difference of two or more logarithms, or using the power property to rewrite  $\log_b(M^p)$  in its equivalent form  $p \log_b M$ , is called **expanding the logarithmic expression**. We illustrate this process in Example 1.

### EXAMPLE 1 » Expand Logarithmic Expressions

Use the properties of logarithms to expand the following logarithmic expressions. Assume all variable expressions represent positive real numbers. When possible, evaluate logarithmic expressions.

a.  $\log_5(xy^2)$     b.  $\ln\left(\frac{e\sqrt{y}}{z^3}\right)$

#### Solution

|  |   |
|--|---|
| <p>a. <math>\log_5(xy^2) = \log_5 x + \log_5 y^2</math><br/> <math>= \log_5 x + 2 \log_5 y</math></p>  | <ul style="list-style-type: none"> <li>• Product property</li> <li>• Power property</li> </ul>  |
| <p>b. <math>\ln\left(\frac{e\sqrt{y}}{z^3}\right) = \ln(e\sqrt{y}) - \ln z^3</math><br/> <math>= \ln e + \ln \sqrt{y} - \ln z^3</math><br/> <math>= \ln e + \ln y^{1/2} - \ln z^3</math><br/> <math>= \ln e + \frac{1}{2} \ln y - 3 \ln z</math><br/> <math>= 1 + \frac{1}{2} \ln y - 3 \ln z</math></p> | <ul style="list-style-type: none"> <li>• Quotient property</li> <li>• Product property</li> <li>• Write <math>\sqrt{y}</math> as <math>y^{1/2}</math>.</li> <li>• Power property</li> <li>• Evaluate <math>\ln e</math>.</li> </ul> |

» Try Exercise 2, page 489

The properties of logarithms are also used to *condense* expressions that involve the sum or difference of logarithms into a single logarithm. For instance, we can use the product property to rewrite  $\log_b M + \log_b N$  as  $\log_b(MN)$ , and the quotient property to rewrite  $\log_b M - \log_b N$  as  $\log_b \frac{M}{N}$ . Before applying the product or quotient properties, use the power property to write all expressions of the form  $p \log_b M$  in their equivalent  $\log_b M^p$  form. See Example 2.

**QUESTION** Does  $\log 2 + \log 5 = 1$ ?

### EXAMPLE 2 » Condense Logarithmic Expressions

Use the properties of logarithms to rewrite each expression as a single logarithm with a coefficient of 1. Assume all variable expressions represent positive real numbers.

$$\text{a. } 2 \ln x + \frac{1}{2} \ln(x+4) \quad \text{b. } \log_5(x^2 - 4) + 3 \log_5 y - \log_5(x-2)^2$$

#### Solution

$$\begin{aligned}\text{a. } 2 \ln x + \frac{1}{2} \ln(x+4) &= \ln x^2 + \ln(x+4)^{1/2} \\ &= \ln[x^2(x+4)^{1/2}] \\ &= \ln[x^2\sqrt{(x+4)}]\end{aligned}$$

- Power property
- Product property
- Rewriting  $(x+4)^{1/2}$  as  $\sqrt{x+4}$  is an optional step.

$$\begin{aligned}\text{b. } \log_5(x^2 - 4) + 3 \log_5 y - \log_5(x-2)^2 &= \log_5(x^2 - 4) + \log_5 y^3 - \log_5(x-2)^2 \\ &= [\log_5(x^2 - 4) + \log_5 y^3] - \log_5(x-2)^2 \\ &= \log_5[(x^2 - 4)y^3] - \log_5(x-2)^2 \\ &= \log_5\left[\frac{(x^2 - 4)y^3}{(x-2)^2}\right] \\ &= \log_5\left[\frac{(x+2)(x-2)y^3}{(x-2)^2}\right] \\ &= \log_5\left[\frac{(x+2)y^3}{x-2}\right]\end{aligned}$$

- Power property
- Order of Operations Agreement
- Product property
- Quotient property
- Factor.
- Simplify.

» Try Exercise 18, page 490

### Change-of-Base Formula

Recall that to determine the value of  $y$  in  $\log_3 81 = y$ , we ask the question, “What power of 3 is equal to 81?” Because  $3^4 = 81$ , we have  $\log_3 81 = 4$ . Now suppose

**ANSWER** Yes. By the product property,  $\log 2 + \log 5 = \log(2 \cdot 5) = \log 10 = 1$ .

that we need to determine the value of  $\log_3 50$ . In this case we need to find the power of 3 that produces 50. Because  $3^3 = 27$  and  $3^4 = 81$ , the value we are seeking is somewhere between 3 and 4. The following procedure can be used to produce an estimate of  $\log_3 50$ .

The exponential form of  $\log_3 50 = y$  is  $3^y = 50$ . Applying logarithmic properties gives us

$$\begin{aligned} 3^y &= 50 \\ \ln 3^y &= \ln 50 && \bullet \text{ Logarithm-of-each-side property} \\ y \ln 3 &= \ln 50 && \bullet \text{ Power property} \\ y &= \frac{\ln 50}{\ln 3} \approx 3.56088 && \bullet \text{ Solve for } y. \end{aligned}$$

Thus  $\log_3 50 \approx 3.56088$ . In the above procedure we could just as well have used logarithms of any base and arrived at the same value. Thus any logarithm can be expressed in terms of logarithms of any base we wish. This general result is summarized in the following formula.

### Change-of-Base Formula

If  $x$ ,  $a$ , and  $b$  are positive real numbers with  $a \neq 1$  and  $b \neq 1$ , then

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Because most calculators use only common logarithms ( $a = 10$ ) or natural logarithms ( $a = e$ ), the change-of-base formula is used most often in the following form.

If  $x$  and  $b$  are positive real numbers and  $b \neq 1$ , then

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

### EXAMPLE 3 » Use the Change-of-Base Formula

Evaluate each logarithm. Round to the nearest ten thousandth.

- a.  $\log_3 18$     b.  $\log_{12} 400$

#### Solution

To approximate these logarithms, we may use the change-of-base formula with  $a = 10$  or  $a = e$ . For this example we choose to use the change-of-base formula with  $a = e$ . That is, we will evaluate these logarithms by using the **[LN]** key on a scientific or graphing calculator.

a.  $\log_3 18 = \frac{\ln 18}{\ln 3} \approx 2.6309$     b.  $\log_{12} 400 = \frac{\ln 400}{\ln 12} \approx 2.4111$

#### take note

If common logarithms had been used for the calculations in Example 3, the final results would have been the same.

$$\begin{aligned} \log_3 18 &= \frac{\log 18}{\log 3} \approx 2.6309 \\ \log_{12} 400 &= \frac{\log 400}{\log 12} \approx 2.4111 \end{aligned}$$

» Try Exercise 34, page 490

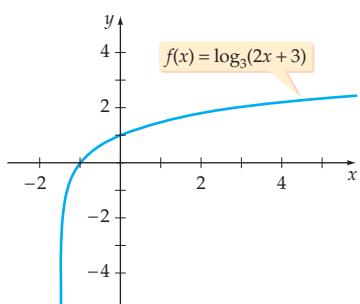


Figure 7.21

The change-of-base formula and a graphing calculator can be used to graph logarithmic functions that have a base other than 10 or  $e$ . For instance, to graph  $f(x) = \log_3(2x + 3)$ , we rewrite the function in terms of base 10 or base  $e$ . Using base 10 logarithms, we have  $f(x) = \log_3(2x + 3) = \frac{\log(2x + 3)}{\log 3}$ . The graph is shown in **Figure 7.21**.

#### EXAMPLE 4 » Use the Change-of-Base Formula to Graph a Logarithmic Function

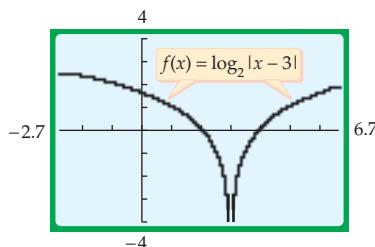


Graph  $f(x) = \log_2|x - 3|$ .

#### Solution

Rewrite  $f$  using the change-of-base formula. We will use the natural logarithm function; however, the common logarithm function could be used instead.

$$f(x) = \log_2|x - 3| = \frac{\ln|x - 3|}{\ln 2}$$



- Enter  $\frac{\ln|x - 3|}{\ln 2}$  into Y1. Note that the domain of  $f(x) = \log_2|x - 3|$  is all real numbers except 3 because  $|x - 3| = 0$  when  $x = 3$  and  $|x - 3|$  is positive for all other values of  $x$ .

#### » Try Exercise 46, page 490

### Logarithmic Scales

Logarithmic functions are often used to scale very large (or very small) numbers into numbers that are easier to comprehend. For instance, the *Richter scale* magnitude of an earthquake uses a logarithmic function to convert the intensity of the earthquake's shock waves  $I$  into a number  $M$ , which for most earthquakes is in the range of 0 to 10. The intensity  $I$  of an earthquake is often given in terms of the constant  $I_0$ , where  $I_0$  is the intensity of the smallest earthquake (called a **zero-level earthquake**) that can be measured on a seismograph near the earthquake's epicenter. The following formula is used to compute the Richter scale magnitude of an earthquake.

#### Math Matters

The Richter scale was created by the seismologist Charles F. Richter in 1935. Notice that a tenfold increase in the intensity level of an earthquake increases the Richter scale magnitude of the earthquake by only 1.

#### The Richter Scale Magnitude of an Earthquake

An earthquake with an intensity of  $I$  has a **Richter scale magnitude** of

$$M = \log\left(\frac{I}{I_0}\right)$$

where  $I_0$  is the measure of the intensity of a zero-level earthquake.

**EXAMPLE 5** » Determine the Magnitude of an Earthquake

Find the Richter scale magnitude (to the nearest 0.1) of the 1999 Joshua Tree, California, earthquake that had an intensity of  $I = 12,589,254I_0$ .

**take note**

Notice in Example 5 that we didn't need to know the value of  $I_0$  to determine the Richter scale magnitude of the quake.

**Solution**

$$M = \log\left(\frac{I}{I_0}\right) = \log\left(\frac{12,589,254I_0}{I_0}\right) = \log(12,589,254) \approx 7.1$$

The 1999 Joshua Tree earthquake had a Richter scale magnitude of 7.1.

» Try Exercise 78, page 491

If you know the Richter scale magnitude of an earthquake, you can determine the intensity of the earthquake.

**EXAMPLE 6** » Determine the Intensity of an Earthquake

Find the intensity of the 1999 Taiwan earthquake, which measured 7.6 on the Richter scale.

**Solution**

$$\log\left(\frac{I}{I_0}\right) = 7.6$$

$$\frac{I}{I_0} = 10^{7.6}$$

• Write in exponential form.

$$I = 10^{7.6}I_0$$

• Solve for  $I$ .

$$I \approx 39,810,717I_0$$

The 1999 Taiwan earthquake had an intensity that was approximately 39,811,000 times the intensity of a zero-level earthquake.

» Try Exercise 80, page 491

In Example 7 we make use of the Richter scale magnitudes of two earthquakes to compare the intensities of the earthquakes.

**EXAMPLE 7** » Compare Earthquakes

The 1960 Chile earthquake had a Richter scale magnitude of 9.5. The 1989 San Francisco earthquake had a Richter scale magnitude of 7.1. Compare the intensities of the earthquakes.

**Continued ►**

**take note**

The results of Example 7 show that if an earthquake has a Richter scale magnitude of  $M_1$  and a smaller earthquake has a Richter scale magnitude of  $M_2$ , then the larger earthquake is  $10^{M_1 - M_2}$  times as intense as the smaller earthquake.

**Solution**

Let  $I_1$  be the intensity of the Chilean earthquake and  $I_2$  the intensity of the San Francisco earthquake. Then

$$\log\left(\frac{I_1}{I_0}\right) = 9.5 \quad \text{and} \quad \log\left(\frac{I_2}{I_0}\right) = 7.1$$

$$\frac{I_1}{I_0} = 10^{9.5} \quad \frac{I_2}{I_0} = 10^{7.1}$$

$$I_1 = 10^{9.5}I_0 \quad I_2 = 10^{7.1}I_0$$

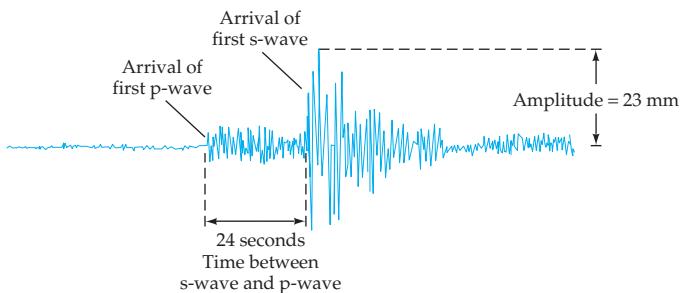
To compare the intensities of the earthquakes, we compute the ratio  $I_1/I_2$ .

$$\frac{I_1}{I_2} = \frac{10^{9.5}I_0}{10^{7.1}I_0} = \frac{10^{9.5}}{10^{7.1}} = 10^{9.5-7.1} = 10^{2.4} \approx 251$$

The earthquake in Chile was approximately 251 times as intense as the San Francisco earthquake.

» Try Exercise 82, page 491

Seismologists generally determine the Richter scale magnitude of an earthquake by examining a **seismogram**. See **Figure 7.22**.



**Figure 7.22**

The magnitude of an earthquake cannot be determined just by examining the amplitude of a seismogram because this amplitude decreases as the distance between the epicenter of the earthquake and the observation station increases. To account for the distance between the epicenter and the observation station, a seismologist examines a seismogram for both small waves called **p-waves** and larger waves called **s-waves**. The Richter scale magnitude  $M$  of the earthquake is a function of both the amplitude  $A$  of the s-waves and the difference in time  $t$  between the occurrence of the s-waves and the occurrence of the p-waves. In the 1950s, Charles Richter developed the following formula to determine the magnitude of an earthquake from the data in a seismogram.

**Amplitude-Time-Difference Formula**

The Richter scale magnitude  $M$  of an earthquake is given by

$$M = \log A + 3 \log 8t - 2.92$$

where  $A$  is the amplitude, in millimeters, of the s-waves on a seismogram and  $t$  is the difference in time, in seconds, between the s-waves and the p-waves.

**EXAMPLE 8** » **Determine the Magnitude of an Earthquake from Its Seismogram**

Find the Richter scale magnitude of the earthquake that produced the seismogram in **Figure 7.22**.

**Solution**

$$\begin{aligned} M &= \log A + 3 \log 8t - 2.92 \\ &= \log 23 + 3 \log[8 \cdot 24] - 2.92 && \text{• Substitute 23 for } A \text{ and 24 for } t. \\ &\approx 1.36173 + 6.84990 - 2.92 \\ &\approx 5.3 \end{aligned}$$

The earthquake had a magnitude of about 5.3 on the Richter scale.

**take note**

The Richter scale magnitude is usually rounded to the nearest tenth.

» **Try Exercise 86, page 492**

Logarithmic scales are also used in chemistry. One example concerns the pH of a liquid, which is a measure of the liquid's **acidity** or **alkalinity**. (You may have tested the pH of the water in a swimming pool or an aquarium.) Pure water, which is considered neutral, has a pH of 7.0. The pH scale ranges from 0 to 14, with 0 corresponding to the most acidic solutions and 14 to the most alkaline. Lemon juice has a pH of about 2, whereas household ammonia measures about 11.

Specifically, the pH of a solution is a function of the hydronium-ion concentration of the solution. Because the hydronium-ion concentration of a solution can be very small (with values such as  $0.00000001$ ), pH uses a logarithmic scale.

**take note**

One mole is equivalent to  $6.022 \times 10^{23}$  ions.

**Definition of the pH of a Solution**

The **pH of a solution** with a hydronium-ion concentration of  $\text{H}^+$  moles per liter is given by

$$\text{pH} = -\log[\text{H}^+]$$

**EXAMPLE 9** » Find the pH of a Solution

Find the pH of each liquid. Round to the nearest tenth.

- Orange juice with  $\text{H}^+ = 2.8 \times 10^{-4}$  mole per liter
- Milk with  $\text{H}^+ = 3.97 \times 10^{-7}$  mole per liter
- Rainwater with  $\text{H}^+ = 6.31 \times 10^{-5}$  mole per liter
- A baking soda solution with  $\text{H}^+ = 3.98 \times 10^{-9}$  mole per liter

**Math Matters**

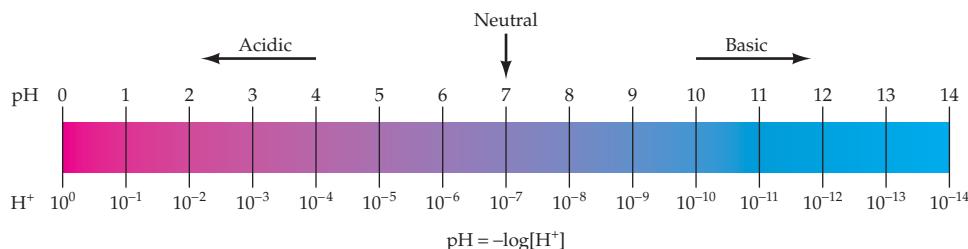
The pH scale was created by the Danish biochemist Søren Sørensen in 1909 to measure the acidity of water used in the brewing of beer. pH is an abbreviation for pondus hydrogenii, which translates as "potential hydrogen."

**Solution**

- $\text{pH} = -\log[\text{H}^+] = -\log(2.8 \times 10^{-4}) \approx 3.6$   
The orange juice has a pH of 3.6.
- $\text{pH} = -\log[\text{H}^+] = -\log(3.97 \times 10^{-7}) \approx 6.4$   
The milk has a pH of 6.4.
- $\text{pH} = -\log[\text{H}^+] = -\log(6.31 \times 10^{-5}) \approx 4.2$   
The rainwater has a pH of 4.2.
- $\text{pH} = -\log[\text{H}^+] = -\log(3.98 \times 10^{-9}) \approx 8.4$   
The baking soda solution has a pH of 8.4.

» Try Exercise 70, page 491

**Figure 7.23** illustrates the pH scale, along with the corresponding hydronium-ion concentrations. A solution on the left half of the scale, with a pH of less than 7, is an **acid**, and a solution on the right half of the scale is an **alkaline solution** or a **base**. Because the scale is logarithmic, a solution with a pH of 5 is 10 times more acidic than a solution with a pH of 6. From Example 9 we see that the orange juice, rainwater, and milk are acids, whereas the baking soda solution is a base.



**Figure 7.23**

**EXAMPLE 10** » Find the Hydronium-Ion Concentration

A sample of blood has a pH of 7.3. Find the hydronium-ion concentration of the blood.

**Solution**

$$\begin{aligned} \text{pH} &= -\log[\text{H}^+] \\ 7.3 &= -\log[\text{H}^+] \\ -7.3 &= \log[\text{H}^+] \\ 10^{-7.3} &= \text{H}^+ \\ 5.0 \times 10^{-8} &\approx \text{H}^+ \end{aligned}$$

- Substitute 7.3 for pH.
- Multiply both sides by  $-1$ .
- Change to exponential form.

The hydronium-ion concentration of the blood is about  $5.0 \times 10^{-8}$  mole per liter.

» Try Exercise 72, page 491


**Topics for Discussion**

1. The function  $f(x) = \log_b x$  is defined only for  $x > 0$ . Explain why this condition is imposed.
2. If  $p$  and  $q$  are positive numbers, explain why  $\ln(p + q)$  isn't normally equal to  $\ln p + \ln q$ .
3. If  $f(x) = \log_b x$  and  $f(c) = f(d)$ , can we conclude that  $c = d$ ?
4. Give examples of situations in which it is advantageous to use logarithmic scales.


**Exercise Set 7.3**

In Exercises 1 to 16, expand the given logarithmic expression. Assume all variable expressions represent positive real numbers. When possible, evaluate logarithmic expressions. Do not use a calculator.

1.  $\log_b(xyz)$

» 2.  $\ln \frac{z^3}{\sqrt{xy}}$

3.  $\ln \frac{x}{z^4}$

4.  $\log_5 \frac{xy^2}{z^4}$

5.  $\log_2 \frac{\sqrt{x}}{y^3}$

6.  $\log_b(x\sqrt[3]{y})$

7.  $\log_7 \frac{\sqrt{xz}}{y^2}$

9.  $\ln(e^2z)$

11.  $\log_4 \left( \frac{\sqrt[3]{z}}{16y^3} \right)$

13.  $\log \sqrt{x\sqrt{z}}$

15.  $\ln(\sqrt[3]{z\sqrt{e}})$

8.  $\ln \sqrt[3]{x^2\sqrt{y}}$

10.  $\ln(x^{1/2}y^{2/3})$

12.  $\log_5 \left( \frac{\sqrt{xz^4}}{125} \right)$

14.  $\ln \left( \frac{\sqrt[3]{x^2}}{z^2} \right)$

16.  $\ln \left[ \frac{x^2\sqrt{z}}{y^{-3}} \right]$

In Exercises 17 to 32, write each expression as a single logarithm with a coefficient of 1. Assume all variable expressions represent positive real numbers.

17.  $\log(x + 5) + 2 \log x$

18.  $3 \log_2 t - \frac{1}{3} \log_2 u + 4 \log_2 v$

19.  $\ln(x^2 - y^2) - \ln(x - y)$

20.  $\frac{1}{2} \log_8(x + 5) - 3 \log_8 y$

21.  $3 \log x + \frac{1}{3} \log y + \log(x + 1)$

22.  $\ln(xz) - \ln(x\sqrt{y}) + 2 \ln \frac{y}{z}$

23.  $\log(xy^2) - \log z$

24.  $\ln(y^{1/2}z) - \ln z^{1/2}$

25.  $2(\log_6 x + \log_6 y^2) - \log_6(x + 2)$

26.  $\frac{1}{2} \log_3 x - \log_3 y + 2 \log_3(x + 2)$

27.  $2 \ln(x + 4) - \ln x - \ln(x^2 - 3)$

28.  $\log(3x) - (2 \log x - \log y)$

29.  $\ln(2x + 5) - \ln y - 2 \ln z + \frac{1}{2} \ln w$

30.  $\log_b x + \log_b(y + 3) + \log_b(y + 2) - \log_b(y^2 + 5y + 6)$

31.  $\ln(x^2 - 9) - 2 \ln(x - 3) + 3 \ln y$

32.  $\log_b(x^2 + 7x + 12) - 2 \log_b(x + 4)$

In Exercises 33 to 44, use the change-of-base formula to approximate the logarithm accurate to the nearest ten thousandths.

33.  $\log_7 20$

34.  $\log_5 37$

35.  $\log_{11} 8$

36.  $\log_{50} 22$

37.  $\log_6 \frac{1}{3}$

38.  $\log_3 \frac{7}{8}$

39.  $\log_9 \sqrt{17}$

40.  $\log_4 \sqrt[3]{7}$

41.  $\log_{\sqrt{2}} 17$

43.  $\log_{\pi} e$

42.  $\log_{\sqrt{3}} 5.5$

44.  $\log_{\pi} \sqrt{15}$



In Exercises 45 to 52, use a graphing utility and the change-of-base formula to graph the logarithmic function.

45.  $f(x) = \log_4 x$

46.  $g(x) = \log_8(5 - x)$

47.  $g(x) = \log_8(x - 3)$

48.  $t(x) = \log_9(5 - x)$

49.  $h(x) = \log_3(x - 3)^2$

50.  $J(x) = \log_{12}(-x)$

51.  $F(x) = -\log_5 |x - 2|$

52.  $n(x) = \log_2 \sqrt{x - 8}$

In Exercises 53 to 62, determine if the statement is true or false for all  $x > 0, y > 0$ . If it is false, write an example that disproves the statement.

53.  $\log_b(x + y) = \log_b x + \log_b y$

54.  $\log_b(xy) = \log_b x \cdot \log_b y$

55.  $\log_b(xy) = \log_b x + \log_b y$

56.  $\log_b x \cdot \log_b y = \log_b x + \log_b y$

57.  $\log_b x - \log_b y = \log_b(x - y), \quad x > y$

58.  $\log_b \frac{x}{y} = \frac{\log_b x}{\log_b y}$

59.  $\frac{\log_b x}{\log_b y} = \log_b x - \log_b y$

60.  $\log_b(x^n) = n \log_b x$

61.  $(\log_b x)^n = n \log_b x$

62.  $\log_b \sqrt{x} = \frac{1}{2} \log_b x$

63. Evaluate the following without using a calculator.

$$\log_3 5 \cdot \log_5 7 \cdot \log_7 9$$

64. Evaluate the following without using a calculator.

$$\log_5 20 \cdot \log_{20} 60 \cdot \log_{60} 100 \cdot \log_{100} 125$$

65. Which is larger,  $500^{501}$  or  $506^{500}$ ? These numbers are too large for most calculators to handle. (They each have 1353 digits!) (Hint: Compare the logarithms of each number.)

- 66.** Which number is smaller,  $\frac{1}{50^{300}}$  or  $\frac{1}{151^{233}}$ ? See hint in Exercise 65.

- 67. ANIMATED MAPS** A software company that creates interactive maps for websites has designed an animated zooming feature such that when a user selects the zoom-in option, the map appears to expand on a location. This is accomplished by displaying several intermediate maps to give the illusion of motion. The company has determined that zooming in on a location is more informative and pleasing to observe when the scale of each step of the animation is determined using the equation

$$S_n = S_0 \cdot 10^{\frac{n}{N}(\log S_f - \log S_0)}$$

where  $S_n$  represents the scale of the current step  $n$  ( $n = 0$  corresponds to the initial scale),  $S_0$  is the starting scale of the map,  $S_f$  is the final scale, and  $N$  is the number of steps in the animation following the initial scale. (If the initial scale of the map is 1:200, then  $S_0 = 200$ .) Determine the scales to be used at each intermediate step if a map is to start with a scale of 1:1,000,000 and proceed through five intermediate steps to end with a scale of 1:500,000.

- 68. ANIMATED MAPS** Use the equation in Exercise 67 to determine the scales for each stage of an animated map zoom that goes from a scale of 1:250,000 to a scale of 1:100,000 in four steps (following the initial scale).

- 69. pH** Milk of magnesia has a hydronium-ion concentration of about  $3.97 \times 10^{-11}$  mole per liter. Determine the pH of milk of magnesia and state whether milk of magnesia is an acid or a base.

- 70. pH** Vinegar has a hydronium-ion concentration of  $1.26 \times 10^{-3}$  mole per liter. Determine the pH of vinegar and state whether vinegar is an acid or a base.

- 71. HYDRONIUM-ION CONCENTRATION** A morphine solution has a pH of 9.5. Determine the hydronium-ion concentration of the morphine solution.

- 72. HYDRONIUM-ION CONCENTRATION** A rainstorm in New York City produced rainwater with a pH of 5.6. Determine the hydronium-ion concentration of the rainwater.

 **DECIBEL LEVEL** The range of sound intensities that the human ear can detect is so large that a special decibel scale (named after Alexander Graham Bell) is used to measure and compare sound intensities. The decibel level  $dB$  of a sound is given by

$$dB(I) = 10 \log \left( \frac{I}{I_0} \right)$$

where  $I_0$  is the intensity of sound that is barely audible to the human ear. Use the decibel level formula to work exercises 73–76.

- 73.** Find the decibel level for the following sounds. Round to the nearest tenth of a decibel.

| <i>Sound</i>          | <i>Intensity</i>                    |
|-----------------------|-------------------------------------|
| a. Automobile traffic | $I = 1.58 \times 10^8 \cdot I_0$    |
| b. Quiet conversation | $I = 10,800 \cdot I_0$              |
| c. Fender guitar      | $I = 3.16 \times 10^{11} \cdot I_0$ |
| d. Jet engine         | $I = 1.58 \times 10^{15} \cdot I_0$ |

- 74. COMPARISON OF SOUND INTENSITIES** A team in Arizona installed a 48,000-watt sound system in a Ford Bronco that it claims can output 175-decibel sound. The human pain threshold for sound is 125 decibels. How many times as great is the intensity of the sound from the Bronco than the human pain threshold?

- 75. COMPARISON OF SOUND INTENSITIES** How many times as great is the intensity of a sound that measures 120 decibels than a sound that measures 110 decibels?

- 76. DECIBEL LEVEL** If the intensity of a sound is doubled, what is the increase in the decibel level? (Hint: Find  $dB(2I) - dB(I)$ .)

- 77. EARTHQUAKE MAGNITUDE** What is the Richter scale magnitude of an earthquake with an intensity of  $I = 100,000I_0$ ?

-  **EARTHQUAKE MAGNITUDE** The Colombia earthquake of 1906 had an intensity of  $I = 398,107,000I_0$ . What did it measure on the Richter scale?

-  **EARTHQUAKE INTENSITY** The Coalinga, California, earthquake of 1983 had a Richter scale magnitude of 6.5. Find the intensity of this earthquake.

-  **EARTHQUAKE INTENSITY** The earthquake that occurred just south of Concepción, Chile, in 1960 had a Richter scale magnitude of 9.5. Find the intensity of this earthquake.

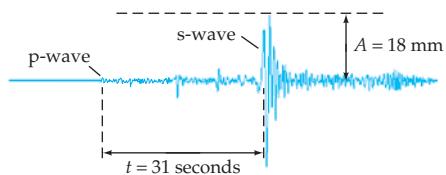
- 81. COMPARISON OF EARTHQUAKES** Compare the intensity of an earthquake that measures 5.0 on the Richter scale to the intensity of an earthquake that measures 3.0 on the Richter scale by finding the ratio of the larger intensity to the smaller intensity.

-  **COMPARISON OF EARTHQUAKES** How many times as great was the intensity of the 1960 earthquake in Chile, which measured 9.5 on the Richter scale, than the San Francisco earthquake of 1906, which measured 8.3 on the Richter scale?

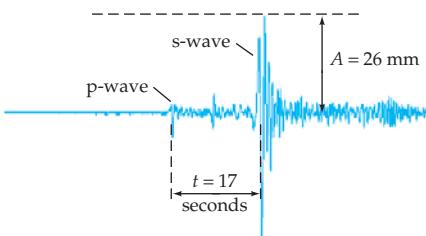
- 83.**  **COMPARISON OF EARTHQUAKES** On March 2, 1933, an earthquake of magnitude 8.9 on the Richter scale struck Japan. In October 1989, an earthquake of magnitude 7.1 on the Richter scale struck San Francisco, California. Compare the intensity of the larger earthquake to the intensity of the smaller earthquake by finding the ratio of the larger intensity to the smaller intensity.

**84.**  **COMPARISON OF EARTHQUAKES** An earthquake that occurred in China in 1978 measured 8.2 on the Richter scale. In 1988, an earthquake in California measured 6.9 on the Richter scale. Compare the intensity of the larger earthquake to the intensity of the smaller earthquake by finding the ratio of the larger intensity to the smaller intensity.

**85.** **EARTHQUAKE MAGNITUDE** Find the Richter scale magnitude of the earthquake that produced the seismogram in the following figure.



86. **EARTHQUAKE MAGNITUDE** Find the Richter scale magnitude of the earthquake that produced the seismogram in the following figure.



- 87.** Prove the quotient property of logarithms:

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

(Hint: See the proof of the product property of logarithms on page 481.)

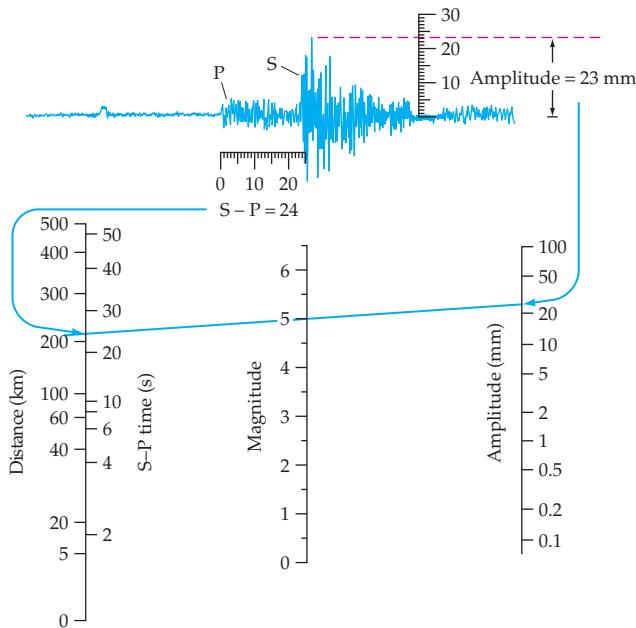
- 88.** Prove the power property of logarithms:

$$\log_b(M^p) = p \log_b M$$

See the hint given in exercise 87.

- 89. NOMOGRAMS AND LOGARITHMIC SCALES** A nomogram is a diagram used to determine a numerical result by drawing a line across numerical scales. The following nomogram, used by Richter, determines the magnitude of an earthquake from its seismogram. To use the nomogram, mark the amplitude of a seismogram on the amplitude scale and mark the time between the s-wave and the p-wave on the S-P scale. Draw a line between these marks. The Richter scale magnitude of the earthquake that produced the seismogram is shown by the intersection of the line and the center scale. The nomogram on the right shows that an earthquake with a seismogram amplitude of 23 millimeters and an S-P time of 24 seconds has a Richter scale magnitude of about 5.

The amplitude and the S-P time are shown on logarithmic scales. On the amplitude scale, the distance from 1 to 10 is the same as the distance from 10 to 100, because  $\log 100 - \log 10 = \log 10 - \log 1$ .



## Richter's earthquake nomogram

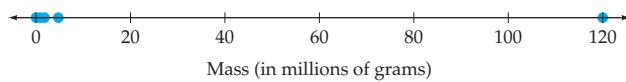
Use the nomogram on page 492 to determine the Richter scale magnitude of an earthquake with a seismogram that has

- a. amplitude of 50 millimeters and S-P time of 40 seconds

- b. amplitude of 1 millimeter and S-P time of 30 seconds
  - c.  How do the results in a. and b. compare with the Richter scale magnitude produced by using the amplitude-time-difference formula on page 487?

- 1.**  **LOGARITHMIC SCALES** Sometimes **logarithmic scales** are used to better view a collection of data that span a wide range of values. For instance, consider the table below, which lists the approximate masses of various marine creatures in grams. Next we have attempted to plot the masses on a number line.

| Animal             | Mass (g)    |
|--------------------|-------------|
| Rotifer            | 0.000000006 |
| Dwarf goby         | 0.30        |
| Lobster            | 15,900      |
| Leatherback turtle | 851,000     |
| Giant squid        | 1,820,000   |
| Whale shark        | 4,700,000   |
| Blue whale         | 120,000,000 |



As you can see, we had to use such a large span of numbers that the data for most of the animals are bunched up at the left. Visually, this number line isn't very helpful for comparisons.

- a. Make a new number line, this time plotting the logarithm (base 10) of each of the masses.
  - b. Which number line is more helpful to compare the masses of the different animals?
  - c. If the data points for two animals on the logarithmic number line are 1 unit apart, how do the animals' masses compare? What if the points are 2 units apart?

- 2.**   **LOGARITHMIC SCALES** The distances of the planets in our solar system from the sun are given in the table in the next column.

| <b>Planet</b> | <b>Distance<br/>(million km)</b> |
|---------------|----------------------------------|
| Mercury       | 58                               |
| Venus         | 108                              |
| Earth         | 150                              |
| Mars          | 228                              |
| Jupiter       | 778                              |
| Saturn        | 1427                             |
| Uranus        | 2871                             |
| Neptune       | 4497                             |

- a. Draw a number line with an appropriate scale to plot the distances.
  - b. Draw a second number line, this time plotting the logarithm (base 10) of each distance.
  - c. Which number line do you find more helpful to compare the different distances?
  - d. If two distances are 3 units apart on the logarithmic number line, how do the distances of the corresponding planets compare?

- 3.**  **BIOLOGIC DIVERSITY** To discuss the variety of species that live in a certain environment, a biologist needs a precise definition of *diversity*. Let  $p_1, p_2, \dots, p_n$  be the proportions of  $n$  species that live in an environment. The biologic diversity  $D$  of this system is

$$D = -(p_1 \log_2 p_1 + p_2 \log_2 p_2 + \cdots + p_n \log_2 p_n)$$

Suppose that an ecosystem has exactly five different varieties of grass: rye (R), bermuda (B), blue (L), fescue (F), and St. Augustine (A).

- a. Calculate the diversity of this ecosystem if the proportions of these grasses are as shown in Table 1. Round to the nearest hundredth.

**Table 1**

| R             | B             | L             | F             | A             |
|---------------|---------------|---------------|---------------|---------------|
| $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |

- b. Because bermuda and St. Augustine are virulent grasses, after a time the proportions will be as shown in Table 2. Calculate the diversity of this system. Does this system have more or less diversity than the system given in Table 1?

**Table 2**

| R             | B             | L              | F             | A              |
|---------------|---------------|----------------|---------------|----------------|
| $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{5}{16}$ |

- c. After an even longer time period, the bermuda and St. Augustine grasses completely overrun the

environment and the proportions are as shown in Table 3. Calculate the diversity of this system. (Note: Although the equation is not technically correct, for purposes of the diversity definition, we may say that  $0 \log_2 0 = 0$ . By using very small values of  $p_i$ , we can demonstrate that this definition makes sense.) Does this system have more or less diversity than the system given in Table 2?

**Table 3**

| R | B             | L | F | A             |
|---|---------------|---|---|---------------|
| 0 | $\frac{1}{4}$ | 0 | 0 | $\frac{3}{4}$ |

- d. Finally, the St. Augustine grasses overrun the bermuda grasses and the proportions are as shown in Table 4. Calculate the diversity of this system. Write a sentence that explains the meaning of the value you obtained.

**Table 4**

| R | B | L | F | A |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |

## Section 7.4

- Solve Exponential Equations
- Solve Logarithmic Equations

## Exponential and Logarithmic Equations

### PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A39.

- PS1.** Use the definition of a logarithm to write the exponential equation  $3^6 = 729$  in logarithmic form. [7.2]
- PS2.** Use the definition of a logarithm to write the logarithmic equation  $\log_5 625 = 4$  in exponential form. [7.2]
- PS3.** Use the definition of a logarithm to write the exponential equation  $a^{x+2} = b$  in logarithmic form. [7.2]
- PS4.** Solve for  $x$ :  $4a = 7bx + 2cx$ . [1.1]
- PS5.** Solve for  $x$ :  $165 = \frac{300}{1 + 12x}$ . [1.1]
- PS6.** Solve for  $x$ :  $A = \frac{100 + x}{100 - x}$ . [1.1]

### Solve Exponential Equations

If a variable appears in the exponent of a term of an equation, such as in  $2^{x+1} = 32$ , then the equation is called an **exponential equation**. Example 1 uses the following Equality of Exponents Theorem to solve  $2^{x+1} = 32$ .

**Equality of Exponents Theorem**

If  $b^x = b^y$ , then  $x = y$ , provided  $b > 0$  and  $b \neq 1$ .

**EXAMPLE 1 »» Solve an Exponential Equation**

Use the Equality of Exponents Theorem to solve  $2^{x+1} = 32$ .

**Solution**

$$2^{x+1} = 32$$

$$2^{x+1} = 2^5$$

$$x + 1 = 5$$

$$x = 4$$

- Write each side as a power of 2.

- Equate the exponents.

- Solve for x.

*Check:* Let  $x = 4$ . Then  $2^{x+1} = 2^{4+1}$

$$= 2^5$$

$$= 32$$

**»» Try Exercise 2, page 501**

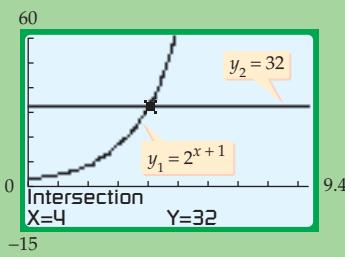
A graphing utility can also be used to find the solutions of an equation of the form  $f(x) = g(x)$ . Either of the following two methods can be employed.

**Using a Graphing Utility to Find the Solutions of**

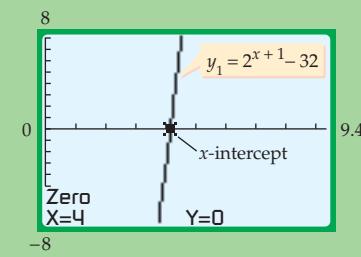
*Intersection Method* Graph  $y_1 = f(x)$  and  $y_2 = g(x)$  on the same screen. The solutions of  $f(x) = g(x)$  are the  $x$ -coordinates of the points of intersection of the graphs.

*Intercept Method* The solutions of  $f(x) = g(x)$  are the  $x$ -coordinates of the  $x$ -intercepts of the graph of  $y = f(x) - g(x)$ .

**Figures 7.24 and 7.25** illustrate the graphical methods for solving  $2^{x+1} = 32$ .



Intersection method



Intercept method

**Figure 7.24**

**Figure 7.25**

In Example 1 we were able to write both sides of the equation as a power of the same base. If you find it difficult to write both sides of an exponential equation in terms of the same base, then try the procedure of taking the logarithm of each side of the equation. This procedure is used in Example 2.

**EXAMPLE 2** » Solve an Exponential Equation

Solve:  $5^x = 40$

**ALGEBRAIC SOLUTION**

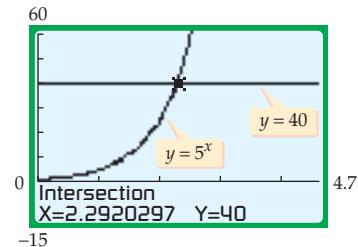
$$\begin{aligned} 5^x &= 40 \\ \log(5^x) &= \log 40 \\ x \log 5 &= \log 40 \\ x &= \frac{\log 40}{\log 5} \\ x &\approx 2.3 \end{aligned}$$

- Take the logarithm of each side.
- Power property
- Exact solution
- Decimal approximation

To the nearest tenth, the solution is 2.3.

**VISUALIZE THE SOLUTION**

*Intersection Method* The solution of  $5^x = 40$  is the  $x$ -coordinate of the point of intersection of  $y = 5^x$  and  $y = 40$ .

**Try Exercise 10, page 501**

An alternative approach to solving the equation in Example 2 is to rewrite the exponential equation in logarithmic form:  $5^x = 40$  is equivalent to the logarithmic equation  $\log_5 40 = x$ . Using the change-of-base formula, we find that  $x = \log_5 40 = \frac{\log 40}{\log 5}$ . In the following example, however, we must take logarithms of both sides to reach a solution.

**EXAMPLE 3** » Solve an Exponential Equation

Solve:  $3^{2x-1} = 5^{x+2}$

**ALGEBRAIC SOLUTION**

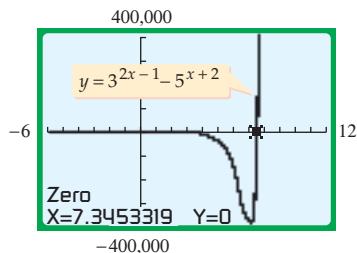
$$\begin{aligned} 3^{2x-1} &= 5^{x+2} \\ \ln 3^{2x-1} &= \ln 5^{x+2} \\ (2x-1) \ln 3 &= (x+2) \ln 5 \\ 2x \ln 3 - \ln 3 &= x \ln 5 + 2 \ln 5 \\ 2x \ln 3 - x \ln 5 &= 2 \ln 5 + \ln 3 \\ x(2 \ln 3 - \ln 5) &= 2 \ln 5 + \ln 3 \\ x &= \frac{2 \ln 5 + \ln 3}{2 \ln 3 - \ln 5} \\ x &\approx 7.3 \end{aligned}$$

- Take the natural logarithm of each side.
- Power property
- Distributive property
- Solve for  $x$ .
- Factor.
- Exact solution
- Decimal approximation

To the nearest tenth, the solution is 7.3.

**VISUALIZE THE SOLUTION**

*Intercept Method* The solution of  $3^{2x-1} = 5^{x+2}$  is the  $x$ -coordinate of the  $x$ -intercept of  $y = 3^{2x-1} - 5^{x+2}$ .

**Try Exercise 18, page 501**

In Example 4 we solve an exponential equation that has two solutions.

### EXAMPLE 4 »» Solve an Exponential Equation Involving $b^x + b^{-x}$

Solve:  $\frac{2^x + 2^{-x}}{2} = 3$

#### ALGEBRAIC SOLUTION

Multiplying each side by 2 produces

$$\begin{aligned} 2^x + 2^{-x} &= 6 \\ 2^{2x} + 2^0 &= 6(2^x) \end{aligned}$$

$$\begin{aligned} (2^x)^2 - 6(2^x) + 1 &= 0 \\ (u)^2 - 6(u) + 1 &= 0 \end{aligned}$$

- Multiply each side by  $2^x$  to clear negative exponents.
- Write in quadratic form.
- Substitute  $u$  for  $2^x$ .

By the quadratic formula,

$$u = \frac{6 \pm \sqrt{36 - 4}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

$$2^x = 3 \pm 2\sqrt{2}$$

$$\log 2^x = \log(3 \pm 2\sqrt{2})$$

$$x \log 2 = \log(3 \pm 2\sqrt{2})$$

$$x = \frac{\log(3 \pm 2\sqrt{2})}{\log 2} \approx \pm 2.54$$

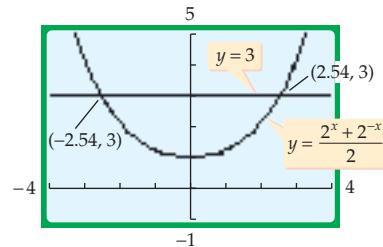
- Replace  $u$  with  $2^x$ .
- Take the common logarithm of each side.
- Power property
- Solve for  $x$ .

The approximate solutions are  $-2.54$  and  $2.54$ .

»» Try Exercise 42, page 501

#### VISUALIZE THE SOLUTION

*Intersection Method* The solutions of  $\frac{2^x + 2^{-x}}{2} = 3$  are the  $x$ -coordinates of the points of intersection of  $y = \frac{2^x + 2^{-x}}{2}$  and  $y = 3$ .



### Solve Logarithmic Equations

Equations that involve logarithms are called **logarithmic equations**. The properties of logarithms, along with the definition of a logarithm, are often used to find the solutions of a logarithmic equation.

### EXAMPLE 5 »» Solve a Logarithmic Equation

Solve:  $\log(3x - 5) = 2$

#### Solution

$$\begin{aligned} \log(3x - 5) &= 2 \\ 3x - 5 &= 10^2 \\ 3x &= 105 \\ x &= 35 \end{aligned}$$

- Definition of a logarithm
- Solve for  $x$ .

Check:  $\log[3(35) - 5] = \log 100 = 2$

»» Try Exercise 22, page 501

**EXAMPLE 6** » Solve a Logarithmic Equation

Solve:  $\log 2x - \log(x - 3) = 1$

**Solution**

$$\log 2x - \log(x - 3) = 1$$

$$\log \frac{2x}{x - 3} = 1$$

$$\frac{2x}{x - 3} = 10^1$$

$$2x = 10x - 30$$

$$-8x = -30$$

- Quotient property

- Definition of a logarithm

- Multiply each side by  $x - 3$ .

- Solve for  $x$ .

$$x = \frac{15}{4}$$

Check the solution by substituting  $\frac{15}{4}$  into the original equation.

» Try Exercise 26, page 501

In Example 7 we make use of the one-to-one property of logarithms to find the solution of a logarithmic equation.

**EXAMPLE 7** » Solve a Logarithmic Equation

Solve:  $\ln(3x + 8) = \ln(2x + 2) + \ln(x - 2)$

**ALGEBRAIC SOLUTION**

$$\ln(3x + 8) = \ln(2x + 2) + \ln(x - 2)$$

- Product property

$$\ln(3x + 8) = \ln[(2x + 2)(x - 2)]$$

$$\ln(3x + 8) = \ln(2x^2 - 2x - 4)$$

- One-to-one property of logarithms

$$3x + 8 = 2x^2 - 2x - 4$$

- Subtract  $3x - 8$  from each side.

$$0 = 2x^2 - 5x - 12$$

- Factor.

$$0 = (2x + 3)(x - 4)$$

- Solve for  $x$ .

$$x = -\frac{3}{2} \quad \text{or} \quad x = 4$$

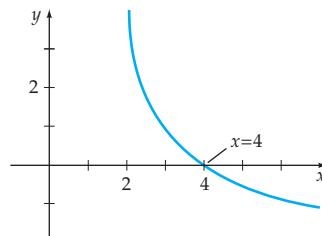
A check will show that 4 is a solution, but  $-\frac{3}{2}$  is not a solution.

**VISUALIZE THE SOLUTION**

The graph of

$$y = \ln(3x + 8) - \ln(2x + 2) - \ln(x - 2)$$

has only one  $x$ -intercept. Thus there is only one real solution.



» Try Exercise 36, page 501

**QUESTION** Why does  $x = -\frac{3}{2}$  not check in Example 7?

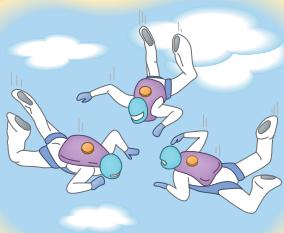
### EXAMPLE 8 » Velocity of a Sky Diver Experiencing Air Resistance

During the free-fall portion of a jump, the time  $t$  in seconds required for a sky diver to reach a velocity  $v$  in feet per second is given by

$$t = -\frac{175}{32} \ln\left(1 - \frac{v}{175}\right), 0 \leq v < 175$$

- Determine the velocity of the diver after 5 seconds.
- The graph of  $t$  has a vertical asymptote at  $v = 175$ . Explain the meaning of the vertical asymptote in the context of this example.

*take note*



If air resistance is not considered, then the time in seconds required for a sky diver to reach a given velocity (in feet per second) is

$$t = \frac{v}{32}. \text{ The function in Example 8}$$

is a more realistic model of the time required to reach a given velocity during the free-fall of a sky diver who is experiencing air resistance.

#### Solution

- Substitute 5 for  $t$  and solve for  $v$ .

$$t = -\frac{175}{32} \ln\left(1 - \frac{v}{175}\right)$$

$$5 = -\frac{175}{32} \ln\left(1 - \frac{v}{175}\right)$$

$$\left(-\frac{32}{175}\right)5 = \ln\left(1 - \frac{v}{175}\right)$$

$$-\frac{32}{35} = \ln\left(1 - \frac{v}{175}\right)$$

$$e^{-32/35} = 1 - \frac{v}{175}$$

$$e^{-32/35} - 1 = -\frac{v}{175}$$

$$v = 175(1 - e^{-32/35})$$

$$v \approx 104.86$$

• Replace  $t$  with 5.

• Multiply each side by  $-\frac{32}{175}$ .

• Simplify.

• Write in exponential form.

• Subtract 1 from each side.

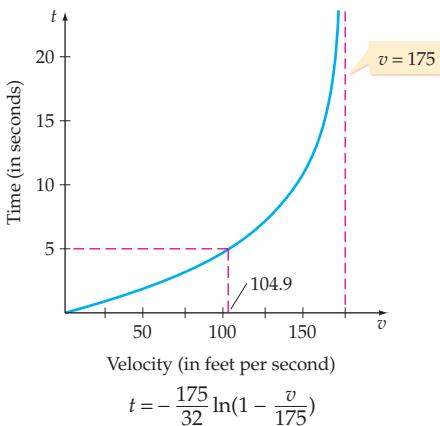
• Multiply each side by  $-175$ .

Continued ►

**ANSWER** If  $x = -\frac{3}{2}$ , the original equation becomes  $\ln\left(\frac{7}{2}\right) = \ln(-1) + \ln\left(-\frac{7}{2}\right)$ .

This cannot be true, because the function  $f(x) = \ln x$  is not defined for negative values of  $x$ .

After 5 seconds the velocity of the sky diver will be about 104.9 feet per second. See **Figure 7.26**.



**Figure 7.26**

- b. The vertical asymptote  $v = 175$  indicates that the velocity of the sky diver approaches, but never reaches or exceeds, 175 feet per second. In **Figure 7.26**, note that as  $v \rightarrow 175$  from the left,  $t \rightarrow \infty$ .

» Try Exercise 74, page 504



### Topics for Discussion

1. Discuss how to solve the equation  $a = \log_b x$  for  $x$ .
2. What is the domain of  $y = \log_4(2x - 5)$ ? Explain why this means that the equation  $\log_4(x - 3) = \log_4(2x - 5)$  has no real number solution.
3.  $-8$  is not a solution of the equation  $\log_2 x + \log_2(x + 6) = 4$ . Discuss at which step in the following solution the extraneous solution  $-8$  was introduced.

$$\begin{aligned} \log_2 x + \log_2(x + 6) &= 4 \\ \log_2 x(x + 6) &= 4 \\ x(x + 6) &= 2^4 \\ x^2 + 6x &= 16 \\ x^2 + 6x - 16 &= 0 \\ (x + 8)(x - 2) &= 0 \\ x = -8 \quad \text{or} \quad x &= 2 \end{aligned}$$


**Exercise Set 7.4**

In Exercises 1 to 48, use algebraic procedures to find the exact solution(s) of the equation.

1.  $2^x = 64$

2.  $3^x = 243$

3.  $49^x = \frac{1}{343}$

4.  $9^x = \frac{1}{243}$

5.  $2^{5x+3} = \frac{1}{8}$

6.  $3^{4x-7} = \frac{1}{9}$

7.  $\left(\frac{2}{5}\right)^x = \frac{8}{125}$

8.  $\left(\frac{2}{5}\right)^x = \frac{25}{4}$

9.  $5^x = 70$

10.  $6^x = 50$

11.  $3^{-x} = 120$

12.  $7^{-x} = 63$

13.  $10^{2x+3} = 315$

14.  $10^{6-x} = 550$

15.  $e^x = 10$

16.  $e^{x+1} = 20$

17.  $2^{1-x} = 3^{x+1}$

18.  $3^{x-2} = 4^{2x+1}$

19.  $2^{2x-3} = 5^{-x-1}$

20.  $5^{3x} = 3^{x+4}$

21.  $\log(4x - 18) = 1$

22.  $\log(x^2 + 19) = 2$

23.  $\ln(x^2 - 12) = \ln x$

24.  $\log(2x^2 + 3x) = \log(10x + 30)$

25.  $\log_2 x + \log_2(x - 4) = 2$

26.  $\log_3 x + \log_3(x + 6) = 3$

27.  $\log(5x - 1) = 2 + \log(x - 2)$

28.  $1 + \log(3x - 1) = \log(2x + 1)$

29.  $\ln(1 - x) + \ln(3 - x) = \ln 8$

30.  $\log(4 - x) = \log(x + 8) + \log(2x + 13)$

31.  $\log \sqrt{x^3 - 17} = \frac{1}{2}$

32.  $\log(x^3) = (\log x)^2$

33.  $\log(\log x) = 1$

34.  $\ln(\ln x) = 2$

35.  $\ln(e^{3x}) = 6$

36.  $\ln x = \frac{1}{2} \ln\left(2x + \frac{5}{2}\right) + \frac{1}{2} \ln 2$

37.  $\log_7(5x) - \log_7 3 = \log_7(2x + 1)$

38.  $\log_4 x + \log_4(x - 2) = \log_4 15$

39.  $e^{\ln(x-1)} = 4$

40.  $10^{\log(2x+7)} = 8$

41.  $\frac{10^x - 10^{-x}}{2} = 20$

42.  $\frac{10^x + 10^{-x}}{2} = 8$

43.  $\frac{10^x + 10^{-x}}{10^x - 10^{-x}} = 5$

44.  $\frac{10^x - 10^{-x}}{10^x + 10^{-x}} = \frac{1}{2}$

45.  $\frac{e^x + e^{-x}}{2} = 15$

46.  $\frac{e^x - e^{-x}}{2} = 15$

47.  $\frac{1}{e^x - e^{-x}} = 4$

48.  $\frac{e^x + e^{-x}}{e^x - e^{-x}} = 3$



In Exercises 49 to 58, use a graphing utility to approximate the solution(s) of the equation to the nearest hundredth.

49.  $2^{-x+3} = x + 1$

50.  $3^{x-2} = -2x - 1$

51.  $e^{3-2x} - 2x = 1$

52.  $2e^{x+2} + 3x = 2$

53.  $3 \log_2(x - 1) = -x + 3$

54.  $2 \log_3(2 - 3x) = 2x - 1$

55.  $\ln(2x + 4) + \frac{1}{2}x = -3$

56.  $2 \ln(3 - x) + 3x = 4$

57.  $2^{x+1} = x^2 - 1$

58.  $\ln x = -x^2 + 4$

**59. POPULATION GROWTH** The population  $P$  of a city grows exponentially according to the function

$$P(t) = 8500(1.1)^t, \quad 0 \leq t \leq 8$$

where  $t$  is measured in years.

a. Find the population at time  $t = 0$  and also at time  $t = 2$ .

b. When, to the nearest year, will the population reach 15,000?

- 60. PHYSICAL FITNESS** After a race, a runner's pulse rate  $R$  in beats per minute decreases according to the function

$$R(t) = 145e^{-0.092t}, \quad 0 \leq t \leq 15$$

where  $t$  is measured in minutes.

- a. Find the runner's pulse rate at the end of the race and also 1 minute after the end of the race.
- b. How long, to the nearest minute, after the end of the race will the runner's pulse rate be 80 beats per minute?

- 61. RATE OF COOLING** A can of soda at 79°F is placed in a refrigerator that maintains a constant temperature of 36°F. The temperature  $T$  of the soda  $t$  minutes after it is placed in the refrigerator is given by

$$T(t) = 36 + 43e^{-0.058t}$$

- a. Find the temperature, to the nearest degree, of the soda 10 minutes after it is placed in the refrigerator.
- b. When, to the nearest minute, will the temperature of the soda be 45°F?

- 62. MEDICINE** During surgery, a patient's circulatory system requires at least 50 milligrams of an anesthetic. The amount of anesthetic present  $t$  hours after 80 milligrams of anesthetic is administered is given by

$$T(t) = 80(0.727)^t$$

- a. How much, to the nearest milligram, of the anesthetic is present in the patient's circulatory system 30 minutes after the anesthetic is administered?
- b. How long, to the nearest minute, can the operation last if the patient does not receive additional anesthetic?

**BERTALANFFY'S EQUATION** In 1938, the biologist Ludwig von Bertalanffy developed the equation

$$L = m - (m - L_0)e^{-rx}$$

which models the length  $L$ , in centimeters, of a fish as it grows under optimal conditions for a period of  $x$  years. In Bertalanffy's equation,  $m$  represents the maximum length, in centimeters, the fish is expected to attain,  $L_0$  is the length, in centimeters, of the fish at birth, and  $r$  is a constant related to the growth rate of the fish species. Use Bertalanffy's equation to predict the age of the fish described in Exercises 63 and 64.

- 63.** A barracuda has a length of 114 centimeters. Use Bertalanffy's equation to predict, to the nearest tenth of a year, the age of the barracuda. Assume  $m = 198$  centimeters,  $L_0 = 0.9$  centimeter, and  $r = 0.23$ .

- 64.** A haddock has a length of 21 centimeters. Use Bertalanffy's equation to predict, to the nearest tenth of a year, the age of the haddock. Assume  $m = 94$  centimeters,  $L_0 = 0.6$  centimeters, and  $r = 0.21$ .

- 65. TYPING SPEED** The following function models the average typing speed  $S$ , in words per minute, for a student who has been typing for  $t$  months.

$$S(t) = 5 + 29 \ln(t + 1), \quad 0 \leq t \leq 9$$

Use  $S$  to determine how long it takes the student to achieve an average typing speed of 65 words per minute. Round to the nearest tenth of a month.

- 66. WALKING SPEED** An approximate relation between the average pedestrian walking speed  $s$ , in miles per hour, and the population  $x$ , in thousands, of a city is given by the formula

$$s(x) = 0.37 \ln x + 0.05$$

Use  $s$  to estimate the population of a city for which the average pedestrian walking speed is 2.9 miles per hour. Round to the nearest hundred thousand.

- 67. DRAG RACING** The quadratic function

$$s_1(x) = -2.25x^2 + 56.26x - 0.28, \quad 0 \leq x \leq 10$$

models the speed of a dragster from the start of a race until the dragster crosses the finish line 10 seconds later. This is the acceleration phase of the race.

The exponential function

$$s_2(x) = 8320(0.73)^x, \quad 10 < x \leq 20$$

models the speed of the dragster during the 10-second period immediately following the time when the dragster crosses the finish line. This is the deceleration period.

How long after the start of the race did the dragster attain a speed of 275 miles per hour? Round to the nearest hundredth of a second.

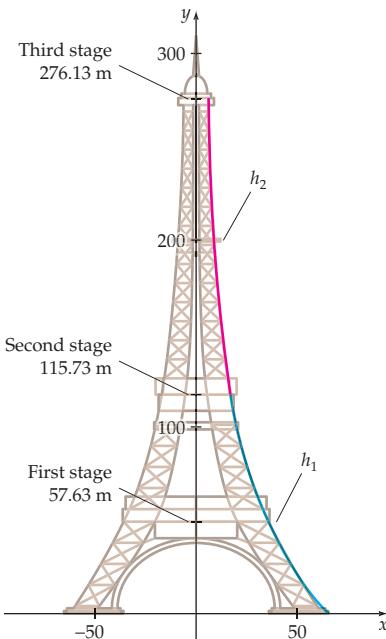
- 68. EIFFEL TOWER** The functions

$$h_1(x) = 363.4 - 88.4 \ln x, \quad 16.47 < x \leq 61.0$$

and

$$h_2(x) = 568.2 - 161.5 \ln x, \quad 6.1 \leq x \leq 16.47$$

approximate the height, in meters, of the Eiffel Tower  $x$  meters to the right of the center line, shown by the  $y$ -axis in the following figure (on page 503).



The graph of  $h_1$  models the shape of the tower from ground level up to the second stage in the figure, and the graph of  $h_2$  models the shape of the tower from the second stage up to the third stage.

Determine the horizontal distance across the Eiffel Tower, rounded to the nearest tenth of a meter, at a height of

a. 50 meters

b. 125 meters

69. **PSYCHOLOGY** Industrial psychologists study employee training programs to assess the effectiveness of the instruction. In one study, the percent score  $P$  on a test for a person who had completed  $t$  hours of training was given by

$$P = \frac{100}{1 + 30e^{-0.088t}}$$

- a. Use a graphing utility to graph the equation for  $t \geq 0$ .
- b. Use the graph to estimate (to the nearest hour) the number of hours of training necessary to achieve a 70% score on the test.
- c. From the graph, determine the horizontal asymptote.
- d. Write a sentence that explains the meaning of the horizontal asymptote.

70. **PSYCHOLOGY** An industrial psychologist has determined that the average percent score for an employee on a test of the employee's knowledge of the company's product is given by

$$P = \frac{100}{1 + 40e^{-0.1t}}$$

where  $t$  is the number of weeks on the job and  $P$  is the percent score.

- a. Use a graphing utility to graph the equation for  $t \geq 0$ .
- b. Use the graph to estimate (to the nearest week) the expected number of weeks of employment that are necessary for an employee to earn a 70% score on the test.
- c. Determine the horizontal asymptote of the graph.
- d. Write a sentence that explains the meaning of the horizontal asymptote.

71. **ECOLOGY** A herd of bison was placed in a wildlife preserve that can support a maximum of 1000 bison. A population model for the bison is given by

$$B = \frac{1000}{1 + 30e^{-0.127t}}$$

where  $B$  is the number of bison in the preserve and  $t$  is time in years, with the year 1999 represented by  $t = 0$ .

- a. Use a graphing utility to graph the equation for  $t \geq 0$ .
- b. Use the graph to estimate (to the nearest year) the number of years before the bison population reaches 500.
- c. Determine the horizontal asymptote of the graph.
- d. Write a sentence that explains the meaning of the horizontal asymptote.

72. **POPULATION GROWTH** A yeast culture grows according to the equation

$$Y = \frac{50,000}{1 + 250e^{-0.305t}}$$

where  $Y$  is the number of yeast and  $t$  is time in hours.

- a. Use a graphing utility to graph the equation for  $t \geq 0$ .

- b.** Use the graph to estimate (to the nearest hour) the number of hours before the yeast population reaches 35,000.
- c.** From the graph, estimate the horizontal asymptote.
- d.**  Write a sentence that explains the meaning of the horizontal asymptote.

- 73.**  **CONSUMPTION OF NATURAL RESOURCES** A model for how long our coal resources will last is given by

$$T = \frac{\ln(300r + 1)}{\ln(r + 1)}$$

where  $r$  is the percent increase in consumption from current levels of use and  $T$  is the time (in years) before the resources are depleted.

- a.** Use a graphing utility to graph this equation.
- b.** If our consumption of coal increases by 3% per year, in how many years will we deplete our coal resources?
- c.** What percent increase in consumption of coal will deplete the resources in 100 years? Round to the nearest tenth of a percent.

- 74.**  **EFFECTS OF AIR RESISTANCE ON VELOCITY** If we assume that air resistance is proportional to the square of the velocity, then the time  $t$  in seconds required for an object to reach a velocity  $v$  in feet per second is given by

$$t = \frac{9}{24} \ln \frac{24 + v}{24 - v}, 0 \leq v < 24$$

- a.** Determine the velocity, to the nearest hundredth of a foot per second, of the object after 1.5 seconds.
- b.** Determine the vertical asymptote for the graph of this function.
- c.**  Write a sentence that explains the meaning of the vertical asymptote in the context of this application.

- 75.**  **TERMINAL VELOCITY WITH AIR RESISTANCE** The velocity  $v$  of an object  $t$  seconds after it has been dropped from a height above the surface of the earth is given by the equation  $v = 32t$  feet per second, assuming no air resistance. If we assume that air resistance is proportional to the square of the velocity, then the velocity after  $t$  seconds is given by

$$v = 100 \left( \frac{e^{0.64t} - 1}{e^{0.64t} + 1} \right)$$

- a.** In how many seconds will the velocity be 50 feet per second?

- b.** Determine the horizontal asymptote for the graph of this function.
- c.**  Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

- 76.**  **EFFECTS OF AIR RESISTANCE ON DISTANCE** The distance  $s$ , in feet, that the object in Exercise 75 will fall in  $t$  seconds is given by

$$s = \frac{100^2}{32} \ln \left( \frac{e^{0.32t} + e^{-0.32t}}{2} \right)$$

- a.** Use a graphing utility to graph this equation for  $t \geq 0$ .
- b.** How long does it take for the object to fall 100 feet? Round to the nearest tenth of a second.

- 77.** **RETIREMENT PLANNING** The retirement account for a graphic designer contains \$250,000 on January 1, 2006, and earns interest at a rate of 0.5% per month. On February 1, 2006, the designer withdraws \$2000 and plans to continue these withdrawals as retirement income each month. The value  $V$  of the account after  $x$  months is

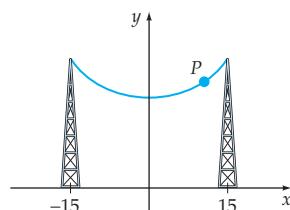
$$V = 400,000 - 150,000(1.005)^x$$

If the designer wishes to leave \$100,000 to a scholarship foundation, what is the maximum number of withdrawals the designer can make from this account and still have \$100,000 to donate?

- 78.** **HANGING CABLE** The height  $h$ , in feet, of any point  $P$  on the cable shown is given by

$$h(x) = 10(e^{x/20} + e^{-x/20}), -15 \leq x \leq 15$$

where  $|x|$  is the horizontal distance in feet between  $P$  and the  $y$ -axis.



- a.** What is the lowest height of the cable?
- b.** What is the height of the cable 10 feet to the right of the  $y$ -axis? Round to the nearest tenth of a foot.
- c.** How far to the right of the  $y$ -axis is the cable 24 feet in height? Round to the nearest tenth of a foot.

## »»» Connecting Concepts »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»» »»»

- 79.** The following argument seems to indicate that  $0.125 > 0.25$ . Find the first incorrect statement in the argument.

$$3 > 2$$

$$3(\log 0.5) > 2(\log 0.5)$$

$$\log 0.5^3 > \log 0.5^2$$

$$0.5^3 > 0.5^2$$

$$0.125 > 0.25$$

- 80.** The following argument seems to indicate that  $4 = 6$ . Find the first incorrect statement in the argument.

$$4 = \log_2 16$$

$$4 = \log_2(8 + 8)$$

$$4 = \log_2 8 + \log_2 8$$

$$4 = 3 + 3$$

$$4 = 6$$

- 81.** A common mistake that students make is to write  $\log(x + y)$  as  $\log x + \log y$ . For what values of  $x$  and  $y$  does  $\log(x + y) = \log x + \log y$ ? (Hint: Solve for  $x$  in terms of  $y$ .)

- 82.** Let  $f(x) = 2 \ln x$  and  $g(x) = \ln x^2$ . Does  $f(x) = g(x)$  for all real numbers  $x$ ?

- 83.** Explain why the functions  $F(x) = 1.4^x$  and  $G(x) = e^{0.336x}$  represent essentially the same function.

- 84.** Find the constant  $k$  that will make  $f(t) = 2.2^t$  and  $g(t) = e^{-kt}$  represent essentially the same function.

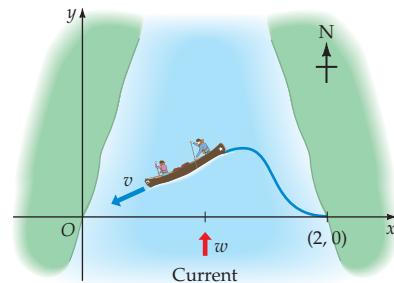
- 1.**  **NAVIGATING** The pilot of a boat is trying to cross a river to a point  $O$  two miles due west of the boat's starting position by always pointing the nose of the boat toward  $O$ . Suppose the speed of the current is  $w$  miles per hour and the speed of the boat is  $v$  miles per hour. If point  $O$  is the origin and the boat's starting position is  $(2, 0)$  (see the diagram at the right), then the equation of the boat's path is given by

$$y = \left(\frac{x}{2}\right)^{1-(w/v)} - \left(\frac{x}{2}\right)^{1+(w/v)}$$

- a. If the speed of the current and the speed of the boat are the same, can the pilot reach point  $O$  by always having the nose of the boat pointed toward  $O'$ ? If not, at what point will the pilot arrive? Explain your answer.
  - b. If the speed of the current is greater than the speed of the boat, can the pilot reach point  $O$  by always pointing-

ing the nose of the boat toward point  $O$ ? If not, where will the pilot arrive? Explain.

- c. If the speed of the current is less than the speed of the boat, can the pilot reach point  $O$  by always pointing the nose of the boat toward point  $O$ ? If not, where will the pilot arrive? Explain.



## Section 7.5

- Exponential Growth and Decay
- Carbon Dating
- Compound Interest Formulas
- Restricted Growth Models

# Exponential Growth and Decay

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A40.

**PS1.** Evaluate  $A = 1000 \left(1 + \frac{0.1}{12}\right)^{12t}$  for  $t = 2$ . Round to the nearest hundredth. [7.1]

**PS2.** Evaluate  $A = 600 \left(1 + \frac{0.04}{4}\right)^{4t}$  for  $t = 8$ . Round to the nearest hundredth. [7.1]

**PS3.** Solve  $0.5 = e^{14k}$  for  $k$ . Round to the nearest ten-thousandth. [7.4]

**PS4.** Solve  $0.85 = 0.5^{t/5730}$  for  $t$ . Round to the nearest ten. [7.4]

**PS5.** Solve  $6 = \frac{70}{5 + 9e^{-kt}}$  for  $k$ . Round to the nearest thousandth. [7.4]

**PS6.** Solve  $2,000,000 = \frac{3^{n+1} - 3}{2}$  for  $n$ . Round to the nearest tenth. [7.4]

## Exponential Growth and Decay

In many applications, a quantity changes at a rate proportional to the amount present. In these applications, the amount present at time  $t$  is given by a special function called an *exponential growth function* or an *exponential decay function*.

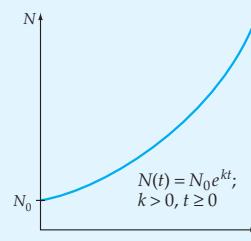
### Definition of Exponential Growth and Decay Functions

If a quantity  $N$  increases or decreases at a rate proportional to the amount present at time  $t$ , then the quantity can be modeled by

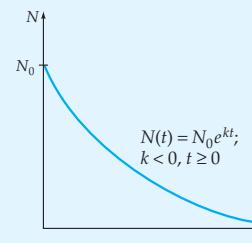
$$N(t) = N_0 e^{kt}$$

where  $N_0$  is the value of  $N$  at time  $t = 0$ , and  $k$  is a constant called the **growth rate constant**.

