Conceptual Trigonometry
Part I

A Companion
to
S. L. Loney's
Plane Trigonometry Part I

By

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Preface

Back in 1990, solving the problems and exercises given in the text-book of **Plane Trigonometry Part I** by **S. L. Loney** had a terrorizing effect on me, irrespective of the outcome of the countless hours, full of perspiration and inspiration, laced with joy and surrendering to the sheer beauty and elegance of each problem, subproblem, ... woven with multi-concepts.. Whenever stuck, I used to revise the concepts embedded in the text-book and related references, monographs, take a break and start all over... an irresistible journey... back n forth between the classics of Loney and others.

As time grew, I ended up stocking a huge pile of sheets comprising of my notes as an endeavor to solve and devour the entire book and beyond (needless to mention that I laid my hands on everything I could in my pursuit).

Somewhere in 2005, I started collating and organizing my notes to instill coherence and capture the elegance in the flow.

The present work is an outcome of this pursuit, which will serve as a complete guide to private students reading the subject with few or no opportunities of instruction. This will save the time and lighten the work of Teachers as well. This book helps in acquiring a better understanding of the basic principles of Plane Trigonometry and in revising a large amount of the subject matter quickly. Care has been taken, as in the forthcoming ones, to present the solutions with multi-concepts and beyond in a simple natural manner, in order to meet the difficulties which are most likely to arise, and to render the work intelligible and instructive.

This work contains several variations of problems, solutions, methods, approaches to enrich, strengthen and enliven the inherent multi-concepts.

Ancient Science Publishers — *Chandra Shekhar Kumar*
July, 2018.

List of Chapters

Chapter 1

Measurement of Angles

1.1 Sexagesimal and Centesimal Measure

Express in terms of a right angle the angles

§ Problem 1.1.1. $60°$. ◊

§§ Solution. $60° = \dfrac{60}{90}$ of a right angle $= \dfrac{2}{3}$ of a right angle. ■

§ Problem 1.1.2. $75°15'$. ◊

§§ Solution. $75°15' = \left(\dfrac{75}{90} + \dfrac{15}{90 \times 60}\right)$ of a right angle $= \dfrac{301}{360}$ of a right angle. ■

§ Problem 1.1.3. $63°17'25''$. ◊

§§ Solution.

$$\begin{aligned} 63°17'25'' &= \left(\frac{63}{90} + \frac{17}{90 \times 60} + \frac{25}{90 \times 60 \times 60}\right) \text{ of a right angle} \\ &= \frac{45569}{64800} \text{ of a right angle.} \end{aligned}$$ ■

§ Problem 1.1.4. $130°30'$. ◊

§§ Solution. $130°30' = \left(\dfrac{130}{90} + \dfrac{30}{90 \times 60}\right)$ of a right angle $= 1\dfrac{9}{20}$ of right angle. ■

§ Problem 1.1.5. $210°30'30''$. ◊

§§ Solution.

$$210^\circ 30' 30'' = \left(\frac{210}{90} + \frac{30}{90\times 60} + \frac{30}{90\times 60\times 60}\right) \text{ of a right angle}$$
$$= 2\frac{3661}{10800} \text{ of a right angle.} \qquad \blacksquare$$

§ Problem 1.1.6. $370^\circ 20' 48''$. ◊

§§ Solution.

$$370^\circ 20' 48'' = \left(\frac{370}{90} + \frac{20}{90\times 60} + \frac{48}{90\times 60\times 60}\right) \text{ of a right angle}$$
$$= 4\frac{388}{3376} \text{ of a right angle.} \qquad \blacksquare$$

Express in grades, minutes, and seconds the angles

§ Problem 1.1.7. 30°. ◊

§§ Solution. $30^\circ = \left(30 + \frac{1}{9}\times 30\right)^g = \frac{100^g}{3} = 33^g 33' 33.\dot{3}''$. ■

§ Problem 1.1.8. 81°. ◊

§§ Solution. $81^\circ = \left(81 + \frac{1}{9}\times 81\right)^g = 90^g$. ■

§ Problem 1.1.9. $138^\circ 30'$. ◊

§§ Solution. $30' = \frac{30^\circ}{60} = .5^\circ$;

$$\therefore 138^\circ 30' = 138.5^\circ = \frac{138.5}{90} \text{ of a right angle}$$
$$= 1.53\dot{8} \text{ of a right angle} = 153.\dot{8}^g = 153^g 88' 88.\dot{8}''. \blacksquare$$

§ Problem 1.1.10. $35^\circ 47' 15''$. ◊

§§ Solution. $15'' = \frac{15'}{60} = .25'$;

$$\therefore 47' 15'' = 47.25' = \frac{47.25^\circ}{60} = .7875^\circ;$$
$$\therefore 35^\circ 47' 15'' = 35.7875^\circ = \frac{35.7875}{90} \text{ of a right angle}$$
$$= .39763\dot{8} \text{ of a right angle } = 39.763\dot{8}^g = 39^g 76' 38.\dot{8}''. \quad \blacksquare$$

§ Problem 1.1.11. $235^\circ 12' 36''$. ◊

§§ Solution. $36'' = \frac{36'}{60} = \frac{3'}{5} = .6'$;

$$\therefore 12' 36'' = 12.6' = \frac{12.6^\circ}{60} = .21^\circ;$$
$$\therefore 235^\circ 12' 36'' = 235.21^\circ = \frac{235.21}{90} \text{ of a right angle}$$
$$= 2.613\dot{4} \text{ of a right angle } = 261.3\dot{4}^g = 261^g 34' 44.\dot{4}''. \quad \blacksquare$$

§ Problem 1.1.12. $475^\circ 13' 48''$. ◊

§§ Solution. $48'' = \dfrac{48'}{60} = \dfrac{4'}{5} = .8'$;

$$\therefore 13'48'' = 13.8' = \frac{13.8^\circ}{60} = .23^\circ;$$

$$\therefore 475^\circ 13'48'' = 475.23^\circ = \frac{475.23}{90} \text{ of a right angle}$$
$$= 5.280\dot{3} \text{ of a right angle } = 528.0\dot{3}^g = 528^g 3' 33.\dot{3}''. \quad \blacksquare$$

Express in terms of right angles, and also in degrees, minutes, and seconds the angles

§ Problem 1.1.13. 120^g. ◊

§§ Solution. $120^g = \dfrac{120}{100}$ of a right angle $= \dfrac{6}{5}$ of a right angle $= \dfrac{6}{5} \times 90^\circ = 108^\circ$.

Otherwise thus : $120^g = \left(120 - \dfrac{1}{10} \times 120\right)^\circ = 108^\circ$. ■

§ Problem 1.1.14. $45^g 35' 24''$. ◊

§§ Solution.

$$45^g 35' 24'' = \left(\frac{45}{100} + \frac{35}{100 \times 100} + \frac{24}{100 \times 100 \times 100}\right) \text{ of a right angle}$$
$$= .453524 \text{ of a right angle}$$
$$= .453524 \times 90^\circ = 40.81716^\circ = 40^\circ + .81716^\circ$$
$$= 40^\circ + .81716 \times 60' = 40^\circ + 49.0296' = 40^\circ + 49' + .0296'$$
$$= 40^\circ + 49' + .0296 \times 60'' = 40^\circ + 49' + 1.776''$$
$$\therefore 45^g 35' 24'' = 40^\circ 49' 1.776''.$$

■

§ Problem 1.1.15. $39^g 45' 36''$. ◊

§§ Solution.

$$39^g 45' 36'' = .394536 \text{ of a right angle}$$
$$= .394536 \times 90^\circ = 35.50824^\circ = 35^\circ + .50824^\circ$$
$$= 35^\circ + .50824 \times 60' = 35^\circ + 30.4944' = 35^\circ + 30' + .4944'$$
$$= 35^\circ + 30' + .4944 \times 60'' = 35^\circ + 30' + 29.664''$$
$$\therefore 39^g 45' 36'' = 35^\circ 30' 29.664''.$$

■

§ Problem 1.1.16. $255^g 8' 9''$. ◊

§§ Solution.

$$255^g 8' 9'' = 2.550809 \text{ of a right angle}$$
$$= 2.550809 \times 90^\circ = 229.57281^\circ = 229^\circ + .57281^\circ$$
$$= 229^\circ + .57281 \times 60' = 229^\circ + 34.3686' = 229^\circ + 34' + .3686'$$
$$= 229^\circ + 34' + .3686 \times 60'' = 229^\circ + 34' + 22.116''$$
$$\therefore 255^g 8' 9'' = 229^\circ 34' 22.116''.$$

■

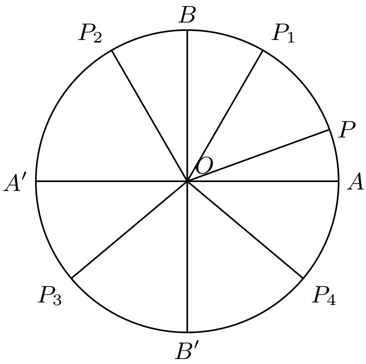
§ Problem 1.1.17. $759^g 0' 5''$. ◊

§§ Solution.

$$\begin{aligned} 759^g 0'5'' &= 7.590005 \text{ of a right angle} \\ &= 7.590005 \times 90^\circ = 683.10045^\circ = 683^\circ + .10045^\circ \\ &= 683^\circ + .10045 \times 60' = 683^\circ + 6.0270' = 683^\circ + 6' + .0270' \\ &= 683^\circ + 6' + .0270 \times 60'' = 683^\circ + 6' + 1.620'' \\ &\therefore 759^g 0'5'' = 683^\circ 6' 1.62''. \end{aligned}$$

■

Mark the position of the revolving line when it has traced out the following angles



§ Problem 1.1.18. $\frac{4}{3}$ *right angle.* ◊

§§ Solution. Since $\frac{4}{3}$ right angle $= 90^\circ + \frac{1}{3} \times 90^\circ = 90^\circ + 30^\circ$, the revolving line has turned through 30° more than a right angle and is therefore in the second quadrant, i.e. is between OB and OA' and makes an angle of 30° with OB. ■

§ Problem 1.1.19. $3\frac{1}{2}$ *right angles.* ◊

§§ Solution. Since $3\frac{1}{2}$ right angles $= 3\times 90^\circ + \frac{1}{2}\times 90^\circ = 3\times 90^\circ + 45^\circ$, the revolving line has turned through 45° more than three right angles and is therefore in the fourth quadrant, i.e. is halfway between OB' and OA. ■

§ Problem 1.1.20. $13\frac{1}{3}$ *right angles.* ◊

§§ Solution. Since $13\frac{1}{3}$ right angles $= 13\times 90^\circ + \frac{1}{3}\times 90^\circ = 13\times 90^\circ + 30^\circ$, the revolving line has turned through 30° more than thirteen right angles and is therefore in the second quadrant, i.e. is between OB and OA', and makes an angle of 30° with OB. ■

§ Problem 1.1.21. 120°. ◊

§§ Solution. Since $120^\circ = 90^\circ + 30^\circ$, we have the same result as in § Problem 1.1.18. ■

§ Problem 1.1.22. 315°. ◊

§§ Solution. Since $315^\circ = 3 \times 90^\circ + 45^\circ$, we have the same result as in § Problem 1.1.19. ■

§ Problem 1.1.23. 745°. ◊

§§ Solution. Since $745^\circ = 8 \times 90^\circ + 25^\circ$, the revolving line has turned through 25° more than eight right angles and is therefore in the first quadrant, i.e. is between OA and OB, and makes an angle of 25° with OA. ■

§ Problem 1.1.24. 1185°. ◊

§§ Solution. Since $1185^\circ = 13 \times 90^\circ + 15^\circ$, the revolving line has turned through 15° more than thirteen right angles and is therefore in the second quadrant, i.e. is between OB and OA', and makes an angle of 15° with OB. ■

§ Problem 1.1.25. 150^g. ◊

§§ Solution. $150^g = 100^g + 50^g$, the revolving line has turned through 50^g more than a right angle, and is therefore in the second quadrant, i.e. is halfway between OB and OA'. ■

§ Problem 1.1.26. 420^g. ◊

§§ Solution. $420^g = 4 \times 100^g + 20^g$, the revolving line has turned through 20^g more than four right angles, and is therefore in the first quadrant, i.e. is between OA and OB and makes an angle of 20^g with OA. ■

§ Problem 1.1.27. 875^g. ◊

§§ Solution. $875^g = 8 \times 100^g + 75^g$, the revolving line has turned through 75^g more than eight right angles, and is therefore in the first quadrant, i.e. is between OA and OB and makes an angle of 75^g with OA. ■

§ Problem 1.1.28. *How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{9}$ minutes by the hour and minute hands of a watch ?* ◊

§§ Solution. A minute-division on the face of a clock $= \dfrac{360^\circ}{60} = 6^\circ$.

Also, the minute-hand moves twelve times as fast as the hour-hand.

Hence, in $11\frac{1}{9}$ minutes, the minute-hand passes over

$$\left(11\frac{1}{9} \times 6\right)^\circ = 66\frac{2}{3}^\circ = 66^\circ 40'.$$

and the hour-hand passes over

$$\frac{66^\circ 40'}{12} = 5^\circ 33' 20''.$$ ■

§ Problem 1.1.29. *The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both the angles in degrees.* $\lozenge$

§§ Solution. If x be the number of degrees in one of the required angles, then $(90-x)$ is the number of degrees in the other angle and therefore $\frac{10}{9}(90-x)$ is the number of grades in the other angle.

$$\therefore x = \frac{10}{9}(90-x), \ \therefore x = 47\frac{7}{19}$$

So that the required angles are $47\frac{7}{19}^\circ$ and $\left(90-47\frac{7}{19}\right)^\circ$, i.e. $47\frac{7}{19}^\circ$ and $42\frac{12}{19}^\circ$. ■

§ Problem 1.1.30. *Prove that the number of Sexagesimal minutes in any angle is to the number of Centesimal minutes in the same angle as* $27:50$. $\lozenge$

§§ Solution. If S be the number of Sexagesimal minutes in any angle and C be the number of Centesimal minutes in the same angle, we have

$$\frac{S}{90\times 60} = \frac{\textit{the given } \angle}{\textit{a right } \angle} = \frac{C}{100\times 100}.$$
$$\therefore \frac{S}{C} = \frac{9\times 6}{10\times 10} = \frac{27}{50}$$
$$\therefore S:C = 27:50.$$

Otherwise thus :

$$\begin{aligned} 1 \text{ Sexagesimal minute } &= \frac{1}{60}\, minute = \frac{10}{9}\times\frac{1}{60}\, grade \\ &= \frac{1}{54}\, grade = \frac{100}{54} \text{ Centesimal minute} \\ &= \frac{50}{27} \text{ Centesimal minute} . \end{aligned}$$ ■

§ Problem 1.1.31. *Divide* $44°8'$ *into two parts such that the number of Sexagesimal seconds in one part may be equal to the number of Centesimal seconds in the other part.* $\lozenge$

§§ Solution. If x be the number of degrees in one part, then $\left(44\frac{2}{15}-x\right)$ is the number of degrees in the other part, i.e. $\frac{10}{9}\left(44\frac{2}{15}-x\right)$ is the number of grades in the other part.

In $x°$, there are $(x\times 60\times 60)$ Sexagesimal seconds and in

$$\frac{10}{9}\left(44\frac{2}{15}-x\right)^g,$$

there are

$$\left[\frac{10}{9}\left(44\frac{2}{15}-x\right)\times 100\times 100\right]$$

Centesimal seconds.

$$\therefore x\times 60\times 60 = \frac{10}{9}\left(44\frac{2}{15}-x\right)\times 100\times 100$$
$$\therefore x = \frac{100°}{3} = 33°20',$$

So that the required parts are $33°20'$ and

$$\left(44°8' - 33°20'\right), \textit{ i.e. } 33°20' \textit{ and } 10°48°.$$ ■

1.2 Circular Measure

§ Problem 1.2.1. *If the radius of the earth be* 4000 *miles, what is the length of its circumference ?* ◊

§§ Solution. The length of the circumference

$$= (2 \times \pi \times 4000) \ miles$$
$$\approx (3.14159265 \times 8000) \ miles \approx 25132.74 \ miles.$$ ■

§ Problem 1.2.2. *The wheel of a railway carriage is* 3 *feet in diameter and makes* 3 *revolutions in a second; how fast is the train going ?* ◊

§§ Solution. The circumference of the wheel $= \left(2 \times \pi \times \frac{3}{2}\right) \ feet = 3\pi \ feet.$

Hence the required rate

$$= (3 \times 3\pi) \ feet \ per \ second \ = \left(9\pi \times \frac{60 \times 60}{1760 \times 3}\right) \ miles \ per \ hour$$
$$= \frac{135}{7} \left[taking \ \pi = \frac{22}{7}\right) \approx 19.28 \ miles \ per \ hour.$$ ■

§ Problem 1.2.3. *A mill sail whose length is* 18 *feet makes* 10 *revolutions per minute. What distance does its end travel in an hour ?* ◊

§§ Solution. The circumference $= (2 \times \pi \times 18) \ feet \ = 36\pi \ feet.$

Hence the end travels $(36\pi \times 10)$ feet per minute, i.e.

$$\left(360\pi \times \frac{60}{1760 \times 3}\right) \ miles \ per \ hour$$
$$i.e. \ \frac{90}{7} \left[taking \ \pi = \frac{22}{7}\right],$$
$$i.e. \ \approx 12.85 \ miles \ per \ hour.$$ ■

§ Problem 1.2.4. *The diameter of a half-penny is an inch; what is the length of a piece of string which would just surround its curved edge ?* ◊

§§ Solution. The required length

$$= \left(2 \times \pi \times \frac{1}{2}\right) \ inches$$
$$= \pi \ inches \ = 3.14159\ldots \ inches.$$ ■

§ Problem 1.2.5. *Assuming that the earth describes in one year a circle of* 92500000 *miles radius, whose center is the sun, how many miles does the earth travel in a year ?* ◊

§§ Solution. The required distance

$$= (2\pi \times 92500000) \ miles$$
$$\approx (3.14159265 \times 185000000) \ miles \approx 581194640 \ miles.$$ ■

§ **Problem 1.2.6.** *The radius of a carriage wheel is* $1\ ft.\ 9\ ins.$ *and in* $\frac{1}{9}$*th of a second it turns through* $80°$ *about its center, which is fixed; how many miles does a point on the rim of the wheel travel in one hour ?* ◊

§§ **Solution**. The wheel makes one complete revolution in

$$\left(\frac{360}{80} \times \frac{1}{9}\right)\ sec.,\ i.e.\ in\ \frac{1}{2}\ sec.$$