- If  $k$  is positive,  $N$  increases as  $t$  increases and  $N(t) = N_0 e^{kt}$  is called an **exponential growth function**. See **Figure 7.27**.
- If  $k$  is negative,  $N$  decreases as  $t$  increases and  $N(t) = N_0 e^{kt}$  is called an **exponential decay function**. See **Figure 7.28**.



**Figure 7.27**  
Exponential growth function



**Figure 7.28**  
Exponential decay function

**QUESTION** Is  $N(t) = 1450e^{0.05t}$  an exponential growth function or an exponential decay function?

In Example 1 we find an exponential growth function that models the population growth of a city.

### EXAMPLE 1 » Find the Exponential Growth Function That Models Population Growth

- The population of a city is growing exponentially. The population of the city was 16,400 in 1995 and 20,200 in 2005. Find the exponential growth function that models the population growth of the city.
- Use the function from a. to predict, to the nearest 100, the population of the city in 2010.

#### Solution

- We need to determine  $N_0$  and  $k$  in  $N(t) = N_0e^{kt}$ . If we represent the year 1995 by  $t = 0$ , then our given data are  $N(0) = 16,400$  and  $N(10) = 20,200$ . Because  $N_0$  is defined to be  $N(0)$ , we know that  $N_0 = 16,400$ . To determine  $k$ , substitute  $t = 10$  and  $N_0 = 16,400$  into  $N(t) = N_0e^{kt}$  to produce

$$N(10) = 16,400e^{k \cdot 10}$$

**20,200** = 16,400 $e^{10k}$  • Substitute 20,200 for  $N(10)$ .

$$\frac{20,200}{16,400} = e^{10k}$$

• Solve for  $e^{10k}$ .

$$\ln \frac{20,200}{16,400} = 10k$$

• Write in logarithmic form.

$$\frac{1}{10} \ln \frac{20,200}{16,400} = k$$

• Solve for  $k$ .

$$0.0208 \approx k$$

The exponential growth function is  $N(t) \approx 16,400e^{0.0208t}$ .

- The year 1995 was represented by  $t = 0$ , so we will use  $t = 15$  to represent the year 2010.

$$N(t) \approx 16,400e^{0.0208t}$$

$$N(15) \approx 16,400e^{0.0208 \cdot 15}$$

$$\approx 22,400 \quad (\text{nearest 100})$$

The exponential growth function yields 22,400 as the approximate population of the city in 2010.

### » Try Exercise 6, page 516

**ANSWER** Because the growth rate constant  $k = 0.05$  is positive, the function is an exponential growth function.



Many radioactive materials *decrease* in mass exponentially over time. This decrease, called radioactive decay, is measured in terms of **half-life**, which is defined as the time required for the disintegration of half the atoms in a sample of a radioactive substance. **Table 7.11** shows the half-lives of selected radioactive isotopes.

**Table 7.11**

| Isotope                        | Half-Life              |
|--------------------------------|------------------------|
| Carbon ( $^{14}\text{C}$ )     | 5730 years             |
| Radium ( $^{226}\text{Ra}$ )   | 1660 years             |
| Polonium ( $^{210}\text{Po}$ ) | 138 days               |
| Phosphorus ( $^{32}\text{P}$ ) | 14 days                |
| Polonium ( $^{214}\text{Po}$ ) | 1/10,000th of a second |

### EXAMPLE 2 » Find an Exponential Decay Function

Find the exponential decay function for the amount of phosphorus ( $^{32}\text{P}$ ) that remains in a sample after  $t$  days.

#### take note

Because  $e^{-0.0495} \approx (0.5)^{1/14}$ , the decay function  $N(t) = N_0 e^{-0.0495t}$  can also be written as

$N(t) = N_0(0.5)^{t/14}$ . In this form it is easy to see that if  $t$  is increased by 14, then  $N$  will decrease by a factor of 0.5.

#### Solution

When  $t = 0$ ,  $N(0) = N_0 e^{k(0)} = N_0$ . Thus  $N(0) = N_0$ . Also, because the phosphorus has a half-life of 14 days (from **Table 7.11**),  $N(14) = 0.5N_0$ . To find  $k$ , substitute  $t = 14$  into  $N(t) = N_0 e^{kt}$  and solve for  $k$ .

$$N(14) = N_0 \cdot e^{k \cdot 14}$$

$$0.5N_0 = N_0 e^{14k}$$

$$0.5 = e^{14k}$$

$$\ln 0.5 = 14k$$

• Substitute  $0.5N_0$  for  $N(14)$ .

• Divide each side by  $N_0$ .

• Write in logarithmic form.

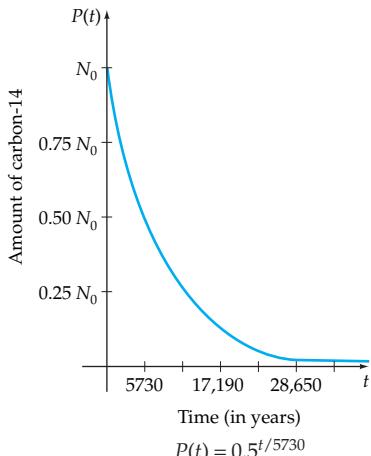
$$\frac{1}{14} \ln 0.5 = k$$

• Solve for  $k$ .

$$-0.0495 \approx k$$

The exponential decay function is  $N(t) \approx N_0 e^{-0.0495t}$ .

### » Try Exercise 8, page 516



**Figure 7.29**

### Carbon Dating

The bone tissue in all living animals contains both carbon-12, which is nonradioactive, and carbon-14, which is radioactive with a half-life of approximately 5730 years. See **Figure 7.29**. As long as the animal is alive, the ratio of carbon-14 to carbon-12 remains constant. When the animal dies ( $t = 0$ ), the carbon-14 begins to decay. Thus a bone that has a smaller ratio of carbon-14 to carbon-12 is older than a bone that has a larger ratio. The percent of carbon-14 present at time  $t$  is

$$P(t) = 0.5^{t/5730}$$

**Math Matters**

The chemist Willard Frank Libby developed the carbon dating process in 1947. In 1960 he was awarded the Nobel Prize in chemistry for this achievement.

The process of using the percent of carbon-14 present at a given time to estimate the age of a bone is called **carbon dating**.

**EXAMPLE 3 » A Carbon Dating Application**

Estimate the age of a bone if it now has 85% of the carbon-14 it had at time  $t = 0$ .

**Solution**

Let  $t$  be the time at which  $P(t) = 0.85$ .

$$\begin{aligned} 0.85 &= 0.5^{t/5730} \\ \ln 0.85 &= \ln 0.5^{t/5730} \\ \ln 0.85 &= \frac{t}{5730} \ln 0.5 \\ 5730 \left( \frac{\ln 0.85}{\ln 0.5} \right) &= t \\ 1340 &\approx t \end{aligned}$$

- Take the natural logarithm of each side.
- Power property.
- Solve for  $t$ .

The bone is about 1340 years old.

**» Try Exercise 12, page 516****Compound Interest Formulas**

**Interest** is money paid for the use of money. The interest  $I$  is called **simple interest** if it is a fixed percent  $r$ , per time period  $t$ , of the amount of money invested. The amount of money invested is called the **principal**  $P$ . Simple interest is computed using the formula  $I = Prt$ . For example, if \$1000 is invested at 12% for 3 years, the simple interest is

$$I = Prt = \$1000(0.12)(3) = \$360$$

The balance after  $t$  years is  $A = P + I = P + Prt$ . In the preceding example, the \$1000 invested for 3 years produced \$360 interest. Thus the balance after 3 years is  $\$1000 + \$360 = \$1360$ .

In many financial transactions, interest is added to the principal at regular intervals so that interest is paid on interest as well as on the principal. Interest earned in this manner is called **compound interest**. For example, if \$1000 is invested at 12% annual interest compounded annually for 3 years, then the total interest after 3 years is

|                      |                                    |
|----------------------|------------------------------------|
| First-year interest  | $\$1000(0.12) = \$120.00$          |
| Second-year interest | $\$1120(0.12) = \$134.40$          |
| Third-year interest  | $\$1254.40(0.12) \approx \$150.53$ |
|                      | \$404.93      • Total interest     |

This method of computing the balance can be tedious and time-consuming. A *compound interest formula* that can be used to determine the balance due after  $t$  years of compounding can be developed as follows.

**Table 7.12**

| Number of Years | Balance            |
|-----------------|--------------------|
| 3               | $A_3 = P(1 + r)^3$ |
| 4               | $A_4 = P(1 + r)^4$ |
| .               | .                  |
| .               | .                  |
| $t$             | $A_t = P(1 + r)^t$ |

Note that if  $P$  dollars is invested at an interest rate of  $r$  per year, then the balance after 1 year is  $A_1 = P + Pr = P(1 + r)$ , where  $Pr$  represents the interest earned for the year. Observe that  $A_1$  is the product of the original principal  $P$  and  $(1 + r)$ . If the amount  $A_1$  is reinvested for another year, then the balance after the second year is

$$A_2 = (A_1)(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$$

Successive reinvestments lead to the results shown in **Table 7.12**. The equation  $A_t = P(1 + r)^t$  is valid if  $r$  is the annual interest rate paid during each of the  $t$  years.

If  $r$  is an annual interest rate and  $n$  is the number of compounding periods per year, then the interest rate each period is  $r/n$ , and the number of compounding periods after  $t$  years is  $nt$ . Thus the compound interest formula is as follows:

### The Compound Interest Formula

A principal  $P$  invested at an annual interest rate  $r$ , expressed as a decimal and compounded  $n$  times per year for  $t$  years, produces the balance

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

### EXAMPLE 4 » Solve a Compound Interest Application

Find the balance if \$1000 is invested at an annual interest rate of 10% for 2 years compounded

- a. monthly
- b. daily

#### Solution

- a. Because there are 12 months in a year, use  $n = 12$ .

$$A = \$1000 \left(1 + \frac{0.1}{12}\right)^{12 \cdot 2} \approx \$1000(1.008333333)^{24} \approx \$1220.39$$

- b. Because there are 365 days in a year, use  $n = 365$ .

$$A = \$1000 \left(1 + \frac{0.1}{365}\right)^{365 \cdot 2} \approx \$1000(1.000273973)^{730} \approx \$1221.37$$

### » Try Exercise 16, page 517

To **compound continuously** means to increase the number of compounding periods per year,  $n$ , without bound.

To derive a continuous compounding interest formula, substitute  $\frac{1}{m}$  for  $\frac{r}{n}$  in the compound interest formula

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

to produce

$$A = P \left(1 + \frac{1}{m}\right)^{nt} \quad (1)$$

This substitution is motivated by the desire to express  $\left(1 + \frac{r}{n}\right)^n$  as  $\left[\left(1 + \frac{1}{m}\right)^m\right]^r$ , which approaches  $e^r$  as  $m$  gets larger without bound.

Solving the equation  $\frac{1}{m} = \frac{r}{n}$  for  $n$  yields  $n = mr$ , so the exponent  $nt$  can be written as  $mrt$ . Therefore Equation (1) can be expressed as

$$A = P \left(1 + \frac{1}{m}\right)^{mrt} = P \left[\left(1 + \frac{1}{m}\right)^m\right]^{rt} \quad (2)$$

By the definition of  $e$ , we know that as  $m$  increases without bound,

$$\left(1 + \frac{1}{m}\right)^m \text{ approaches } e$$

Thus, using continuous compounding, Equation (2) simplifies to  $A = Pe^{rt}$ .

### Continuous Compounding Interest Formula

If an account with principal  $P$  and annual interest rate  $r$  is compounded continuously for  $t$  years, then the balance is  $A = Pe^{rt}$ .

#### EXAMPLE 5 » Solve a Continuous Compound Interest Application

Find the balance after 4 years on \$800 invested at an annual rate of 6% compounded continuously.

##### ALGEBRAIC SOLUTION

Use the continuous compounding formula with  $P = 800$ ,  $r = 0.06$ , and  $t = 4$ .

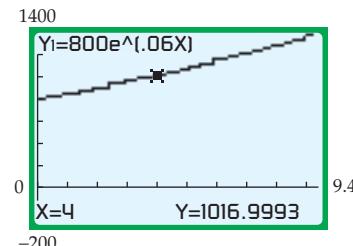
$$\begin{aligned} A &= Pe^{rt} \\ &= 800e^{0.06(4)} \\ &= 800e^{0.24} \\ &\approx 800(1.27124915) \\ &\approx 1017.00 \end{aligned}$$

- Substitute given values.
- Simplify.
- Round to the nearest cent.

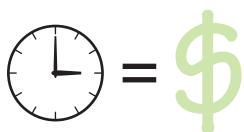
The balance after 4 years will be \$1017.00.

##### VISUALIZE THE SOLUTION

The following graph of  $A = 800e^{0.06t}$  shows that the balance is about \$1017.00 when  $t = 4$ .



» Try Exercise 18, page 517



You have probably heard it said that time is money. In fact, many investors ask the question “How long will it take to double my money?” The following example answers this question for two different investments.

### EXAMPLE 6 » Double Your Money

Find the time required for money invested at an annual rate of 6% to double in value if the investment is compounded.

- a. semiannually      b. continuously

#### Solution

- a. Use  $A = P \left(1 + \frac{r}{n}\right)^{nt}$  with  $r = 0.06$ ,  $n = 2$ , and the balance  $A$  equal to twice the principal ( $A = 2P$ ).

$$2P = P \left(1 + \frac{0.06}{2}\right)^{2t}$$

$$2 = \left(1 + \frac{0.06}{2}\right)^{2t}$$

$$\ln 2 = \ln \left(1 + \frac{0.06}{2}\right)^{2t}$$

$$\ln 2 = 2t \ln \left(1 + \frac{0.06}{2}\right)$$

$$2t = \frac{\ln 2}{\ln \left(1 + \frac{0.06}{2}\right)}$$

$$t = \frac{1}{2} \cdot \frac{\ln 2}{\ln \left(1 + \frac{0.06}{2}\right)}$$

$$t \approx 11.72$$

• Divide each side by  $P$ .

• Take the natural logarithm of each side.

• Apply the power property.

• Solve for  $t$ .

If the investment is compounded semiannually, it will double in value in about 11.72 years.

- b. Use  $A = Pe^{rt}$  with  $r = 0.06$  and  $A = 2P$ .

$$2P = Pe^{0.06t}$$

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06}$$

$$t \approx 11.55$$

• Divide each side by  $P$ .

• Write in logarithmic form.

• Solve for  $t$ .

If the investment is compounded continuously, it will double in value in about 11.55 years.

### » Try Exercise 22, page 517

## ■ Restricted Growth Models

The exponential growth function  $N(t) = N_0 e^{kt}$  is an *unrestricted growth model* that does not consider any limited resources that eventually will curb population growth.

The **logistic model** is a *restricted growth model* that takes into consideration the effects of limited resources. The logistic model was developed by Pierre Verhulst in 1836.

### Definition of the Logistic Model (A Restricted Growth Model)

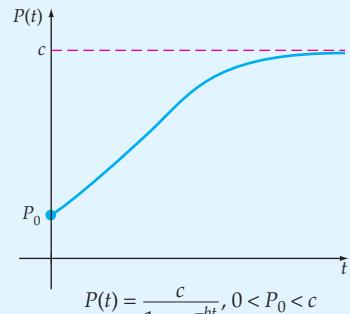
The magnitude of a population at time  $t \geq 0$  is given by

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

where  $c$  is the **carrying capacity** (the maximum population that can be supported by available resources as  $t \rightarrow \infty$ ) and  $b$  is a positive constant called the **growth rate constant**.

The **initial population** is  $P_0 = P(0)$ . The constant  $a$  is related to the initial population  $P_0$  and the carrying capacity  $c$  by the formula

$$a = \frac{c - P_0}{P_0}$$



In the following example we determine a logistic growth model for a coyote population.

### EXAMPLE 7 » Find and Use a Logistic Model

At the beginning of 2005, the coyote population in a wilderness area was estimated at 200. By the beginning of 2007, the coyote population had increased to 250. A park ranger estimates that the carrying capacity of the wilderness area is 500 coyotes.

- Use the given data to determine the growth rate constant for the logistic model of this coyote population.
- Use the logistic model determined in **a.** to predict the year in which the coyote population will first reach 400.

#### Solution



- If we represent the beginning of the year 2005 by  $t = 0$ , then the beginning of the year 2007 will be represented by  $t = 2$ . In the logistic model, make the following substitutions:  $P(2) = 250$ ,  $c = 500$ , and

$$a = \frac{c - P_0}{P_0} = \frac{500 - 200}{200} = 1.5.$$

$$\begin{aligned}
 P(t) &= \frac{c}{1 + ae^{-bt}} \\
 P(2) &= \frac{500}{1 + 1.5e^{-b \cdot 2}} && \bullet \text{ Substitute the given values.} \\
 250 &= \frac{500}{1 + 1.5e^{-b \cdot 2}} && \bullet P(2) = 250 \\
 250(1 + 1.5e^{-b \cdot 2}) &= 500 && \bullet \text{ Solve for the growth rate constant } b. \\
 1 + 1.5e^{-b \cdot 2} &= \frac{500}{250} \\
 1.5e^{-b \cdot 2} &= 2 - 1 \\
 e^{-b \cdot 2} &= \frac{1}{1.5} \\
 -2b &= \ln\left(\frac{1}{1.5}\right) && \bullet \text{ Write in logarithmic form.} \\
 b &= -\frac{1}{2} \ln\left(\frac{1}{1.5}\right) \\
 b &\approx 0.20273255
 \end{aligned}$$

Using  $a = 1.5$ ,  $b = 0.20273255$ , and  $c = 500$  gives us the following logistic model.

$$P(t) = \frac{500}{1 + 1.5e^{-0.20273255t}}$$

- b.** To determine during what year the logistic model predicts the coyote population will first reach 400, replace  $P(t)$  with 400 and solve for  $t$ .

$$\begin{aligned}
 400 &= \frac{500}{1 + 1.5e^{-0.20273255t}} \\
 400(1 + 1.5e^{-0.20273255t}) &= 500 \\
 1 + 1.5e^{-0.20273255t} &= \frac{500}{400} \\
 1.5e^{-0.20273255t} &= 1.25 - 1 \\
 e^{-0.20273255t} &= \frac{0.25}{1.5} \\
 -0.20273255t &= \ln\left(\frac{0.25}{1.5}\right) && \bullet \text{ Write in logarithmic form.} \\
 t &= \frac{1}{-0.20273255} \ln\left(\frac{0.25}{1.5}\right) && \bullet \text{ Solve for } t. \\
 &\approx 8.8
 \end{aligned}$$

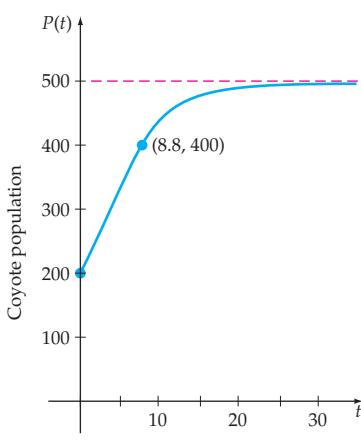


Figure 7.30

According to the logistic model, the coyote population will reach 400 about 8.8 years after the beginning of 2005, which is during the year 2013. The graph of the logistic model is shown in **Figure 7.30**. Note that  $P(8.8) \approx 400$  and that as  $t \rightarrow \infty$ ,  $P(t) \rightarrow 500$ .

» Try Exercise 38, page 518

In Example 8, we use a function of the form  $v = a(1 - e^{-kt})$  to model the velocity of an object that has been dropped from a high elevation.

### EXAMPLE 8 » Application to Air Resistance

Assuming that air resistance is proportional to the velocity of a falling object, the velocity (in feet per second) of the object  $t$  seconds after it has been dropped is given by  $v = 82(1 - e^{-0.39t})$ .

- Determine when the velocity will be 70 feet per second.
- The graph of  $v$  has  $v = 82$  as a horizontal asymptote. Explain the meaning of this asymptote in the context of this example.

#### ALGEBRAIC SOLUTION

a.  $v = 82(1 - e^{-0.39t})$

$$70 = 82(1 - e^{-0.39t})$$

$$\frac{70}{82} = 1 - e^{-0.39t}$$

$$e^{-0.39t} = 1 - \frac{70}{82}$$

$$-0.39t = \ln \frac{6}{41}$$

$$t = \frac{\ln(6/41)}{-0.39} \approx 4.9277246$$

• Substitute 70 for  $v$ .

• Divide each side by 82.

• Solve for  $e^{-0.39t}$ .

• Write in logarithmic form.

• Solve for  $t$ .

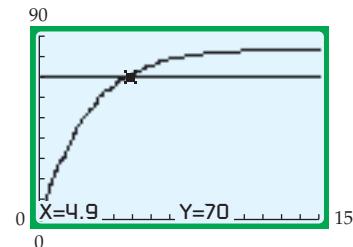
The velocity will be 70 feet per second after approximately 4.9 seconds.

- The horizontal asymptote  $v = 82$  means that as time increases, the velocity of the object will approach, but never reach or exceed, 82 feet per second.

### » Try Exercise 48, page 519

#### VISUALIZE THE SOLUTION

- a. A graph of  $y = 82(1 - e^{-0.39x})$  and  $y = 70$  shows that the  $x$ -coordinate of the point of intersection is about 4.9.



$$y = 82(1 - e^{-0.39x})$$

Note: The  $x$  value shown is rounded to the nearest tenth.



#### Topics for Discussion

- Explain the difference between compound interest and simple interest.
- What is an exponential growth function? Give an example of an application for which an exponential growth function might be an appropriate model.
- What is an exponential decay function? Give an example of an application for which an exponential decay function might be an appropriate model.
- Consider the exponential function  $P(t) = P_0 e^{kt}$  and the logistic function  $P(t) = \frac{c}{1 + ae^{-bt}}$ . Explain the similarities and differences between the graphs of the two functions.

## Exercise Set 7.5

- 1. POPULATION GROWTH** The number of bacteria  $N(t)$  present in a culture at time  $t$  hours is given by

$$N(t) = 2200(2)^t$$

Find the number of bacteria present when

- a.  $t = 0$  hours    b.  $t = 3$  hours

- 2. POPULATION GROWTH** The population of a city grows exponentially according to the function

$$f(t) = 12,400(1.14)^t$$

for  $0 \leq t \leq 5$  years. Find, to the nearest hundred, the population of the city when  $t$  is

- a. 3 years    b. 4.25 years

- 3. POPULATION GROWTH** A city had a population of 22,600 in 1997 and a population of 24,200 in 2002.

- a. Find the exponential growth function for the city. Use  $t = 0$  to represent the year 1997.

- b. Use the growth function to predict the population of the city in 2012. Round to the nearest hundred.

- 4. POPULATION GROWTH** A city had a population of 53,700 in 1999 and a population of 58,100 in 2003.

- a. Find the exponential growth function for the city. Use  $t = 0$  to represent the year 1999.

- b. Use the growth function to predict the population of the city in 2011. Round to the nearest hundred.

- 5. POPULATION GROWTH** The population of Miami, Florida, is growing exponentially. The population of Miami was 362,300 in the year 2000 and 379,700 in 2004. Find the exponential growth function that models the population growth of Miami and use it to predict the population of Miami in 2009. Use  $t = 0$  to represent the year 2000. Round to the nearest hundred.

- 6. POPULATION GROWTH** The population of Aurora, Colorado, is growing exponentially. The population of Aurora was 276,400 in the year 2000 and 291,800 in 2004. Find the exponential growth function that models the population growth of the Aurora and use it to predict the population of Aurora in 2008. Use  $t = 0$  to represent the year 2000. Round to the nearest hundred.

- 7. MEDICINE** Sodium-24 is a radioactive isotope of sodium that is used to study circulatory dysfunction. Assuming that

4 micrograms of sodium-24 is injected into a person, the amount  $A$  in micrograms remaining in that person after  $t$  hours is given by the equation  $A = 4e^{-0.046t}$ .

- a. Graph this equation.  
 b. What amount of sodium-24 remains after 5 hours?  
 c. What is the half-life of sodium-24?  
 d. In how many hours will the amount of sodium-24 be 1 microgram?

In Exercises 8–12, use the half-life information from Table 7.11, page 508, to work each exercise.

- 8. RADIOACTIVE DECAY** Find the decay function for the amount of polonium ( $^{210}\text{Po}$ ) that remains in a sample after  $t$  days.

- 9. GEOLOGY** Geologists have determined that Crater Lake in Oregon was formed by a volcanic eruption. Chemical analysis of a wood chip assumed to be from a tree that died during the eruption has shown that it contains approximately 45% of its original carbon-14. Estimate how long ago the volcanic eruption occurred.

- 10. RADIOACTIVE DECAY** Estimate the percentage of polonium ( $^{210}\text{Po}$ ) that remains in a sample after 2 years. Round to the nearest hundredth of a percent.

- 11. ARCHEOLOGY** The Rhind papyrus, named after A. Henry Rhind, contains most of what we know today of ancient Egyptian mathematics. A chemical analysis of a sample from the papyrus has shown that it contains approximately 75% of its original carbon-14. Estimate the age of the Rhind papyrus.

- 12. ARCHEOLOGY** Estimate the age of a bone if it now contains 65% of its original amount of carbon-14. Round to the nearest 100 years.

- 13. COMPOUND INTEREST** If \$8000 is invested at an annual interest rate of 5% and compounded annually, find the balance after

- a. 4 years    b. 7 years

- 14. COMPOUND INTEREST** If \$22,000 is invested at an annual interest rate of 4.5% and compounded annually, find the balance after

- a. 2 years    b. 10 years

- 15. COMPOUND INTEREST** If \$38,000 is invested at an annual interest rate of 6.5% for 4 years, find the balance if the interest is compounded

a. annually    b. daily    c. hourly

- 16. COMPOUND INTEREST** If \$12,500 is invested at an annual interest rate of 8% for 10 years, find the balance if the interest is compounded

a. annually    b. daily    c. hourly

- 17. COMPOUND INTEREST** Find the balance if \$15,000 is invested at an annual rate of 10% for 5 years, compounded continuously.

- 18. COMPOUND INTEREST** Find the balance if \$32,000 is invested at an annual rate of 8% for 3 years, compounded continuously.

- 19. COMPOUND INTEREST** How long will it take \$4000 to double if it is invested in a certificate of deposit that pays 7.84% annual interest compounded continuously? Round to the nearest tenth of a year.

- 20. COMPOUND INTEREST** How long will it take \$25,000 to double if it is invested in a savings account that pays 5.88% annual interest compounded continuously? Round to the nearest tenth of a year.

- 21. CONTINUOUS COMPOUNDING INTEREST** Use the Continuous Compounding Interest Formula to derive an expression for the time it will take money to triple when invested at an annual interest rate of  $r$  compounded continuously.

- 22. CONTINUOUS COMPOUNDING INTEREST** How long will it take \$1000 to triple if it is invested at an annual interest rate of 5.5% compounded continuously? Round to the nearest year.

- 23. CONTINUOUS COMPOUNDING INTEREST** How long will it take \$6000 to triple if it is invested in a savings account that pays 7.6% annual interest compounded continuously? Round to the nearest year.

- 24. CONTINUOUS COMPOUNDING INTEREST** How long will it take \$10,000 to triple if it is invested in a savings account that pays 5.5% annual interest compounded continuously? Round to the nearest year.

**In Exercises 25 to 30, determine the following constants for the given logistic growth model.**

- a. The carrying capacity  
b. The growth rate constant  
c. The initial population  $P_0$

25.  $P(t) = \frac{1900}{1 + 8.5e^{-0.16t}}$

26.  $P(t) = \frac{32,550}{1 + 0.75e^{-0.08t}}$

27.  $P(t) = \frac{157,500}{1 + 2.5e^{-0.04t}}$

28.  $P(t) = \frac{51}{1 + 1.04e^{-0.03t}}$

29.  $P(t) = \frac{2400}{1 + 7e^{-0.12t}}$

30.  $P(t) = \frac{320}{1 + 15e^{-0.12t}}$

**In Exercises 31 to 34, use algebraic procedures to find the logistic growth model for the data.**

31.  $P_0 = 400$ ,  $P(2) = 780$ , and the carrying capacity is 5500.

32.  $P_0 = 6200$ ,  $P(8) = 7100$ , and the carrying capacity is 9500.

33.  $P_0 = 18$ ,  $P(3) = 30$ , and the carrying capacity is 100.

34.  $P_0 = 3200$ ,  $P(22) \approx 5565$ , and the growth rate constant is 0.056.

- 35. REVENUE** The annual revenue  $R$ , in dollars, of a new company can be closely modeled by the logistic function

$$R(t) = \frac{625,000}{1 + 3.1e^{-0.045t}}$$

where the *natural* number  $t$  is the time, in years, since the company was founded.

- a. According to the model, what will be the company's annual revenue for its first year and its second year ( $t = 1$  and  $t = 2$ ) of operation? Round to the nearest \$1000.

- b. According to the model, what will the company's annual revenue approach in the long-term future?

- 36. NEW CAR SALES** The number of cars  $A$  sold annually by an automobile dealership can be closely modeled by the logistic function

$$A(t) = \frac{1650}{1 + 2.4e^{-0.055t}}$$

where the *natural* number  $t$  is the time, in years, since the dealership was founded.

- a. According to the model, what number of cars will the dealership sell during its first year and its second year ( $t = 1$  and  $t = 2$ ) of operation? Round to the nearest unit.

- b. According to the model, what will the dealership's annual car sales approach in the long-term future?

- 37. POPULATION GROWTH** The population of wolves in a preserve satisfies a logistic model in which  $P_0 = 312$  in the year 2005,  $c = 1600$ , and  $P(6) = 416$ .

- a. Determine the logistic model for this population, where  $t$  is the number of years after 2005.
- b. Use the logistic model from a. to predict the size of the wolf population in 2015.

**38. POPULATION GROWTH** The population of groundhogs on a ranch satisfies a logistic model in which  $P_0 = 240$  in the year 2004,  $c = 3400$ , and  $P(1) = 310$ .

- a. Determine the logistic model for this population, where  $t$  is the number of years after 2004.
- b. Use the logistic model from a. to predict the size of the groundhog population in 2011.

**39. POPULATION GROWTH** The population of squirrels in a nature preserve satisfies a logistic model in which  $P_0 = 1500$  in the year 2007. The carrying capacity of the preserve is estimated at 8500 squirrels, and  $P(2) = 1900$ .

- a. Determine the logistic model for this population, where  $t$  is the number of years after 2007.
- b. Use the logistic model from a. to predict the year in which the squirrel population will first exceed 4000.

**40. POPULATION GROWTH** The population of walruses on an island satisfies a logistic model in which  $P_0 = 800$  in the year 2003. The carrying capacity of the island is estimated at 5500 walruses, and  $P(1) = 900$ .

- a. Determine the logistic model for this population, where  $t$  is the number of years after 2003.
- b. Use the logistic model from a. to predict the year in which the walrus population will first exceed 2000.

**41. PHYSICS** Newton's Law of Cooling states that if an object at temperature  $T_0$  is placed into an environment at constant temperature  $A$ , then the temperature of the object,  $T(t)$  (in degrees Fahrenheit), after  $t$  minutes is given by  $T(t) = A + (T_0 - A)e^{-kt}$ , where  $k$  is a constant that depends on the object.

- a. Determine the constant  $k$  (to the nearest thousandth) for a canned soda drink that takes 5 minutes to cool from 75°F to 65°F after being placed in a refrigerator that maintains a constant temperature of 34°F.
- b. What will be the temperature (to the nearest degree) of the soda drink after 30 minutes?
- c. When (to the nearest minute) will the temperature of the soda drink be 36°F?

**42. PSYCHOLOGY** According to a software company, the users of its typing tutorial can expect to type  $N(t)$  words per minute after  $t$  hours of practice with the product, according to the function  $N(t) = 100(1.04 - 0.99^t)$ .

- a. How many words per minute can a student expect to type after 2 hours of practice?
- b. How many words per minute can a student expect to type after 40 hours of practice?
- c. According to the function  $N$ , how many hours (to the nearest hour) of practice will be required before a student can expect to type 60 words per minute?

**43. PSYCHOLOGY** In the city of Whispering Palms, which has a population of 80,000 people, the number of people  $P(t)$  exposed to a rumor in  $t$  hours is given by the function  $P(t) = 80,000(1 - e^{-0.0005t})$ .

- a. Find the number of hours until 10% of the population have heard the rumor.
- b. Find the number of hours until 50% of the population have heard the rumor.

**44. LAW** A lawyer has determined that the number of people  $P(t)$  in a city of 1,200,000 people who have been exposed to a news item after  $t$  days is given by the function

$$P(t) = 1,200,000(1 - e^{-0.03t})$$

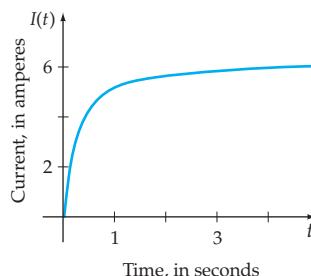
- a. How many days after a major crime has been reported have 40% of the population heard of the crime?
- b. A defense lawyer knows it will be very difficult to pick an unbiased jury after 80% of the population have heard of the crime. After how many days will 80% of the population have heard of the crime?

**45. DEPRECIATION** An automobile depreciates according to the function  $V(t) = V_0(1 - r)^t$ , where  $V(t)$  is the value in dollars after  $t$  years,  $V_0$  is the original value, and  $r$  is the yearly depreciation rate. A car has a yearly depreciation rate of 20%. Determine, to the nearest 0.1 year, in how many years the car will depreciate to half its original value.

**46. PHYSICS** The current  $I(t)$  (measured in amperes) of a circuit is given by the function  $I(t) = 6(1 - e^{-2.5t})$ , where  $t$  is the number of seconds after the switch is closed.

- a. Find the current when  $t = 0$ .
- b. Find the current when  $t = 0.5$ .

- c. Solve the equation for  $t$ .



- 47. AIR RESISTANCE** Assuming that air resistance is proportional to velocity, the velocity  $v$ , in feet per second, of a falling object after  $t$  seconds is given by  $v = 32(1 - e^{-t})$ .

- Graph this equation for  $t \geq 0$ .
- Determine algebraically, to the nearest 0.01 second, when the velocity is 20 feet per second.
- Determine the horizontal asymptote of the graph of  $v$ .
- Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

- 48. AIR RESISTANCE** Assuming that air resistance is proportional to velocity, the velocity  $v$ , in feet per second, of a falling object after  $t$  seconds is given by  $v = 64(1 - e^{-t/2})$ .

- Graph this equation for  $t \geq 0$ .
- Determine algebraically, to the nearest 0.1 second, when the velocity is 50 feet per second.
- Determine the horizontal asymptote of the graph of  $v$ .
- Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

- 49.** The distance  $s$  (in feet) that the object in Exercise 47 will fall in  $t$  seconds is given by the function  $s = 32t + 32(e^{-t} - 1)$ .

- Graph this equation for  $t \geq 0$ .

- b. Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet.

- c. Calculate the slope of the secant line through  $(1, s(1))$  and  $(2, s(2))$ .

- d. Write a sentence that explains the meaning of the slope of the secant line you calculated in c.

- 50.** The distance  $s$  (in feet) that the object in Exercise 48 will fall in  $t$  seconds is given by the function  $s = 64t + 128(e^{-t/2} - 1)$ .

- Graph this equation for  $t \geq 0$ .
- Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet.
- Calculate the slope of the secant line through  $(1, s(1))$  and  $(2, s(2))$ .
- Write a sentence that explains the meaning of the slope of the secant line you calculated in c.

- 51. LEARNING THEORY** The logistic model is also used in learning theory. Suppose that historical records from employee training at a company show that the percent score on a product information test is given by

$$P = \frac{100}{1 + 25e^{-0.095t}}$$

where  $t$  is the number of hours of training. What is the number of hours (to the nearest hour) of training needed before a new employee will answer 75% of the questions correctly?

- 52. LEARNING THEORY** A company provides training in the assembly of a computer circuit to new employees. Past experience has shown that the number of correctly assembled circuits per week can be modeled by

$$N = \frac{250}{1 + 249e^{-0.503t}}$$

where  $t$  is the number of weeks of training. What is the number of weeks (to the nearest week) of training needed before a new employee will correctly make 140 circuits?

- 53. MEDICATION LEVEL** A patient is given three doses of aspirin. Each dose contains 1 gram of aspirin. The second and third doses are each taken 3 hours after the previous dose is administered. The half-life of the aspirin is 2 hours. The amount of aspirin  $A$  in the patient's body  $t$  hours after the first dose is administered is

$$A(t) = \begin{cases} 0.5^{t/2} & 0 \leq t < 3 \\ 0.5^{t/2} + 0.5^{(t-3)/2} & 3 \leq t < 6 \\ 0.5^{t/2} + 0.5^{(t-3)/2} + 0.5^{(t-6)/2} & t \geq 6 \end{cases}$$

Find, to the nearest hundredth of a gram, the amount of aspirin in the patient's body when

- a**  $t \equiv 1$       **b**  $t \equiv 4$       **c**  $t \equiv 9$

- 54.**  **MEDICATION LEVEL** Use the dosage formula in Exercise 53 to determine when, to the nearest tenth of an hour, the amount of aspirin in the patient's body first reaches 0.25 gram.

- 55.**  **ANNUAL GROWTH RATE** The exponential growth function for the population of a city is  $N(t) = 78,245e^{0.0245t}$ , where  $t$  is in years. Because

$$e^{0.0245t} = (e^{0.0245})^t \approx (1.0248)^t$$

we can write the growth function as

$$N(t) = 78,245(1.0248)^t \approx 78,245 \left(1 + \frac{0.0248}{1}\right)^{1+t}$$

In this form we can see that the city's population is growing by 2.48% per year.

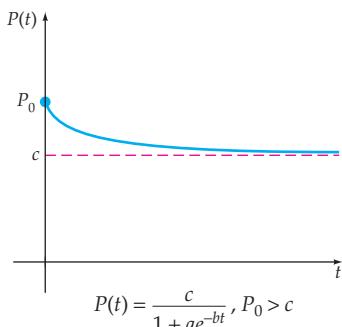
The population of the city of Lake Tahoe, Nevada, can be modeled by the exponential growth function  $N(t) = 22,755e^{0.0287t}$ . Find the annual growth rate, expressed as a percent, of Lake Tahoe. Round to the nearest hundredth of a percent.

- 56. OIL SPILLS** Crude oil leaks from a tank at a rate that depends on the amount of oil that remains in the tank.

Because  $\frac{1}{8}$  of the oil in the tank leaks out every 2 hours, the volume  $V(t)$  of oil in the tank after  $t$  hours is given by  $V(t) = V_0(0.875)^{t/2}$ , where  $V_0 = 350,000$  gallons is the number of gallons in the tank at the time the tank started to leak ( $t = 0$ ).

- a. How many gallons does the tank hold after 3 hours?
  - b. How many gallons does the tank hold after 5 hours?
  - c. How long, to the nearest hour, will it take until 90% of the oil has leaked from the tank?

**A DECLINING LOGISTIC MODEL** If  $P_0 > c$  (which implies that  $-1 < a < 0$ ), then the logistic function  $P(t) = \frac{c}{1 + ae^{-bt}}$  decreases as  $t$  increases. Biologists often use this type of logistic function to model populations that decrease over time. See the following figure.



- 1. A DECLINING FISH POPULATION** A biologist finds that the fish population in a small lake can be closely modeled by the logistic function

$$P(t) = \frac{1000}{1 + (-0.3333)e^{-0.05t}}$$

where  $t$  is the time, in years, since the lake was first stocked with fish.

- a. What was the fish population when the lake was first stocked with fish?
  - b. According to the logistic model, what will the fish population approach in the long-term future?

- 2. A DECLINING DEER POPULATION** The deer population in a reserve is given by the logistic function

$$P(t) = \frac{1800}{1 + (-0.25)e^{-0.07t}}$$

where  $t$  is the time, in years, since July 1, 2001.

- a. What was the deer population on July 1, 2001? What was the deer population on July 1, 2003?
  - b. According to the logistic model, what will the deer population approach in the long-term future?


**MODELING WORLD  
RECORD TIMES IN  
THE MEN'S MILE**

**RACE** In the early 1950s, many people speculated that no runner would ever run a mile race in under 4 minutes. During the period from 1913 to 1945, the world record in the mile event had been reduced from 4.14.4 (4 minutes, 14.4 seconds) to 4.01.4, but no one seemed capable of running a sub-four-minute mile. Then, in 1954, Roger Bannister broke through the four-minute barrier by



running a mile in 3.59.6. In 1999, the current record of 3.43.13 was established. It is fun to think about future record times in the mile race. Will they ever go below 3 minutes, 30 seconds? Below 3 minutes, 20 seconds? What about a sub-three-minute mile?

A declining logistic function that closely models the world record times  $WR$ , in seconds, in the men's mile run from 1913 ( $t = 0$ ) to 1999 ( $t = 86$ ) is given by

$$WR(t) = \frac{199.13}{1 + (-0.21726)e^{-0.0079889t}}$$

- a. Use the above logistic function to predict the world record time for the men's mile run in the year 2020 and the year 2050.
- b. According to the logistic function, what time will the world record in the men's mile event approach but never break through?

## Section 7.6

- Analyze Scatter Plots
- Model Data
- Find a Logistic Growth Model

# Modeling Data with Exponential and Logarithmic Functions

## PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A40.

**PS1.** Determine whether  $N(t) = 4 - \ln t$  is an increasing or a decreasing function. [7.2]

**PS2.** Determine whether  $P(t) = 1 - 2(1.05^t)$  is an increasing or a decreasing function. [7.1]

**PS3.** Evaluate  $P(t) = \frac{108}{1 + 2e^{-0.1t}}$  for  $t = 0$ . [7.1]

**PS4.** Evaluate  $N(t) = 840e^{1.05t}$  for  $t = 0$ . [7.1]

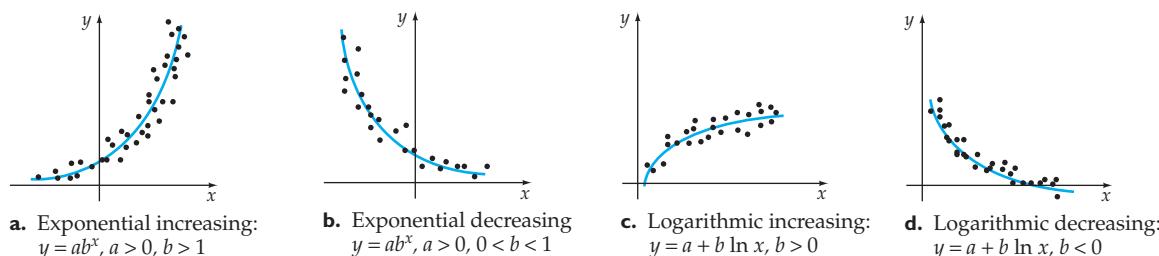
**PS5.** Solve  $10 = \frac{20}{1 + 2.2e^{-0.05t}}$  for  $t$ . Round to the nearest tenth. [7.4]

**PS6.** Determine the horizontal asymptote of the graph of  $P(t) = \frac{55}{1 + 3e^{-0.08t}}$ , for  $t \geq 0$ . [7.5]

## Analyze Scatter Plots

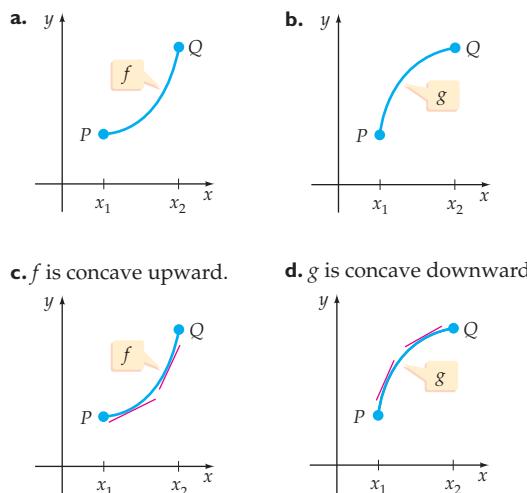
In Section 1.7 we used linear and quadratic functions to model several data sets. However, in some applications, data can be modeled more closely by using

exponential or logarithmic functions. For instance, **Figure 7.31** illustrates some scatter plots that can be modeled effectively by exponential and logarithmic functions.



**Figure 7.31**  
Exponential and Logarithmic Models

The terms *concave upward* and *concave downward* are often used to describe a graph. For instance, **Figures 7.32a** and **7.32b** show the graphs of two increasing functions that join the points  $P$  and  $Q$ . The graphs of  $f$  and  $g$  differ in that they bend in different directions. We can distinguish between these two types of “bending” by examining the positions of *tangent lines* to the graphs. In **Figures 7.32c** and **7.32d**, tangent lines (in red) have been drawn to the graphs of  $f$  and  $g$ . The graph of  $f$  lies above its tangent lines and the graph of  $g$  lies below its tangent lines. The function  $f$  is said to be *concave upward*, and  $g$  is *concave downward*.



**Figure 7.32**

### Definition of Concavity

If the graph of  $f$  lies above all of its tangents on an interval  $[x_1, x_2]$ , then  $f$  is **concave upward** on  $[x_1, x_2]$ .

If the graph of  $f$  lies below all of its tangents on an interval  $[x_1, x_2]$ , then  $f$  is **concave downward** on  $[x_1, x_2]$ .

An examination of the graphs in **Figure 7.31** shows that the graphs of all exponential functions of the form  $y = ab^x$ ,  $a > 0$ ,  $b > 0$ ,  $b \neq 1$  are concave upward. The graphs of increasing logarithmic functions of the form  $y = a + b \ln x$ ,  $b > 0$  are concave downward, and the graphs of decreasing logarithmic functions of the form  $y = a + b \ln x$ ,  $b < 0$ , are concave upward.

In Example 1 we analyze scatter plots to determine whether the shape of the scatter plot can be best approximated by a function that is concave upward or concave downward.

**QUESTION** Is the graph of  $y = 5 - 2 \ln x$  concave upward or concave downward?

### EXAMPLE 1 » Analyze Scatter Plots



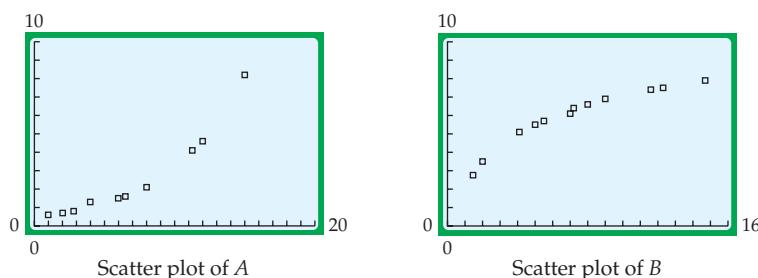
For each of the following data sets, determine whether the most suitable model of the data would be an increasing exponential function or an increasing logarithmic function.

$$A = \{(1, 0.6), (2, 0.7), (2.8, 0.8), (4, 1.3), (6, 1.5), (6.5, 1.6), (8, 2.1), (11.2, 4.1), (12, 4.6), (15, 8.2)\}$$

$$B = \{(1.5, 2.8), (2, 3.5), (4.1, 5.1), (5, 5.5), (5.5, 5.7), (7, 6.1), (7.2, 6.4), (8, 6.6), (9, 6.9), (11.6, 7.4), (12.3, 7.5), (14.7, 7.9)\}$$

#### Solution

For each set, construct a scatter plot of the data. See **Figure 7.33**.



**Figure 7.33**

The scatter plot of  $A$  suggests that  $A$  is an increasing function that is concave upward. **Thus an increasing exponential function would be the most suitable model for data set  $A$ .**

The scatter plot of  $B$  suggests that  $B$  is an increasing function that is concave downward. **Thus an increasing logarithmic function would be the most suitable model for data set  $B$ .**

### » Try Exercise 4, page 529

**ANSWER** The graph of  $y = 5 - 2 \ln x$  is concave upward because the  $b$ -value,  $-2$ , is less than zero. See **Figure 7.31d**.



## Integrating Technology

Most graphing utilities have built-in routines that can be used to determine the exponential or logarithmic regression function that models a set of data. On a TI-83/TI-83 Plus/TI-84 Plus, the ExpReg instruction is used to find the exponential regression function and the LnReg instruction is used to find the logarithmic regression function. The TI-83/TI-83 Plus/TI-84 Plus does not show the value of the regression coefficient  $r$  or the coefficient of determination unless the DiagnosticOn command has been entered. The DiagnosticOn command is in the CATALOG menu.

### take note

The value of a diamond is generally determined by its color, cut, clarity, and carat weight. These characteristics of a diamond are known as the four c's. In Example 2 we have assumed that the color, cut, and clarity of all the diamonds are similar. This assumption enables us to model the value of each diamond as a function of just its carat weight.

## Model Data

The methods used to model data using exponential or logarithmic functions are similar to the methods used in Section 1.7 to model data using linear or quadratic functions. Here is a summary of the modeling process.

### The Modeling Process

Use a graphing utility to:

1. Construct a *scatter plot* of the data to determine which type of function will effectively model the data.
2. Find the *equation* of the modeling function and the correlation coefficient or the coefficient of determination for the equation.
3. Examine the *correlation coefficient* or the *coefficient of determination* and *view a graph* that displays both the modeling function and the scatter plot to determine how well your function fits the data.

In the following example we use the modeling process to find an exponential function that closely models the value of a diamond as a function of its weight.

### EXAMPLE 2 » Model an Application with an Exponential Function

A diamond merchant has determined the values of several white diamonds that have different weights (measured in carats), but are *similar in quality*. See **Table 7.13**.

**Table 7.13**

| 0.50 ct | 0.75 ct | 1.00 ct | 1.25 ct | 1.50 ct | 1.75 ct | 2.00 ct | 3.00 ct  | 4.00 ct  |
|---------|---------|---------|---------|---------|---------|---------|----------|----------|
|         |         |         |         |         |         |         |          |          |
| \$4,600 | \$5,000 | \$5,800 | \$6,200 | \$6,700 | \$7,300 | \$7,900 | \$10,700 | \$14,500 |

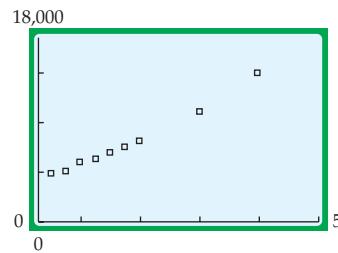
Find a function that models the values of the diamonds as a function of their weights, and use the function to predict the value of a 3.5-carat diamond of similar quality.

### Solution

1. Construct a scatter plot of the data.

| L1   | L2   | L3    | 2 |
|------|------|-------|---|
| .5   | 4600 | ----- |   |
| .75  | 5000 |       |   |
| 1    | 5800 |       |   |
| 1.25 | 6200 |       |   |
| 1.5  | 6700 |       |   |
| 1.75 | 7300 |       |   |
| 2    | 7900 |       |   |

L2(6) = 7300



**Figure 7.34**

**Math Matters**

The Hope Diamond, shown below, is the world's largest deep blue diamond. It has a weight of 45.52 carats. We should not expect the function  $y \approx 4067.6 \times 1.3816^x$  in Example 2 to yield an accurate value of the Hope Diamond because the Hope Diamond is not the same type of diamond as the diamonds in **Table 7.13**, and its weight is much larger than the weights of the diamonds in **Table 7.13**.



The Hope Diamond is on display at the Smithsonian Museum of Natural History in Washington, D.C.

From the scatter plot in **Figure 7.34**, it appears that the data can be closely modeled by an exponential function of the form  $y = ab^x$ ,  $a > 0$  and  $b > 1$ .

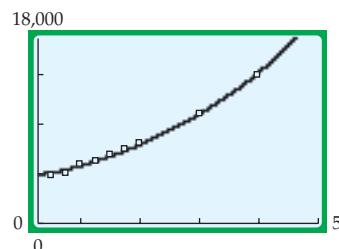
- Find the equation of the model.** The calculator display in **Figure 7.35** shows that the exponential regression equation is  $y \approx 4067.6(1.3816)^x$ , where  $x$  is the carat weight of the diamond and  $y$  is the value of the diamond.

```
ExpReg
y=a*b^x
a=4067.641145
b=1.381644186
r^2=.994881215
r=.9974373238
```

**Figure 7.35**

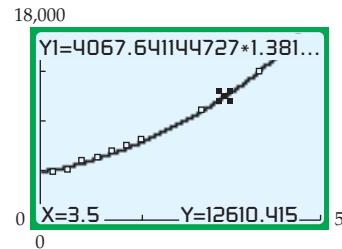
ExpReg display (DiagnosticOn)

- Examine the correlation coefficient or the coefficient of determination.** The correlation coefficient  $r \approx 0.9974$  is close to 1. This indicates that the exponential regression function  $y \approx 4067.6(1.3816)^x$  provides a good fit for the data. The graph in **Figure 7.36** also shows that the exponential regression function provides a good model for the data.



**Figure 7.36**

To estimate the value of a 3.5-carat diamond, substitute 3.5 for  $x$  in the exponential regression function or use the VALUE command in the CALCULATE menu to evaluate the exponential regression function at  $x = 3.5$ . See **Figure 7.37**.



**Figure 7.37**

According to the exponential regression function, the value of a 3.5-carat diamond of similar quality is about \$12,610.

» Try Exercise 22, page 531

When you are selecting a function to model a given set of data, try to find a function that provides a good fit to the data *and* is likely to produce realistic predictions. The following guidelines may facilitate the selection process.

### Guidelines for Selecting a Modeling Function

1. Use a graphing utility to construct a scatter plot of the data.
2. Compare the graphical features of the scatter plot with the graphical features of the basic modeling functions available on the graphing utility: linear, quadratic, cubic, exponential, logarithmic, or logistic. Pay particular attention to the concave nature of each function. Eliminate those functions that do not display the desired concavity.
3. Use the graphing utility to find the equation of each type of function you identified in Step 2 as a possible model.
4. Determine how well each function fits the given data, and compare the graphs of the functions to determine which function is most likely to produce realistic predictions.

### EXAMPLE 3 » Select a Modeling Function and Make a Prediction



**Table 7.14** shows the winning times in the women's Olympic 100-meter freestyle event for the years 1960 to 2004.

**Table 7.14 Women's Olympic 100-Meter Freestyle, 1960 to 2004**

| Year | Time (in seconds) |
|------|-------------------|
| 1960 | 61.2              |
| 1964 | 59.5              |
| 1968 | 60.0              |
| 1972 | 58.59             |
| 1976 | 55.65             |
| 1980 | 54.79             |

| Year | Time (in seconds) |
|------|-------------------|
| 1984 | 55.92             |
| 1988 | 54.93             |
| 1992 | 54.64             |
| 1996 | 54.50             |
| 2000 | 53.83             |
| 2004 | 53.84             |

Source: Time Almanac 2006.

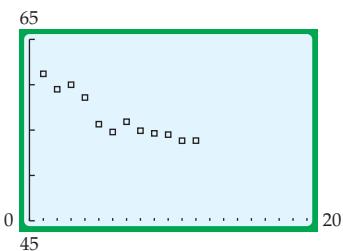
#### take note

When you use a graphing utility to find a logarithmic model, remember that the domain of  $y = a + b \ln x$  is the set of positive numbers. Thus zero must not be used as an  $x$  value of a data point. This is the reason we have used  $x = 1$  to represent the year 1960 in Example 3.

Find a function to model the data, and use the function to predict the winning time in the women's Olympic 100-meter freestyle event for the year 2012.

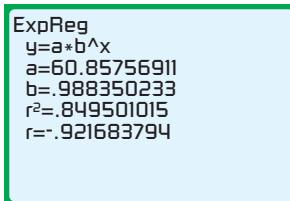
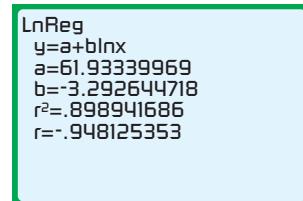
#### Solution

Construct a scatter plot of the data. See **Figure 7.38.** (Note: This scatter plot was produced using  $x = 1$  to represent the year 1960,  $x = 2$  to represent the year 1964, . . . , and  $x = 12$  to represent the year 2004.) The general shape of the scatter plot suggests that we consider functions whose graphs are decreasing and concave upward. Thus we



**Figure 7.38**

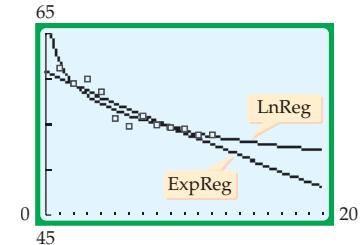
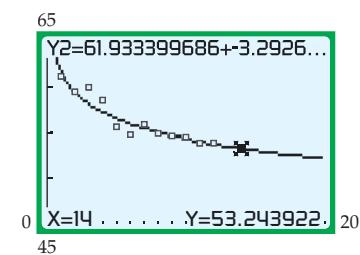
consider a decreasing exponential function and a decreasing logarithmic function as possible models. Use a graphing utility to find the exponential regression function and the logarithmic regression function for the data. See **Figures 7.39 and 7.40**.

**Figure 7.39****Figure 7.40**

The exponential function is  $y \approx 60.85757(0.98835)^x$  and the logarithmic function is  $y \approx 61.93340 - 3.29264 \ln x$ . The coefficient of determination  $r^2$  for the logarithmic regression is larger than the coefficient of determination for the exponential regression. See **Figures 7.39 and 7.40**. Thus the logarithmic regression function provides a better fit to the data than the exponential regression function. The correlation coefficients  $r$  can also be used to determine which function provides the better fit. For decreasing functions, the function with correlation coefficient closest to  $-1$  provides the better fit.

Notice that the graph of the logarithmic function has the desired behavior to the right of the scatter plot. That is, it is a *gradually decreasing curve*, and this is the *general behavior* we would expect for future winning times in the 100-meter freestyle event. The graph of the exponential function is almost linear and is decreasing at a rapid pace, which is not what we would expect for results in an established Olympic event. See **Figure 7.41**. Thus we select the logarithmic function as our modeling function.

To predict the winning time for this event in the year 2012 (represented by  $x = 14$ ), substitute 14 for  $x$  in the equation of the logarithmic function or use the VALUE command in the CALCULATE menu to produce the approximate time of **53.24 seconds**, as shown in **Figure 7.42**.

**Figure 7.41****Figure 7.42**

### Find a Logistic Growth Model

If a scatter plot of a set of data suggests that the data can be effectively modeled by a logistic growth model, then you can use the Logistic feature of a graphing utility to find the logistic growth model. This process is illustrated in Example 4.

#### EXAMPLE 4 » Find a Logistic Growth Model



**Table 7.15** shows the population of deer in an animal preserve for the years 1990 to 2004.

**Table 7.15 Deer Population at the Wild West Animal Preserve**

| Year | Population | Year | Population | Year | Population |
|------|------------|------|------------|------|------------|
| 1990 | 320        | 1995 | 1150       | 2000 | 2620       |
| 1991 | 410        | 1996 | 1410       | 2001 | 2940       |
| 1992 | 560        | 1997 | 1760       | 2002 | 3100       |
| 1993 | 730        | 1998 | 2040       | 2003 | 3300       |
| 1994 | 940        | 1999 | 2310       | 2004 | 3460       |

Find a logistic model that approximates the deer population as a function of the year. Use the model to predict the deer population in the year 2010.



#### Integrating Technology

On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, the logistic growth model is given in the form

$$y = \frac{c}{1 + ae^{-bx}}$$

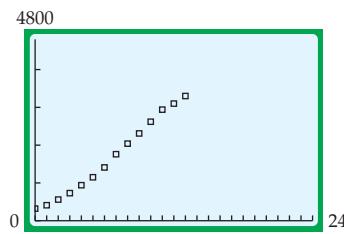
Think of the variable  $x$  as the time  $t$  and the variable  $y$  as  $P(t)$ .

#### Solution

##### 1. Construct a scatter plot of the data.

Enter the data into a graphing utility, and then use the utility to display a scatter plot of the data. In this example we represent the year 1990 by  $x = 0$ , the year 2004 by  $x = 14$ , and the deer population by  $y$ .

**Figure 7.43** shows that the data can be closely approximated by a logistic growth model.



**Figure 7.43**

##### 2. Find the equation of the model.

On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, select B: Logistic, which is in the STAT CALC menu. The logistic function for the data is

$$y \approx \frac{3965.3}{1 + 11.445e^{-0.31152x}}. \text{ See } \textbf{Figure 7.44}.$$

```
Logistic
y=c/(1+ae^(-bx))
a=11.44466821
b=-.3115234553
c=3965.337214
```

**Figure 7.44**

##### 3. Examine the fit.

A TI-83/TI-83 Plus/TI-84 Plus calculator does not compute the coefficient of determination or the correlation coefficient for a logistic model. However, **Figure 7.45** shows that the logistic model provides a good fit to the data. The VALUE command in the CALCULATE menu shows that the logistic model predicts a deer population of about 3878 in the year 2010 ( $x = 20$ ). See **Figure 7.46**.

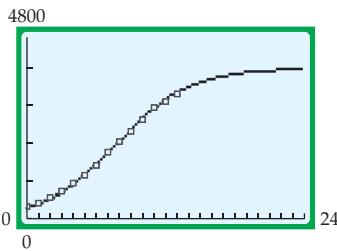


Figure 7.45

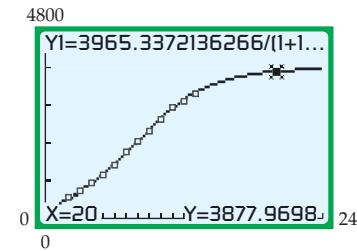


Figure 7.46

» Try Exercise 26, page 532



### Topics for Discussion

1. A student tries to determine the exponential regression equation for the following data.

|          |   |   |   |      |    |
|----------|---|---|---|------|----|
| <b>x</b> | 1 | 2 | 3 | 4    | 5  |
| <b>y</b> | 8 | 2 | 0 | -1.5 | -2 |

The student's calculator displays an ERROR message. Explain why the calculator was unable to determine the exponential regression equation for the data.

2. Consider the logarithmic model  $h(x) = 6 - 2 \ln x$ .
  - a. Is  $h$  an increasing or a decreasing function?
  - b. Is  $h$  concave up or concave down on the interval  $(0, \infty)$ ?
  - c. Find, if possible,  $h(0)$  and  $h(e)$ .
  - d. Does  $h$  have a horizontal asymptote? Explain.



## Exercises Set 7.6



In Exercises 1 to 6, use a scatter plot of the given data to determine which of the following types of functions might provide a suitable model of the data.

- An increasing exponential function  
 $y = ab^x$ ,  $a > 0$ ,  $b > 1$  (See Figure 7.31a.)
- An increasing logarithmic function  
 $y = a + b \ln x$ ,  $b > 0$  (See Figure 7.31c.)
- A decreasing exponential function  
 $y = ab^x$ ,  $a > 0$ ,  $0 < b < 1$  (See Figure 7.31b.)
- A decreasing logarithmic function  
 $y = a + b \ln x$ ,  $b < 0$  (See Figure 7.31d.)

(Note: Some data sets can be closely modeled by more than one type of function.)

1.  $\{(1, 3), (1.5, 4), (2, 6), (3, 13), (3.5, 19), (4, 27)\}$
2.  $\{(1.0, 1.12), (2.1, 0.87), (3.2, 0.68), (3.5, 0.63), (4.4, 0.52)\}$
3.  $\{(1, 2.4), (2, 1.1), (3, 0.5), (4, 0.2), (5, 0.1)\}$
4.  $\{(5, 2.3), (7, 3.9), (9, 4.5), (12, 5.0), (16, 5.4), (21, 5.8), (26, 6.1)\}$
5.  $\{(1, 2.5), (1.5, 1.7), (2, 0.7), (3, -0.5), (3.5, -1.3), (4, -1.5)\}$
6.  $\{(1, 3), (1.5, 3.8), (2, 4.4), (3, 5.2), (4, 5.8), (6, 6.6)\}$



**In Exercises 7 to 10, find the exponential regression function for the data. State the correlation coefficient  $r$ . Round  $a$ ,  $b$ , and  $r$  to the nearest hundred thousandth.**

7.  $\{(10, 6.8), (12, 6.9), (14, 15.0), (16, 16.1), (18, 50.0), (19, 20.0)\}$
8.  $\{(2.6, 16.2), (3.8, 48.8), (5.1, 160.1), (6.5, 590.2), (7, 911.2)\}$
9.  $\{(0, 1.83), (1, 0.92), (2, 0.51), (3, 0.25), (4, 0.13), (5, 0.07)\}$
10.  $\{(4.5, 1.92), (6.0, 1.48), (7.5, 1.14), (10.2, 0.71), (12.3, 0.49)\}$



**In Exercises 11 to 14, find the logarithmic regression function for the data. State the correlation coefficient  $r$ . Round  $a$ ,  $b$ , and  $r$  to the nearest hundred thousandth.**

11.  $\{(5, 2.7), (6, 2.5), (7.2, 2.2), (9.3, 1.9), (11.4, 1.6), (14.2, 1.3)\}$
12.  $\{(11, 15.75), (14, 15.52), (17, 15.34), (20, 15.18), (23, 15.05)\}$
13.  $\{(3, 16.0), (4, 16.5), (5, 16.9), (7, 17.5), (8, 17.7), (9.8, 18.1)\}$
14.  $\{(8, 67.1), (10, 67.8), (12, 68.4), (14, 69.0), (16, 69.4)\}$



**In Exercises 15 to 18, find the logistic regression function for the data. Round the constants  $a$ ,  $b$ , and  $c$  to the nearest hundred thousandth.**

15.  $\{(0, 81), (2, 87), (6, 98), (10, 110), (15, 125)\}$
16.  $\{(0, 175), (5, 195), (10, 217), (20, 264), (35, 341)\}$
17.  $\{(0, 955), (10, 1266), (20, 1543), (30, 1752)\}$
18.  $\{(0, 1588), (5, 2598), (10, 3638), (25, 5172)\}$



**MOVIE TICKET PRICES** The following table shows the average U.S. movie theater ticket prices for selected years from 1994 to 2004.

| Year | Price, $P$ |
|------|------------|
| 1994 | \$4.08     |
| 1996 | \$4.42     |
| 1998 | \$4.69     |
| 2000 | \$5.39     |
| 2002 | \$5.80     |
| 2004 | \$6.21     |

Source: National Association of Theatre Owners

- a. Determine an exponential regression model and a linear regression model for the data. Use  $t = 0$  to represent

the year 1994 and  $t = 2$  to represent the year 1996. Round the constants  $a$  and  $b$  to the nearest hundred thousandth. State the correlation coefficient  $r$  for each model.

- b. Examine the correlation coefficients to determine which model provides a better fit for the data.
- c. Use the model you selected in b. to predict the average U.S. movie theater ticket price for the year 2010.



#### GENERATION OF GARBAGE

According to the U.S. Environmental Protection Agency, the amount of garbage generated per person has been increasing over the last few decades. The following table shows the per capita garbage, in pounds per day, generated in the United States.

| Year                | 1960 | 1970 | 1980 | 1990 | 2003 |
|---------------------|------|------|------|------|------|
| Pounds per day, $p$ | 2.7  | 3.3  | 3.7  | 4.5  | 4.5  |

- a. Find a linear model and a logarithmic model for the data. Use  $t$  as the independent variable (domain) and  $p$  as the dependent variable (range). Represent the year 1960 by  $t = 60$  and the year 1970 by  $t = 70$ .
- b. Examine the correlation coefficients to determine which model provides a better fit for the data.
- c. Use the model you selected in b. to predict the amount of garbage that will be generated per capita per day in 2009. Round to the nearest tenth of a pound.



#### HYPOTHERMIA

The following table shows the time  $T$ , in hours, before a scuba diver wearing a 3-millimeter-thick wet suit reaches hypothermia ( $95^{\circ}\text{F}$ ) for various water temperatures  $F$ , in degrees Fahrenheit.

| Water temperature, $^{\circ}\text{F}$ | Time $T$ , hours |
|---------------------------------------|------------------|
| 41                                    | 1.1              |
| 46                                    | 1.4              |
| 50                                    | 1.8              |
| 59                                    | 3.7              |

- a. Find an exponential regression model for the data. Round the constants  $a$  and  $b$  to the nearest hundred thousandth.

- b. Use the model from a. to estimate the time it takes for the diver to reach hypothermia in water that has a temperature of  $65^{\circ}\text{F}$ . Round to the nearest tenth of an hour.



**ATMOSPHERIC PRESSURE** The following table shows the earth's atmospheric pressure  $y$  (in newtons per square centimeter) at an altitude of  $x$  kilometers. Find a suitable function that models the atmospheric pressure as a function of the altitude. Use the function to estimate the atmospheric pressure at an altitude of 24 kilometers. Round to the nearest tenth of a newton per square centimeter.

| Altitude $x$ , kilometers | Pressure $y$ , newtons/cm <sup>2</sup> |
|---------------------------|--|
| 0                         | 10.3                                   |
| 2                         | 8.0                                    |
| 4                         | 6.4                                    |
| 6                         | 5.1                                    |
| 8                         | 4.0                                    |
| 10                        | 3.2                                    |
| 12                        | 2.5                                    |
| 14                        | 2.0                                    |
| 16                        | 1.6                                    |
| 18                        | 1.3                                    |



**HYPOTHERMIA** The following table shows the time  $T$ , in hours, before a scuba diver wearing a 4-millimeter-thick wet suit reaches hypothermia ( $95^{\circ}\text{E}$ ) for various water temperatures  $F$ , in degrees Fahrenheit.

| Water temperature, $^{\circ}\text{F}$ | Time $T$ , hours |
|---------------------------------------|------------------|
| 41                                    | 1.5              |
| 46                                    | 1.9              |
| 50                                    | 2.4              |
| 59                                    | 5.2              |

- a. Find an exponential regression model for the data. Round the constants  $a$  and  $b$  to the nearest hundred thousandth.
- b. Use the model from a. to estimate the time it takes for the diver to reach hypothermia in water that has a temperature of  $65^{\circ}\text{F}$ . Round to the nearest tenth of an hour.

an hour. How much greater is this result compared with the answer to Exercise 21b.?



#### 400-METER RACE

The following table lists the progression of world record times in the men's 400-meter race for the years from 1948 to 2005. (Note: No new world record times were set during the time period from 2000 to 2005.)

**World Record Times in the Men's 400-Meter Race, 1948 to 2005**

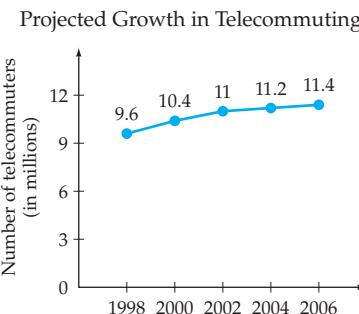
| Year | Time, in seconds | Year | Time, in seconds |
|------|------------------|------|------------------|
| 1948 | 45.9             | 1964 | 44.9             |
| 1950 | 45.8             | 1967 | 44.5             |
| 1955 | 45.4             | 1968 | 44.1             |
| 1956 | 45.2             | 1968 | 43.86            |
| 1960 | 44.9             | 1988 | 43.29            |
| 1963 | 44.9             | 1999 | 43.18            |

Source: Track and Field Statistics, <http://trackfield.brinkster.net/Main.asp>.

- a. Determine whether the data can best be modeled by a decreasing exponential function or a decreasing logarithmic function. Let  $x = 48$  represent the year 1948 and  $x = 50$  represent the year 1950.
- b. Assume a new world record time will be established in 2008. Use the function you chose in a. to predict the world record time in the men's 400-meter race for the year 2008. Round to the nearest hundredth of a second.



**TELECOMMUTING** The graph below shows the projected growth in the number of telecommuters.



Which type of model, an increasing exponential model or an increasing logarithmic model, is more likely to provide a better fit for the data? Explain.



**POPULATION OF HAWAII** The following table shows the population of the state of Hawaii for selected years from 1950 to 2004.

Population of the State of Hawaii

| Year | Population, $P$ |
|------|-----------------|
| 1950 | 499,000         |
| 1955 | 529,000         |
| 1960 | 642,000         |
| 1965 | 704,000         |
| 1970 | 762,920         |
| 1975 | 875,052         |
| 1980 | 967,710         |

| Year | Population, $P$ |
|------|-----------------|
| 1985 | 1,039,698       |
| 1990 | 1,113,491       |
| 1995 | 1,196,854       |
| 2000 | 1,212,670       |
| 2001 | 1,227,024       |
| 2002 | 1,244,898       |
| 2004 | 1,262,840       |

Source: economagic.com, <http://www.economagic.com/em-cgi/data.exe/beapi/a15300>.

- Find a logistic growth model that approximates the population of the state of Hawaii as a function of the year. Use  $t = 0$  to represent the year 1950.
- Use the model from a. to predict the population of the state of Hawaii for the year 2010. Round to the nearest ten thousand.
- What is the carrying capacity of the model? Round to the nearest thousand.



**OPTOMETRY** The *near point*  $p$  of a person is the closest distance at which the person can see an object distinctly. As one grows older, one's near point increases. The table below shows data for the average near point of various people with normal eyesight.

| Age $y$ , years | Near point $p$ (cm) |
|-----------------|---------------------|
| 15              | 11                  |
| 20              | 13                  |
| 25              | 15                  |
| 30              | 17                  |
| 35              | 20                  |
| 40              | 23                  |
| 50              | 26                  |

- Find an exponential regression model for these data. Round each constant to the nearest thousandth.
- What near point does this equation predict for a person 60 years old? Round to the nearest centimeter.



**CHEMISTRY** The amount of oxygen  $x$ , in milliliters per liter, that can be absorbed by water at a certain temperature  $T$ , in degrees Fahrenheit, is given in the following table.

| Temperature, °F | Oxygen absorbed, ml/L |
|-----------------|-----------------------|
| 32              | 10.5                  |
| 38              | 8.4                   |
| 46              | 7.6                   |
| 52              | 7.1                   |
| 58              | 6.8                   |
| 64              | 6.5                   |

- Find a logarithmic regression equation for these data. Round each constant to the nearest thousandth.
- Using your model, how much oxygen, to the nearest tenth of a milliliter per liter, can be absorbed in water that is 50°F?



**THE HENDERSON-HASSELBACH FUNCTION** The scientists Henderson and Hasselbach determined that the pH of blood is a function of the ratio  $q$  of the amounts of bicarbonate and carbonic acid in the blood.

- Determine a linear model and a logarithmic model for the data. Use  $q$  as the independent variable (domain) and pH as the dependent variable (range). State the correlation coefficient for each model. Round  $a$  and  $b$  to five decimal places and  $r$  to six decimal places. Which model provides the better fit for the data?

|                       |     |      |      |      |      |
|-----------------------|-----|------|------|------|------|
| <b><math>q</math></b> | 7.9 | 12.6 | 31.6 | 50.1 | 79.4 |
| <b>pH</b>             | 7.0 | 7.2  | 7.6  | 7.8  | 8.0  |

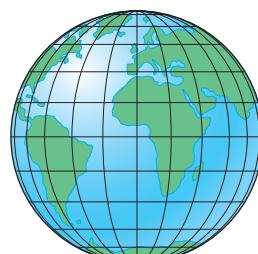
- Use the model you chose in a. to find the  $q$ -value associated with a pH of 8.2. Round to the nearest tenth.



**WORLD POPULATION** The following table lists the years in which the world's population first reached 3, 4, 5, and 6 billion.

#### World Population Milestones

| Year | Population |
|------|------------|
| 1960 | 3 billion  |
| 1974 | 4 billion  |
| 1987 | 5 billion  |
| 1999 | 6 billion  |



Source: *The World Almanac 2006*, p. 851.

- Find an exponential model for the data in the table. Let  $x = 0$  represent the year 1960.
- Use the model to predict the year in which the world's population will first reach 8 billion.



- PANDA POPULATION** One estimate gives the world panda population as 3200 in 1980 and 590 in 2000.

- Find an exponential model for the data and use the model to predict the year in which the panda population  $p$  will be reduced to 200. (Let  $t = 0$  represent the year 1980.)
- Because the exponential model in a. fits the data perfectly, does this mean that the model will accurately predict future panda populations? Explain.



- OLYMPIC HIGH JUMP** The following table shows the Olympic gold medal heights for the women's high jump from 1968 to 2004.



**Women's Olympic High Jump, 1968 to 2004**

| Year | Height         | Year | Height        |
|------|----------------|------|---------------|
| 1968 | 5 ft 11.75 in. | 1988 | 6 ft 8 in.    |
| 1972 | 6 ft 3.65 in.  | 1992 | 6 ft 7.5 in.  |
| 1976 | 6 ft 4 in.     | 1996 | 6 ft 8.75 in. |
| 1980 | 6 ft 5.5 in    | 2000 | 6 ft 7 in.    |
| 1984 | 6 ft 7.5 in.   | 2004 | 6 ft 9.1 in.  |

Source: Time Almanac 2006.

- Determine a linear model and a logarithmic model for the data, with the height measured in inches. State the correlation coefficient  $r$  for each model. Represent the year 1968 by  $x = 1$ , the year 1972 by  $x = 2$ , and the year 2004 by  $x = 10$ .
- Examine the correlation coefficients to determine which model provides a better fit for the data.

- Use the model you selected in b. to predict the women's Olympic gold medal high jump height in 2012. Round to the nearest tenth of an inch.

33.



#### NUMBER OF CINEMA SITES

The following table shows the number of U.S. indoor cinema sites for the years 1999 to 2004.

| Year | Number of Indoor Cinema Sites, $S$ |
|------|------------------------------------|
| 1999 | 7031                               |
| 2000 | 6550                               |
| 2001 | 5813                               |
| 2002 | 5712                               |
| 2003 | 5700                               |
| 2004 | 5629                               |

Source: National Association of Theatre Owners

- Determine an exponential regression model and a logarithmic regression model for the data. Use  $t = 1$  to represent the year 1999. Round the constants  $a$  and  $b$  to the nearest hundred thousandth. State the correlation coefficient  $r$  for each model.
- Examine the correlation coefficients to determine which model provides a better fit for the data.
- Use the model you selected in b. to predict the number of U.S. indoor cinema sites for the year 2009.

34.



#### TEMPERATURE OF COFFEE

A cup of coffee is placed in a room that maintains a constant temperature of 70°F. The following table shows both the coffee temperature  $T$  after  $t$  minutes and the difference between the coffee temperature and the room temperature after  $t$  minutes.

| Time $t$ (minutes)    | 0    | 5    | 10   | 15   | 20  | 25  |
|-----------------------|------|------|------|------|-----|-----|
| Coffee temp. $T$ (°F) | 165° | 140° | 121° | 107° | 97° | 89° |
| $T - 70$ °            | 95°  | 70°  | 51°  | 37°  | 27° | 19° |

- Find an exponential model for the difference  $T - 70$ ° as a function of  $t$ .
- Use the model in a. to predict how long it will take (to the nearest minute) for the coffee to cool to 80°F.

- 35.**   **WORLD POPULATION** The following table lists the years in which the world's population first reached 3, 4, 5, and 6 billion.

| World Population Milestones |            |
|-----------------------------|------------|
| Year                        | Population |
| 1960                        | 3 billion  |
| 1974                        | 4 billion  |
| 1987                        | 5 billion  |
| 1999                        | 6 billion  |

Source: *The World Almanac 2006*, p. 851.

- a. Find a logistic growth model  $P(t)$  for the data in the table. Let  $t$  represent the number of years after 1960 ( $t = 0$  represents the year 1960).

- 37.**  **DUPLICATE DATA POINTS** An engineer needs to model the data in set  $A$  with an exponential function.

$$A = \{(2, 5), (3, 10), (4, 17), (4, 17), (5, 28)\}$$

Because the ordered pair  $(4, 17)$  is listed twice, the engineer decides to eliminate one of these ordered pairs and model the data in set  $B$ .

$$B = \{(2, 5), (3, 10), (4, 17), (5, 28)\}$$

Determine whether  $A$  and  $B$  both have the same exponential regression function.

- 38.**   **DOMAIN ERROR** A scientist needs to model the data in set  $A$ .

$$A = \{(0, 1.2), (1, 2.3), (2, 2.8), (3, 3.1), (4, 3.3), (5, 3.4)\}$$

The scientist views a scatter plot of the data and decides to model the data with a logarithmic function of the form  $y = a + b \ln x$ .

- a. When the scientist attempts to use a graphing calculator to determine the logarithmic regression equation, the calculator displays the message

## “ERR:DOMAIN”

Explain why the calculator was unable to determine the logarithmic regression equation for the data.

- b. Explain what the scientist could do so that the data in set A could be modeled by a logarithmic function of the form  $y = a + b \ln x$ .

- 39.**  **POWER FUNCTIONS** A function that can be written in the form  $y = ax^b$  is said to be a **power function**.

- b. According to the logistic growth model, what will the world's population approach as  $t \rightarrow \infty$ ? Round to the nearest billion.

- 36. A CORRELATION COEFFICIENT OF 1** A scientist uses a graphing calculator to model the data set  $\{(2, 5), (4, 6)\}$  with a logarithmic function. The following display shows the results.

LnReg  
 $y=a+b\ln x$   
 $a=4$   
 $b=1.442695041$   
 $r^2=1$   
 $r=1$

What is the significance of the fact that the correlation coefficient for the regression equation is  $r = 1$ ?

Some data sets can best be modeled by a power function. On a TI-83/TI-83 Plus/TI-84 Plus, the PwrReg instruction is used to produce a power regression function for a set of data.

- a. Find an exponential regression function and a power regression function for the following data. State the correlation coefficient  $r$  for each model.

|          |     |     |     |      |      |      |
|----------|-----|-----|-----|------|------|------|
| <b>x</b> | 1   | 2   | 3   | 4    | 5    | 6    |
| <b>y</b> | 2.1 | 5.5 | 9.8 | 14.6 | 20.1 | 25.8 |

- b. Which of the two regression functions provides the better fit for the data?

- 40.**  **PERIOD OF A PENDULUM** The following table shows the time  $t$  (in seconds) of the period of a pendulum of length  $l$  (in feet). (Note: The period of a pendulum is the time it takes the pendulum to complete a swing from the right to the left and back.)

| <b>Length /</b> | 1    | 2    | 3    | 4    | 6    | 8    |
|-----------------|------|------|------|------|------|------|
| <b>Time t</b>   | 1.11 | 1.57 | 1.92 | 2.25 | 2.72 | 3.14 |

- a. Determine the equation of the best model for the data. Your model must be a power function or an exponential function.
  - b. According to the model you chose in a. what is the length of a pendulum, to the nearest tenth of a foot, that has a period of 12 seconds?

## »»» Projects



### A MODELING PROJECT

The purpose of this Project is for you to find data that can be modeled by an exponential or a logarithmic function. Choose data from a *real-life* situation that you find interesting. Search for the data in a magazine, in a newspaper, in an almanac, or on the Internet. If you wish, you can collect your data by performing an experiment.

- List the source of your data. Include the date, page number, and any other specifics about the source. If your data were collected by performing an experiment, then provide all the details of the experiment.
- Explain what you have chosen as your variables. Which variable is the dependent variable and which variable is the independent variable?

- Use the three-step modeling process to find a regression equation that models the data.
- Graph the regression equation on the scatter plot of the data. What is the correlation coefficient for the model? Do you think your regression equation accurately models your data? Explain.
- Use the regression equation to predict the value of
  - the dependent variable for a specific value of the independent variable
  - the independent variable for a specific value of the dependent variable
- Write a few comments about what you have learned from this Project.



## Exploring Concepts with Technology

**Table 7.16**

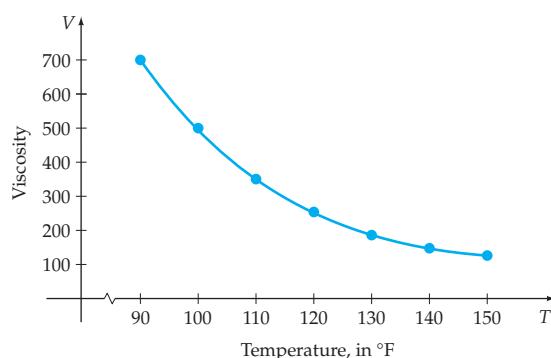
| T   | V   |
|-----|-----|
| 90  | 700 |
| 100 | 500 |
| 110 | 350 |
| 120 | 250 |
| 130 | 190 |
| 140 | 150 |
| 150 | 120 |

### Using a Semilog Graph to Model Exponential Decay

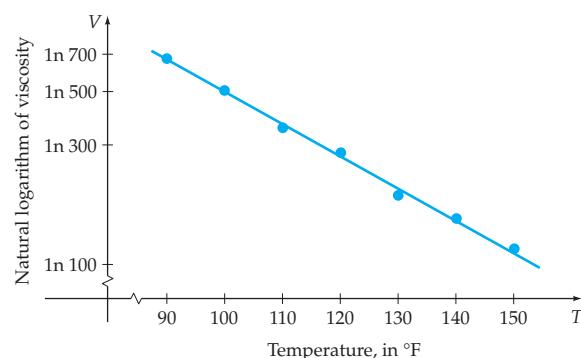
Consider the data in **Table 7.16**, which shows the viscosity  $V$  of SAE 40 motor oil at various temperatures  $T$  in degrees Fahrenheit. The graph of these data is shown below, along with a curve that passes through the points. The graph in **Figure 7.47** appears to have the shape of an exponential decay model.

One way to determine whether the graph in **Figure 7.47** is the graph of an exponential function is to plot the data on *semilog* graph paper. On this graph paper, the horizontal axis remains the same, but the vertical axis uses a logarithmic scale.

The data in **Table 7.16** are graphed again in **Figure 7.48**, but this time the vertical axis is a natural logarithm axis. This graph is approximately a straight line.



**Figure 7.47**



**Figure 7.48**

The slope of the line in **Figure 7.48**, to the nearest ten-thousandth, is

$$m = \frac{\ln 500 - \ln 120}{100 - 150} \approx -0.0285$$

Using this slope and the point-slope formula with  $V$  replaced by  $\ln V$ , we have

$$\begin{aligned}\ln V - \ln 120 &= -0.0285(T - 150) \\ \ln V &\approx -0.0285T + 9.062\end{aligned}\quad (1)$$

Equation (1) is the equation of the line on a semilog coordinate grid.

Now solve Equation (1) for  $V$ .

$$\begin{aligned}e^{\ln V} &= e^{-0.0285T+9.062} \\ V &= e^{-0.0285T}e^{9.062} \\ V &\approx 8621e^{-0.0285T}\end{aligned}\quad (2)$$

Equation (2) is a model of the data in the rectangular coordinate system shown in **Figure 7.47**.

1. A chemist wishes to determine the decay characteristics of iodine-131. A 100-milligram sample of iodine-131 is observed over a 30-day period. **Table 7.17** shows the amount  $A$  (in milligrams) of iodine-131 remaining after  $t$  days.

- a. Graph the ordered pairs  $(t, A)$  on semilog paper. (Note: Semilog paper comes in different varieties. Our calculations are based on semilog paper that has a natural logarithm scale on the vertical axis.)
- b. Use the points  $(4, 4.3)$  and  $(15, 3.3)$  to approximate the slope of the line that passes through the points.
- c. Using the slope calculated in b. and the point  $(4, 4.3)$ , determine the equation of the line.
- d. Solve the equation you derived in c. for  $A$ .
- e. Graph the equation you derived in d. in a rectangular coordinate system.
- f. What is the half-life of iodine-131?

2. The birth rates  $B$  per thousand people in the United States are given in **Table 7.18** for the years 1986 through 1990 ( $t = 0$  corresponds to 1986).

- a. Graph the ordered pairs  $(t, \ln B)$ . (You will need to adjust the scale so that you can discriminate between plotted points. A suggestion is given in **Figure 7.49**.)
- b. Use the points  $(1, 2.754)$  and  $(3, 2.785)$  to approximate the slope of the line that passes through the points.
- c. Using the slope calculated in b. and the point  $(1, 2.754)$ , determine the equation of the line.
- d. Solve the equation you derived in c. for  $B$ .
- e. Graph the equation you derived in d. in a rectangular coordinate system.
- f. If the birth rate continues as predicted by your model, in what year will the birth rate be 17.5 per thousand?

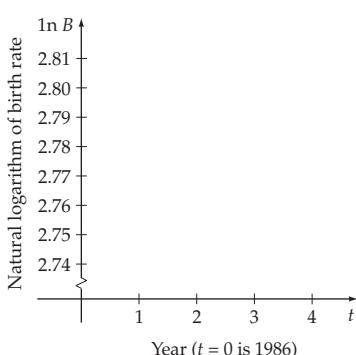


Figure 7.49

The difference in graphing strategies between Exercise 1 and Exercise 2 is that in Exercise 1, semilog paper was used. When a point is graphed on this coordinate paper, the  $y$ -coordinate is  $\ln y$ . In Exercise 2, graphing a point  $(x, \ln y)$  in a rectangular coordinate system has the same effect as graphing  $(x, y)$  in a semilog coordinate system.

## Chapter 7 Summary

### 7.1 Exponential Functions and Their Applications

- For all positive real numbers  $b$ ,  $b \neq 1$ , the exponential function defined by  $f(x) = b^x$  has the following properties:
  - $f$  has the set of real numbers as its domain.
  - $f$  has the set of positive real numbers as its range.
  - $f$  has a graph with a  $y$ -intercept of  $(0, 1)$ .
  - $f$  has a graph that is asymptotic to the  $x$ -axis.
  - $f$  is a one-to-one function.
  - $f$  is an increasing function if  $b > 1$ .
  - $f$  is a decreasing function if  $0 < b < 1$ .
- As  $n$  increases without bound,  $(1 + 1/n)^n$  approaches an irrational number denoted by  $e$ . The value of  $e$  accurate to eight decimal places is 2.71828183.
- The function defined by  $f(x) = e^x$  is called the natural exponential function.

### 7.2 Logarithmic Functions and Their Applications

- Definition of a Logarithm* If  $x > 0$  and  $b$  is a positive constant ( $b \neq 1$ ), then

$$y = \log_b x \text{ if and only if } b^y = x$$

- For all positive real numbers  $b$ ,  $b \neq 1$ , the function defined by  $f(x) = \log_b x$  has the following properties:
  - $f$  has the set of positive real numbers as its domain.
  - $f$  has the set of real numbers as its range.
  - $f$  has a graph with an  $x$ -intercept of  $(1, 0)$ .
  - $f$  has a graph that is asymptotic to the  $y$ -axis.
  - $f$  is a one-to-one function.
  - $f$  is an increasing function if  $b > 1$ .
  - $f$  is a decreasing function if  $0 < b < 1$ .
- The exponential form of  $y = \log_b x$  is  $b^y = x$ .
- The logarithmic form of  $b^y = x$  is  $y = \log_b x$ .
- Basic Logarithmic Properties*
- 1.  $\log_b b = 1$
- 2.  $\log_b 1 = 0$
- 3.  $\log_b (b^x) = x$
- 4.  $b^{\log_b x} = x$
- The function  $f(x) = \log_{10} x$  is the common logarithmic function. It is customarily written as  $f(x) = \log x$ .
- The function  $f(x) = \log_e x$  is the natural logarithmic function. It is customarily written as  $f(x) = \ln x$ .

### 7.3 Properties of Logarithms and Logarithmic Scales

- If  $b$ ,  $M$ , and  $N$  are positive real numbers ( $b \neq 1$ ), and  $p$  is any real number, then
 
$$\log_b(MN) = \log_b M + \log_b N$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b(M^p) = p \log_b M$$

$$\log_b M = \log_b N \text{ implies } M = N$$

$$M = N \text{ implies } \log_b M = \log_b N$$
- Change-of-Base Formula* If  $x$ ,  $a$ , and  $b$  are positive real numbers with  $a \neq 1$  and  $b \neq 1$ , then

$$\log_b x = \frac{\log_a x}{\log_a b}$$

- An earthquake with an intensity of  $I$  has a Richter scale magnitude of  $M = \log\left(\frac{I}{I_0}\right)$ , where  $I_0$  is the measure of the intensity of a zero-level earthquake.
- The pH of a solution with a hydronium-ion concentration of  $H^+$  mole per liter is given by  $\text{pH} = -\log[H^+]$ .

### 7.4 Exponential and Logarithmic Equations

- Equality of Exponents Theorem* If  $b$  is a positive real number ( $b \neq 1$ ) such that  $b^x = b^y$ , then  $x = y$ .
- Exponential equations of the form  $b^x = b^y$  can be solved by using the Equality of Exponents Theorem.
- Exponential equations of the form  $b^x = c$  can be solved by taking either the common logarithm or the natural logarithm of each side of the equation.
- Logarithmic equations can often be solved by using the properties of logarithms and the definition of a logarithm.

### 7.5 Exponential Growth and Decay

- The function defined by  $N(t) = N_0 e^{kt}$  is called an exponential growth function if  $k$  is a positive constant, and it is called an exponential decay function if  $k$  is a negative constant.
- The Compound Interest Formula* A principal  $P$  invested at an annual interest rate  $r$ , expressed as a decimal and compounded  $n$  times per year for  $t$  years, produces the balance

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- Continuous Compounding Interest Formula* If an account with principal  $P$  and annual interest rate  $r$  is compounded continuously for  $t$  years, then the balance is  $A = Pe^{rt}$ .
- The Logistic Model* The magnitude of a population at time  $t$  is given by

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

where  $P_0 = P(0)$  is the population at time  $t = 0$ ,  $c$  is the carrying capacity of the population, and  $b$  is a constant called the growth rate constant.

## 7.6 Modeling Data with Exponential and Logarithmic Functions

- If the graph of  $f$  lies above all of its tangents on  $[x_1, x_2]$ , then  $f$  is concave upward on  $[x_1, x_2]$ .

- If the graph of  $f$  lies below all of its tangents on  $[x_1, x_2]$ , then  $f$  is concave downward on  $[x_1, x_2]$ .
- The Modeling Process* Use a graphing utility to
  - construct a scatter plot of the data to determine which type of function will best model the data.
  - find the equation of the modeling function and the correlation coefficient or the coefficient of determination for the equation.
  - examine the correlation coefficient or the coefficient of determination and view a graph that displays both the function and the scatter plot to determine how well the function fits the data.

## Chapter 7 Assessing Concepts

In Exercises 1 to 5, determine whether the statement is true or false. If the statement is false, give an example or state a reason to demonstrate that the statement is false.

- Every function has an inverse function.
- If  $7^x = 40$ , then  $\log_7 40 = x$ .
- If  $\log_4 x = 3.1$ , then  $4^{3.1} = x$ .
- The exponential function  $h(x) = b^x$  is an increasing function.
- The logarithmic function  $j(x) = \log_b x$  is an increasing function.

In Exercises 6 to 12, match an expression to each description. A letter may be used more than once. Some letters may not be needed.

- A function which is symmetric with respect to the  $y$ -axis. \_\_\_\_\_
  - An increasing function which is symmetric with respect to the origin. \_\_\_\_\_
  - A logistic function. \_\_\_\_\_
  - An increasing function with a vertical asymptote. \_\_\_\_\_
  - A decreasing function with a horizontal asymptote. \_\_\_\_\_
  - A decreasing function with a vertical asymptote. \_\_\_\_\_
  - An increasing function which is concave upward. \_\_\_\_\_
- a.  $f(x) = \ln(x - 4)$
- b.  $f(x) = \frac{e^x - e^{-x}}{2}$
- c.  $f(x) = \frac{e^x + e^{-x}}{2}$
- d.  $f(x) = \log_{1/2} x$
- e.  $f(x) = \frac{1}{2}e^{-x}$
- f.  $f(x) = \frac{485}{1 + 9.5e^{-0.019x}}, x \geq 0$
- g.  $f(x) = e^x$



## Chapter 7 Review Exercises

In Exercises 1 to 12, solve each equation. Do not use a calculator.

1.  $\log_5 25 = x$

2.  $\log_3 81 = x$

3.  $\ln e^3 = x$

4.  $\ln e^\pi = x$

5.  $3^{2x+7} = 27$

6.  $5^{x-4} = 625$

7.  $2^x = \frac{1}{8}$

8.  $27(3^x) = 3^{-1}$

9.  $\log x^2 = 6$

10.  $\frac{1}{2} \log |x| = 5$

11.  $10^{\log 2x} = 14$

12.  $e^{\ln x^2} = 64$

In Exercises 13 to 22, sketch the graph of each function.

13.  $f(x) = (2.5)^x$

14.  $f(x) = \left(\frac{1}{4}\right)^x$

15.  $f(x) = 3^{|x|}$

16.  $f(x) = 4^{-|x|}$

17.  $f(x) = 2^x - 3$

18.  $f(x) = 2^{(x-3)}$

19.  $f(x) = \frac{1}{3} \log x$

20.  $f(x) = 3 \log x^{1/3}$

21.  $f(x) = -\frac{1}{2} \ln x$

22.  $f(x) = -\ln |x|$



In Exercises 23 and 24, use a graphing utility to graph each function.

23.  $f(x) = \frac{4^x + 4^{-x}}{2}$

24.  $f(x) = \frac{3^x - 3^{-x}}{2}$

In Exercises 25 to 28, change each logarithmic equation to its exponential form.

25.  $\log_4 64 = 3$

26.  $\log_{1/2} 8 = -3$

27.  $\log_{\sqrt{2}} 4 = 4$

28.  $\ln 1 = 0$

In Exercises 29 to 32, change each exponential equation to its logarithmic form.

29.  $5^3 = 125$

30.  $2^{10} = 1024$

31.  $10^0 = 1$

32.  $8^{1/2} = 2\sqrt{2}$

In Exercises 33 to 36, expand the given logarithmic expression.

33.  $\log_b \frac{x^2 y^3}{z}$

34.  $\log_b \frac{\sqrt{x}}{y^2 z}$

35.  $\ln xy^3$

36.  $\ln \frac{\sqrt{xy}}{z^4}$

In Exercises 37 to 40, write each logarithmic expression as a single logarithm with a coefficient of 1.

37.  $2 \log x + \frac{1}{3} \log(x+1)$

38.  $5 \log x - 2 \log(x+5)$

39.  $\frac{1}{2} \ln 2xy - 3 \ln z$

40.  $\ln x - (\ln y - \ln z)$



In Exercises 41 to 44, use the change-of-base formula and a calculator to approximate each logarithm accurate to six significant digits.

41.  $\log_5 101$

42.  $\log_3 40$

43.  $\log_4 0.85$

44.  $\log_8 0.3$

In Exercises 45 to 60, solve each equation for  $x$ . Give exact answers. Do not use a calculator.

45.  $4^x = 30$

46.  $5^{x+1} = 41$

47.  $\ln 3x - \ln(x-1) = \ln 4$

48.  $\ln 3x + \ln 2 = 1$

49.  $e^{\ln(x+2)} = 6$

50.  $10^{\log(2x+1)} = 31$

51.  $\frac{4^x + 4^{-x}}{4^x - 4^{-x}} = 2$

52.  $\frac{5^x + 5^{-x}}{2} = 8$

53.  $\log(\log x) = 3$

54.  $\ln(\ln x) = 2$

55.  $\log \sqrt{x-5} = 3$

56.  $\log x + \log(x-15) = 1$

57.  $\log_4(\log_3 x) = 1$

58.  $\log_7(\log_5 x^2) = 0$

59.  $\log_5 x^3 = \log_5 16x$

60.  $25 = 16^{\log_4 x}$

61. **EARTHQUAKE MAGNITUDE** Determine, to the nearest 0.1, the Richter scale magnitude of an earthquake with an intensity of  $I = 51,782,000I_0$ .

- 62. EARTHQUAKE MAGNITUDE** A seismogram has an amplitude of 18 millimeters and the difference in time between the s-wave and the p-wave is 21 seconds. Find, to the nearest tenth, the Richter scale magnitude of the earthquake that produced the seismogram.

- 63. COMPARISON OF EARTHQUAKES** An earthquake had a Richter scale magnitude of 7.2. Its aftershock had a Richter scale magnitude of 3.7. Compare the intensity of the earthquake to the intensity of the aftershock by finding, to the nearest unit, the ratio of the larger intensity to the smaller intensity.

- 64. COMPARISON OF EARTHQUAKES** An earthquake has an intensity 600 times the intensity of a second earthquake. Find, to the nearest tenth, the difference between the Richter scale magnitudes of the earthquakes.

- 65. CHEMISTRY** Find the pH of tomatoes that have a hydronium-ion concentration of  $6.28 \times 10^{-5}$ . Round to the nearest tenth.

- 66. CHEMISTRY** Find the hydronium-ion concentration of rainwater that has a pH of 5.4.

- 67. COMPOUND INTEREST** Find the balance when \$16,000 is invested at an annual rate of 8% for 3 years if the interest is compounded

- monthly
- continuously

- 68. COMPOUND INTEREST** Find the balance when \$19,000 is invested at an annual rate of 6% for 5 years if the interest is compounded

- daily
- continuously

- 69. DEPRECIATION** The scrap value  $S$  of a product with an expected life span of  $n$  years is given by  $S(n) = P(1 - r)^n$ , where  $P$  is the original purchase price of the product and  $r$  is the annual rate of depreciation. A taxicab is purchased for \$12,400 and is expected to last 3 years. What is its scrap value if it depreciates at a rate of 29% per year?

- 70. MEDICINE** A skin wound heals according to the function given by  $N(t) = N_0 e^{-0.12t}$ , where  $N$  is the number of square centimeters of unhealed skin  $t$  days after the injury, and  $N_0$  is the number of square centimeters covered by the original wound.

- What percentage of the wound will be healed after 10 days?
- How many days, to the nearest day, will it take for 50% of the wound to heal?

- c. How long, to the nearest day, will it take for 90% of the wound to heal?

In Exercises 71 to 74, find the exponential growth or decay function  $N(t) = N_0 e^{kt}$  that satisfies the given conditions.

71.  $N(0) = 1, N(2) = 5$

72.  $N(0) = 2, N(3) = 11$

73.  $N(1) = 4, N(5) = 5$

74.  $N(-1) = 2, N(0) = 1$

**75. POPULATION GROWTH**

- a. Find the exponential growth function for a city whose population was 25,200 in 2005 and 26,800 in 2006. Use  $t = 0$  to represent the year 2005.

- b. Use the growth function to predict, to the nearest hundred, the population of the city in 2012.

- 76. CARBON DATING** Determine, to the nearest ten years, the age of a bone if it now contains 96% of its original amount of carbon-14. The half-life of carbon-14 is 5730 years.

77.   **CELLULAR TELEPHONE SUBSCRIBERSHIP** The following table shows the number of U.S. cellular telephone subscriptions, in thousands, for selected years from 1990 to 2004.

| Year | Number of Cellular Telephone Subscriptions<br>(in thousands) |
|------|--|
| 1990 | 5283   |
| 1992 | 11,033   |
| 1994 | 24,134   |
| 1996 | 44,043   |
| 1998 | 69,209   |
| 2000 | 109,478  |
| 2002 | 140,767  |
| 2004 | 182,140  |

Source: *The World Almanac 2006*

Find the equation of the mathematical model that you believe will most accurately predict the number of U.S. cellular telephone subscriptions for the year 2008. Explain the reasoning you used to select your model.



- 78.** **MORTALITY RATE** The following table shows the infant mortality rate in the United States for selected years from 1960 to 2003. (Source: *The World Almanac 2006*.)

**U.S. Infant Mortality Rate, 1960–2003 (per 1000 live births)**

| Year | Rate, $R$ |
|------|-----------|
| 1960 | 26.0      |
| 1970 | 20.0      |
| 1980 | 12.6      |
| 1990 | 9.2       |
| 1995 | 7.6       |
| 2000 | 6.9       |
| 2001 | 6.8       |
| 2002 | 7.0       |
| 2003 | 6.9       |

- a. Find an exponential model and a logarithmic model for the infant mortality rate,  $R$ , as a function of the year. Represent the year 1960 by  $t = 60$ .

- b. Examine the correlation coefficients of the regression models to determine which model provides the better fit.

- c. Use the model you selected in b. to predict, to the nearest 0.1, the infant mortality rate in 2008.

- 79. LOGISTIC GROWTH** The population of coyotes in a national park satisfies the logistic model with  $P_0 = 210$  in 1997,  $c = 1400$ , and  $P(3) = 360$  (the population in 2000).

- a. Determine the logistic model.

- b. Use the model to predict, to the nearest 10, the coyote population in 2010.

- 80.** Consider the logistic function

$$P(t) = \frac{128}{1 + 5e^{-0.27t}}$$

- a. Find  $P_0$ .

- b. What does  $P(t)$  approach as  $t \rightarrow \infty$ ?

## »»» Quantitative Reasoning: Sales »»»



- DIGITAL CAMERA SALES** The following table shows the worldwide sales, in millions, of digital cameras for the years 1999 to 2003. (Source: *Digital Photography Review*)

| Year   | 1999 | 2000 | 2001 | 2002 | 2003 |
|--|------|------|------|------|------|
| Worldwide Sales of Digital Cameras (in millions) | 5.5  | 11.0 | 18.5 | 30.5 | 50.0 |

- QR1.** Find an exponential model and a logistic model for the data. Let  $t = 0$  represent the year 1999. Round constants to the nearest hundred thousandth.

- QR2.** Use each of the models to predict the number of sales of digital cameras for the year 2009. Round to the nearest tenth of a million.

A business analyst thinks that the digital camera market has started to reach its saturation point. The analyst predicts that during the 2004 to 2007 period, the sales of digital cameras will be as shown in the following table.

| Year   | 2004 | 2005 | 2006 | 2007 |
|--|------|------|------|------|
| Projected Worldwide Sales of Digital Cameras (in millions) | 59.3 | 69.2 | 77.2 | 82.5 |

- QR3.** Find a logarithmic model for the data in the above table. Let  $t = 1$  represent the year 2004. Round constants to the nearest hundred thousandth.
- QR4.** Use the model from Exercise QR3 to predict the number of sales of digital cameras for the year 2009. Round to the nearest tenth of a million.
- QR5.** Find a logistic model for all the data by combining the tables. Let  $t = 0$  represent the year 1999 and  $t = 5$  represent the year 2004. Use this model to predict the number of sales for the year 2009.
- QR6.** The answers in Exercises QR2, QR4, and QR5 provide us with four predictions for the number of digital camera sales in the year 2009. Which of these predicted values do you think is the most realistic?

## Chapter 7 Test

1. a. Write  $\log_b(5x - 3) = c$  in exponential form.  
 b. Write  $3^{x/2} = y$  in logarithmic form.
2. Expand  $\log_b \frac{z^2}{y^3 \sqrt{x}}$ .
3. Write  $\log(2x + 3) - 3\log(x - 2)$  as a single logarithm with a coefficient of 1.
4. Use the change-of-base formula and a calculator to approximate  $\log_4 12$ . Round your result to the nearest ten thousandth.
5. Graph:  $f(x) = 3^{-x/2}$
6. Graph:  $f(x) = -\ln(x + 1)$
7. Solve:  $5^x = 22$ . Round your solution to the nearest ten thousandth.
8. Find the exact solution of  $4^{5-x} = 7^x$ .
9. Solve:  $\log(x + 99) - \log(3x - 2) = 2$
10. Solve:  $\ln(2 - x) + \ln(5 - x) = \ln(37 - x)$
11. Find the balance on \$20,000 invested at an annual interest rate of 7.8% for 5 years, compounded  
 a. monthly  
 b. continuously
12. **COMPOUND INTEREST** Find the time required for money invested at an annual rate of 4% to double in value if the investment is compounded monthly. Round to the nearest hundredth of a year.
13. **EARTHQUAKE MAGNITUDE**  
 a. What, to the nearest tenth, will an earthquake measure on the Richter scale if it has an intensity of  $I = 42,304,000I_0$ ?  
 b. Compare the intensity of an earthquake that measures 6.3 on the Richter scale with the intensity of an earthquake that measures 4.5 on the Richter scale by finding the ratio of the larger intensity to the smaller intensity. Round to the nearest whole number.

- 14. a.** Find the exponential growth function for a city whose population was 34,600 in 2001 and 39,800 in 2004. Use  $t = 0$  to represent the year 2001.

- b.** Use the growth function in **a.** to predict the population of the city in 2011. Round to the nearest thousand.

- 15.** Determine, to the nearest ten years, the age of a bone if it now contains 92% of its original amount of carbon-14. The half-life of carbon-14 is 5730 years.

- 16. a.**  Find the exponential regression function for the following data.

$$\{(2.5, 16), (3.7, 48), (5.0, 155), (6.5, 571), (6.9, 896)\}$$

- b.** Use the function to predict, to the nearest whole number, the  $y$  value associated with  $x = 7.8$ .

- 17.**  **WOMEN'S JAVELIN THROW** The following table shows the progression of the world record distances for the women's javelin throw from 1999 to 2005. (Source: <http://www.athletix.org/Statistics/wrjt-women.html>)

**World Record Progression in the Women's Javelin Throw**

| Year | Distance in meters, $d$ |
|------|-------------------------|
| 1999 | 67.09                   |
| 2000 | 68.22                   |
| 2000 | 69.48                   |
| 2001 | 71.54                   |
| 2005 | 71.70                   |

- a.** Find a logarithmic model and a logistic model for the data. Use  $t = 1$  to represent the year 1999 and  $t = 7$  to represent the year 2005.

- b.** Use each of the models from **a.** to predict the women's world record javelin throw distance for the year 2010. Round to the nearest hundredth of a meter.

- 18. POPULATION GROWTH** The population of raccoons in a state park satisfies a logistic growth model with  $P_0 = 160$  in 2003 and  $P(1) = 190$ . A park ranger has estimated the carrying capacity of the park to be 1100 raccoons.

- a.** Determine the logistic growth model for the raccoon population where  $t$  is the number of years after 2003.

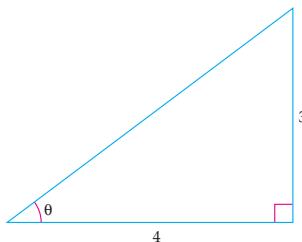
- b.** Use the logistic model from **a.** to predict the raccoon population in 2010.

## Cumulative Review Exercises

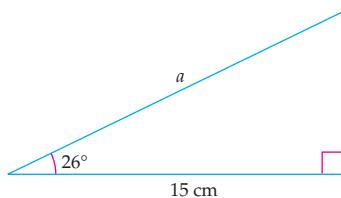
- 1.** Given  $f(x) = \cos x$  and  $g(x) = x^2 + 1$ , find  $(f \circ g)(x)$ .

- 2.** Given  $f(x) = 2x + 8$ , find  $f^{-1}(x)$ .

- 3.** For the right triangle shown at the right, find  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$ .



- 4.** For the right triangle shown at the right, find  $a$ . Round to the nearest centimeter.



- 5.** What are the amplitude, period, and phase shift of the graph of  $y = 4 \cos\left(2x - \frac{\pi}{2}\right)$ ?

6. What is the amplitude and period of the graph of  $y = \sin x - \cos x$ ?
7. Is the sine function an even function, an odd function, or neither an even nor an odd function?
8. Verify the identity  $\frac{1}{\sin x} - \sin x = \cos x \cot x$ .
9. Evaluate  $\tan\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$ .
10. Solve  $2\cos^2 x + \sin x - 1 = 0$  for  $0 \leq x < 2\pi$ .
11. Find the magnitude and direction angle for the vector  $\langle -3, 4 \rangle$ . Round the angle to the nearest tenth of a degree.
12. Find the angle between the vectors  $\mathbf{v} = \langle 2, -3 \rangle$  and  $\mathbf{w} = \langle -3, 4 \rangle$ . Round to the nearest tenth of a degree.
13. **GROUND SPEED AND COURSE OF A PLANE** An airplane is traveling with an airspeed of 400 mph at a heading of  $48^\circ$ . A wind of 55 mph is blowing at a heading of  $115^\circ$ . Find the ground speed and the course of the plane.
14. For triangle  $ABC$ ,  $B = 32^\circ$ ,  $a = 42$  feet, and  $b = 50$  feet. Find the measure of angle  $A$ . Round to the nearest degree.
15. Use De Moivre's Theorem to find  $(1 - i)^8$ .
16. Find the two square roots of the imaginary number  $i$ .
17. Transform the point  $(1, 1)$  in a rectangular coordinate system to a point in a polar coordinate system.
18. Sketch the graph of the polar equation  $r = 2 - 2\cos\theta$ .
19. Solve  $5^x = 10$ . Round to the nearest hundredth.
20. **RADIOACTIVE DECAY** The half-life of an isotope of polonium is 138 days. If an ore sample originally contained 3 milligrams of polonium, how many milligrams of polonium remain after 100 days? Round to the nearest tenth of a milligram.

# Solutions to the Try Exercises

## Exercise Set 1.1, page 12

6.  $6(5s - 11) - 12(2s + 5) = 0$   
 $30s - 66 - 24s - 60 = 0$   
 $6s - 126 = 0$   
 $s = \frac{126}{6} = 21$

20.  $3x - 5y = 15$   
 $-5y = -3x + 15$  • Subtract 3x from each side.  
 $y = \frac{3}{5}x - 3$  • Divide each side by -5.

30.  $x^2 + x - 2 = 0$   
 $a = 1 \quad b = 1 \quad c = -2$   
 $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)}$   
 $= \frac{-1 \pm \sqrt{1 + 8}}{2} = \frac{-1 \pm 3}{2}$   
 $x = \frac{-1 + 3}{2} = 1 \quad \text{or} \quad x = \frac{-1 - 3}{2} = -2$

The solutions are 1 and -2.

44.  $12w^2 - 41w + 24 = 0$   
 $(4w - 3)(3w - 8) = 0$   
 $4w - 3 = 0 \quad \text{or} \quad 3w - 8 = 0$   
 $w = \frac{3}{4} \quad w = \frac{8}{3}$

The solutions are  $\frac{3}{4}$  and  $\frac{8}{3}$ .

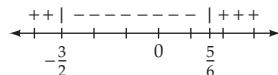
58.  $3(x + 7) \leq 5(2x - 8)$   
 $3x + 21 \leq 10x - 40$   
 $-7x \leq -61$   
 $x \geq \frac{61}{7}$

The solution set is  $\left[\frac{61}{7}, \infty\right)$ .

66.  $12x^2 + 8x \geq 15$   
 $12x^2 + 8x - 15 \geq 0$   
 $(6x - 5)(2x + 3) \geq 0$   
 $x = \frac{5}{6} \quad \text{and} \quad x = -\frac{3}{2}$  • Critical values

Use a test value from each of the intervals

$\left(-\infty, -\frac{3}{2}\right)$ ,  $\left(-\frac{3}{2}, \frac{5}{6}\right)$ , and  $\left(\frac{5}{6}, \infty\right)$  to determine where  $12x^2 + 8x - 15$  is positive.



The solution set is  $\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{5}{6}, \infty\right)$ .

78.  $|2x - 5| \geq 1$   
 $2x - 5 \leq -1 \quad \text{or} \quad 2x - 5 \geq 1$   
 $2x \leq 4 \quad 2x \geq 6$   
 $x \leq 2 \quad x \geq 3$

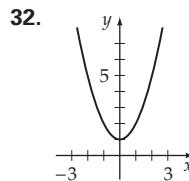
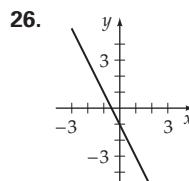
The solution set is  $(-\infty, 2] \cup [3, \infty)$ .

80.  $|3 - 2x| \leq 5$   
 $-5 \leq 3 - 2x \leq 5$   
 $-8 \leq -2x \leq 2$   
 $4 \geq x \geq -1$

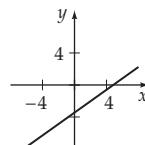
The solution set is  $[-1, 4]$ .

## Exercise Set 1.2, page 27

6.  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$   
 $d = \sqrt{[-10 - (-5)]^2 + (14 - 8)^2}$   
 $= \sqrt{(-5)^2 + 6^2} = \sqrt{25 + 36}$   
 $= \sqrt{61}$



40.  $y\text{-intercept: } \left(0, -\frac{15}{4}\right)$   
 $x\text{-intercept: } (5, 0)$



64.  $r = \sqrt{(1 - (-2))^2 + (7 - 5)^2}$   
 $= \sqrt{9 + 4} = \sqrt{13}$

Using the standard form

$$(x - h)^2 + (y - k)^2 = r^2$$

with  $h = -2$ ,  $k = 5$ , and  $r = \sqrt{13}$  yields

$$(x + 2)^2 + (y - 5)^2 = (\sqrt{13})^2$$

66.  $x^2 + y^2 - 6x - 4y + 12 = 0$

$$x^2 - 6x + y^2 - 4y = -12$$

$$x^2 - 6x + 9 + y^2 - 4y + 4 = -12 + 9 + 4$$

$$(x - 3)^2 + (y - 2)^2 = 1^2$$

center  $(3, 2)$ , radius 1

### Exercise Set 1.3, page 46

2. Given  $g(x) = 2x^2 + 3$

a.  $g(3) = 2(3)^2 + 3 = 18 + 3 = 21$

b.  $g(-1) = 2(-1)^2 + 3 = 2 + 3 = 5$

c.  $g(0) = 2(0)^2 + 3 = 0 + 3 = 3$

d.  $g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 + 3 = \frac{1}{2} + 3 = \frac{7}{2}$

e.  $g(c) = 2(c)^2 + 3 = 2c^2 + 3$

f.  $g(c + 5) = 2(c + 5)^2 + 3 = 2c^2 + 20c + 50 + 3 = 2c^2 + 20c + 53$

10. a. Because  $0 \leq 0 \leq 5$ ,  $Q(0) = 4$ .

b. Because  $6 < a < 7$ ,  $Q(a) = -a + 9$ .

c. Because  $1 < n < 2$ ,  $Q(n) = 4$ .

d. Because  $1 < m \leq 2$ ,  $8 < m^2 + 7 \leq 11$ . Thus

$$Q(m^2 + 7) = \sqrt{(m^2 + 7) - 7} = \sqrt{m^2} = m$$

14.  $x^2 - 2y = 2$

• Solve for  $y$ .

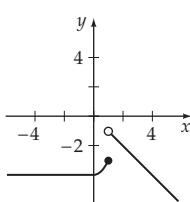
$$-2y = -x^2 + 2$$

$$y = \frac{1}{2}x^2 - 1$$

$y$  is a function of  $x$  because each  $x$  value will yield one and only one  $y$  value.

28. Domain is the set of all real numbers.

40. Domain is the set of all real numbers.



54. a.  $[0, \infty)$

b. Since \$31,250 is between \$30,650 and \$74,200 use  $T(x) = 0.25(x - 30,650) + 4220$ . Then,  $T(31,250) = 0.25(31,250 - 30,650) + 4220 = \$4370$ .

c. Since \$78,900 is between \$74,200 and \$154,800, use  $T(x) = 0.28(x - 74,200) + 15,107.50$  Then,  $T(78,900) = 0.28(78,900 - 74,200) + 15,107.50 = \$16,423.50$ .

56. a. This is the graph of a function. Every vertical line intersects the graph in at most one point.

b. This is not the graph of a function. Some vertical lines intersect the graph at two points.

c. This is not the graph of a function. The vertical line at  $x = -2$  intersects the graph at more than one point.

d. This is the graph of a function. Every vertical line intersects the graph at exactly one point.

72.  $v(t) = 44,000 - 4200t$ ,  $0 \leq t \leq 8$

74. a.  $V(x) = (30 - 2x)^2x$

$$= (900 - 120x + 4x^2)x \\ = 900x - 120x^2 + 4x^3$$

b. Domain:  $\{x | 0 < x < 15\}$

78.  $AB = \sqrt{1 + x^2}$ . The time required to swim from  $A$  to  $B$  at 2 mph is  $\frac{\sqrt{1 + x^2}}{2}$  hours.

$BC = 3 - x$ . The time required to run from  $B$  to  $C$  at 8 mph is  $\frac{3 - x}{8}$  hours.

Thus the total time to reach point  $C$  is

$$t = \frac{\sqrt{1 + x^2}}{2} + \frac{3 - x}{8} \text{ hours}$$

### Exercise Set 1.4, page 64

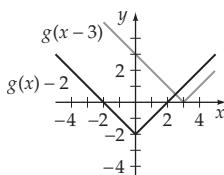
14. a. The graph is symmetric with respect to the  $x$ -axis because replacing  $y$  with  $-y$  leaves the equation unaltered.

b. The graph is not symmetric with respect to the  $y$ -axis because replacing  $x$  with  $-x$  alters the equation.

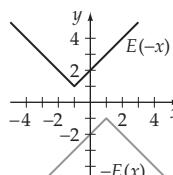
24. The graph is symmetric with respect to the origin because  $(-y) = (-x)^3 - (-x)$  simplifies to  $-y = -x^3 + x$ , which is equivalent to the original equation  $y = x^3 - x$ .

44. Even, because  $h(-x) = (-x)^2 + 1 = x^2 + 1 = h(x)$ .

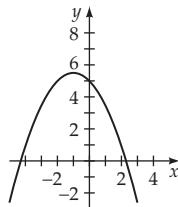
58.



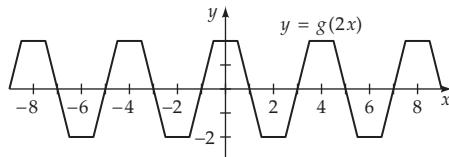
68.



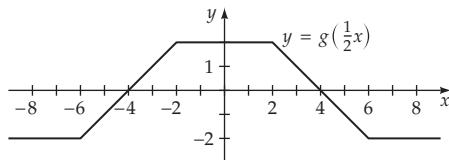
70.



72. a.



b.



## Exercise Set 1.5, page 77

10.  $f(x) + g(x) = \sqrt{x-4} - x$  Domain:  $\{x | x \geq 4\}$

$f(x) - g(x) = \sqrt{x-4} + x$  Domain:  $\{x | x \geq 4\}$

$f(x) \cdot g(x) = -x\sqrt{x-4}$  Domain:  $\{x | x \geq 4\}$

$$\frac{f(x)}{g(x)} = -\frac{\sqrt{x-4}}{x}$$
 Domain:  $\{x | x \geq 4\}$

14.  $(f+g)(x) = (x^2 - 3x + 2) + (2x - 4) = x^2 - x - 2$

$(f+g)(-7) = (-7)^2 - (-7) - 2 = 49 + 7 - 2 = 54$

30. 
$$\frac{f(x+h) - f(x)}{h} = \frac{[4(x+h) - 5] - (4x - 5)}{h}$$
$$= \frac{4x + 4(h) - 5 - 4x + 5}{h}$$
$$= \frac{4(h)}{h} = 4$$

38.  $(g \circ f)(x) = g[f(x)] = g[2x - 7]$ 
$$= 3[2x - 7] + 2 = 6x - 19$$

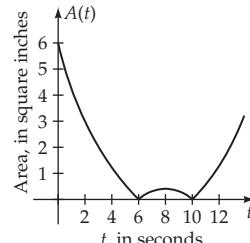
$(f \circ g)(x) = f[g(x)] = f[3x + 2]$ 
$$= 2[3x + 2] - 7 = 6x - 3$$

50.  $(f \circ g)(4) = f[g(4)] = f[4^2 - 5(4)]$ 
$$= f[-4] = 2(-4) + 3 = -5$$

66. a.  $l = 3 - 0.5t$  for  $0 \leq t \leq 6$ , and  $l = -3 + 0.5t$  for  $t > 6$ . In either case,  $l = |3 - 0.5t|$ . The width is  $w = |2 - 0.2t|$  as in Example 7.

b.  $A(t) = |3 - 0.5t||2 - 0.2t|$

- c.  $A$  is decreasing on  $[0, 6]$  and on  $[8, 10]$ .  
 $A$  is increasing on  $[6, 8]$  and on  $[10, 14]$ .



- d. The highest point on the graph of  $A$  for  $0 \leq t \leq 14$  occurs when  $t = 0$  seconds.

72. a. On  $[2, 3]$ ,

$a = 2$

$\Delta t = 3 - 2 = 1$

$s(a + \Delta t) = s(3) = 6 \cdot 3^2 = 54$

$s(a) = s(2) = 6 \cdot 2^2 = 24$

$$\text{Average velocity} = \frac{s(a + \Delta t) - s(a)}{\Delta t}$$

$$= \frac{s(3) - s(2)}{1}$$

$$= 54 - 24 = 30 \text{ feet per second}$$

This is identical to the slope of the line through  $(2, s(2))$  and  $(3, s(3))$  because

$$m = \frac{s(3) - s(2)}{3 - 2} = s(3) - s(2) = 54 - 24 = 30$$

b. On  $[2, 2.5]$ ,

$a = 2$

$\Delta t = 2.5 - 2 = 0.5$

$s(a + \Delta t) = s(2.5) = 6(2.5)^2 = 37.5$

$$\text{Average velocity} = \frac{s(2.5) - s(2)}{0.5} = \frac{37.5 - 24}{0.5}$$

$$= \frac{13.5}{0.5} = 27 \text{ feet per second}$$

c. On  $[2, 2.1]$ ,

$a = 2$

$\Delta t = 2.1 - 2 = 0.1$

$s(a + \Delta t) = s(2.1) = 6(2.1)^2 = 26.46$

$$\text{Average velocity} = \frac{s(2.1) - s(2)}{0.1} = \frac{26.46 - 24}{0.1}$$

$$= \frac{2.46}{0.1} = 24.6 \text{ feet per second}$$

- d. On  $[2, 2.01]$ ,

$$a = 2$$

$$\Delta t = 2.01 - 2 = 0.01$$

$$s(a + \Delta t) = s(2.01) = 6(2.01)^2 = 24.2406$$

$$\begin{aligned}\text{Average velocity} &= \frac{s(2.01) - s(2)}{0.01} \\ &= \frac{24.2406 - 24}{0.01} \\ &= \frac{0.2406}{0.01} = 24.06 \text{ feet per second}\end{aligned}$$

- e. On  $[2, 2.001]$ ,

$$a = 2$$

$$\Delta t = 2.001 - 2 = 0.001$$

$$s(a + \Delta t) = s(2.001) = 6(2.001)^2 = 24.024006$$

$$\begin{aligned}\text{Average velocity} &= \frac{s(2.001) - s(2)}{0.001} \\ &= \frac{24.024006 - 24}{0.001} \\ &= \frac{0.024006}{0.001} = 24.006 \text{ feet per second}\end{aligned}$$

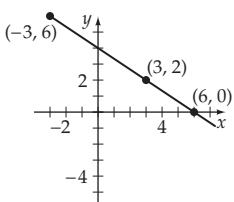
- f. On  $[2, 2 + \Delta t]$ ,

$$\begin{aligned}\frac{s(2 + \Delta t) - s(2)}{\Delta t} &= \frac{6(2 + \Delta t)^2 - 24}{\Delta t} \\ &= \frac{6(4 + 4(\Delta t) + (\Delta t)^2) - 24}{\Delta t} \\ &= \frac{24 + 24(\Delta t) + 6(\Delta t)^2 - 24}{\Delta t} \\ &= \frac{24\Delta t + 6(\Delta t)^2}{\Delta t} = 24 + 6(\Delta t)\end{aligned}$$

As  $\Delta t$  approaches zero, the average velocity approaches 24 feet per second.

### Exercise Set 1.6, page 90

10. Because the graph of the given function is a line that passes through  $(0, 6)$ ,  $(2, 3)$ , and  $(6, -3)$ , the graph of the inverse will be a line that passes through  $(6, 0)$ ,  $(3, 2)$ , and  $(-3, 6)$ . See the following figure. Notice that the line shown below is a reflection of the line given in Exercise 10 across the line given by  $y = x$ . Yes, the inverse relation is a function.



20. Check to see if  $f[g(x)] = x$  for all  $x$  in the domain of  $g$  and  $g[f(x)] = x$  for all  $x$  in the domain of  $f$ . The following shows that  $f[g(x)] = x$  for all real numbers  $x$ .

$$\begin{aligned}f[g(x)] &= f[2x + 3] \\ &= \frac{1}{2}(2x + 3) - \frac{3}{2} \\ &= x + \frac{3}{2} - \frac{3}{2} = x\end{aligned}$$

The following shows that  $g[f(x)] = x$  for all real numbers  $x$ .

$$\begin{aligned}g[f(x)] &= g\left[\frac{1}{2}x - \frac{3}{2}\right] \\ &= 2\left(\frac{1}{2}x - \frac{3}{2}\right) + 3 \\ &= x - 3 + 3 = x\end{aligned}$$

Thus  $f$  and  $g$  are inverses.

32.  $f(x) = 4x - 8$

$y = 4x - 8$  • Replace  $f(x)$  by  $y$ .

$x = 4y - 8$  • Interchange  $x$  and  $y$ .

$x + 8 = 4y$  • Solve for  $y$ .

$$\frac{1}{4}(x + 8) = y$$

$$y = \frac{1}{4}x + 2$$

$$f^{-1}(x) = \frac{1}{4}x + 2 \quad \bullet \text{Replace } y \text{ by } f^{-1}(x).$$

38.  $f(x) = \frac{x}{x-2}, x \neq 2$

$y = \frac{x}{x-2}$  • Replace  $f(x)$  by  $y$ .

$x = \frac{y}{y-2}$  • Interchange  $x$  and  $y$ .

$$x(y-2) = y$$

$$xy - 2x = y$$

$$xy - y = 2x$$

$$y(x-1) = 2x$$

$$y = \frac{2x}{x-1}$$

$$f^{-1}(x) = \frac{2x}{x-1}, x \neq 1$$

• Replace  $y$  by  $f^{-1}(x)$  and indicate any restrictions.

44.  $f(x) = \sqrt{4-x}, x \leq 4$

$y = \sqrt{4-x}$  • Replace  $f(x)$  by  $y$ .

$x = \sqrt{4-y}$  • Interchange  $x$  and  $y$ .

$x^2 = 4 - y$  • Solve for  $y$ .

$$\begin{aligned}x^2 - 4 &= -y \\-x^2 + 4 &= y \\f^{-1}(x) &= -x^2 + 4, x \geq 0\end{aligned}$$

• Replace  $y$  by  $f^{-1}(x)$  and indicate any restrictions.

The range of  $f$  is  $\{y | y \geq 0\}$ . Therefore, the domain of  $f^{-1}$  is  $\{x | x \geq 0\}$ , as indicated above.

**54.**  $K(x) = 1.3x - 4.7$

$$\begin{aligned}y &= 1.3x - 4.7 && \bullet \text{Replace } K(x) \text{ by } y. \\x &= 1.3y - 4.7 && \bullet \text{Interchange } x \text{ and } y. \\x + 4.7 &= 1.3y && \bullet \text{Solve for } y. \\x + 4.7 &= y \\1.3 &= \frac{x + 4.7}{1.3} \\K^{-1}(x) &= \frac{x + 4.7}{1.3} && \bullet \text{Replace } y \text{ by } K^{-1}(x).\end{aligned}$$

The function  $K^{-1}(x) = \frac{x + 4.7}{1.3}$  can be used to convert a United Kingdom men's shoe size to its equivalent U.S. shoe size.

### Exercise Set 1.7, page 103

- 18. a.** Enter the data in the table. Then use your calculator to find the linear regression equation:

$$y = 3.410344828x + 65.09359606$$

- b.** Evaluate the linear regression equation when  $x = 58$ .

$$\begin{aligned}y &= 3.410344828(58) + 65.09359606 \\&\approx 263\end{aligned}$$

The ball will travel approximately 263 feet.

- 32. a.** Enter the data in the table. Then use your calculator to find the quadratic regression model:

$$y = 0.05208x^2 - 3.56026x + 82.32999$$

- b.** Evaluate the quadratic regression model when  $x = 40$ .

$$\begin{aligned}y &= 0.05208(40^2) - 3.5602(40) + 82.3299 \\&\approx 23\end{aligned}$$

The bird will consume approximately 23 milliliters of oxygen per minute.

### Exercise Set 2.1, page 130

- 2.** The measure of the complement of an angle of  $87^\circ$  is  $(90^\circ - 87^\circ) = 3^\circ$

The measure of the supplement of an angle of  $87^\circ$  is  $(180^\circ - 87^\circ) = 93^\circ$

- 14.** Because  $765^\circ = 2 \cdot 360^\circ + 45^\circ$ ,  $\angle \alpha$  is coterminal with an angle that has a measure of  $45^\circ$ .  $\angle \alpha$  is a Quadrant I angle.

**32.**  $-45^\circ = -45^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = -\frac{\pi}{4} \text{ radian}$

**44.**  $\frac{\pi}{4} \text{ radian} = \frac{\pi}{4} \text{ radian} \left(\frac{180^\circ}{\pi \text{ radians}}\right) = 45^\circ$

**68.**  $s = r\theta = 5 \left(144^\circ \cdot \frac{\pi}{180^\circ}\right) = 4\pi \approx 12.57 \text{ meters}$

- 72.** Let  $\theta_2$  be the angle through which the pulley with a diameter of 0.8 meter turns. Let  $\theta_1$  be the angle through which the pulley with a diameter of 1.2 meters turns. Let  $r_2 = 0.4$  meter be the radius of the smaller pulley, and let  $r_1 = 0.6$  meter be the radius of the larger pulley.

$$\theta_1 = 240^\circ = \frac{4}{3}\pi \text{ radians}$$

Thus  $r_2\theta_2 = r_1\theta_1$

$$0.4\theta_2 = 0.6 \left(\frac{4}{3}\pi\right)$$

$$\theta_2 = \frac{0.6}{0.4} \left(\frac{4}{3}\pi\right) = 2\pi \text{ radians or } 360^\circ$$

- 74.** The earth makes one revolution ( $\theta = 2\pi$ ) in 1 day.

$$t = 24 \cdot 3600 = 86,400 \text{ seconds}$$

$$\omega = \frac{\theta}{t} = \frac{2\pi}{86,400} \approx 7.27 \times 10^{-5} \text{ radian/second}$$

- 80.**  $C = 2\pi r = 2\pi(18 \text{ inches}) = 36\pi \text{ inches}$

Thus one conversion factor is  $(36\pi \text{ inches}/1 \text{ revolution})$ .

$$\begin{aligned}\frac{500 \text{ revolutions}}{1 \text{ minute}} &= \frac{500 \text{ revolutions}}{1 \text{ minute}} \left(\frac{36\pi \text{ inches}}{1 \text{ revolution}}\right) \\&= \frac{18,000\pi \text{ inches}}{1 \text{ minute}}$$

Now convert inches to miles and minutes to hours.

$$\begin{aligned}\frac{18,000\pi \text{ inches}}{1 \text{ minute}} &= \frac{18,000\pi \text{ inches}}{1 \text{ minute}} \left(\frac{1 \text{ foot}}{12 \text{ inches}}\right) \left(\frac{1 \text{ mile}}{5280 \text{ feet}}\right) \left(\frac{60 \text{ minutes}}{1 \text{ hour}}\right) \\&\approx 54 \text{ miles per hour}\end{aligned}$$

### Exercise Set 2.2, page 142

**6.**  $\text{adj} = \sqrt{8^2 - 5^2}$

$$\text{adj} = \sqrt{64 - 25} = \sqrt{39}$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{5}{8} \quad \csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{8}{5}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{39}}{8} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{8}{\sqrt{39}} = \frac{8\sqrt{39}}{39}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{5}{\sqrt{39}} = \frac{5\sqrt{39}}{39} \quad \cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{39}}{5}$$

18. Because  $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$ , let opp = 4 and adj = 3.

$$\text{hyp} = \sqrt{3^2 + 4^2} = 5$$

$$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$$

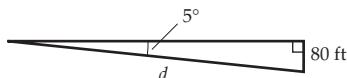
$$\begin{aligned} 34. \quad \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \tan \frac{\pi}{4} &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - 1 \\ &= \frac{\sqrt{6}}{4} - 1 = \frac{\sqrt{6} - 4}{4} \end{aligned}$$

56.



$$\begin{aligned} \tan 68.9^\circ &= \frac{h}{116} \\ h &= 116 \tan 68.9^\circ \\ h &\approx 301 \text{ meters} \\ &\text{(three significant digits)} \end{aligned}$$

58.



$$\sin 5^\circ = \frac{80}{d}$$

$$d = \frac{80}{\sin 5^\circ}$$

$$d \approx 917.9 \text{ feet}$$

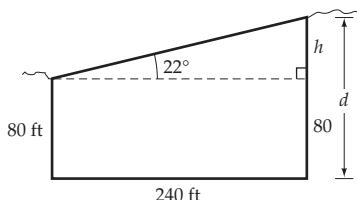
Change 9 miles per hour to feet per minute.

$$\begin{aligned} r &= 9 \frac{\text{miles}}{\text{hour}} = \frac{9 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \\ &= \frac{9(5280)}{60} \frac{\text{feet}}{\text{minute}} = 792 \frac{\text{feet}}{\text{minute}} \end{aligned}$$

$$t = \frac{d}{r}$$

$$t \approx \frac{917.9 \text{ feet}}{792 \text{ feet per minute}} \approx 1.2 \text{ minutes (to the nearest tenth of a minute)}$$

66.



$$\tan 22^\circ = \frac{h}{240}$$

$$h = 240 \tan 22^\circ$$

$$d = 80 + h$$

$$d = 80 + 240 \tan 22^\circ$$

$$d \approx 180 \text{ feet (two significant digits)}$$

### Exercise Set 2.3, page 153

$$6. \quad x = -6, y = -9, r = \sqrt{(-6)^2 + (-9)^2} = \sqrt{117} = 3\sqrt{13}$$

$$\sin \theta = \frac{y}{r} = \frac{-9}{3\sqrt{13}} = -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13} \quad \csc \theta = -\frac{\sqrt{13}}{3}$$

$$\cos \theta = \frac{x}{r} = \frac{-6}{3\sqrt{13}} = -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13} \quad \sec \theta = -\frac{\sqrt{13}}{2}$$

$$\tan \theta = \frac{y}{x} = \frac{-9}{-6} = \frac{3}{2} \quad \cot \theta = \frac{2}{3}$$

$$30. \quad \sec \theta = \frac{2\sqrt{3}}{3} = \frac{r}{x} \quad \bullet \text{ Let } r = 2\sqrt{3} \text{ and } x = 3.$$

$$y = \pm \sqrt{(2\sqrt{3})^2 - 3^2} = \pm \sqrt{3}$$

$y = -\sqrt{3}$  because  $y < 0$  in Quadrant IV.

$$\sin \theta = \frac{-\sqrt{3}}{2\sqrt{3}} = -\frac{1}{2}$$

$$38. \quad \theta' = 255^\circ - 180^\circ = 75^\circ$$

$$50. \quad \cos 300^\circ > 0, \theta' = 360^\circ - 300^\circ = 60^\circ$$

$$\text{Thus } \cos 300^\circ = \cos 60^\circ = \frac{1}{2}.$$

### Exercise Set 2.4, page 166

$$10. \quad t = -\frac{7\pi}{4}; W(t) = P(x, y) \text{ where}$$

$$\begin{aligned} x &= \cos t & y &= \sin t \\ &= \cos\left(-\frac{7\pi}{4}\right) & &= \sin\left(-\frac{7\pi}{4}\right) \\ &= \cos \frac{\pi}{4} & &= \sin \frac{\pi}{4} \\ &= \frac{\sqrt{2}}{2} & &= \frac{\sqrt{2}}{2} \\ W\left(-\frac{7\pi}{4}\right) &= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \end{aligned}$$

$$16. \quad \text{The reference angle for } -\frac{5\pi}{6} \text{ is } \frac{\pi}{6}.$$

$$\begin{aligned} \sec\left(-\frac{5\pi}{6}\right) &= -\sec \frac{\pi}{6} \quad \bullet \text{ sec } t < 0 \text{ for } t \text{ in Quadrant III} \\ &= -\frac{2\sqrt{3}}{3} \end{aligned}$$

$$44. \quad F(-x) = \tan(-x) + \sin(-x)$$

$$= -\tan x - \sin x$$

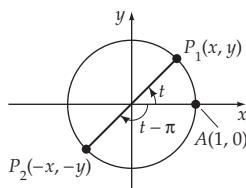
$$= -(\tan x + \sin x)$$

$$= -F(x)$$

• tan x and sin x are odd functions.

Because  $F(-x) = -F(x)$ , the function defined by  $F(x) = \tan x + \sin x$  is an odd function.

56.



$$\tan t = \frac{y}{x}$$

$$\tan(t - \pi) = \frac{-y}{-x} = \frac{y}{x}$$

• From the unit circle

Therefore,  $\tan t = \tan(t - \pi)$ .

$$\begin{aligned} 72. \quad \frac{1}{1 - \sin t} + \frac{1}{1 + \sin t} &= \frac{1 + \sin t + 1 - \sin t}{(1 - \sin t)(1 + \sin t)} \\ &= \frac{2}{1 - \sin^2 t} \\ &= \frac{2}{\cos^2 t} \\ &= 2 \sec^2 t \end{aligned}$$

$$78. \quad 1 + \tan^2 t = \sec^2 t$$

$$\tan^2 t = \sec^2 t - 1$$

$$\tan t = \pm \sqrt{\sec^2 t - 1}$$

Because  $\frac{3\pi}{2} < t < 2\pi$ ,  $\tan t$  is negative. Thus  
 $\tan t = -\sqrt{\sec^2 t - 1}$ .

82. March 5 is represented by  $t = 2$ .

$$\begin{aligned} T(2) &= -41 \cos\left(\frac{\pi}{6} \cdot 2\right) + 36 \\ &= -41 \cos\left(\frac{\pi}{3}\right) + 36 \\ &= -41(0.5) + 36 \\ &= 15.5^\circ\text{F} \end{aligned}$$

July 20 is represented by  $t = 6.5$ .

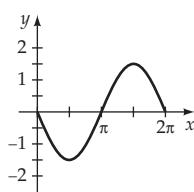
$$\begin{aligned} T(6.5) &= -41 \cos\left(\frac{\pi}{6} \cdot 6.5\right) + 36 \\ &\approx -41(-0.9659258263) + 36 \\ &\approx 75.6^\circ\text{F} \end{aligned}$$

**Exercise Set 2.5, page 177**

$$22. \quad y = -\frac{3}{2} \sin x$$

$$a = \left| -\frac{3}{2} \right| = \frac{3}{2}$$

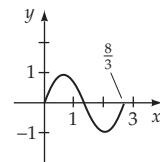
$$\text{period} = 2\pi$$



$$32. \quad y = \sin \frac{3\pi}{4} x$$

$$a = 1$$

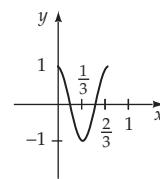
$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{3\pi/4} = \frac{8}{3}$$



$$34. \quad y = \cos 3\pi x$$

$$a = 1$$

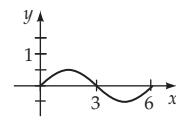
$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{3\pi} = \frac{2}{3}$$



$$40. \quad y = \frac{1}{2} \sin \frac{\pi x}{3}$$

$$a = \frac{1}{2}$$

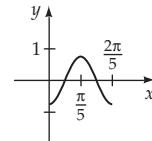
$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{\pi/3} = 6$$



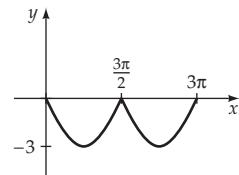
$$48. \quad y = -\frac{3}{4} \cos 5x$$

$$a = \left| -\frac{3}{4} \right| = \frac{3}{4}$$

$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{5}$$



$$54. \quad y = -\left| 3 \sin \frac{2}{3} x \right|$$



60. Because the graph passes through the origin, we start with an equation of the form  $y = a \sin bx$ . The graph completes one cycle in  $\frac{4\pi}{3}$  units. Thus the period is  $\frac{4\pi}{3}$ .

Use the equation  $\frac{2\pi}{b} = \frac{4\pi}{3}$  to solve for  $b$ .

$$\frac{2\pi}{b} = \frac{4\pi}{3}$$

$6\pi = 4b\pi$  • Multiply each side by 3b.