Hence it makes $(60 \times 60 \times 2)$ revolutions per hour.
The circumference of the wheel

$$= (2 \times \pi \times 21)\ inches \approx (22 \times 6)\ inches\ \left[taking\ \pi = \frac{22}{7}\right].$$

Hence the required number of miles

$$= (22 \times 6) \times \frac{60 \times 60 \times 2}{1760 \times 3 \times 12} = 22 \times 6 \times \frac{5}{44} = 15.$$

If we take $\pi = 3.14159265$, the required number ≈ 14.994. ■

1.3 The Radian

Express in degrees, minutes and seconds the angles :

§ **Problem 1.3.1.** $\frac{\pi^c}{3}$. ◊

§§ **Solution**. $\frac{\pi^c}{3} = \frac{1}{3} \times 180° = 60°$. ■

§ **Problem 1.3.2.** $\frac{4\pi^c}{3}$. ◊

§§ **Solution**. $\frac{4\pi^c}{3} = \frac{4}{3} \times 180° = 240°$. ■

§ **Problem 1.3.3.** $10\pi^c$. ◊

§§ **Solution**. $10\pi^c = 10 \times 180° = 1800°$. ■

§ **Problem 1.3.4.** 1^c. ◊

§§ **Solution**. $1^c = \frac{1}{\pi} \times \pi^c = \frac{1}{\pi} \times 180° = 57°17'44.8''$ $[see\ Art.\ 16]$. ■

§ **Problem 1.3.5.** 8^c. ◊

§§ **Solution**. $8^c = \frac{8}{\pi} \times \pi^c = \frac{8}{\pi} \times 180° = 8 \times (57°17'44.8'') = 458°21'58.4''$. ■

Express in grades, minutes and seconds the angles :

§ **Problem 1.3.6.** $\frac{4\pi^c}{5}$. ◊

§§ **Solution**. $\frac{4\pi^c}{5} = \frac{4}{5} \times 200^g = 160^g$. ■

§ **Problem 1.3.7.** $\frac{7\pi^c}{6}$. ◊

§§ **Solution**. $\frac{7\pi^c}{6} = \frac{7}{6} \times 200^g = 233\frac{1}{3}^g = 233^g 33' 33.\dot{3}''$. ■

§ Problem 1.3.8. $10\pi^c$. ◊

§§ Solution. $10\pi^c = 10 \times 200^g = 2000^g$. ■

Express in radians the following angles :

§ Problem 1.3.9. $60°$. ◊

§§ Solution. $60° = 60 \times \dfrac{\pi^c}{180} = \dfrac{\pi}{3}\ radians.$ ■

§ Problem 1.3.10. $110°30'$. ◊

§§ Solution. $110°30' = 110\frac{1}{2} \times \dfrac{\pi^c}{180} = \dfrac{221}{360}\pi\ radians.$ ■

§ Problem 1.3.11. $175°45'$. ◊

§§ Solution. $175°45' = 175\frac{3}{4} \times \dfrac{\pi^c}{180} = \dfrac{703}{720}\pi\ radians.$ ■

§ Problem 1.3.12. $47°25'36''$. ◊

§§ Solution.

$$\begin{aligned}47°25'36'' &= 47°25\frac{3}{5}' = \left(47\frac{25\frac{3}{5}}{60}\right)^{\circ} \\ &= 47\frac{32}{75} \times \frac{\pi^c}{180} = \frac{3557}{13500}\pi\ radians.\end{aligned}$$ ■

§ Problem 1.3.13. $395°$. ◊

§§ Solution. $395° = 395 \times \dfrac{\pi^c}{180} = \dfrac{79}{36}\pi\ radians.$ ■

§ Problem 1.3.14. 60^g. ◊

§§ Solution. $60^g = 60 \times \dfrac{\pi^c}{200} = \dfrac{3}{10}\pi\ radians.$ ■

§ Problem 1.3.15. $110^g30'$. ◊

§§ Solution. $110^g30' = 110.30^g = 110.30 \times \dfrac{\pi^c}{200} = \dfrac{1103}{2000}\pi\ radians.$ ■

§ Problem 1.3.16. $345^g25'36''$. ◊

§§ Solution.

$$\begin{aligned}345^g25'36'' &= 345.2536^g \\ &= 345.2536 \times \frac{\pi^c}{200} = 1.726268\pi\ radians.\end{aligned}$$ ■

§ Problem 1.3.17. *The difference between the two acute angles of a right-angled triangle is* $\frac{2}{5}\pi$ *radians; express the angles in degrees.* ◊

§§ Solution. $\frac{2}{5}\pi$ radians $= \frac{2}{5} \times 180° = 72°$.

Hence, if x be the number of degrees in the larger of the required angles, then $(90 - x)$ is the number of degrees in the other angle.

Hence $x-(90 - x) = 72$, whence $x = 81$, so that the required angles are $81°$ and $9°$. ■

§ Problem 1.3.18. *One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees, whilst the third is $\frac{\pi x}{75}$ radians; express them all in degrees.* ◊

§§ Solution. Since the sum of the three angles of a triangle is $180°$, we have, by reducing the given angles to degrees,

$$\frac{9}{10}\cdot\frac{2x}{3}+\frac{3x}{2}+\frac{x}{75}\times 180=180$$
$$\therefore \frac{x}{5}+\frac{x}{2}+\frac{4x}{5}=60,\ \therefore x=40$$

so that the angles are $24°$, $60°$ and $96°$. ■

§ Problem 1.3.19. *The circular measure of two angles of a triangle are respectively $\frac{1}{2}$ and $\frac{1}{3}$; what is the number of degrees in the third angle ?* ◊

§§ Solution. The circular measure of the third angle

$$=\pi-\frac{1}{2}-\frac{1}{3}=\pi-\frac{5}{6}.$$

Hence the required number of degrees

$$=180°-\frac{5}{6}\cdot\frac{180°}{\pi}=180°-\frac{5}{6}\times 57°17'44.8''\ [Art.\ 16]\,.$$
$$=180°-47°44'47.\dot{3}''=132°15'12.\dot{6}''.$$ ■

§ Problem 1.3.20. *The angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as 60 to π; find the angles in degrees.* ◊

§§ Solution. Let the angles be $(x-y)°$, $x°$ and $(x+y)°$.
Since the sum of the three angles of a triangle is $180°$, we have

$$x-y+x+x+y=180°$$
$$\therefore 3x=180°,\ \therefore x=60°.$$

The required angles are therefore

$$(60-y)°\ ,\ 60°\ and\ (60+y)°\ .$$

Now
$$(60+y)°=\frac{\pi}{180}(60+y)\ \ radians.$$
$$\therefore 60-y:\frac{\pi}{180}(60+y)::60:\pi.$$
$$\therefore \frac{180}{\pi}\cdot\frac{60-y}{60+y}=\frac{60}{\pi}$$
$$\therefore 3\,(60-y)=60+y,\ \therefore y=30.$$

The angles are therefore $30°$, $60°$ and $90°$. ■

§ Problem 1.3.21. *The angles of a triangle are in A. P. and the number of radians in the least angle is to the number of degrees in the mean angle as $1:120$. Find the angles in radians.* ◊

§§ Solution. Let $(x-y)$, x and $(x+y)$ be the number of radians in the three angles respectively.

$$\therefore x-y+x+x+y=\pi.$$
$$\therefore 3x=\pi,\ \therefore x=\frac{\pi}{3}.\ \ \therefore\ \frac{\pi}{3}-y:60::1:120.$$
$$\therefore \frac{2\pi}{3}-2y=1,\ \therefore y=\frac{\pi}{3}-\frac{1}{2}.$$

Thus the angles contain $\frac{1}{2}$, $\frac{\pi}{3}$ and $\frac{2\pi}{3} - \frac{1}{2}$ radians. ■

§ **Problem 1.3.22.** *Find the magnitude, in radians and degrees, of the interior angle of*

(1) a regular pentagon

(2) a regular heptagon

(3) a regular octagon

(4) a regular duodecagon, and

(5) a regular polygon of 17 *sides.* ◊

§§ **Solution**. Proceeding as in $Ex.\ 2,\ Art.\ 20$, we have

(1) $5x + 4 = 10,\ \therefore x = \frac{6}{5}$ right angle.

Hence the required angle
$$= \frac{6}{5} \times \frac{\pi}{2} = \frac{3\pi}{5}\ radians\ = \frac{6}{5} \times 90^\circ = 108^\circ.$$

(2) $7x + 4 = 14,\ \therefore x = \frac{10}{7}$ right angle.

Hence the required angle
$$= \frac{10}{7} \times \frac{\pi}{2} = \frac{5\pi}{7}\ radians\ = \frac{10}{7} \times 90^\circ = 128\frac{4}{7}^\circ.$$

(3) $8x + 4 = 16,\ \therefore x = \frac{12}{8}\ right\ angle = \frac{3}{2}\ right\ angle.$

Hence the required angle
$$= \frac{3}{2} \times \frac{\pi}{2} = \frac{3\pi}{4}\ radians\ = \frac{3}{2} \times 90^\circ = 135^\circ.$$

(4) $12x + 4 = 24,\ \therefore x = \frac{20}{12}\ right\ angle = \frac{5}{3}\ right\ angle.$

Hence the required angle
$$= \frac{5}{3} \times \frac{\pi}{2} = \frac{5\pi}{6}\ radians\ = \frac{5}{3} \times 90^\circ = 150^\circ.$$

(5) $17x + 4 = 34,\ \therefore x = \frac{30}{17}\ right\ angle.$

Hence the required angle
$$= \frac{30}{17} \times \frac{\pi}{2} = \frac{15\pi}{17}\ radians\ = \frac{30}{17} \times 90^\circ = 158\frac{14}{17}^\circ.$$
■

§ **Problem 1.3.23.** *The angle in one regular polygon is to that in another as* $3:2$*; also the number of sides in the first is twice that in the second; how many sides have the polygons ?* ◊

§§ Solution. Let $2n$ and n be the number of sides in the two polygons respectively. Since all the angles of the first polygon $= (4n-4)$ right angles, therefore each angle contains $\frac{(4n-4)90}{2n}$ degrees.

Similarly, each angle of the second polygon contains $\frac{(2n-4)90}{n}$ degrees. Hence we have

$$\frac{(4n-4)90}{2n} : \frac{(2n-4)90}{n} = 3 : 2$$
$$\therefore 4n - 4 = 6n - 12; \ \therefore n = 4;$$

Hence the polygons have 8 sides and 4 sides respectively. ■

§ Problem 1.3.24. *The number of sides in two regular polygons are as* $5 : 4$ *and the difference between their angles is* 9°*; find the number of sides in the polygons.* ♢

§§ Solution. Let $5n$ and $4n$ be the number of sides in the two polygons respectively.

Since all the angles of the first polygon $= (10n-4)$ right angles, therefore each angle contains $\frac{(10n-4)90}{5n}$ degrees.

Similarly, each angle of the second polygon contains $\frac{(8n-4)90}{4n}$ degrees. Hence we have

$$\frac{(10n-4)90}{5n} - \frac{(8n-4)90}{4n} = 9, \ \therefore n = 2,$$

so that the polygons have 10 sides and 8 sides respectively. ■

§ Problem 1.3.25. *Find two regular polygons such that the number of their sides may be as* 3 *to* 4 *and the number of degrees in an angle of the first to the number of grades in an angle of the second as* 4 *to* 5. ♢

§§ Solution. Let $3n$ and $4n$ be the number of sides in the two polygons respectively.

Since all the angles of the first polygon $= (6n-4)$ right angles, therefore each angle contains $\frac{(6n-4)90}{3n}$ degrees.

Similarly, each angle of the second polygon contains $\frac{(8n-4)90}{4n}$ degrees, i.e. $\frac{(8n-4)100}{4n}$ grades.

Hence we have

$$\frac{(6n-4)90}{3n} : \frac{(8n-4)100}{4n} = 4 : 5, \ \therefore n = 2,$$

so that the polygons have 6 sides and 8 sides respectively. ■

§ Problem 1.3.26. *The angles of a quadrilateral are in* $A.\ P.$ *and the greatest is double the least; express the least angle in radians.* ♢

§§ Solution. Let $x-3y, \ x-y, \ x+y$ and $x+3y$ be the number of radians in the angles. Then

$$x - 3y + x - y + x + y + x + 3y = 2\pi;$$
$$\therefore 4x = 2\pi, \ \therefore x = \frac{\pi}{2}.$$

Also, we have $$x + 3y = 2(x - 3y).$$

$$\therefore 9y = x = \frac{\pi}{2}, \therefore y = \frac{\pi}{18}.$$

Hence the least angle contains $\left(\frac{\pi}{2} - \frac{3\pi}{18}\right)$ radians, i.e. $\frac{\pi}{3}$ radians. ■

§ Problem 1.3.27. *Find in radians, degrees and grades the angle between the hour-hand and the minute-hand of a clock at*

(1) half-past three

(2) twenty minutes to six

(3) a quarter past eleven. ◊

§§ Solution. Since the minute-hand moves twelve times as fast as the hour-hand, it gains 11 minute-divisions in 12 minutes on the hour-hand, i.e. 1 minute-division in $\frac{12}{11}$ minutes.

(1) At 3 o'clock there are 15 minute-divisions between the hands and in 30 minutes the minute-hand gains $\frac{11 \times 30}{12}$, i.e. $\frac{55}{2}$, minute-divisions on the hour-hand.

Hence at half-past 3, there are $\left(\frac{55}{2} - 15\right)$, i.e. $12\frac{1}{2}$, minute-divisions between the hands.

Now a minute-division on the face of a clock

$$= \frac{\pi^c}{30} = 6^\circ = \frac{20^g}{3} \ [cf\ \S Problem\ 1.1.28].$$

Hence the required answers are

$$12\frac{1}{2} \times \frac{\pi^c}{30},\ i.e.\ \frac{5\pi^c}{12};\ 12\frac{1}{2} \times 6^\circ,\ i.e.\ 75^\circ;\ and\ 12\frac{1}{2} \times \frac{20^g}{3},\ i.e.\ 83\frac{1}{3}^g.$$

(2) Here there are $11\frac{2}{3}$ minute-divisions between the hands. Hence the required answers are

$$11\frac{2}{3} \times \frac{\pi^c}{30},\ i.e.\ \frac{7\pi^c}{18};\ 11\frac{2}{3} \times 6^\circ,\ i.e.\ 70^\circ;\ and\ 11\frac{2}{3} \times \frac{20^g}{3},\ i.e.\ 77\frac{7}{9}^g.$$

(3) Here there are $18\frac{3}{4}$ minute-divisions between the hands. Hence the required answers are

$$18\frac{3}{4} \times \frac{\pi^c}{30},\ i.e.\ \frac{5\pi^c}{8};\ 18\frac{3}{4} \times 6^\circ,\ i.e.\ 112\frac{1}{2}^\circ;\ and$$
$$18\frac{3}{4} \times \frac{20^g}{3},\ i.e.\ 125^g.$$
■

§ Problem 1.3.28. *Find the times*

(1) between four and five o'clock when the angle between the minute-hand and the hour-hand is 78°

(2) between seven and eight o'clock when this angle is 54°. ◊

§§ Solution. (1) At 4 o'clock the minute-hand is 20 minute-divisions behind the hour-hand.

Since a minute-division on the face of a clock $= \dfrac{360^\circ}{60} = 6^\circ$, the hands are at an angle of 78° when they are separated by $\dfrac{78}{6}$, i.e. 13, minute-divisions; and this will be the case when the minute-hand has gained $(20 - 13)$, i.e. 7, and also when it has gained $(20 + 13)$, i.e. 33 minute-divisions.

But the minute-hand gains

$$11 \; minute - divisions \; in \; 12 \; minutes$$
$$\therefore 7 \; minute - divisions \; in \; \left(\frac{12}{11} \times 7\right) = 7\frac{7}{11} \; minutes$$
$$\therefore 30 \; minute - divisions \; in \; \left(\frac{12}{11} \times 33\right) = 36 \; minutes.$$

Hence the angle between the hands is 78° after intervals of $7\dfrac{7}{11}$ minutes and 36 minutes, i.e. at $7\dfrac{7}{11}$ and 36 minutes past 4.

(2) At 7 o'clock the minute-hand is 35 minute-divisions behind the hour-hand.

Since a minute-division on the face of a clock $= \dfrac{360^\circ}{60} = 6^\circ$, the hands are at angle of 54° when they are separated by $\dfrac{54}{6} = 9$ minute-divisions and this will be the case when the minute-hand has gained $(35 - 9) = 26$, and also when it has gained $(35 + 9 = 44)$ minute-divisions. But the minute-hand gains

$$11 \; minute - divisions \; in \; 12 \; minutes$$
$$\therefore 26 \; minute - divisions \; in \; 28\frac{4}{11} \; minutes$$
$$\therefore 44 \; minute - divisions \; in \; \left(\frac{12}{11} \times 44\right) = 48 \; minutes.$$

Hence the angle between the hands is 54° after intervals of $28\dfrac{4}{11}$ minutes and 48 minutes, i.e. at $28\dfrac{4}{11}$ and 48 minutes past 7. ■

1.4 Measurement of Any Angle in Radians

§ Problem 1.4.1. *Find the number of degrees subtended at the center of a circle by an arc whose length is $.357$ times the radius.* ◊

§§ Solution. The number of radians in the angle $= \dfrac{.357r}{r} = .357$.

Hence the angle

$$= .357 \; radian \; = .357 \times \frac{2}{\pi} \; right \; angle$$
$$= \frac{.714}{\pi} \times 90^\circ = (.714 \times .3183 \times 90)^\circ \approx 20.454^\circ.$$ ■

§ Problem 1.4.2. *Express in radians and degrees the angle subtended at the center of a circle by an arc whose length is 15 feet, the radius of the circle being 25 feet.* ◊

§§ Solution. The number of radians in the angle $= \dfrac{15}{25} = \dfrac{3}{5}$.