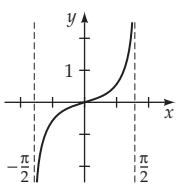
$$\frac{3}{2} = b$$
 • Divide each side by 4π.

The graph has a maximum height of  $\frac{3}{2}$  and a minimum height of  $-\frac{3}{2}$ . Thus its amplitude is  $a = \frac{3}{2}$ . Substitute  $\frac{3}{2}$  for  $a$  and  $\frac{3}{2}$  for  $b$  in  $y = a \sin bx$  to produce  $y = \frac{3}{2} \sin \frac{3}{2}x$ .

## Exercise Set 2.6, page 189

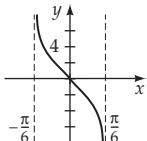
24.  $y = \frac{1}{3} \tan x$

period  $= \frac{\pi}{b} = \pi$



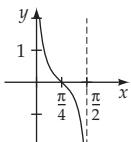
32.  $y = -3 \tan 3x$

period  $= \frac{\pi}{b} = \frac{\pi}{3}$



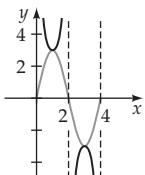
34.  $y = \frac{1}{2} \cot 2x$

period  $= \frac{\pi}{b} = \frac{\pi}{2}$



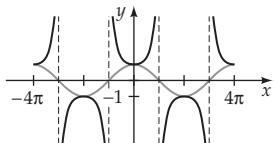
40.  $y = 3 \csc \frac{\pi x}{2}$

period  $= \frac{2\pi}{b} = \frac{2\pi}{\pi/2} = 4$



44.  $y = \sec \frac{x}{2}$

period  $= \frac{2\pi}{b} = \frac{2\pi}{1/2} = 4\pi$



## Exercise Set 2.7, page 199

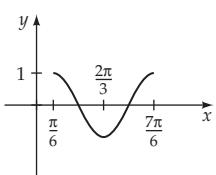
20.  $y = \cos\left(2x - \frac{\pi}{3}\right)$

$a = 1$

period  $= \pi$

phase shift  $= -\frac{c}{b}$

$$= -\frac{-\pi/3}{2} = \frac{\pi}{6}$$

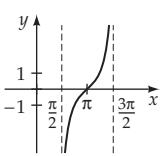


22.  $y = \tan(x - \pi)$

period  $= \pi$

phase shift  $= -\frac{c}{b}$

$$= -\frac{-\pi}{1} = \pi$$



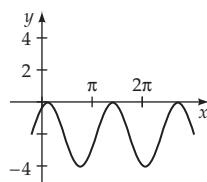
40.  $y = 2 \sin\left(\frac{\pi x}{2} + 1\right) - 2$

$a = 2$

period  $= 4$

phase shift  $= -\frac{c}{b}$

$$= -\frac{1}{2/\pi} = -\frac{2}{\pi}$$

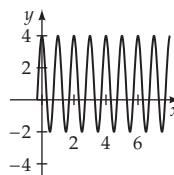


42.  $y = -3 \cos(2\pi x - 3) + 1$

$a = 3$

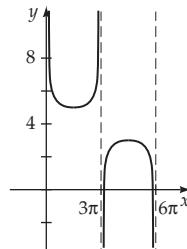
period  $= 1$

phase shift  $= -\frac{c}{b} = \frac{3}{2\pi}$

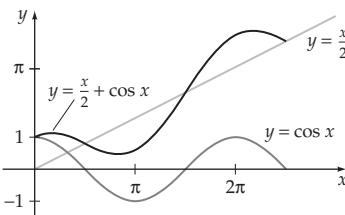


48.  $y = \csc \frac{x}{3} + 4$

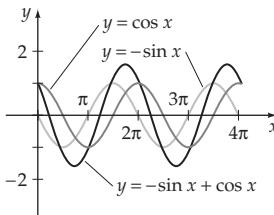
period  $= 6\pi$



52.  $y = \frac{x}{2} + \cos x$



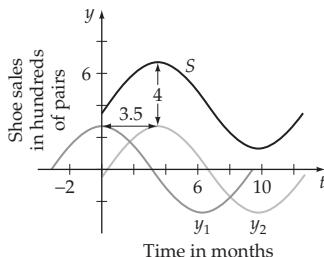
56.  $y = \cos x - \sin x$



**64. a.** Phase shift =  $-\frac{c}{b} = -\frac{\left(-\frac{7}{12}\pi\right)}{\left(\frac{\pi}{6}\right)} = 3.5$  months,

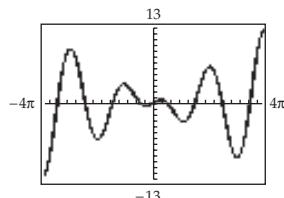
$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{\pi/6} = 12 \text{ months}$$

**b.** First graph  $y_1 = 2.7 \cos\left(\frac{\pi}{6}t\right)$ . Because the phase shift is 3.5 months, shift the graph of  $y_1$  3.5 units to the right to produce the graph of  $y_2$ . Now shift the graph of  $y_2$  upward 4 units to produce the graph of  $S$ .



**c.** 3.5 months after January 1 is the middle of April.

**78.**  $y = x \cos x$



### Exercise Set 2.8, page 208

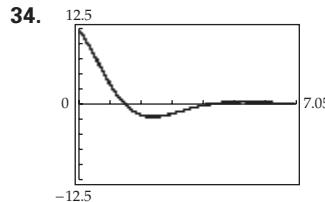
**20.** Amplitude = 3, frequency =  $\frac{1}{\pi}$ , period =  $\pi$

Because  $\frac{2\pi}{b} = \pi$ , we have  $b = 2$ . Thus  $y = 3 \cos 2t$ .

**28.** Amplitude =  $|-1.5| = 1.5$

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{3}{27}} = \frac{1}{2\pi} \cdot \frac{1}{3} = \frac{1}{6\pi}, \text{ period} = 6\pi$$

$$y = a \cos 2\pi ft = -1.5 \cos\left[2\pi\left(\frac{1}{6\pi}\right)t\right] = -1.5 \cos \frac{1}{3}t$$

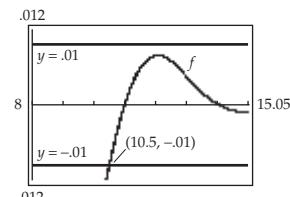


a.  $f$  has pseudoperiod  $\frac{2\pi}{1} = 2\pi$ .

$$10 \div (2\pi) \approx 1.59$$

Thus  $f$  completes only one full oscillation on  $0 \leq t \leq 10$ .

b. The following graph of  $f$  shows that  $|f(t)| < 0.01$  for  $t > 10.5$ .



### Exercise Set 3.1, page 222

2. We will try to verify the identity by rewriting the left side so that it involves only sines and cosines.

$$\begin{aligned} \tan x \sec x \sin x &= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \cdot \sin x \\ &= \frac{\sin^2 x}{\cos^2 x} \\ &= \left(\frac{\sin x}{\cos x}\right)^2 \\ &= (\tan x)^2 \\ &= \tan^2 x \end{aligned}$$

12.  $\sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)$   
 $= 1(\sin^2 x - \cos^2 x) = \sin^2 x - \cos^2 x$

24. 
$$\begin{aligned} \frac{2 \sin x \cot x + \sin x - 4 \cot x - 2}{2 \cot x + 1} \\ &= \frac{(\sin x)(2 \cot x + 1) - 2(2 \cot x + 1)}{2 \cot x + 1} \\ &= \frac{(2 \cot x + 1)(\sin x - 2)}{2 \cot x + 1} = \sin x - 2 \end{aligned}$$

34. 
$$\begin{aligned} \frac{\frac{1}{\sin x} + \frac{1}{\cos x}}{\frac{1}{\sin x} - \frac{1}{\cos x}} &= \frac{\frac{1}{\sin x} + \frac{1}{\cos x}}{\frac{1}{\sin x} - \frac{1}{\cos x}} \cdot \frac{\sin x \cos x}{\sin x \cos x} \\ &= \frac{\cos x + \sin x}{\cos x - \sin x} \\ &= \frac{\cos x + \sin x}{\cos x - \sin x} \cdot \frac{\cos x - \sin x}{\cos x - \sin x} \\ &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x - 2 \sin x \cos x + \sin^2 x} \\ &= \frac{\cos^2 x - \sin^2 x}{1 - 2 \sin x \cos x} \end{aligned}$$

44. Rewrite the left side so that it involves only sines and cosines.

$$\begin{aligned} \frac{2 \cot x}{\cot x + \tan x} &= \frac{2 \cdot \frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} \\ &= \frac{2 \cdot \frac{\cos x}{\sin x}}{\frac{\cos x \cos x}{\sin x \cos x} + \frac{\sin x \sin x}{\cos x \sin x}} \\ &= \frac{2 \cdot \frac{\cos x}{\sin x}}{\frac{\cos^2 x + \sin^2 x}{\sin x \cos x}} \\ &= \frac{2 \cdot \frac{\cos x}{\sin x}}{\frac{1}{\sin x \cos x}} \\ &= 2 \cdot \frac{\cos x}{\sin x} \cdot \frac{\sin x \cos x}{1} \\ &= 2 \cos^2 x \end{aligned}$$

### Exercise Set 3.2, page 233

4. Use the identity  $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$  with  $\alpha = 120^\circ$  and  $\beta = 45^\circ$ .

$$\begin{aligned} \cos(120^\circ - 45^\circ) &= \cos 120^\circ \cos 45^\circ + \sin 120^\circ \sin 45^\circ \\ &= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

20. The value of a given trigonometric function of  $\theta$ , measured in degrees, is equal to its cofunction of  $90^\circ - \theta$ . Thus

$$\begin{aligned} \cos 80^\circ &= \sin(90^\circ - 80^\circ) \\ &= \sin 10^\circ \end{aligned}$$

26.  $\sin x \cos 3x + \cos x \sin 3x = \sin(x + 3x) = \sin 4x$

38.  $\tan \alpha = \frac{24}{7}$ , with  $0^\circ < \alpha < 90^\circ$ ;  $\sin \alpha = \frac{24}{25}$ ,  $\cos \alpha = \frac{7}{25}$

$$\sin \beta = -\frac{8}{17}, \text{ with } 180^\circ < \beta < 270^\circ$$

$$\cos \beta = -\frac{15}{17}, \tan \beta = \frac{8}{15}$$

a.  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$\begin{aligned} &= \left(\frac{24}{25}\right)\left(-\frac{15}{17}\right) + \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) \\ &= -\frac{360}{425} - \frac{56}{425} = -\frac{416}{425} \end{aligned}$$

b.  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$$\begin{aligned} &= \left(\frac{7}{25}\right)\left(-\frac{15}{17}\right) - \left(\frac{24}{25}\right)\left(-\frac{8}{17}\right) \\ &= -\frac{105}{425} + \frac{192}{425} = \frac{87}{425} \end{aligned}$$

c.  $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

$$\begin{aligned} &= \frac{\frac{24}{7} - \frac{8}{15}}{1 + \left(\frac{24}{7}\right)\left(\frac{8}{15}\right)} = \frac{\frac{24}{7} - \frac{8}{15}}{1 + \frac{192}{105}} \cdot \frac{105}{105} \\ &= \frac{360 - 56}{105 + 192} = \frac{304}{297} \end{aligned}$$

50.  $\cos(\theta + \pi) = \cos \theta \cos \pi - \sin \theta \sin \pi$

$$= (\cos \theta)(-1) - (\sin \theta)(0) = -\cos \theta$$

62.  $\cos 5x \cos 3x + \sin 5x \sin 3x = \cos(5x - 3x) = \cos 2x$

$$\begin{aligned} &= \cos(x + x) = \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

76.  $\sin(\theta + 2\pi) = \sin \theta \cos 2\pi + \cos \theta \sin 2\pi$

$$= (\sin \theta)(1) + (\cos \theta)(0) = \sin \theta$$

### Exercise Set 3.3, page 243

2.  $2 \sin 3\theta \cos 3\theta = \sin[2(3\theta)] = \sin 6\theta$

10.  $\cos \alpha = \frac{24}{25}$  with  $270^\circ < \alpha < 360^\circ$

$$\sin \alpha = -\sqrt{1 - \left(\frac{24}{25}\right)^2}$$

$$= -\frac{7}{25}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$= 2\left(-\frac{7}{25}\right)\left(\frac{24}{25}\right)$$

$$= -\frac{336}{625}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$= \frac{2\left(-\frac{7}{24}\right)}{1 - \left(-\frac{7}{24}\right)^2}$$

$$= \frac{-\frac{7}{12}}{1 - \frac{49}{576}} = \frac{-\frac{7}{12}}{\frac{527}{576}} = -\frac{336}{527}$$

22.  $\sin^2 x \cos^4 x$

$$= \sin^2 x (\cos^2 x)^2$$

$$= \left(\frac{1 - \cos 2x}{2}\right) \left(\frac{1 + \cos 2x}{2}\right)^2$$

$$= \left(\frac{1 - \cos 2x}{2}\right) \left(\frac{1 + 2 \cos 2x + \cos^2 2x}{4}\right)$$

$$= \frac{1}{8}(1 - \cos 2x) \left(1 + 2 \cos 2x + \frac{1 + \cos 4x}{2}\right)$$

$$= \frac{1}{8}(1 - \cos 2x) \left(\frac{2 + 4 \cos 2x + 1 + \cos 4x}{2}\right)$$

$$= \frac{1}{16}(1 - \cos 2x)(3 + 4 \cos 2x + \cos 4x)$$

$$= \frac{1}{16}(3 + 4 \cos 2x + \cos 4x - 3 \cos 2x - 4 \cos^2 2x - \cos 2x \cos 4x)$$

$$= \frac{1}{16}(3 + \cos 2x + \cos 4x - 4 \cos^2 2x - \cos 2x \cos 4x)$$

$$= \frac{1}{16}(3 + \cos 2x + \cos 4x - 4\left(\frac{1 + \cos 4x}{2}\right) - \cos 2x \cos 4x)$$

$$= \frac{1}{16}(3 + \cos 2x + \cos 4x - 2(1 + \cos 4x) - \cos 2x \cos 4x)$$

$$= \frac{1}{16}(3 + \cos 2x + \cos 4x - 2 - 2 \cos 4x - \cos 2x \cos 4x)$$

$$= \frac{1}{16}(1 + \cos 2x - \cos 4x - \cos 2x \cos 4x)$$

28. Because  $165^\circ = \frac{1}{2}(330^\circ)$ , we can find  $\cos 165^\circ$  by using the half-angle identity for  $\cos \frac{\alpha}{2}$  with  $\alpha = 330^\circ$ . The angle  $\frac{\alpha}{2} = 165^\circ$  lies in Quadrant II and the cosine function is negative in Quadrant II. Thus  $\cos 165^\circ < 0$ , and we must select the minus sign that precedes the radical

$$\text{in } \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\cos 165^\circ = -\sqrt{\frac{1 + \cos 330^\circ}{2}}$$

$$= -\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

$$= -\sqrt{\frac{\frac{2}{2} + \frac{\sqrt{3}}{2}}{2}}$$

$$= -\sqrt{\left(\frac{2 + \sqrt{3}}{2}\right) \cdot \frac{1}{2}}$$

$$= -\sqrt{\frac{2 + \sqrt{3}}{4}}$$

$$= -\frac{\sqrt{2 + \sqrt{3}}}{2}$$

38. Because  $\alpha$  is in Quadrant III,  $\cos \alpha < 0$ . We now solve for  $\cos \alpha$ .

$$\begin{aligned} \cos \alpha &= -\sqrt{1 - \sin^2 \alpha} = -\sqrt{1 - \left(-\frac{7}{25}\right)^2} \\ &= -\sqrt{1 - \frac{49}{625}} \\ &= -\sqrt{\frac{49}{625}} \\ &= -\frac{24}{25} \end{aligned}$$

Because  $180^\circ < \alpha < 270^\circ$ , we know that  $90^\circ < \frac{\alpha}{2} < 135^\circ$ .

Thus  $\frac{\alpha}{2}$  is in Quadrant II,  $\sin \frac{\alpha}{2} > 0$ ,  $\cos \frac{\alpha}{2} < 0$ , and

$\tan \frac{\alpha}{2} < 0$ .

Use the half-angle formulas.

$$\begin{aligned} \sin \frac{\alpha}{2} &= \sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1 - \left(-\frac{24}{25}\right)}{2}} \\ &= \sqrt{\frac{25 + 24}{50}} = \sqrt{\frac{49}{50}} = \frac{7\sqrt{2}}{10} \end{aligned}$$

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{1 + \cos \alpha}{2}} = -\sqrt{\frac{1 + \left(-\frac{24}{25}\right)}{2}}$$

$$= -\sqrt{\frac{25 - 24}{50}} = -\sqrt{\frac{1}{50}} = -\frac{\sqrt{2}}{10}$$

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{-\frac{7}{25}}{1 + \left(-\frac{24}{25}\right)}$$

$$= \frac{-\frac{7}{25}}{\frac{1}{25}} = -7$$

$$50. \frac{1}{1 - \cos 2x} = \frac{1}{1 - 1 + 2 \sin^2 x}$$

$$= \frac{1}{2 \sin^2 x} = \frac{1}{2} \csc^2 x$$

$$68. \cos^2 \frac{x}{2} = \left[ \pm \sqrt{\frac{1 + \cos x}{2}} \right]^2$$

$$= \frac{1 + \cos x}{2}$$

$$= \frac{1 + \cos x}{2} \cdot \frac{\sec x}{\sec x}$$

$$= \frac{\sec x + 1}{2 \sec x}$$

### Exercise Set 3.4, page 251

$$22. \cos 3\theta + \cos 5\theta = 2 \cos \frac{3\theta + 5\theta}{2} \cos \frac{3\theta - 5\theta}{2}$$

$$= 2 \cos 4\theta \cos(-\theta) = 2 \cos 4\theta \cos \theta$$

$$36. \sin 5x \cos 3x = \frac{1}{2} [\sin(5x + 3x) + \sin(5x - 3x)]$$

$$= \frac{1}{2} (\sin 8x + \sin 2x)$$

$$= \frac{1}{2} (2 \sin 4x \cos 4x + 2 \sin x \cos x)$$

$$= \sin 4x \cos 4x + \sin x \cos x$$

$$44. \frac{\cos 5x - \cos 3x}{\sin 5x + \sin 3x} = \frac{-2 \sin \frac{5x + 3x}{2} \sin \frac{5x - 3x}{2}}{2 \sin \frac{5x + 3x}{2} \cos \frac{5x - 3x}{2}}$$

$$= -\frac{\sin 4x \sin x}{\sin 4x \cos x} = -\tan x$$

62.  $a = 1, b = \sqrt{3}, k = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ . Thus  $\alpha$  is a first-quadrant angle.

$$\sin \alpha = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos \alpha = \frac{1}{2}$$

$$\text{Thus } \alpha = \frac{\pi}{3}.$$

$$y = k \sin(x + \alpha)$$

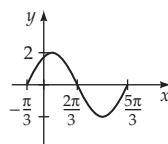
$$y = 2 \sin\left(x + \frac{\pi}{3}\right)$$

70. From Exercise 62, we know that

$$y = \sin x + \sqrt{3} \cos x = 2 \sin\left(x + \frac{\pi}{3}\right)$$

The graph of  $y = \sin x + \sqrt{3} \cos x$  has an amplitude of 2, and a phase shift of  $-\frac{\pi}{3}$ . It is the graph of

$y = 2 \sin x$  shifted  $\frac{\pi}{3}$  units to the left.



### Exercise Set 3.5, page 265

$$2. y = \sin^{-1} \frac{\sqrt{2}}{2} \text{ implies}$$

$$\sin y = \frac{\sqrt{2}}{2} \quad \text{for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\text{Thus } y = \frac{\pi}{4}.$$

28. Because  $\tan(\tan^{-1} x) = x$  for all real numbers  $x$ , we have

$$\tan\left[\tan^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{2}.$$

50. Let  $x = \cos^{-1} \frac{3}{5}$ . Thus

$$\cos x = \frac{3}{5} \quad \text{and} \quad \sin x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$y = \tan\left(\cos^{-1} \frac{3}{5}\right) = \tan x = \frac{\sin x}{\cos x} = \frac{4/5}{3/5} = \frac{4}{3}$$

$$56. y = \cos\left(\sin^{-1} \frac{3}{4} + \cos^{-1} \frac{5}{13}\right)$$

$$\text{Let } \alpha = \sin^{-1} \frac{3}{4}, \sin \alpha = \frac{3}{4}, \cos \alpha = \sqrt{1 - \left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}.$$

$$\beta = \cos^{-1} \frac{5}{13}, \cos \beta = \frac{5}{13}, \sin \beta = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{12}{13}.$$

$$\begin{aligned}y &= \cos(\alpha + \beta) \\&= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\&= \frac{\sqrt{7}}{4} \cdot \frac{5}{13} - \frac{3}{4} \cdot \frac{12}{13} = \frac{5\sqrt{7}}{52} - \frac{36}{52} = \frac{5\sqrt{7} - 36}{52}\end{aligned}$$

66.  $\sin^{-1} x + \cos^{-1} \frac{4}{5} = \frac{\pi}{6}$

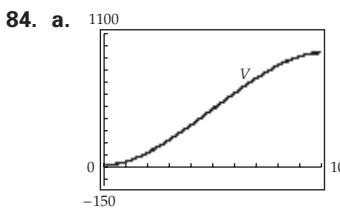
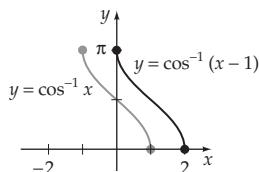
$$\sin^{-1} x = \frac{\pi}{6} - \cos^{-1} \frac{4}{5}$$

$$\begin{aligned}\sin(\sin^{-1} x) &= \sin\left(\frac{\pi}{6} - \cos^{-1} \frac{4}{5}\right) \\x &= \sin \frac{\pi}{6} \cos\left(\cos^{-1} \frac{4}{5}\right) - \cos \frac{\pi}{6} \sin\left(\cos^{-1} \frac{4}{5}\right) \\&= \frac{1}{2} \cdot \frac{4}{5} - \frac{\sqrt{3}}{2} \cdot \frac{3}{5} = \frac{4 - 3\sqrt{3}}{10}\end{aligned}$$

72. Let  $\alpha = \cos^{-1} x$  and  $\beta = \cos^{-1}(-x)$ . Thus  $\cos \alpha = x$  and  $\cos \beta = -x$ . We know that  $\sin \alpha = \sqrt{1 - x^2}$  and  $\sin \beta = \sqrt{1 - x^2}$  because  $\alpha$  is in Quadrant I and  $\beta$  is in Quadrant II.

$$\begin{aligned}\cos^{-1} x + \cos^{-1}(-x) &= \alpha + \beta \\&= \cos^{-1}[\cos(\alpha + \beta)] \\&= \cos^{-1}(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \\&= \cos^{-1}[x(-x) - \sqrt{1 - x^2} \cdot \sqrt{1 - x^2}] \\&= \cos^{-1}(-x^2 - 1 + x^2) \\&= \cos^{-1}(-1) = \pi\end{aligned}$$

76. The graph of  $y = f(x - a)$  is a horizontal shift of the graph of  $y = f(x)$ . Therefore, the graph of  $y = \cos^{-1}(x - 1)$  is the graph of  $y = \cos^{-1} x$  shifted 1 unit to the right.

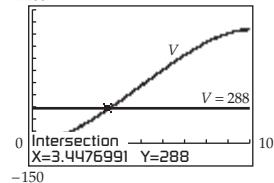


- b. Although the water rises 0.1 foot in each case, there is more surface area (and thus more volume of water) at the 4.9- to 5.0-foot level near the diameter of the cylinder than at the 0.1- to 0.2-foot level near the bottom.

c.

$$\begin{aligned}V(4) &= 12 \left[ 25 \cos^{-1} \left( \frac{5 - (4)}{5} \right) - [5 - (4)] \sqrt{10(4) - (4)^2} \right] \\&= 12 \left[ 25 \cos^{-1} \left( \frac{1}{5} \right) - \sqrt{24} \right] \\&\approx 352.04 \text{ cubic feet}\end{aligned}$$

d.



When  $V = 288$  cubic feet,  $x \approx 3.45$  feet.

### Exercise Set 3.6, page 278

$$\begin{aligned}14. \quad 2 \cos^2 x + 1 &= -3 \cos x \\2 \cos^2 x + 3 \cos x + 1 &= 0 \\(2 \cos x + 1)(\cos x + 1) &= 0 \\2 \cos x + 1 &= 0 \quad \text{or} \quad \cos x + 1 = 0 \\ \cos x &= -\frac{1}{2} \quad \cos x = -1 \\x &= \frac{2\pi}{3}, \frac{4\pi}{3} \quad x = \pi\end{aligned}$$

The solutions in the interval  $0 \leq x < 2\pi$  are  $\frac{2\pi}{3}$ ,  $\pi$ , and  $\frac{4\pi}{3}$ .

$$\begin{aligned}52. \quad \sin x + 2 \cos x &= 1 \\ \sin x &= 1 - 2 \cos x \\(\sin x)^2 &= (1 - 2 \cos x)^2 \\ \sin^2 x &= 1 - 4 \cos x + 4 \cos^2 x \\1 - \cos^2 x &= 1 - 4 \cos x + 4 \cos^2 x \\0 &= \cos x(5 \cos x - 4) \\ \cos x &= 0 \quad \text{or} \quad 5 \cos x - 4 = 0 \\x &= 90^\circ, 270^\circ \quad \cos x = \frac{4}{5} \\x &\approx 36.9^\circ, 323.1^\circ\end{aligned}$$

The solutions in the interval  $0 \leq x < 360^\circ$  are  $90^\circ$  and  $323.1^\circ$ . (Note:  $x = 270^\circ$  and  $x = 36.9^\circ$  are extraneous solutions. Neither of these values satisfies the original equation.)

56.  $2 \cos^2 x - 5 \cos x - 5 = 0$

$$\cos x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-5)}}{2(2)} = \frac{5 \pm \sqrt{65}}{4}$$

$$\cos x \approx 3.27 \quad \text{or} \quad \cos x \approx -0.7656$$

$$\text{no solution} \quad x \approx 140.0^\circ, 220.0^\circ$$

The solutions in the interval  $0^\circ \leq x < 360^\circ$  are  $140.0^\circ$  and  $220.0^\circ$ .

66.  $\cos 2x = -\frac{\sqrt{3}}{2}$

$$2x = \frac{5\pi}{6} + 2k\pi \quad \text{or} \quad 2x = \frac{7\pi}{6} + 2k\pi, k \text{ an integer}$$

$$x = \frac{5\pi}{12} + k\pi \quad \text{or} \quad x = \frac{7\pi}{12} + k\pi, k \text{ an integer}$$

84.  $2 \sin x \cos x - 2\sqrt{2} \sin x - \sqrt{3} \cos x + \sqrt{6} = 0$

$$2 \sin x(\cos x - \sqrt{2}) - \sqrt{3}(\cos x - \sqrt{2}) = 0$$

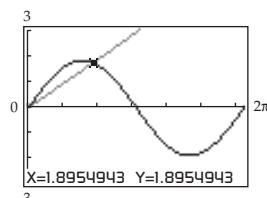
$$(\cos x - \sqrt{2})(2 \sin x - \sqrt{3}) = 0$$

$$\cos x = \sqrt{2} \quad \text{or} \quad \sin x = \frac{\sqrt{3}}{2}$$

$$\text{no solution} \quad x = \frac{\pi}{3}, \frac{2\pi}{3}$$

The solutions in the interval  $0 \leq x < 2\pi$  are  $\frac{\pi}{3}$  and  $\frac{2\pi}{3}$ .

86. The following graph shows that the solutions in the interval  $[0, 2\pi)$  are  $x = 0$  and  $x = 1.895$ .



92. When  $\theta = 45^\circ$ ,  $d$  attains its maximum of 4394.5 feet.

94. a. The following work shows the sine regression procedure for a TI-83/TI-83 Plus/TI-84 Plus calculator. Note that each time (given in hours:minutes) must be converted to hours. Thus

$$17:06 \text{ is } 17 + \frac{6}{60} = 17.1 \text{ hours.}$$

| L1        | L2     | L3                             |
|-----------|--------|--------------------------------|
| 1         | 17.1   | SinReg [F6, L1, L2, 35.25, Y1] |
| 32        | 17.683 |                                |
| 60        | 18.25  |                                |
| 91        | 18.817 |                                |
| 121       | 19.367 |                                |
| 152       | 19.85  |                                |
| 182       | 20.017 |                                |
| <hr/>     |        |                                |
| L1(1) = 1 |        |                                |

```

SinReg [F6, L1, L2, 35.25, Y1]
y = a·sin(bx+c)+d
a = 1.5346578145031
b = .0162395434
c = -1.131646526
d = 18.41181546

```

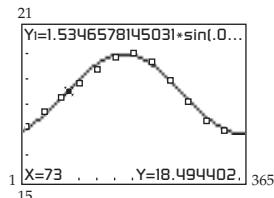
The sine regression function is

$$Y_1 \approx 1.5347 \sin(0.01624x - 1.1316) + 18.4118$$

b. Using the "value" command in the CALC menu gives

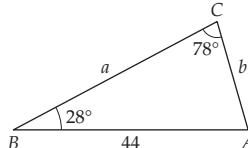
$$Y_1(73) \approx 18.494402$$

$\approx 18:30$  (to the nearest minute)



### Exercise Set 4.1, page 298

4.



$$A = 180^\circ - 78^\circ - 28^\circ = 74^\circ$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 28^\circ} = \frac{44}{\sin 78^\circ}$$

$$b = \frac{44 \sin 28^\circ}{\sin 78^\circ} \approx 21$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 74^\circ} = \frac{44}{\sin 78^\circ}$$

$$a = \frac{44 \sin 74^\circ}{\sin 78^\circ} \approx 43$$

20.  $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\frac{13.8}{\sin A} = \frac{5.55}{\sin 22.6^\circ}$$

$$\sin A \approx 0.9555$$

$$A \approx 72.9^\circ \quad \text{or} \quad 107.1^\circ$$

$$\text{If } A = 72.9^\circ, C \approx 180^\circ - 72.9^\circ - 22.6^\circ = 84.5^\circ.$$

$$\frac{c}{\sin 84.5^\circ} = \frac{5.55}{\sin 22.6^\circ}$$

$$c = \frac{5.55 \sin 84.5^\circ}{\sin 22.6^\circ} \approx 14.4$$

$$\text{If } A = 107.1^\circ, C \approx 180^\circ - 107.1^\circ - 22.6^\circ = 50.3^\circ.$$

$$\frac{c}{\sin 50.3^\circ} = \frac{5.55}{\sin 22.6^\circ}$$

$$c = \frac{5.55 \sin 50.3^\circ}{\sin 22.6^\circ} \approx 11.1$$

$$\text{Case 1: } A = 72.9^\circ, C = 84.5^\circ, \text{ and } c = 14.4$$

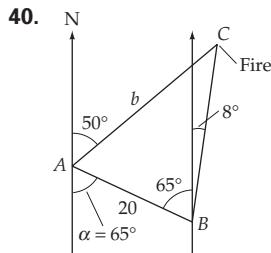
$$\text{Case 2: } A = 107.1^\circ, C = 50.3^\circ, \text{ and } c = 11.1$$

30. The angle with its vertex at the position of the helicopter measures  $180^\circ - (59.0^\circ + 77.2^\circ) = 43.8^\circ$ . Let the distance from the helicopter to the carrier be  $x$ . Using the Law of Sines, we have

$$\frac{x}{\sin 77.2^\circ} = \frac{7620}{\sin 43.8^\circ}$$

$$x = \frac{7620 \sin 77.2^\circ}{\sin 43.8^\circ}$$

$$\approx 10,700 \text{ feet} \quad (\text{to three significant digits})$$



$$\alpha = 65^\circ$$

$$B = 65^\circ + 8^\circ = 73^\circ$$

$$A = 180^\circ - 50^\circ - 65^\circ = 65^\circ$$

$$C = 180^\circ - 65^\circ - 73^\circ = 42^\circ$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 73^\circ} = \frac{20}{\sin 42^\circ}$$

$$b = \frac{20 \sin 73^\circ}{\sin 42^\circ}$$

$$b \approx 29 \text{ miles}$$

### Exercise Set 4.2, page 308

12.  $c^2 = a^2 + b^2 - 2ab \cos C$

$$c^2 = 14.2^2 + 9.30^2 - 2(14.2)(9.30) \cos 9.20^\circ$$

$$c = \sqrt{14.2^2 + 9.30^2 - 2(14.2)(9.30) \cos 9.20^\circ}$$

$$c \approx 5.24$$

18.  $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\cos A = \frac{132^2 + 160^2 - 108^2}{2(132)(160)} \approx 0.7424$$

$$A \approx \cos^{-1}(0.7424) \approx 42.1^\circ$$

30.  $K = \frac{1}{2}ac \sin B$

$$K = \frac{1}{2}(32)(25) \sin 127^\circ \approx 320 \text{ square units}$$

32.  $A = 180^\circ - 102^\circ - 27^\circ = 51^\circ$

$$K = \frac{a^2 \sin B \sin C}{2 \sin A}$$

$$K = \frac{8.5^2 \sin 102^\circ \sin 27^\circ}{2 \sin 51^\circ} \approx 21 \text{ square units}$$

40.  $s = \frac{1}{2}(a + b + c)$

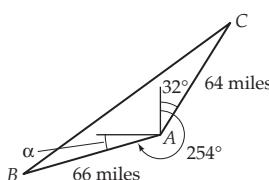
$$= \frac{1}{2}(10.2 + 13.3 + 15.4) = 19.45$$

$$K = \sqrt{s(s - a)(s - b)(s - c)}$$

$$= \sqrt{19.45(19.45 - 10.2)(19.45 - 13.3)(19.45 - 15.4)}$$

$$\approx 66.9 \text{ square units}$$

52.



$$\alpha = 270^\circ - 254^\circ = 16^\circ$$

$$A = 16^\circ + 90^\circ + 32^\circ = 138^\circ$$

$$b = 4 \cdot 16 = 64 \text{ miles}$$

$$c = 3 \cdot 22 = 66 \text{ miles}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 64^2 + 66^2 - 2(64)(66) \cos 138^\circ$$

$$a = \sqrt{64^2 + 66^2 - 2(64)(66) \cos 138^\circ}$$

$$a \approx 120 \text{ miles}$$

60.  $S = \frac{1}{2}(324 + 412 + 516) = 626$

$$K = \sqrt{626(626 - 324)(626 - 412)(626 - 516)}$$

$$= \sqrt{4,450,284,080}$$

$$\text{Cost} = 4.15(\sqrt{4,450,284,080}) \approx \$276,848, \text{ or } \$277,000 \text{ to the nearest } \$1000$$

### Exercise Set 4.3, page 325

10.  $a = 3 - 3 = 0$

$$b = 0 - (-2) = 2$$

A vector equivalent to  $\mathbf{P}_1\mathbf{P}_2$  is  $\mathbf{v} = \langle 0, 2 \rangle$ .

12.  $\|\mathbf{v}\| = \sqrt{6^2 + 10^2}$

$$= \sqrt{36 + 100} = \sqrt{136} = 2\sqrt{34}$$

$$\theta = \tan^{-1} \frac{10}{6} = \tan^{-1} \frac{5}{3} \approx 59.0^\circ$$

Thus  $\mathbf{v}$  has a direction of about  $59^\circ$  as measured from the positive  $x$ -axis.

A unit vector in the direction of  $\mathbf{v}$  is

$$\mathbf{u} = \left\langle \frac{6}{2\sqrt{34}}, \frac{10}{2\sqrt{34}} \right\rangle = \left\langle \frac{3\sqrt{34}}{34}, \frac{5\sqrt{34}}{34} \right\rangle$$

$$\begin{aligned} 24. \quad \frac{3}{4}\mathbf{u} - 2\mathbf{v} &= \frac{3}{4}\langle -2, 4 \rangle - 2\langle -3, -2 \rangle \\ &= \left\langle -\frac{3}{2}, 3 \right\rangle - \langle -6, -4 \rangle \\ &= \left\langle \frac{9}{2}, 7 \right\rangle \end{aligned}$$

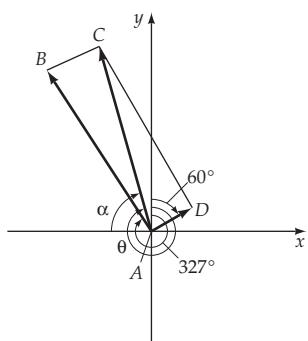
$$\begin{aligned} 30. \quad 3\mathbf{u} + 2\mathbf{v} &= 3(3\mathbf{i} - 2\mathbf{j}) + 2(-2\mathbf{i} + 3\mathbf{j}) \\ &= (9\mathbf{i} - 6\mathbf{j}) + (-4\mathbf{i} + 6\mathbf{j}) \\ &= (9 - 4)\mathbf{i} + (-6 + 6)\mathbf{j} \\ &= 5\mathbf{i} + 0\mathbf{j} \\ &= 5\mathbf{i} \end{aligned}$$

$$40. \quad a_1 = 2 \cos \frac{8\pi}{7} \approx -1.8$$

$$a_2 = 2 \sin \frac{8\pi}{7} \approx -0.9$$

$$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} \approx -1.8\mathbf{i} - 0.9\mathbf{j}$$

44.



$$\mathbf{AB} = 18 \cos 123^\circ \mathbf{i} + 18 \sin 123^\circ \mathbf{j} \approx -9.8\mathbf{i} + 15.1\mathbf{j}$$

$$\mathbf{AD} = 4 \cos 30^\circ \mathbf{i} + 4 \sin 30^\circ \mathbf{j} \approx 3.5\mathbf{i} + 2\mathbf{j}$$

$$\begin{aligned} \mathbf{AC} &= \mathbf{AB} + \mathbf{AD} \approx (-9.8\mathbf{i} + 15.1\mathbf{j}) + (3.5\mathbf{i} + 2\mathbf{j}) \\ &= -6.3\mathbf{i} + 17.1\mathbf{j} \end{aligned}$$

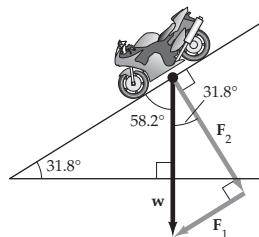
$$\|\mathbf{AC}\| = \sqrt{(-6.3)^2 + (17.1)^2} \approx 18$$

$$\alpha = \tan^{-1} \left| \frac{17.1}{-6.3} \right| = \tan^{-1} \frac{17.1}{6.3} \approx 70^\circ$$

$$\theta \approx 270^\circ + 70^\circ = 340^\circ$$

The speed of the boat is about 18 miles per hour at an approximate heading of  $340^\circ$ .

48. In the following figure, we need to find  $\|\mathbf{-F}_1\|$  and  $\|\mathbf{F}_2\|$ .



From the right triangle formed by  $\mathbf{w}$ ,  $\mathbf{F}_1$ , and  $\mathbf{F}_2$ , we see that  $\sin 31.8^\circ = \frac{\|\mathbf{F}_1\|}{\|\mathbf{w}\|}$  and  $\cos 31.8^\circ = \frac{\|\mathbf{F}_2\|}{\|\mathbf{w}\|}$ .

a.  $\sin 31.8^\circ = \frac{\|\mathbf{F}_1\|}{\|\mathbf{w}\|}$

$$\sin 31.8^\circ = \frac{\|\mathbf{F}_1\|}{811} \cdot \|\mathbf{w}\| = 811$$

$$\|\mathbf{F}_1\| = 811 \sin 31.8^\circ \approx 427$$

Now  $\|\mathbf{-F}_1\| = \|\mathbf{F}_1\|$ . Thus the magnitude of the force needed to keep the motorcycle from rolling down the ramp is approximately 427 pounds.

b.  $\cos 31.8^\circ = \frac{\|\mathbf{F}_2\|}{\|\mathbf{w}\|}$

$$\cos 31.8^\circ = \frac{\|\mathbf{F}_2\|}{811} \cdot \|\mathbf{w}\| = 811$$

$$\|\mathbf{F}_2\| = 811 \cos 31.8^\circ \approx 689$$

The magnitude of the force the motorcycle exerts against the ramp is approximately 689 pounds.

$$\begin{aligned} 60. \quad \mathbf{v} \cdot \mathbf{w} &= (5\mathbf{i} + 3\mathbf{j}) \cdot (4\mathbf{i} - 2\mathbf{j}) \\ &= 5(4) + 3(-2) = 20 - 6 = 14 \end{aligned}$$

$$70. \quad \cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

$$\cos \theta = \frac{(3\mathbf{i} - 4\mathbf{j}) \cdot (6\mathbf{i} - 12\mathbf{j})}{\sqrt{3^2 + (-4)^2} \sqrt{6^2 + (-12)^2}}$$

$$\cos \theta = \frac{3(6) + (-4)(-12)}{\sqrt{25} \sqrt{180}}$$

$$\cos \theta = \frac{66}{5\sqrt{180}} \approx 0.9839$$

$$\theta \approx 10.3^\circ$$

$$72. \quad \text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}$$

$$\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\langle -7, 5 \rangle \cdot \langle -4, 1 \rangle}{\sqrt{(-4)^2 + 1^2}} = \frac{33}{\sqrt{17}} = \frac{33\sqrt{17}}{17} \approx 8.0$$

$$80. \quad W = \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha$$

$$W = 100 \cdot 25 \cdot \cos 42^\circ$$

$$W \approx 1858 \text{ foot-pounds}$$

**Exercise Set 5.1, page 340**

8.  $6 - \sqrt{-1} = 6 - i$

18.  $(5 - 3i) - (2 + 9i) = 5 - 3i - 2 - 9i = 3 - 12i$

34.  $(5 + 2\sqrt{-16})(1 - \sqrt{-25}) = [5 + 2(4i)][1 - 5i]$   
 $= (5 + 8i)(1 - 5i)$   
 $= 5 - 25i + 8i - 40i^2$   
 $= 5 - 25i + 8i - 40(-1)$   
 $= 5 - 25i + 8i + 40$   
 $= 45 - 17i$

48.  $\frac{8 - i}{2 + 3i} = \frac{8 - i}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i}$   
 $= \frac{16 - 24i - 2i + 3i^2}{2^2 + 3^2}$   
 $= \frac{16 - 24i - 2i + 3(-1)}{4 + 9}$   
 $= \frac{16 - 26i - 3}{13}$   
 $= \frac{13 - 26i}{13} = \frac{13(1 - 2i)}{13} = 1 - 2i$

60.  $\frac{1}{i^{83}} = \frac{1}{i^{80}i^3} = \frac{1}{i^3} = \frac{1}{i^3} \cdot \frac{i}{i} = \frac{i}{i^4} = i$

**Exercise Set 5.2, page 349**

12.  $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\alpha = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \tan^{-1}\sqrt{3} = 60^\circ$

$\theta = \alpha = 60^\circ, z = 2 \text{ cis } 60^\circ$

30.  $z = 4\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right) = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2 - 2i\sqrt{3}$

56.  $\sqrt{3} - i = 2 \text{ cis}(-30^\circ); 1 + i\sqrt{3} = 2 \text{ cis } 60^\circ$

$$\begin{aligned} (\sqrt{3} - i)(1 + i\sqrt{3}) &= 2 \text{ cis}(-30^\circ) \cdot 2 \text{ cis } 60^\circ \\ &= 2 \cdot 2 \text{ cis}(-30^\circ + 60^\circ) \\ &= 4 \text{ cis } 30^\circ \\ &= 4(\cos 30^\circ + i \sin 30^\circ) \\ &= 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\ &= 2\sqrt{3} + 2i \end{aligned}$$

60.  $z_1 = 1 + i$

$r_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\alpha_1 = \tan^{-1}\left|\frac{1}{1}\right| = 45^\circ; \theta_1 = 45^\circ$

$z_1 = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ) = \sqrt{2} \text{ cis } 45^\circ$

$z_2 = 1 - i$

$r_2 = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$$\alpha_2 = \tan^{-1}\left|\frac{-1}{1}\right| = 45^\circ; \theta_2 = 315^\circ$$

$z_2 = \sqrt{2}(\cos 315^\circ + i \sin 315^\circ) = \sqrt{2} \text{ cis } 315^\circ$

$$\frac{z_1}{z_2} = \frac{\sqrt{2} \text{ cis } 45^\circ}{\sqrt{2} \text{ cis } 315^\circ}$$

$= \text{cis}(-270^\circ)$

$= \cos 270^\circ - i \sin 270^\circ = 0 - (-i) = i$

**Exercise Set 5.3, page 354**

6.  $(2 \text{ cis } 330^\circ)^4 = 2^4 \text{ cis}(4 \cdot 330^\circ)$

$= 16 \text{ cis } 1320^\circ$

$= 16 \text{ cis } 240^\circ$

$= 16(\cos 240^\circ + i \sin 240^\circ)$

$$= 16\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$= -8 - 8i\sqrt{3}$

16.  $\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^{12} = [1(\cos 135^\circ + i \sin 135^\circ)]^{12}$

$= 1^{12}[\cos(12 \cdot 135^\circ) + i \sin(12 \cdot 135^\circ)]$

$= \cos 1620^\circ + i \sin 1620^\circ$

$= -1 + 0i, \text{ or } -1$

28.  $-2 + 2i\sqrt{3} = 4(\cos 120^\circ + i \sin 120^\circ)$

$$w_k = 4^{1/3}\left(\cos \frac{120^\circ + 360^\circ k}{3} + i \sin \frac{120^\circ + 360^\circ k}{3}\right), \quad k = 0, 1, 2$$

$$w_0 = 4^{1/3}\left(\cos \frac{120^\circ}{3} + i \sin \frac{120^\circ}{3}\right) \approx 1.216 + 1.020i$$

$$w_1 = 4^{1/3}\left(\cos \frac{120^\circ + 360^\circ}{3} + i \sin \frac{120^\circ + 360^\circ}{3}\right) \approx -1.492 + 0.543i$$

$$w_2 = 4^{1/3}\left(\cos \frac{120^\circ + 360^\circ \cdot 2}{3} + i \sin \frac{120^\circ + 360^\circ \cdot 2}{3}\right) \approx 0.276 - 1.563i$$

30.  $-1 + i\sqrt{3} = 2(\cos 120^\circ + i \sin 120^\circ)$

$$w_k = 2^{1/2}\left(\cos \frac{120^\circ + 360^\circ k}{2} + i \sin \frac{120^\circ + 360^\circ k}{2}\right), \quad k = 0, 1$$

$$w_0 = 2^{1/2}(\cos 60^\circ + i \sin 60^\circ) = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i \quad (k = 0)$$

$$w_1 = 2^{1/2}\left(\cos \frac{120^\circ + 360^\circ}{2} + i \sin \frac{120^\circ + 360^\circ}{2}\right) \quad (k = 1)$$

$= 2^{1/2}(\cos 240^\circ + i \sin 240^\circ)$

$$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$$

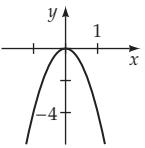
## Exercise Set 6.1, page 370

6. Comparing  $x^2 = 4py$  with  $x^2 = -\frac{1}{4}y$ , we have  $4p = -\frac{1}{4}$ , or  $p = -\frac{1}{16}$ .

Vertex:  $(0, 0)$

$$\text{Focus: } \left(0, -\frac{1}{16}\right)$$

$$\text{Directrix: } y = \frac{1}{16}$$



22.  $x^2 + 5x - 4y - 1 = 0$

$$x^2 + 5x = 4y + 1$$

$$x^2 + 5x + \frac{25}{4} = 4y + 1 + \frac{25}{4}$$

$$\left(x + \frac{5}{2}\right)^2 = 4\left(y + \frac{29}{16}\right)$$

$$4p = 4$$

$$p = 1$$

$$\text{Vertex: } \left(-\frac{5}{2}, -\frac{29}{16}\right)$$

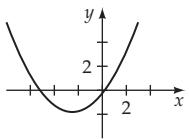
$$\text{Focus: } (h, k + p) = \left(-\frac{5}{2}, -\frac{13}{16}\right)$$

$$\text{Directrix: } y = k - p = -\frac{45}{16}$$

• Complete the square.

$$\bullet h = -\frac{5}{2}, k = -\frac{29}{16}$$

$$\bullet \text{Compare to } (x - h)^2 = 4p(y - k)^2.$$



30. Vertex:  $(0, 0)$ , focus:  $(5, 0)$ ;  $p = 5$  because focus is  $(p, 0)$ .

$$y^2 = 4px$$

$$y^2 = 4(5)x$$

$$y^2 = 20x$$

32. Vertex:  $(2, -3)$ , focus:  $(0, -3)$

$$(h, k) = (2, -3), \text{ so } h = 2 \text{ and } k = -3.$$

$$\text{Focus is } (h + p, k) = (2 + p, -3) = (0, -3).$$

Therefore,  $2 + p = 0$  and  $p = -2$ .

$$(y - k)^2 = 4p(x - h)$$

$$(y + 3)^2 = 4(-2)(x - 2)$$

$$(y + 3)^2 = -8(x - 2)$$

40.  $x^2 = 4py$

$$40.5^2 = 4p(16)$$

$$p = \frac{40.5^2}{64}$$

$$p \approx 25.6 \text{ feet}$$

## Exercise Set 6.2, page 385

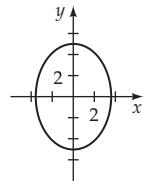
22.  $25x^2 + 12y^2 = 300 \quad \bullet a^2 = 25, b^2 = 12, c^2 = 25 - 12$

$$\frac{x^2}{25} + \frac{y^2}{12} = 1 \quad \bullet a = 5, b = 2\sqrt{3}, c = \sqrt{13}$$

Center:  $(0, 0)$

Vertices:  $(0, 5)$  and  $(0, -5)$

Foci:  $(0, \sqrt{13})$  and  $(0, -\sqrt{13})$



28.  $9x^2 + 16y^2 + 36x - 16y - 104 = 0$

$$9x^2 + 36x + 16y^2 - 16y - 104 = 0$$

$$9(x^2 + 4x) + 16(y^2 - y) = 104$$

$$9(x^2 + 4x + 4) + 16\left(y^2 - y + \frac{1}{4}\right) = 104 + 36 + 4$$

$$9(x + 2)^2 + 16\left(y - \frac{1}{2}\right)^2 = 144$$

$$\frac{(x + 2)^2}{16} + \frac{\left(y - \frac{1}{2}\right)^2}{9} = 1$$

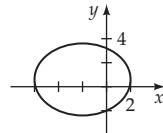
Center:  $(-2, \frac{1}{2})$

$$a = 4, b = 3,$$

$$c = \sqrt{4^2 - 3^2} = \sqrt{7}$$

$$\text{Vertices: } \left(2, \frac{1}{2}\right) \text{ and } \left(-6, \frac{1}{2}\right)$$

$$\text{Foci: } \left(-2 + \sqrt{7}, \frac{1}{2}\right) \text{ and } \left(-2 - \sqrt{7}, \frac{1}{2}\right)$$



44. Center  $(-4, 1) = (h, k)$ . Therefore,  $h = -4$  and  $k = 1$ .

Length of minor axis is 8, so  $2b = 8$ , or  $b = 4$ . The equation of the ellipse is of the form

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\frac{(x + 4)^2}{a^2} + \frac{(y - 1)^2}{16} = 1$$

$$\bullet h = -4, k = 1, b = 4$$

$$\frac{(0 + 4)^2}{a^2} + \frac{(4 - 1)^2}{16} = 1$$

• The point  $(0, 4)$  is on the graph. Thus  $x = 0$  and  $y = 4$  satisfy the equation.

$$\frac{16}{a^2} + \frac{9}{16} = 1$$

$$\frac{16}{a^2} = \frac{7}{16}$$

$$a^2 = \frac{256}{7}$$

$$\frac{(x + 4)^2}{256/7} + \frac{(y - 1)^2}{16} = 1$$

50. Because the foci are  $(0, -3)$  and  $(0, 3)$ ,  $c = 3$  and the center is  $(0, 0)$ , the midpoint of the line segment between  $(0, -3)$  and  $(0, 3)$ .

$$e = \frac{c}{a}$$

$$\frac{1}{4} = \frac{3}{a}$$

$$a = 12$$

$$3^2 = 12^2 - b^2$$

$$b^2 = 144 - 9 = 135$$

$$\cdot e = \frac{1}{4}$$

$$\cdot c^2 = a^2 - b^2$$

**Solve for  $b^2$ .**

$$\text{The equation of the ellipse is } \frac{x^2}{135} + \frac{y^2}{144} = 1.$$

58. The mean distance is  $a = 67.08$  million miles.

$$\text{Aphelion} = a + c = 67.58 \text{ million miles}$$

$$\text{Thus } c = 67.58 - a = 0.50 \text{ million miles.}$$

$$b = \sqrt{a^2 - c^2} = \sqrt{67.08^2 - 0.50^2} \approx 67.078$$

An equation of the orbit of Venus is

$$\frac{x^2}{67.08^2} + \frac{y^2}{67.078^2} = 1.$$

60. The length of the semimajor axis is 50 feet. Thus

$$c^2 = a^2 - b^2$$

$$32^2 = 50^2 - b^2$$

$$b^2 = 50^2 - 32^2$$

$$b = \sqrt{50^2 - 32^2}$$

$$b \approx 38.4 \text{ feet}$$

### Exercise Set 6.3, page 399

6.  $\frac{y^2}{25} - \frac{x^2}{36} = 1$

$$a^2 = 25 \quad b^2 = 36 \quad c^2 = a^2 + b^2 = 25 + 36 = 61$$

$$a = 5 \quad b = 6 \quad c = \sqrt{61}$$

Transverse axis is on  $y$ -axis because  $y^2$  term is positive.

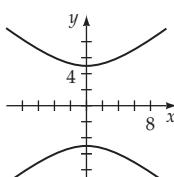
Center:  $(0, 0)$

Foci:  $(0, \sqrt{61})$  and  $(0, -\sqrt{61})$

Asymptotes:  $y = \frac{5}{6}x$  and

$$y = -\frac{5}{6}x$$

Vertices:  $(0, 5)$  and  $(0, -5)$



28.  $16x^2 - 9y^2 - 32x - 54y + 79 = 0$

$$16(x^2 - 2x + 1) - 9(y^2 + 6y + 9) = -79 + 16 - 81$$

$$= -144$$

$$\frac{(y+3)^2}{16} - \frac{(x-1)^2}{9} = 1$$

Transverse axis is parallel to  $y$ -axis because  $y^2$  term is positive. Center is at  $(1, -3)$ ;  $a^2 = 16$  so  $a = 4$ .

Vertices:  $(h, k + a) = (1, 1)$

$(h, k - a) = (1, -7)$

$$c^2 = a^2 + b^2 = 16 + 9 = 25$$

$$c = \sqrt{25} = 5$$

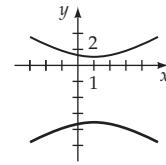
Foci:  $(h, k + c) = (1, 2)$

$(h, k - c) = (1, -8)$

Because  $b^2 = 9$  and  $b = 3$ , the asymptotes are

$$y + 3 = \frac{4}{3}(x - 1) \text{ and}$$

$$y + 3 = -\frac{4}{3}(x - 1).$$



50. Because the vertices are  $(2, 3)$  and  $(-2, 3)$ ,  $a = 2$  and the center is  $(0, 3)$ .

$$e = \frac{c}{a} \quad c^2 = a^2 + b^2$$

$$\frac{5}{2} = \frac{c}{2} \quad 5^2 = 2^2 + b^2$$

$$c = 5$$

Substituting into the standard equation yields

$$\frac{x^2}{4} - \frac{(y-3)^2}{21} = 1.$$

56. a. Because the transmitters are 300 miles apart,  $2c = 300$  and  $c = 150$ .

$$2a = \text{rate} \times \text{time}$$

$$2a = 0.186 \times 800 = 148.8 \text{ miles}$$

$$\text{Thus } a = 74.4 \text{ miles.}$$

$$b = \sqrt{c^2 - a^2}$$

$$= \sqrt{150^2 - 74.4^2} \approx 130.25 \text{ miles}$$

The ship is located on the hyperbola given by

$$\frac{x^2}{74.4^2} - \frac{y^2}{130.25^2} = 1$$

- b. The ship will reach the coastline when  $x < 0$  and  $y = 0$ . Thus

$$\frac{x^2}{74.4^2} - \frac{0^2}{130.25^2} = 1$$

$$\frac{x^2}{74.4^2} = 1$$

$$x^2 = 74.4^2$$

$$x = -74.4$$

The ship reaches the coastline 74.4 miles to the left of the origin, at the point  $(-74.4, 0)$ .

**Exercise Set 6.4, page 410**

10.  $xy = -10$

$$xy + 10 = 0$$

$$A = 0, B = 1, C = 0, F = 10$$

$$\cot 2\alpha = \frac{A - C}{B} = \frac{0 - 0}{1} = 0$$

Thus  $2\alpha = 90^\circ$  and  $\alpha = 45^\circ$ .

$$A' = A \cos^2 \alpha + B \cos \alpha \sin \alpha + C \sin^2 \alpha$$

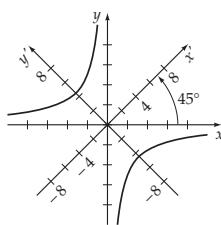
$$= 0\left(\frac{\sqrt{2}}{2}\right)^2 + 1\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 0\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

$$C' = A \sin^2 \alpha - B \cos \alpha \sin \alpha + C \cos^2 \alpha$$

$$= 0\left(\frac{\sqrt{2}}{2}\right)^2 - 1\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 0\left(\frac{\sqrt{2}}{2}\right)^2 = -\frac{1}{2}$$

$$F' = F = 10$$

$$\frac{1}{2}(x')^2 - \frac{1}{2}(y')^2 + 10 = 0 \text{ or } \frac{(y')^2}{20} - \frac{(x')^2}{20} = 1$$



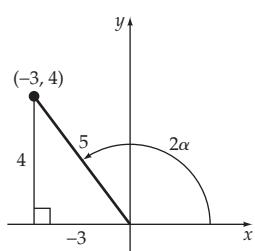
20.  $x^2 + 4xy + 4y^2 - 2\sqrt{5}x + \sqrt{5}y = 0$

$$A = 1, B = 4, C = 4, D = -2\sqrt{5}, E = \sqrt{5}, F = 0$$

$$\cot 2\alpha = \frac{A - C}{B} = \frac{1 - 4}{4} = -\frac{3}{4}$$

The following figure shows the second quadrant angle  $2\alpha$  for which  $\cot 2\alpha = -\frac{3}{4}$ .

$$2\alpha \text{ for which } \cot 2\alpha = -\frac{3}{4}$$



From the figure we see that  $\cos 2\alpha = -\frac{3}{5}$ .

Use the half-angle formulas to find  $\sin \alpha$  and  $\cos \alpha$ .

$$\sin \alpha = \sqrt{\frac{1 - \left(-\frac{3}{5}\right)}{2}} = \frac{2\sqrt{5}}{5}$$

$$\cos \alpha = \sqrt{\frac{1 + \left(-\frac{3}{5}\right)}{2}} = \frac{\sqrt{5}}{5}$$

$$\alpha \approx 63.4^\circ$$

$$A' = A \cos^2 \alpha + B \cos \alpha \sin \alpha + C \sin^2 \alpha$$

$$= 1\left(\frac{\sqrt{5}}{5}\right)^2 + 4\left(\frac{\sqrt{5}}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) + 4\left(\frac{2\sqrt{5}}{5}\right)^2 = 5$$

$$C' = A \sin^2 \alpha - B \cos \alpha \sin \alpha + C \cos^2 \alpha$$

$$= 1\left(\frac{2\sqrt{5}}{5}\right)^2 - 4\left(\frac{\sqrt{5}}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) + 4\left(\frac{\sqrt{5}}{5}\right)^2 = 0$$

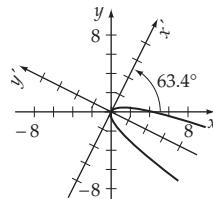
$$D' = D \cos \alpha + E \sin \alpha$$

$$= -2\sqrt{5}\left(\frac{\sqrt{5}}{5}\right) + \sqrt{5}\left(\frac{2\sqrt{5}}{5}\right) = 0$$

$$E' = -D \sin \alpha + E \cos \alpha$$

$$= 2\sqrt{5}\left(\frac{2\sqrt{5}}{5}\right) + \sqrt{5}\left(\frac{\sqrt{5}}{5}\right) = 5$$

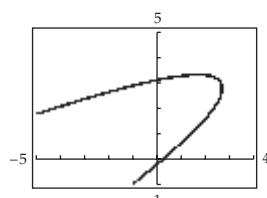
$$5(x')^2 + 5y' = 0 \text{ or } y' = -(x')^2$$



26. Use  $y_1$  and  $y_2$  as in Equations (12) and (13) that precede Example 4. Store the following constants.

$$A = 2, B = -8, C = 8, D = 20, E = -24, F = -3$$

Graph  $y_1$  and  $y_2$  on the same screen to produce the following parabola.



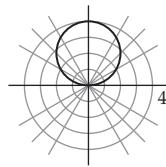
32. Because

$$\begin{aligned} B^2 - 4AC &= (-10\sqrt{3})^2 - 4(11)(1) \\ &= 300 - 44 > 0 \end{aligned}$$

the graph is a hyperbola.

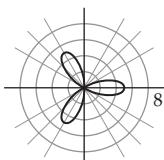
## Exercise Set 6.5, page 423

14.

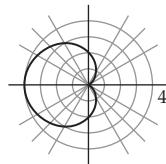


16.  $r = 5 \cos 3\theta$

Because 3 is odd, this is a rose with three petals.

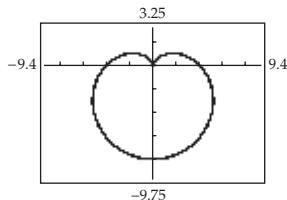


20. Because  $|a| = |b| = 2$ , the graph of  $r = 2 - 2 \cos \theta$ ,  $-\pi \leq \theta \leq \pi$ , is a cardioid.



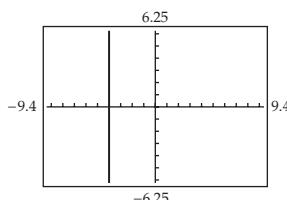
26.  $r = 4 - 4 \sin \theta$

$\theta_{\min} = 0$   
 $\theta_{\max} = 2\pi$   
 $\theta_{\text{step}} = 0.1$



32.  $r = -4 \sec \theta$

$\theta_{\min} = 0$   
 $\theta_{\max} = 2\pi$   
 $\theta_{\text{step}} = 0.1$



44.  $x = r \cos \theta$

$$\begin{aligned} &= (2) \left[ \cos \left( -\frac{\pi}{3} \right) \right] \\ &= (2) \left( \frac{1}{2} \right) = 1 \end{aligned}$$

$y = r \sin \theta$

$$\begin{aligned} &= (2) \left[ \sin \left( -\frac{\pi}{3} \right) \right] \\ &= (2) \left( -\frac{\sqrt{3}}{2} \right) = -\sqrt{3} \end{aligned}$$

The rectangular coordinates of the point are  $(1, -\sqrt{3})$ .

$$\begin{aligned} 48. \quad r &= \sqrt{x^2 + y^2} & \alpha &= \tan^{-1} \left| \frac{y}{x} \right| & \bullet \alpha \text{ is the reference angle for } \theta. \\ &= \sqrt{(12)^2 + (-5)^2} \\ &= \sqrt{144 + 25} \\ &= \sqrt{169} & &= \tan^{-1} \left| \frac{-5}{12} \right| \approx 22.6^\circ \\ &= 13 \end{aligned}$$

$\theta$  is a Quadrant IV angle. Thus  $\theta \approx 360^\circ - 22.6^\circ = 337.4^\circ$ .

The approximate polar coordinates of the point are  $(13, 337.4^\circ)$ .

50.

$$\begin{aligned} r &= 2 \sin \theta & r^2 &= 2r \sin \theta & \bullet \text{Multiply each side by } r. \\ r^2 &= 2r \sin \theta & x^2 + y^2 &= 2y & \bullet r^2 = x^2 + y^2; y = r \sin \theta \\ x^2 + y^2 &= 2y & x^2 + (y^2 - 2y) &= 0 & \bullet \text{Subtract } 2y \text{ from each side.} \\ x^2 + (y^2 - 2y + 1) &= 1 & x^2 + (y - 1)^2 &= 1 & \bullet \text{Complete the square.} \\ x^2 + (y - 1)^2 &= 1 & & & \end{aligned}$$

A rectangular form of  $r = 2 \sin \theta$  is  $x^2 + (y - 1)^2 = 1^2$ . The graph of each of these equations is a circle with center  $(0, 1)$  and radius 1.

58.

$$\begin{aligned} r &= \cot \theta & r \sin \theta &= \cos \theta & \bullet \cot \theta = \frac{\cos \theta}{\sin \theta} \\ r &= \frac{\cos \theta}{\sin \theta} & r(r \sin \theta) &= r \cos \theta & \bullet \text{Multiply each side by } r. \\ & & (\sqrt{x^2 + y^2})y &= x & \bullet y = r \sin \theta; x = r \cos \theta \\ & & (x^2 + y^2)y^2 &= x^2 & \bullet \text{Square each side.} \\ & & y^4 + x^2y^2 - x^2 &= 0 & \end{aligned}$$

70.

$$\begin{aligned} 2x - 3y &= 6 & 2r \cos \theta - 3r \sin \theta &= 6 & \bullet x = r \cos \theta; y = r \sin \theta \\ 2r \cos \theta - 3r \sin \theta &= 6 & r(2 \cos \theta - 3 \sin \theta) &= 6 & \\ r(2 \cos \theta - 3 \sin \theta) &= 6 & & & \\ r &= \frac{6}{2 \cos \theta - 3 \sin \theta} & & & \end{aligned}$$

## Exercise Set 6.6, page 431

$$\begin{aligned} 2. \quad r &= \frac{8}{2 - 4 \cos \theta} \\ r &= \frac{4}{1 - 2 \cos \theta} & \bullet \text{Divide numerator and denominator by 2.} \end{aligned}$$

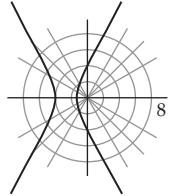
$e = 2$ , so the graph is a hyperbola. The transverse axis is on the polar axis because the equation involves  $\cos \theta$ . Let  $\theta = 0$ .

$$r = \frac{8}{2 - 4 \cos 0} = \frac{8}{2 - 4} = -4$$

Let  $\theta = \pi$ .

$$r = \frac{8}{2 - 4 \cos \pi} = \frac{8}{2 + 4} = \frac{4}{3}$$

The vertices are  $(-4, 0)$  and  $\left(\frac{4}{3}, \pi\right)$ .



4.  $r = \frac{6}{3 + 2 \cos \theta}$

$$r = \frac{2}{1 + \left(\frac{2}{3}\right) \cos \theta}$$

• Divide numerator and denominator by 3.

$e = \frac{2}{3}$ , so the graph is an ellipse. The major axis is on the polar axis because the equation involves  $\cos \theta$ . Let  $\theta = 0$ .

$$r = \frac{6}{3 + 2 \cos 0} = \frac{6}{3 + 2} = \frac{6}{5}$$

Let  $\theta = \pi$ .

$$r = \frac{6}{3 + 2 \cos \pi} = \frac{6}{3 - 2} = 6$$

The vertices on the major axis are  $\left(\frac{6}{5}, 0\right)$  and  $(6, \pi)$ . Let

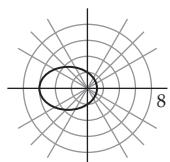
$$\theta = \frac{\pi}{2}.$$

$$r = \frac{6}{3 + 2 \cos\left(\frac{\pi}{2}\right)} = \frac{6}{3 + 0} = 2$$

$$\text{Let } \theta = \frac{3\pi}{2}.$$

$$r = \frac{6}{3 + 2 \cos\left(\frac{3\pi}{2}\right)} = \frac{6}{3 + 0} = 2$$

Two additional points on the ellipse are  $\left(2, \frac{\pi}{2}\right)$  and  $\left(2, \frac{3\pi}{2}\right)$ .



24.  $e = 1, r \cos \theta = -2$

Thus the directrix is  $x = -2$ . The distance from the focus (pole) to the directrix is  $d = 2$ .

$$r = \frac{ed}{1 - e \cos \theta}$$

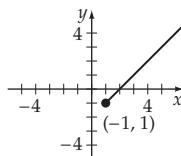
• Use Eq (2) because the vertical directrix is to the left of the pole.

$$r = \frac{(1)(2)}{1 - (1) \cos \theta} \quad \bullet e = 1, d = 2$$

$$r = \frac{2}{1 - \cos \theta}$$

### Exercise Set 6.7, page 440

6. Plotting points for several values of  $t$  yields the following graph.



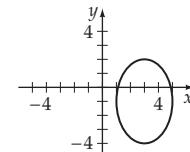
12.  $x = 3 + 2 \cos t, y = -1 - 3 \sin t, 0 \leq t < 2\pi$

$$\cos t = \frac{x - 3}{2}, \sin t = -\frac{y + 1}{3}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x - 3}{2}\right)^2 + \left(-\frac{y + 1}{3}\right)^2 = 1$$

$$\frac{(x - 3)^2}{4} + \frac{(y + 1)^2}{9} = 1$$



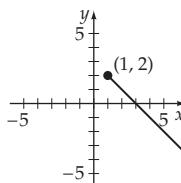
14.  $x = 1 + t^2, y = 2 - t^2, t \in R$

$$x = 1 + t^2$$

$$t^2 = x - 1$$

$$y = 2 - (x - 1)$$

$$y = -x + 3$$



Because  $x = 1 + t^2$  and  $t^2 \geq 0$  for all real numbers  $t$ ,  $x \geq 1$  for all  $t$ . Similarly,  $y \leq 2$  for all  $t$ .

24. Eliminate the parameter to determine an equation in  $x$  and  $y$  that is equivalent to the two parametric equations  $x = t + 1, y = \sqrt{t}$ . First, solve  $x = t + 1$  for  $t$ .

$$x = t + 1$$

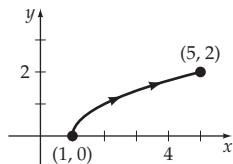
$$t = x - 1$$

Substitute  $x - 1$  for  $t$  in the equation  $y = \sqrt{t}$ .

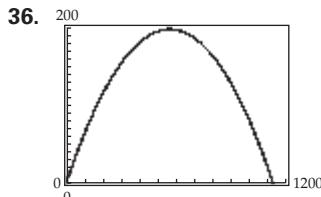
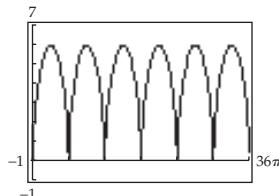
$$y = \sqrt{x - 1}$$

Squaring both sides produces  $y^2 = x - 1$ , with  $y \geq 0$ .

(Note:  $y$  is nonnegative because of the information provided by the parametric equation  $y = \sqrt{t}$ .) Thus the graph of the parametric equations  $x = t + 1$ ,  $y = \sqrt{t}$ , with  $t \geq 0$ , is the upper half of a parabola. When  $t = 0$ ,  $P$  is at  $(1, 0)$ ; when  $t = 4$ ,  $P$  is at  $(5, 2)$ . Thus  $P$  starts at  $(1, 0)$  and moves along the upper portion of the parabola  $y^2 = x - 1$  until it reaches  $(5, 2)$  at time  $t = 4$ .



32. The maximum height of the cycloid is  $2a = 2(3) = 6$ . The cycloid intersects the  $x$ -axis every  $2\pi a = 2\pi(3) = 6\pi$  units.



The maximum height is about 195 feet when  $t \approx 3.50$  seconds. The range is 1117 feet when  $t \approx 6.99$  seconds.

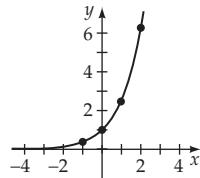
### Exercise Set 7.1, page 461

2.  $f(3) = 5^3 = 5 \cdot 5 \cdot 5 = 125$

$$f(-2) = 5^{-2} = \frac{1}{5^2} = \frac{1}{5 \cdot 5} = \frac{1}{25}$$

22. The graph of  $f(x) = \left(\frac{5}{2}\right)^x$  has a  $y$ -intercept of  $(0, 1)$  and the graph passes through  $\left(1, \frac{5}{2}\right)$ . Plot a few additional points, such as  $\left(-1, \frac{2}{5}\right)$  and  $\left(2, \frac{25}{4}\right)$ . Because the base  $\frac{5}{2}$  is greater than 1, we know that the graph must have all the properties of an increasing exponential function. Draw a smooth increasing curve through

the points. The graph should be asymptotic to the negative portion of the  $x$ -axis, as shown in the following figure.



28. Because  $F(x) = 6^{x+5} = f(x + 5)$ , the graph of  $F(x)$  can be produced by shifting the graph of  $f$  horizontally to the left 5 units.

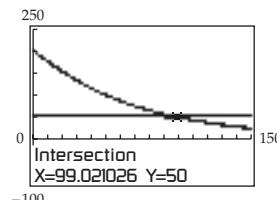
30. Because  $F(x) = -\left[\left(\frac{5}{2}\right)^x\right] = -f(x)$ , the graph of  $F(x)$  can be produced by reflecting the graph of  $f$  across the  $x$ -axis.

48. a.  $A(45) = 200e^{-0.014(45)}$   
 $\approx 106.52$

After 45 minutes the patient will have about 107 milligrams of medication in his or her bloodstream.

- b. Use a graphing calculator to graph  $y = 200e^{-0.014x}$  and  $y = 50$  in the same viewing window, as shown below.

|                                     |       |       |
|-------------------------------------|-------|-------|
| Plot1                               | Plot2 | Plot3 |
| Y1 = $200 \cdot e^{-0.014 \cdot X}$ |       |       |
| Y2 = 50                             |       |       |
| Y3 =                                |       |       |
| Y4 =                                |       |       |
| Y5 =                                |       |       |
| Y6 =                                |       |       |
| Y7 =                                |       |       |



The  $x$ -coordinate (which represents time in minutes) of the point of intersection is about 99.02. Thus it will take about 99 minutes before the patient's medication level is reduced to 50 milligrams.

54. a.  $P(0) = \frac{3600}{1 + 7e^{-0.05(0)}}$   
 $= \frac{3600}{1 + 7}$   
 $= \frac{3600}{8}$   
 $= 450$

Immediately after the lake was stocked, the lake contained 450 bass.

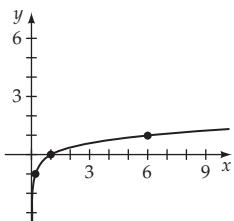
b.  $P(12) = \frac{3600}{1 + 7e^{-0.05(12)}}$   
 $\approx 743.54$

After 1 year (12 months) there were about 744 bass in the lake.

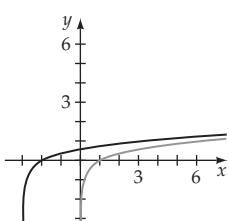
- c. As  $t \rightarrow \infty$ ,  $7e^{-0.05t} = \frac{7}{e^{0.05t}}$  approaches 0. Thus, as  $t \rightarrow \infty$ ,  $P(t) = \frac{3600}{1 + 7e^{-0.05t}}$  will approach  $\frac{3600}{1 + 0} = 3600$ . As time goes by the bass population will increase, approaching 3600.

### Exercise Set 7.2, page 476

4. The exponential form of  $y = \log_b x$  is  $b^y = x$ . Thus the exponential form of  $3 = \log_4 64$  is  $4^3 = 64$ .
14. The logarithmic form of  $b^y = x$  is  $y = \log_b x$ . Thus the logarithmic form of  $5^3 = 125$  is  $3 = \log_5 125$ .
32.  $\log 1,000,000 = \log_{10} 10^6 = 6$
44. To graph  $y = \log_6 x$ , use the equivalent exponential equation  $x = 6^y$ . Choose some  $y$  values, such as  $-1, 0, 1$ , and calculate the corresponding  $x$  values. This yields the ordered pairs  $(\frac{1}{6}, -1), (1, 0)$ , and  $(6, 1)$ . Plot these ordered pairs and draw a smooth curve through the points to produce the following graph.



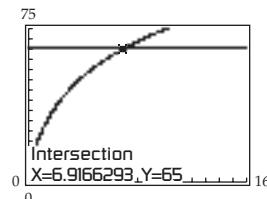
52.  $\log_4(5 - x)$  is defined only for  $5 - x > 0$ , which is equivalent to  $x < 5$ . Using interval notation, the domain of  $k(x) = \log_4(5 - x)$  is  $(-\infty, 5)$ .
66. The graph of  $f(x) = \log_6(x + 3)$  can be produced by shifting the graph of  $f(x) = \log_6 x$  (from Exercise 44) 3 units to the left. See the following figure.



86. a.  $S(0) = 5 + 29 \ln(0 + 1) = 5 + 0 = 5$ . When starting, the student had an average typing speed of 5 words per minute.  $S(3) = 5 + 29 \ln(3 + 1) \approx 45.2$ . After

3 months the student's average typing speed was about 45 words per minute.

- b. Use the intersection feature of a graphing utility to find the  $x$ -coordinate of the point of intersection of the graphs of  $y = 5 + 29 \ln(x + 1)$  and  $y = 65$ .



The graphs intersect at about  $(6.9, 65)$ . The student will achieve a typing speed of 65 words per minute in about 6.9 months.

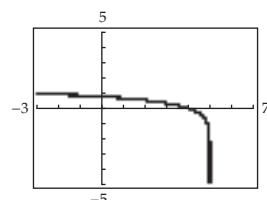
### Exercise Set 7.3, page 489

$$\begin{aligned} 2. \quad \ln \frac{z^3}{\sqrt{xy}} &= \ln z^3 - \ln \sqrt{xy} \\ &= \ln z^3 - \ln(xy)^{1/2} \\ &= 3 \ln z - \frac{1}{2} \ln(xy) \\ &= 3 \ln z - \frac{1}{2} (\ln x + \ln y) \\ &= 3 \ln z - \frac{1}{2} \ln x - \frac{1}{2} \ln y \end{aligned}$$

$$\begin{aligned} 18. \quad 3 \log_2 t - \frac{1}{3} \log_2 u + 4 \log_2 v &= \log_2 t^3 - \log_2 u^{1/3} + \log_2 v^4 \\ &= \log_2 \frac{t^3}{u^{1/3}} + \log_2 v^4 \\ &= \log_2 \frac{t^3 v^4}{u^{1/3}} \\ &= \log_2 \frac{t^3 v^4}{\sqrt[3]{u}} \end{aligned}$$

34.  $\log_5 37 = \frac{\log 37}{\log 5} \approx 2.2436$

46.  $\log_8(5 - x) = \frac{\ln(5 - x)}{\ln 8}$ , so enter  $\frac{\ln(5 - x)}{\ln 8}$  into Y1 on a graphing calculator.



70.  $\text{pH} = -\log[\text{H}^+] = -\log(1.26 \times 10^{-3}) \approx 2.9$

72.  $\text{pH} = -\log[\text{H}^+]$

$$5.6 = -\log[\text{H}^+]$$

$$-5.6 = \log[\text{H}^+]$$

$$10^{-5.6} = \text{H}^+$$

The hydronium-ion concentration is  $10^{-5.6} \approx 2.51 \times 10^{-6}$  mole per liter.

78.  $M = \log\left(\frac{I}{I_0}\right) = \log\left(\frac{398,107,000I_0}{I_0}\right) = \log 398,107,000$   
 $\approx 8.6$

80.  $\log\left(\frac{I}{I_0}\right) = 9.5$

$$\frac{I}{I_0} = 10^{9.5}$$

$$I = 10^{9.5}I_0$$

$$I \approx 3,162,277,660I_0$$

82. In Example 7 we noticed that if an earthquake has a Richter scale magnitude of  $M_1$  and a smaller earthquake has a Richter scale magnitude of  $M_2$ , then the larger earthquake is  $10^{M_1-M_2}$  times as intense as the smaller earthquake. In this exercise,  $M_1 = 9.5$  and  $M_2 = 8.3$ . Thus  $10^{M_1-M_2} = 10^{9.5-8.3} = 10^{1.2} \approx 15.8$ . The 1960 earthquake in Chile was about 15.8 times as intense as the San Francisco earthquake of 1906.

86.  $M = \log A + 3 \log 8t - 2.92$

$$= \log 26 + 3 \log[8 \cdot 17] - 2.92 \\ \approx 1.4150 + 6.4006 - 2.92$$

$$\approx 4.9$$

• Substitute 26 for A and 17 for t.

### Exercise Set 7.4, page 501

2.  $3^x = 243$

$$3^x = 3^5$$

$$x = 5$$

10.  $6^x = 50$

$$\log(6^x) = \log 50$$

$$x \log 6 = \log 50$$

$$x = \frac{\log 50}{\log 6} \approx 2.18$$

18.

$$3^{x-2} = 4^{2x+1}$$

$$\log 3^{x-2} = \log 4^{2x+1}$$

$$(x-2)\log 3 = (2x+1)\log 4$$

$$x \log 3 - 2 \log 3 = 2x \log 4 + \log 4$$

$$x \log 3 - 2 \log 3 - 2x \log 4 = \log 4$$

$$x \log 3 - 2x \log 4 = \log 4 + 2 \log 3$$

$$x(\log 3 - 2 \log 4) = \log 4 + 2 \log 3$$

$$x = \frac{\log 4 + 2 \log 3}{\log 3 - 2 \log 4}$$

$$x \approx -2.141$$

22.  $\log(x^2 + 19) = 2$

$$x^2 + 19 = 10^2$$

$$x^2 + 19 = 100$$

$$x^2 = 81$$

$$x = \pm 9$$

A check shows that 9 and -9 are both solutions of the original equation.

26.  $\log_3 x + \log_3(x+6) = 3$

$$\log_3[x(x+6)] = 3$$

$$3^3 = x(x+6)$$

$$27 = x^2 + 6x$$

$$x^2 + 6x - 27 = 0$$

$$(x+9)(x-3) = 0$$

$$x = -9 \quad \text{or} \quad x = 3$$

Because  $\log_3 x$  is defined only for  $x > 0$ , the only solution is  $x = 3$ .

36.  $\ln x = \frac{1}{2} \ln\left(2x + \frac{5}{2}\right) + \frac{1}{2} \ln 2$

$$\ln x = \frac{1}{2} \left[ \ln\left(2x + \frac{5}{2}\right) + \ln 2 \right]$$

$$\ln x = \frac{1}{2} \ln\left[2\left(2x + \frac{5}{2}\right)\right]$$

$$\ln x = \frac{1}{2} \ln(4x + 5)$$

$$\ln x = \ln(4x + 5)^{1/2}$$

$$x = \sqrt{4x + 5}$$

$$x^2 = 4x + 5$$

$$0 = x^2 - 4x - 5$$

$$0 = (x-5)(x+1)$$

$$x = 5 \quad \text{or} \quad x = -1$$

Check:  $\ln 5 = \frac{1}{2} \ln\left(10 + \frac{5}{2}\right) + \frac{1}{2} \ln 2$

$$1.6094 \approx 1.2629 + 0.3466$$

Because  $\ln(-1)$  is not defined, -1 is not a solution. Thus the only solution is  $x = 5$ .

42.  $\frac{10^x + 10^{-x}}{2} = 8$

$$10^x + 10^{-x} = 16$$

$$10^x(10^x + 10^{-x}) = (16)10^x$$

$$10^{2x} + 1 = 16(10^x)$$

$$10^{2x} - 16(10^x) + 1 = 0$$

$$u^2 - 16u + 1 = 0$$

• Multiply each side by  $10^x$ .

• Let  $u = 10^x$ .

$$u = \frac{16 \pm \sqrt{16^2 - 4(1)(1)}}{2} = 8 \pm 3\sqrt{7}$$

$$10^x = 8 \pm 3\sqrt{7} \quad \text{• Replace } u \text{ with } 10^x.$$

$$\log 10^x = \log(8 \pm 3\sqrt{7})$$

$$x = \log(8 \pm 3\sqrt{7}) \approx \pm 1.20241$$

74. a.  $t = \frac{9}{24} \ln \frac{24+v}{24-v}$

$$1.5 = \frac{9}{24} \ln \frac{24+v}{24-v}$$

$$4 = \ln \frac{24+v}{24-v}$$

$$e^4 = \frac{24+v}{24-v}$$

•  $N = \ln M$  means  $e^N = M$ .

$$(24-v)e^4 = 24+v$$

$$-v - ve^4 = 24 - 24e^4$$

$$v(-1 - e^4) = 24 - 24e^4$$

$$v = \frac{24 - 24e^4}{-1 - e^4} \approx 23.14$$

The velocity is about 23.14 feet per second.

b. The vertical asymptote is  $v = 24$ .

c. Due to the air resistance, the object can never reach or exceed a velocity of 24 feet per second.

### Exercise Set 7.5, page 516

6. Let  $t = 0$  represent the year 2000 and let  $t = 4$  represent the year 2004. Start by substituting 4 for  $t$  in  $N(t) = N_0 e^{kt}$ .

$$N(4) = N_0 e^{k \cdot 4}$$

$$291,800 = 276,400 e^{4k}$$

• Substitute.

$$\frac{291,800}{276,400} = e^{4k}$$

• Solve for  $k$ .

$$\ln\left(\frac{291,800}{276,400}\right) = 4k$$

$$\frac{1}{4} \ln\left(\frac{291,800}{276,400}\right) = k$$

$$0.013555 \approx k$$

The growth function is  $N(t) \approx 276,400 e^{0.013555t}$ . The year 2008 is represented by  $t = 8$ .

$$N(8) \approx 276,400 e^{0.013555 \cdot 8}$$

$$\approx 308,100$$

The exponential growth function yields 308,100 as the approximate population of Aurora in 2008.

8.  $N(t) = N_0 e^{kt}$

$$N(138) = N_0 e^{138k}$$

$$0.5N_0 = N_0 e^{138k}$$

$$0.5 = e^{138k}$$

$$\ln 0.5 = 138k$$

$$k = \frac{\ln 0.5}{138} \approx -0.005023$$

$$N(t) = N_0(0.5)^{t/138} \approx N_0 e^{-0.005023t}$$

12.  $N(t) = N_0(0.5)^{t/5730}$

$$0.65N_0 = N_0(0.5)^{t/5730}$$

$$0.65 = (0.5)^{t/5730}$$

$$\ln 0.65 = \ln(0.5)^{t/5730}$$

$$t = 5730 \frac{\ln 0.65}{\ln 0.5} \approx 3600$$

The bone is approximately 3600 years old.

16. a.  $P = 12,500, r = 0.08, t = 10, n = 1$

$$A = 12,500 \left(1 + \frac{0.08}{1}\right)^{10} \approx \$26,986.56$$

b.  $n = 365$

$$A = 12,500 \left(1 + \frac{0.08}{365}\right)^{3650} \approx \$27,816.82$$

c.  $n = 8760$

$$A = 12,500 \left(1 + \frac{0.08}{8760}\right)^{87600} \approx \$27,819.16$$

18.  $P = 32,000, r = 0.08, t = 3$

$$A = Pe^{rt} = 32,000e^{3(0.08)} \approx \$40,679.97$$

22.  $t = \frac{\ln 3}{r}, r = 0.055$

$$t = \frac{\ln 3}{0.055}$$

$t \approx 20$  years (to the nearest year)

38. a. Represent the year 2004 by  $t = 0$ ; then the year 2005 will be represented by  $t = 1$ . Use the following substitutions:  $P_0 = 240$ ,  $P(1) = 310$ ,  $c = 3400$ , and

$$a = \frac{c - P_0}{P_0} = \frac{3400 - 240}{240} \approx 13.16667.$$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

$$P(1) = \frac{3400}{1 + 13.16667e^{-b(1)}}$$

$$310 = \frac{3400}{1 + 13.16667e^{-b}}$$

$$310(1 + 13.16667e^{-b}) = 3400$$

$$1 + 13.16667e^{-b} = \frac{3400}{310}$$

$$13.16667e^{-b} = \frac{3400}{310} - 1$$

$$13.16667e^{-b} \approx 9.96774$$

$$e^{-b} \approx \frac{9.96774}{13.16667}$$

$$-b \approx \ln \frac{9.96774}{13.16667}$$

$$b \approx 0.27833$$

Using  $a = 13.16667$ ,  $b = 0.27833$ , and  $c = 3400$  gives the following logistic model.

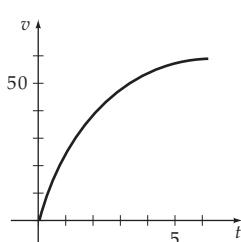
$$P(t) \approx \frac{3400}{1 + 13.16667e^{-0.27833t}}$$

- b.** Because 2011 is 7 years past 2004, the year 2011 is represented by  $t = 7$ .

$$P(7) \approx \frac{3400}{1 + 13.16667e^{-0.27833(7)}} \approx 1182$$

According to the model there will be about 1182 groundhogs in 2011.

**48. a.**



- b.** Here is an algebraic solution. An approximate solution can be obtained from the graph.

$$v = 64(1 - e^{-t/2})$$

$$50 = 64(1 - e^{-t/2})$$

$$\frac{50}{64} = (1 - e^{-t/2})$$

$$1 - \frac{50}{64} = e^{-t/2}$$

$$\ln\left(1 - \frac{50}{64}\right) = -\frac{t}{2}$$

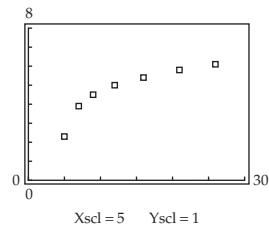
$$t = -2 \ln\left(1 - \frac{50}{64}\right) \approx 3.0$$

The velocity is 50 feet per second in approximately 3.0 seconds.

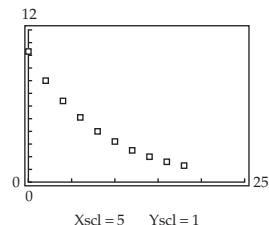
- c.** As  $t \rightarrow \infty$ ,  $e^{-t/2} \rightarrow 0$ . Therefore,  $64(1 - e^{-t/2}) \rightarrow 64$ . The horizontal asymptote is  $v = 64$ .
- d.** Because of air resistance, the velocity of the object will approach, but never reach or exceed, 64 feet per second.

### Exercise Set 7.6, page 529

- 4.** The following scatter plot suggests that the data can be modeled by an increasing function that is concave down. Thus the most suitable model for the data is an increasing logarithmic function.

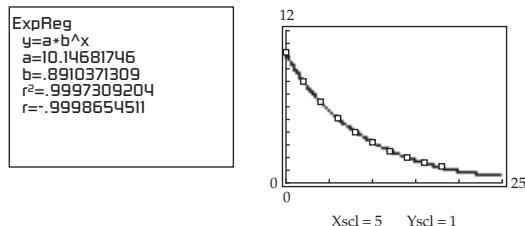


- 22.** From the scatter plot in the following figure, it appears that the data can be closely modeled by a decreasing exponential function of the form  $y = ab^x$ , with  $b < 1$ .



The calculator display in the following figure shows that the exponential regression equation is  $y \approx 10.1468(0.89104)^x$ , where  $x$  is the altitude in kilometers and  $y$  is the pressure in newtons per square centimeter.

```
ExpReg
y=a*b^x
a=10.14681746
b=.8910371309
r=.9997309204
r=.9998654511
```



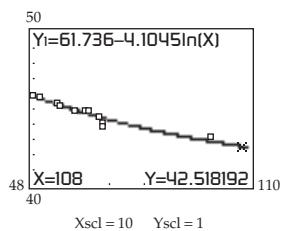
The correlation coefficient  $r \approx -0.99987$  is close to  $-1$ . This indicates that the function  $y \approx 10.1468(0.89104)^x$  provides a good fit for the data. The graph of  $y$  also

indicates that the regression function provides a good model for the data. When  $x = 24$  kilometers, the atmospheric pressure is about  $10.1468(0.89104)^{24} \approx 0.6$  newton per square centimeter.

- 24. a.** Use a graphing utility to perform an exponential regression and a logarithmic regression. For the given data, the logarithmic function  $y = 61.735786 - 4.1044761 \ln x$  provides a slightly better fit than does the exponential regression function, as determined by comparing the correlation coefficients. See the calculator displays below.

```
ExpReg
y=a+b^x
a=48.55569695
b=.9987196271
r^2=.8923900275
r=-.944663976
```

```
LnReg
y=a+blnx
a=61.73578555
b=-4.104476112
r^2=.9302117833
r=-.9644748744
```



- b.** To predict the world record time in 2008, evaluate  $y = 61.735786 - 4.104476 \ln x$  at  $x = 108$ . The graph on the right above shows that the predicted world record time in the men's 400-meter race for the year 2008 is about 42.52 seconds.

- 26. a.** Use a graphing utility to perform a logistic regression on the data. The following figure shows the results obtained by using a TI-83/TI-83 Plus/TI-84 Plus graphing calculator.

|   |
|---|
| <b>Logistic</b><br>$y=c/(1+ae^{-bx})$<br>$a=2.233147667$<br>$b=.043679087$<br>$c=1534426.531$ |
|---|

The logistic regression function for the data is

$$P(t) \approx \frac{1,534,427}{1 + 2.233148 e^{-0.043679t}}.$$

- b.** The year 2010 is represented by  $t = 60$ .
- $$P(60) = \frac{1,534,427}{1 + 2.233148 e^{-0.043679(60)}} \approx 1,320,000$$

The logistic regression function predicts that Hawaii's population will be about 1,320,000 in 2010.

- c.** The carrying capacity, to the nearest thousand, of the logistic model is 1,534,000 people.

# Answers to Selected Exercises

## Exercise Set 1.1, page 12

1. 15   3.  $-4$    5.  $\frac{9}{2}$    7.  $\frac{2}{9}$    9. 12   11. 16   13. 75   15.  $\frac{1}{2}$    17. 1200   19.  $y = -\frac{x}{2} + 4$    21.  $x = -\frac{5}{2}y + 5$    23.  $y = \frac{c}{a-b}$   
 25.  $y = \frac{x}{1+x}$    27.  $-3, 5$    29.  $\frac{-1 \pm \sqrt{5}}{2}$    31.  $\frac{-2 \pm \sqrt{2}}{2}$    33.  $\frac{5 \pm \sqrt{61}}{6}$    35.  $\frac{-3 \pm \sqrt{41}}{4}$    37.  $-\frac{\sqrt{2}}{2}, -\sqrt{2}$    39.  $\frac{3 \pm \sqrt{29}}{2}$   
 41.  $-3, 5$    43.  $\frac{3}{8}, -24$    45.  $0, \frac{7}{3}$    47.  $2, 8$    49.  $(-\infty, 4)$    51.  $(-\infty, -6)$    53.  $(-\infty, -3]$    55.  $\left[-\frac{13}{8}, \infty\right)$    57.  $(-\infty, 2)$   
 59.  $(-\infty, -7) \cup (0, \infty)$    61.  $(-5, -2)$    63.  $(-\infty, -4] \cup [7, \infty)$    65.  $\left[-\frac{1}{2}, \frac{4}{3}\right]$    67.  $(-4, 4)$    69.  $(-8, 10)$    71.  $(-\infty, -33) \cup (27, \infty)$   
 73.  $\left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$    75.  $(-\infty, -8] \cup [2, \infty)$    77.  $\left[-\frac{4}{3}, 8\right]$    79.  $(-\infty, -4] \cup \left[\frac{28}{5}, \infty\right)$    81.  $(-\infty, \infty)$    83. 4  
 85. 3.5 centimeters by 10 centimeters   87. 100 feet by 150 feet   89. at least 58 checks   91. at least 34 sales   93.  $20^\circ \leq C \leq 40^\circ$   
 95. 22   97.  $(0, 210)$    99. 1 second  $< t < 3$  seconds   101. a.  $|s - 4.25| \leq 0.01$    b.  $4.24 \leq s \leq 4.26$

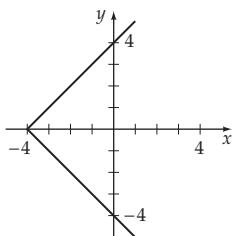
## Prepare for This Section (1.2), page 15

- PS1.  $-\frac{3}{2}$    PS2.  $5\sqrt{2}$    PS3. No   PS4. 16   PS5. 2   PS6. 5

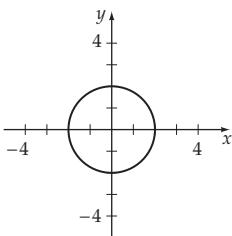
## Exercise Set 1.2, page 27

1.   
 3. a. \$31,500   b. \$34,100   c. \$34,111   5.  $7\sqrt{5}$    7.  $\sqrt{1261}$    9.  $\sqrt{89}$    11.  $\sqrt{38 - 12\sqrt{6}}$    13.  $2\sqrt{a^2 + b^2}$   
 15.  $-x\sqrt{10}$    17.  $(12, 0), (-4, 0)$    19.  $(3, 2)$    21.  $(6, 4)$    23.  $(-0.875, 3.91)$    25.   
 27.   
 29.   
 31.   
 33.   
 35.   
 37.   
 39.  $(6, 0), \left(0, \frac{12}{5}\right)$    41.  $(5, 0); (0, \sqrt{5}), (0, -\sqrt{5})$    43.

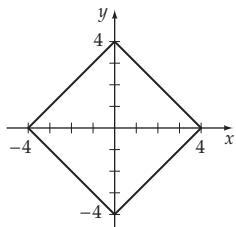
43.  $(-4, 0); (0, 4), (0, -4)$



45.  $(\pm 2, 0), (0, \pm 2)$



47.  $(\pm 4, 0), (0, \pm 4)$



73.  $(x + 1)^2 + (y - 7)^2 = 25$

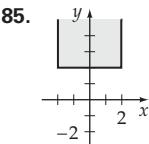
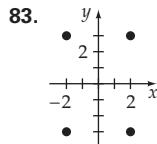
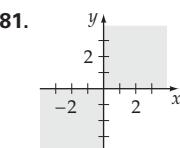
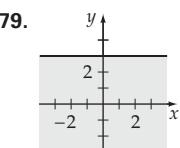
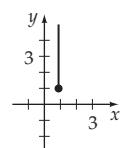
75.  $(x - 7)^2 + (y - 11)^2 = 121$

49. center  $(0, 0)$ , radius 655. center  $(8, 0)$ , radius  $\frac{1}{2}$ 59.  $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{4}\right)^2 = (\sqrt{5})^2$ 63.  $(x - 1)^2 + (y - 3)^2 = 5^2$ 69. center  $\left(-\frac{1}{2}, 0\right)$ , radius 451. center  $(1, 3)$ , radius 757.  $(x - 4)^2 + (y - 1)^2 = 2^2$ 61.  $(x - 0)^2 + (y - 0)^2 = 5^2$ 65. center  $(3, 0)$ , radius 271. center  $\left(\frac{1}{2}, -\frac{3}{2}\right)$ , radius  $\frac{5}{2}$ 53. center  $(-2, -5)$ , radius 557.  $(x - 4)^2 + (y - 1)^2 = 2^2$ 61.  $(x - 0)^2 + (y - 0)^2 = 5^2$ 65. center  $(7, -4)$ , radius 371. center  $\left(\frac{1}{2}, -\frac{3}{2}\right)$ , radius  $\frac{5}{2}$ 

73.  $(x + 1)^2 + (y - 7)^2 = 25$

75.  $(x - 7)^2 + (y - 11)^2 = 121$

77.



87.  $(13, 5)$

89.  $(7, -6)$

91.  $x^2 - 6x + y^2 - 8y = 0$

93.  $9x^2 + 25y^2 = 225$

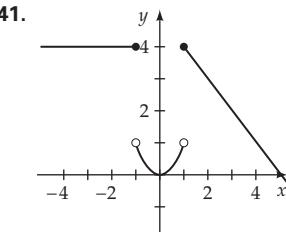
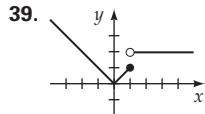
95.  $(x + 3)^2 + (y - 3)^2 = 3^2$

## Prepare for This Section (1.3), page 30

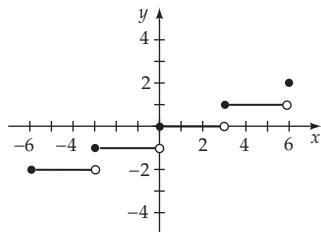
PS1. -4 PS2. 1 and 3 PS3. -3 and 3 PS4.  $\sqrt{58}$  PS5. -4 PS6. decrease

## Exercise Set 1.3, page 46

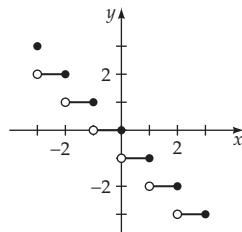
1. a. 5 b. -4 c. -1 d. 1 e.  $3k - 1$  f.  $3k + 5$  3. a.  $\sqrt{5}$  b. 3 c. 3 d.  $\sqrt{21}$  e.  $\sqrt{r^2 + 2r + 6}$  f.  $\sqrt{c^2 + 5}$   
 5. a.  $\frac{1}{2}$  b.  $\frac{1}{2}$  c.  $\frac{5}{3}$  d. 1 e.  $\frac{1}{c^2 + 4}$  f.  $\frac{1}{|2 + h|}$  7. a. 1 b. 1 c. -1 d. -1 e. 1 f. -1 9. a. -11 b. 6  
 c.  $3c + 1$  d.  $-k^2 - 2k + 10$  11. Yes 13. No 15. No 17. Yes 19. No 21. Yes 23. Yes 25. Yes 27. all real numbers  
 29. all real numbers 31.  $\{x|x \neq -2\}$  33.  $\{x|x \geq -7\}$  35.  $\{x|-2 \leq x \leq 2\}$  37.  $\{x|x > -4\}$



43.

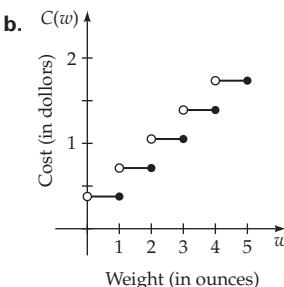


45.



47.  $\frac{\text{int}[10^2(2.3458) + 0.5]}{10^2} = 2.35$  49.  $\frac{\text{int}[10^3(34.05622) + 0.5]}{10^3} = 34.056$  51.  $\frac{\text{int}[10^4(0.08951) + 0.5]}{10^4} = 0.0895$

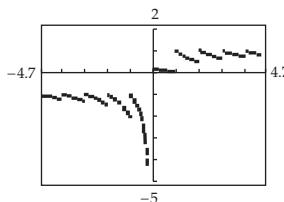
53. a. \$1.07

b.  $C(w)$ 

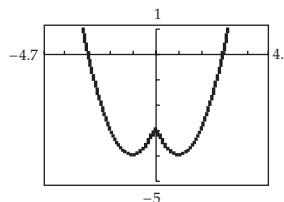
55. a, b, and d.

57. decreasing on  $(-\infty, 0]$ ; increasing on  $[0, \infty)$ 59. increasing on  $(-\infty, \infty)$  61. decreasing on  $(-\infty, -3]$ ; increasing on  $[-3, 0]$ ; decreasing on  $[0, 3]$ ; increasing on  $[3, \infty)$  63. constant on  $(-\infty, 0]$ ; increasing on  $[0, \infty)$ 65. decreasing on  $(-\infty, 0]$ ; constant on  $[0, 1]$ ; increasing on  $[1, \infty)$  67. g and F69. a.  $w = 25 - l$  b.  $A = 25l - l^2$  71.  $v(t) = 80,000 - 6500t$ ,  $0 \leq t \leq 10$ 73. a.  $C(x) = 2000 + 22.80x$  b.  $R(x) = 37.00x$  c.  $P(x) = 14.20x - 2000$ 75.  $h = 15 - 5r$  77.  $d = \sqrt{9t^2 + 2500}$  79.  $d = \sqrt{(45 - 8t)^2 + (6t)^2}$ 81. a.  $A(x) = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - \frac{5}{2}x + 25$  b. 25, 17.27, 14.09, 15.46, 21.37, 31.83c.  $[0, 20]$  83. a.  $L(x) = \sqrt{900 + x^2} + \sqrt{400 + (40 - x)^2}$ b. 74.72, 67.68, 64.34, 64.79, 70 c.  $[0, 40]$  85. 275, 375, 385, 390, 39487.  $c = -2$  or  $c = 3$ 89. 1 is not in the range of  $f$ .

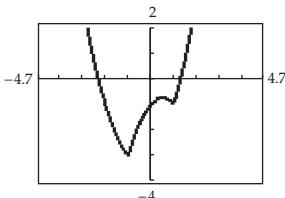
91.



93.



95.



97. 4

99. 2

101.

a. 36

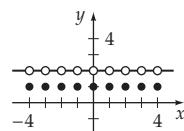
b. 13

c. 12

d. 30 e.  $13k - 2$ f.  $8k - 11$ 103.  $4\sqrt{21}$ 

105. 1, -3

107.



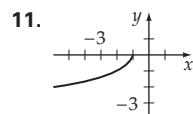
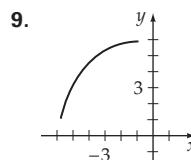
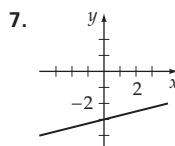
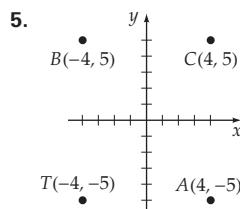
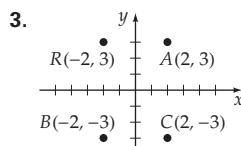
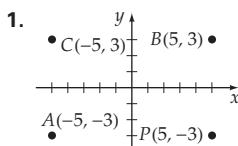
## Prepare for This Section (1.4), page 53

PS1. The graph of  $g$  is one unit above the graph of  $f$ .

PS4. 3, -1, -3, -3, -1

PS5.  $(0, b)$ PS6.  $(0, 0)$ 

## Exercise Set 1.4, page 64



13. a. No

b. Yes

15. a. No

b. No

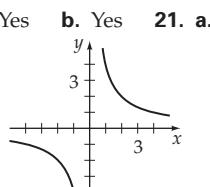
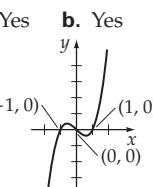
17. a. Yes

b. Yes

19. a. Yes

b. Yes

35.

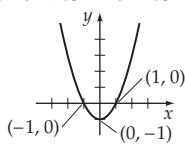
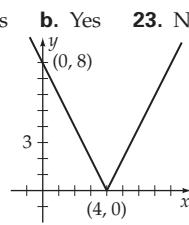


21. a. Yes

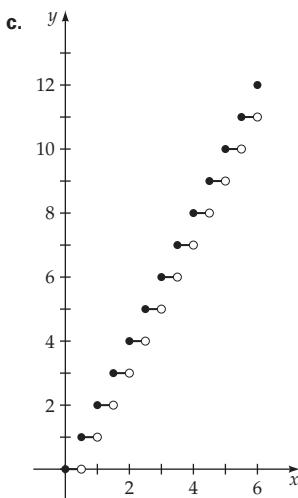
b. Yes

23. No

25. Yes



- 39.**
- 41.**
- 43. even** **45. odd** **47. even** **49. even** **51. even** **53. even** **55. neither**
- 57. a., b.** intercepts:  $(a, 0)$ ,  $a \geq 0$
- 59. a.**
- b.**
- 61. a.**  $(-5, 5), (-3, -2), (-2, 0)$  **b.**  $(-2, 6), (0, -1), (1, 1)$  **63. a.**
- b.**
- 65. a.**  $(1, 3), (-2, -4)$  **b.**  $(-1, -3), (2, 4)$  **67. a., b.**
- 69.**
- 71. a.**
- b.**
- 73. a.**
- b.**
- 75.**
- 77.**
- 79.**
- 81.**
- 83. a.**
- b.**



**85. a.**  $f(x) = \frac{2}{(x+1)^2 + 1} + 1$     **b.**  $f(x) = -\frac{2}{(x-2)^2 + 1}$

### Prepare for This Section (1.5), page 69

**PS1.** 2    **PS2.**  $-90$     **PS3.**  $18a^2 - 15a + 2$     **PS4.**  $2h^2 + 3h$     **PS5.** all real numbers except  $x = 1$     **PS6.**  $[4, \infty)$

### Exercise Set 1.5, page 77

- 1.**  $f(x) + g(x) = x^2 - x - 12$ , Domain is the set of all real numbers.  
 $f(x) - g(x) = x^2 - 3x - 18$ , Domain is the set of all real numbers.  
 $f(x) \cdot g(x) = x^3 + x^2 - 21x - 45$ , Domain is the set of all real numbers.  
 $\frac{f(x)}{g(x)} = x - 5$ , Domain  $\{x | x \neq -3\}$

- 5.**  $f(x) + g(x) = x^3 - 2x^2 + 8x$ , Domain is the set of all real numbers.  
 $f(x) - g(x) = x^3 - 2x^2 + 6x$ , Domain is the set of all real numbers.  
 $f(x) \cdot g(x) = x^4 - 2x^3 + 7x^2$ , Domain is the set of all real numbers.  
 $\frac{f(x)}{g(x)} = x^2 - 2x + 7$ , Domain  $\{x | x \neq 0\}$

- 9.**  $f(x) + g(x) = \sqrt{x-3} + x$ , Domain  $\{x | x \geq 3\}$   
 $f(x) - g(x) = \sqrt{x-3} - x$ , Domain  $\{x | x \geq 3\}$   
 $f(x) \cdot g(x) = x\sqrt{x-3}$ , Domain  $\{x | x \geq 3\}$   
 $\frac{f(x)}{g(x)} = \frac{\sqrt{x-3}}{x}$ , Domain  $\{x | x \geq 3\}$

- 13.** 18    **15.**  $-\frac{9}{4}$     **17.** 30    **19.** 12    **21.** 300    **23.**  $-\frac{384}{125}$     **25.**  $-\frac{5}{2}$     **27.**  $-\frac{1}{4}$     **29.** 2    **31.**  $2x + h$     **33.**  $4x + 2h + 4$

- 35.**  $-8x - 4h$     **37.**  $(g \circ f)(x) = 6x + 3$     **39.**  $(g \circ f)(x) = x^2 + 4x + 1$   
 $(f \circ g)(x) = 6x - 16$      $(f \circ g)(x) = x^2 + 8x + 11$

- 45.**  $(g \circ f)(x) = \frac{\sqrt{1-x^2}}{|x|}$     **47.**  $(g \circ f)(x) = -\frac{2|5-x|}{3}$     **49.** 66    **51.** 51    **53.**  $-4$     **55.** 41    **57.**  $-\frac{3848}{625}$     **59.**  $6 + 2\sqrt{3}$   
 $(f \circ g)(x) = \frac{1}{x-1}$      $(f \circ g)(x) = \frac{3|x|}{|5x+2|}$

- 61.**  $16c^2 + 4c - 6$     **63.**  $9k^4 + 36k^3 + 45k^2 + 18k - 4$     **65. a.**  $A(t) = \pi(1.5t)^2$ ,  $A(2) = 9\pi$  square feet  $\approx 28.27$  square feet  
**b.**  $V(t) = 2.25\pi t^3$ ,  $V(3) = 60.75\pi$  cubic feet  $\approx 190.85$  cubic feet    **67. a.**  $d(t) = \sqrt{(48-t)^2 - 4^2}$     **b.**  $s(35) = 13$  feet,  $d(35) \approx 12.37$  feet

- 3.**  $f(x) + g(x) = 3x + 12$ , Domain is the set of all real numbers.  
 $f(x) - g(x) = x + 4$ , Domain is the set of all real numbers.  
 $f(x) \cdot g(x) = 2x^2 + 16x + 32$ , Domain is the set of all real numbers.  
 $\frac{f(x)}{g(x)} = 2$ , Domain  $\{x | x \neq -4\}$

- 7.**  $f(x) + g(x) = 2x^2 + 7x - 12$ , Domain is the set of all real numbers.  
 $f(x) - g(x) = -2x^2 + x - 2$ , Domain is the set of all real numbers.  
 $f(x) \cdot g(x) = 8x^3 - 2x^2 - 41x + 35$ , Domain is the set of all real numbers.  
 $\frac{f(x)}{g(x)} = \frac{4x-7}{2x^2+3x-5}$ , Domain  $\left\{x | x \neq 1, x \neq -\frac{5}{2}\right\}$

- 11.**  $f(x) + g(x) = \sqrt{4-x^2} + 2 + x$ , Domain  $\{x | -2 \leq x \leq 2\}$   
 $f(x) - g(x) = \sqrt{4-x^2} - 2 - x$ , Domain  $\{x | -2 \leq x \leq 2\}$   
 $f(x) \cdot g(x) = (\sqrt{4-x^2})(2+x)$ , Domain  $\{x | -2 \leq x \leq 2\}$   
 $\frac{f(x)}{g(x)} = \frac{\sqrt{4-x^2}}{2+x}$ , Domain  $\{x | -2 < x \leq 2\}$
- 41.**  $(g \circ f)(x) = -5x^3 - 10x$     **43.**  $(g \circ f)(x) = \frac{1-5x}{x+1}$   
 $(f \circ g)(x) = -125x^3 - 10x$   
 $(f \circ g)(x) = \frac{2}{3x-4}$

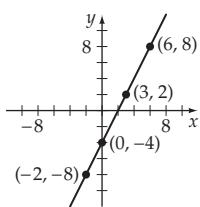
- 69.**  $(Y \circ F)(x)$  converts  $x$  inches to yards.   **71.** **a.** 99.8; this is identical to the slope of the line through  $(0, C(0))$  and  $(1, C(1))$ .   **b.** 156.2  
**c.** -49.7   **d.** -30.8   **e.** -16.4   **f.** 0

### Prepare for This Section (1.6), page 81

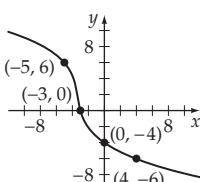
**PS1.**  $y = -\frac{2}{5}x + 3$    **PS2.**  $y = \frac{1}{x-1}$    **PS3.** -1   **PS4.**  $(3, 7)$    **PS5.** all real numbers   **PS6.**  $\{x | x \geq -2\}$

### Exercise Set 1.6, page 90

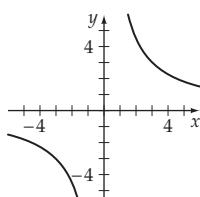
- 1.** 3   **3.** -3   **5.** 3   **7.** range   **9.** Yes



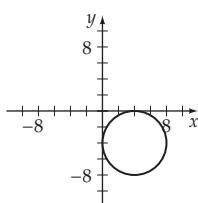
- 11.** Yes



- 13.** Yes



- 15.** No



- 17.** Yes

- 19.** Yes

- 21.** No

- 23.** Yes

- 25.** Yes

- 27.**  $\{(1, -3), (2, -2), (5, 1), (-7, 4)\}$

**29.**  $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4)\}$    **31.**  $f^{-1}(x) = \frac{1}{2}x - 2$    **33.**  $f^{-1}(x) = \frac{1}{3}x + \frac{7}{3}$    **35.**  $f^{-1}(x) = -\frac{1}{2}x + \frac{5}{2}$    **37.**  $f^{-1}(x) = \frac{x}{x-2}, x \neq 2$

**39.**  $f^{-1}(x) = \frac{x+1}{1-x}, x \neq 1$    **41.**  $f^{-1}(x) = \sqrt{x-1}, x \geq 1$    **43.**  $f^{-1}(x) = x^2 + 2, x \geq 0$    **45.**  $f^{-1}(x) = \sqrt{x+4} - 2, x \geq -4$

**47.**  $f^{-1}(x) = -\sqrt{x+5} - 2, x \geq -5$    **49.**  $V^{-1}(x) = \sqrt[3]{x}$ .  $V^{-1}$  finds the length of a side of a cube given the volume.   **51.**  $f^{-1}(x) = \frac{9}{5}x + 32$ ;

$f^{-1}(x)$  is used to convert  $x$  degrees Celsius to its equivalent Fahrenheit temperature.   **53.**  $s^{-1}(x) = \frac{1}{2}x - 12$

**55.**  $E^{-1}(s) = 20s - 50,000$ . The executive can determine the value of the software that must be sold in order to achieve a given monthly income.

**57.** **a.**  $p(10) \approx 0.12 = 12\%$ ;  $p(30) \approx 0.71 = 71\%$    **b.** The graph of  $p$ , for  $1 \leq n \leq 60$ , is an increasing function. Thus  $p$  (with  $1 \leq n \leq 60$ ) has an inverse that is a function.   **c.** Answers will vary.   **59.** **a.** 25 47 71 67 47 59 53 71 33 47 43 27 63 47 53 39   **b.** PHONE HOME   **c.** Answers will vary.   **61.** Because the function is increasing and 4 is between 2 and 5,  $c$  must be between 7 and 12.   **63.** between 2 and 5

**65.** between 3 and 7   **67.** slope:  $\frac{1}{m}$ ;  $y$ -intercept:  $\left(0, -\frac{b}{m}\right)$    **69.** The reflection of  $f$  across the line given by  $y = x$  yields  $f$ . Thus  $f$  is its own inverse.

- 71.** Yes   **73.** No

### Prepare for This Section (1.7), page 95

**PS1.** slope:  $-\frac{3}{4}$ ;  $y$ -intercept: 4   **PS2.** slope:  $\frac{3}{4}$ ;  $y$ -intercept: -3   **PS3.**  $y = -0.45x + 2.3$    **PS4.** 19   **PS5.** 69   **PS6.** 3

### Exercise Set 1.7, page 103

- 1.** no linear relationship   **3.** linear   **5.** Figure A   **7.**  $y = 2.00862069x + 0.5603448276$    **9.**  $y = -0.7231182796x + 9.233870968$   
**11.**  $y = 2.222641509x - 7.364150943$    **13.**  $y = 1.095779221x^2 - 2.69642857x + 1.136363636$   
**15.**  $y = -0.2987274717x^2 - 3.20998141x + 3.416463667$    **17.** **a.**  $y = 23.55706665x - 24.4271215$    **b.** 1248 centimeters  
**19.** **a.**  $y = 0.1094224924x + 0.7978723404$    **b.** 4.3 meters per second   **21.** **a.**  $y = 0.1628623408x - 0.6875682232$    **b.** 25   **23.** No, because the linear correlation coefficient is close to 0.   **25.** **a.** Yes, there is a strong linear correlation.   **b.**  $y = -0.9033088235x + 78.62573529$   
**c.** 56 years   **27.** **a.** positively   **b.** 1098 calories   **29.**  $y = -0.6328671329x^2 + 33.61608392x - 379.4405594$   
**31.** **a.**  $y = -0.0165034965x^2 + 1.366713287x + 5.685314685$    **b.** 32.8 miles per gallon  
**33.** **a.** 5-pound:  $s = 0.6130952381t^2 - 0.0714285714t + 0.1071428571$   
10-pound:  $s = 0.6091269841t^2 - 0.0011904762t - 0.3$   
15-pound:  $s = 0.5922619048t^2 + 0.3571428571t - 1.520833333$   
**b.** All the regression equations are approximately the same. Therefore, the equations of motion of the three masses are the same.

## Chapter 1 Assessing Concepts, page 111

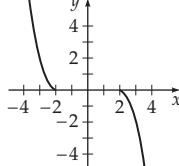
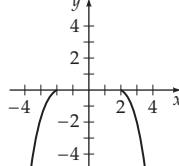
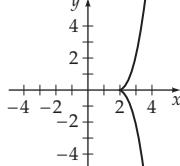
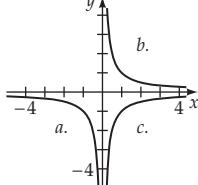
1. a, c, d, and e    2. No    3. 3    4.  $|x + 2| < 3$     5. It is the slope of the line between  $(a, f(a))$  and  $(b, f(b))$ .    6.  $(3, -2)$     7.  $(7, 3)$   
 8.  $(-3, 6)$     9.  $(3, 4)$     10. Yes. The slope of the regression line is negative.

## Chapter 1 Review Exercises, page 112

1.  $-\frac{9}{4}$  [1.1]    2.  $-4$  [1.1]    3.  $-2$  [1.1]    4.  $-\frac{2}{3}$  [1.1]    5.  $-3, 6$  [1.1]    6.  $\frac{1}{2}, 4$  [1.1]    7.  $\frac{-1 \pm \sqrt{13}}{6}$  [1.1]    8.  $\frac{-3 \pm \sqrt{41}}{4}$  [1.1]  
 9.  $c \geq -6$  [1.1]    10.  $a > 1$  [1.1]    11.  $(-\infty, -3] \cup [4, \infty)$  [1.1]    12.  $-\frac{1}{2} < x < 1$  [1.1]    13.  $(-\infty, 1) \cup (4, \infty)$  [1.1]    14.  $-1 \leq x \leq \frac{5}{3}$  [1.1]

15.  $10$  [1.2]    16.  $3\sqrt{31}$  [1.2]    17.  $5$  [1.2]    18.  $7\sqrt{5}$  [1.2]    19.  $\sqrt{181}$  [1.2]    20.  $4\sqrt{5}$  [1.2]    21.  $\left(-\frac{1}{2}, 10\right)$  [1.2]    22.  $(2, -2)$  [1.2]

23. [1.4]    24. a. [1.4]



25.  $y$ -axis [1.4]    26.  $x$ -axis [1.4]    27. origin [1.4]    28.  $x$ -axis,  $y$ -axis, origin [1.4]    29.  $x$ -axis,  $y$ -axis, origin [1.4]    30. origin [1.4]

31.  $x$ -axis,  $y$ -axis, origin [1.4]    32. origin [1.4]    33. center  $(3, -4)$ , radius 9 [1.2]    34. center  $(-5, -2)$ , radius = 3 [1.2]  
 35.  $(x - 2)^2 + (y + 3)^2 = 5^2$  [1.2]  
 d.  $3x^2 + 6xh + 3h^2 + 4x + 4h - 5$   
 e.  $2\sqrt{64 - t^2}$     f.  $2\sqrt{16 - t^2}$  [1.3]

- c.  $2x^2 - 4x + 9$     d.  $2x^2 + 6$  [1.5]    36.  $(x + 5)^2 + (y - 1)^2 = 64$  [1.2]    37. a. 2    b. 10    c.  $3t^2 + 4t - 5$   
 38. a.  $\sqrt{55}$     b.  $\sqrt{39}$     c. 0    d.  $\sqrt{64 - x^2}$

39. a. 5    b. -11    c.  $x^2 - 12x + 32$     d.  $x^2 + 4x - 8$  [1.5]    40. a. 79    b. 56

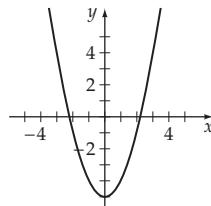
41.  $8x + 4h - 3$  [1.5]    42.  $3x^2 + 3xh + h^2 - 1$  [1.5]    43.  $(-\infty, \infty)$  [1.3]

44.  $x \leq 6$  [1.3]    45.  $[-5, 5]$  [1.3]    46. all real numbers except -3 and 5 [1.3]

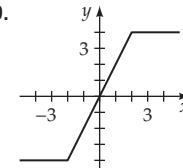
47. [1.3]    48. [1.3]



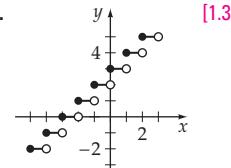
increasing on  $[3, \infty)$   
decreasing on  $(-\infty, 3]$



decreasing on  $(-\infty, 0]$   
increasing on  $[0, \infty)$

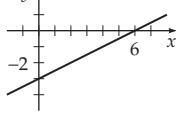


increasing on  $[-2, 2]$   
constant on  $(-\infty, -2] \cup [2, \infty)$



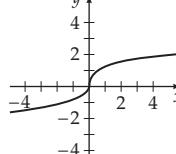
constant on  $[n, n + 1)$ , where  $n$  is an integer

51. [1.3]



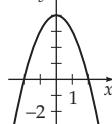
increasing on  $(-\infty, \infty)$

52. [1.3]



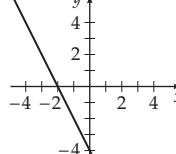
increasing on  $(-\infty, \infty)$

53. [1.3/1.4]



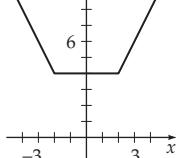
- a. domain  $(-\infty, \infty)$   
range  $\{y \mid y \leq 4\}$   
b. even

54. [1.3/1.4]



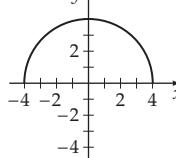
- a. domain  $(-\infty, \infty)$   
range  $(-\infty, \infty)$   
b. neither

55. [1.3/1.4]



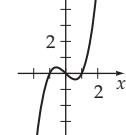
- a. domain  $(-\infty, \infty)$   
range  $\{y \mid y \geq 6\}$   
b. even

56. [1.3/1.4]



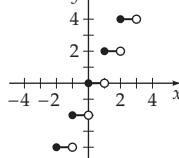
- a. domain  $[-4, 4]$   
range  $[0, 4]$   
b. even

57. [1.3/1.4]



- a. domain  $(-\infty, \infty)$   
range  $(-\infty, \infty)$   
b. odd

58. [1.3/1.4]



- a. domain  $(-\infty, \infty)$   
range: even integers  
b. neither

59.  $f(x) + g(x) = x^2 + x - 6$ , domain:  $(-\infty, \infty)$

$f(x) - g(x) = x^2 - x - 12$ , domain:  $(-\infty, \infty)$

$f(x) \cdot g(x) = x^3 + 3x^2 - 9x - 27$ , domain:  $(-\infty, \infty)$

$\frac{f(x)}{g(x)} = x - 3$ , domain  $\{x | x \neq -3\}$  [1.5]

60.  $f(x) + g(x) = x^3 + x^2 - 2x + 12$ , domain:  $(-\infty, \infty)$

$f(x) - g(x) = x^3 - x^2 + 2x + 4$ , domain:  $(-\infty, \infty)$

$f(x) \cdot g(x) = x^5 - 2x^4 + 4x^3 + 8x^2 - 16x + 32$ , domain:  $(-\infty, \infty)$

$\frac{f(x)}{g(x)} = x + 2$ , domain:  $(-\infty, \infty)$  [1.5]

63. Yes [1.6]

64. No [1.6]

65.  $f^{-1}(x) = \frac{x+4}{3}$  [1.6]

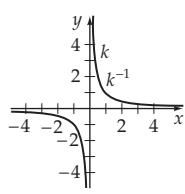
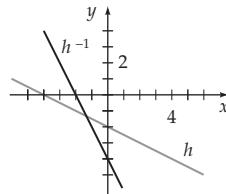
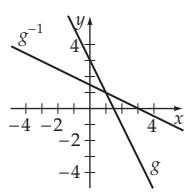
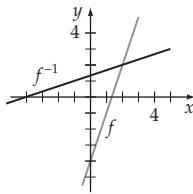
66.  $g^{-1}(x) = -\frac{1}{2}x + \frac{3}{2}$  [1.6]

67.  $h^{-1}(x) = -2x - 4$  [1.6]

61. Yes [1.6]

62. Yes [1.6]

68.  $k^{-1}(x) = \frac{1}{x}$  [1.6]

69.  $t \approx 3.7$  seconds [1.3]

70. a. 150 feet

b. 525 feet [1.3]

71. a.  $y = 0.018024687x + 0.00050045744$ b. Yes.  $r \approx 0.999$ , which is very close to 1.

c. 1.8 seconds [1.7]

72. a.  $y = 0.0047952048x^2 - 1.756843157x + 180.4065934$ b. The graph of the regression equation never crosses the  $x$ -axis. Therefore, the model predicts that the can will never be empty.

c. A regression model only approximates a situation. [1.7]

## Chapter 1 Quantitative Reasoning Exercises, page 114

QR1. a. 3

b. 2

c. 52

QR2. Factor the modulus,  $m$ .

QR3. Answer will vary.

## Chapter 1 Test, page 116

1.  $\frac{1}{2}$  [1.1]

2.  $x \leq 1$  [1.1]

3.  $-\frac{1}{2}, 2$  [1.1]

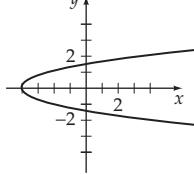
4.  $-\frac{2}{3}, 1$  [1.1]

5.  $(-\infty, -\frac{2}{5}) \cup (2, \infty)$  [1.1]

6.  $\sqrt{85}$  [1.2]

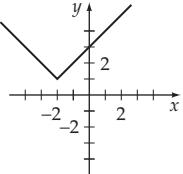
7. midpoint (1, 1); length  $2\sqrt{13}$  [1.2]

8.  $(0, \sqrt{2}), (0, -\sqrt{2}), (-4, 0)$



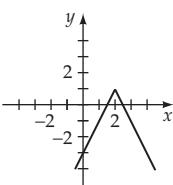
[1.2]

9.



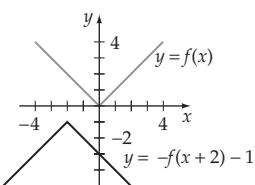
[1.2]

13.



[1.3/1.4]

14.



[1.4]

15. b [1.4]

16.  $x^2 + x - 3; \frac{x^2 - 1}{x - 2}, x \neq 2$  [1.5]

17.  $2x + h$  [1.5]

increasing on  $(-\infty, 2]$   
decreasing on  $[2, \infty)$ 

18.  $x - 2\sqrt{x-2} - 1$  [1.5]

19.  $f^{-1}(x) = \frac{x}{1-x}$  [1.6]

20. a.  $y = -7.98245614x + 767.122807$

b. 56.7 calories [1.7]

## Exercise Set 2.1, page 130

1.  $75^\circ, 165^\circ$

3.  $1945', 10945'$

5.  $33^\circ 26' 45'', 123^\circ 26' 45''$

7.  $\frac{\pi}{2} - 1, \pi - 1$

9.  $\frac{\pi}{4}, \frac{3\pi}{4}$

11.  $\frac{\pi}{10}, \frac{3\pi}{5}$

13.  $250^\circ$ , Quadrant III

15.  $105^\circ$ , Quadrant II

17.  $296^\circ$ , Quadrant IV

19.  $24^\circ 33' 36''$

21.  $64^\circ 9' 28.8''$

23.  $3^\circ 24' 7.2''$

25.  $25.42^\circ$

27.  $183.56^\circ$

29.  $211.78^\circ$

31.  $\frac{\pi}{6}$

33.  $\frac{\pi}{2}$

35.  $\frac{11\pi}{12}$

37.  $\frac{7\pi}{3}$

39.  $\frac{13\pi}{4}$

41.  $-\frac{\pi}{20}$

43.  $420^\circ$

45.  $36^\circ$

47.  $30^\circ$

49.  $67.5^\circ$

51.  $660^\circ$

53.  $-75^\circ$

55.  $85.94^\circ$

57. 2.32

59.  $472.69^\circ$

61.  $4, 229.18^\circ$

63.  $2.38, 136.63^\circ$

65. 6.28 inches

67. 18.33 centimeters

69.  $3\pi$

71.  $\frac{5\pi}{12}$  radians or  $75^\circ$

- 73.**  $\frac{\pi}{30}$  radian per second  $\approx 0.105$  radian per second    **75.**  $\frac{5\pi}{3}$  radians per second  $\approx 5.24$  radians per second    **77.**  $\frac{10\pi}{9}$  radians per second  $\approx 3.49$  radians per second    **79.** 40 mph    **81.** 1885 feet    **83.** 6.9 mph    **85.** 840,000 miles    **87.** **a.** 3.9 radians per hour  
**b.** 27,300 kilometers per hour    **89.** **a.**  $B$     **b.** Both points have the same linear velocity.    **91.** **a.** 1.15 statute miles    **b.** 10%  
**93.** 13 square inches    **95.** 4680 square centimeters    **97.** 1780 miles

### Prepare for This Section (2.2), page 134

- PS1.**  $\frac{\sqrt{3}}{3}$     **PS2.**  $\sqrt{2}$     **PS3.** 2    **PS4.**  $\frac{\sqrt{3}}{3}$     **PS5.** 3.54    **PS6.** 10.39

### Exercise Set 2.2, page 142

- 1.**  $\sin \theta = \frac{12}{13}$      $\csc \theta = \frac{13}{12}$     **3.**  $\sin \theta = \frac{4}{7}$      $\csc \theta = \frac{7}{4}$     **5.**  $\sin \theta = \frac{5\sqrt{29}}{29}$      $\csc \theta = \frac{\sqrt{29}}{5}$   
 $\cos \theta = \frac{5}{13}$      $\sec \theta = \frac{13}{5}$      $\cos \theta = \frac{\sqrt{33}}{7}$      $\sec \theta = \frac{7\sqrt{33}}{33}$      $\cos \theta = \frac{2\sqrt{29}}{29}$      $\sec \theta = \frac{\sqrt{29}}{2}$   
 $\tan \theta = \frac{12}{5}$      $\cot \theta = \frac{5}{12}$      $\tan \theta = \frac{4\sqrt{33}}{33}$      $\cot \theta = \frac{\sqrt{33}}{4}$      $\tan \theta = \frac{5}{2}$      $\cot \theta = \frac{2}{5}$
- 7.**  $\sin \theta = \frac{\sqrt{21}}{7}$      $\csc \theta = \frac{\sqrt{21}}{3}$     **9.**  $\sin \theta = \frac{\sqrt{3}}{2}$      $\csc \theta = \frac{2\sqrt{3}}{3}$     **11.**  $\sin \theta = \frac{6\sqrt{61}}{61}$      $\csc \theta = \frac{\sqrt{61}}{6}$   
 $\cos \theta = \frac{2\sqrt{7}}{7}$      $\sec \theta = \frac{\sqrt{7}}{2}$      $\cos \theta = \frac{1}{2}$      $\sec \theta = 2$      $\cos \theta = \frac{5\sqrt{61}}{61}$      $\sec \theta = \frac{\sqrt{61}}{5}$   
 $\tan \theta = \frac{\sqrt{3}}{2}$      $\cot \theta = \frac{2\sqrt{3}}{3}$      $\tan \theta = \sqrt{3}$      $\cot \theta = \frac{\sqrt{3}}{3}$      $\tan \theta = \frac{6}{5}$      $\cot \theta = \frac{5}{6}$
- 13.**  $\frac{3}{4}$     **15.**  $\frac{4}{5}$     **17.**  $\frac{3}{4}$     **19.**  $\frac{12}{13}$     **21.**  $\frac{13}{5}$     **23.**  $\frac{3}{2}$     **25.**  $\sqrt{2}$     **27.**  $-\frac{3}{4}$     **29.**  $\frac{5}{4}$     **31.**  $\sqrt{3}$     **33.**  $\frac{3\sqrt{2} + 2\sqrt{3}}{6}$     **35.**  $\frac{3 - \sqrt{3}}{3}$   
**37.**  $2\sqrt{2} - \sqrt{3}$     **39.** 0.6249    **41.** 0.4488    **43.** 0.8221    **45.** 1.0053    **47.** 0.4816    **49.** 1.0729    **51.** 9.5 feet    **53.** 92.9 inches  
**55.** 5.1 feet    **57.** 1.7 miles    **59.** 74.6 feet    **61.** 686,000,000 kilometers    **65.** 612 feet    **67.** 560 feet    **69.** **a.** 559 feet    **b.** 193 feet  
**71.**  $\sqrt{27}$  meters  $\approx 5.2$  meters    **73.**  $\approx 8.5$  feet

### Prepare for This Section (2.3), page 147

- PS1.**  $-\frac{4}{3}$     **PS2.**  $\frac{\sqrt{5}}{2}$     **PS3.** 60    **PS4.**  $\frac{\pi}{5}$     **PS5.**  $\pi$     **PS6.**  $\sqrt{34}$

### Exercise Set 2.3, page 153

- 1.**  $\sin \theta = \frac{3\sqrt{13}}{13}$      $\csc \theta = \frac{\sqrt{13}}{3}$     **3.**  $\sin \theta = \frac{3\sqrt{13}}{13}$      $\csc \theta = \frac{\sqrt{13}}{3}$     **5.**  $\sin \theta = -\frac{5\sqrt{89}}{89}$      $\csc \theta = -\frac{\sqrt{89}}{5}$   
 $\cos \theta = \frac{2\sqrt{13}}{13}$      $\sec \theta = \frac{\sqrt{13}}{2}$      $\cos \theta = -\frac{2\sqrt{13}}{13}$      $\sec \theta = -\frac{\sqrt{13}}{2}$      $\cos \theta = -\frac{8\sqrt{89}}{89}$      $\sec \theta = -\frac{\sqrt{89}}{8}$   
 $\tan \theta = \frac{3}{2}$      $\cot \theta = \frac{2}{3}$      $\tan \theta = -\frac{3}{2}$      $\cot \theta = -\frac{2}{3}$      $\tan \theta = \frac{5}{8}$      $\cot \theta = \frac{8}{5}$
- 7.**  $\sin \theta = 0$      $\csc \theta$  is undefined.    **9.** 0    **11.** 0    **13.** 1    **15.** 0    **17.** undefined    **19.** 1    **21.** Quadrant I    **23.** Quadrant IV  
 $\cos \theta = -1$      $\sec \theta = -1$   
 $\tan \theta = 0$      $\cot \theta$  is undefined.
- 25.** Quadrant III    **27.**  $\frac{\sqrt{3}}{3}$     **29.** -1    **31.**  $-\frac{\sqrt{3}}{3}$     **33.**  $\frac{2\sqrt{3}}{3}$     **35.**  $-\frac{\sqrt{3}}{3}$     **37.**  $20^\circ$     **39.**  $9^\circ$     **41.**  $\frac{\pi}{5}$     **43.**  $\pi - \frac{8}{3}$     **45.**  $34^\circ$   
**47.**  $65^\circ$     **49.**  $-\frac{\sqrt{2}}{2}$     **51.** 1    **53.**  $-\frac{2\sqrt{3}}{3}$     **55.**  $\frac{\sqrt{2}}{2}$     **57.**  $\sqrt{2}$     **59.** cot  $540^\circ$  is undefined.    **61.** 0.798636    **63.** -0.438371  
**65.** -1.26902    **67.** -0.587785    **69.** -1.70130    **71.** -3.85522    **73.** 0    **75.** 1    **77.**  $-\frac{3}{2}$     **79.** 1    **81.**  $30^\circ, 150^\circ$     **83.**  $150^\circ, 210^\circ$   
**85.**  $225^\circ, 315^\circ$     **87.**  $\frac{3\pi}{4}, \frac{7\pi}{4}$     **89.**  $\frac{5\pi}{6}, \frac{11\pi}{6}$     **91.**  $\frac{\pi}{3}, \frac{2\pi}{3}$

## Prepare for This Section (2.4), page 155

**PS1.** Yes   **PS2.** Yes   **PS3.** No   **PS4.**  $2\pi$    **PS5.** even function   **PS6.** neither

## Exercise Set 2.4, page 166

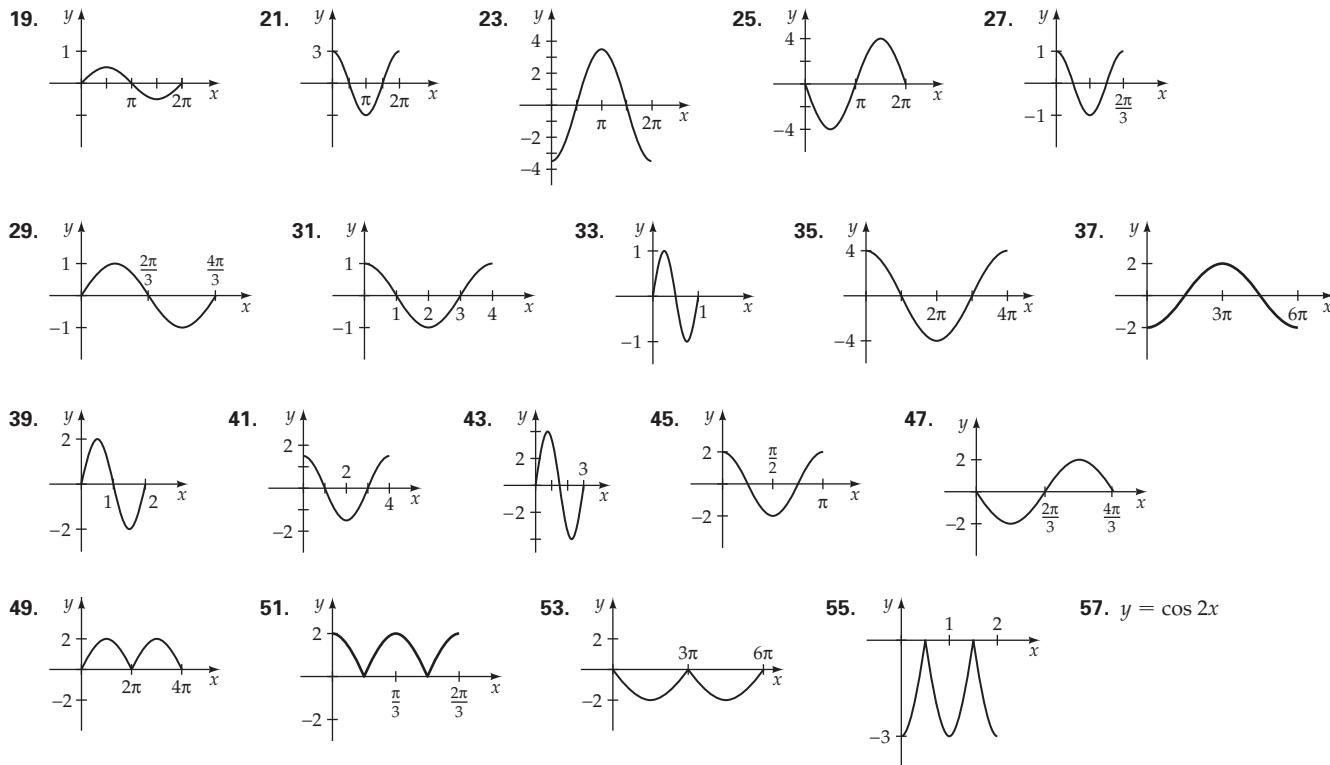
1.  $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
3.  $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
5.  $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
7.  $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
9.  $(-1, 0)$
11.  $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
13.  $-\frac{\sqrt{3}}{3}$
15.  $-\frac{1}{2}$
17.  $-\frac{2\sqrt{3}}{3}$
19.  $-1$
21.  $-\frac{2\sqrt{3}}{3}$
23. 0.9391
25.  $-1.1528$
27.  $-0.2679$
29. 0.8090
31. 48.0889
33. a. 0.9   b.  $-0.4$
35. a.  $-0.8$    b. 0.6
37. 0.4, 2.7
39. 3.4, 6.0
41. odd
43. neither
45. even
47. odd
49.  $2\pi$
51.  $\pi$
53.  $2\pi$
61.  $\sin t$
63.  $\sec t$
65.  $-\tan^2 t$
67.  $-\cot t$
69.  $\cos^2 t$
71.  $2 \csc^2 t$
73.  $\csc^2 t$
75. 1
77.  $\sqrt{1 - \cos^2 t}$
79.  $\sqrt{1 + \cot^2 t}$
81. 750 miles
83.  $-\frac{\sin^2 t}{\cos t}$
85.  $\csc t \sec t$
87.  $1 - 2 \sin t + \sin^2 t$
89.  $1 - 2 \sin t \cos t$
91.  $\cos^2 t$
93.  $2 \csc t$
95.  $(\cos t - \sin t)(\cos t + \sin t)$
97.  $(\tan t + 2)(\tan t - 3)$
99.  $(2 \sin t + 1)(\sin t - 1)$
101.  $\frac{\sqrt{2}}{2}$
103.  $-\frac{\sqrt{3}}{3}$

## Prepare for This Section (2.5), page 169

**PS1.** 0.7   **PS2.**  $-0.7$    **PS3.** Reflect the graph of  $y = f(x)$  across the  $x$ -axis.   **PS4.** Compress each point on the graph of  $y = f(x)$  toward the  $y$ -axis by a factor of  $\frac{1}{2}$ .   **PS5.**  $6\pi$    **PS6.**  $5\pi$

## Exercise Set 2.5, page 177

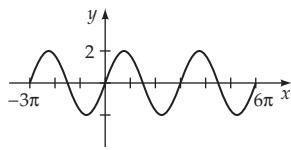
1.  $2, 2\pi$
3.  $1, \pi$
5.  $\frac{1}{2}, 1$
7.  $2, 4\pi$
9.  $\frac{1}{2}, 2\pi$
11.  $1, 8\pi$
13.  $2, 6$
15.  $3, 3\pi$
17.  $4.7, 2.5$



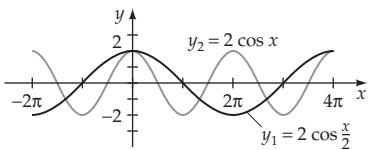
57.  $y = \cos 2x$

59.  $y = 2 \sin \frac{2}{3}x$     61.  $y = -2 \cos \pi x$     63. a.  $V = 4 \sin \pi t$ ,  $0 \leq t \leq 8$  milliseconds    b.  $\frac{1}{2}$  cycle per millisecond

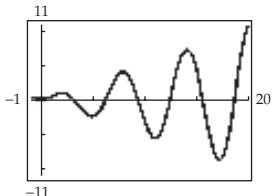
65.