Hence the angle

$$\begin{aligned}
&= \frac{3}{5}\ radian = \frac{3}{5} \times \frac{2}{\pi}\ right\ angle \\
&= \frac{6}{5\pi} \times 90^\circ = \frac{6}{\pi} \times 18^\circ = 108^\circ \times .31831 \\
&= 34.37748^\circ = 34^\circ + .37748 \times 60' = 34^\circ + 22.6488' \\
&= 34^\circ + 22' + .6488 \times 60'' = 34^\circ + 22' + 38.928'' \\
&\approx 34^\circ 22' 38.9''.
\end{aligned}$$

■

§ Problem 1.4.3. *The value of the divisions on the outer rim of a graduated circle is* $5'$ *and the distance between successive graduations is* $.1$ *inch. Find the radius of the circle.* ◊

§§ Solution. If r be the number of inches in the radius, we have

$$\frac{.1}{r} = \text{ the number of radians in } 5'$$

$$\therefore \frac{1}{10r} = \frac{5}{60} \times \frac{\pi}{180}$$

$$\therefore r = \frac{6}{5} \times \frac{180}{\pi} = 216 \times .31831 \approx 68.75''.$$

■

§ Problem 1.4.4. *The diameter of a graduated circle is* 6 *feet and the graduations on its rim are* $5'$ *apart; find the distance from one graduation to another.* ◊

§§ Solution. If x be the required distance in inches, we have

$$\frac{x}{36} = \text{ the number of radians in } 5' = \frac{5}{60} \times \frac{\pi}{180}.$$

$$\therefore x = \frac{\pi}{60} = \frac{3.14159}{60} \approx .05236.$$

■

§ Problem 1.4.5. *Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by* $1^\circ 10'$ *may be half-an-inch.* ◊

§§ Solution. If r be the number of inches in the radius, we have

$$\frac{\frac{1}{2}}{r} = \text{ the number of radians in } 1^\circ 10' = 1\frac{1}{6} \times \frac{\pi}{180}.$$

$$\therefore r = \frac{1}{2} \times \frac{6}{7} \times 180 \times \frac{1}{\pi} = \frac{3}{7} \times 180 \times .31831 \approx 24.555''.$$

■

§ Problem 1.4.6. *Taking the radius of the earth as* 4000 *miles, find the difference in latitude of two places, one of which is* 100 *miles north of the other.* ◊

§§ Solution. The number of radians in the required difference $= \dfrac{100}{4000} = \dfrac{1}{40}$.

Hence the difference

$$\begin{aligned}
&= \frac{1}{40}\ radian = \frac{1}{40} \times \frac{2}{\pi}\ right\ angle \\
&= \frac{1}{20\pi} \times 90^\circ = \left(\frac{9}{2} \times .31831\right)^\circ \\
&= 1.43239^\circ = 1^\circ + .43239 \times 60' = 1^\circ + 25.9434' \\
&= 1^\circ + 25' + .9434 \times 60'' = 1^\circ + 25' + 56.604'' \\
&\approx 1^\circ 25' 57''.
\end{aligned}$$

■

§ **Problem 1.4.7.** *Assuming the earth to be a sphere and the distance between two parallels of latitude, which subtends an angle of* 1° *at the earth's center, to be* $69\frac{1}{9}$ *miles, find the radius of the earth.* ♢

§§ **Solution**. If r be the number of miles in the radius, we have

$$\frac{69\frac{1}{9}}{r} = \text{the number of radians in } 1^\circ = \frac{\pi}{180}.$$

$$\therefore r = \frac{622}{9} \times \frac{180}{\pi} = 622 \times 20 \times .31831 \approx 3959.8 \; miles.$$ ■

§ **Problem 1.4.8.** *The radius of a certain circle is* 3 *feet; find approximately the length of an arc of this circle, if the length of the chord of the arc be* 3 *feet also.* ♢

§§ **Solution**. The arc subtends an angle of 60° at the center of the circle, so that if x feet be the required length, we have

$$\frac{x}{3} = \text{the number of radians in } 60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3}.$$

$$\therefore x = \pi = 3.14159\ldots \; feet.$$ ■

§ **Problem 1.4.9.** *What is the ratio of the radii of two circles at the center of which two arcs of the same length subtend angles of* 60° *and* 75° *?* ♢

§§ **Solution**. If r and r' be the radii respectively and x be the length of an arc, we have

$$\frac{x}{r} = \text{the number of radians in } 60^\circ = 60 \times \frac{\pi}{180}$$

and

$$\frac{x}{r'} = \text{the number of radians in } 75^\circ = 75 \times \frac{\pi}{180}.$$

Hence, by division, we have $\frac{r}{r'} = \frac{75}{60} = \frac{5}{4}$, i.e. the required ratio is $5:4$. ■

§ **Problem 1.4.10.** *If an arc, of length* 10 *feet, on a circle of* 8 *feet diameter subtend at the center an angle of* $143^\circ 14' 22''$*; find the value of* π *to* 4 *places of decimals.* ♢

§§ **Solution**. We have

$$\frac{10}{4} = \text{the number of radians in } 143^\circ 14' 22'' \left[i.e.\ in\ 515662''\right]$$

$$= \frac{515662}{60 \times 60} \times \frac{\pi}{180}.$$

$$\therefore \pi = \frac{10}{4} \times \frac{60 \times 60 \times 180}{515662} = 3.1416.$$ ■

§ **Problem 1.4.11.** *If the circumference of a circle be divided into* 5 *parts which are in* $A.\ P.$ *and if the greatest part be* 6 *times the least, find in radians the magnitude of the angles that the parts subtend at the center of the circle.* ♢

§§ **Solution**. Let $x-2y$, $x-y$, x, $x+y$ and $x+2y$ be the parts into which the circumference is divided.

Their sum $= 5x = 2\pi r, \; \therefore x = \frac{2\pi r}{5}.$

Also, we have $x + 2y = 6(x - 2y)$

$$\therefore 14y = 5x, \therefore y = \frac{5x}{14} = \frac{2\pi r}{14} = \frac{\pi r}{7}.$$

Hence the arcs are

$$\frac{4\pi r}{35}, \frac{9\pi r}{35}, \frac{14\pi r}{35}, \frac{19\pi r}{35} \textit{ and } \frac{24\pi r}{35}.$$

Thus the angles contain

$$\frac{4\pi}{35}, \frac{9\pi}{35}, \frac{14\pi}{35}, \frac{19\pi}{35} \textit{ and } \frac{24\pi}{35}. \quad \blacksquare$$

§ **Problem 1.4.12.** *The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius; express the angle of the sector in degrees, minutes and seconds.* $\Diamond$

§§ **Solution**. If r be the radius of the circle and θ be the number of radians in the angle, then the perimeter of the sector $= r\theta + 2r$.

$$\therefore r\theta + 2r = \pi r, \therefore \theta = (\pi - 2) \textit{ radians}.$$

Hence the angle

$$\begin{aligned} &= (\pi - 2) \times \frac{2}{\pi} \textit{ right angle} \\ &= \frac{2\pi - 4}{\pi} \times 90^\circ = 180^\circ - 2 \times 57^\circ 17' 44.8'' \\ &= 180^\circ - 114^\circ 35' 29.6'' = 65^\circ 24' 30.4''. \end{aligned} \quad \blacksquare$$

§ **Problem 1.4.13.** *At what distance does a man, whose height is* 6 *feet, subtend an angle of* $10'$ *?* $\Diamond$

§§ **Solution**. If x be the required distance in feet, we have

$$\frac{6}{x} = \text{ the number of radians in } 10' = \frac{10}{60} \times \frac{\pi}{180}.$$

$$\therefore x = 6 \times 6 \times \frac{180}{\pi} = 6 \times 6 \times 180 \times .31831 \approx 2062.65 \textit{ feet}. \quad \blacksquare$$

§ **Problem 1.4.14.** *Find the length which at a distance of one mile will subtend an angle of* $1'$ *at the eye.* $\Diamond$

§§ **Solution**. If x be the required length in feet, we have

$$\frac{x}{1760 \times 3} = \text{ the number of radians in } 1' = \frac{1}{60} \times \frac{\pi}{180}.$$

$$\therefore x = \frac{1760 \times 3 \times 3.14159}{60 \times 180} \approx 1.5359 \textit{ feet}. \quad \blacksquare$$

§ **Problem 1.4.15.** *Find approximately the distance at which a globe,* $5\frac{1}{2}$ *inches in diameter, will subtend an angle of* $6'$. $\Diamond$

§§ **Solution**. If x be the required distance in feet, we have

$$\frac{5\frac{1}{2}}{12x} = \text{ the number of radians in } 6' = \frac{6}{60} \times \frac{\pi}{180}.$$

$$\therefore x = \frac{11}{24} \times 10 \times 180 \times .31831 \approx 262.6 \textit{ feet}. \quad \blacksquare$$

§ **Problem 1.4.16.** *Find approximately the distance of a tower whose height is* 51 *feet and which subtends at the eye an angle of* $5\frac{5}{11}'$. $\Diamond$

§§ **Solution**. If x be the required distance in feet, we have

$$\frac{51}{x} = \text{ the number of radians in } 5\frac{5}{11}' = \frac{\frac{60}{11}}{60} \times \frac{\pi}{180} = \frac{1}{11} \times \frac{\pi}{180}.$$

$$\therefore x = 51 \times 11 \times 180 \times .31831 \approx 32142.9 \textit{ feet}. \quad \blacksquare$$

§ **Problem 1.4.17.** *A church spire, whose height is known to be* 100 *feet, subtends an angle of* $9'$ *at the eye; find approximately its distance.* ♢

§§ **Solution**. If x be the required distance in feet, we have

$$\frac{100}{x} = \text{ the number of radians in } 9' = \frac{9}{60} \times \frac{\pi}{180} = \frac{\pi}{60 \times 20}.$$
$$\therefore x = 100 \times 60 \times 20 \times .31831 = 3183.1 \times 12 \approx 38197.2 \ feet.$$ ■

§ **Problem 1.4.18.** *Find approximately in minutes the inclination to the horizon of an incline which rises* $3\frac{1}{2}$ *feet in* 210 *yards.* ♢

§§ **Solution**. The number of radians in the angle

$$= \frac{3\frac{1}{2}}{210 \times 3} = \frac{7}{210 \times 6} = \frac{1}{180}.$$

Hence the angle

$$= \frac{1}{180} \ radian \ = \frac{1}{180} \times \frac{2}{\pi} \ right \ angle$$
$$= \frac{1}{90\pi} \times 90^\circ = .31831^\circ = (60 \times .31831)' = 19.099'.$$ ■

§ **Problem 1.4.19.** *The radius of the earth being taken to be* 3960 *miles and the distance of the moon from the earth being* 60 *times the radius of the earth, find approximately the radius of the moon which subtends at the earth an angle of* $16'$*.* ♢

§§ **Solution**. If r be the required radius in miles, we have

$$\frac{r}{3960 \times 60} = \text{ the number of radians in } 16' = \frac{16}{60} \times \frac{\pi}{180}.$$
$$\therefore r = \frac{3960 \times 16 \times \pi}{180} = 22 \times 16 \times 3.14159 \approx 1105.8 \ miles.$$ ■

§ **Problem 1.4.20.** *When the moon is setting at any given place, the angle that is subtended at its center by the radius of the earth passing through the given place is* $57'$*. If the earth's radius be* 3960 *miles, find approximately the distance of the moon.* ♢

§§ **Solution**. If x be the required distance in miles, we have

$$\frac{3960}{x} = \text{ the number of radians in } 57' = \frac{57}{60} \times \frac{\pi}{180}.$$
$$\therefore x = 3960 \times \frac{20}{19} \times 180 \times .31831 = 238833 \ miles.$$ ■

§ **Problem 1.4.21.** *Prove that the distance of the sun is about* 81 *million geographical miles, assuming that the angle which the earth's radius subtends at the distance of the sun is* $8.76''$ *and that a geographical mile subtends* $1'$ *at the earth's center. Find also the circumference and diameter of the earth in geographical miles.* ♢

§§ **Solution**. If d be the distance of the sun and r be the radius of the earth, we have

$$\frac{r}{d} = \text{ the number of radians in } 8.76'' = \frac{8.76}{60 \times 60} \times \frac{\pi}{180}, \ and$$
$$\frac{1}{r} = \text{ the number of radians in } 1' = \frac{1}{60} \times \frac{\pi}{180}.$$

Hence, by multiplication, we have

$$\frac{1}{d} = \frac{8.76}{60 \times 60 \times 60} \times \frac{\pi^2}{180 \times 180}.$$

$$\therefore d = \frac{6 \times 6 \times 6}{876} \times \left(\frac{180}{\pi}\right)^2 \times 10^5 = \frac{6 \times 3}{730} \times \left(\frac{180}{\pi}\right)^2 \times 10^6 \approx 81 \times 10^6.$$

Again, if r be the number of geographical miles in the radius of the earth, we have

$$\frac{1}{r} = \text{ the number of radians in } 1' = \frac{1}{60} \times \frac{\pi}{180}$$

$$\therefore r = 60 \times \frac{180}{\pi} = 60 \times 57.2957795 \approx 3437.75 \; miles.$$

$$\therefore \; the \; diameter \; \approx 6875.5 \; miles, \; and$$

$$the \; circumference \; = 2\pi r = (2 \times 60 \times 180) \, miles \; = 21600 \; miles. \quad \blacksquare$$

§ Problem 1.4.22. *The radius of the earth's orbit, which is about* 92700000 *miles, subtends at the star Sirius an angle of about* $.4''$*; find roughly the distance of Sirius.* $\Diamond$

§§ Solution. If d be the required distance in miles, we have

$$\frac{92700000}{d} = \text{ the number of radians in } .4'' = \frac{.4}{60 \times 60} \times \frac{\pi}{180}$$

$$\therefore d = 92700000 \times \frac{60 \times 600}{4} \times \frac{180}{\pi} = 92700000 \times 9000 \times 180 \times .31831$$

$$\approx 478019 \times 10^8 \approx 478 \times 10^{11} \; miles. \quad \blacksquare$$

Chapter 2

Trigonometrical Ratios for Angles Less Than A Right Angle

2.1 Trigonometrical Ratios

Prove the following statements.

§ Problem 2.1.1. $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A.$ ◊

§§ Solution.

$$\begin{aligned}\cos^4 A - \sin^4 A + 1 &= \left(\cos^2 A + \sin^2 A\right)\left(\cos^2 A - \sin^2 A\right) + 1\\ &= \cos^2 A - \sin^2 A + 1, \because \cos^2 A + \sin^2 A = 1\\ &= \cos^2 - \left(1 - \cos^2 A\right) + 1 = 2\cos^2 A.\end{aligned}$$ ■

§ Problem 2.1.2. $(\sin A + \cos A)(1 - \sin A \cos A) = \sin^3 A + \cos^3 A.$ ◊

§§ Solution.

$$\begin{aligned}&(\sin A + \cos A)(1 - \sin A \cos A)\\ &= (\sin A + \cos A)\left(\sin^2 A + \cos^2 A - \sin A \cos A\right)\\ &= \sin^3 A + \cos^3 A.\end{aligned}$$ ■

§ Problem 2.1.3. $\dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A} = 2\,\mathrm{cosec}\ A.$ ◊

§§ Solution.

$$\begin{aligned}\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} &= \frac{\sin^2 A + \cos^2 A + 2\cos A + 1}{(1+\cos A)\sin A}\\ &= \frac{2(1+\cos A)}{(1+\cos A)\sin A}\\ &= \frac{2}{\sin A} = 2 \times \frac{1}{\sin A} = 2\,\mathrm{cosec}\ A.\end{aligned}$$ ■

§ Problem 2.1.4. $\cos^6 A + \sin^6 A = 1 - 3\sin^2 A\cos^2 A.$ ◊

§§ Solution.

$$\begin{aligned}\cos^6 A + \sin^6 A &= \left(\cos^2 A + \sin^2 A\right)\left(\cos^4 A + \sin^4 A - \sin^2 A\cos^2 A\right)\\ &= \cos^4 A + \sin^4 A - \sin^2 A\cos^2 A\\ &= \left(\cos^2 A + \sin^2 A\right)^2 - 3\sin^2 A\cos^2 A\\ &= 1 - 3\sin^2 A\cos^2 A. \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.5. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A.$ ◊

§§ Solution.

$$\begin{aligned}\sqrt{\frac{1-\sin A}{1+\sin A}} &= \sqrt{\frac{(1-\sin A)^2}{1-\sin^2 A}} = \frac{1-\sin A}{\sqrt{1-\sin^2 A}}\\ &= \frac{1-\sin A}{\cos A} = \frac{1}{\cos A} - \frac{\sin A}{\cos A} = \sec A - \tan A. \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.6. $\dfrac{\text{cosec } A}{\text{cosec } A - 1} + \dfrac{\text{cosec } A}{\text{cosec } A + 1} = 2\sec^2 A.$ ◊

§§ Solution.

$$\begin{aligned}\frac{\text{cosec } A}{\text{cosec } A - 1} + \frac{\text{cosec } A}{\text{cosec } A + 1} &= \frac{\frac{1}{\sin A}}{\frac{1}{\sin A} - 1} + \frac{\frac{1}{\sin A}}{\frac{1}{\sin A} + 1}\\ &= \frac{1}{1-\sin A} + \frac{1}{1+\sin A} = \frac{2}{1-\sin^2 A}\\ &= \frac{2}{\cos^2 A} = 2\sec^2 A. \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.7. $\dfrac{\text{cosec } A}{\cot A + \tan A} = \cos A.$ ◊

§§ Solution.

$$\begin{aligned}\frac{\text{cosec } A}{\cot A + \tan A} &= \frac{\frac{1}{\sin A}}{\frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}}\\ &= \frac{\frac{1}{\sin A}}{\frac{\cos^2 A + \sin^2 A}{\sin A\cos A}}\\ &= \frac{1}{\sin A} \div \frac{1}{\sin A\cos A} = \cos A. \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.8. $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A.$ ◊

§§ Solution.

$$\begin{aligned}(\sec A + \cos A)(\sec A - \cos A) &= \sec^2 A - \cos^2 A\\ &= 1 + \tan^2 A - \left(1 - \sin^2 A\right)\\ &= \tan^2 A + \sin^2 A. \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.9. $\dfrac{1}{\cot A + \tan A} = \sin A\cos A.$ ◊

§§ Solution.

$$\frac{1}{\cot A+\tan A}=\frac{1}{\dfrac{\cos A}{\sin A}+\dfrac{\sin A}{\cos A}}$$
$$=\frac{\sin A\cos A}{\cos^2 A+\sin^2 A}=\sin A\cos A. \qquad \blacksquare$$

§ Problem 2.1.10. $\dfrac{1}{\sec A-\tan A}=\sec A+\tan A.$ ◊

§§ Solution.

$$\frac{1}{\sec A-\tan A}=\frac{\sec A+\tan A}{\sec^2 A-\tan^2 A}$$
$$=\sec A+\tan A,\ \because \sec^2 A-\tan^2 A=1\,[By\ Art.\ 27(3)]\,. \qquad \blacksquare$$

§ Problem 2.1.11. $\dfrac{1-\tan A}{1+\tan A}=\dfrac{\cot A-1}{\cot A+1}.$ ◊

§§ Solution. $\dfrac{1-\tan A}{1+\tan A}=\dfrac{1-\dfrac{1}{\cot A}}{1+\dfrac{1}{\cot A}}=\dfrac{\cot A-1}{\cot A+1}.$ ■