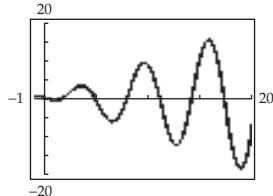
67.



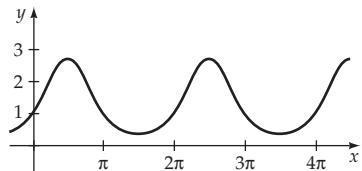
69.



71.



73.



75.  $y = 2 \sin \frac{2}{3}x$     77.  $y = 2.5 \sin \frac{5\pi}{8}x$

maximum =  $e$ , minimum =  $\frac{1}{e} \approx 0.3679$ , period =  $2\pi$

79.  $y = 3 \cos \frac{4\pi}{5}x$

### Prepare for This Section (2.6), page 180

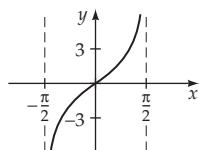
**PS1.** 1.7    **PS2.** 0.6    **PS3.** Stretch each point on the graph of  $y = f(x)$  away from the  $x$ -axis by a factor of 2.

**PS4.** Shift the graph of  $y = f(x)$  2 units to the right and 3 units up.    **PS5.**  $2\pi$     **PS6.**  $\frac{4}{3}\pi$

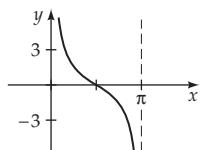
### Exercise Set 2.6, page 189

1.  $\frac{\pi}{2} + k\pi$ ,  $k$  an integer    3.  $\frac{\pi}{2} + k\pi$ ,  $k$  an integer    5.  $2\pi$     7.  $\pi$     9.  $2\pi$     11.  $\frac{2\pi}{3}$     13.  $\frac{\pi}{3}$     15.  $8\pi$     17. 1    19. 5    21. 8.5

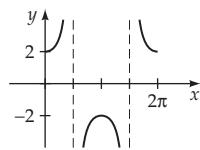
23.



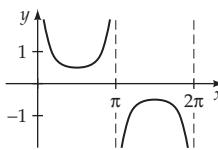
25.



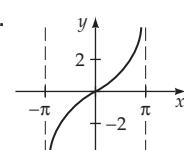
27.



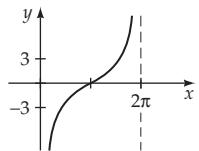
29.



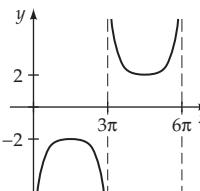
31.



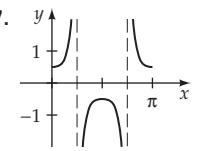
33.



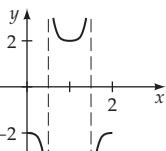
35.



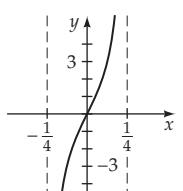
37.



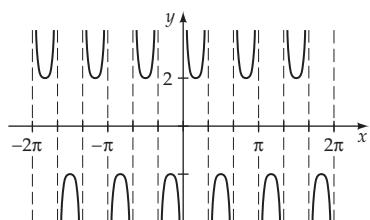
39.



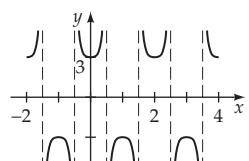
41.



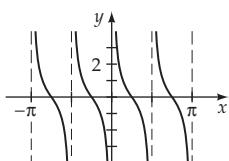
43.



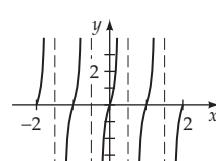
45.



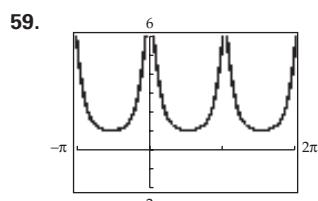
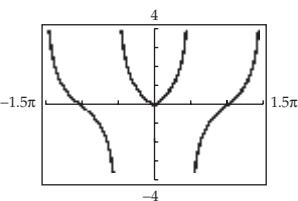
47.



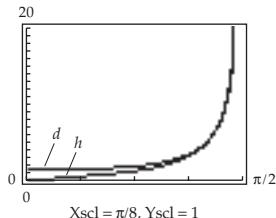
49.



51.  $y = \cot \frac{3}{2}x$     53.  $y = \csc \frac{2}{3}x$     55.  $y = \sec \frac{3}{4}x$     57.



61. a.  $h = 1.4 \tan x$     b.  $d = 1.4 \sec x$



- d. The graph of  $d$  is above the graph of  $h$ , but the distance between the graphs approaches 0 as  $x$  approaches  $\frac{\pi}{2}$ .

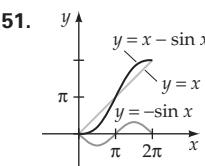
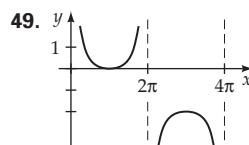
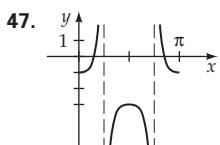
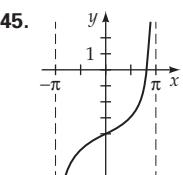
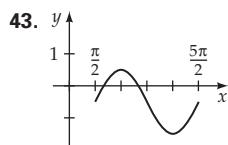
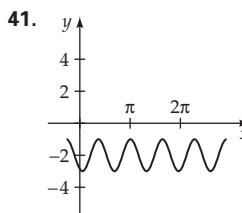
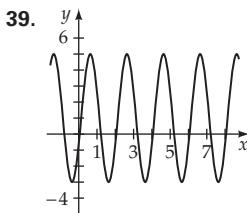
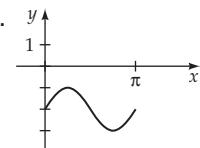
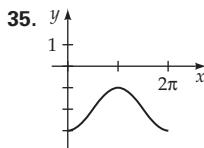
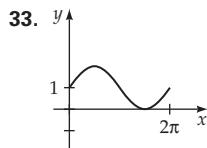
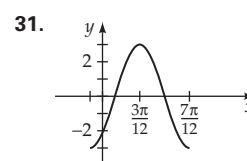
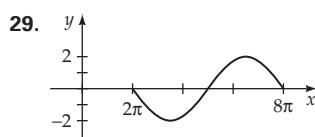
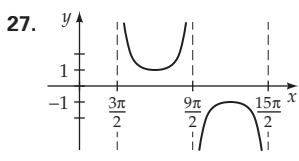
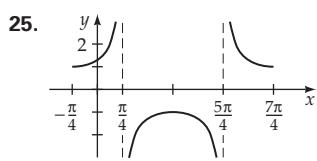
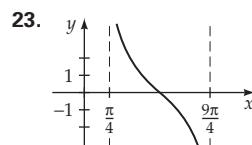
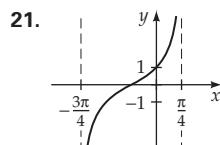
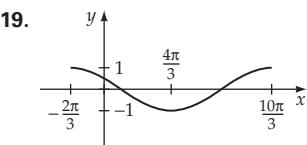
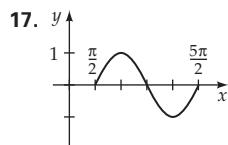
63.  $y = \tan 3x$     65.  $y = \sec \frac{8}{3}x$   
67.  $y = \cot \frac{\pi}{2}x$     69.  $y = \csc \frac{4\pi}{3}x$

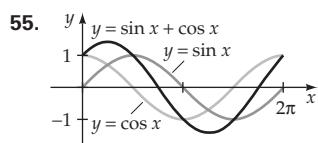
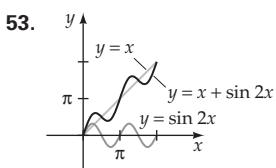
### Prepare for This Section (2.7), page 192

- PS1. amplitude 2, period  $\pi$     PS2. amplitude  $\frac{2}{3}$ , period  $6\pi$     PS3. amplitude 4, period 1    PS4. 2    PS5. -3    PS6.  $y$ -axis

### Exercise Set 2.7, page 199

1.  $2, \frac{\pi}{2}, 2\pi$     3.  $1, \frac{\pi}{8}, \pi$     5.  $4, -\frac{\pi}{4}, 3\pi$     7.  $\frac{5}{4}, \frac{2\pi}{3}, \frac{2\pi}{3}$     9.  $\frac{\pi}{8}, \frac{\pi}{2}$     11.  $-3\pi, 6\pi$     13.  $\frac{\pi}{16}, \pi$     15.  $-12\pi, 4\pi$

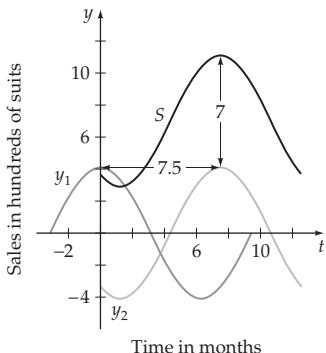




57.  $y = \sin\left(2x - \frac{\pi}{3}\right)$     59.  $y = \csc\left(\frac{x}{2} - \pi\right)$     61.  $y = \sec\left(x - \frac{\pi}{2}\right)$

63. a. 7.5 months, 12 months

b.



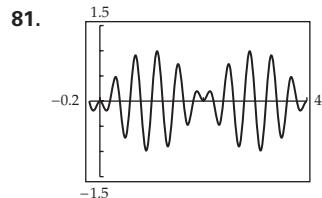
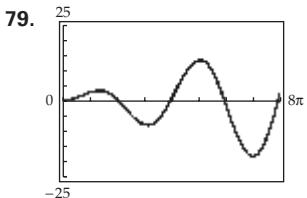
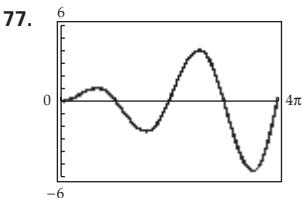
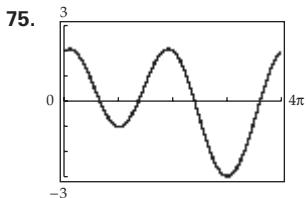
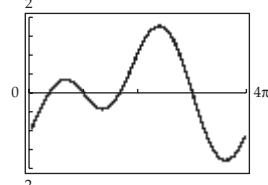
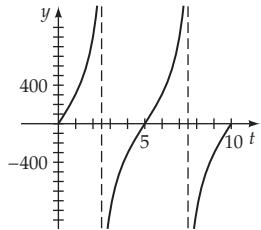
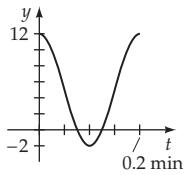
c. August    65.  $\approx 20$  parts per million

67.  $s = 7 \cos 10\pi t + 5$

69.  $s = 400 \tan \frac{\pi}{5} t$ ,  $t$  in seconds

71.  $y = 3 \cos \frac{\pi}{6}t + 9$ , 12 feet at 6:00 P.M.

73.

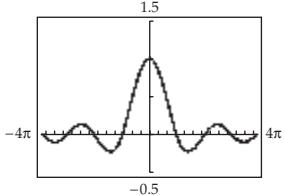


83.  $y = 2 \sin\left(2x - \frac{2\pi}{3}\right)$

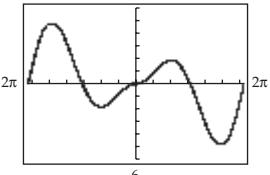
85.  $y = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)$

87.  $\cos^2 x + 2$

89.



91.



The graph above does not show that the function is undefined at  $x = 0$ .

### Prepare for This Section (2.8), page 203

PS1.  $\frac{3}{2\pi}$     PS2.  $\frac{5}{2}$     PS3. 4    PS4. 3    PS5. 4    PS6.  $y = 4 \cos \pi x$

## Exercise Set 2.8, page 208

1.  $2, \pi, \frac{1}{\pi}$     3.  $3, 3\pi, \frac{1}{3\pi}$     5.  $4, 2, \frac{1}{2}$     7.  $\frac{3}{4}, 4, \frac{1}{4}$

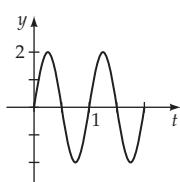
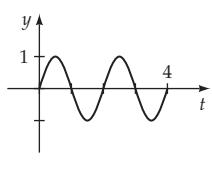
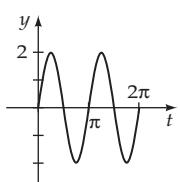
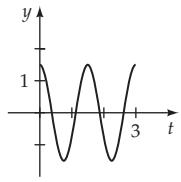
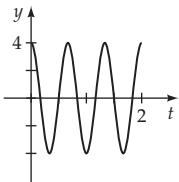
9.  $y = 4 \cos 3\pi t$

11.  $y = \frac{3}{2} \cos \frac{4\pi}{3}t$

13.  $y = 2 \sin 2t$

15.  $y = \sin \pi t$

17.  $y = 2 \sin 2\pi t$



19.  $y = \frac{1}{2} \cos 4t$     21.  $y = 2.5 \cos \pi t$     23.  $y = \frac{1}{2} \cos \frac{2\pi}{3}t$     25.  $y = 4 \cos 4t$     27.  $4\pi, \frac{1}{4\pi}$ , 2 feet;  $y = -2 \cos \frac{t}{2}$

29. a. 196 cycles per second;  $\frac{1}{196}$  second    b. The amplitude needs to increase.    31.  $h = -37 \cos \left( \frac{\pi}{22.5}t \right) + 41$     33. a. 3    b. 59.8 seconds

35. a. 10    b. 71.0 seconds    37. a. 10    b. 9.1 seconds    39. a. 10    b. 6.1 seconds    41. The new period is 3 times the original period.

43. Yes    45. Yes

## Chapter 2 Assessing Concepts, page 212

1. True    2. False    3. False    4. True    5.  $\frac{\pi}{4}$     6.  $(0, 1)$     7.  $\frac{8}{3}$     8. Shift the graph of  $y_1$  to the left  $\frac{\pi}{2}$  units.

9. all real numbers except multiples of  $\pi$     10.  $x = \frac{\pi}{2}$  and  $x = \frac{3\pi}{2}$

## Chapter 2 Review Exercises, page 212

1. complement measures  $25^\circ$ ; supplement measures  $115^\circ$ . [2.1]    2.  $80^\circ$  [2.3]    3.  $114.59^\circ$  [2.1]    4.  $\frac{7\pi}{4}$  [2.1]    5. 3.93 meters [2.1]    6. 0.3 [2.1]

7. 55 radians per second [2.1]    8.  $\frac{\sqrt{5}}{3}$  [2.2]    9.  $\frac{\sqrt{5}}{2}$  [2.2]    10.  $\frac{2}{3}$  [2.2]    11.  $\frac{3\sqrt{5}}{5}$  [2.2]    12.  $\sin \theta = -\frac{3\sqrt{10}}{10}$      $\csc \theta = -\frac{\sqrt{10}}{3}$  [2.3]

$\cos \theta = \frac{\sqrt{10}}{10}$      $\sec \theta = \sqrt{10}$

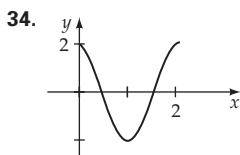
$\tan \theta = -3$      $\cot \theta = -\frac{1}{3}$

13. a.  $-\frac{2\sqrt{3}}{3}$     b. 1    c. -1    d.  $-\frac{1}{2}$  [2.3]    14. a. -0.5446    b. 0.5365    c. -3.2361    d. 3.0777 [2.3]    15. a.  $-\frac{1}{2}$     b.  $\frac{\sqrt{3}}{3}$  [2.3]

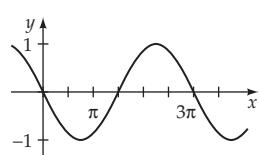
16. a.  $-\frac{2\sqrt{3}}{3}$     b. 2 [2.3]    17. a.  $\frac{\sqrt{2}}{2}$     b. -1 [2.3]    18. a.  $(-1, 0)$     b.  $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$     c.  $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$     d.  $(1, 0)$  [2.4]

19. even [2.4]    22.  $\sec^2 \phi$  [2.4]    23.  $\tan \phi$  [2.4]    24.  $\sin \phi$  [2.4]    25.  $\tan^2 \phi$  [2.4]    26.  $\csc^2 \phi$  [2.4]    27. 0 [2.4]    28.  $3, \pi, \frac{\pi}{2}$  [2.5]

29. no amplitude,  $\frac{\pi}{3}, 0$  [2.6]    30.  $2, \frac{2\pi}{3}, -\frac{\pi}{9}$  [2.5]    31.  $1, \pi, \frac{\pi}{3}$  [2.5]    32. no amplitude,  $\frac{\pi}{2}, \frac{3\pi}{8}$  [2.6]    33. no amplitude,  $2\pi, \frac{\pi}{4}$  [2.6]

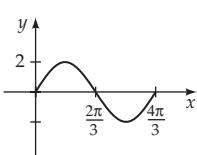


[2.5]

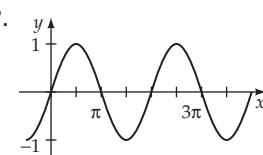


[2.5]

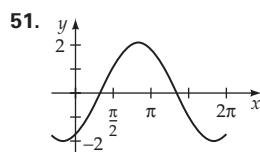
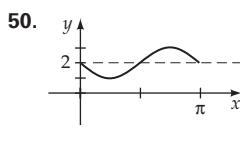
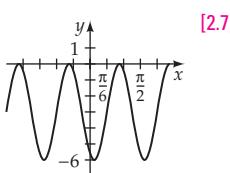
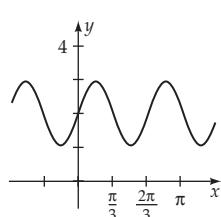
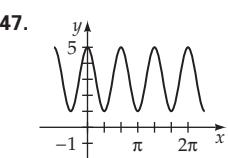
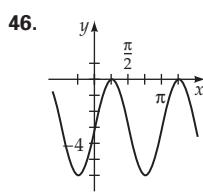
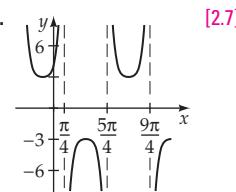
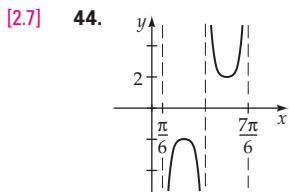
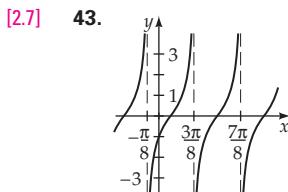
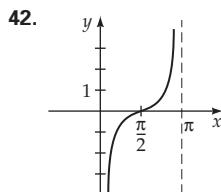
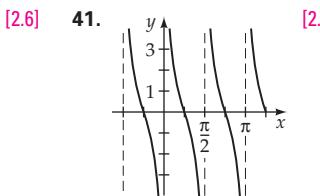
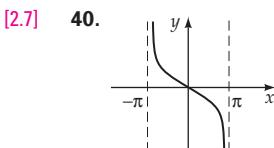
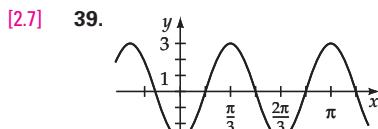
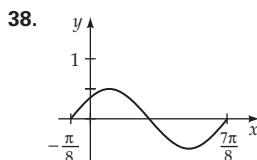
36. [2.5]



[2.5]



[2.7]



52. 0.089 mile [2.2] 53. 12.3 feet [2.2] 54. 1.7 feet per second [2.1]

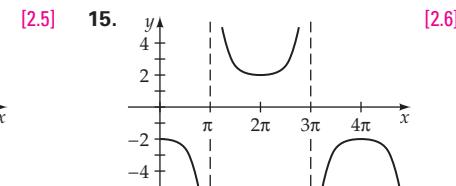
55. 46 feet [2.2] 56.  $2.5, \frac{\pi}{25}, \frac{25}{\pi}$  [2.8] 57. amplitude = 0.5,  $f = \frac{1}{\pi}$ ,  $p = \pi$ ,  $y = -0.5 \cos 2t$  [2.8] 58. 7.2 seconds [2.8]

## Chapter 2 Quantitative Reasoning Exercises, page 214

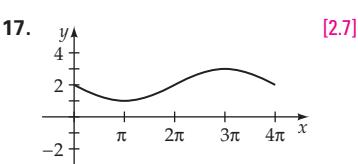
- QR1.** a.  $6\pi$  b. 4 c.  $2\pi$  d.  $24\pi$  e. 7.5 f.  $4\pi$  **QR2.** 15 seconds **QR3.** 11.25 seconds **QR4.** 54 seconds

## Chapter 2 Test, page 214

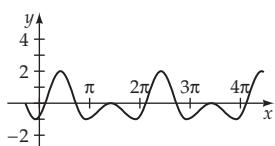
1.  $\frac{5\pi}{6}$  [2.1] 2.  $\frac{\pi}{12}$  [2.1] 3. 13.1 centimeters [2.1] 4.  $12\pi$  radians/second [2.1] 5. 80 centimeters/second [2.1] 6.  $\frac{\sqrt{58}}{7}$  [2.2]  
 7. 1.0864 [2.2] 8.  $\frac{\sqrt{3}-6}{6}$  [2.3] 9.  $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$  [2.4] 10.  $\sin^2 t$  [2.4] 11.  $\frac{\pi}{3}$  [2.6] 12. amplitude 3, period  $\pi$ , phase shift  $-\frac{\pi}{4}$  [2.7]  
 13. period 3, phase shift  $-\frac{1}{2}$  [2.7] 14.



- 16.** Shift the graph of  $y = 2 \sin 2x$ ,  $\frac{\pi}{4}$  units to the right and 1 unit down. [2.7]



- 18.** [2.7] **19.** 25.5 meters [2.2] **20.**  $y = 13 \sin \frac{2\pi}{5} t$  [2.8]



### Cumulative Review Exercises, page 215

1.  $5\sqrt{2}$  [1.2]
2.  $\frac{\sqrt{3}}{2}$  [1.2]
3.  $y$ -intercept  $(0, -9)$ ,  $x$ -intercepts  $(-3, 0)$  and  $(3, 0)$  [1.2]
4. odd function [1.4]
5.  $f^{-1}(x) = \frac{3x}{2x - 1}$  [1.6]
6.  $(-\infty, 4) \cup (4, \infty)$  [1.3]
7. 2,  $-3$  [1.1]
8. Shift the graph of  $y = f(x)$  horizontally 3 units to the right. [1.4]
9. Reflect the graph of  $y = f(x)$  across the  $y$ -axis. [1.4]
10.  $\frac{5\pi}{3}$  [2.1]
11.  $225^\circ$  [2.1]
12. 1 [2.3]
13.  $\frac{\sqrt{3} + 1}{2}$  [2.2]
14.  $\frac{5}{4}$  [2.2]
15. negative [2.3]
16.  $30^\circ$  [2.3]
17.  $\frac{\pi}{3}$  [2.3]
18.  $(-\infty, \infty)$  [2.5]
19.  $[-1, 1]$  [2.5]
20.  $\frac{3}{5}$  [2.2]

### Exercise Set 3.1, page 222

- 57.** identity    **59.** identity    **61.** identity    **63.** not an identity    **65.** If  $x = \frac{\pi}{4}$ , the left side is 2 and the right side is 1.    **67.** If  $x = 0^\circ$ , the left side is  $\frac{\sqrt{3}}{2}$  and the right side is  $\frac{2 + \sqrt{3}}{2}$ .    **69.** If  $x = 0$ , the left side is  $-1$  and the right side is 1.

### Prepare for This Section (3.2), page 225

- PS1.** Both function values equal  $\frac{1}{2}$ .    **PS2.** Both function values equal  $\frac{1}{2}$ .    **PS3.** For each of the given values of  $\theta$ , the function values are equal.    **PS4.** For each of the given values of  $\theta$ , the function values are equal.    **PS5.** Both function values equal  $\frac{\sqrt{3}}{3}$ .    **PS6.** 0

### Exercise Set 3.2, page 233

1.  $\frac{\sqrt{6} + \sqrt{2}}{4}$
3.  $\frac{\sqrt{6} + \sqrt{2}}{4}$
5.  $2 - \sqrt{3}$
7.  $\frac{-\sqrt{6} + \sqrt{2}}{4}$
9.  $-\frac{\sqrt{6} + \sqrt{2}}{4}$
11.  $2 + \sqrt{3}$
13. 0
15.  $\frac{1}{2}$
17.  $\sqrt{3}$
19.  $\cos 48^\circ$
21.  $\cot 75^\circ$
23.  $\csc 65^\circ$
25.  $\sin 5x$
27.  $\cos x$
29.  $\sin 4x$
31.  $\cos 2x$
33.  $\sin x$
35.  $\tan 7x$
37. a.  $-\frac{77}{85}$
- b.  $\frac{84}{85}$
- c.  $\frac{77}{36}$
39. a.  $-\frac{63}{65}$
- b.  $-\frac{56}{65}$
- c.  $-\frac{63}{16}$
41. a.  $\frac{63}{65}$
- b.  $\frac{56}{65}$
- c.  $\frac{33}{56}$
43. a.  $-\frac{77}{85}$
- b.  $-\frac{84}{85}$
- c.  $-\frac{13}{84}$
45. a.  $-\frac{33}{65}$
- b.  $-\frac{16}{65}$
- c.  $\frac{63}{16}$
47. a.  $-\frac{56}{65}$
- b.  $-\frac{63}{65}$
- c.  $\frac{16}{63}$
75.  $-\cos \theta$
77.  $\tan \theta$
79.  $\sin \theta$
81. identity
83. identity

### Prepare for This Section (3.3), page 236

- PS1.**  $2 \sin \alpha \cos \alpha$     **PS2.**  $\cos^2 \alpha - \sin^2 \alpha$     **PS3.**  $\frac{2 \tan \alpha}{1 - \tan^2 \alpha}$     **PS4.** For each of the given values of  $\alpha$ , the function values are equal.    **PS5.** Let  $\alpha = 45^\circ$ ; then the left side of the equation is 1, and the right side of the equation is  $\sqrt{2}$ .    **PS6.** Let  $\alpha = 60^\circ$ ; then the left side of the equation is  $\frac{\sqrt{3}}{2}$ , and the right side of the equation is  $\frac{1}{4}$ .

### Exercise Set 3.3, page 243

1.  $\sin 4\alpha$
3.  $\cos 10\beta$
5.  $\cos 6\alpha$
7.  $\tan 6\alpha$
9.  $\sin 2\alpha = -\frac{24}{25}$ ,  $\cos 2\alpha = \frac{7}{25}$ ,  $\tan 2\alpha = -\frac{24}{7}$
11.  $\sin 2\alpha = -\frac{240}{289}$ ,  $\cos 2\alpha = \frac{161}{289}$ ,  $\tan 2\alpha = -\frac{240}{161}$

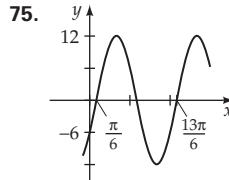
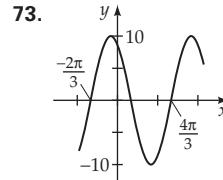
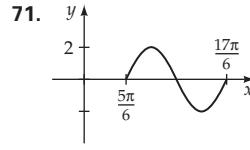
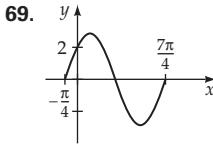
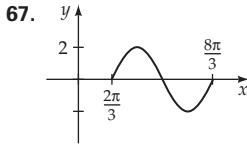
13.  $\sin 2\alpha = -\frac{336}{625}$ ,  $\cos 2\alpha = -\frac{527}{625}$ ,  $\tan 2\alpha = \frac{336}{527}$     15.  $\sin 2\alpha = \frac{240}{289}$ ,  $\cos 2\alpha = -\frac{161}{289}$ ,  $\tan 2\alpha = -\frac{240}{161}$   
 17.  $\sin 2\alpha = -\frac{720}{1681}$ ,  $\cos 2\alpha = \frac{1519}{1681}$ ,  $\tan 2\alpha = -\frac{720}{1519}$     19.  $3(1 + \cos 2x)$     21.  $\frac{1}{8}(3 + 4 \cos 2x + \cos 4x)$   
 23.  $\frac{1}{16}(1 - \cos 2x - \cos 4x + \cos 2x \cos 4x)$     25.  $\frac{\sqrt{2 + \sqrt{3}}}{2}$     27.  $\sqrt{2} + 1$     29.  $-\frac{\sqrt{2 + \sqrt{2}}}{2}$     31.  $\frac{\sqrt{2 - \sqrt{2}}}{2}$     33.  $\frac{\sqrt{2 - \sqrt{2}}}{2}$   
 35.  $\frac{\sqrt{2 - \sqrt{3}}}{2}$     37.  $\sin \frac{\alpha}{2} = \frac{5\sqrt{26}}{26}$ ,  $\cos \frac{\alpha}{2} = \frac{\sqrt{26}}{26}$ ,  $\tan \frac{\alpha}{2} = 5$     39.  $\sin \frac{\alpha}{2} = \frac{5\sqrt{34}}{34}$ ,  $\cos \frac{\alpha}{2} = -\frac{3\sqrt{34}}{34}$ ,  $\tan \frac{\alpha}{2} = -\frac{5}{3}$   
 41.  $\sin \frac{\alpha}{2} = \frac{\sqrt{5}}{5}$ ,  $\cos \frac{\alpha}{2} = \frac{2\sqrt{5}}{5}$ ,  $\tan \frac{\alpha}{2} = \frac{1}{2}$     43.  $\sin \frac{\alpha}{2} = \frac{\sqrt{2}}{10}$ ,  $\cos \frac{\alpha}{2} = -\frac{7\sqrt{2}}{10}$ ,  $\tan \frac{\alpha}{2} = -\frac{1}{7}$   
 91. a.  $\frac{2}{\sqrt{2 - \sqrt{2}}} \approx 2.61$     b.  $\alpha = 2 \sin^{-1}\left(\frac{1}{M}\right)$     c.  $\alpha$  decreases.    93. identity    95. identity

### Prepare for This Section (3.4), page 246

- PS1.  $\sin \alpha \cos \beta$     PS2.  $\cos \alpha \cos \beta$     PS3. Both function values equal  $-\frac{1}{2}$ .    PS4.  $\sin x + \cos x$     PS5. Answers will vary.    PS6. 2

### Exercise Set 3.4, page 251

1.  $\sin 3x - \sin x$     3.  $\frac{1}{2}(\sin 8x - \sin 4x)$     5.  $\sin 8x + \sin 2x$     7.  $\frac{1}{2}(\cos 4x - \cos 6x)$     9.  $\frac{1}{4}$     11.  $-\frac{\sqrt{2}}{4}$     13.  $-\frac{1}{4}$     15.  $\frac{\sqrt{3} - 2}{4}$   
 17.  $2 \sin 3\theta \cos \theta$     19.  $2 \cos 2\theta \cos \theta$     21.  $-2 \sin 4\theta \sin 2\theta$     23.  $2 \cos 4\theta \cos 3\theta$     25.  $2 \sin 7\theta \cos 2\theta$     27.  $-2 \sin \frac{3}{2}\theta \sin \frac{1}{2}\theta$   
 29.  $2 \sin \frac{3}{4}\theta \sin \frac{\theta}{4}$     31.  $2 \cos \frac{5}{12}\theta \sin \frac{1}{12}\theta$     49.  $y = \sqrt{2} \sin(x - 135^\circ)$     51.  $y = \sin(x - 60^\circ)$     53.  $y = \frac{\sqrt{2}}{2} \sin(x - 45^\circ)$   
 55.  $y = 3\sqrt{2} \sin(x + 135^\circ)$     57.  $y = \pi\sqrt{2} \sin(x - 45^\circ)$     59.  $y = \sqrt{2} \sin\left(x + \frac{3\pi}{4}\right)$     61.  $y = \sin\left(x + \frac{\pi}{6}\right)$     63.  $y = 20 \sin\left(x + \frac{2\pi}{3}\right)$   
 65.  $y = 5\sqrt{2} \sin\left(x + \frac{3\pi}{4}\right)$



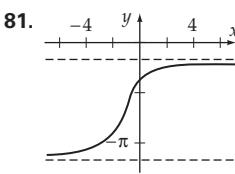
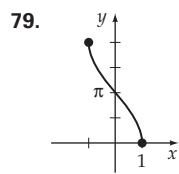
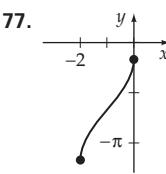
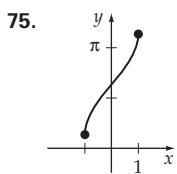
77. a.  $p(t) = \sin(2\pi \cdot 1336t) + \sin(2\pi \cdot 770t)$     b.  $p(t) = 2 \sin(2106\pi t) \sin(566\pi t)$     c. 1053 cycles per second    79. identity    81. identity  
 83. identity

### Prepare for This Section (3.5), page 255

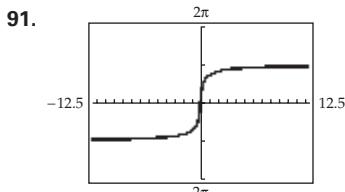
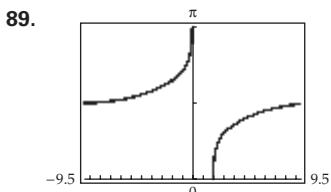
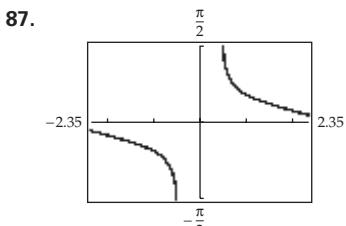
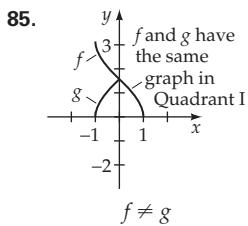
- PS1. A one-to-one function is a function for which each range value ( $y$  value) is paired with one and only one domain value ( $x$  value).  
 PS2. If every horizontal line intersects the graph of a function at most once, then the function is a one-to-one function.  
 PS3.  $f[g(x)] = x$     PS4.  $f[f^{-1}(x)] = x$     PS5. The graph of  $f^{-1}$  is the reflection of the graph of  $f$  across the line given by  $y = x$ .    PS6. No

### Exercise Set 3.5, page 265

1.  $\frac{\pi}{2}$     3.  $\frac{5\pi}{6}$     5.  $-\frac{\pi}{4}$     7.  $\frac{\pi}{3}$     9.  $\frac{\pi}{3}$     11.  $-\frac{\pi}{4}$     13.  $-\frac{\pi}{3}$     15.  $\frac{2\pi}{3}$     17.  $\frac{\pi}{6}$     19. a. 1.0014    b. 0.2341    21. a. 1.1102    b. 0.2818  
 23.  $\theta = \cos^{-1}\left(\frac{x}{7}\right)$     25.  $\frac{1}{2}$     27. 2    29.  $\frac{3}{5}$     31. 1    33.  $\frac{1}{2}$     35.  $\frac{\pi}{6}$     37.  $\frac{\pi}{4}$     39. not defined    41. 0.4636    43.  $-\frac{\pi}{6}$     45.  $\frac{\sqrt{3}}{3}$   
 47.  $\frac{4\sqrt{15}}{15}$     49.  $\frac{24}{25}$     51. 0    53.  $\frac{24}{25}$     55.  $\frac{2 + \sqrt{15}}{6}$     57.  $\frac{1}{5}(3\sqrt{7} - 4\sqrt{3})$     59.  $\frac{12}{13}$     61. 2    63.  $\frac{2 - \sqrt{2}}{2}$     65.  $\frac{7\sqrt{2}}{10}$   
 67.  $\cos \frac{5\pi}{12} \approx 0.2588$     69.  $\frac{\sqrt{1 - x^2}}{x}$



83. a.  $s = 3960 \cos^{-1} \left( \frac{3960}{a + 3960} \right)$  b. 17,930 miles



97.  $y = \frac{1}{3} \tan 5x$  99.  $y = 3 + \cos \left( x - \frac{\pi}{3} \right)$

### Prepare for This Section (3.6), page 268

PS1.  $x = \frac{5 \pm \sqrt{73}}{6}$

PS2.  $1 - \cos^2 x$

PS3.  $\frac{5}{2}\pi, \frac{9}{2}\pi$ , and  $\frac{13}{2}\pi$

PS4.  $(x+1) \left( x - \frac{\sqrt{3}}{2} \right)$

PS5.

PS6. 0, 1

### Exercise Set 3.6, page 278

1.  $\frac{\pi}{4}, \frac{7\pi}{4}$  3.  $\frac{\pi}{3}, \frac{4\pi}{3}$  5.  $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}$  7.  $\frac{\pi}{2}, \frac{3\pi}{2}$  9.  $\frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{6}$  11.  $\frac{\pi}{4}, \frac{3\pi}{4}$  13.  $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$  15.  $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

17.  $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$  19.  $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}$  21.  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$  23.  $41.4^\circ, 318.6^\circ$  25. no solution 27.  $68.0^\circ, 292.0^\circ$  29. no solution

31.  $12.8^\circ, 167.2^\circ$  33.  $15.5^\circ, 164.5^\circ$  35.  $0^\circ, 33.7^\circ, 180^\circ, 213.7^\circ$  37. no solution 39. no solution 41.  $0^\circ, 120^\circ, 240^\circ$  43.  $70.5^\circ, 289.5^\circ$

45.  $68.2^\circ, 116.6^\circ, 248.2^\circ, 296.6^\circ$  47.  $19.5^\circ, 90^\circ, 160.5^\circ, 270^\circ$  49.  $60^\circ, 90^\circ, 300^\circ$  51.  $53.1^\circ, 180^\circ$  53.  $72.4^\circ, 220.2^\circ$  55.  $50.1^\circ, 129.9^\circ, 205.7^\circ, 334.3^\circ$

57. no solution 59.  $22.5^\circ, 157.5^\circ$  61.  $\frac{\pi}{8} + \frac{k\pi}{2}$ , where  $k$  is an integer 63.  $\frac{\pi}{10} + \frac{2k\pi}{5}$ , where  $k$  is an integer 65.  $0 + 2k\pi$ ,

$\frac{\pi}{3} + 2k\pi, \pi + 2k\pi, \frac{5\pi}{3} + 2k\pi$ , where  $k$  is an integer 67.  $\frac{\pi}{2} + k\pi, \frac{5\pi}{6} + k\pi$ , where  $k$  is an integer 69.  $0 + 2k\pi$ , where  $k$  is an integer

71.  $0, \pi$  73.  $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$  75.  $0, \frac{\pi}{2}, \frac{3\pi}{2}$  77.  $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$  79.  $\frac{4\pi}{3}, \frac{5\pi}{3}$  81.  $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$

83.  $\frac{\pi}{6}, \frac{5\pi}{6}, \pi$  85. 0.7391 87.  $-3.2957, 3.2957$  89. 1.16 91.  $14.99^\circ$  and  $75.01^\circ$

The sine regression functions in Exercises 93, 95, and 97 were obtained on a TI-83/TI-83 Plus/TI-84 Plus calculator by using an iteration factor of 16. The use of a different iteration factor may produce a sine regression function that varies from the regression functions listed below.

93. a.  $y \approx 1.1213 \sin(0.01595x + 1.8362) + 6.6257$  b.  $6:49$  95. a.  $y \approx 49.2125 \sin(0.2130x - 1.4576) + 48.0550$  b.  $3\%$

97. a.  $y \approx 35.185 \sin(0.30395x - 2.1630) + 2.1515$  b.  $24.8^\circ$  99. b.  $42^\circ$  and  $79^\circ$  c.  $60^\circ$  101. 0.93 foot, 1.39 feet

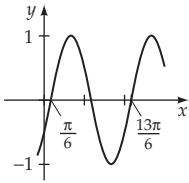
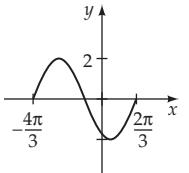
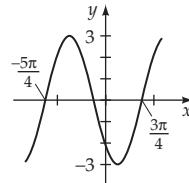
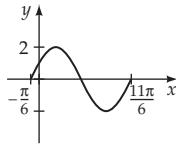
103.  $\frac{\pi}{6}, \frac{\pi}{2}$  105.  $\frac{5\pi}{3}, 0$  107.  $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

### Chapter 3 Assessing Concepts, page 286

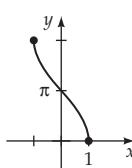
1. True
2. False
3. False
4. True
5. 4
6.  $-\frac{1}{2} \leq x \leq \frac{1}{2}$
7.  $0 \leq y \leq \pi$
8.  $\frac{\pi}{3}$
9.  $\sqrt{2}$
10.  $2 + \sqrt{3}$

## Chapter 3 Review Exercises, page 286

1.  $\frac{\sqrt{6} - \sqrt{2}}{4}$  [3.2]    2.  $\sqrt{3} - 2$  [3.2]    3.  $\frac{\sqrt{6} - \sqrt{2}}{4}$  [3.2]    4.  $\sqrt{2} - \sqrt{6}$  [3.2]    5.  $-\frac{\sqrt{6} + \sqrt{2}}{4}$  [3.2]    6.  $\frac{\sqrt{2} + \sqrt{6}}{4}$  [3.2]    7.  $\frac{\sqrt{2} - \sqrt{2}}{2}$  [3.3]
8.  $-\frac{\sqrt{2} - \sqrt{3}}{2}$  [3.3]    9.  $\sqrt{2} + 1$  [3.3]    10.  $\frac{\sqrt{2} + \sqrt{2}}{2}$  [3.3]    11. a. 0    b.  $\sqrt{3}$     c.  $\frac{1}{2}$  [3.2/3.3]    12. a. 0    b.  $-2$     c.  $\frac{1}{2}$  [3.2/3.3]
13. a.  $\frac{\sqrt{3}}{2}$     b.  $-\sqrt{3}$     c.  $-\frac{\sqrt{2} - \sqrt{3}}{2}$  [3.2/3.3]    14. a.  $\frac{\sqrt{6} - \sqrt{2}}{4}$     b.  $-\sqrt{3}$     c. 1 [3.2/3.3]    15.  $\sin 6x$  [3.3]    16.  $\tan 3x$  [3.2]
17.  $\sin 3x$  [3.2]    18.  $\cos 4\theta$  [3.3]    19.  $\tan 2\theta$  [3.1]    20.  $\tan \theta$  [3.3]    21.  $2 \sin 3\theta \sin \theta$  [3.4]    22.  $-2 \cos 4\theta \sin \theta$  [3.4]
23.  $2 \sin 4\theta \cos 2\theta$  [3.4]    24.  $2 \cos 3\theta \sin 2\theta$  [3.4]    43.  $\frac{13}{5}$  [3.5]    44.  $\frac{4}{5}$  [3.5]    45.  $\frac{56}{65}$  [3.5]    46.  $\frac{7}{25}$  [3.5]    47.  $\frac{3}{2}$  [3.6]    48.  $\frac{4}{5}$  [3.6]
49.  $30^\circ, 150^\circ, 240^\circ, 300^\circ$  [3.6]    50.  $0^\circ, 45^\circ, 135^\circ$  [3.6]    51.  $\frac{\pi}{2} + 2k\pi, 3.8713 + 2k\pi, 5.553 + 2k\pi$ , where  $k$  is an integer. [3.6]    52.  $-\frac{\pi}{4} + k\pi,$   
1.2490 +  $k\pi$ , where  $k$  is an integer. [3.6]    53.  $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$  [3.6]    54.  $\frac{7\pi}{12}, \frac{19\pi}{12}, \frac{3\pi}{4}, \frac{7\pi}{4}$  [3.6]
55.  $y = 2 \sin\left(x + \frac{\pi}{6}\right)$  [3.4]    56.  $y = 2\sqrt{2} \sin\left(x + \frac{5\pi}{4}\right)$  [3.4]    57.  $y = 2 \sin\left(x + \frac{4\pi}{3}\right)$  [3.4]    58.  $y = \sin\left(x - \frac{\pi}{6}\right)$  [3.4]

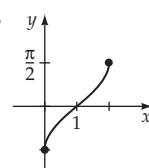


59.



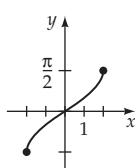
[3.5]

60.



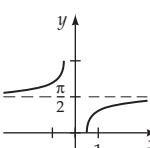
[3.5]

61.



[3.5]

62.

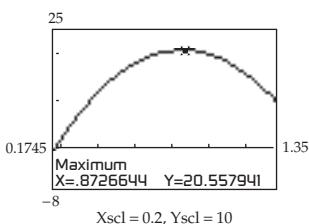


[3.5]

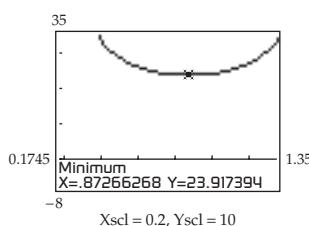
63. a.  $y \approx 1.1835 \sin(0.01600x + 1.8497) + 6.4394$     b. 6:01 [3.6]

## Chapter 3 Quantitative Reasoning Exercises, page 288

QR1. a.

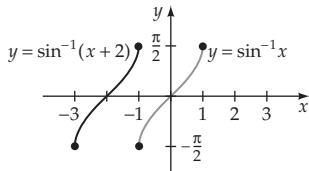
 $\alpha \approx 0.87266$ 

QR2. a.

 $\alpha \approx 0.87266$ 

## Chapter 3 Test, page 289

5.  $\frac{-\sqrt{6} + \sqrt{2}}{4}$  [3.2]    6.  $-\frac{\sqrt{2}}{10}$  [3.2]    8.  $\sin 9x$  [3.2]    9.  $-\frac{7}{25}$  [3.3]    12.  $\frac{2 - \sqrt{3}}{4}$  [3.4]    13.  $y = \sin\left(x + \frac{5\pi}{6}\right)$  [3.4]    14. 0.701 [3.5]
15.  $\frac{5}{13}$  [3.5]    16. [3.5]    17.  $41.8^\circ, 138.2^\circ$  [3.6]    18.  $0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}$  [3.6]    19.  $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$  [3.6]

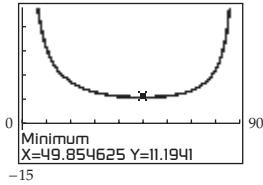


20. a.  $y \approx 1.7569 \sin(0.01675x + 1.3056) + 12.1100$     b. 12 hours 1 minute [3.6]

**Cumulative Review Exercises, page 290**

- 1.**  $x > -3$  [1.1]   **2.** Shift the graph of  $y = f(x)$  horizontally 1 unit to the left and up 2 units. [1.4]   **3.** Reflect the graph of  $y = f(x)$  across the  $x$ -axis. [1.4]   **4.** odd function [1.4/2.5]   **5.**  $f^{-1}(x) = \frac{x}{x-5}$  [1.6]   **6.**  $\frac{4\pi}{3}$  [2.1]   **7.**  $300^\circ$  [2.1]   **8.**  $\frac{\sqrt{3}}{2}$  [2.2]   **9.**  $\frac{2\sqrt{3}}{3}$  [2.2]   **10.**  $\frac{2\sqrt{5}}{5}$  [2.2]
- 11.** positive [2.3]   **12.**  $50^\circ$  [2.3]   **13.**  $\frac{\pi}{3}$  [2.3]   **14.**  $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$  [2.4]   **15.**  $0.43, \pi, \frac{\pi}{12}$  [2.7]   **16.**  $\frac{\pi}{6}$  [3.5]   **17.** 2.498 [3.5]
- 18.**  $[-1, 1]$  [3.5]   **19.**  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  [3.5]   **20.**  $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$  [3.6]

**Exercise Set 4.1, page 298**

- 1.**  $C = 77^\circ, b \approx 16, c \approx 17$    **3.**  $B = 38^\circ, a \approx 18, c \approx 10$    **5.**  $C \approx 15^\circ, B \approx 33^\circ, c \approx 7.8$    **7.**  $C = 45.1^\circ, b \approx 39.4, c \approx 30.2$    **9.**  $C = 32.6^\circ, c \approx 21.6, a \approx 39.8$    **11.**  $B = 47.7^\circ, a \approx 57.4, b \approx 76.3$    **13.**  $A \approx 58.5^\circ, B \approx 7.3^\circ, a \approx 81.5$    **15.**  $C = 59^\circ, B = 84^\circ, b \approx 46$  or  $C = 121^\circ, B = 22^\circ, b \approx 17$    **17.** No triangle is formed.   **19.** No triangle is formed.   **21.**  $C = 19.8^\circ, B = 145.4^\circ, b \approx 10.7$  or  $C = 160.2^\circ, B = 5.0^\circ, b \approx 1.64$    **23.** No triangle is formed.   **25.**  $C = 51.21^\circ, A = 11.47^\circ, c \approx 59.00$    **27.**  $B \approx 130.9^\circ, C \approx 28.6^\circ, b \approx 22.2$  or  $B \approx 8.1^\circ, C \approx 151.4^\circ, b \approx 4.17$    **29.**  $\approx 68.8$  miles   **31.** 231 yards   **33.**  $\approx 110$  feet   **35.** 4840 feet   **37.**  $\approx 96$  feet   **39.**  $\approx 33$  feet   **41.**  $\approx 8.1$  miles   **43.**  $\approx 1200$  miles   **45.**  $\approx 260$  meters   **49.** 
- minimum value of  $L \approx 11.19$  meters

**Prepare for This Section (4.2), page 302**

- PS1.** 20.7   **PS2.** 25.5 square inches   **PS3.**  $C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$    **PS4.** 12.5 meters   **PS5.** 6   **PS6.**  $c^2 = a^2 + b^2$

**Exercise Set 4.2, page 308**

- 1.**  $\approx 13$    **3.**  $\approx 150$    **5.**  $\approx 29$    **7.**  $\approx 9.5$    **9.**  $\approx 10$    **11.**  $\approx 40.1$    **13.**  $\approx 90.7$    **15.**  $\approx 39^\circ$    **17.**  $\approx 90^\circ$    **19.**  $\approx 47.9^\circ$    **21.**  $\approx 116.67^\circ$    **23.**  $\approx 80.3^\circ$    **25.**  $a \approx 11.1, B \approx 62.0^\circ, C \approx 78.6^\circ$    **27.**  $A \approx 34.2^\circ, B \approx 104.6^\circ, C \approx 41.3^\circ$    **29.**  $\approx 140$  square units   **31.**  $\approx 53$  square units   **33.**  $\approx 81$  square units   **35.**  $\approx 299$  square units   **37.**  $\approx 36$  square units   **39.**  $\approx 7.3$  square units   **41.**  $\approx 710$  miles   **43.**  $\approx 74$  feet   **45.**  $\approx 60.9^\circ$    **47.**  $\approx 350$  miles   **49.** 40 centimeters   **51.**  $\approx 2800$  feet   **53.** 402 miles, S62°E   **55.**  $\approx 47,500$  square meters   **57.** 162 square inches   **59.**  $\approx \$41,000$    **61.**  $\approx 6.23$  acres   **63.** Triangle DEF has an incorrect dimension.   **65.**  $\approx 12.5^\circ$    **69.**  $\approx 140$  cubic inches

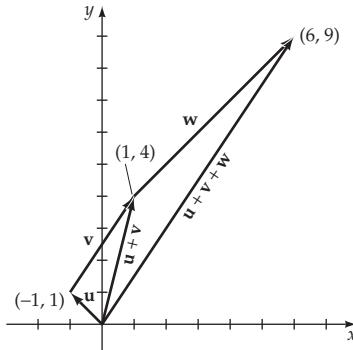
**Prepare for This Section (4.3), page 312**

- PS1.** 1   **PS2.**  $-6.691$    **PS3.**  $30^\circ$    **PS4.**  $157.6^\circ$    **PS5.**  $\frac{\sqrt{5}}{5}$    **PS6.**  $\frac{14\sqrt{17}}{17}$

**Exercise Set 4.3, page 325**

- 1.**  $a = 4, b = 2; \langle 4, 2 \rangle$    **3.**  $a = -5, b = 4; \langle -5, 4 \rangle$    **5.**  $a = 7, b = -1; \langle 7, -1 \rangle$    **7.**  $a = -7, b = -5; \langle -7, -5 \rangle$    **9.**  $a = 0, b = 8; \langle 0, 8 \rangle$    **11.** 5,  $\approx 126.9^\circ, \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$    **13.**  $\approx 44.7, \approx 296.6^\circ, \left\langle \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5} \right\rangle$    **15.**  $\approx 4.5, \approx 296.6^\circ, \left\langle \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5} \right\rangle$    **17.**  $\approx 45.7, \approx 336.8^\circ, \left\langle \frac{7\sqrt{58}}{58}, -\frac{3\sqrt{58}}{58} \right\rangle$    **19.**  $\langle -6, 12 \rangle$    **21.**  $\langle -1, 10 \rangle$    **23.**  $\left\langle -\frac{11}{6}, \frac{7}{3} \right\rangle$    **25.**  $2\sqrt{5}$    **27.**  $2\sqrt{109}$    **29.**  $-8i + 12j$    **31.**  $14i - 6j$    **33.**  $\frac{11}{12}i + \frac{1}{2}j$    **35.**  $\sqrt{113}$    **37.**  $a_1 \approx 4.5, a_2 \approx 2.3, 4.5i + 2.3j$    **39.**  $a_1 \approx 2.8, a_2 \approx 2.8, 2.8i + 2.8j$    **41.**  $\approx 380$  miles per hour   **43.**  $\approx 250$  miles per hour at a heading of  $86^\circ$    **45.** 293 pounds   **47.** **a.** 131 pounds   **b.** 319 pounds   **49.** The forces are in equilibrium.   **51.** The forces are not in equilibrium.  $F_4 = 0i + 10j$    **53.** The forces are in equilibrium.   **55.**  $-3$    **57.** 0   **59.** 1   **61.** 0   **63.**  $\approx 79.7^\circ$    **65.**  $45^\circ$    **67.**  $90^\circ$ , orthogonal
- 69.**  $180^\circ$    **71.**  $\frac{46}{5}$    **73.**  $\frac{14\sqrt{29}}{29} \approx 2.6$    **75.**  $\sqrt{5} \approx 2.2$    **77.**  $-\frac{11\sqrt{5}}{5} \approx -4.9$    **79.**  $\approx 954$  foot-pounds   **81.**  $\approx 779$  foot-pounds

83.

85. the vector from  $P_1(3, -1)$  to  $P_2(5, -4)$ 87. Because  $\mathbf{v} \cdot \mathbf{w} = 0$ , the vectors are perpendicular. 89.  $\langle 7, 2 \rangle$  is one example. 91. No 95. The same amount of work is done.

## Chapter 4 Assessing Concepts, page 329

1. a triangle that does not contain a right angle 2. the Law of Cosines 3. SSA 4. the semiperimeter of a triangle 5. a scalar  
 6. a scalar 7. True 8. False 9. True 10. True

## Chapter 4 Review Exercises, page 329

1.  $B = 51^\circ, a \approx 11, c \approx 18$  [4.1] 2.  $A = 8.6^\circ, a \approx 1.77, b \approx 11.5$  [4.1] 3.  $B \approx 48^\circ, C \approx 95^\circ, A \approx 37^\circ$  [4.2] 4.  $A \approx 47^\circ, B \approx 76^\circ, C \approx 58^\circ$  [4.2]  
 5.  $c \approx 13, A \approx 55^\circ, B \approx 90^\circ$  [4.2] 6.  $a \approx 169, B \approx 37^\circ, C \approx 61^\circ$  [4.2] 7. No triangle is formed. [4.1] 8. No triangle is formed. [4.1]  
 9.  $C = 45^\circ, a \approx 29, b \approx 35$  [4.1] 10.  $A = 115^\circ, a \approx 56, b \approx 26$  [4.1] 11.  $\approx 360$  square units [4.2] 12.  $\approx 31$  square units [4.2]  
 13.  $\approx 920$  square units [4.2] 14.  $\approx 46$  square units [4.2] 15.  $\approx 790$  square units [4.2] 16.  $\approx 210$  square units [4.2]  
 17.  $\approx 170$  square units [4.2] 18.  $\approx 140$  square units [4.2] 19.  $a_1 = 5, a_2 = 3, \langle 5, 3 \rangle$  [4.3] 20.  $a_1 = 1, a_2 = 6, \langle 1, 6 \rangle$  [4.3]  
 21.  $\approx 4.5, 153.4^\circ$  [4.3] 22.  $\approx 6.7, 333.4^\circ$  [4.3] 23.  $\approx 3.6, 123.7^\circ$  [4.3] 24.  $\approx 8.1, 240.3^\circ$  [4.3] 25.  $\left\langle -\frac{8\sqrt{89}}{89}, \frac{5\sqrt{89}}{89} \right\rangle$  [4.3]  
 26.  $\left\langle \frac{7\sqrt{193}}{193}, -\frac{12\sqrt{193}}{193} \right\rangle$  [4.3] 27.  $\frac{5\sqrt{26}}{26}\mathbf{i} + \frac{\sqrt{26}}{26}\mathbf{j}$  [4.3] 28.  $\frac{3\sqrt{34}}{34}\mathbf{i} - \frac{5\sqrt{34}}{34}\mathbf{j}$  [4.3] 29.  $\langle -7, -3 \rangle$  [4.3] 30.  $\langle 18, 7 \rangle$  [4.3]  
 31.  $-6\mathbf{i} - \frac{17}{2}\mathbf{j}$  [4.3] 32.  $-\frac{13}{6}\mathbf{i} - \frac{47}{6}\mathbf{j}$  [4.3] 33. 420 mph [4.3] 34.  $\approx 7^\circ$  [4.3] 35. 18 [4.3] 36.  $-21$  [4.3] 37.  $-9$  [4.3]  
 38. 20 [4.3] 39.  $\approx 86^\circ$  [4.3] 40.  $\approx 138^\circ$  [4.3] 41.  $\approx 125^\circ$  [4.3] 42.  $\approx 157^\circ$  [4.3] 43.  $\frac{10\sqrt{41}}{41}$  [4.3] 44.  $\frac{27\sqrt{29}}{29}$  [4.3]

45.  $\approx 662$  foot-pounds [4.3]

## Chapter 4 Quantitative Reasoning Exercises, page 330

- QR1. 2210 miles QR2.  $289^\circ$  QR3. 1620 miles QR4.  $82^\circ$

## Chapter 4 Test, page 332

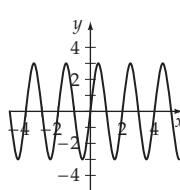
1.  $B = 94^\circ, a \approx 48, b \approx 51$  [4.1] 2.  $\approx 11^\circ$  [4.1] 3.  $\approx 14$  [4.2] 4.  $\approx 48^\circ$  [4.2] 5.  $\approx 39$  square units [4.2] 6.  $\approx 93$  square units [4.2]  
 7.  $\approx 260$  square units [4.2] 8.  $\sqrt{13}$  [4.3] 9.  $-9.193\mathbf{i} - 7.713\mathbf{j}$  [4.3] 10.  $-19\mathbf{i} - 29\mathbf{j}$  [4.3] 11.  $-1$  [4.3] 12.  $103^\circ$  [4.3]  
 13.  $\approx 27$  miles [4.2] 14.  $\approx 21$  miles [4.1] 15.  $\approx \$65,800$  [4.2]

## Cumulative Review Exercises, page 332

1.  $\sqrt{74}$  units [1.2] 2.  $\sin x + \cos x$  [1.5] 3.  $\sec(\cos x)$  [1.5] 4.  $f^{-1}(x) = 2x + 6$  [1.6] 5. shifted 2 units to the right and 3 units up [1.4]

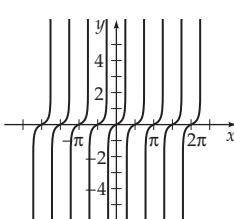
6. 33 centimeters [2.2]

7.



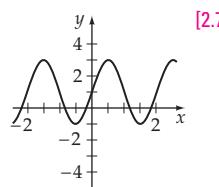
[2.5]

8.



[2.6]

9.



[2.7]

10. amplitude: 3; period:  $6\pi$ , phase shift:  $\frac{3\pi}{2}$  [2.7]    11. amplitude:  $\sqrt{2}$ , period:  $2\pi$ , phase shift:  $\frac{\pi}{4}$  [2.7]    12. 9.5 centimeters [4.2]  
 13. See [3.1]    14.  $\frac{\pi}{3}$  [3.5]    15.  $\frac{5}{12}$  [3.5]    16.  $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$  [3.6]    17. magnitude: 5; angle:  $323.1^\circ$  [4.3]    18.  $60.3^\circ$  [4.3]    19.  $289.5^\circ$  [4.3]  
 20. ground speed: 592 mph, heading:  $67.4^\circ$  [4.3]

### Exercise Set 5.1, page 340

1.  $9i$     3.  $7i\sqrt{2}$     5.  $4 + 9i$     7.  $5 + 7i$     9.  $8 - 3i\sqrt{2}$     11.  $11 - 5i$     13.  $-7 + 4i$     15.  $8 - 5i$     17.  $-10$     19.  $-2 + 16i$     21.  $-40$   
 23.  $-10$     25.  $19i$     27.  $20 - 10i$     29.  $22 - 29i$     31.  $41$     33.  $12 - 5i$     35.  $-114 + 42i\sqrt{2}$     37.  $-6i$     39.  $3 - 6i$     41.  $\frac{7}{53} - \frac{2}{53}i$   
 43.  $1 + i$     45.  $\frac{15}{41} - \frac{29}{41}i$     47.  $\frac{5}{13} + \frac{12}{13}i$     49.  $2 + 5i$     51.  $-16 - 30i$     53.  $-11 - 2i$     55.  $-i$     57.  $-1$     59.  $-i$     61.  $-1$   
 63.  $z_1 = 2 + 4i, z_2 = 6 + 7i, z_3 = 10 + 10i, z_4 = 14 + 13i, z_5 = 18 + 16i$     65.  $z_1 = 1 + i, z_2 = -1 + i, z_3 = -1 - i, z_4 = 1 - i, z_5 = z_1, z_6 = z_2, z_7 = z_3, z_8 = z_4$     67.  $\frac{1}{2} + \frac{\sqrt{3}}{2}i$     69.  $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$   
 71.  $\frac{1}{2} + \frac{1}{2}i$     73.  $(x + 4i)(x - 4i)$     75.  $(z + 5i)(z - 5i)$     77.  $(2x + 9i)(2x - 9i)$     83.  $0$

### Prepare for This Section (5.2), page 343

- PS1.  $1 + 3i$     PS2.  $\frac{1}{2} + \frac{1}{2}i$     PS3.  $2 - 3i$     PS4.  $3 + 5i$     PS5.  $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$     PS6.  $-3i, 3i$

### Exercise Set 5.2, page 349

- 1–7.
- 
9.  $\sqrt{2} \text{ cis } 315^\circ$     11.  $2 \text{ cis } 330^\circ$     13.  $3 \text{ cis } 90^\circ$     15.  $5 \text{ cis } 180^\circ$     17.  $16 \text{ cis } 120^\circ$     19.  $4 \text{ cis } 240^\circ$   
 21.  $\sqrt{2} + i\sqrt{2}$     23.  $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$     25.  $-3\sqrt{2} + 3i\sqrt{2}$     27.  $8$     29.  $-\sqrt{3} + i$     31.  $-3i$     33.  $-4\sqrt{2} + 4i\sqrt{2}$   
 35.  $\frac{9\sqrt{3}}{2} - \frac{9}{2}i$     37.  $\approx -0.832 + 1.819i$     39.  $6 \text{ cis } 255^\circ$     41.  $12 \text{ cis } 335^\circ$     43.  $10 \text{ cis } \frac{16\pi}{15}$     45.  $24 \text{ cis } 6.5$   
 47.  $-4 - 4i\sqrt{3}$     49.  $3i$     51.  $-\frac{3\sqrt{3}}{2} + \frac{3i}{2}$     53.  $\approx -2.081 + 4.546i$     55.  $\approx 2.732 - 0.732i$     57.  $6 + 0i = 6$   
 59.  $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$     61.  $0 - \sqrt{2}i = -\sqrt{2}i$     63.  $16 - 16i$     65.  $-\frac{3}{8} + \frac{\sqrt{3}}{8}i$     67.  $59.0 + 43.0i$     69.  $r^2 \text{ or } a^2 + b^2$

### Prepare for This Section (5.3), page 350

- PS1.  $i$     PS2.  $2$     PS3.  $3$     PS4.  $2\sqrt{2} \text{ cis } \frac{\pi}{4}$     PS5.  $-\sqrt{3} + i$     PS6.  $1$

### Exercise Set 5.3, page 354

1.  $-128 - 128i\sqrt{3}$     3.  $-16 + 16i\sqrt{3}$     5.  $16\sqrt{2} + 16i\sqrt{2}$     7.  $64 + 0i = 64$     9.  $0 - 32i = -32i$     11.  $-4 + 0i = -4$     13.  $1024 - 1024i$   
 15.  $0 - 1i = -i$     17.  $3 + 0i = 3$     19.  $2 + 0i = 2$     21.  $0.809 + 0.588i$     23.  $1 + 0i = 1$     25.  $1.070 + 0.213i$     27.  $-0.276 + 1.563i$   
 $-3 + 0i = -3$      $1 + i\sqrt{3}$      $-0.309 + 0.951i$      $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$      $-0.213 + 1.070i$   
 $-1 + i\sqrt{3}$      $-1 + 0i = -1$      $\frac{1}{2} + \frac{i\sqrt{3}}{2}$      $-1.070 - 0.213i$   
 $-2 + 0i = -2$      $-0.309 - 0.951i$      $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$      $1.492 - 0.543i$   
 $-1 - i\sqrt{3}$      $0.809 - 0.588i$      $\frac{1}{2} - \frac{i\sqrt{3}}{2}$      $0.213 - 1.070i$   
 $1 - i\sqrt{3}$
29.  $2\sqrt{2} + 2i\sqrt{6}$     31.  $2 \text{ cis } 60^\circ$     33.  $\text{cis } 67.5^\circ$     35.  $3 \text{ cis } 0^\circ$     37.  $3 \text{ cis } 45^\circ$     39.  $\sqrt[4]{2} \text{ cis } 75^\circ$     41.  $\sqrt[3]{2} \text{ cis } 80^\circ$   
 $-2\sqrt{2} - 2i\sqrt{6}$      $2 \text{ cis } 180^\circ$      $\text{cis } 157.5^\circ$      $3 \text{ cis } 120^\circ$      $3 \text{ cis } 135^\circ$      $\sqrt[4]{2} \text{ cis } 165^\circ$      $\sqrt[3]{2} \text{ cis } 200^\circ$   
 $2 \text{ cis } 300^\circ$      $\text{cis } 247.5^\circ$      $3 \text{ cis } 240^\circ$      $3 \text{ cis } 225^\circ$      $\sqrt[4]{2} \text{ cis } 255^\circ$      $\sqrt[3]{2} \text{ cis } 320^\circ$   
 $\text{cis } 337.5^\circ$      $3 \text{ cis } 315^\circ$      $3 \text{ cis } 345^\circ$

47. For  $n \geq 2$ , the sum of the  $n$ th roots of 1 is 0.

## Chapter 5 Assessing Concepts, page 358

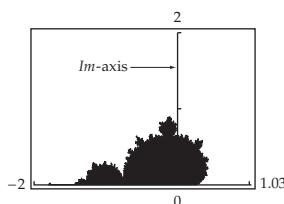
1. True    2. True    3. True    4. False    5. The four roots are equally spaced around a circle with center  $(0, 0)$  and radius 1.    6. 1    7. 5  
8. No    9.  $-3 + 5i$     10. 2

## Chapter 5 Review Exercises, page 358

1.  $3 - 8i, 3 + 8i$  [5.1]    2.  $6 + 2i, 6 - 2i$  [5.1]    3.  $-2 + i\sqrt{5}, -2 - i\sqrt{5}$  [5.1]    4.  $-5 - 3i\sqrt{3}, -5 + 3i\sqrt{3}$  [5.1]    5.  $-4$  [5.1]    6.  $9$  [5.1]  
7.  $5 + 2i$  [5.1]    8.  $-3 + 4i$  [5.1]    9.  $-3 + 3i$  [5.1]    10.  $-5 - 15i$  [5.1]    11.  $25 - 19i$  [5.1]    12.  $29 - 2i$  [5.1]    13.  $\frac{8}{25} - \frac{6}{25}i$  [5.1]  
14.  $\frac{26}{53} + \frac{15}{53}i$  [5.1]    15.  $-2 - 2i$  [5.1]    16.  $2 - 11i$  [5.1]    17.  $6 + 6i$  [5.1]    18.  $-5 - 6i$  [5.1]    19.  $7$  [5.1]    20.  $11 + i\sqrt{5}$  [5.1]  
21.  $-i$  [5.1]    22.  $i$  [5.1]    23.  $1$  [5.1]    24.  $-1$  [5.1]    25.  $8$  [5.1]    26.  $\sqrt{13}$  [5.1]    27.  $\sqrt{41}$  [5.1]    28.  $\sqrt{2}$  [5.1]    29.  $2\sqrt{2} \text{ cis } 315^\circ$  [5.2]  
30.  $2 \text{ cis } 150^\circ$  [5.2]    31.  $\approx \sqrt{13} \text{ cis } 146.3^\circ$  [5.2]    32.  $\sqrt{17} \text{ cis } 345.96^\circ$  [5.2]    33.  $\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i \approx 3.536 - 3.536i$  [5.2]    34.  $-3 - 3i\sqrt{3}$  [5.2]  
35.  $\approx -0.832 + 1.819i$  [5.2]    36.  $-1.27 + 2.72i$  [5.2]    37.  $0 - 30i = -30i$  [5.2]    38.  $-5\sqrt{2} - 5i\sqrt{2}$  [5.2]    39.  $\approx -8.918 + 8.030i$  [5.2]  
40.  $\approx -0.968 + 3.881i$  [5.2]    41.  $\approx -6.012 - 13.742i$  [5.2]    42.  $\approx 13.2 + 27.0i$  [5.2]    43.  $3 \text{ cis } (-100^\circ)$  [5.2]    44.  $3 \text{ cis } 110^\circ$  [5.2]  
45.  $5 \text{ cis } (-59^\circ)$  [5.2]    46.  $\sqrt{2} \text{ cis } (-50^\circ)$  [5.2]    47.  $\frac{5}{3} \text{ cis } 1.9$  [5.2]    48.  $\frac{1}{2} \text{ cis } (-4)$  [5.2]    49.  $-\frac{243\sqrt{2}}{2} - \frac{243\sqrt{2}}{2}i \approx -171.827 - 171.827i$  [5.3]  
50.  $-1$  [5.3]    51.  $64 - 64\sqrt{3}i \approx 64 - 110.851i$  [5.3]    52.  $32,768i$  [5.3]    53.  $-16\sqrt{2} + 16i\sqrt{2} \approx -22.627 + 22.627i$  [5.3]  
54.  $-237 + 3116i$  [5.3]    55.  $3 \text{ cis } 30^\circ, 3 \text{ cis } 150^\circ, 3 \text{ cis } 270^\circ$  [5.3]    56.  $\sqrt[4]{8} \text{ cis } 22.5^\circ, \sqrt[4]{8} \text{ cis } 112.5^\circ, \sqrt[4]{8} \text{ cis } 202.5^\circ, \sqrt[4]{8} \text{ cis } 292.5^\circ$  [5.3]  
57.  $4 \text{ cis } 0^\circ, 4 \text{ cis } 90^\circ, 4 \text{ cis } 180^\circ, 4 \text{ cis } 270^\circ$  [5.3]    58.  $2 \text{ cis } 45^\circ, 2 \text{ cis } 117^\circ, 2 \text{ cis } 189^\circ, 2 \text{ cis } 261^\circ, 2 \text{ cis } 333^\circ$  [5.3]  
59.  $3 \text{ cis } 0^\circ, 3 \text{ cis } 90^\circ, 3 \text{ cis } 180^\circ, 3 \text{ cis } 270^\circ$  [5.3]    60.  $5 \text{ cis } 60^\circ, 5 \text{ cis } 180^\circ, 5 \text{ cis } 300^\circ$  [5.3]

## Chapter 5 Quantitative Reasoning Exercises, page 359

QR1.