§ Problem 2.1.12. $\dfrac{1+\tan^2 A}{1+\cot^2 A}=\dfrac{\sin^2 A}{\cos^2 A}.$ ◊

§§ Solution. $\dfrac{1+\tan^2 A}{1+\cot^2 A}=\dfrac{\sec^2 A}{\operatorname{cosec}^2 A}=\dfrac{1}{\cos^2 A}\div\dfrac{1}{\sin^2 A}=\dfrac{\sin^2 A}{\cos^2 A}.$ ■

§ Problem 2.1.13. $\dfrac{\sec A-\tan A}{\sec A+\tan A}=1-2\sec A\tan A+2\tan^2 A.$ ◊

§§ Solution.

$$\frac{\sec A-\tan A}{\sec A+\tan A}=\frac{(\sec A-\tan A)^2}{\sec^2 A-\tan^2 A}=(\sec A-\tan A)^2$$
$$=\sec^2 A-2\sec A\tan A+\tan^2 A$$
$$=1+\tan^2 A-2\sec A\tan A+\tan^2 A$$
$$=1-2\sec A\tan A+2\tan^2 A. \qquad \blacksquare$$

§ Problem 2.1.14. $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=\sec A\operatorname{cosec} A+1.$ ◊

§§ Solution.

$$\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}=\frac{\dfrac{\sin A}{\cos A}}{1-\dfrac{\cos A}{\sin A}}+\frac{\dfrac{\cos A}{\sin A}}{1-\dfrac{\sin A}{\cos A}}$$
$$=\frac{\sin^2 A}{\cos A\,(\sin A-\cos A)}-\frac{\cos^2 A}{\sin A\,(\sin A-\cos A)}$$
$$=\frac{\sin^3 A-\cos^3 A}{\cos A\sin A\,(\sin A-\cos A)}$$
$$=\frac{\sin^2 A+\cos^2 A+\sin A\cos A}{\cos A\sin A}$$
$$=\frac{1+\sin A\cos A}{\cos A\sin A}$$
$$=\frac{1}{\cos A}\cdot\frac{1}{\sin A}+1=\sec A\operatorname{cosec} A+1. \qquad \blacksquare$$

§ Problem 2.1.15. $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A.$ ◊

§§ Solution.

$$\begin{aligned}\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}&=\frac{\cos A}{1-\dfrac{\sin A}{\cos A}}+\frac{\sin A}{1-\dfrac{\cos A}{\sin A}}\\&=\frac{\cos^2 A}{\cos A-\sin A}+\frac{\sin^2 A}{\sin A-\cos A}\\&=\frac{\cos^2 A-\sin^2 A}{\cos A-\sin A}=\cos A+\sin A.\end{aligned}$$ ■

§ Problem 2.1.16. $(\sin A+\cos A)(\cot A+\tan A)=\sec A+\operatorname{cosec} A.$ ◊

§§ Solution.

$$\begin{aligned}(\sin A+\cos A)(\cot A+\tan A)&=(\sin A+\cos A)\left(\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)\\&=(\sin A+\cos A)\left(\frac{\cos^2 A+\sin^2 A}{\sin A\cos A}\right)\\&=(\sin A+\cos A)\left(\frac{1}{\sin A\cos A}\right)\\&=\frac{\sin A}{\sin A\cos A}+\frac{\cos A}{\sin A\cos A}\\&=\frac{1}{\cos A}+\frac{1}{\sin A}=\sec A+\operatorname{cosec} A.\end{aligned}$$ ■

§ Problem 2.1.17. $\sec^4 A-\sec^2 A=\tan^4 A+\tan^2 A.$ ◊

§§ Solution.

$$\begin{aligned}\sec^4 A-\sec^2 A&=\sec^2 A\left(\sec^2 A-1\right)\\&=\left(1+\tan^2 A\right)\tan^2 A=\tan^2 A+\tan^4 A.\end{aligned}$$ ■

§ Problem 2.1.18. $\cot^4 A+\cot^2 A=\operatorname{cosec}^4 A-\operatorname{cosec}^2 A.$ ◊

§§ Solution.

$$\begin{aligned}\cot^4 A+\cot^2 A&=\cot^2 A\left(\cot^2 A+1\right)\\&=\left(\operatorname{cosec}^2 A-1\right)\operatorname{cosec}^2 A\\&=\operatorname{cosec}^4 A-\operatorname{cosec}^2 A.\end{aligned}$$ ■

§ Problem 2.1.19. $\sqrt{\operatorname{cosec}^2 A-1}=\cos A\operatorname{cosec} A.$ ◊

§§ Solution.

$$\begin{aligned}\sqrt{\operatorname{cosec}^2 A-1}=\sqrt{\cot^2 A}&=\frac{\cos A}{\sin A}\\&=\cos A\times\frac{1}{\sin A}=\cos A\operatorname{cosec} A.\end{aligned}$$ ■

§ Problem 2.1.20. $\sec^2 A\operatorname{cosec}^2 A=\tan^2 A+\cot^2 A+2.$ ◊

§§ Solution.

$$\begin{aligned}\sec^2 A\operatorname{cosec}^2 A&=\left(1+\tan^2 A\right)\left(1+\cot^2 A\right)\\&=1+\tan^2 A+\cot^2 A+\tan^2 A\cot^2 A\\&=1+\tan^2 A+\cot^2 A+1=\tan^2 A+\cot^2 A+2.\end{aligned}$$ ■

§ Problem 2.1.21. $\tan^2 A-\sin^2 A=\sin^4 A\sec^2 A.$ ◊

§§ Solution.

$$\begin{aligned}\tan^2 A - \sin^2 A &= \frac{\sin^2 A}{\cos^2 A} - \sin^2 A \\ &= \frac{\sin^2 A\left(1-\cos^2 A\right)}{\cos^2 A} = \frac{\sin^2 A \sin^2 A}{\cos^2 A} \\ &= \sin^4 A \times \frac{1}{\cos^2 A} = \sin^4 A \sec^2 A.\end{aligned}$$ ■

§ Problem 2.1.22. $(1+\cot A - \operatorname{cosec} A)(1+\tan A + \sec A) = 2.$ ◊

§§ Solution.

$$\begin{aligned}&(1+\cot A - \operatorname{cosec} A)(1+\tan A + \sec A) \\ &= \left(1+\frac{\cos A}{\sin A} - \frac{1}{\sin A}\right)\left(1+\frac{\sin A}{\cos A}+\frac{1}{\cos A}\right) \\ &= \left(\frac{\sin A + \cos A - 1}{\sin A}\right)\left(\frac{\sin A + \cos A + 1}{\cos A}\right) \\ &= \frac{(\sin A + \cos A)^2 - 1}{\sin A \cos A} \\ &= \frac{\sin^2 A + \cos^2 A + 2\sin A\cos A - 1}{\sin A \cos A} \\ &= \frac{2\sin A \cos A}{\sin A \cos A} = 2.\end{aligned}$$ ■

§ Problem 2.1.23. $\dfrac{1}{\operatorname{cosec} A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A + \cot A}.$ ◊

§§ Solution.

$$\begin{aligned}\frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A} &= \frac{1}{\dfrac{1}{\sin A} - \dfrac{\cos A}{\sin A}} - \frac{1}{\sin A} \\ &= \frac{\sin A}{1-\cos A} - \frac{1}{\sin A} = \frac{\sin^2 A - 1 + \cos A}{\sin A\,(1-\cos A)} \\ &= \frac{-\cos^2 A + \cos A}{\sin A\,(1-\cos A)} = \frac{\cos A\,(1-\cos A)}{\sin A\,(1-\cos A)} \\ &= \frac{\cos A}{\sin A} = \frac{\cos A\,(1+\cos A)}{\sin A\,(1+\cos A)} \\ &= \frac{\cos A + \cos^2 A}{\sin A\,(1+\cos A)} = \frac{\cos A + 1 - \sin^2 A}{\sin A\,(1+\cos A)} \\ &= \frac{1}{\sin A} - \frac{\sin A}{1+\cos A} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A}.\end{aligned}$$

Otherwise thus :

$$\begin{aligned}&\frac{1}{\operatorname{cosec} A - \cot A} + \frac{1}{\operatorname{cosec} A + \cot A} \\ &= \frac{\operatorname{cosec} A + \cot A + \operatorname{cosec} A - \cot A}{\operatorname{cosec}^2 A - \cot^2 A} \\ &= \frac{2\operatorname{cosec} A}{\operatorname{cosec}^2 A - \cot^2 A} = 2\operatorname{cosec} A \left[\because \operatorname{cosec}^2 A - \cot^2 A = 1\right] \\ &= \frac{2}{\sin A} = \frac{1}{\sin A} + \frac{1}{\sin A}\end{aligned}$$

$$\therefore \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A}.$$ ■

§ Problem 2.1.24. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}$. ◊

§§ Solution.

$$\frac{\cot A \cos A}{\cot A + \cos A} = \frac{\cot A \cos A\,(\cot A - \cos A)}{\cot^2 A - \cos^2 A}.$$

$$\begin{aligned}
\text{Now } \cot^2 A - \cos^2 A &= \frac{\cos^2 A}{\sin^2 A} - \cos^2 A \\
&= \frac{\cos^2 A\left(1 - \sin^2 A\right)}{\sin^2 A} \\
&= \frac{\cos^2 A}{\sin^2 A} \times \cos^2 A = \cot^2 A \cos^2 A
\end{aligned}$$

$$\therefore \frac{\cot A \cos A}{\cot A + \cos A} = \frac{\cot A \cos A\,(\cot A - \cos A)}{\cot^2 A \cos^2 A} = \frac{\cot A - \cos A}{\cot A \cos A}. \quad \blacksquare$$

§ Problem 2.1.25. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B$. ◊

§§ Solution.

$$\begin{aligned}
\frac{\cot A + \tan B}{\cot B + \tan A} &= \frac{\dfrac{1}{\tan A} + \tan B}{\dfrac{1}{\tan B} + \tan A} \\
&= \left[\frac{1 + \tan A \tan B}{\tan A}\right] \div \left[\frac{1 + \tan A \tan B}{\tan B}\right] \\
&= \frac{\tan B}{\tan A} = \cot A \tan B. \quad \blacksquare
\end{aligned}$$

§ Problem 2.1.26.

$$\left(\frac{1}{\sec^2 \alpha - \cos^2 \alpha} + \frac{1}{\operatorname{cosec}^2 \alpha - \sin^2 \alpha}\right) \cos^2 \alpha \sin^2 \alpha = \frac{1 - \cos^2 \alpha \sin^2 \alpha}{2 + \cos^2 \alpha \sin^2 \alpha}. \quad ◊$$

§§ Solution.

$$\begin{aligned}
&\left(\frac{1}{\sec^2 \alpha - \cos^2 \alpha} + \frac{1}{\operatorname{cosec}^2 \alpha - \sin^2 \alpha}\right) \cos^2 \alpha \sin^2 \alpha \\
&= \left(\frac{1}{\dfrac{1}{\cos^2 \alpha} - \cos^2 \alpha} + \frac{1}{\dfrac{1}{\sin^2 \alpha} - \sin^2 \alpha}\right) \cos^2 \alpha \sin^2 \alpha \\
&= \left(\frac{\cos^2 \alpha}{1 - \cos^4 \alpha} + \frac{\sin^2 \alpha}{1 - \sin^4 \alpha}\right) \cos^2 \alpha \sin^2 \alpha \\
&= \frac{\cos^4 \alpha}{1 + \cos^2 \alpha} + \frac{\sin^4 \alpha}{1 + \sin^2 \alpha} \\
&= \frac{\cos^2 \alpha\left(1 - \sin^2 \alpha\right)}{1 + \cos^2 \alpha} + \frac{\sin^2 \alpha\left(1 - \cos^2 \alpha\right)}{1 + \sin^2 \alpha} \\
&= \frac{\cos^2 \alpha\left(1 - \sin^4 \alpha\right) + \sin^2 \alpha\left(1 - \cos^4 \alpha\right)}{(1 + \cos^2 \alpha)\left(1 + \sin^2 \alpha\right)} \\
&= \frac{\left(\cos^2 \alpha + \sin^2 \alpha\right)\left(1 - \cos^2 \alpha \sin^2 \alpha\right)}{1 + \left(\cos^2 \alpha + \sin^2 \alpha\right) + \cos^2 \alpha \sin^2 \alpha} \\
&= \frac{1 - \cos^2 \alpha \sin^2 \alpha}{2 + \cos^2 \alpha \sin^2 \alpha}. \quad \blacksquare
\end{aligned}$$

§ Problem 2.1.27.

$$\sin^8 A - \cos^8 A = \left(\sin^2 A - \cos^2 A\right) \times \left(1 - 2\sin^2 A \cos^2 A\right). \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}\sin^8 A - \cos^8 A &= \left(\sin^4 A - \cos^4 A\right)\left(\sin^4 A + \cos^4 A\right)\\ &= \left(\sin^2 A + \cos^2 A\right)\left(\sin^2 A - \cos^2 A\right)\\ &\times \left[\left(\sin^2 A + \cos^2 A\right)^2 - 2\sin^2 A\cos^2 A\right]\\ &= \left(\sin^2 A - \cos^2 A\right)\left(1 - 2\sin^2 A\cos^2 A\right). \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.28. $\dfrac{\cos A \operatorname{cosec} A - \sin A \sec A}{\cos A + \sin A} = \operatorname{cosec} A - \sec A.$ $\lozenge$

§§ Solution.

$$\begin{aligned}\frac{\cos A \operatorname{cosec} A - \sin A \sec A}{\cos A + \sin A} &= \frac{\dfrac{\cos A}{\sin A} - \dfrac{\sin A}{\cos A}}{\cos A + \sin A}\\ &= \frac{\cos^2 A - \sin^2 A}{\sin A \cos A\,(\cos A + \sin A)}\\ &= \frac{\cos A - \sin A}{\sin A \cos A}\\ &= \frac{1}{\sin A} - \frac{1}{\cos A} = \operatorname{cosec} A - \sec A. \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.29. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}.$ $\lozenge$

§§ Solution.

$$\begin{aligned}\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} &= \frac{[\tan A + (\sec A - 1)]^2}{\tan^2 A - (\sec A - 1)^2}\\ &= \frac{\tan^2 A + \sec^2 A - 2\sec A + 1 + 2\tan A\,(\sec A - 1)}{\tan^2 A - \sec^2 A + 2\sec A - 1}\\ &= \frac{2\sec^2 A - 2\sec A + 2\tan A\,(\sec A - 1)}{2\sec A - 2}\\ &= \frac{2\sec A\,(\sec A - 1) + 2\tan A\,(\sec A - 1)}{2\,(\sec A - 1)}\\ &= \sec A + \tan A = \frac{1 + \sin A}{\cos A}.\end{aligned}$$

Otherwise thus :

$$\begin{aligned}\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} &= \frac{\tan A + \sec A - \left(\sec^2 A - \tan^2 A\right)}{\tan A - \sec A + 1}\\ &= \frac{\tan A + \sec A + \tan^2 A - \sec^2 A}{\tan A - \sec A + 1}\\ &= \frac{(\tan A + \sec A)(1 + \tan A - \sec A)}{\tan A - \sec A + 1}\\ &= \tan A + \sec A = \frac{1 + \sin A}{\cos A}. \qquad \blacksquare\end{aligned}$$

§ Problem 2.1.30.

$$(\tan\alpha + \operatorname{cosec}\beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha\cot\beta\,(\operatorname{cosec}\alpha + \sec\beta). \qquad \lozenge$$

§§ Solution.

$$
\begin{aligned}
&(\tan\alpha+\operatorname{cosec}\beta)^2-(\cot\beta-\sec\alpha)^2\\
&=\tan^2\alpha+\operatorname{cosec}^2\beta+2\tan\alpha\operatorname{cosec}\beta-\cot^2\beta-\sec^2\alpha+2\cot\beta\sec\alpha\\
&=\tan^2\alpha-\sec^2\alpha+\operatorname{cosec}^2\beta-\cot^2\beta+2\left(\tan\alpha\operatorname{cosec}\beta+\cot\beta\sec\alpha\right)\\
&=-1+1+2\left(\frac{\sin\alpha}{\cos\alpha}\cdot\frac{1}{\sin\beta}+\frac{\cos\beta}{\sin\beta}\cdot\frac{1}{\cos\alpha}\right)\\
&=2\left(\frac{\sin\alpha+\cos\beta}{\cos\alpha\sin\beta}\right)=\frac{2\sin\alpha\cos\beta}{\cos\alpha\sin\beta}\left(\frac{\sin\alpha+\cos\beta}{\sin\alpha\cos\beta}\right)\\
&=2\tan\alpha\cot\beta\left(\sec\beta+\operatorname{cosec}\alpha\right). \qquad\blacksquare
\end{aligned}
$$

§ Problem 2.1.31.

$$
\begin{aligned}
2\sec^2\alpha-\sec^4\alpha-2\operatorname{cosec}^2\alpha+\operatorname{cosec}^4\alpha&\\
=\cot^4\alpha-\tan^4\alpha.&\qquad\lozenge
\end{aligned}
$$

§§ Solution.

$$
\begin{aligned}
2\sec^2\alpha-\sec^4\alpha&-2\operatorname{cosec}^2\alpha+\operatorname{cosec}^4\alpha\\
&=\operatorname{cosec}^4\alpha-2\operatorname{cosec}^2\alpha+1-\left(\sec^4\alpha-2\sec^2\alpha+1\right)\\
&=\left(\operatorname{cosec}^2\alpha-1\right)^2-\left(\sec^2\alpha-1\right)^2=\cot^4\alpha-\tan^4\alpha. \qquad\blacksquare
\end{aligned}
$$

§ Problem 2.1.32.

$$
\begin{aligned}
(\sin\alpha+\operatorname{cosec}\alpha)^2+(\cos\alpha+\sec\alpha)^2&\\
=\tan^2\alpha+\cot^2\alpha+7.&\qquad\lozenge
\end{aligned}
$$

§§ Solution.

$$
\begin{aligned}
&(\sin\alpha+\operatorname{cosec}\alpha)^2+(\cos\alpha+\sec\alpha)^2\\
&=\sin^2\alpha+2\sin\alpha\operatorname{cosec}\alpha+\operatorname{cosec}^2\alpha+\cos^2\alpha+2\cos\alpha\sec\alpha+\sec^2\alpha\\
&=\sin^2\alpha+\cos^2\alpha+2+2+\operatorname{cosec}^2\alpha+\sec^2\alpha\\
&=5+1+\cot^2\alpha+1+\tan^2\alpha=\tan^2\alpha+\cot^2\alpha+7. \qquad\blacksquare
\end{aligned}
$$