QR2.  $-0.25 + 0.25i, -1 + 0.1i$ , and  $0.1 + 0.2i$ QR3.  $-2$  is an element of the Mandelbrot set because all of its iterates equal  $2$ .QR4. The first iterate of  $2i$  is  $z_1 = -4 + 2i$ .  $2i$  is not an element of the Mandelbrot set because  $| -4 + 2i | > 2$ .QR5.  $(Z^2 + S) \rightarrow Z$ 

QR6. Answers will vary.

## Chapter 5 Test, page 360

1.  $6 + 3i$  [5.1]    2.  $3i\sqrt{2}$  [5.1]    3.  $10 - i$  [5.1]    4.  $-9 + 9i$  [5.1]    5.  $-6$  [5.1]    6.  $-i$  [5.1]    7.  $5 + 16i$  [5.1]    8.  $3 - 54i$  [5.1]  
9.  $16 + 30i$  [5.1]    10.  $-5 - 4i$  [5.1]    11.  $-\frac{13}{25} - \frac{34}{25}i$  [5.1]    12.  $2 + 4i$  [5.1]    13.  $\sqrt{34}$  [5.1]    14.  $3\sqrt{2} \text{ cis } 315^\circ$  [5.2]    15.  $6 \text{ cis } 270^\circ$  [5.2]  
16.  $-2 + 2i\sqrt{3}$  [5.2]    17.  $-\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}i}{2}$  [5.2]    18.  $6\sqrt{2} + 6i\sqrt{2}$  [5.2]    19.  $\approx -11,472 + 16,383i$  [5.2]    20.  $-4$  [5.2]    21.  $-6i$  [5.2]  
22.  $16,777,216$  [5.3]    23.  $2 \text{ cis } 0^\circ, 2 \text{ cis } 60^\circ, 2 \text{ cis } 120^\circ, 2 \text{ cis } 180^\circ, 2 \text{ cis } 240^\circ, 2 \text{ cis } 300^\circ$  [5.3]    24.  $\sqrt[3]{2} \text{ cis } 40^\circ, \sqrt[3]{2} \text{ cis } 160^\circ, \sqrt[3]{2} \text{ cis } 280^\circ$  [5.3]  
25.  $2 \text{ cis } 36^\circ, 2 \text{ cis } 108^\circ, 2 \text{ cis } 180^\circ, 2 \text{ cis } 252^\circ, 2 \text{ cis } 324^\circ$  [5.3]

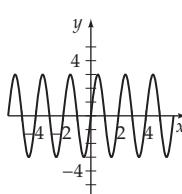
## Cumulative Review Exercises, page 361

1.  $[-2, 3]$  [1.1]    2. All real numbers except  $-2$  and  $2$  [1.3]    3.  $-2$  [1.3]    4.  $(f \circ g)(x) = \sin(x^2 - 1)$  [1.5]    5.  $\frac{3}{2}$  [1.6]    6.  $270^\circ$  [2.1]

7.  $\approx 25$  centimeters [2.2]

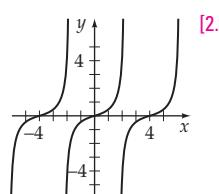
8.  $a = -1, b = 1$  [2.4]

- 9.



- [2.5]

- 10.



- [2.6]

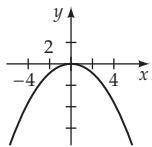
11. See [3.1]    12.  $-\sin x$  [3.2]    13.  $\frac{56}{65}$  [3.2]    14.  $\frac{33}{65}$  [3.5]    15.  $0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}$  [3.6]    16.  $\approx 170$  centimeters [4.2]    17.  $64.7^\circ$  [4.3]

18. 1449 foot-pounds [4.3]    19.  $2\sqrt{2} \text{ cis } 45^\circ$  [5.2]    20.  $-3, \frac{3}{2} + \frac{3\sqrt{3}}{2}i, \frac{3}{2} - \frac{3\sqrt{3}}{2}i$  [5.3]

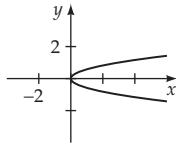
## Exercise Set 6.1, page 370

1. a. iii b. i c. iv d. ii

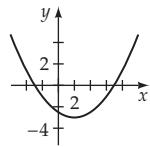
3. vertex:  $(0, 0)$   
focus:  $(0, -1)$   
directrix:  $y = 1$



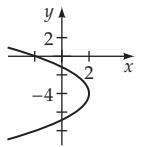
5. vertex:  $(0, 0)$   
focus:  $\left(\frac{1}{12}, 0\right)$   
directrix:  $x = -\frac{1}{12}$



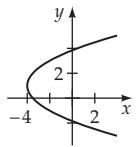
7. vertex:  $(2, -3)$   
focus:  $(2, -1)$   
directrix:  $y = -5$



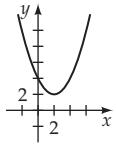
9. vertex:  $(2, -4)$   
focus:  $(1, -4)$   
directrix:  $x = 3$



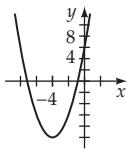
11. vertex:  $(-4, 1)$   
focus:  $\left(-\frac{7}{2}, 1\right)$   
directrix:  $x = -\frac{9}{2}$



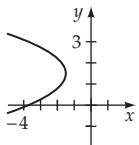
13. vertex:  $(2, 2)$   
focus:  $\left(2, \frac{5}{2}\right)$   
directrix:  $y = \frac{3}{2}$



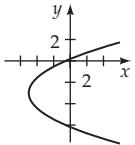
15. vertex:  $(-4, -10)$   
focus:  $\left(-4, -\frac{39}{4}\right)$   
directrix:  $y = -\frac{41}{4}$



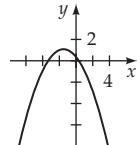
17. vertex:  $\left(-\frac{7}{4}, \frac{3}{2}\right)$   
focus:  $\left(-2, \frac{3}{2}\right)$   
directrix:  $x = -\frac{3}{2}$



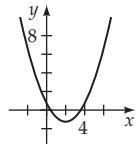
19. vertex:  $(-5, -3)$   
focus:  $\left(-\frac{9}{2}, -3\right)$   
directrix:  $x = -\frac{11}{2}$



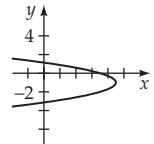
21. vertex:  $\left(-\frac{3}{2}, \frac{13}{12}\right)$   
focus:  $\left(-\frac{3}{2}, \frac{1}{3}\right)$   
directrix:  $y = \frac{11}{6}$



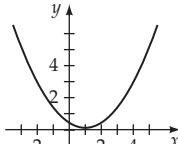
23. vertex:  $\left(2, -\frac{5}{4}\right)$   
focus:  $\left(2, -\frac{3}{4}\right)$   
directrix:  $y = -\frac{7}{4}$



25. vertex:  $\left(\frac{9}{2}, -1\right)$   
focus:  $\left(\frac{35}{8}, -1\right)$   
directrix:  $x = \frac{37}{8}$



27. vertex:  $\left(1, \frac{1}{9}\right)$   
focus:  $\left(1, \frac{31}{36}\right)$   
directrix:  $y = -\frac{23}{36}$



29.  $x^2 = -16y$     31.  $(x + 1)^2 = 4(y - 2)$

33.  $(x - 3)^2 = 4(y + 4)$     35.  $(x + 4)^2 = 4(y - 1)$     37. vertex:  $(250, 20)$ , focus:  $\left(\frac{3240}{13}, 20\right)$     39. on axis of symmetry 4 feet

above vertex

41. 6.0 inches

43. a. 5900 square feet

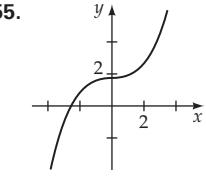
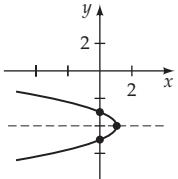
b. 56,800 square feet

45.  $a = 1.5$  inches

47. a.  $4\left(\frac{800}{84}\right)^2(y - 32) = x^2$     b. 117 millimeters    49. 4

51.  $4|p|$

53.

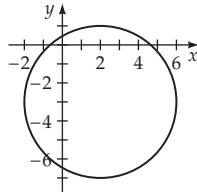


57.  $x^2 + y^2 - 8x - 8y - 2xy = 0$

**Prepare for This Section (6.2), page 374**

**PS1.** midpoint:  $(2, 3)$ ; length:  $2\sqrt{13}$     **PS2.**  $-8, 2$     **PS3.**  $1 \pm \sqrt{3}$     **PS4.**  $x^2 - 8x + 16 = (x - 4)^2$     **PS5.**  $y = \pm\sqrt{4 - (x - 2)^2}$

**PS6.**



**Exercise Set 6.2, page 385**

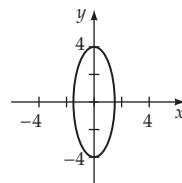
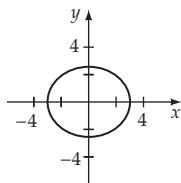
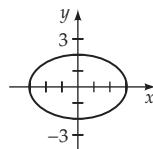
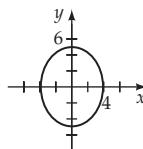
1. a. iv    b. i    c. ii    d. iii

3. center:  $(0, 0)$   
vertices:  $(0, 5), (0, -5)$   
foci:  $(0, 3), (0, -3)$

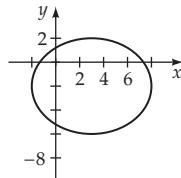
5. center:  $(0, 0)$   
vertices:  $(3, 0), (-3, 0)$   
foci:  $(\sqrt{5}, 0), (-\sqrt{5}, 0)$

7. center:  $(0, 0)$   
vertices:  $(3, 0), (-3, 0)$   
foci:  $(\sqrt{2}, 0), (-\sqrt{2}, 0)$

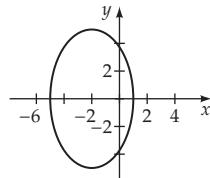
9. center:  $(0, 0)$   
vertices:  $(0, 4), (0, -4)$   
foci:  $\left(0, \frac{\sqrt{55}}{2}\right), \left(0, -\frac{\sqrt{55}}{2}\right)$



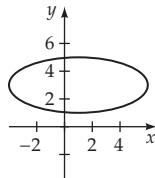
11. center:  $(3, -2)$   
vertices:  $(8, -2), (-2, -2)$   
foci:  $(6, -2), (0, -2)$



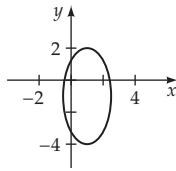
13. center:  $(-2, 0)$   
vertices:  $(-2, 5), (-2, -5)$   
foci:  $(-2, 4), (-2, -4)$



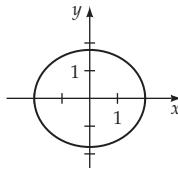
15. center:  $(1, 3)$   
vertices:  $(1 + \sqrt{21}, 3), (1 - \sqrt{21}, 3)$   
foci:  $(1 + \sqrt{17}, 3), (1 - \sqrt{17}, 3)$



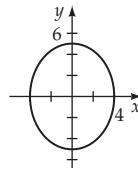
17. center:  $(1, -1)$   
vertices:  $(1, 2), (1, -4)$   
foci:  $\left(1, -1 + \frac{\sqrt{65}}{3}\right), \left(1, -1 - \frac{\sqrt{65}}{3}\right)$



19. center:  $(0, 0)$   
vertices:  $(2, 0), (-2, 0)$   
foci:  $(1, 0), (-1, 0)$

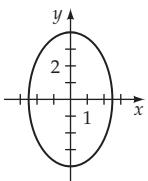


21. center:  $(0, 0)$   
vertices:  $(0, 5), (0, -5)$   
foci:  $(0, 3), (0, -3)$

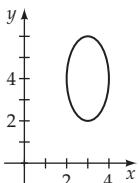


23. center:  $(0, 0)$ vertices:  $(0, 4), (0, -4)$ 

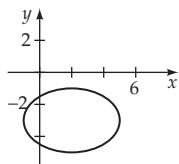
$$\text{foci: } \left(0, \frac{\sqrt{39}}{2}\right), \left(0, -\frac{\sqrt{39}}{2}\right)$$

25. center:  $(3, 4)$ vertices:  $(3, 6), (3, 2)$ 

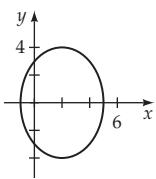
$$\text{foci: } (3, 4 + \sqrt{3}), (3, 4 - \sqrt{3})$$

27. center:  $(2, -3)$ vertices:  $(-1, -3), (5, -3)$ 

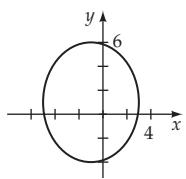
$$\text{foci: } (0, -3), (4, -3)$$

29. center:  $(2, 0)$ vertices:  $(2, 4), (2, -4)$ 

$$\text{foci: } (2, \sqrt{7}), (2, -\sqrt{7})$$

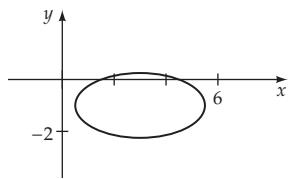
31. center:  $(-1, 1)$ vertices:  $(-1, 6), (-1, -4)$ 

$$\text{foci: } (-1, 4), (-1, -2)$$

33. center:  $(3, -1)$ 

$$\text{vertices: } \left(\frac{11}{2}, -1\right), \left(\frac{1}{2}, -1\right)$$

$$\text{foci: } \left(3 + \frac{\sqrt{17}}{2}, -1\right), \left(3 - \frac{\sqrt{17}}{2}, -1\right)$$



$$35. \frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$37. \frac{x^2}{36} + \frac{y^2}{16} = 1$$

$$39. \frac{x^2}{36} + \frac{y^2}{81/8} = 1$$

$$41. \frac{(x+2)^2}{16} + \frac{(y-4)^2}{7} = 1$$

$$43. \frac{(x-2)^2}{25/24} + \frac{(y-4)^2}{25} = 1$$

$$45. \frac{(x-5)^2}{16} + \frac{(y-1)^2}{25} = 1$$

$$47. \frac{x^2}{25} + \frac{y^2}{21} = 1$$

$$49. \frac{x^2}{20} + \frac{y^2}{36} = 1$$

$$51. \frac{(x-1)^2}{25} + \frac{(y-3)^2}{21} = 1$$

$$53. \frac{x^2}{80} + \frac{y^2}{144} = 1$$

55. on the major axis of the ellipse, 41 centimeters from the emitter

$$57. \frac{x^2}{884.74^2} + \frac{y^2}{883.35^2} = 1$$

59. 40 feet

$$61. \frac{\left(x - \frac{9\sqrt{15}}{2}\right)^2}{324} + \frac{y^2}{81/4} = 1$$

63. 24 feet

65. a.  $\sqrt{7}$  feet to the right and left of  $O$ .

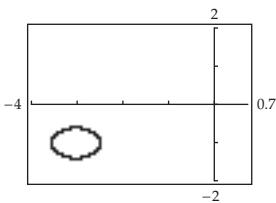
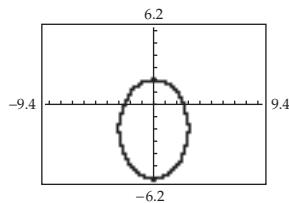
b. 8 feet

$$67. y = \frac{-36 \pm \sqrt{1296 - 36(16x^2 - 108)}}{18}$$

$$69. y = \frac{-18 \pm \sqrt{324 - 36(4x^2 + 24x + 44)}}{18}$$

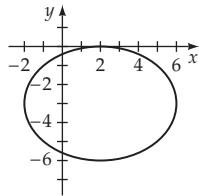
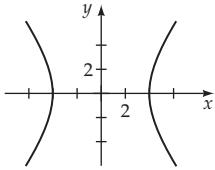
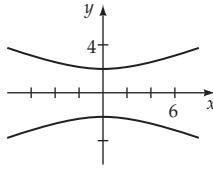
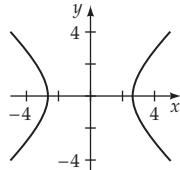
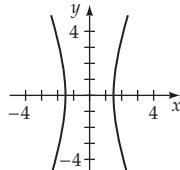
$$71. \frac{x^2}{36} + \frac{y^2}{27} = 1$$

$$73. \frac{(x-1)^2}{16} + \frac{(y-2)^2}{12} = 1$$

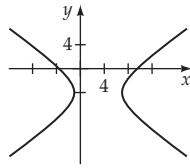


## Prepare for This Section (6.3), page 390

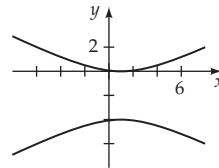
- PS1.** midpoint:  $(1, -1)$ ; length:  $2\sqrt{13}$     **PS2.**  $-4, 2$     **PS3.**  $\sqrt{2}$     **PS4.**  $4(x^2 + 6x + 9) = 4(x + 3)^2$     **PS5.**  $y = \pm \frac{3}{2} \sqrt{x^2 - 4}$

**PS6.****Exercise Set 6.3 page 399****1. a. iii   b. ii   c. i   d. iv****3. center:  $(0, 0)$** vertices:  $(\pm 4, 0)$ foci:  $(\pm \sqrt{41}, 0)$ asymptotes:  $y = \pm \frac{5}{4}x$ **5. center:  $(0, 0)$** vertices:  $(0, \pm 2)$ foci:  $(0, \pm \sqrt{29})$ asymptotes:  $y = \pm \frac{2}{5}x$ **7. center:  $(0, 0)$** vertices:  $(\pm \sqrt{7}, 0)$ foci:  $(\pm 4, 0)$ asymptotes:  $y = \pm \frac{3\sqrt{7}}{7}x$ **9. center:  $(0, 0)$** vertices:  $(\pm \frac{3}{2}, 0)$ foci:  $(\pm \frac{\sqrt{73}}{2}, 0)$ asymptotes:  $y = \pm \frac{8}{3}x$ 

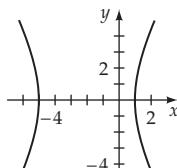
**11. center:  $(3, -4)$**   
 vertices:  $(7, -4), (-1, -4)$   
 foci:  $(8, -4), (-2, -4)$   
 asymptotes:  $y + 4 = \pm \frac{3}{4}(x - 3)$



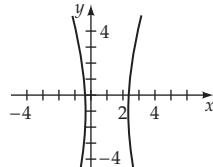
**13. center:  $(1, -2)$**   
 vertices:  $(1, 0), (1, -4)$   
 foci:  $(1, -2 \pm 2\sqrt{5})$   
 asymptotes:  $y + 2 = \pm \frac{1}{2}(x - 1)$



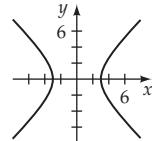
**15. center:  $(-2, 0)$**   
 vertices:  $(1, 0), (-5, 0)$   
 foci:  $(-2 \pm \sqrt{34}, 0)$   
 asymptotes:  $y = \pm \frac{5}{3}(x + 2)$



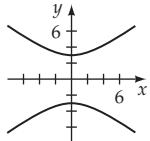
**17. center:  $(1, -1)$**   
 vertices:  $(\frac{7}{3}, -1), (-\frac{1}{3}, -1)$   
 foci:  $(1 \pm \frac{\sqrt{97}}{3}, -1)$   
 asymptotes:  $y + 1 = \pm \frac{9}{4}(x - 1)$



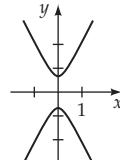
**19. center:  $(0, 0)$**   
 vertices:  $(\pm 3, 0)$   
 foci:  $(\pm 3\sqrt{2}, 0)$   
 asymptotes:  $y = \pm x$



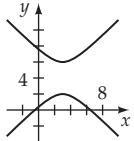
**21. center:  $(0, 0)$**   
 vertices:  $(0, \pm 3)$   
 foci:  $(0, \pm 5)$   
 asymptotes:  $y = \pm \frac{3}{4}x$



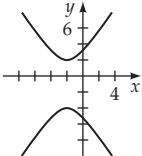
**23. center:  $(0, 0)$**   
 vertices:  $(0, \pm \frac{2}{3})$   
 foci:  $(0, \pm \frac{\sqrt{5}}{3})$   
 asymptotes:  $y = \pm 2x$



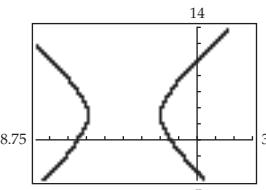
25. center:  $(3, 4)$   
 vertices:  $(3, 6), (3, 2)$   
 foci:  $(3, 4 \pm 2\sqrt{2})$   
 asymptotes:  $y - 4 = \pm(x - 3)$



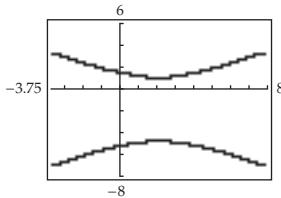
27. center:  $(-2, -1)$   
 vertices:  $(-2, 2), (-2, -4)$   
 foci:  $(-2, -1 \pm \sqrt{13})$   
 asymptotes:  $y + 1 = \pm \frac{3}{2}(x + 2)$



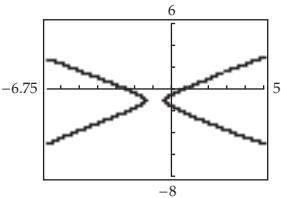
29.  $y = \frac{-6 \pm \sqrt{36 + 4(4x^2 + 32x + 39)}}{-2}$



31.  $y = \frac{64 \pm \sqrt{4096 + 64(9x^2 - 36x + 116)}}{-32}$



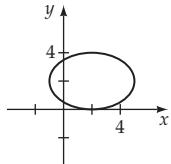
33.  $y = \frac{18 \pm \sqrt{324 + 36(4x^2 + 8x - 6)}}{-18}$



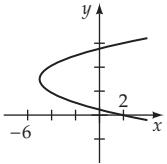
35.  $\frac{x^2}{9} - \frac{y^2}{7} = 1$     37.  $\frac{y^2}{20} - \frac{x^2}{5} = 1$

39.  $\frac{y^2}{9} - \frac{x^2}{36/7} = 1$     41.  $\frac{y^2}{16} - \frac{x^2}{64} = 1$     43.  $\frac{(x-4)^2}{4} - \frac{(y-3)^2}{5} = 1$     45.  $\frac{(x-4)^2}{144/41} - \frac{(y+2)^2}{225/41} = 1$     47.  $\frac{(y-2)^2}{3} - \frac{(x-7)^2}{12} = 1$   
 49.  $\frac{(y-7)^2}{1} - \frac{(x-1)^2}{3} = 1$     51.  $\frac{x^2}{4} - \frac{y^2}{12} = 1$     53.  $\frac{(x-4)^2}{36/7} - \frac{(y-1)^2}{4} = 1$  and  $\frac{(y-1)^2}{36/7} - \frac{(x-4)^2}{4} = 1$     55. a.  $\frac{x^2}{2162.25} - \frac{y^2}{13,462.75} = 1$   
 b. 221 miles    57.  $y^2 - x^2 = 10,000^2$ , hyperbola    59. a.  $\frac{x^2}{2^2} - \frac{y^2}{0.5^2} = 1$     b. 6.25 inches

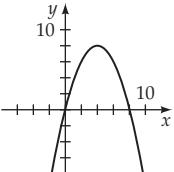
61. ellipse



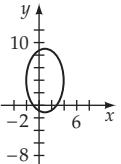
63. parabola



65. parabola

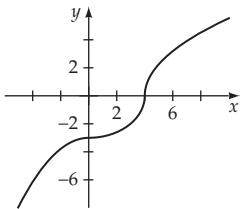


67. ellipse



69.  $\frac{x^2}{1} - \frac{y^2}{3} = 1$

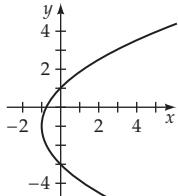
71.  $\frac{y^2}{9} - \frac{x^2}{7} = 1$     73.



### Prepare for This Section (6.4), page 403

- PS1.  $\cos \alpha \cos \beta - \sin \alpha \sin \beta$     PS2.  $\sin \alpha \cos \beta + \cos \alpha \sin \beta$     PS3.  $\frac{\pi}{6}$     PS4.  $150^\circ$     PS5. hyperbola

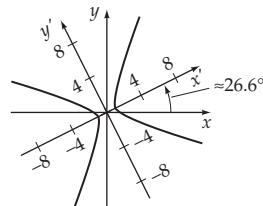
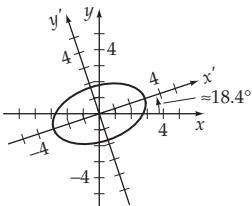
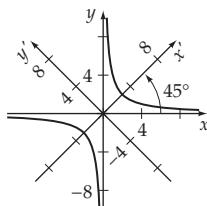
PS6.



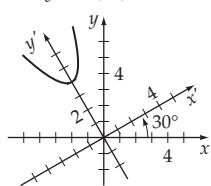
## Exercise Set 6.4, page 410

1.  $45^\circ$ 

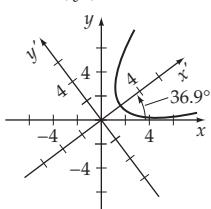
3.  $36.9^\circ$  5.  $73.5^\circ$  7.  $22.5^\circ$  9.  $45^\circ, \frac{(x')^2}{8} - \frac{(y')^2}{8} = 1$  11.  $18.4^\circ, \frac{(x')^2}{9} + \frac{(y')^2}{3} = 1$  13.  $26.6^\circ, \frac{(x')^2}{1/2} - \frac{(y')^2}{1/3} = 1$



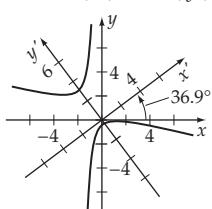
15.  $30^\circ, y' = (x')^2 + 4$



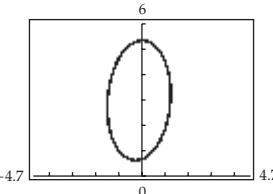
17.  $36.9^\circ, (y')^2 = 2(x' - 2)$



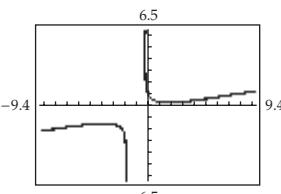
19.  $36.9^\circ, 15(x')^2 - 10(y')^2 + 6x' + 28y' + 11 = 0$



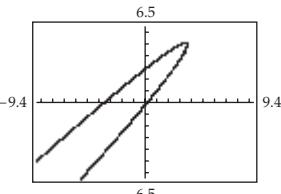
21.



23.



25.



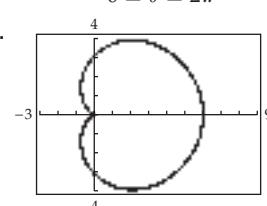
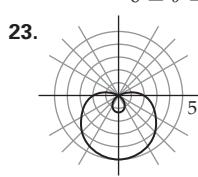
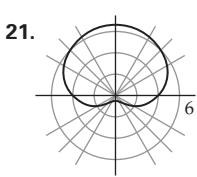
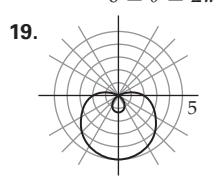
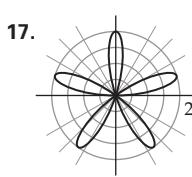
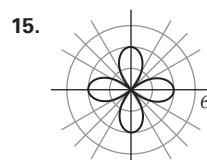
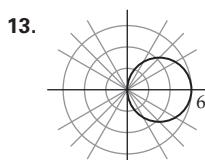
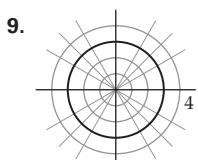
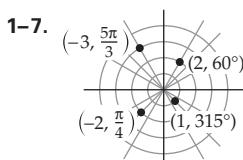
27.  $y = \frac{\sqrt{3} + 2\sqrt{2}}{2\sqrt{3} - \sqrt{2}}x$  and  $y = \frac{\sqrt{3} - 2\sqrt{2}}{2\sqrt{3} + \sqrt{2}}x$  29.  $\left(\frac{3\sqrt{15}}{5}, \frac{\sqrt{15}}{5}\right)$  and  $\left(-\frac{3\sqrt{15}}{5}, -\frac{\sqrt{15}}{5}\right)$  31. hyperbola 33. parabola 35. parabola

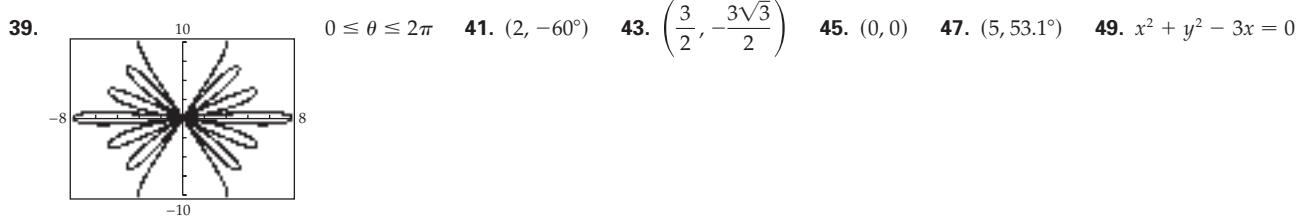
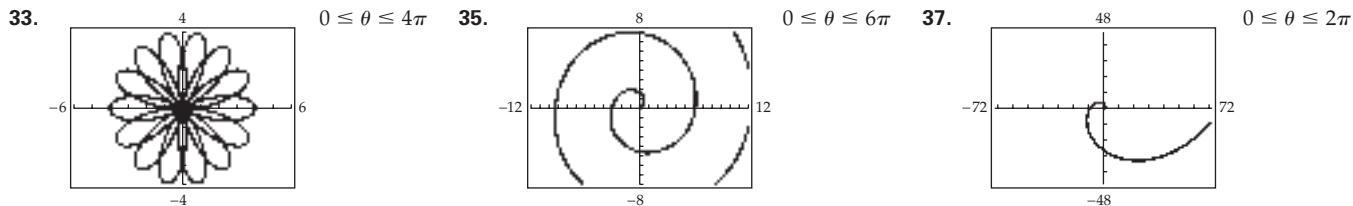
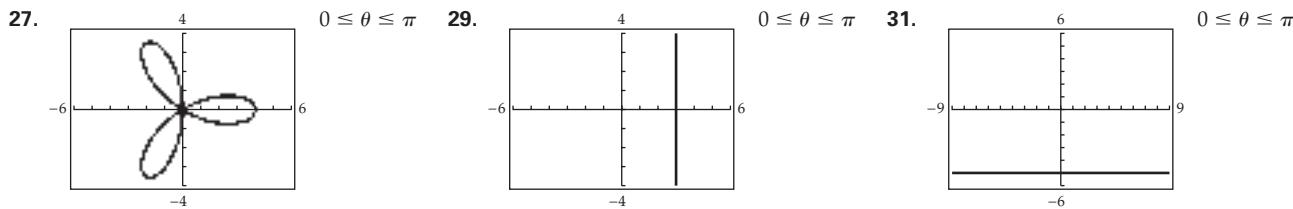
37. hyperbola 39. ellipse 43.  $9x^2 - 4xy + 6y^2 = 100$ 

## Prepare for This Section (6.5), page 412

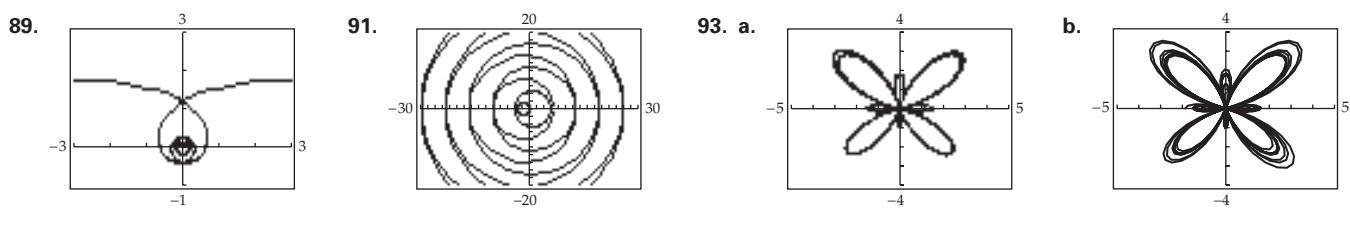
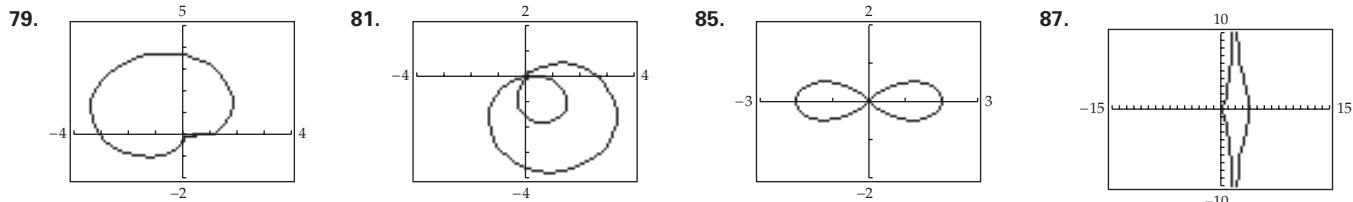
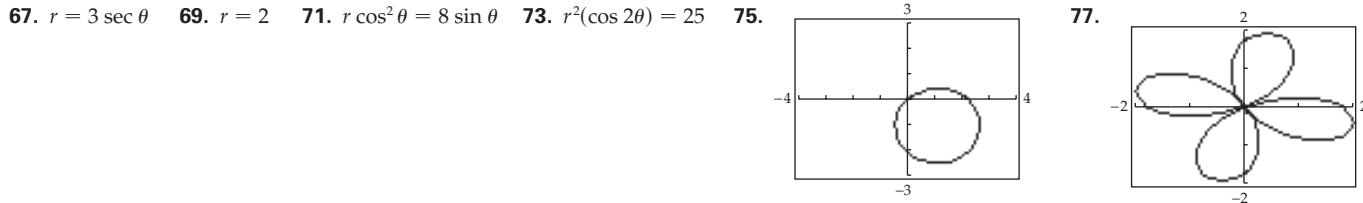
PS1. odd PS2. even PS3.  $\frac{2\pi}{3}, \frac{5\pi}{3}$  PS4.  $240^\circ$  PS5.  $r^2$  PS6. (4.2, 2.6)

## Exercise Set 6.5, page 423

 $0 \leq \theta \leq \pi$  $0 \leq \theta \leq 2\pi$  $0 \leq \theta \leq 2\pi$  $0 \leq \theta \leq 2\pi$  $0 \leq \theta \leq 2\pi$



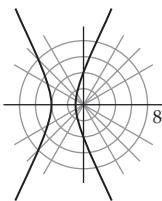
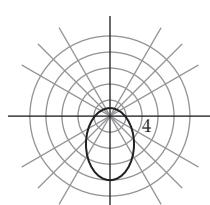
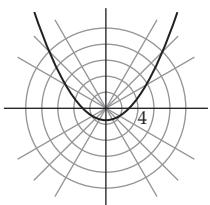
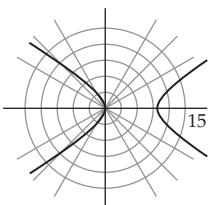
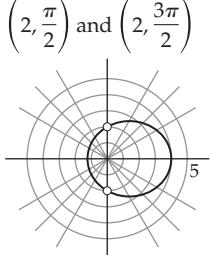
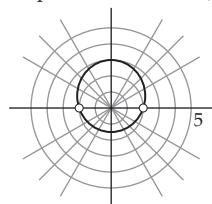
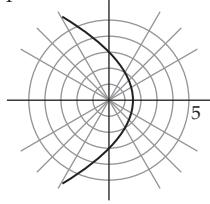
51.  $x = 3$     53.  $x^2 + y^2 = 16$     55.  $y = \frac{\sqrt{3}}{3}x$     57.  $x^4 - y^2 + x^2y^2 = 0$     59.  $y^2 + 4x - 4 = 0$     61.  $y = 2x + 6$     63.  $r = 2 \csc \theta$     65.  $\theta = \frac{\pi}{3}$



## Prepare for This Section (6.6), page 426

**PS1.**  $\frac{3}{5}$    **PS2.**  $x = -1$    **PS3.**  $y = \frac{2}{1 - 2x}$    **PS4.**  $\frac{3\pi}{2}$    **PS5.**  $e > 1$    **PS6.**  $\frac{4}{2 - \cos x}$

## Exercise Set 6.6, page 431

**1.** hyperbola**3.** ellipse**5.** parabola**7.** hyperbola**9.** ellipse with holes at  $(2, \frac{\pi}{2})$  and  $(2, \frac{3\pi}{2})$ **11.** ellipse with holes at  $(2, 0)$  and  $(2, \pi)$ **13.** parabola**15.**  $3x^2 - y^2 + 16x + 16 = 0$ **17.**  $16x^2 + 7y^2 + 48y - 64 = 0$ 

**19.**  $x^2 - 6y - 9 = 0$

**21.**  $r = \frac{2}{1 - 2 \cos \theta}$

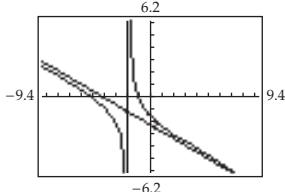
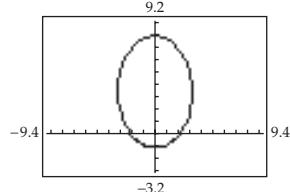
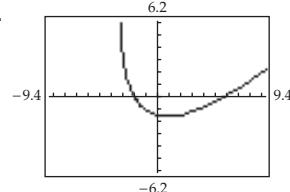
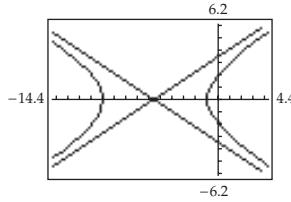
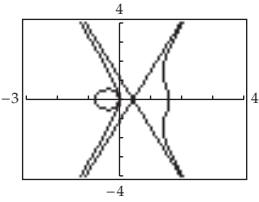
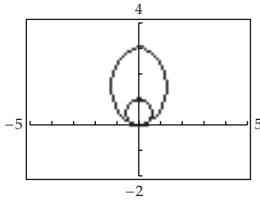
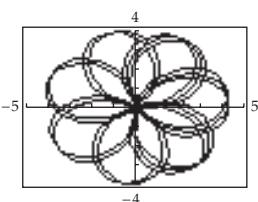
**23.**  $r = \frac{2}{1 + \sin \theta}$

**25.**  $r = \frac{8}{3 - 2 \sin \theta}$

**27.**  $r = \frac{6}{2 + 3 \cos \theta}$

**29.**  $r = \frac{4}{1 - \cos \theta}$

**31.**  $r = \frac{3}{1 - 2 \sin \theta}$

**33.****35.****37.**Rotate the graph in Exercise 1  
counterclockwise  $\frac{\pi}{6}$  radian.Rotate the graph in Exercise 3  
counterclockwise  $\pi$  radians.Rotate the graph in Exercise 5  
clockwise  $\frac{\pi}{6}$  radian.**39.****41.****43.****45.**Rotate the graph in Exercise 7  
clockwise  $\pi$  radians.

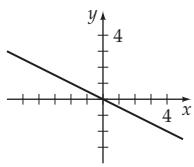
$0 \leq \theta \leq 12\pi$

## Prepare for This Section (6.7), page 432

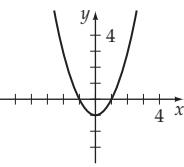
**PS1.**  $y^2 + 3y + \frac{9}{4} = \left(y + \frac{3}{2}\right)^2$    **PS2.**  $y = 4t^2 + 4t + 1$    **PS3.** ellipse   **PS4.** 1   **PS5.**  $(6, -2)$    **PS6.** domain:  $(-\infty, \infty)$ ; range:  $[-3, 3]$

## Exercise Set 6.7, page 440

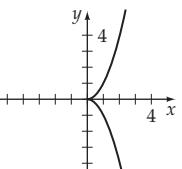
1.



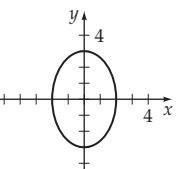
3.



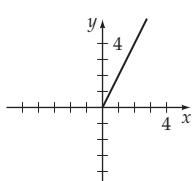
5.



7.

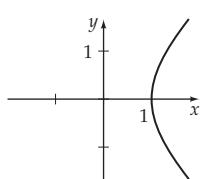


9.



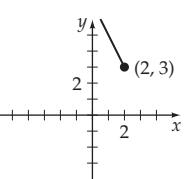
11.

$$x^2 - y^2 - 1 = 0 \\ x \geq 1 \\ y \in \mathbb{R}$$

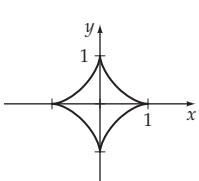


13.

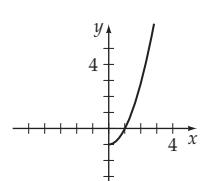
$$y = -2x + 7 \\ x \leq 2 \\ y \geq 3$$



15.  $x^{2/3} + y^{2/3} = 1$   
 $-1 \leq x \leq 1$   
 $-1 \leq y \leq 1$



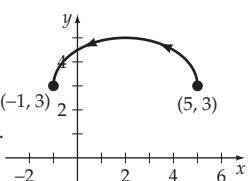
17.  $y = x^2 - 1$   
 $x \geq 0$   
 $y \geq -1$



19. The point traces the top half of the ellipse

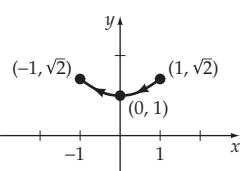
$$\frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1,$$
 as shown in

the figure. The point starts at  $(5, 3)$  and moves counterclockwise along the ellipse until it reaches the point  $(-1, 3)$  at time  $t = \pi.$

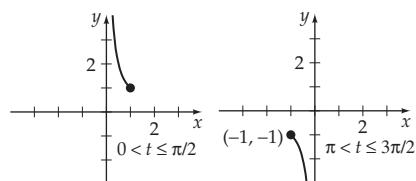


23. The point traces a portion of the top branch of the hyperbola  $y^2 - x^2 = 1,$  as shown in the figure. The point starts at  $(1, \sqrt{2})$  and moves along the hyperbola until it reaches the point  $(-1, \sqrt{2})$  at time  $t = \frac{\pi}{2}.$

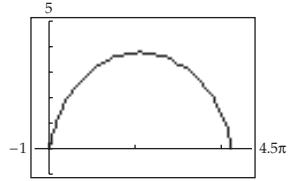
$$\text{the point } (-1, \sqrt{2}) \text{ at time } t = \frac{\pi}{2}.$$



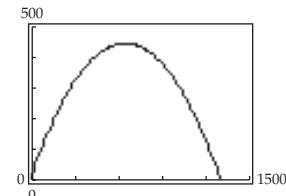
27.



29.

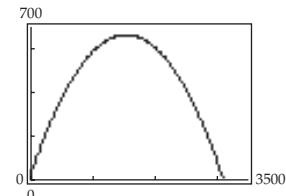
31. a.  $x = 6, y = 60t,$  for  $t \geq 0$    b. the Hummer

33.



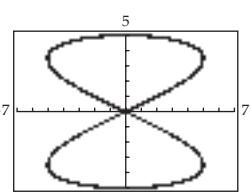
Max height (nearest foot) of 462 feet is attained when  $t \approx 5.38$  seconds.  
Range (nearest foot) of 1295 feet is attained when  $t \approx 10.75$  seconds.

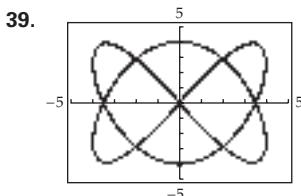
35.



Max height (nearest foot) of 694 feet is attained when  $t \approx 6.59$  seconds.  
Range (nearest foot) of 3084 feet is attained when  $t \approx 13.17$  seconds.

37.





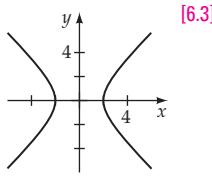
43.  $x = a \cos \theta + a\theta \sin \theta$     45.  $x = (b-a) \cos \theta + a \cos\left(\frac{b-a}{a}\theta\right)$   
 $y = a \sin \theta - a\theta \cos \theta$      $y = (b-a) \sin \theta - a \sin\left(\frac{b-a}{a}\theta\right)$

### Chapter 6 Assessing Concepts, page 445

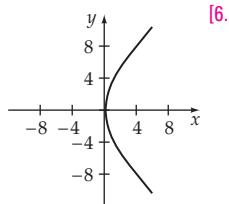
1. d    2. b    3. e    4. c    5. a    6. f    7. g    8. i    9. h    10. j    11. k    12. k

### Chapter 6 Review Exercises, page 446

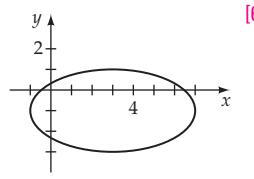
1. center:  $(0, 0)$   
 vertices:  $(\pm 2, 0)$   
 foci:  $(\pm 2\sqrt{2}, 0)$   
 asymptotes:  $y = \pm x$



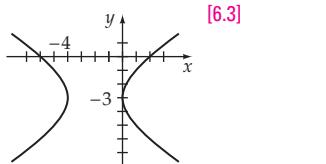
2. vertex:  $(0, 0)$   
 focus:  $(4, 0)$   
 directrix:  $x = -4$



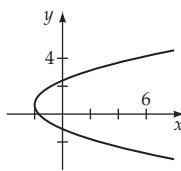
3. center:  $(3, -1)$   
 vertices:  $(-1, -1), (7, -1)$   
 foci:  $(3 \pm 2\sqrt{3}, -1)$



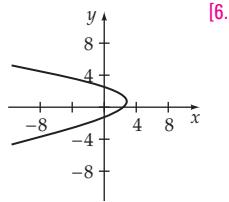
4. center:  $(-2, -3)$   
 vertices:  $(0, -3), (-4, -3)$   
 foci:  $(-2 \pm \sqrt{7}, -3)$   
 asymptotes:  $y + 3 = \pm \frac{\sqrt{3}}{2}(x + 2)$



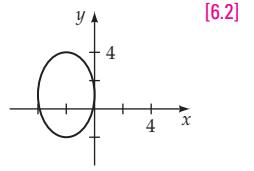
5. vertex:  $(-2, 1)$   
 focus:  $\left(-\frac{29}{16}, 1\right)$   
 directrix:  $x = -\frac{35}{16}$



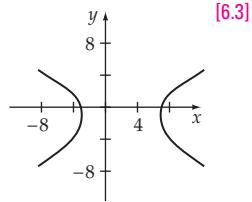
6. vertex:  $(3, 1)$   
 focus:  $\left(\frac{21}{8}, 1\right)$   
 directrix:  $x = \frac{27}{8}$



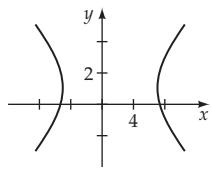
7. center:  $(-2, 1)$   
 vertices:  $(-2, -2), (-2, 4)$   
 foci:  $(-2, 1 \pm \sqrt{5})$



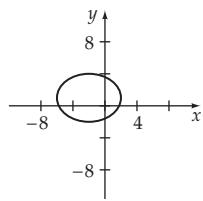
8. center:  $(2, -1)$   
 vertices:  $(7, -1), (-3, -1)$   
 foci:  $(8, -1), (-4, -1)$   
 asymptotes:  $y + 1 = \pm \frac{\sqrt{11}}{5}(x - 2)$



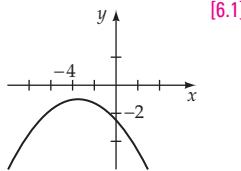
9. center:  $\left(1, \frac{2}{3}\right)$   
 vertices:  $\left(-5, \frac{2}{3}\right), \left(7, \frac{2}{3}\right)$   
 foci:  $\left(1 \pm 2\sqrt{13}, \frac{2}{3}\right)$   
 asymptotes:  $y - \frac{2}{3} = \pm \frac{2}{3}(x - 1)$



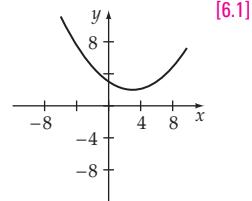
10. center:  $\left(-2, \frac{1}{2}\right)$   
 vertices:  $\left(2, \frac{1}{2}\right), \left(-6, \frac{1}{2}\right)$   
 foci:  $\left(-2 \pm \sqrt{7}, \frac{1}{2}\right)$



11. vertex:  $\left(-\frac{7}{2}, -1\right)$   
 focus:  $\left(-\frac{7}{2}, -3\right)$   
 directrix:  $y = 1$



12. vertex:  $(3, 2)$   
 focus:  $\left(3, \frac{17}{4}\right)$   
 directrix:  $y = -\frac{1}{4}$

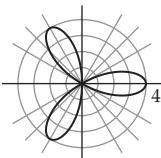


13.  $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{16} = 1$  [6.2]    14.  $\frac{(x-1)^2}{9} - \frac{(y-1)^2}{7} = 1$  [6.3]    15.  $\frac{(x+2)^2}{4} - \frac{(y-2)^2}{5} = 1$  [6.3]    16.  $(y+3)^2 = -8(x-4)$  [6.1]

17.  $x^2 = \frac{3(y+2)}{2}$  or  $(y+2)^2 = 12x$  [6.1]    18.  $\frac{(x+2)^2}{9} + \frac{(y+1)^2}{5} = 1$  [6.2]    19.  $\frac{x^2}{36} - \frac{y^2}{4/9} = 1$  [6.3]    20.  $y = (x-1)^2$  [6.1]

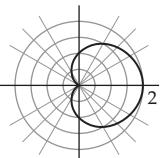
21.  $(x')^2 + 2(y')^2 - 4 = 0$ , ellipse [6.4]    22.  $6(x')^2 + \frac{\sqrt{2}}{2}x' + \frac{9\sqrt{2}}{2}y' - 12 = 0$ , parabola [6.4]    23.  $(x')^2 - 4y' + 8 = 0$ , parabola [6.4]

24.  $\frac{1}{2}(x')^2 - \frac{1}{2}(y')^2 - \sqrt{2}x' - 1 = 0$ , hyperbola [6.4]    25.



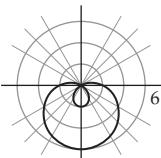
[6.5]

26.



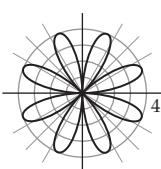
[6.5]

27.



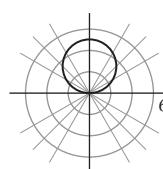
[6.5]

28.



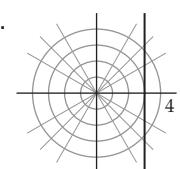
[6.5]

29.



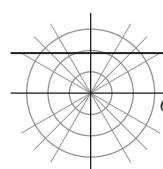
[6.5]

30.



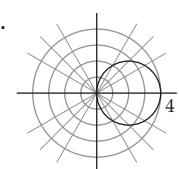
[6.5]

31.



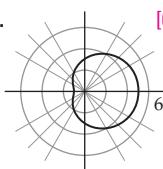
[6.5]

32.



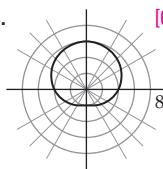
[6.5]

33.



[6.5]

34.



[6.5]

35.  $r \sin^2 \theta = 16 \cos \theta$  [6.5]

36.  $r + 4 \cos \theta + 3 \sin \theta = 0$  [6.5]

37.  $3r \cos \theta - 2r \sin \theta = 6$  [6.5]

38.  $r^2 \sin 2\theta = 8$  [6.5]

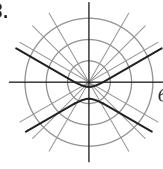
39.  $y^2 = 8x + 16$  [6.5]

40.  $x^2 - 3x + y^2 + 4y = 0$  [6.5]

41.  $x^4 + y^4 + 2x^2y^2 - x^2 + y^2 = 0$  [6.5]

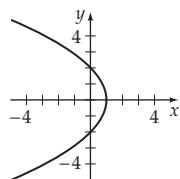
42.  $y = (\tan 1)x$  [6.5]

43.



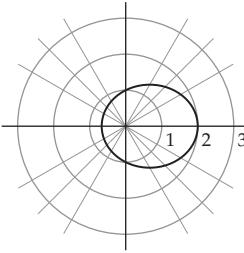
[6.6]

44.



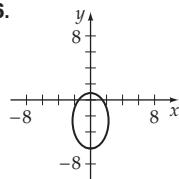
[6.6]

45.



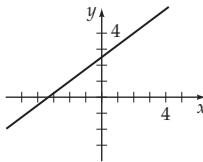
[6.6]

46.

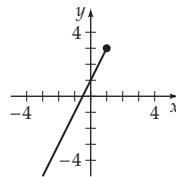


[6.6]

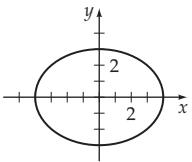
47.  $y = \frac{3}{4}x + \frac{5}{2}$  [6.7]



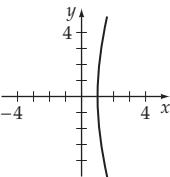
48.  $y = 2x + 1$ ,  $x \leq 1$  [6.7]



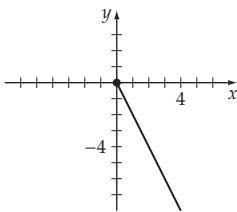
49.  $\frac{x^2}{16} + \frac{y^2}{9} = 1$  [6.7]



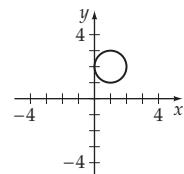
50.  $\frac{x^2}{1} - \frac{y^2}{16} = 1$  [6.7]



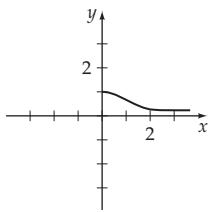
51.  $y = -2x$ ,  $x > 0$  [6.7]



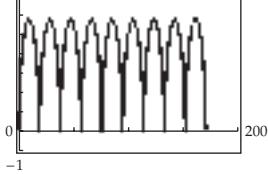
52.  $(x-1)^2 + (y-2)^2 = 1$  [6.7]

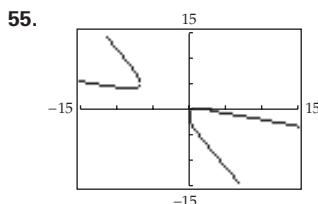


53.  $y = 2^{-x^2}$ ,  $x \geq 0$  [6.7]

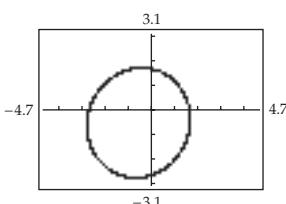


54. [6.7]

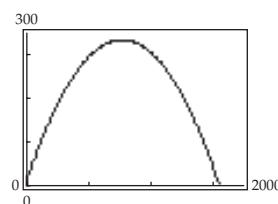




[6.4]



[6.5]



[6.7]

Max. height (nearest foot) of 278 feet  
is attained when  $t \approx 4.17$  seconds.

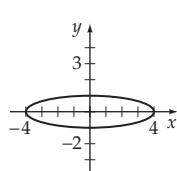
## Chapter 6 Quantitative Reasoning Exercises, page 447

**QR1.** They appear to be the same.    **QR2.** They appear to be the same.    **QR3.** 5.2 units

## Chapter 6 Test, page 449

1. vertex:  $(0, 0)$ focus:  $(0, 2)$ directrix:  $y = -2$  [6.1]

2.

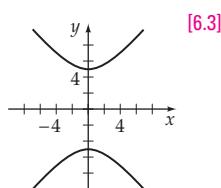


[6.2]

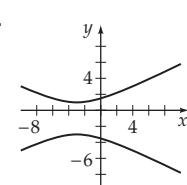
3. vertices:  $(3, 4), (3, -6)$   
foci:  $(3, 3), (3, -5)$  [6.2]

4.  $\frac{x^2}{45} + \frac{(y+3)^2}{9} = 1$  [6.2]

5.

6. vertices:  $(6, 0), (-6, 0)$ foci:  $(-10, 0), (10, 0)$ asymptotes:  $y = \pm \frac{4x}{3}$  [6.3]

7.



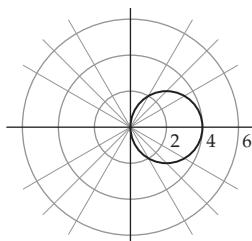
[6.3]

8.  $73.15^\circ$  [6.4]

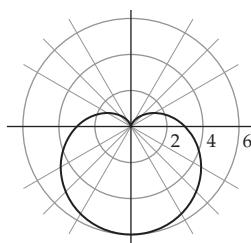
9. ellipse [6.4]

10.  $P(2, 300^\circ)$  [6.5]

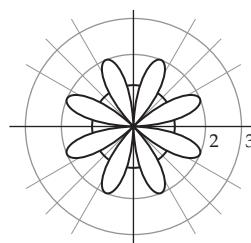
11.



[6.5] 12.



[6.5] 13.



[6.5]

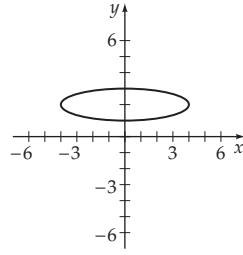
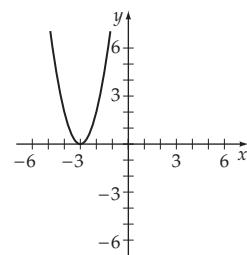
14.  $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$  [6.5]

15.  $y^2 - 8x - 16 = 0$  [6.5]

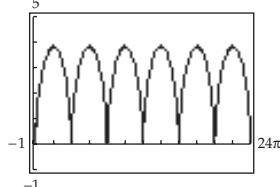
16.  $x^2 + 8y - 16 = 0$  [6.6]

17.  $(x+3)^2 = \frac{1}{2}y$  [6.7]

18.  $\frac{x^2}{16} + \frac{(y-2)^2}{1} = 1$  [6.7]



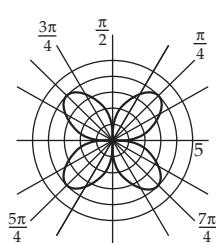
19. [6.7]    20. 443 feet [6.7]

Xscl =  $2\pi$ 

### Cumulative Review Exercises, page 450

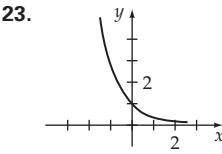
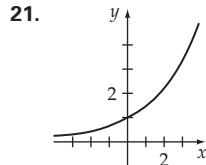
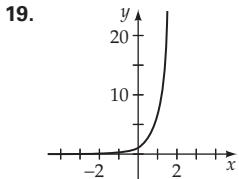
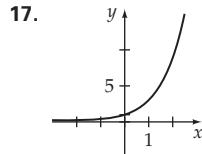
1.  $-2 \pm \sqrt{10}$  [1.1]    2. origin [1.4]    3.  $(g \circ f)(x) = 3 \sin x - 2$  [1.5]    4.  $\frac{4\pi}{3}$  [2.1]    5. 11 mph [2.1]    6.  $\tan(t) = -\sqrt{3}$  [2.4]  
 7. 19 centimeters [2.2]    8. amplitude:  $\frac{1}{2}$ , period: 6 [2.5]    9. 3 [2.6]    10. See [3.1].    11.  $-\frac{63}{65}$  [3.2]    12.  $\frac{2\sqrt{6}}{5}$  [3.5]    13.  $\frac{5}{13}$  [3.6]  
 14. 298 feet [4.1]    15.  $38^\circ$  [4.2]    16.  $-24.6i + 17.2j$  [4.3]    17. No.  $\mathbf{v} \cdot \mathbf{w} = 1 \neq 0$  [4.3]    18.  $4 \operatorname{cis}\left(\frac{2\pi}{3}\right)$  [5.2]

19. [6.5]    20.  $y = x^2 + 2x + 2$  [6.7]

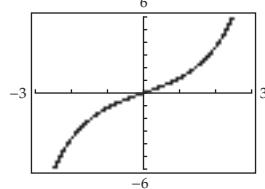
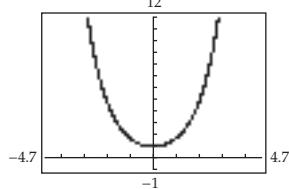


### Exercise Set 7.1, page 461

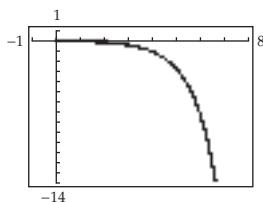
1.  $f(0) = 1; f(4) = 81$     3.  $g(-2) = \frac{1}{100}; g(3) = 1000$     5.  $h(2) = \frac{9}{4}; h(-3) = \frac{8}{27}$     7.  $j(-2) = 4; j(4) = \frac{1}{16}$     9. 9.19    11. 9.03    13. 9.74  
 15. a.  $k(x)$     b.  $g(x)$     c.  $h(x)$     d.  $f(x)$



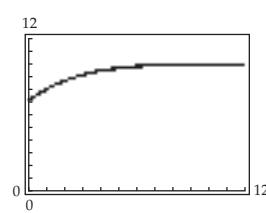
25. Shift the graph of  $f$  vertically upward 2 units.    27. Shift the graph of  $f$  horizontally to the right 2 units.  
 29. Reflect the graph of  $f$  across the  $y$ -axis.    31. Stretch the graph of  $f$  vertically away from the  $x$ -axis by a factor of 2.  
 33. Reflect the graph of  $f$  across the  $y$ -axis and then shift this graph vertically upward 2 units.    35. Shift the graph of  $f$  horizontally to the right 4 units and then reflect this graph across the  $x$ -axis.    37. Reflect the graph of  $f$  across the  $y$ -axis and then shift this graph vertically upward 3 units.  
 39. no horizontal asymptote



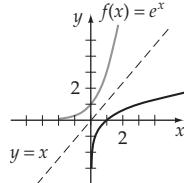
- 43.** horizontal asymptote:  $y = 0$



- 45.** horizontal asymptote:  $y = 10$



- 47. a.** 442 million connections   **b.** 2008   **49. a.** 233 items per month; 59 items per month   **b.** The demand will approach 25 items per month.  
**51. a.** 6400 bacteria; 409,600 bacteria   **b.** 11.6 hours   **53. a.** 69.2%   **b.** 7.6   **55. a.** 363 beneficiaries; 88,572 beneficiaries  
**b.** 13 rounds   **57. a.** 141°F   **b.** after 28.3 minutes   **59. a.** 261.63 vibrations per second   **b.** No. The function  $f(n)$  is not a linear function. Therefore, the graph of  $f(n)$  does not increase at a constant rate.   **63.**



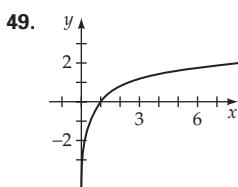
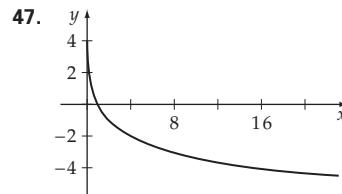
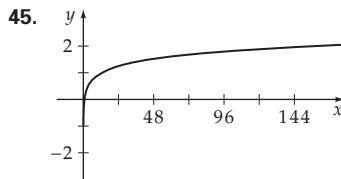
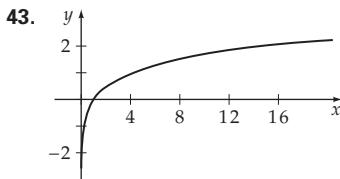
- 69.** Let  $f(x) = e^x$  and  $g(x) = 2x - 5$ . Then  $h(x) = e^{2x-5} = f[2x - 5] = f[g(x)] = (f \circ g)(x)$ .

### Prepare for This Section (7.2), page 467

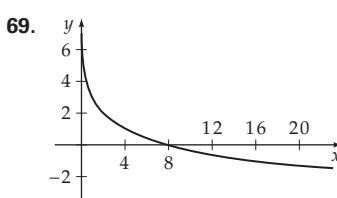
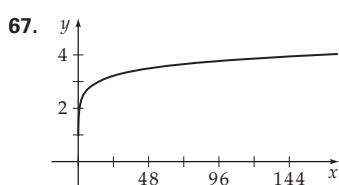
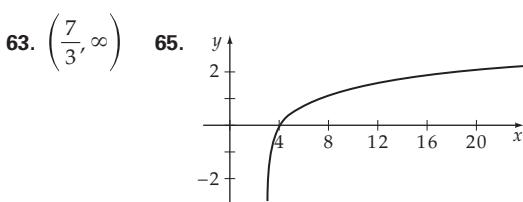
- PS1.** 4   **PS2.** 3   **PS3.** 5   **PS4.**  $f^{-1}(x) = \frac{3x}{2-x}$    **PS5.**  $\{x|x \geq 2\}$    **PS6.** the set of all positive real numbers

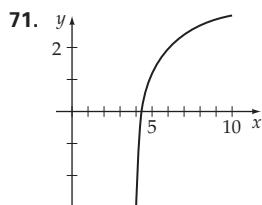
### Exercise Set 7.2, page 476

- 1.**  $10^1 = 10$    **3.**  $8^2 = 64$    **5.**  $7^0 = x$    **7.**  $e^4 = x$    **9.**  $e^0 = 1$    **11.**  $10^2 = 3x + 1$    **13.**  $\log_3 9 = 2$    **15.**  $\log_4 \frac{1}{16} = -2$    **17.**  $\log_b y = x$   
**19.**  $\ln y = x$    **21.**  $\log 100 = 2$    **23.**  $2 = \ln(x + 5)$    **25.** 2   **27.** -5   **29.** 3   **31.** -2   **33.** -4   **35.** 12   **37.** 8   **39.**  $\frac{2}{5}$    **41.**  $\frac{10}{3}$



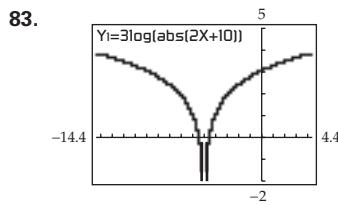
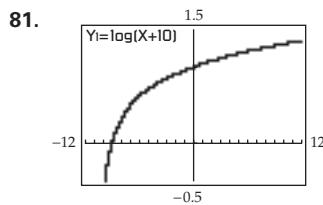
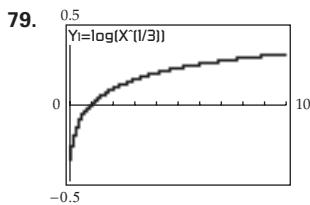
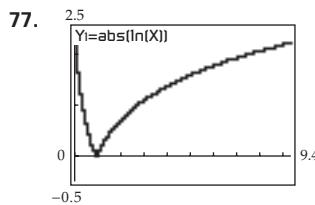
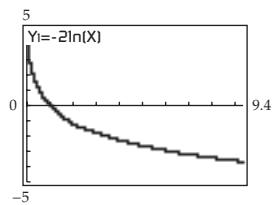
- 51.**  $(3, \infty)$    **53.**  $(-\infty, 11)$    **55.**  $(-\infty, -2) \cup (2, \infty)$    **57.**  $(4, \infty)$    **59.**  $(-1, 0) \cup (1, \infty)$    **61.**  $\left(\frac{11}{2}, \infty\right)$





73. a.  $k(x)$    b.  $f(x)$    c.  $g(x)$    d.  $h(x)$

75.



85. a. 2.0%   b. 45 months

87. a. 3298 units; 3418 units; 3490 units   b. 2750 units

89. 2.05 square meters   91. a. Answers will vary.   b. 96 digits   c. 3385 digits   d. 6,320,430 digits   93.  $f$  and  $g$  are inverse functions.

95. range of  $f$ :  $\{y \mid -1 < y < 1\}$ ; range of  $g$ : all real numbers

### Prepare for This Section (7.3), page 480

**PS1.**  $\approx 0.77815$  for each expression

**PS2.**  $\approx 0.98083$  for each expression

**PS3.**  $\approx 1.80618$  for each expression

**PS4.**  $\approx 3.21888$  for each expression

**PS5.**  $\approx 1.60944$  for each expression

**PS6.**  $\approx 0.90309$  for each expression

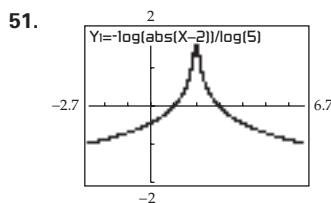
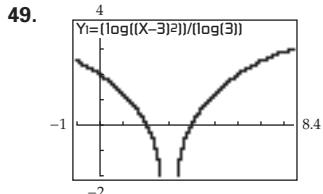
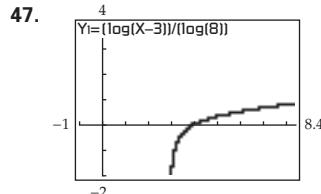
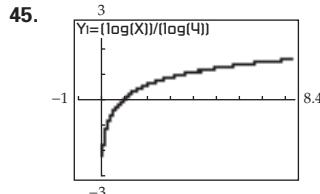
### Exercise Set 7.3, page 489

1.  $\log_b x + \log_b y + \log_b z$    3.  $\ln x - 4 \ln z$    5.  $\frac{1}{2} \log_2 x - 3 \log_2 y$    7.  $\frac{1}{2} \log_7 x + \frac{1}{2} \log_7 z - 2 \log_7 y$    9.  $2 + \ln z$

11.  $\frac{1}{3} \log_4 z - 2 - 3 \log_4 z$    13.  $\frac{1}{2} \log x + \frac{1}{4} \log z$    15.  $\frac{1}{3} \ln z + \frac{1}{6}$    17.  $\log[x^2(x+5)]$    19.  $\ln(x+y)$

21.  $\log[x^3 \cdot \sqrt[3]{y}(x+1)]$    23.  $\log\left(\frac{xy^2}{z}\right)$    25.  $\log_6\left(\frac{x^2y^4}{x+2}\right)$    27.  $\ln\left[\frac{(x+4)^2}{x(x^2-3)}\right]$    29.  $\ln\left[\frac{(2x+5)\sqrt{w}}{yz^2}\right]$    31.  $\ln\left[\frac{(x+3)y^3}{x-3}\right]$

33. 1.5395   35. 0.8672   37. -0.6131   39. 0.6447   41. 8.1749   43. 0.8735



53. False;  $\log 10 + \log 10 = 2$  but  $\log(10 + 10) = \log 20 \neq 2$ .   55. True

**57.** False;  $\log 100 - \log 10 = 1$  but  $\log(100 - 10) = \log 90 \neq 1$ . **59.** False;  $\frac{\log 100}{\log 10} = \frac{2}{1} = 2$  but  $\log 100 - \log 10 = 1$ .

**61.** False;  $(\log 10)^2 = 1$  but  $2 \log 10 = 2$ . **63.** 2 **65.**  $500^{501}$  **67.** 1:870,551; 1:757,858; 1:659,754; 1:574,349; 1:500,000 **69.** 10.4; base

**71.**  $3.16 \times 10^{-10}$  mole per liter **73.** **a.** 82.0 decibels **b.** 40.3 decibels **c.** 115.0 decibels **d.** 152.0 decibels **75.** 10 times as great

**77.** 5 **79.**  $10^{6.5}I_0$  or about 3,162,277.7 $I_0$  **81.** 100 to 1 **83.**  $10^{1.8}$  to 1 or about 63 to 1 **85.** 5.5 **89.** **a.**  $M \approx 6$  **b.**  $M \approx 4$  **c.** The results are close to the magnitudes produced by using the amplitude-time-difference formula.