§ Problem 2.1.33.

$$
\begin{aligned}
(\operatorname{cosec}A+\cot A)\ \textit{covers}\ A-(\sec A+\tan A)\ \textit{vers}\ A&\\
=(\operatorname{cosec}A-\sec A)\left(2-\ \textit{vers}\ A\ \textit{covers}\ A\right).&\lozenge
\end{aligned}
$$

§§ Solution.

$$
\begin{aligned}
&(\operatorname{cosec}A+\cot A)\ \textit{covers}\ A-(\sec A+\tan A)\ \textit{vers}\ A\\
&=\left(\frac{1}{\sin A}+\frac{\cos A}{\sin A}\right)(1-\sin A)-\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}\right)(1-\cos A)\\
&=\frac{(1+\cos A)(1-\sin A)}{\sin A}-\frac{(1+\sin A)(1-\cos A)}{\cos A}\\
&=\frac{1-\sin A\cos A+(\cos A-\sin A)}{\sin A}-\frac{1-\sin A\cos A-(\cos A-\sin A)}{\cos A}\\
&=(1-\sin A\cos A)\left(\frac{1}{\sin A}-\frac{1}{\cos A}\right)+(\cos A-\sin A)\left(\frac{1}{\sin A}+\frac{1}{\cos A}\right)\\
&=\frac{\cos A-\sin A}{\sin A\cos A}\left(1-\sin A\cos A+\cos A+\sin A\right)\\
&=\left(\frac{1}{\sin A}-\frac{1}{\cos A}\right)\left[2-(1-\cos A)(1-\sin A)\right]\\
&=(\operatorname{cosec}A-\sec A)\left(2-\ \textit{vers}\ A\ \textit{covers}\ A\right). \qquad\blacksquare
\end{aligned}
$$

§ Problem 2.1.34.

$$(1+\cot A+\tan A)(\sin A-\cos A) = \frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A}. \qquad \Diamond$$

§§ Solution.

$$\begin{aligned}
(1+\cot A+\tan A)&(\sin A-\cos A)\\
&=\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)\\
&=\frac{\left(\sin^2 A+\sin A\cos A+\cos^2 A\right)(\sin A-\cos A)}{\sin A\cos A}\\
&=\frac{\sin^3 A-\cos^3 A}{\sin A\cos A}=\frac{\sin^2 A}{\cos A}-\frac{\cos^2 A}{\sin A}\\
&=\frac{1}{\cos A}\cdot\frac{1}{\operatorname{cosec}^2 A}-\frac{1}{\sin A}\cdot\frac{1}{\sec^2 A}\\
&=\frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A}. \qquad \blacksquare
\end{aligned}$$

§ Problem 2.1.35. $2\,versin\,A+\cos^2 A=1+\,versin^2 A.$ $\Diamond$

§§ Solution.

$$\begin{aligned}
2\,versin\,A+\cos^2 A&=2(1-\cos A)+\cos^2 A\\
&=1+1-2\cos A+\cos^2 A\\
&=1+(1-\cos A)^2=1+\,versin^2 A. \qquad \blacksquare
\end{aligned}$$

2.2 Relations and Trigonometrical Ratios

§ Problem 2.2.1. *Express all the other trigonometrical ratios in terms of the cosine.* $\Diamond$

§§ Solution. Taking the figure in *Art.* 31, let the length OP be unity and let the corresponding length of OM be c.

Then $MP=\sqrt{OP^2-OM^2}=\sqrt{1-c^2}$.

$$\begin{aligned}
\therefore \cos\theta&=\frac{OM}{OP}=\frac{c}{1}=c\\
\sin\theta&=\frac{MP}{OP}=\frac{\sqrt{1-c^2}}{1}=\sqrt{1-c^2}=\sqrt{1-\cos^2\theta}\\
\tan\theta&=\frac{MP}{OM}=\frac{\sqrt{1-c^2}}{c}=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\\
\cot\theta&=\frac{OM}{MP}=\frac{c}{\sqrt{1-c^2}}=\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\\
\operatorname{cosec}\theta&=\frac{OP}{MP}=\frac{1}{\sqrt{1-c^2}}=\frac{1}{\sqrt{1-\cos^2\theta}}
\end{aligned}$$

and
$$\sec\theta=\frac{OP}{OM}=\frac{1}{c}=\frac{1}{\cos\theta}. \qquad \blacksquare$$

§ Problem 2.2.2. *Express all the ratios in terms of the tangent.* $\Diamond$

§§ Solution. Here let the length OM be unity and let the corresponding length of MP be t.

Then
$$OP=\sqrt{OM^2+MP^2}=\sqrt{1+t^2}.$$

$$\therefore \tan\theta=\frac{MP}{OM}=\frac{t}{1}=t$$

$$\sin\theta = \frac{MP}{OP} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\cos\theta = \frac{OM}{OP} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2\theta}}$$

$$\cot\theta = \frac{OM}{MP} = \frac{1}{t} = \frac{1}{\tan\theta}$$

$$\text{cosec }\theta = \frac{OP}{MP} = \frac{\sqrt{1+t^2}}{t} = \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

and
$$\sec\theta = \frac{OP}{OM} = \frac{\sqrt{1+t^2}}{1} = \sqrt{1+\tan^2\theta}.$$
■

§ **Problem 2.2.3.** *Express all the ratios in terms of the cosecant.* ◊

§§ **Solution**. Here let the length MP be unity and let the corresponding length of OP be x.

Then
$$OM = \sqrt{OP^2 - MP^2} = \sqrt{x^2-1}.$$

$$\therefore \text{cosec }\theta = \frac{OP}{MP} = \frac{x}{1} = x$$

$$\sin\theta = \frac{MP}{OP} = \frac{1}{x} = \frac{1}{\text{cosec }\theta}$$

$$\cos\theta = \frac{OM}{OP} = \frac{\sqrt{x^2-1}}{x} = \frac{\sqrt{\text{cosec }^2\theta - 1}}{\text{cosec }\theta}$$

$$\tan\theta = \frac{MP}{OM} = \frac{1}{\sqrt{x^2-1}} = \frac{1}{\sqrt{\text{cosec }^2\theta - 1}}$$

$$\cot\theta = \frac{OM}{OP} = \frac{\sqrt{x^2-1}}{1} = \sqrt{\text{cosec }^2\theta - 1}$$

and
$$\sec\theta = \frac{OP}{OM} = \frac{x}{\sqrt{x^2-1}} = \frac{\text{cosec }\theta}{\sqrt{\text{cosec }^2\theta - 1}}.$$
■

§ **Problem 2.2.4.** *Express all the ratios in terms of the secant.* ◊

§§ **Solution**. Here let the length OM be unity and let the corresponding length of OP be x.

Then
$$MP = \sqrt{OP^2 - OM^2} = \sqrt{x^2-1}.$$

$$\therefore \sec\theta = \frac{OP}{OM} = \frac{x}{1} = x$$

$$\sin\theta = \frac{MP}{OP} = \frac{\sqrt{x^2-1}}{x} = \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}$$

$$\cos\theta = \frac{OM}{OP} = \frac{1}{x} = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{MP}{OM} = \frac{\sqrt{x^2-1}}{1} = \sqrt{\sec^2\theta - 1}$$

$$\cot\theta = \frac{OM}{MP} = \frac{1}{\sqrt{x^2-1}} = \frac{1}{\sqrt{\sec^2\theta - 1}}$$

and
$$\text{cosec }\theta = \frac{OP}{MP} = \frac{x}{\sqrt{x^2-1}} = \frac{\sec\theta}{\sqrt{\sec^2\theta - 1}}.$$
■

§ **Problem 2.2.5.** *The sine of a certain angle is $\frac{1}{4}$; find the numerical values of the other trigonometrical ratios of this angle.* ◊

§§ Solution. With the same figure, take P so that the length of OP is 4.

Then, since $\sin\theta = \dfrac{MP}{OP} = \dfrac{1}{4}$, we have $MP = 1$, and

$$OM = \sqrt{OP^2 - MP^2} = \sqrt{16-1} = \sqrt{15}.$$

$$\therefore \cos\theta = \frac{OM}{OP} = \frac{\sqrt{15}}{4}, \ \tan\theta = \frac{MP}{OM} = \frac{1}{\sqrt{15}}$$

$$\cot\theta = \frac{OM}{MP} = \frac{\sqrt{15}}{1} = \sqrt{15}, \ \sec\theta = \frac{OP}{OM} = \frac{4}{\sqrt{15}}$$

and
$$\operatorname{cosec}\,\theta = \frac{OP}{MP} = \frac{4}{1} = 4.$$

Otherwise thus :

Proceed as in $Ex.$ 4, $Art.$ 31, with $\sin\theta = \dfrac{1}{4}$. ■

§ Problem 2.2.6. *If* $\sin\theta = \dfrac{12}{13}$, *find* $\tan\theta$ *and* $versin\ \theta$. ◊

§§ Solution. Here we have $\sin\theta = \dfrac{MP}{OP} = \dfrac{12}{13}$.

$$\therefore OP = 13,\ MP = 12;$$

$$\therefore OM = \sqrt{OP^2 - MP^2} = \sqrt{13^2 - 12^2} = \sqrt{25} = 5.$$

$$\therefore \tan\theta = \frac{MP}{OM} = \frac{12}{5}.$$

$$\therefore\ versin\ \theta = 1 - \cos\theta = 1 - \frac{OM}{OP} = 1 - \frac{5}{13} = \frac{8}{13}.$$

Otherwise thus :

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{12}{13} \div \sqrt{1-\left(\frac{12}{13}\right)^2} = \frac{12}{\sqrt{25}} = \frac{12}{5}.$$

$$\therefore\ versin\ \theta = 1 - \cos\theta = 1 - \sqrt{1-\sin^2\theta}$$

$$= 1 - \sqrt{1-\left(\frac{12}{13}\right)^2} = 1 - \frac{5}{13} = \frac{8}{13}.$$ ■

§ Problem 2.2.7. *If* $\sin A = \dfrac{11}{61}$, *find* $\tan A,\ \cos A$ *and* $\sec A$. ◊

§§ Solution. Here take $OP = 61$ and $MP = 11$ and let the angle θ be denoted by A.

$$\therefore OM = \sqrt{OP^2 - MP^2} = \sqrt{61^2 - 11^2}$$

$$= \sqrt{(61+11)(61-11)} = \sqrt{72 \times 50} = \sqrt{36 \times 100} = 60.$$

$$\therefore \tan A = \frac{MP}{OM} = \frac{11}{60}$$

$$\therefore \cos A = \frac{OM}{OP} = \frac{60}{61}\ \textit{and}\ \sec A = \frac{OP}{OM} = \frac{61}{60}.$$

Otherwise thus :

$$\tan A = \frac{\sin A}{\sqrt{1-\sin^2 A}} = \frac{11}{61} \div \sqrt{1-\left(\frac{11}{61}\right)^2} = \frac{11}{61} \div \frac{60}{61} = \frac{11}{60}$$

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{11}{61}\right)^2} = \frac{60}{61}$$

and $$\sec A = \frac{1}{\sqrt{1 - \sin^2 A}} = \frac{1}{\sqrt{1 - \left(\frac{11}{61}\right)^2}} = \frac{61}{60}.$$ ■

§ **Problem 2.2.8.** *If* $\cos\theta = \frac{4}{5}$*, find* $\sin\theta$ *and* $\cot\theta$*.* ◊

§§ **Solution**. Here take $OM = 4$ and proceed as in $Ex.\ 3,\ Art.\ 31.$

Otherwise thus :

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

and $$\cot\theta = \frac{\cos\theta}{\sqrt{1 - \cos^2\theta}} = \frac{4}{5} \div \frac{3}{5} = \frac{4}{3}.$$ ■

§ **Problem 2.2.9.** *If* $\cos A = \frac{9}{41}$*, find* $\tan A$ *and* $\operatorname{cosec} A$*.* ◊

§§ **Solution**.

Here we have $OM = 9,\ OP = 41,\ and$

$$MP = \sqrt{OP^2 - OM^2} = \sqrt{41^2 - 9^2} = \sqrt{50 \times 32} = \sqrt{100 \times 16} = 40.$$

$$\therefore \tan A = \frac{MP}{OM} == \frac{40}{9}$$

and $$\operatorname{cosec} A = \frac{OP}{MP} = \frac{41}{40},$$

the angle $\angle POM$ being denoted by A. ■

§ **Problem 2.2.10.** *If* $\tan\theta = \frac{3}{4}$*, find the sine, cosine, versine and cosecant of* θ*.* ◊

§§ **Solution**. Here take $OM = 4$ and $MP = 3$. Then

$$OP = \sqrt{OM^2 + MP^2} = \sqrt{16 + 9} = 5.$$

$$\therefore \sin\theta = \frac{MP}{OP} = \frac{3}{5},\ \cos\theta = \frac{OM}{OP} = \frac{4}{5}$$

$$versin\ \theta = 1 - \cos\theta = 1 - \frac{4}{5} = \frac{1}{5}\ and\ \operatorname{cosec}\theta = \frac{OP}{MP} = \frac{5}{3}.$$

Otherwise thus :

$$\sin\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = \frac{3}{4} \div \sqrt{1 + \frac{9}{16}} = \frac{3}{4} \div \frac{5}{4} = \frac{3}{5}$$

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = 1 \div \frac{5}{4} = \frac{4}{5},\ versin\ \theta = 1 - \cos\theta = \frac{1}{5}$$

and $$\operatorname{cosec}\theta = \frac{\sqrt{1 + \tan^2\theta}}{\tan\theta} = \frac{5}{4} \div \frac{3}{4} = \frac{5}{3}.$$ ■

§ **Problem 2.2.11.** *If* $\tan\theta = \frac{1}{\sqrt{7}}$*, find the value of* $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$*.* ◊

§§ Solution.

We have $$\operatorname{cosec}^2\theta = 1+\cot^2\theta = 1+\frac{1}{\tan^2\theta} = 1+7 = 8$$

and $$\sec^2\theta = 1+\tan^2\theta = 1+\frac{1}{7} = \frac{8}{7}$$

$$\therefore \frac{\operatorname{cosec}^2\theta-\sec^2\theta}{\operatorname{cosec}^2\theta+\sec^2\theta} = \frac{8-\frac{8}{7}}{8+\frac{8}{7}} = \frac{48}{64} = \frac{3}{4}.$$

Otherwise thus :

$$\frac{\operatorname{cosec}^2\theta-\sec^2\theta}{\operatorname{cosec}^2\theta+\sec^2\theta} = \frac{\left(\operatorname{cosec}^2\theta-\sec^2\theta\right)\sin^2\theta}{(\operatorname{cosec}^2\theta+\sec^2\theta)\sin^2\theta}$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{1}{7}}{1+\frac{1}{7}} = \frac{6}{8} = \frac{3}{4}.$$ ■

§ Problem 2.2.12. *If* $\cot\theta = \dfrac{15}{8}$, *find* $\cos\theta$ *and* $\operatorname{cosec}\theta$. ◊

§§ Solution. Here take $OM = 15$ and $MP = 8$. Then

$$OP = \sqrt{OM^2+MP^2} = \sqrt{15^2+8^2} = \sqrt{289} = 17.$$

$$\therefore \cos\theta = \frac{OM}{OP} = \frac{15}{17}$$

and $$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{17}{8}.$$

Otherwise thus :

$$\cos\theta = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}} = \frac{15}{8} \div \sqrt{1+\left(\frac{15}{8}\right)^2} = \frac{15}{8} \div \frac{17}{8} = \frac{15}{17}$$

and $\operatorname{cosec}\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+\left(\frac{15}{8}\right)^2} = \sqrt{\frac{289}{64}} = \frac{17}{8}.$ ■

§ Problem 2.2.13. *If* $\sec A = \dfrac{3}{2}$, *find* $\tan A$ *and* $\operatorname{cosec} A$. ◊

§§ Solution.

Here take $OM = 2$ *and* $OP = 3\,[cf.\ Ex.\ 3,\ Art.\ 31]$.

Then $$MP = \sqrt{OP^2-OM^2} = \sqrt{3^2-2^2} = \sqrt{5}.$$

$$\therefore \tan A = \frac{MP}{OM} = \frac{\sqrt{5}}{2}$$

and $$\operatorname{cosec} A = \frac{OP}{MP} = \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5},$$

the angle $\angle POM$ being denoted by A.