### Prepare for This Section (7.4), page 494

**PS1.**  $\log_3 729 = 6$  **PS2.**  $5^4 = 625$  **PS3.**  $\log_a b = x + 2$  **PS4.**  $x = \frac{4a}{7b + 2c}$  **PS5.**  $x = \frac{3}{44}$  **PS6.**  $x = \frac{100(A - 1)}{A + 1}$

### Exercise Set 7.4, page 501

**1.** 6 **3.**  $-\frac{3}{2}$  **5.**  $-\frac{6}{5}$  **7.** 3 **9.**  $\frac{\log 70}{\log 5}$  **11.**  $-\frac{\log 120}{\log 3}$  **13.**  $\frac{\log 315 - 3}{2}$  **15.**  $\ln 10$  **17.**  $\frac{\ln 2 - \ln 3}{\ln 6}$  **19.**  $\frac{3 \log 2 - \log 5}{2 \log 2 + \log 5}$

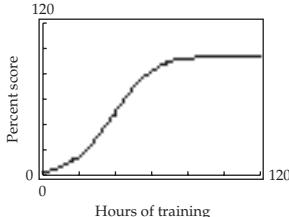
**21.** 7 **23.** 4 **25.**  $2 + 2\sqrt{2}$  **27.**  $\frac{199}{95}$  **29.** -1 **31.** 3 **33.**  $10^{10}$  **35.** 2 **37.** no solution **39.** 5 **41.**  $\log(20 + \sqrt{401})$

**43.**  $\frac{1}{2} \log\left(\frac{3}{2}\right)$  **45.**  $\ln(15 \pm 4\sqrt{14})$  **47.**  $\ln(1 + \sqrt{65}) - \ln 8$  **49.** 1.61 **51.** 0.96 **53.** 2.20 **55.** -1.93 **57.** -1.34

**59.** **a.** 8500, 10,285 **b.** in 6 years **61.** **a.** 60°F **b.** 27 minutes **63.** 3.7 years **65.** 6.9 months

**67.** 6.67 seconds and 10.83 seconds

**69.** **a.**

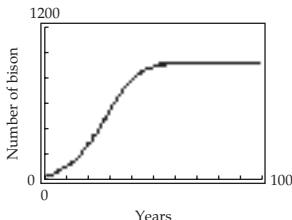


**b.** 48 hours

**c.**  $P = 100$

**d.** As the number of hours of training increases, the test scores approach 100%.

**71.** **a.**

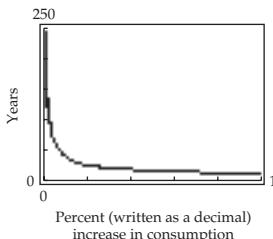


**b.** in 27 years, or the year 2026

**c.**  $B = 1000$

**d.** As the number of years increases, the bison population approaches, but never reaches or exceeds, 1000.

**73.** **a.**



**b.** 78 years

**c.** 1.9%

**75.** **a.** 1.72 seconds

**b.**  $v = 100$

**c.** The object cannot fall faster than 100 feet per second.

**77.** 138 withdrawals

**79.** The second step. Because  $\log 0.5 < 0$ , the inequality sign must be reversed.

**81.**  $x = \frac{y}{y - 1}$

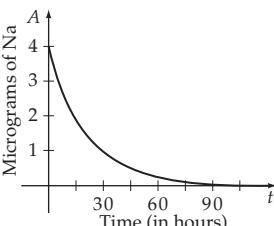
**83.**  $e^{0.336} \approx 1.4$

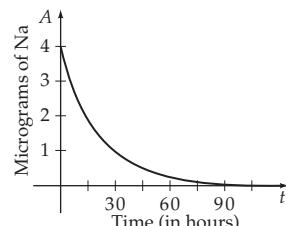
## Prepare for This Section (7.5), page 506

PS1. 1220.39 PS2. 824.96 PS3. -0.0495 PS4. 1340 PS5. 0.025 PS6. 12.8

## Exercise Set 7.5, page 516

1. a. 2200 bacteria b. 17,600 bacteria 3. a.  $N(t) \approx 22,600e^{0.01368t}$  b. 27,700 5.  $N(t) \approx 362,300 e^{0.011727t}$ ; 402,600

7. a.  b. 3.18 micrograms c.  $\approx 15.07$  hours d.  $\approx 30.14$  hours 9.  $\approx 6601$  years ago

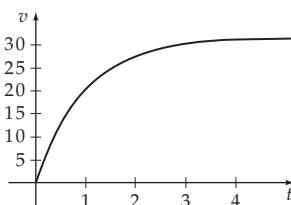


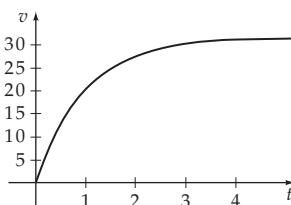
11.  $\approx 2378$  years old 13. a. \$9724.05 b. \$11,256.80 15. a. \$48,885.72 b. \$49,282.20 c. \$49,283.30 17. \$24,730.82 19. 8.8 years

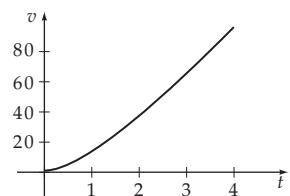
$$21. t = \frac{\ln 3}{r} \quad 23. 14 \text{ years} \quad 25. \begin{array}{lll} \text{a. } 1900 & \text{b. } 0.16 & \text{c. } 200 \end{array} \quad 27. \begin{array}{lll} \text{a. } 157,500 & \text{b. } 0.04 & \text{c. } 45,000 \end{array} \quad 29. \begin{array}{lll} \text{a. } 2400 & \text{b. } 0.12 & \text{c. } 300 \end{array}$$

$$31. P(t) \approx \frac{5500}{1 + 12.75e^{-0.37263t}} \quad 33. P(t) \approx \frac{100}{1 + 4.55556e^{-0.22302t}} \quad 35. \begin{array}{lll} \text{a. } \$158,000, \$163,000 & \text{b. } \$625,000 & \text{c. } P(t) \approx \frac{1600}{1 + 4.12821e^{-0.06198t}} \end{array}$$

- b. 497 wolves 39. a.  $P(t) \approx \frac{8500}{1 + 4.66667e^{-0.14761t}}$  b. 2016 41. a. 0.056 b. 42°F c. after 54 minutes 43. a. 211 hours b. 1386 hours

45. 3.1 years 47. a.  b. 0.98 second c.  $v = 32$  d. As time increases, the velocity approaches, but never reaches or exceeds, 32 feet per second.



49. a.  b. 2.5 seconds c.  $\approx 24.56$  feet per second d. The average speed of the object was approximately 24.56 feet per second during the period from  $t = 1$  to  $t = 2$  seconds.

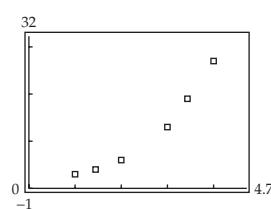
51. 45 hours 53. a. 0.71 gram b. 0.96 gram c. 0.52 gram 55. 2.91%

## Prepare for This Section (7.6), page 521

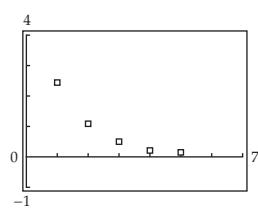
PS1. decreasing PS2. decreasing PS3. 36 PS4. 840 PS5. 15.8 PS6.  $P = 55$ 

## Exercise Set 7.6, page 529

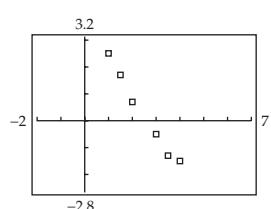
1. increasing exponential function



3. decreasing exponential function; decreasing logarithmic function



5. decreasing logarithmic function

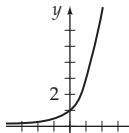
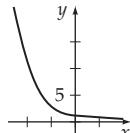
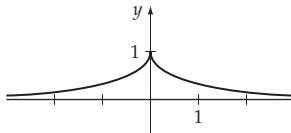
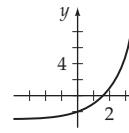
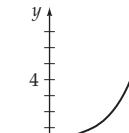
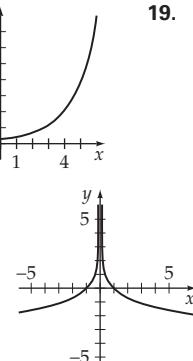
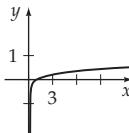
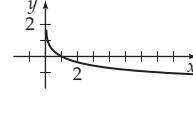
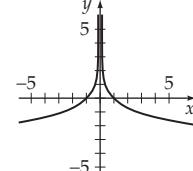
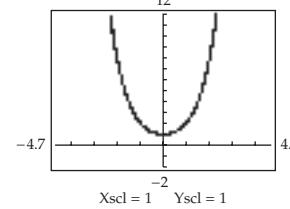
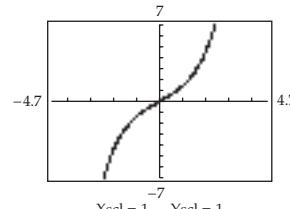


7.  $y \approx 0.99628(1.20052)^x$ ;  $r \approx 0.85705$     9.  $y \approx 1.81505(0.51979)^x$ ;  $r \approx -0.99978$     11.  $y \approx 4.89060 - 1.35073 \ln x$ ;  $r \approx -0.99921$   
 13.  $y \approx 14.05858 + 1.76393 \ln x$ ;  $r \approx 0.99983$     15.  $y \approx \frac{235.58598}{1 + 1.90188e^{-0.05101x}}$     17.  $y \approx \frac{2098.68307}{1 + 1.19794e^{-0.06004x}}$   
 19. a. LinReg:  $P \approx 0.22129t + 3.99190$ ,  $r \approx 0.99288$ ; ExpReg:  $P \approx 4.05326(1.04460)^t$ ,  $r = 0.99412$  b. the exponential model c. \$8.15  
 21. a.  $T \approx 0.06273(1.07078)^F$  b. 5.3 hours    23. a.  $T \approx 0.07881(1.07259)^F$  b. 7.5 hours; 2.2 hours    25. An increasing logarithmic model provides a better fit because of the concave-downward nature of the graph.    27. a.  $p \approx 7.862(1.026)^y$  b. 36 centimeters  
 29. a. LinReg:  $\text{pH} \approx 0.01353q + 7.02852$ ,  $r \approx 0.956627$ ; LnReg:  $\text{pH} \approx 6.10251 + 0.43369 \ln q$ ,  $r \approx 0.999998$ . The logarithmic model provides a better fit.    b. 126.0    31. a.  $p \approx 3200(0.91894)^t$ ; 2012    b. No. The model fits the data perfectly because there are only two data points.  
 33. a. ExpReg:  $S \approx 7062.46390(0.956776)^t$ ,  $r \approx -0.89618$ ; LnReg:  $S \approx 6995.50673 - 841.74326 \ln t$ ,  $r \approx -0.96240$  b. the logarithmic model  
 c. 4977    35. a.  $P(t) \approx \frac{11.26828}{1 + 2.74965e^{-0.02924t}}$  b. 11 billion people    37. A and B have different exponential regression functions.  
 39. a. ExpReg:  $y \approx 1.81120(1.61740)^x$ ,  $r \approx 0.96793$ ; PwrReg:  $y \approx 2.09385(x)^{1.40246}$ ,  $r \approx 0.99999$  b. The power regression function provides the better fit.

## Chapter 7 Assessing Concepts, page 538

1. False;  $f(x) = x^2$  does not have an inverse function.    2. True    3. True    4. False;  $h(x)$  is not an increasing function for  $0 < b < 1$ .  
 5. False;  $j(x)$  is not an increasing function for  $0 < b < 1$ .    6. c    7. b    8. f    9. a    10. e    11. d    12. g

## Chapter 7 Review Exercises, page 539

1. 2 [7.2]    2. 4 [7.2]    3. 3 [7.2]    4.  $\pi$  [7.2]    5.  $-2$  [7.4]    6. 8 [7.4]    7.  $-3$  [7.4]    8.  $-4$  [7.4]    9.  $\pm 1000$  [7.4]    10.  $\pm 10^{10}$  [7.4]    11. 7 [7.4]  
 12.  $\pm 8$  [7.4]    13. [7.1]
14. [7.1] 
15. [7.1] 
16. [7.1] 
17. [7.1] 
18. [7.1] 
19. [7.2] 
20. [7.2] 
21. [7.2] 
22. [7.2] 
23. [7.1] 
24. [7.1] 
25.  $4^3 = 64$  [7.2]    26.  $\left(\frac{1}{2}\right)^{-3} = 8$  [7.2]    27.  $(\sqrt{2})^4 = 4$  [7.2]    28.  $e^0 = 1$  [7.2]

29.  $\log_5 125 = 3$  [7.2]    30.  $\log_2 1024 = 10$  [7.2]    31.  $\log_{10} 1 = 0$  [7.2]    32.  $\log_8 2\sqrt{2} = \frac{1}{2}$  [7.2]    33.  $2 \log_b x + 3 \log_b y - \log_b z$  [7.3]

34.  $\frac{1}{2} \log_b x - 2 \log_b y - \log_b z$  [7.3]    35.  $\ln x + 3 \ln y$  [7.3]    36.  $\frac{1}{2} \ln x + \frac{1}{2} \ln y - 4 \ln z$  [7.3]    37.  $\log(x^2\sqrt[3]{x+1})$  [7.3]

38.  $\log \frac{x^5}{(x+5)^2}$  [7.3]    39.  $\ln \frac{\sqrt{2xy}}{z^3}$  [7.3]    40.  $\ln \frac{xz}{y}$  [7.3]    41. 2.86754 [7.3]    42. 3.35776 [7.3]    43.  $-0.117233$  [7.3]    44.  $-0.578989$  [7.3]

45.  $\frac{\ln 30}{\ln 4}$  [7.4]    46.  $\frac{\log 41}{\log 5} - 1$  [7.4]    47. 4 [7.4]    48.  $\frac{1}{6}e$  [7.4]    49. 4 [7.4]    50. 15 [7.4]    51.  $\frac{\ln 3}{2 \ln 4}$  [7.4]    52.  $\frac{\ln(8 \pm 3\sqrt{7})}{\ln 5}$  [7.4]
53.  $10^{1000}$  [7.4]    54.  $e^{(c^2)}$  [7.4]    55. 1,000,005 [7.4]    56.  $\frac{15 + \sqrt{265}}{2}$  [7.4]    57. 81 [7.4]    58.  $\pm\sqrt{5}$  [7.4]    59. 4 [7.4]    60. 5 [7.4]
61. 7.7 [7.3]    62. 5.0 [7.3]    63. 3162 to 1 [7.3]    64. 2.8 [7.3]    65. 4.2 [7.3]    66.  $\approx 3.98 \times 10^{-6}$  [7.3]    67. a. \$20,323.79    b. \$20,339.99 [7.5]
68. a. \$25,646.69    b. \$25,647.32 [7.5]    69. \$4,438.10 [7.5]    70. a. 69.9%    b. 6 days    c. 19 days [7.5]    71.  $N(t) \approx e^{0.8047t}$  [7.5]
72.  $N(t) \approx 2e^{0.5682t}$  [7.5]    73.  $N(t) \approx 3.783e^{0.0558t}$  [7.5]    74.  $N(t) \approx e^{-0.6931t}$  [7.5]    75. a.  $P(t) \approx 25,200e^{0.06155789t}$     b. 38,800 [7.5]
76. 340 years [7.5]    77. Answers will vary. [7.6]    78. a. ExpReg:  $R \approx 179.94943(0.968094)^t$ ,  $r \approx -0.99277$ ; LnReg:  $R \approx 171.19665 - 35.71341 \ln t$ ,  $r \approx -0.98574$     b. The exponential equation provides a better fit for the data.    c. 5.4 per 1000 live births [7.6]    79. a.  $P(t) \approx \frac{1400}{1 + \frac{17}{3}e^{-0.22458t}}$     b. 1070 coyotes [7.5]    80. a.  $21\frac{1}{3}$     b.  $P(t) \rightarrow 128$  [7.5]

## Chapter 7 Quantitative Reasoning Exercises, page 541

**QR1.** ExpReg:  $S \approx 5.94860(1.72192)^t$ ; logistic:  $S \approx \frac{244.56468}{1 + 39.38651e^{-0.57829t}}$  [7.6]    **QR2.** exponential: 1363.1 million; logistic: 218.1 million [7.6]

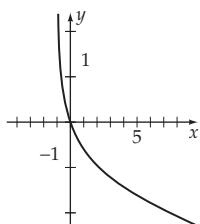
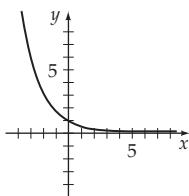
**QR3.**  $S \approx 58.73554 + 16.75801 \ln t$  [7.6]    **QR4.** 88.8 million [7.6]    **QR5.**  $y \approx \frac{85.24460}{1 + 14.0040e^{-0.70591t}}$ , 84.2 million [7.6]

**QR6.** Answers will vary. [7.6]

## Chapter 7 Test, page 542

1. a.  $b^c = 5x - 3$  [7.2]    b.  $\log_3 y = \frac{x}{2}$  [7.2]    2.  $2 \log_b z - 3 \log_b y - \frac{1}{2} \log_b x$  [7.3]    3.  $\log \frac{2x+3}{(x-2)^3}$  [7.3]    4. 1.7925 [7.3]

5. [7.1]    6. [7.2]    7. 1.9206 [7.4]    8.  $\frac{5 \ln 4}{\ln 28}$  [7.4]    9. 1 [7.4]    10. -3 [7.4]



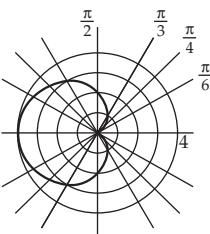
11. a. \$29,502.36    b. \$29,539.62 [7.5]    12. 17.36 years [7.5]    13. a. 7.6    b. 63 to 1 [7.3]    14. a.  $P(t) \approx 34,600e^{0.04667108t}$     b. 55,000 [7.5]

15. 690 years [7.5]    16. a.  $y \approx 1.67199(2.47188)^x$     b. 1945 [7.6]    17. a. LnReg:  $d \approx 67.35501 + 2.54015 \ln t$ ; logistic:  $d \approx \frac{72.03783}{1 + 0.15279e^{-0.67752t}}$     b. logarithmic: 73.67 meters; logistic: 72.03 meters [7.6]    18. a.  $P(t) \approx \frac{1100}{1 + 5.875e^{-0.20429t}}$     b. about 457 raccoons [7.5]

## Cumulative Review Exercises, page 543

1.  $\cos(x^2 + 1)$  [1.5]    2.  $f^{-1}(x) = \frac{1}{2}x - 4$  [1.6]    3.  $\sin \theta = \frac{3}{5}$ ,  $\cos \theta = \frac{4}{5}$ ,  $\tan \theta = \frac{3}{4}$  [2.2]    4. 17 centimeters [2.2]
5. amplitude: 4, period:  $\pi$ , phase shift:  $\frac{\pi}{4}$  [2.5]    6. amplitude:  $\sqrt{2}$ , period:  $2\pi$  [3.4]    7. odd [2.4]    9.  $\frac{12}{5}$  [3.5]    10.  $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$  [3.6]
11. magnitude: 5, direction:  $126.9^\circ$  [4.3]    12.  $176.8^\circ$  [4.3]    13. ground speed:  $\approx 420$  mph; course:  $\approx 55^\circ$  [4.1/4.2]    14.  $26^\circ$  [4.1]    15. 16 [5.3]
16.  $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$  [5.3]    17.  $(\sqrt{2}, 45^\circ)$  [6.5]    18. 

A unit circle centered at the origin of a Cartesian coordinate system. The circle is divided into four quadrants by the x and y axes. Angles are labeled in radians:  $\frac{\pi}{2}$  at the top,  $\frac{\pi}{3}$  in the first quadrant,  $\frac{\pi}{4}$  in the first quadrant,  $\frac{\pi}{6}$  in the first quadrant, and  $\pi$  at the bottom. The angle  $\pi$  is also labeled on the negative x-axis. The angle  $4$  is labeled on the positive x-axis.
19. 1.43 [7.4]    20. 1.8 milligrams [7.5]



# Index

- Abscissa, 15  
Absolute value  
of a complex number, 343–344  
of a real number, 3  
Absolute value function, 22, 34  
Absolute value inequalities, 10–11  
Acidity, 487–489  
Acute angles, 119  
Addition  
of complex numbers, 336  
of functions, 69–70, 197  
of ordinates, 197  
of real numbers to inequalities, 7  
of vectors, 313, 315, 316  
Additive inverse, of a vector, 315  
Adjacent side, 134, 135  
Agnesi, Maria, 19  
Air resistance, 206, 499–500, 515  
Airspeed, 318  
Alkalinity, 487–489  
Amplitude  
of cosine functions, 174, 175  
of simple harmonic motion, 204  
of sine functions, 170–171, 172–173  
Amplitude-Time-Difference Formula, 487  
Analytic geometry, 15, 17  
*See also* Conic sections  
Angle of depression, 139, 140  
Angle of elevation, 139–140, 141  
Angle of rotation, 404–405, 406  
Angles  
classification of, 119–122  
definition of, 118  
degree measure of, 118–123, 124–126  
naming of, 118  
negative, 118  
positive, 118  
radian measure of, 123–126  
reference angle of, 150–153  
standard position of, 120  
between vectors, 322  
Angular speed, 127–129  
Aphelion, 383  
Appollonius, 363  
Arc, 123, 124, 133  
Archimedes, 2, 306  
Arc length, 126–127  
 $\arcsin x$ , 256  
Area  
of polygon, 146  
of triangle, 305–307  
Argand diagram, 342  
Argument, of complex number, 344, 354  
Asymptotes  
of hyperbolas, 392, 394  
vertical, 38–39  
Average velocity, 71–72
- Axis, coordinate. *See* Coordinate axis  
Axis of symmetry, 54–55, 57  
of conic in polar coordinates, 427  
of ellipse, 375  
of parabola, 363, 364, 367  
*See also* Symmetries of graphs
- Base  
of exponential function, 452–453  
of logarithmic function, 467, 482–484
- Bearing, 297
- Beats, 202, 254
- Benford's Law, 479
- Bernoulli, Johann, 437
- Brachistochrone problem, 437–438
- Butterfly curve, 425
- Calculators. *See* Graphing calculators
- Carbon dating, 508–509
- Cardioid, 416, 417
- Carrying capacity, 513
- Cartesian (rectangular) coordinates, 15–16  
rotation of, 404  
transformation to/from polar coordinates, 419–423
- Catenary, 466
- Central angle, 123, 124
- Change-of-base formula, 482–484
- Circles  
arc length of, 126–127  
arc of, 123, 124  
as conic sections, 363, 375  
definition of, 24  
equations of, 24–26, 407  
great circle, 330–331  
identifying with discriminant, 407–408  
involute of, 442  
parametric form of, 435  
polar equations of, 413, 415, 432  
sector of, 133  
unit circle, 155–157, 159–160
- Circular functions, 157
- cis notation, for complex number, 344
- Closed intervals, 2, 3
- Coefficient of determination, 100, 101, 102, 524, 525, 527
- Cofunction identities, 227–228
- Common logarithms, 473, 474
- Complementary angles, 120
- Completing the square, 26
- Complex conjugates, 338–339
- Complex numbers, 335–340  
absolute value of, 343–344  
addition of, 336  
argument of, 344, 354  
division of, 338–339, 347, 348  
graphs of, 342, 343
- modulus of, 344  
multiplication of, 337–338, 346, 348  
powers of, 350–351  
roots of, 352–354, 443  
standard form of, 343  
trigonometric form of, 343–348
- Complex plane, 342, 343
- Components of a vector, 314, 315, 317–318, 319
- Composition of functions, 72–76  
of function with its inverse, 84–85  
logarithmic and exponential, 468  
trigonometric, 259–263
- Compound interest, 509–512
- Compressing graphs, 61–63
- Computer algebra systems, 389
- Concavity, 522–523, 526
- Conic sections, 363, 374–375, 390  
degenerate, 375, 390, 403, 407  
focus-directrix definitions of, 426–428  
general equation of, 403, 407  
graphing with calculator, 408–409  
identification theorem for, 407–408  
polar equations of, 426–430  
rotation of, 403–407  
*See also* Circles; Ellipses; Hyperbolas; Parabolas
- Conjugate  
of complex number, 338–339  
of trigonometric expression, 219–220
- Conjugate axis, of hyperbola, 391
- Constant functions, 40
- Continuous functions, 41
- Conversion factors, 122
- Coordinate axis  
polar, 412  
real number line, 2  
rotation of, 404  
symmetry with respect to, 54–55, 57
- Coordinate plane, 15, 16
- Coordinates  
of point in a plane, 15–16, 412–413  
of point on real number line, 2  
*See also* Cartesian (rectangular) coordinates; Polar coordinates
- Correlation coefficient, 99–100, 524, 525, 527
- Cosecant  
of acute angles, 134–135  
of any angle, 147–148  
domain and range of, 160  
graph of, 185–187, 196  
inverse function of, 257, 258–259  
as odd function, 160  
period of, 161–162  
of quadrantal angles, 148–149  
of a real number, 157–158  
sign of, 149  
of special angles, 136–138

- Cosine  
 of acute angles, 134–135  
 of any angle, 147–148  
 approximating with calculator, 139  
 domain and range of, 159, 160  
 dot product and, 321–322  
 of double angle, 237  
 as even function, 160  
 graph of, 173–177  
 graph translations of, 192–193, 195–196  
 of half angle, 240–241  
 inverse function of, 256–257, 259–263  
**Law of Cosines**, 302–305  
 period of, 161, 162  
 of quadrant angles, 148–149  
 of a real number, 157–162  
 reference angle and, 152  
 sign of, 149  
 of special angles, 136–138  
 of sum or difference of angles, 226–227, 231
- Cotangent**  
 of acute angles, 134–135  
 of any angle, 147–148  
 domain and range of, 160  
 graph of, 184–185  
 graph translations of, 194  
 inverse function of, 257, 258, 259  
 as odd function, 160  
 period of, 162  
 of quadrant angles, 148–149  
 of a real number, 157–158  
 sign of, 149  
 of special angles, 136–138
- Coterminal angles**, 121–122
- Critical value method**, 8–10
- Cryptography**, 114–115
- Cube roots**, 352
- Cubic equations**, 5
- Curve**  
 orientation of, 437  
 parametric equations for, 433–434
- Cycle**, of sine graph, 170
- Cycloid**, 437–438
- Damped harmonic motion**, 206–207
- Damping factor**, 198
- Day of the week**, 53
- Decibels (dB)**, 491
- Decimal degree method**, 122–123
- Decimals**, 2
- Decreasing functions**, 40, 82
- Degree measure**, 118–123  
 conversion to/from radians, 124–126
- de Moivre**, Abraham, 351
- De Moivre's Theorem**, 350–354
- Dependent variable**, 32
- Descartes**, René, 15, 335
- Difference**. *See Subtraction*
- Difference quotient**, 70–72
- Direction angle**, 315, 317
- Directrix**  
 of conic sections, 426–428  
 of parabola, 363, 364, 367
- Dirichlet**, Peter Gustav, 68
- Dirichlet function**, 68
- Discontinuities**, 41, 42
- Discriminant**, of second-degree equation in two variables, 407–408
- Displacement**  
 in simple harmonic motion, 204  
 as vector quantity, 313  
 work and, 324
- Distance**  
 in Cartesian coordinates, 17  
 between points on a line, 3, 17  
 in polar coordinates, 425  
 speed and, 128  
*See also Displacement*
- Distance formula**, 17
- Division**  
 of complex numbers, 338–339, 347, 348  
 of functions, 69–70  
 of inequalities by real numbers, 7
- DMS (Degree, Minute, Second) method**, 122–123
- Domain**  
 of function, 32, 36  
 of inverse function, 82  
 of sum or product of functions, 69–70
- Dot product**, 320–324
- Double-angle identities**, 236–238
- e** (base of natural exponential function), 458
- Earthquakes**, 484–487, 492–493
- Eccentricity**  
 of conic sections, 426–428  
 of ellipse, 381–382  
 of hyperbola, 396
- Ellipses**, 363, 374–384  
 applications of, 382–384  
 with center at  $(0, 0)$ , 375–378  
 with center at  $(h, k)$ , 378–381  
 definition of, 374–375  
 degenerate, 375  
 eccentricity of, 381–382  
 general equation of, 403, 405–406, 407  
 identifying with discriminant, 407–408  
 latus rectum of, 389  
 parametric form of, 436  
 polar equations of, 426–428, 429  
 reflective property of, 384
- Endpoint**, of ray, 118
- Epicycloid**, 442
- Epirochoid**, 448–449
- Equality**  
 of ordered pairs, 17  
 of real numbers, 2
- Equality of Exponents Theorem**, 495
- Equations**  
 cubic, 5  
 definition of, 4  
 exponential, 494–497  
 functions represented by, 32  
 linear in one variable, 4  
 literal, 4–5  
 logarithmic, 497–500, 512  
 parametric, 432–440, 447–449  
 polar, 413–419, 421–423, 426–430  
 quadratic, 5–6  
 quartic, 5  
 second-degree in two variables, 403, 407  
 trigonometric, 262, 268–275, 288–289, 328  
 in two variables, 19–26  
*See also Solutions*
- Equilibrium point**, 204
- Equivalent inequalities**, 7
- Equivalent vectors**, 313
- Euler**, Leonhard, 2, 32, 458
- Evaluating a function**, 33, 70
- Even and odd functions**, 56–57  
 trigonometric, 160–161
- Exponential decay**, 506, 508–509, 535–536
- Exponential equations**, 494–497
- Exponential functions**, 452–461  
 definition of, 452–453  
 evaluating, 453  
 graphs of, 453–457  
 logarithmic functions and, 467–469  
 models based on, 451, 459–461, 521–527  
 natural, 458–459  
 properties of, 455
- Exponential growth**, 506–507, 511, 512
- Exponential regression**, 524, 525, 527
- Exponents**  
 equality of, 495  
*See also Powers*
- Factoring quadratic equations**, 6
- Falling objects**, 499–500, 515
- Family of curves**, 58–59, 210
- Fermat**, Pierre de, 15
- Floor function**, 41–44
- Focus**  
 of conic sections, 426–428  
 of ellipse, 375, 376, 377, 379  
 of hyperbola, 390, 391, 394, 396  
 of parabola, 363, 364, 367, 369
- Force vectors**, 318, 319–320, 323–324
- Formulas**, 45
- Fractals**, 334, 341
- Fractions**  
 as conversion factors, 122  
 as rational numbers, 2
- Frequency**, of harmonic motion, 204, 205, 207
- Functions**, 31–32  
 algebraic operations on, 69–70, 197  
 composition of, 72–76, 84–85  
 continuous, 41  
 definition of, 32  
 difference quotient of, 70–72  
 domain of, 32, 36  
 evaluating, 33, 70  
 even and odd, 56–57  
 families of, 58–59, 210  
 graphs of, 37–41  
 identifying, 35–36  
 inverse, 81–90  
 notation for, 33  
 one-to-one, 40–41, 82, 84, 86  
 piecewise-defined, 33–35, 108–109

- range of, 32  
 vertical line test for, 39–40
- Golden rectangles, 14  
 Graphing calculators  
     adjusting settings, 44  
     angle measure, 123, 125, 126  
     complex numbers, 338  
     connected mode, 42  
     dot mode, 42, 109  
     EPIT program, 448  
     estimating  $x$ -intercepts, 23–24  
     evaluating exponential expressions, 458  
     evaluating inverse trigonometric  
         functions, 283–284  
     evaluating logarithms, 474  
     evaluating trigonometric functions,  
         138–139  
     finding  $n$ th roots of complex number, 443  
     graphing absolute value equation, 22  
     graphing Colosseum model, 389  
     graphing conic sections, 408–409  
     graphing cycloid, 438  
     graphing damped harmonic motion, 207  
     graphing ellipses, 380  
     graphing equations in two variables, 22  
     graphing exponential functions, 459, 460  
     graphing families of curves, 58–59, 210  
     graphing functions, 37, 42, 44  
     graphing hyperbolas, 395  
     graphing inverse functions, 85–86, 90  
     graphing inverse trigonometric  
         functions, 263, 264  
     graphing logarithmic functions, 474, 484  
     graphing Mandelbrot set, 359–360  
     graphing parabolas, 366  
     graphing parametric equations, 438, 439  
     graphing piecewise functions, 108–109  
     graphing polar equations, 417, 419  
     graphing second-degree equations in two  
         variables, 408–409  
     greatest integer function, 41, 42  
     holes in graphs, 68  
     isolated points in graphs, 68  
     LIST feature, 58–59  
     logistic models, 528–529  
     Mandelbrot iteration procedure, 356–357  
     median–median line, 108  
     modeling guidelines, 526–527  
     POWERMOD program, 115  
     REFANG program, 151  
     regression analysis, 97–98, 101–102,  
         275–277, 524, 525, 527  
     ROTATE program, 408  
     solving exponential equations, 495–497  
     solving trigonometric equations, 273–275,  
         328  
     SQUARE viewing window, 86  
     TABLE feature, 20–21  
     viewing window, 85–86  
     WANKEL program, 447–448  
     ZERO feature, 23–24
- Graphs  
     of circles, 413, 415  
     of complex numbers, 342, 343  
     compressing, 61–63  
     of conic sections, 403, 405–408, 426–429  
     of cosecant functions, 185–187, 196  
     of cosine functions, 173–177, 192–193,  
         195–196  
     of cotangent functions, 184–185, 194  
     of difference of functions, 197  
     of ellipses, 375–376, 427, 429  
     of equations in two variables, 19–22  
     of even and odd functions, 57  
     of exponential functions, 453–457  
     of floor function, 41–42  
     of functions, 37–41  
     of hyperbolas, 391–392, 427, 428–429  
     intercepts of, 22–24  
     of inverse functions, 83–84, 85–86, 90  
     of inverse trigonometric functions, 263  
     of logarithmic functions, 470–472  
     of parabolas, 22, 364, 366, 367, 427, 430  
     of parametric curves, 433–434  
     of polar equations, 413–419, 427, 428–429  
     reflections of, 60  
     scatter diagrams, 16  
     of secant functions, 187–189  
     semilog, 535–536  
     of sets of real numbers, 2–3  
     of sine functions, 169–173  
     stretching, 61–63  
     of sum of functions, 197  
     of tangent functions, 180–183  
     translations of, 58–59, 60  
     See also Symmetries of graphs
- Great circle distance, 330–331  
 Greater than, 2  
 Greatest integer function, 41–44  
 Ground speed, 318  
 Growth models  
     exponential, 506–507, 511, 512  
     logistic, 513–514, 528–529
- Half-angle identities, 240–242  
 Half-life, 508  
 Half-line, 118  
 Half-open intervals, 2, 3  
 Harmonic motion  
     damped, 206–207  
     simple, 204–206  
 Head, of a vector, 313  
 Heading, 297, 331  
 Heron's formula, 306–307  
 Herschel, Caroline, 397  
 Holes in graphs, 68  
 Horizontal component, 317–318  
 Horizontal compressing and stretching,  
         62–63  
 Horizontal lines, 413  
 Horizontal line test, 40–41  
 Horizontal translations, 58, 59  
 Hyperbolas, 363, 390–398  
     applications of, 396–398  
     asymptotes of, 392, 394  
     with center at  $(0, 0)$ , 391–393  
     with center at  $(h, k)$ , 393–395
- definition of, 390  
 degenerate, 390  
 eccentricity of, 396  
 general equation of, 403, 407  
 identifying with discriminant, 407–408  
 polar equations of, 426–429  
 reflective property of, 398  
 Hypocycloid, 442
- i* (imaginary unit), 335  
     powers of, 339–340  
*i* (unit vector), 316  
 Identities, trigonometric. *See* Trigonometric identities  
 Imaginary axis, 343  
 Imaginary numbers, 335, 336  
 Imaginary part, 335  
 Imaginary unit, 335, 339–340  
 Increasing functions, 40, 82  
 Independent variable, 32  
 Inequalities, 6–8  
     with absolute values, 10–11  
     equivalent, 7  
     with polynomials, 8–10  
     properties of, 7  
     solution sets of, 6–7, 8, 10  
     solving by critical value method, 8–10  
 Inequality symbols, 2  
 Infinity, approach of function to, 38  
 Initial point, of vector, 313  
 Initial side, of angle, 118  
 Inner product, 320–324  
 Integers, 2  
 Intercepts, 22–24  
 Interest  
     compound, 509–512  
     simple, 509  
 Internet, 1  
 Interval notation, 2–3  
 Inverse functions, 81–83  
     composition of a function with, 84–85  
     finding, 86–89  
     graphs of, 83–84, 85–86, 90  
     logarithmic and exponential, 468  
     with restricted domain, 88  
     trigonometric, 255–265, 283–284  
 Inverse relations, 82, 84  
 Inverse trigonometric functions, 255–265  
     approximating with polynomials,  
         283–284  
     composition with, 259–263  
     equations with, 262  
     graphing with calculator, 263, 264  
     identities with, 259, 262–263  
 Involute of a circle, 442  
 Irrational numbers, 2, 458  
 Isolated points in graphs, 68  
 Iterates, 341, 356  
 Iteration, 341  
     Mandelbrot procedure, 356–357
- j* (unit vector), 316  
 Kepler's Laws, 389

- Latitude, 330–331  
 Latus rectum  
     of ellipse, 389  
     of parabola, 373  
 Law of Cosines, 302–305  
 Law of Sines, 293–298, 304  
 Least-squares regression line. *See* Linear regression  
 Less than, 2  
 Libby, Willard Frank, 509  
 Limaçon, 416–417  
 Line(s)  
     graphing an equation of, 19  
     parametric forms of, 435  
     polar equations of, 413, 432  
     symmetry with respect to, 54–55, 57, 414, 415, 416  
 Linear correlation coefficient, 99–100  
 Linear equations in one variable, 4  
 Linear regression, 95–100, 102  
 Linear speed, 127–129  
 Line of best fit. *See* Linear regression  
 Line segment, midpoint of, 18  
 Lissajous figures, 442  
 Literal equations, 4–5  
 Logarithmic decrement, 209  
 Logarithmic equations, 497–500, 512  
 Logarithmic functions, 467–475  
     applications of, 474–475, 484–489, 492–493  
     common, 473–474  
     definition of, 467  
     domains of, 472–473  
     exponential functions and, 467–468  
     graphs of, 470–472, 474, 484  
     modeling data with, 521–522, 523, 524, 526–527  
     natural, 473–474  
     properties of, 472  
 Logarithmic regression, 524, 526–527  
 Logarithmic scales, 484–489, 492–493  
 Logarithms  
     change-of-base formula for, 482–484  
     changing to exponential form, 468–469  
     common, 473, 474  
     definition of, 467  
     natural, 473, 474  
     properties of, 469–470, 480–482  
 Logistic models, 513–514, 520–521, 528–529  
 Longitude, 330–331  
 Mach numbers, 245  
 Magnitude of a vector, 313, 315, 321  
 Major axis, 375, 376, 377, 379, 381  
 Mandelbrot, Benoit B., 334  
 Mandelbrot iteration procedure, 356–357  
 Mandelbrot set, 357, 359–360  
 Maple, 389  
 Maximum value, 186  
 Measure of an angle, 118  
     in degrees, 118–123, 124–126  
     in radians, 123–126  
 Median–median line, 108  
 Midpoint formula, 18  
 Minimum value, 186  
 Minor axis, 375, 376  
 Minute (angle measure), 122–123  
 Modeling data, 524–529  
     linear, 95–100, 102  
     quadratic, 100–102  
     sinusoidal, 275–277  
 Modular functions, 1, 114–115  
 Modulus, of complex number, 344  
 Mollweide's formulas, 311  
 Motion  
     of falling objects, 499–500, 515  
     parametric equations for, 436–437  
     projectile, 273–275, 439  
     vibratory, 204–207  
     *See also* Force vectors; Speed  
 Multiplication  
     of complex numbers, 337–338, 346, 348  
     of functions, 69–70  
     of inequalities by real numbers, 7  
     of vectors (dot product), 320–324  
     of vectors (scalar), 313, 315, 316  
 Napier, John, 467  
 Natural exponential function, 458–461  
 Natural logarithms, 473, 474  
 Natural numbers, 2  
 Negative angles, 118  
 Never-negative function, 80–81  
 Nomograms, 492–493  
 Oblique triangle, 293, 304  
 Obtuse angle, 119  
 Odd and even functions, 56–57  
     trigonometric, 160–161  
 One-to-one function, 40–41  
     as inverse function, 82, 84, 86  
 Open intervals, 2, 3  
 Opposite side, 134, 135  
 Ordered pairs  
     as coordinates, 15, 16, 412–413  
     equality of, 17  
     of a function, 32, 37  
     graph of, 37  
     of inverse function, 82  
     of a relation, 31  
     as solutions of equation, 19  
 Ordinate, 15  
 Orientation of curve, 437  
 Origin, 15  
     symmetry with respect to, 55–56, 57  
 Orthogonal vectors, 323  
 Oscillation. *See* Harmonic motion  
 Parabolas, 363–370  
     applications of, 368–370  
     definition of, 363  
     general equation of, 403, 407  
     graphs of, 22, 364, 366, 367, 427, 430  
     identifying with discriminant, 407–408  
     latus rectum of, 373  
     parametric form of, 434  
 polar equations of, 426–428, 429–430  
     reflective property of, 369  
     with vertex at  $(0, 0)$ , 363–366  
     with vertex at  $(h, k)$ , 366–368  
     vertex of, 363, 364, 367, 369  
 Paraboloid, 369  
 Parallelogram method, of vector addition, 313  
 Parallel vectors, 323  
 Parameter, of family of functions, 58  
 Parametric equations, 432–440  
     for rotary engine, 447–449  
 Perihelion, 383  
 Period  
     of  $\cos bx$ , 174, 175  
     of  $\cot bx$ , 184  
     of  $\csc bx$ , 186  
     definition of, 161  
     of harmonic motion, 204  
     of  $\sec bx$ , 188  
     of  $\sin bx$ , 172  
     of sum of functions, 214  
     of  $\tan bx$ , 182  
     of trigonometric functions, 161–162  
 Periodic functions, 161–162  
 Perpendicular vectors, 323  
 Phase shift, 192–193, 194, 249–250  
 pH of a solution, 487–489  
 Pi ( $\pi$ ), 2  
 Piecewise-defined functions, 33–35  
     graphing with calculator, 108–109  
 Plotting points, 16  
     in graphing a function, 37–38  
     in graphing an equation, 20, 21–22  
 Point(s)  
     of coordinate plane, 15  
     of real number line, 2  
     symmetry with respect to, 55–56, 57, 414  
     *See also* Plotting points  
 Poiseuille, Jean Louis, 328  
 Poiseuille's Law, 328  
 Polar axis, 412  
 Polar coordinates, 412–413  
     distance in, 425  
     symmetries in, 414, 415, 416, 427  
     transformation to/from rectangular coordinates, 419–421  
 Polar equations, 413  
     of conic sections, 426–430  
     graphs of, 413–419  
     writing as rectangular equations, 421–423  
 Polar form, of complex number, 344  
 Pole, 412, 414  
 Polygons, areas and perimeters of, 146  
 Polynomials  
     approximating inverse trigonometric functions with, 283–284  
     inequalities with, 8–10  
     sign property of, 8  
     zeros of, 8  
 Population growth, 507, 513–514  
 Positive angles, 118  
 Power functions, 534

- Power-reducing identities, 239  
 Powers  
   of a complex number, 350–351  
   Equality of Exponents Theorem, 495  
   of  $i$ , 339–340  
 Principal, 509  
 Product. *See* Multiplication  
 Product-to-sum identities, 246–247  
 Projectile motion, 273–275, 439  
 Projection, scalar, 322–323  
 Protractor, 119, 124  
 Pseudoperiod, 207  
 Public key cryptography, 114–115  
 Pythagorean identities, 163, 217, 218–219  
 Pythagorean Theorem, 17
- Quadrantal angles, 121  
   trigonometric functions of, 148–149  
 Quadrants, 15  
 Quadratic equations, 5–6  
 Quadratic formula, 5–6  
 Quadratic regression, 100–102  
 Quartic equations, 5  
 Quotient. *See* Difference quotient; Division
- Radian measure, 123–126  
 Radioactive decay, 508–509  
 Radius of a circle, 24, 25, 26  
 Range  
   of a function, 32  
   of an inverse function, 82  
   of trigonometric functions, 159–160  
 Ratio identities, 163, 217  
 Rational numbers, 2  
 Ray, 118  
 Real axis, 343  
 Real number line, 2–3  
 Real numbers, 2  
 Real part, of complex number, 335  
 Reciprocal functions, 138  
 Reciprocal identities, 163, 217  
 Rectangular (Cartesian) coordinates, 15–16  
   rotation of, 404  
   transformation to/from polar  
     coordinates, 419–421  
 Rectangular form, of complex number,  
   343  
 Reduction formulas, trigonometric, 232  
 Reference angle, 150–153  
 Reflections of graphs, 60  
   of exponential functions, 457  
 Reflective property  
   of ellipse, 384  
   of hyperbola, 398  
   of parabola, 369  
 Regression analysis, 524–529  
   exponential, 524, 525, 527  
   linear, 95–100, 102  
   logarithmic, 524, 526–527  
   quadratic, 100–102  
   sine, 275–277  
 Relations, 31  
   inverse of, 82, 84
- Relative maximum, 186  
 Relative minimum, 186  
 Resolving a vector, 319  
 Resultant vector, 313  
 Richter, Charles F., 484  
 Richter scale, 484–487  
 Right angles, 119  
 Right triangles  
   applications of, 139–141  
   Pythagorean Theorem, 17  
   trigonometric expressions evaluated  
     with, 260–263  
   trigonometric functions defined with,  
     134–138  
 Roots  
   of a complex number, 352–354, 443  
   of an equation, 4  
   *See also* Solutions; Zero of a polynomial  
 Rose curves, 418  
 Rotary engine, 447–449  
 Rotation of axes, 404  
 Rotation Theorem for Conics, 404–407  
 Rounding numbers, 42–43, 140
- Scalar, 312  
 Scalar multiplication, of a vector, 313, 315,  
   316  
 Scalar product, of two vectors, 320–324  
 Scalar projection, 322–323  
 Scalar quantities, 312  
 Scatter diagrams, 16, 524, 526  
   *See also* Regression analysis  
 Secant  
   of acute angles, 134–135  
   of any angle, 147–148  
   domain and range of, 160  
   as even function, 160  
   graph of, 187–189  
   inverse function of, 257, 258, 259  
   period of, 161–162  
   of quadrantal angles, 148–149  
   of a real number, 157–158  
   sign of, 149  
   of special angles, 136–138  
 Second (angle measure), 122–123  
 Sector, 133  
 Seed, 341, 356  
 Seismograms, 486–487, 492–493  
 Semilog graphs, 535–536  
 Semimajor or semiminor axis, 375  
 Semiperimeter, 307  
 Set-builder notation, 2–3  
 Sets of real numbers, 2–3  
 Sign diagram, 9  
 Significant digits, 140  
 Simple harmonic motion, 204–206  
 Simple interest, 509  
 Sine  
   of acute angles, 134–135  
   of any angle, 147–148  
   domain and range of, 159, 160  
   of double angle, 236, 237–238  
   graph of, 169–173
- graph translations of, 192–193, 194  
 of half angle, 240  
 inverse function of, 255–256, 259–262,  
   263, 283–284  
 Law of Sines, 293–298, 304  
 modeling data with, 275–277  
 as odd function, 160  
 period of, 161, 162  
 of quadrantal angles, 148–149  
 of a real number, 157–158  
 reference angle and, 152  
 sign of, 149–150  
 of special angles, 136–138  
 of sum or difference of angles, 226, 229,  
   230, 232
- Sinusoidal data, 275–277  
 Solutions  
   of an equation in one variable, 4  
   of an equation in two variables, 19  
   extraneous, 270, 498–499  
   of a polar equation, 413  
   of a trigonometric equation, 268–269, 271,  
   273
- Solution set of an inequality, 6–7, 8, 10  
 Sørensen, Søren, 488  
 Sound  
   beats, 202, 254  
   decibels, 491  
   whispering galleries, 383–384  
 Sound tracks, 214  
 Special angles, 136–138  
 Speed  
   airspeed vs. ground speed, 318  
   angular vs. linear, 127–129  
   Mach number for, 245  
   as magnitude of velocity, 313  
   *See also* Velocity  
 Spring constant, 205  
 Square roots, of negative numbers, 335  
 Step function, 42  
 Straight angle, 119  
 Stretching graphs, 61–63  
   of exponential functions, 457  
 Subtending an angle, 123, 124  
 Subtraction  
   of complex numbers, 336  
   of functions, 69–70, 197  
   of real numbers from inequalities, 7  
   of vectors, 313  
 Sum. *See* Addition  
 Sum-to-product identities, 247–249  
 Supplementary angles, 120  
 Symmetries of graphs  
   of function and its inverse, 84, 85  
   in polar coordinates, 414, 415, 416, 427  
   with respect to coordinate axis, 54–55,  
   57  
   with respect to line, 54–55, 57, 414, 415,  
   416  
   with respect to point, 55–56, 57, 414  
   *See also* Axis of symmetry
- Tail, of vector, 313

- Tangent  
 of acute angles, 134–136  
 of any angle, 147–148  
 domain and range of, 159, 160  
 of double angle, 237  
 graphs of, 180–183  
 graph translations of, 194  
 of half angle, 240, 241  
 inverse function of, 257, 258, 259, 260, 264  
 as odd function, 160  
 period of, 162  
 of quadrantal angles, 148–149  
 of a real number, 157–158  
 reference angle and, 152  
 sign of, 149–150  
 of special angles, 136–138  
 of sum or difference of angles, 226, 229, 230
- Tangent lines, concavity and, 522
- Terminal point, of vector, 313
- Terminal side, of angle, 118
- Test value, 8–9
- Time  
 as parameter, 436–437  
 speed and, 128
- Transformation equations, 366, 378, 393
- Translations of graphs, 58–59, 60  
 of conic sections, 366, 378, 393, 403  
 of exponential functions, 456–457  
 of inverse trigonometric functions, 263  
 of logarithmic functions, 473  
 of trigonometric functions, 192–196
- Transverse axis, of hyperbola, 391, 394, 396
- Triangle method, of vector addition, 313
- Triangles  
 area of, 305–307  
 checking solutions with Mollweide's formulas, 311  
 oblique, 293, 304  
 solving with Law of Cosines, 302–305  
 solving with Law of Sines, 293–298, 304  
*See also* Right triangles
- Trigonometric equations, 268–275  
 applications of, 273–275, 288–289, 328  
 with inverse functions, 262
- Trigonometric expressions, evaluating, 260–261
- Trigonometric form, of complex numbers, 343–348
- Trigonometric functions  
 of acute angles, 134–136
- of any angle, 147–148  
 approximating with calculator, 138–139  
 composition with inverses, 259–263  
 domain and range of, 159–160  
 even and odd, 160–161  
 graphs of, 169–189  
 inverse, 255–265, 283–284  
 periodic properties of, 161–162  
 properties of, 159–162  
 of quadrantal angles, 148–149  
 of real numbers, 157–162  
 reciprocal, 138  
 reference angles and, 153  
 signs of, 149–150  
 of special angles, 136–138  
 translations of, 192–196
- Trigonometric identities  
 cofunctions, 227–228  
 double-angle, 236–238  
 fundamental, 163–165, 217  
 half-angle, 240–242  
 with inverse functions, 259, 262–263  
 odd-even, 160, 217  
 power-reducing, 239  
 product-to-sum, 246–247  
 Pythagorean, 163, 217, 218–219  
 ratio, 163, 217  
 reciprocal, 163, 217  
 reduction formulas, 232  
 sum of sines and cosines, 249–251  
 sum or difference of two angles, 226–227, 229–231  
 sum-to-product, 247–249  
 verification of, 217–222
- Unit circle, 155–157, 159–160
- Unit fraction, 122
- Unit vectors, 315–318
- Variable  
 dependent, 32  
 independent, 32
- Vectors, 312–324  
 addition of, 313, 315, 316  
 additive inverse of, 315  
 angle between, 322  
 applications of, 318–320, 323–324  
 components of, 314, 315, 317–318, 319  
 definition of, 313  
 direction angle of, 315, 317
- dot product of, 320–324  
 equivalent, 313  
 magnitude of, 313, 315, 321  
 orthogonal, 323  
 parallel, 323  
 scalar multiplication of, 313, 315, 316  
 scalar projection of, 322–323  
 subtraction of, 313  
 unit vectors, 315–318  
 zero vector, 315
- Velocity  
 average, 71–72  
 as vector, 313, 318  
*See also* Speed
- Vertex  
 of angle, 118  
 of ellipse, 375, 376, 377, 379  
 of hyperbola, 391, 394  
 of parabola, 363, 364, 367, 369
- Vertical asymptote, 38–39
- Vertical component, 317–318
- Vertical compressing and stretching, 61
- Vertical lines, 413
- Vertical line test, 39–40
- Vertical translations, 58–59, 60
- Vibratory motion, 204–207
- Wankel, Felix, 447, 448
- Wessel, Caspar, 342
- Whispering galleries, 383–384
- Work, force and, 323–324
- World Wide Web, 1
- Wrapping function, 155–157
- x*-axis, 15  
 reflection across, 60  
 symmetry with respect to, 54–55
- x*-coordinate, 15
- x*-intercepts, 22–24
- xy*-plane, 15, 16
- y*-axis, 15  
 reflection across, 60  
 symmetry with respect to, 54–55, 57
- y*-coordinate, 15
- y*-intercept, 22–23
- Zeller's Congruence, 53
- Zero of a polynomial, 8
- Zero product principle, 6
- Zero vector, 315

# Index of Applications

## • Agriculture

Area of a pasture, 310  
Crop yield, 51  
Dimensions of a fenced region, 13  
Dimensions of a garden, 13

## • Archeology

Age of a bone, 516  
Age of the Rhind Papyrus, 516  
Height of the Cheops pyramid, 144  
Pterosaur wingspan data, 103

## • Architecture and Art

Colosseum in Rome, 389  
Eiffel Tower, 502  
Eiffel Tower replica, 145  
Golden Gate Bridge cables, 113  
London bridge clearance, 388  
Observation angle, 282  
Petronas Towers, 145  
Saint Louis Gateway Arch, 466  
Washington Monument, 145

## • Astronomy and Space Science

Altitude of the sun, 281  
Apparent magnitude of stars, 477  
Cepheid variable stars, 179  
Diameter of the sun, 131  
Dimensions of telescopes, 372  
Distance from Earth to Jupiter, 144  
Focus of a radio telescope, 372  
Focus of an optical telescope, 372  
Height of the Eiffel Tower, 143  
Height of the space shuttle, 141, 299  
Kepler's Laws, 389  
Moons of Saturn, 283  
Neptune, 389  
Orbits of planets/comets, 144, 382, 383, 387, 388, 396, 397  
Path of a satellite, 167  
Percent of the moon illuminated, 275, 280  
Period-luminosity relationship, 179  
Radius of Earth, 132  
Recession velocities, 107  
Rotation time of the space shuttle, 133  
Satellite coverage, 266  
Velocity of the Hubble Space Telescope, 132

## • Automotive

Braking distance, 106  
Fuel efficiency, 106  
Headlight design, 373  
Movement of a bus, 282  
Speed of a car, 131

## • Biology and Life Sciences

Animal populations, 464, 503, 513, 517, 518, 520, 528, 533, 541, 543  
Bacteria, 104, 516  
Bertalanffy's equation, 502  
Biologic diversity, 493  
Drag resistance on a fish, 224  
Flying speeds, 105  
Larvae population, 106  
Logistic growth model, 513, 517  
Mosquito population, 81  
Oxygen consumption of a bird, 107  
Predator-prey interactions, 203

## • Business and Economics

Cellular telephone subscriptions, 540  
Cost and sales or profit, 49, 52  
Cost of a commercial lot, 310  
Demand for a product, 463  
Depreciation, 518, 540  
Digital camera sales, 541  
Money market rates, 477  
Monthly revenue, 14  
Postage rates, 48, 92  
Retail/wholesale price of a bracelet, 89  
Revenue, 16, 460, 517  
Sales, 27, 92, 200, 463, 517  
Telecommuting, 531

## • Chemistry

Henderson-Hasselbach function, 532  
Hydronium-ion concentration, 489, 491, 540  
Absorption of oxygen, 532  
pH of a solution, 487, 488, 491, 540  
Viscosity of oil, 535

## • Computers and Computer Science

Animated maps, 491  
Cryptology, 93, 94  
Internet connections, 463  
Moore's Law, 452  
Public key cryptology, 114

Speed of a disk, 128

Time to download a file, 113

## • Construction

Area of a housing tract, 310  
Construction of a box, 45  
Cooling tower design, 401, 402  
Cost of glass for the Luxor Hotel, 307  
Cross-sectional area of a rain gutter, 281  
Dimensions of trusses, 311  
Inclination angle of a ramp, 326, 330  
Length of a brace, 300  
Length of a guy wire, 51, 300  
Length of a street, 309  
Length of an airport runway, 299  
Fountain design, 371  
Placement of a light, 143  
Semielliptical plywood form, 388  
Vent pipe design, 386  
Whispering gallery, 384, 387

## • Consumer Applications

Clothing sizes, 92  
Cost of a car rental, 13  
Cost to paint a house, 52  
Cost of parking, 43  
Depreciation of a boat, 49  
Photochromatic eyeglass lenses, 463  
Value of a car over time, 44  
Value of a diamond, 524  
Video rental plans, 13

## • Earth Science

Amplitude-time distance formula, 487  
Earthquakes, 484, 485, 486, 491, 492, 540, 542  
Richter scale, 484, 492, 540, 542  
Seismogram, 486, 492, 540

## • Ecology and Demographics

Carbon dioxide levels, 200  
Generation of garbage, 530  
Natural resources, 504  
Oil spill, 72, 520  
Population growth of  
Aurora, Colorado, 516  
Hawaii, 532  
Miami, Florida, 516  
Population of a city, 501, 507, 516, 520, 540, 542  
World population, 532, 534

## • Entertainment

Amusement park ride, 131, 209  
Elliptical pool table, 387  
Horse racing, 131  
Linear speed on a carousel, 213  
Millennium Wheel, 158  
Movie ticket prices, 530  
Number of cinema sites, 533  
Photography, 132

## • Finance

Compensation, 92  
Compound interest, 509, 510, 511, 512, 516, 517, 540, 542  
Cost of a checking account, 13  
Currency conversion, 133  
Income tax, 48  
Payment plans, 13  
Per capita income, 27  
Retirement planning, 504

## • General Interest

3-D optical illusion, 374  
Average rate of change, 79  
Benford's Law, 479  
Birthday problem, 92  
Cubic formula, 14  
Dirichlet function, 68  
Exponential reward, 466  
Heron's formula, 306  
Mandelbrot iteration procedure, 356  
Median-median line, 108  
Mollweide's formulas, 311  
Never-negative function, 80  
Nomogram, 492  
Number of digits in  $b^x$ , 478  
Pay It Forward model, 464  
Zeller's Congruence, 53

## • Geography

Latitude, 133

## • Geometry

Epicycloid, 442  
Epitrochoid, 448  
Height of an inscribed cylinder, 50  
Hypocycloid, 442  
Involute of a circle, 442  
Perimeter of a scaled rectangle, 76, 78, 79

## • Medicine, Health, and Nutrition

Anesthesiology, 477  
Average remaining lifetime, 105  
Blood pressure, 195  
Calories, 106, 116

Concentration of a medication, 79, 93, 463, 516, 520  
*E. Coli* infection, 463  
Lithotripsy, 386  
Mortality rate, 541  
Optimal branching of arteries, 282, 328  
Optometry, 532  
Skin wound, 540

## • Meteorology

Average high temperature in Fairbanks, Alaska, 167  
Distance from a hurricane to a city, 298  
Hours of daylight, 201, 281, 290  
Minimum-maximum temperature range, 13  
Sunrise/sunset times, 279, 280, 288  
Temperatures, 106, 202  
Tidal cycle for Gray's Harbor, 208  
Tides, 202

## • Miscellaneous

Angle between the boundaries of a lot, 310  
Area of  
    a hexagon, 144  
    a regular  $n$ -gon, 146  
    a triangular lot, 310  
    a triangle, 306  
    an isosceles triangle, 144  
Average walking speed in a city, 474, 502  
Conversion functions, 79, 92  
Depth of a submarine, 309  
Dimensions of a  
    rectangle, 13  
    triangular plot of land, 300  
Distance between houses, 301  
Distance to  
    a plane, 310  
    an aircraft carrier, 299  
    a control tower, 191  
    a fire, 300  
    a hot air balloon, 299  
    a radar station, 140  
Distance across a  
    lake, 144, 310  
    marsh, 143  
    ravine, 450  
Height of a  
    building, 143, 145, 213, 296  
    hill, 300  
    kite, 300  
    paddle, 201  
    tree, 139, 213, 215  
    tower, 145  
Motion of a point, 437  
Number of diagonals of a polygon, 14

Perimeter of a regular  $n$ -gon, 146  
Volume of water in a  
    box, 49  
    cone, 50, 78  
    cylindrical tank, 266  
Wingspan of an airplane, 309

## • Music

Musical scales, 465  
Period of combined musical sound tracks, 214  
Sound waves and beats, 202, 254

## • Physics

Angular rotation and angular speed, 131  
Atmospheric pressure, 531  
Average velocity, 71  
Brachistochrone problem, 437  
Carbon dating, 508, 509, 540  
Current in a circuit, 518  
Damped harmonic motion, 206, 209, 213  
Decibel level, 491  
Distance an object falls in a given time, 31, 504, 519  
Distance a ball rolls down a ramp in a given time, 79, 107  
Fermat's Principle, 301  
Frequency of a sound wave, 178  
Hanging cable, 504  
Intensity of light/sound, 464  
Mach numbers, 245  
Newton's Law of Cooling, 518  
Oscilloscope and sound waves, 178  
Pendulum, 31, 534  
Projectiles, 14, 273, 274, 279, 439, 447  
Pulleys, 127, 130, 131  
Radiation, 463  
Radioactive decay, 508, 516, 544  
Rocket launch, 190  
Rotation versus lift distance of a winch, 131  
Simple harmonic motion, 205, 206, 208, 209, 213  
Snell's Law, 301  
Sonic boom, 401  
Temperature of a fluid, 459, 465, 533  
Time of descent, 113, 143  
Velocity, 504, 515, 519  
Voltage and amperage, 201  
Water waves, 402  
Work, 324, 326, 327, 330, 361

## • Psychology and Learning

Exposure to a rumor, 518  
Learning theory, 519  
Training/test scores, 503  
Typing speed, 477, 502

## • Sports and Recreation

Baseball bat speed, 104  
Dimensions of a baseball field, 34  
Distance from a golf tee to the pin, 299  
Distance of a drive by a golfer, 299  
Distance to first base, 309  
Exercise heart rate, 502  
Focus of a parabolic microphone, 372  
Launch angle of a basketball, 288  
Navigation of a boat, 505  
Number of professional sports teams in a city, 95, 98  
Olympic events, 526, 533  
Parabolic ski design, 373  
Scuba diving and hypothermia, 530, 531  
Sky diving, 499  
Speed of a boat, 104  
Speed of a dragster, 502  
Towing a boat, 78  
Width of a skateboard jump ramp, 143  
World records, 521, 531, 543

## • Technology and Engineering

Angle between vectors, 322, 361  
Angular speed of a wheel, 214  
Direct-current motors, 105  
Elliptical gears, 388  
Engine design, 310  
Focus of a satellite dish, 369, 372  
Forces in equilibrium, 326  
Hyperbolic gear, 401  
Linear speed of a point on a rotating wheel, 214, 450  
Logarithmic decrement of damped harmonic motion, 209  
LORAN, 397, 400  
Magnitude of a force, 319, 325, 326  
Reflective property of a hyperbola, 398  
Reflective property of a parabola, 369  
Reflective property of an ellipse, 384  
Rotary engine design, 447  
Scotch yoke, 201

Structural defects, 371

Tones on a touch-tone phone, 252, 253

Uniform motion simulations, 441

## • Travel

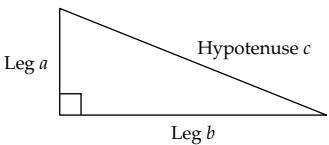
Bearing of a boat, 303  
Distance and bearing from a starting point, 310  
Distance between a ship and a lighthouse, 45, 297, 300  
Distance between airports, 300, 309  
Distance between ships, 50, 310  
Distance of a great circle route, 331  
Great circle routes, 330  
Ground speed/course of an airplane, 318, 325, 330, 333, 544  
Heading of a boat, 325, 333  
Initial heading required to fly a great circle route, 331  
Lighthouse beacon, 201  
Nautical miles and statute miles, 132

*This page intentionally left blank*

## Important Formulas

*Pythagorean Theorem*

$$c^2 = a^2 + b^2$$



The *distance* between  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The *slope*  $m$  of a line through  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

*Quadratic Formula*

If  $a \neq 0$ , the solutions of  $ax^2 + bx + c = 0$  are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## Properties of Functions

A *function* is a set of ordered pairs in which no two ordered pairs that have the same first coordinate have different second coordinates.

If  $a$  and  $b$  are elements of an interval  $I$  that is a subset of the domain of a function  $f$ , then

- $f$  is an *increasing* function on  $I$  if  $f(a) < f(b)$  whenever  $a < b$ .
- $f$  is a *decreasing* function on  $I$  if  $f(a) > f(b)$  whenever  $a < b$ .
- $f$  is a *constant* function on  $I$  if  $f(a) = f(b)$  for all  $a$  and  $b$ .

A *one-to-one* function satisfies the additional condition that given any  $y$ , there is one and only one  $x$  that can be paired with that given  $y$ .

## Graphing Concepts

*Odd Functions*

A function  $f$  is an odd function if  $f(-x) = -f(x)$  for all  $x$  in the domain of  $f$ . The graph of an odd function is symmetric with respect to the origin.

*Even Functions*

A function is an even function if  $f(-x) = f(x)$  for all  $x$  in the domain of  $f$ . The graph of an even function is symmetric with respect to the  $y$ -axis.

*Vertical and Horizontal Translations*

If  $f$  is a function and  $c$  is a positive constant, then the graph of

- $y = f(x) + c$  is the graph of  $y = f(x)$  shifted up vertically  $c$  units.
- $y = f(x) - c$  is the graph of  $y = f(x)$  shifted down vertically  $c$  units.
- $y = f(x + c)$  is the graph of  $y = f(x)$  shifted left horizontally  $c$  units.
- $y = f(x - c)$  is the graph of  $y = f(x)$  shifted right horizontally  $c$  units.

*Reflections*

If  $f$  is a function then the graph of

- $y = -f(x)$  is the graph of  $y = f(x)$  reflected across the  $x$ -axis.
- $y = f(-x)$  is the graph of  $y = f(x)$  reflected across the  $y$ -axis.

*Vertical Shrinking and Stretching*

- If  $c > 0$  and the graph of  $y = f(x)$  contains the point  $(x, y)$ , then the graph of  $y = c \cdot f(x)$  contains the point  $(x, cy)$ .
- If  $c > 1$ , the graph of  $y = c \cdot f(x)$  is obtained by stretching the graph of  $y = f(x)$  away from the  $x$ -axis by a factor of  $c$ .
- If  $0 < c < 1$ , the graph of  $y = c \cdot f(x)$  is obtained by shrinking the graph of  $y = f(x)$  toward the  $x$ -axis by a factor of  $c$ .

*Horizontal Shrinking and Stretching*

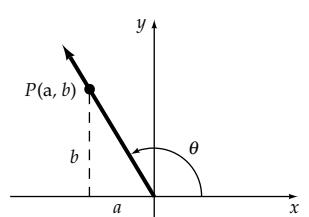
- If  $a > 0$  and the graph of  $y = f(x)$  contains the point  $(x, y)$ , then the graph of  $y = f(ax)$  contains the point  $\left(\frac{1}{a}x, y\right)$ .
- If  $a > 1$ , the graph of  $y = f(ax)$  is a *horizontal shrinking* of the graph of  $y = f(x)$ .
- If  $0 < a < 1$ , the graph of  $y = f(ax)$  is a *horizontal stretching* of the graph of  $y = f(x)$ .

## Definitions of Trigonometric Functions

$$\sin \theta = \frac{b}{r} \quad \csc \theta = \frac{r}{b}$$

$$\cos \theta = \frac{a}{r} \quad \sec \theta = \frac{r}{a}$$

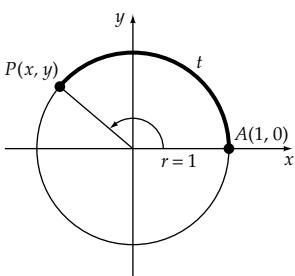
$$\tan \theta = \frac{b}{a} \quad \cot \theta = \frac{a}{b}$$



where  $r = \sqrt{a^2 + b^2}$

## Definitions of Circular Functions

$$\begin{array}{ll} \sin t = y & \csc t = \frac{1}{y} \\ \cos t = x & \sec t = \frac{1}{x} \\ \tan t = \frac{y}{x} & \cot t = \frac{x}{y} \end{array}$$



## Formulas for Triangles

For any triangle  $ABC$ , the following formulas can be used.

### Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

### Law of Cosines

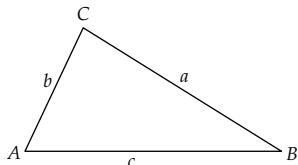
$$c^2 = a^2 + b^2 - 2ab \cos C$$

### Area of a Triangle

$$K = \frac{1}{2}ab \sin C \quad K = \frac{a^2 \sin B \sin C}{2 \sin A}$$

### Heron's Formula

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}$$



## Fundamental Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

## Formulas for Negatives

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

## Reciprocal Identities

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

## Sum of Two Angle Identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

## Difference of Two Angle Identities

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

## Double-Angle Identities

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

## Power-Reducing Identities

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\tan^2 \alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$$

## Half-Angle Identities

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

## Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

## Product-to-Sum Identities

$$2 \sin \alpha \cos \beta = \sin (\alpha + \beta) + \sin (\alpha - \beta)$$

$$2 \cos \alpha \sin \beta = \sin (\alpha + \beta) - \sin (\alpha - \beta)$$

$$2 \cos \alpha \cos \beta = \cos (\alpha + \beta) + \cos (\alpha - \beta)$$

$$2 \sin \alpha \sin \beta = \cos (\alpha - \beta) - \cos (\alpha + \beta)$$

## Sum-to-Product Identities

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

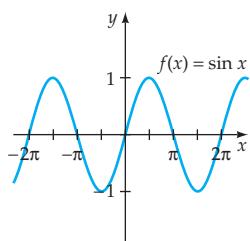
$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

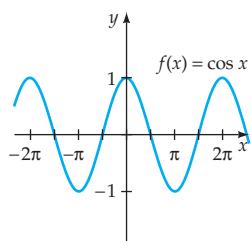
$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

# The Trigonometric Functions

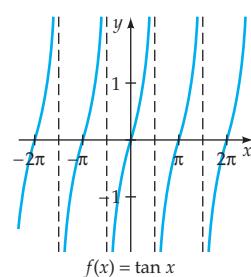
Sine function



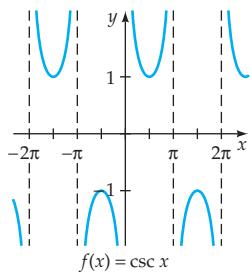
Cosine function



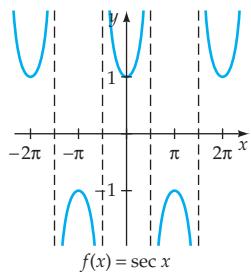
Tangent function



Cosecant function



Secant function



Cotangent function