Otherwise thus :

$$\tan A = \sqrt{\sec^2 A-1} = \sqrt{\frac{9}{4}-1} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

and $$\operatorname{cosec} A = \frac{\sec A}{\sqrt{\sec^2 A-1}} = \frac{3}{2} \div \frac{\sqrt{5}}{2} = \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}.$$ ■

§ Problem 2.2.14. *If* $2\sin\theta = 2-\cos\theta$*, find* $\sin\theta$*.* $\lozenge$

§§ Solution.

$$2\sin\theta = 2-\cos\theta$$
$$\therefore \cos\theta = 2\left(1-\sin\theta\right);\ \therefore \cos^2\theta = 4\left(1-\sin\theta\right)^2$$
$$\therefore 1-\sin^2\theta = 4\left(1-\sin\theta\right)^2$$
$$\therefore \left(1-\sin\theta\right)\left(1+\sin\theta\right) = 4\left(1-\sin\theta\right)^2$$
$$\therefore 1-\sin\theta = 0 \ or\ 1+\sin\theta = 4\left(1-\sin\theta\right).$$

If $\quad 1-\sin\theta = 0,\ then\ \sin\theta = 1.$

If $\quad 1+\sin\theta = 4\left(1-\sin\theta\right),\ then\ 1+\sin\theta = 4-4\sin\theta$

$$\therefore 5\sin\theta = 3,\ \therefore \sin\theta = \frac{3}{5}.$$ ■

§ Problem 2.2.15. *If* $8\sin\theta = 4+\cos\theta$*, find* $\sin\theta$*.* $\lozenge$

§§ Solution.

$$8\sin\theta = 4+\cos\theta$$
$$\therefore 4\left(2\sin\theta-1\right) = \cos\theta = \sqrt{1-\sin^2\theta}$$
$$\therefore 16\left(4\sin^2\theta-4\sin\theta+1\right) = 1-\sin^2\theta$$
$$\therefore 65\sin^2\theta-64\sin\theta+15 = 0$$
$$\therefore \left(13\sin\theta-5\right)\left(5\sin\theta-3\right) = 0$$
$$\therefore 13\sin\theta-5 = 0 \ or\ 5\sin\theta-3 = 0.$$

If $\quad 13\sin\theta-5 = 0\ then\ \sin\theta = \frac{5}{13}.$

If $\quad 5\sin\theta-3 = 0\ then\ \sin\theta = \frac{3}{5}.$ ■

§ Problem 2.2.16. *If* $\tan\theta+\sec\theta = 1.5$*, find* $\sin\theta$*.* $\lozenge$

§§ Solution.

$$\tan\theta+\sec\theta = 1.5$$
$$\therefore \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta} = \frac{3}{2};\ \therefore \frac{\left(1+\sin\theta\right)^2}{\cos^2\theta} = \frac{9}{4};$$
$$\therefore \frac{\left(1+\sin\theta\right)^2}{1-\sin^2\theta} = \frac{9}{4};\ \therefore \frac{1+\sin\theta}{1-\sin\theta} = \frac{9}{4};\ \therefore \sin\theta = \frac{9-4}{9+4} = \frac{5}{13}.$$ ■

§ Problem 2.2.17. *If* $\cot\theta+\operatorname{cosec}\theta = 5$*, find* $\cos\theta$*.* $\lozenge$

§§ Solution.

$$\cot\theta+\operatorname{cosec}\theta = 5$$
$$\therefore \frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta} = 5;\ \therefore \frac{\left(1+\cos\theta\right)^2}{\sin^2\theta} = 25;\ \therefore \frac{\left(1+\cos\theta\right)^2}{1-\cos^2\theta} = 25$$
$$\therefore \frac{1+\cos\theta}{1-\cos\theta} = 25;\ \therefore \cos\theta = \frac{25-1}{25+1} = \frac{24}{26} = \frac{12}{13}.$$ ■

§ Problem 2.2.18. *If* $3\sec^4\theta+8 = 10\sec^2\theta$*, find the values of* $\tan\theta$*.* $\lozenge$

§§ Solution.

$$3\sec^4\theta+8 = 10\sec^2\theta$$
$$\therefore 3\left(1+\tan^2\theta\right)^2+8 = 10\left(1+\tan^2\theta\right)$$
$$\therefore 3\tan^4\theta-4\tan^2\theta+1 = 0$$
$$\therefore \left(3\tan^2\theta-1\right)\left(\tan^2\theta-1\right) = 0$$

$$\therefore 3\tan^2\theta - 1 = 0 \textit{ or } \tan^2\theta - 1 = 0.$$

If $\quad 3\tan^2\theta - 1 = 0, \textit{ then } \tan^2\theta = \dfrac{1}{3}, \ \therefore \tan\theta = \dfrac{1}{\sqrt{3}}.$

If $\quad \tan^2\theta - 1 = 0, \textit{ then } \tan^2\theta = 1, \ \therefore \tan\theta = 1.$

Note : Negative values of the ratios have not yet been discussed; hence, in this set of solutions, only positive ones are given.

Otherwise thus :

We have
$$3\sec^4\theta - 10\sec^2\theta + 8 = 0$$
$$\therefore \left(3\sec^2\theta - 4\right)\left(\sec^2\theta - 2\right) = 0$$
$$\therefore 3\sec^2\theta - 4 = 0 \textit{ or } \sec^2\theta - 2 = 0.$$

If $\quad 3\sec^2\theta - 4 = 0, \textit{ then } \sec^2\theta = \dfrac{4}{3},$

and $\quad \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\dfrac{4}{3} - 1} = \dfrac{1}{\sqrt{3}}.$

If $\quad \sec^2\theta - 2 = 0, \textit{ then } \sec^2\theta = 2, \textit{ and } \tan\theta = \sqrt{2 - 1} = 1.$ ■

§ Problem 2.2.19. *If* $\tan^2\theta + \sec\theta = 5$*, find* $\cos\theta$*.* ◊

§§ Solution.

$$\tan^2\theta + \sec\theta = 5$$
$$\therefore \frac{\sin^2\theta}{\cos^2\theta} + \frac{1}{\cos\theta} = 5$$
$$\therefore 1 - \cos^2\theta + \cos\theta = 5\cos^2\theta$$
$$\therefore 6\cos^2\theta - \cos\theta - 1 = 0$$
$$\therefore (2\cos\theta - 1)(3\cos\theta + 1) = 0$$
$$\therefore 2\cos\theta - 1 = 0 \textit{ or } 3\cos\theta + 1 = 0.$$

If $2\cos\theta - 1 = 0, \textit{ then } \cos\theta = \dfrac{1}{2}.$

If $3\cos\theta + 1 = 0, \textit{ then } \cos\theta = -\dfrac{1}{3}.$ [Refer note to the last solution] .

Otherwise thus :

$$\sec^2\theta - 1 + \sec\theta = 5$$
$$\therefore \sec^2\theta + \sec\theta - 6 = 0; \ \therefore (\sec\theta - 2)(\sec\theta + 3) = 0$$
$$\therefore \sec\theta - 2 = 0 \textit{ or } \sec\theta + 3 = 0.$$

If $\quad \sec\theta - 2 = 0, \textit{ then } \sec\theta = 2 \textit{ and } \therefore \cos\theta = \dfrac{1}{\sec\theta} = \dfrac{1}{2}.$

If $\quad [\sec\theta + 3 = 0, \textit{ then } \sec\theta = -3 \textit{ and } \therefore \cos\theta = -\dfrac{1}{3}.]$ ■

§ Problem 2.2.20. *If* $\tan\theta + \cot\theta = 2$*, find* $\sin\theta$*.* ◊

§§ Solution.

$$\tan\theta + \cot\theta = 2; \ \therefore \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 2$$
$$\therefore \sin^2\theta + \cos^2\theta = 2\sin\theta\cos\theta$$
$$\therefore 1 = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1 - \sin^2\theta}$$
$$\therefore 1 = 4\sin^2\theta - 4\sin^4\theta; \ \therefore 4\sin^4\theta - 4\sin^2\theta + 1 = 0$$

$$\therefore \left(2\sin^2\theta - 1\right)^2 = 0;\ \therefore 2\sin^2\theta - 1 = 0,\ \therefore \sin\theta = \frac{1}{\sqrt{2}}.$$ ■

§ Problem 2.2.21. *If* $\sec^2\theta = 2 + 2\tan\theta$, *find* $\tan\theta$. ◊

§§ Solution.

$$\sec^2\theta = 2 + 2\tan\theta,\ \therefore 1 + \tan^2\theta = 2 + 2\tan\theta$$
$$\therefore \tan^2\theta - 2\tan\theta + 1 = 2;\ \therefore (\tan\theta - 1)^2 = 2$$
$$\therefore \tan\theta - 1 = \sqrt{2};\ \therefore \tan\theta = 1 + \sqrt{2}.$$ ■

§ Problem 2.2.22. *If* $\tan\theta = \frac{2x(x+1)}{2x+1}$, *find* $\sin\theta$ *and* $\cos\theta$. ◊

§§ Solution. We have

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{2x(x+1)}{2x+1} \div \sqrt{1 + \frac{4x^2(x+1)^2}{4x^2+4x+1}}$$
$$= \frac{2x(x+1)}{2x+1} \div \sqrt{\frac{4x^4+8x^3+8x^2+4x+1}{4x^2+4x+1}}$$
$$= \frac{2x(x+1)}{2x+1} \div \frac{2x^2+2x+1}{2x+1} = \frac{2x(x+1)}{2x^2+2x+1}$$

and $$\cos\theta = \sin\theta \div \tan\theta = \frac{2x+1}{2x^2+2x+1}.$$

Otherwise thus :

With the figure of *Art.* 31, take $OM = 2x+1$ and $MP = 2x(x+1)$. Then

$$OP = \sqrt{OM^2 + MP^2} = \sqrt{(2x+1)^2 + 4x^2(x+1)^2}$$
$$= \sqrt{4x^4+8x^3+8x^2+4x+1} = 2x^2+2x+1.$$
$$\therefore \sin\theta = \frac{MP}{OP} = \frac{2x(x+1)}{2x^2+2x+1}$$

and $$\cos\theta = \frac{OM}{OP} = \frac{2x+1}{2x^2+2x+1}.$$ ■

2.3 Values of Trigonometrical Ratios

§ Problem 2.3.1. *If* $A = 30^\circ$, *verify that*

(1) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

(2) $\sin 2A = 2\sin A\cos A$

(3) $\cos 3A = 4\cos^3 A - 3\cos A$

(4) $\sin 3A = 3\sin A - 4\sin^3 A$, *and*

(5) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$. ◊

§§ Solution. (1) $\cos 2A = \cos 60^\circ = \frac{1}{2}$

$$\cos^2 A - \sin^2 A = \cos^2 30^\circ - \sin^2 30^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$
$$= \frac{3-1}{4} = \frac{1}{2}$$

and $$2\cos^2 A - 1 = 2\cos^2 30^\circ - 1 = 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$
$$= \frac{3}{2} - 1 = \frac{1}{2}.$$

(2) $\sin 2A = \sin 60^\circ = \dfrac{\sqrt{3}}{2}$
$$2\sin A\cos A = 2\sin 30^\circ \cos 30^\circ = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

(3) $\cos 3A = \cos 90^\circ = 0$, and
$$\begin{aligned}4\cos^3 A - 3\cos A &= \cos A\left(4\cos^2 A - 3\right)\\ &= \cos 30^\circ\left(4\cos^2 30^\circ - 3\right) = \frac{\sqrt{3}}{2}\left[4\left(\frac{\sqrt{3}}{2}\right)^2 - 3\right]\\ &= \frac{\sqrt{3}}{2}(3-3) = 0.\end{aligned}$$

(4) $\sin 3A = \sin 90^\circ = 1$, and
$$\begin{aligned}3\sin A - 4\sin^3 A &= \sin A\left(3 - 4\sin^2 A\right)\\ &= \sin 30^\circ\left(3 - 4\sin^2 30^\circ\right)\\ &= \frac{1}{2}\left[3 - 4\left(\frac{1}{2}\right)^2\right] = \frac{1}{2}(3-1) = 1.\end{aligned}$$

(5) $\tan 2A = \tan 60^\circ = \sqrt{3}$, and
$$\begin{aligned}\frac{2\tan A}{1-\tan^2 A} &= \frac{2\tan 30^\circ}{1-\tan^2 30^\circ}\\ &= \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2\sqrt{3}}{2} = \sqrt{3}.\end{aligned}$$
■

§ **Problem 2.3.2.** *If* $A = 45^\circ$, *verify that*

(1) $\sin 2A = 2\sin A\cos A$

(2) $\cos 2A = 1 - 2\sin^2 A$, *and*

(3) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ ◊

§§ **Solution.** (1) $\sin 2A = \sin 90^\circ = 1$, and
$$2\sin A\cos A = 2\sin 45^\circ\cos 45^\circ = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1.$$

(2) $\cos 2A = \cos 90^\circ = 0$, and
$$1 - 2\sin^2 A = 1 - 2\sin^2 45^\circ = 1 - 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 = 1 - 1 = 0.$$

(3) $\tan 2A = \tan 90^\circ = \infty$, and
$$\frac{2\tan A}{1-\tan^2 A} = \frac{2\tan 45^\circ}{1-\tan^2 45^\circ} = \frac{2\times 1}{1-1} = \frac{2}{0} = \infty.$$
■

Verify that

§ Problem 2.3.3. $\sin^2 30° + \sin^2 45° + \sin^2 60° = \dfrac{3}{2}$. ◊

§§ Solution.

$$\begin{aligned}\sin^2 30° + \sin^2 45° + \sin^2 60° &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= \frac{1}{4} + \frac{1}{2} + \frac{3}{4} = \frac{3}{2}.\end{aligned}$$ ■

§ Problem 2.3.4. $\tan^2 30° + \tan^2 45° + \tan^2 60° = 4\dfrac{1}{3}$. ◊

§§ Solution.

$$\begin{aligned}&\tan^2 30° + \tan^2 45° + \tan^2 60° \\ &\quad= \left(\frac{1}{\sqrt{3}}\right)^2 + 1 + \left(\sqrt{3}\right)^2 = \frac{1}{3} + 1 + 3 = 4\frac{1}{3}.\end{aligned}$$ ■

§ Problem 2.3.5. $\sin 30° \cos 60° + \cos 30° \sin 60° = 1$. ◊

§§ Solution.

$$\begin{aligned}&\sin 30° \cos 60° + \cos 30° \sin 60° \\ &\quad= \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1.\end{aligned}$$ ■

§ Problem 2.3.6. $\cos 45° \cos 60° - \sin 45° \sin 60° = -\dfrac{\sqrt{3}-1}{2\sqrt{2}}$. ◊

§§ Solution.

$$\begin{aligned}&\cos 45° \cos 60° - \sin 45° \sin 60° \\ &\quad= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}-1}{2\sqrt{2}}.\end{aligned}$$ ■

§ Problem 2.3.7. $\dfrac{4}{3} \cot^2 30° + 3 \sin^2 60° - 2 \operatorname{cosec}^2 60° - \dfrac{3}{4} \tan^2 30° = 3\dfrac{1}{3}$. ◊

§§ Solution.

$$\begin{aligned}&\frac{4}{3} \cot^2 30° + 3 \sin^2 60° - 2 \operatorname{cosec}^2 60° - \frac{3}{4} \tan^2 30° \\ &\quad= \frac{4}{3}\left(\sqrt{3}\right)^2 + 3\left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{2}{\sqrt{3}}\right)^2 - \frac{3}{4}\left(\frac{1}{\sqrt{3}}\right)^2 \\ &\quad= 4 + \frac{9}{4} - \frac{8}{3} - \frac{1}{4} = 6 - 2\frac{2}{3} = 3\frac{1}{3}.\end{aligned}$$ ■

§ Problem 2.3.8. $\operatorname{cosec}^2 45° \cdot \sec^2 30° \cdot \sin^3 90° \cdot \cos 60° = 1\dfrac{1}{3}$. ◊

§§ Solution.

$$\begin{aligned}\operatorname{cosec}^2 45° \cdot \sec^2 30° \cdot \sin^3 90° \cdot \cos 60° &= \left(\sqrt{2}\right)^2 \times \left(\frac{2}{\sqrt{3}}\right)^2 \times 1^3 \times \frac{1}{2} \\ &= 2 \times \frac{4}{3} \times 1 \times \frac{1}{2} = \frac{4}{3} = 1\frac{1}{3}.\end{aligned}$$ ■

§ Problem 2.3.9. $4 \cot^2 45° - \sec^2 60° + \sin^3 30° = \dfrac{1}{3}$. ◊

§§ Solution.

$$\begin{aligned} 4\cot^2 45^\circ - \sec^2 60^\circ + \sin^3 30^\circ &= 4 \times 1^2 - 2^2 + \left(\frac{1}{2}\right)^3 \\ &= 4 - 4 + \frac{1}{8} = \frac{1}{8}. \end{aligned}$$ ■

Chapter 3

Simple Problems in Heights and Distances

3.1 Simple Problems

§ Problem 3.1.1. *A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is* $60°$*; when he retires* 40 *feet from the bank he finds the angle to be* $30°$ *; find the height of the tree and the breadth of the river.* $\lozenge$

§§ Solution. Take the figure of $Ex.\ 2,\ Art.\ 45$. Let $PM(x\ feet,\ say)$ be the tree, and B and A be the two positions of the observer respectively, so that BM represents the breadth of the river. We are given $BA = 40\ feet$, the $\angle MBP = 60°$ and the $\angle MAP = 30°$. We then have

$$\frac{AM}{x} = \cot 30° = \sqrt{3},\ and\ \frac{BM}{x} = \cot 60° = \frac{1}{\sqrt{3}}.$$

$$\therefore AM = x\sqrt{3},\ and\ BM = \frac{x}{\sqrt{3}}.$$

$$\therefore 40 = AM - BM = x\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = x\left(\frac{3-1}{\sqrt{3}}\right) = \frac{2x}{\sqrt{3}}.$$

$$\therefore x = 20\sqrt{3} = 20 \times 1.73205 = 34.64\ feet.$$

$$\therefore BM = \frac{x}{\sqrt{3}} = \frac{20\sqrt{3}}{\sqrt{3}} = 20\ ft.$$

■

§ Problem 3.1.2. *At a certain point the angle of elevation of a tower is found to be such that its cotangent is* $\frac{3}{5}$ *; on walking* 32 *feet directly toward the tower its angle of elevation is an angle whose cotangent is* $\frac{2}{5}$*. Find the height of the tower.* $\lozenge$

§§ **Solution**. Taking the same figure, let $PM(x\ feet,\ say)$ be the tower and A and B be the two points at which the angles of elevation are taken respectively.

We are given $AB = 32\ feet,\ \cot\angle MAP = \frac{3}{5}\ and\ \cot\angle MBP = \frac{2}{5}$.

We then have $\frac{AM}{x} = \frac{3}{5},\ and\ \frac{BM}{x} = \frac{2}{5}$.

$$\therefore AM = \frac{3}{5}x\ and\ BM = \frac{2}{5}x.$$

$$\therefore 32 = AM - BM = \frac{x}{5}$$

$$\therefore x = 32 \times 5\ feet = 160\ feet.$$

■

§ **Problem 3.1.3.** *At a point A, the angle of elevation of a tower is found to be such that its tangent is $\frac{5}{12}$; on walking 240 feet nearer the tower the tangent of the angle of elevation is found to be $\frac{3}{4}$; what is the height of the tower?* ♢

§§ **Solution**. Taking the same figure, let $PM(x\ feet,\ say)$ be the tower and A and B be the two points at which the angles of elevation are taken respectively.

We are given $AB = 240\ feet,\ \tan\angle MAP = \frac{5}{12}\ and\ \tan\angle MBP = \frac{3}{4}$.

We then have $\frac{x}{AM} = \frac{5}{12},\ and\ \frac{x}{BM} = \frac{3}{4}$.

$$\therefore AM = \frac{12}{5}x\ and\ BM = \frac{4}{3}x.$$

$$\therefore 240 = AM - BM = \left(\frac{12}{5} - \frac{4}{3}\right)x = \frac{16}{15}x.$$

$$\therefore x = 15 \times 15\ feet = 225\ feet.$$

■

§ **Problem 3.1.4.** *Find the height of a chimney when it is found that, on walking towards it 100 feet in a horizontal line through its base, the angular elevation of its top changes from $30°$ to $45°$.* ♢

§§ **Solution**. Taking the same figure, with $PM(x\ feet,\ say)$ representing the chimney and A and B being the two points at which the angles of elevation are taken respectively, we are given $AB = 100\ feet$, the $\angle MAP = 30°$, and the $\angle MBP = 45°$. We then have

$$\frac{AM}{x} = \cot 30° = \sqrt{3},\ and\ \frac{BM}{x} = \cot 45° = 1.$$

$$\therefore AM = x\sqrt{3},\ and\ BM = x;$$

$$\therefore 100 = AM - BM = \left(\sqrt{3} - 1\right)x.$$

$$\therefore x = \frac{100}{\sqrt{3} - 1} = 50\left(\sqrt{3} + 1)\right) = 50 \times 2.73205 = 136.6\ feet.$$

■

§ **Problem 3.1.5.** *An observer on the top of a cliff, 200 feet above the sea-level, observes the angles of depression of two ships at anchor to be $45°$ and $30°$ respectively ; find the distances between the ships if the line joining them points to the base of the cliff.* ♢

§§ Solution. Take the same figure as before. Draw PN parallel to MA, so that PN is the horizontal line passing through P. Let P be the position of the observer on the cliff PM and B and A be the positions of the ships respectively.

We are given $PM = 200\ feet$, the $\angle NPB = 45° = \angle PBM$ $(Euc.\ I.\ 29)$ and the $\angle NPA = 30° = \angle PAM$.

We then have

$$\frac{AM}{200} = \cot 30° = \sqrt{3},\ and\ \frac{BM}{200} = \cot 45° = 1.$$

$$\therefore AM = 200\sqrt{3},\ and\ BM = 200.$$

$$\therefore AB = AM - BM = 200\left(\sqrt{3} - 1\right) = 200 \times .73205 = 146.4\ feet.$$ ■

§ Problem 3.1.6. *From the top of a cliff an observer finds that the angles of depression of two buoys in the sea are* $39°$ *and* $26°$ *respectively; the buoys are* 300 *yards apart and the line joining them points straight at the foot of the cliff ; find the height of the cliff and the distance of the nearest buoy from the foot of the cliff, given that* $\cot 26° = 2.0503$*, and* $\cot 39° = 1.2349$. ◊

§§ Solution. Construct a figure as in the last example, with B and A as the positions of the two buoys.

We are given $BA = 300\ yards$, the $\angle NPB = 39° = \angle PBM$, and the $\angle NPA = 26° = \angle PAM$.

Let x yards be the height of the cliff PM. We then have $BM = x \cot 39° = x \times 1.2349$, and

$$\begin{aligned} x \times 2.0503 = x \cot 26° = AM &= 300 + BM \\ &= 300 + x \times 1.2349. \\ \therefore 300 &= x(2.0503 - 1.2349) = .8154x. \\ \therefore x &= \frac{300}{.8154} = 367.9\ yards. \end{aligned}$$

Also, the required distance

$$\begin{aligned} &= BM = x \cot 39° \\ &= (367.9 \times 1.2349)\ yards \\ &= 454.3\ yards. \end{aligned}$$ ■

§ Problem 3.1.7. *The upper part of a tree broken over by the wind makes an angle of* $30°$ *with the ground, and the distance from the root to the point where the top of the tree touches the ground is* $60\ feet$ *; what was the height of the tree?* ◊

§§ Solution. Let y feet be the length of the broken part of the tree and x feet be the height of the stump. We then have

$$x = 50 \tan 30° = \frac{50}{\sqrt{3}},\ and\ y = x \operatorname{cosec} 30° = 2x = \frac{100}{\sqrt{3}}.$$

Hence the original height of the tree

$$\begin{aligned} &= x + y \\ &= \frac{150}{\sqrt{3}} = 50\sqrt{3} \\ &= 50 \times 1.73205 = 86.6\ feet. \end{aligned}$$ ■

§ Problem 3.1.8. *The horizontal distance between two towers is* 60 *feet and the angular depression of the top of the first as seen from the top of the second, which is* 150 *feet high, is* 30° *; find the height of the first.* ◊

§§ Solution. Take the figure of $Ex.\ 3,\ Art.\ 45$. Let AB be the second tower, 150 feet high and CD be the first tower, x feet high.

We are given the $\angle EAC = 30°$, $CE = 150 - x$ and $DB = 60\ feet = EA$.

We then have

$$150 - x = 60 \tan 30° = \frac{60}{\sqrt{3}} = 20\sqrt{3} = 34.641$$
$$\therefore x = 150 - 34.641 = 115.359\ldots\ ft.$$ ■

§ Problem 3.1.9. *The angle of elevation of the top of an unfinished tower at a point distant* 120 *feet from its base is* 45° *; how much higher must the tower be raised so that its angle of elevation at the same point may be* 60° *?* ◊

§§ Solution. Let h feet be the present height of the tower and x feet be the required height. We then have

$$\frac{h}{120} = \tan 45° = 1,\ and\ \frac{h+x}{120} = \tan 60° = \sqrt{3}.$$
$$\therefore x = \left(\sqrt{3} - 1\right) 120 = .73205 \times 120 = 87.846\ feet.$$ ■

§ Problem 3.1.10. *Two pillars of equal height stand on either side of a roadway which is* 100 *feet wide ; at a point in the roadway between the pillars the elevations of the tops of the pillars are* 60° *and* 30° *; find their height and the position of the point.* ◊

§§ Solution. Let h feet be the height of the pillars and x feet and y feet be the distances of the point respectively. We then have

$$x = h \cot 60° = \frac{h}{\sqrt{3}} = \frac{h\sqrt{3}}{3},\ and\ y = h \cot 30° = h\sqrt{3}$$
$$\therefore \frac{h\sqrt{3}}{3} + h\sqrt{3} = 100$$
$$\therefore 4h\sqrt{3} = 300$$
$$\therefore h = \frac{100\sqrt{3}}{4} = \frac{173.205}{4} = 43.3\ feet.$$
$$\text{Also, } x = \frac{100\sqrt{3}}{4} \times \frac{\sqrt{3}}{3} = 25\ feet$$
$$\text{and } y = \frac{100\sqrt{3}}{4} \times \sqrt{3} = 75\ feet.$$ ■

§ Problem 3.1.11. *The angle of elevation of the top of a tower is observed to be* 60° *; at a point* 40 *feet above the first point of observation the elevation is found to be* 45° *; find the height of the tower and its horizontal distance from the points of observation.* ◊

§§ Solution. Let h feet be the height of the tower and x feet be the required horizontal distance. We then have

$$h = x \tan 60° = x\sqrt{3},\ and\ h - 40 = x \tan 45° = x.$$

Hence, by subtraction, we have

$$40 = x\left(\sqrt{3} - 1\right).$$

$$\therefore x = \frac{40}{\sqrt{3}-1} = 20\left(\sqrt{3}+1\right) = 20 \times 2.73205 = 54.641 \ feet.$$
$$\therefore h = x + 40 = 94.641 \ feet. \qquad \blacksquare$$

§ **Problem 3.1.12.** *At the foot of a mountain the elevation of its summit is found to be* $45°$*; after ascending one mile towards the mountain up a slope of* $30°$ *inclination the elevation is found to be* $60°$*. Find the height of the mountain .* ♢

§§ **Solution**. Let A be the foot of the mountain and C be its summit, AB be the slope and AD be the horizontal line through A and D being vertically below C.

Join BC. We are given the $\angle CAD = 45°$, $AB = 1 \ mile$, $\angle BAD = 30°$ and the $\angle CBF = 60°$, where BF is drawn parallel to AD, meeting CD in F.

We then have

$$\angle CAB = 45° - 30° = 15°, \ and \ \angle ACD = 45°$$
$$\therefore \angle ACB = 45° - \angle BCF = 45° - 30° = 15° = \angle CAB$$
$$\therefore BC = AB = 1 \ mile.$$

Draw BE perpendicular to AD.

$$\therefore CF = BC \sin 60° = \frac{\sqrt{3}}{2}, \ and$$
$$FD = BE = AB \sin 30° = \frac{1}{2}.$$
$$\therefore CD = CF + FD = \frac{\sqrt{3}+1}{2} = \frac{2.73205}{2} = 1.366\ldots \ mile. \qquad \blacksquare$$

§ **Problem 3.1.13.** *What is the angle of elevation of the sun when the length of the shadow of a pole is* $\sqrt{3}$ *times the height of the pole ?* ♢

§§ **Solution**. In the figure, $Art.$ 23, let PM represent the pole and OM its shadow.

$$\therefore \tan \angle POM = \frac{PM}{OM} = \frac{1}{\sqrt{3}}$$
$$\therefore \text{ The angle of elevation } = 30°. \qquad \blacksquare$$

§ **Problem 3.1.14.** *The shadow of a tower standing on a level plane is found to be* 60 *feet longer when the sun's altitude is* $30°$ *than when it is* $45°$*. Prove that the height of the tower is* $30(1+\sqrt{3}) \ feet$ *.* ♢

§§ **Solution**. Take the figure of $Ex.$ 2, $Art.$ 45. Let $PM(x \ feet, \ say)$ be the tower.

We are given the $\angle PBM = 45°$, $BA = 60 \ feet$ and the $\angle PAM = 30°$.

We then have

$$\frac{BM}{x} = \cot 45° = 1, \ and \ \frac{AM}{x} = \cot 30° = \sqrt{3}.$$
$$\therefore BM = x, \ and \ AM = AB + BM = 60 + x = x\sqrt{3}.$$
$$\therefore \left(\sqrt{3}-1\right) x = 60$$
$$x = \frac{60}{\sqrt{3}-1} = 30\left(1+\sqrt{3}\right) \ feet. \qquad \blacksquare$$

§ **Problem 3.1.15.** *On a straight coast there are three objects* A*,* B*, and* C *such that* $AB = BC = 2\ miles$*. A vessel approaches* B *in a line perpendicular to the coast and at a certain point* AC *is found to subtend an angle of* $60°$ *; after sailing in the same direction for ten minutes* AC *is found to subtend* $120°$ *; find the rate at which the ship is going.* ◊

§§ **Solution**. Let P and Q be the points at which AC subtends angles of $60°$ and $120°$ respectively.

We are given the $\angle APC = 60°$, $\angle AQC = 120°$ and $AB = BC = 2\ miles$.

We then have

$$BP = AB \cot 30° = 2\sqrt{3}\ miles,\ and$$
$$BQ = AB \cot 60° = \frac{2}{\sqrt{3}}\ miles.$$

Hence in 10 minutes the ship sails a distance

$$\begin{aligned} &= PQ = BP - BQ = \left(2\sqrt{3} - \frac{2}{\sqrt{3}}\right)\ miles \\ &= \frac{4}{\sqrt{3}}\ miles. \\ \therefore \text{ the required rate } &= \left(6 \times \frac{4}{\sqrt{3}}\right)\ miles\ per\ hour \\ &= 8\sqrt{3} = 8 \times 1.73205 \\ &= 13.8564\ miles\ per\ hour. \end{aligned}$$

■

§ **Problem 3.1.16.** *Two flagstaffs stand on a horizontal plane.* A *and* B *are two points on the line joining the bases of the flagstaffs and between them. The angles of elevation of the tops of the flagstaffs as seen from* A *are* $30°$ *and* $60°$ *and, as seen from* B*, they are* $60°$ *and* $45°$*. If the length* AB *be* 30 *feet, find the heights of the flagstaffs and the distance between them.* ◊

§§ **Solution**. Let $EC(h\ feet,\ say)$ be the flagstaff near A, C being its base and $FD(h'\ feet,\ say)$ be the flagstaff near B, D being its base.

Let the

$$\angle FAD = 30°,\ \angle EAC = 60°,\ \angle FBD = 60°\ and\ \angle EBC = 45°.$$

We then have

$$CA = h \cot 60° = \frac{h}{\sqrt{3}},\ CB = h \cot 45° = h,$$
$$DA = h' \cot 30° = h'\sqrt{3},\ and\ DB = h' \cot 60° = \frac{h'}{\sqrt{3}}.$$
$$\because CB = CA + AB,\ \therefore h = \frac{h}{\sqrt{3}} + 30$$
$$\begin{aligned} \therefore h = \frac{30\sqrt{3}}{\sqrt{3}-1} &= 15\sqrt{3}\left(\sqrt{3}+1\right) = 15\left(3+\sqrt{3}\right) \\ &= 15 \times 4.73205 = 70.98\ldots\ feet. \end{aligned}$$

Also, since $AD = AB + BD,\ we\ have\ h'\sqrt{3} = 30 + \dfrac{h'}{\sqrt{3}}$

$$\therefore 2h' = 30\sqrt{3};\ \therefore h' = 15\sqrt{3} = 25.98\ldots\ feet.$$
$$\therefore CD = CA + AB + BD = \frac{h}{\sqrt{3}} + 30 + \frac{h'}{\sqrt{3}}$$

$$= 15\left(\sqrt{3}+1\right) + 30 + 15 = 85.98\ldots\ feet. \qquad \blacksquare$$

§ Problem 3.1.17. *P is the top and Q the foot of a tower standing on a horizontal plane. A and B are two points on this plane such that AB is 32 feet and $\angle QAB$ is a right angle. It is found that $\cot\angle PAQ = \dfrac{2}{5}$ and*

$$\cot\angle PBQ = \frac{3}{5}$$

find the height of the tower. $\Diamond$

§§ Solution. If h feet be the height of the tower, we have

$$AQ = PQ\cot\angle PAQ = \frac{2}{5}h,\ and$$

$$BQ = PQ\cot\angle PBQ = \frac{3}{5}h.$$

Also,
$$BQ^2 = BA^2 + AQ^2$$

$$\therefore \frac{9}{25}h^2 = (32)^2 + \frac{4}{25}h^2$$

$$\therefore 5h^2 = (32)^2 \times 25$$

$$\therefore h = 32\sqrt{5} = 32 \times 2.236\ldots = 71.55\ldots\ feet. \qquad \blacksquare$$

§ Problem 3.1.18. *A square tower stands upon a horizontal plane. From a point in this plane, from which three of its upper corners are visible, their angular elevations are respectively 45°, 60°, and 45°. Show that the height of the tower is to the breadth of one of its sides as $\sqrt{6}\left(\sqrt{5}+1\right)$ to 4.* $\Diamond$

§§ Solution. Let P, Q and R be the three upper corners whose angular elevations are 45°, 60° and 45° respectively at the point D.

Let C, B and A be the three lower corners vertically under P, Q and R respectively.

Also, let h be the height of the tower, a be the breadth of one of its sides and E be the middle point of its base. We then have

$$AD = h\cot 45^\circ = h = CD$$

$$BD = h\cot 60^\circ = \frac{h}{\sqrt{3}},\ and$$

$$BE = EA = a\cos 45^\circ = \frac{a}{\sqrt{2}}.$$

Also,
$$DA^2 = DE^2 + EA^2 = (DB+BE)^2 + EA^2$$

$$\therefore h^2 = \left(\frac{h}{\sqrt{3}} + \frac{a}{\sqrt{2}}\right)^2 + \frac{a^2}{2} = \frac{h^2}{3} + \frac{2ah}{\sqrt{6}} + \frac{a^2}{2} + \frac{a^2}{2}$$

$$\therefore 2h^2 - ah\sqrt{6} - 3a^2 = 0$$

$$\therefore h = \frac{a\sqrt{6} \pm \sqrt{6a^2 + 24a^2}}{4} = \frac{a\sqrt{6}}{4}\left(1+\sqrt{5}\right)$$

since the positive sign must obviously be taken.

$$\therefore h : a = \sqrt{6}\left(\sqrt{5}+1\right) : 4. \qquad \blacksquare$$

§ Problem 3.1.19. *A lighthouse, facing north, sends out a fan-shaped beam of light extending from north-east to north-west. An observer on a steamer, sailing due west, first sees the light when he is 5 miles* *away from the lighthouse and continues to see it for* $30\sqrt{2}$ *minutes. What is the speed of the steamer ?* ◊

§§ Solution. Let L be the lighthouse and S and H be respectively the two positions of the steamer at which the light is visible. We then have

$$LS = LH = 5 \; miles, \; and \; \angle SLH = 90^\circ.$$

Hence in $30\sqrt{2}$ minutes the steamer sails a distance

$$= SH = 5 \sec 45^\circ = 5\sqrt{2} \; miles$$

$$\begin{aligned} \therefore \text{ The required speed } &= \left(\frac{5\sqrt{2}}{30\sqrt{2}} \times 60 \right) \; miles \; per \; hour \\ &= 10 \; miles \; per \; hour. \end{aligned}$$ ■

§ Problem 3.1.20. *A man stands at a point* X *on the bank* XY *of a river with straight and parallel banks and observes that the line joining* X *to a point* Z *on the opposite bank makes an angle of* 30° *with* XY. *He then goes along the bank a distance of* 200 *yards to* Y *and finds that the* $\angle ZYX$ *is* 60°. *Find the breadth of the river.* ◊

§§ Solution. Let A be the point in XY opposite Z.

The $\angle XZY = 90^\circ$ and we have

$$XZ = XY \sin 60^\circ = 200 \times \frac{\sqrt{3}}{2} = 100\sqrt{3} \; yards.$$

Hence the breadth of the river

$$\begin{aligned} &= AZ = XZ \sin 30^\circ \\ &= 100\sqrt{3} \times \frac{1}{2} = 50\sqrt{3} \; yards \\ &= 50 \times 1.73205 = 86.6 \ldots \; yards. \end{aligned}$$

Otherwise thus : We have

$$\frac{AY}{AZ} = \cot 60^\circ = \frac{1}{\sqrt{3}}$$

$$\text{and } \frac{AX}{AZ} = \cot 30^\circ = \sqrt{3}.$$

Hence, by addition, we have

$$\frac{XY}{AZ} = \frac{1}{\sqrt{3}} + \sqrt{3} = \frac{4}{\sqrt{3}}$$

$$\therefore AZ = \frac{XY\sqrt{3}}{4} = \frac{200\sqrt{3}}{4} = 50\sqrt{3} = 86.6 \ldots \; yards.$$ ■

§ Problem 3.1.21. *A man, walking due north, observes that the elevation of a balloon, which is due east of him and is sailing toward the north-west, is then* 60° *; after he has walked* 400 *yards the balloon is vertically over his head ; find its height supposing it to have always remained the same.* ◊

§§ Solution. Let M and M' be the two positions of the man, so that $MM' = 400$ yards.

Let B and B' be two positions of the balloon and h yards be its height above the ground, so that $BA = h = B'M'$, where A is the point of the ground vertically below B.

Then ΔBAM and $\Delta B'M'M$ are right angled triangles in two vertical planes and $\Delta AMM'$ a right angled triangle in the horizontal plane,

and we have $$AM = h \cot 60^\circ = \frac{h}{\sqrt{3}}.$$

Also, $$\angle AMM' = 90^\circ, \; and \; \angle MAM' = 45^\circ$$

$$\therefore AM = MM'$$

$$\therefore \frac{h}{\sqrt{3}} = 400$$

$$\therefore h = 400\sqrt{3} \; yards = 4 \times 173.205\ldots = 692.8\ldots \; yards.$$ ■

Chapter 4

Applications of Algebraic Signs to Trigonometry

4.1 Tracing the changes in the ratios

§ Problem 4.1.1. *In a triangle one angle contains as many grades as another contains degrees, and the third contains as many centesimal seconds as there are sexagesimal seconds in the sum of the other two; find the number of radians in each angle.* ◊

§§ Solution. Let x, y and z be the number of radians in each angle respectively.

We then have

$$x + y + z = \pi \tag{4.1}$$

$$x \times \frac{200}{\pi} = y \times \frac{180}{\pi} \tag{4.2}$$

$$z \times \frac{200 \times 100 \times 100}{\pi} = (x + y)\frac{180 \times 60 \times 60}{\pi} \tag{4.3}$$

From Eq. (4.2), we have $10x = 9y$.

From Eq. (4.3), we have $250z = 81(x + y)$.

$$\therefore \frac{x}{9} = \frac{y}{10} = \frac{x + y}{19} = \frac{250z}{81 \times 19}.$$

$$\therefore \frac{x}{2250} = \frac{y}{2500} = \frac{z}{1539} = \frac{x + y + z}{6289} = \frac{\pi}{6289}$$

So that the angles contain

$$\frac{2250}{6289}\pi, \ \frac{2500}{6289}\pi, \ and \ \frac{1539}{6289}\pi \left(i.e. \ \frac{81}{331}\pi \right) \ radians.$$ ■

§ Problem 4.1.2. *Find the number of degrees, minutes, and seconds in the angle at the center of a circle, whose radius is* 5 *feet, which is subtended by an arc of length* 6 *feet.* $\Diamond$

§§ Solution. The number of radians in the angle $= \dfrac{6}{5}$.

Hence, in degrees, the angle

$$= \left(\frac{6}{5} \times \frac{180}{\pi}\right)^{\circ} = \frac{6}{5} \times 57^{\circ}17'44.8''[Art.\ 16] = 68^{\circ}45'17.8''. \qquad \blacksquare$$

§ Problem 4.1.3. *To turn radians into seconds, prove that we must multiply by* 206265 *nearly, and to turn seconds into radians the multiplier must be* $.0000048$. $\Diamond$

§§ Solution. If θ be the number of radians in any angle and s be the number of seconds in the same angle, we have

$$\frac{\theta}{\pi} = \frac{s}{180 \times 60 \times 60}$$

$$\therefore s = \theta \times \frac{180}{\pi} \times 60 \times 60 \approx 206265$$

and
$$\theta = s \times \frac{\pi}{180 \times 60 \times 60} \approx .0000048. \qquad \blacksquare$$

§ Problem 4.1.4. *If* $\sin\theta = \dfrac{x^2 - y^2}{x^2 + y^2}$, *find the values of* $\cos\theta$ *and* $\cot\theta$. $\Diamond$

§§ Solution. We have

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2}$$

$$= \sqrt{\frac{4x^2y^2}{(x^2 + y^2)^2}} = \frac{2xy}{x^2 + y^2}$$

and
$$\cot\theta = \frac{\sqrt{1 - \sin^2\theta}}{\sin\theta} = \frac{2xy}{x^2 + y^2} \div \frac{x^2 - y^2}{x^2 + y^2} = \frac{2xy}{x^2 - y^2}.$$

Otherwise thus : Taking the figure of *Art.* 31, let the length OP be $x^2 + y^2$ and let the corresponding length of MP be $x^2 - y^2$. Then

$$OM = \sqrt{OP^2 - MP^2} = \sqrt{(x^2 + y^2)^2 - (x^2 - y^2)^2} = \sqrt{4x^2y^2} = 2xy.$$

$$\therefore \cos\theta = \frac{OM}{OP} = \frac{2xy}{x^2 + y^2}$$

and
$$\cot\theta = \frac{OM}{MP} = \frac{2xy}{x^2 - y^2}. \qquad \blacksquare$$

§ Problem 4.1.5. *If* $\sin\theta = \dfrac{m^2 + 2mn}{m^2 + 2mn + 2n^2}$,

prove that
$$\tan\theta = \frac{m^2 + 2mn}{2mn + 2n^2}. \qquad \Diamond$$

§§ Solution. With the same figure, let the length OP be $m^2 + 2mn + 2n^2$ and let the corresponding length of MP be $m^2 + 2mn$. Then

$$OM = \sqrt{OP^2 - MP^2} = \sqrt{(m^2 + 2mn + 2n^2)^2 - (m^2 + 2mn)^2}$$
$$= \sqrt{(2m^2 + 4mn + 2n^2)\, 2n^2} \left[\because a^2 - b^2 = (a + b)(a - b)\right]$$
$$= \sqrt{(m^2 + 2mn + n^2)\, 4n^2} = (m + n)2n.$$

$$\therefore \tan\theta = \frac{MP}{OM} = \frac{m^2+2mn}{2mn+2n^2}.$$

Otherwise thus : Substitute for $\sin\theta$ in the formula

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$$

■

§ Problem 4.1.6.

If $$\cos\theta - \sin\theta = \sqrt{2}\sin\theta$$

prove that $$\cos\theta + \sin\theta = \sqrt{2}\cos\theta.$$ ◊

§§ Solution. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, then $\cos\theta = \left(\sqrt{2}+1\right)\sin\theta$.

$\therefore$ multiplying both sides by $\left(\sqrt{2}-1\right)$, we have

$$\left(\sqrt{2}-1\right)\cos\theta = \left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)\sin\theta = \sin\theta$$

$$\therefore \cos\theta + \sin\theta = \sqrt{2}\cos\theta.$$ ■

§ Problem 4.1.7.

$$\operatorname{cosec}^6\alpha - \cot^6\alpha = 3\operatorname{cosec}^2\alpha\cot^2\alpha + 1.$$ ◊

§§ Solution.

$\operatorname{cosec}^6\alpha - \cot^6\alpha$

$$\begin{aligned}
&= \left(\operatorname{cosec}^2\alpha - \cot^2\alpha\right)\left(\operatorname{cosec}^4\alpha + \operatorname{cosec}^2\alpha\cot^2\alpha + \cot^4\alpha\right)\\
&= \left(\operatorname{cosec}^2\alpha - \cot^2\alpha\right)\left[\left(\operatorname{cosec}^2\alpha - \cot^2\alpha\right)^2 + 3\operatorname{cosec}^2\alpha\cot^2\alpha\right]\\
&= 1 + 3\operatorname{cosec}^2\alpha\cot^2\alpha, \ \because \operatorname{cosec}^2\alpha - \cot^2\alpha = 1.
\end{aligned}$$ ■

§ Problem 4.1.8. *Express*

$$2\sec^2 A - \sec^4 A - 2\operatorname{cosec}^2 A + \operatorname{cosec}^4 A$$

in terms of $\tan A$. ◊

§§ Solution.

$2\sec^2 A - \sec^4 A - 2\operatorname{cosec}^2 A + \operatorname{cosec}^4 A$

$$\begin{aligned}
&= \left(\operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A + 1\right) - \left(\sec^4 A - 2\sec^2 A + 1\right)\\
&= \left(\operatorname{cosec}^2 A - 1\right)^2 - \left(\sec^2 A - 1\right)^2\\
&= \left(\cot^2 A)\right)^2 - \left(\tan^2 A\right)^2\\
&= \cot^4 A - \tan^4 A\\
&= \frac{1}{\tan^4 A} - \tan^4 A.
\end{aligned}$$ ■

§ Problem 4.1.9. *Solve the equation* $3\operatorname{cosec}^2\theta = 2\sec\theta$. ◊

§§ Solution.

$$\begin{aligned}
3\operatorname{cosec}^2\theta &= 2\sec\theta\\
\therefore \frac{3}{\sin^2\theta} &= \frac{2}{\cos\theta}\\
\therefore 3\cos\theta &= 2\left(1-\cos^2\theta\right)\\
\therefore 2\cos^2\theta + 3\cos\theta - 2 &= 0\\
\therefore (2\cos\theta - 1)(\cos\theta + 2) &= 0\\
\therefore 2\cos\theta - 1 = 0, \ &\textit{or} \ \cos\theta + 2 = 0.
\end{aligned}$$

If $2\cos\theta - 1 = 0$ then $\cos\theta = \dfrac{1}{2} = \cos 60^\circ$, i.e. $\theta = 60^\circ$.

If $\cos\theta + 2 = 0$ then $\cos\theta = -2$. This value is inadmissible, since the cosine of an angle cannot be numerically greater than unity. ■

§ Problem 4.1.10. *A man on a cliff observes a boat at an angle of depression of* $30°$*, which is making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is* $60°$*. How soon will it reach the shore ?* ♢

§§ Solution. Take the figure of $Ex.\ 2$, $Art.\ 45$. Draw PN parallel to MA, so that PN is the horizontal line passing through P.

Let P be the position of the man and A and B be the positions of the boat respectively.

We are given

$$\angle NPA = 30° = \angle PAM$$
$$\angle NPB = 60° = \angle PBM.$$
$$\because \angle BAP = 30° = \angle APB$$
$$\therefore BP = BA.$$
$$\therefore MB = BP \cos 60° = \frac{1}{2}BP = \frac{1}{2}BA.$$
$$\therefore \text{ The required time } = \left(\frac{1}{2} \times 3\right) \ min. = 1\frac{1}{2} \ min.$$ ■

§ Problem 4.1.11. *Prove that the equation* $\sin\theta = x + \dfrac{1}{x}$ *is impossible if* x *be real.* ♢

§§ Solution. By Algebra, $(x-1)^2 > 0$, so that $x^2+1 > 2x$, $\therefore x + \dfrac{1}{x} > 2$.

$\therefore \sin\theta = x + \dfrac{1}{x}$ is impossible if x be real, since the sine of an angle cannot be numerically greater than unity. ■

§ Problem 4.1.12. *Show that the equation* $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ *is only possible when* $x = y$*.* ♢

§§ Solution. By Algebra, we know that if x and y be unequal, then $(x-y)^2 > 0$,

$$\therefore x^2 + y^2 > 2xy$$
$$\therefore (x+y)^2 > 4xy$$
$$\therefore \frac{4xy}{(x+y)^2} < 1.$$

But $\sec\theta$ cannot be < 1.

Hence the equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is only possible when $x = y$, in which case $\sec^2\theta = \dfrac{4}{4} = 1$. ■

Chapter 5

Trigonometrical Functions of Angles of Any Size and Sign

5.1 Angles of Any Size and Sign

Prove that

§ Problem 5.1.1. $\sin 420^\circ \cos 390^\circ + \cos(-300^\circ)\sin(-330^\circ) = 1.$ ◊

§§ Solution.

$$\begin{aligned}
\sin 420^\circ &= \sin(360^\circ + 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2} \\
\cos 390^\circ &= \cos(360^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2} \\
\cos(-300^\circ) &= \cos 300^\circ = \cos(180^\circ + 120^\circ) = -\cos 120^\circ \\
&= -\cos(90^\circ + 30^\circ) = \sin 30^\circ = \frac{1}{2} \\
\sin(-330^\circ) &= -\sin 330^\circ = -\sin(180^\circ + 150^\circ) = \sin 150^\circ \\
&= \sin(90^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}.
\end{aligned}$$

Hence the expression $= \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{3+1}{4} = 1.$ ■

§ Problem 5.1.2. $\cos 570^\circ \sin 510^\circ - \sin 330^\circ \cos 390^\circ = 0.$ ◊

§§ Solution.

$$\begin{aligned}
\cos 570^\circ &= \cos(360^\circ + 210^\circ) = \cos 210^\circ = \cos(180^\circ + 30^\circ) \\
&= -\cos 30^\circ = -\frac{\sqrt{3}}{2} \\
\sin 510^\circ &= \sin(360^\circ + 150^\circ) = \sin 150^\circ = \sin(90^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}\sin 330^\circ &= \sin(180^\circ + 150^\circ) = -\sin 150^\circ \\ &= -\sin(90^\circ + 60^\circ) = -\cos 60^\circ = -\frac{1}{2} \\ \cos 390^\circ &= \cos(360^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}.\end{aligned}$$

Hence the expression $= -\frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \left(-\frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0.$ ■

§ Problem 5.1.3. $\tan 225^\circ \cot 405^\circ + \tan 765^\circ \cot 675^\circ = 0.$ ◊

§§ Solution.

$$\begin{aligned}\tan 225^\circ &= \tan(180^\circ + 45^\circ) = \tan 45^\circ = 1 \\ \cot 405^\circ &= \cot(360^\circ + 45^\circ) = \cot 45^\circ = 1 \\ \tan 765^\circ &= \tan(2 \times 360^\circ + 45^\circ) = \tan 45^\circ = 1 \\ \cot 675^\circ &= \cot(360^\circ + 315^\circ) = \cot 315^\circ \\ &= \cot(180^\circ + 135^\circ) = \cot 135^\circ \\ &= \cot(90^\circ + 45^\circ) = -\tan 45^\circ = -1.\end{aligned}$$

Hence the expression $= 1 - 1 = 0.$ ■

What are the values of $\cos A - \sin A$ **and** $\tan A + \cot A$ **when** A **has the values**

§ Problem 5.1.4. $\frac{\pi}{3}$ ◊

§§ Solution.

$$\begin{aligned}\cos\frac{\pi}{3} - \sin\frac{\pi}{3} &= \frac{1}{2} - \frac{\sqrt{3}}{2} = -\frac{.73205}{2} = -.366\ldots \\ \tan\frac{\pi}{3} + \cot\frac{\pi}{3} &= \sqrt{3} + \frac{1}{\sqrt{3}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \\ &= \frac{4 \times 1.73205\ldots}{3} = 2.3094\ldots.\end{aligned}$$ ■

§ Problem 5.1.5. $\frac{2\pi}{3}$ ◊

§§ Solution.

$$\begin{aligned}\cos\frac{2\pi}{3} - \sin\frac{2\pi}{3} &= -\frac{1}{2} - \frac{\sqrt{3}}{2} \\ &= -\frac{1+\sqrt{3}}{2} = -\frac{2.73205\ldots}{2} = -1.366\ldots \\ \tan\frac{2\pi}{3} + \cot\frac{2\pi}{3} &= -\sqrt{3} - \frac{1}{\sqrt{3}} = -\frac{4\sqrt{3}}{3} = -2.3094\ldots.\end{aligned}$$ ■

§ Problem 5.1.6. $\frac{5\pi}{4}$ ◊

§§ Solution.

$$\begin{aligned}\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} &= \cos\left(\pi + \frac{\pi}{4}\right) - \sin\left(\pi + \frac{\pi}{4}\right) \\ &= -\cos\frac{\pi}{4} + \sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0 \\ \tan\frac{5\pi}{4} + \cot\frac{5\pi}{4} &= \tan\left(\pi + \frac{\pi}{4}\right) + \cot\left(\pi + \frac{\pi}{4}\right) \\ &= \tan\frac{\pi}{4} + \cot\frac{\pi}{4} = 1 + 1 = 2.\end{aligned}$$ ■

§ Problem 5.1.7. $\frac{7\pi}{4}$ ◊

§§ Solution.

$$\begin{aligned}
\cos\frac{7\pi}{4}-\sin\frac{7\pi}{4} &= \cos\left(\pi+\frac{3\pi}{4}\right)-\sin\left(\pi+\frac{3\pi}{4}\right)\\
&= -\cos\frac{3\pi}{4}+\sin\frac{3\pi}{4} = -\cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right)\\
&= \sin\frac{\pi}{4}+\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} = 1.414\ldots\\
\tan\frac{7\pi}{4}+\cot\frac{7\pi}{4} &= \tan\left(\pi+\frac{3\pi}{4}\right)+\cot\left(\pi+\frac{3\pi}{4}\right)\\
&= \tan\frac{3\pi}{4}+\cot\frac{3\pi}{4} = \tan\left(\frac{\pi}{2}+\frac{\pi}{4}\right)+\cot\left(\frac{\pi}{2}+\frac{\pi}{4}\right)\\
&= -\cot\frac{\pi}{4}-\tan\frac{\pi}{4} = -1-1 = -2.
\end{aligned}$$

■

§ Problem 5.1.8. $\frac{11\pi}{3}$ ◊

§§ Solution.

$$\begin{aligned}
\cos\frac{11\pi}{3}-\sin\frac{11\pi}{3} &= \cos\left(2\pi+\frac{5\pi}{3}\right)-\sin\left(2\pi+\frac{5\pi}{3}\right)\\
&= \cos\frac{5\pi}{3}-\sin\frac{5\pi}{3} = \cos\left(\pi+\frac{2\pi}{3}\right)-\sin\left(\pi+\frac{2\pi}{3}\right)\\
&= -\cos\frac{2\pi}{3}+\sin\frac{2\pi}{3} = +\frac{1}{2}+\frac{\sqrt{3}}{2}\\
&= \frac{2.73205}{2} = 1.366\ldots\\
\tan\frac{11\pi}{3}+\cot\frac{11\pi}{3} &= \tan\left(2\pi+\frac{5\pi}{3}\right)+\cot\left(2\pi+\frac{5\pi}{3}\right)\\
&= \tan\frac{5\pi}{3}+\cot\frac{5\pi}{3} = \tan\left(\pi+\frac{2\pi}{3}\right)+\cot\left(\pi+\frac{2\pi}{3}\right)\\
&= \tan\frac{2\pi}{3}+\cot\frac{2\pi}{3} = -\sqrt{3}-\frac{1}{\sqrt{3}}\\
&= -\frac{4\sqrt{3}}{3} = -2.3094\ldots.
\end{aligned}$$

■

What values between $0°$ and $360°$ may A have when

§ Problem 5.1.9. $\sin A = \frac{1}{\sqrt{2}}$ ◊

§§ Solution. Since $\sin A$ is positive, A must be in the first or second quadrant.

Now $\sin A = \frac{1}{\sqrt{2}} = \sin 45° = \sin(180° - 45°)$.

Hence $A = 45°,$ *or* $180° - 45°,$ *i.e.* $45°$ *or* $135°$. ■

§ Problem 5.1.10. $\cos A = -\frac{1}{2}$ ◊

§§ Solution. Since $\cos A$ is negative, A must be in the second or third quadrant.

Now $\cos A = -\frac{1}{2} = -\cos 60° = \cos(180° - 60°)$ or $\cos(180° + 60°)$.

Hence $A = 120°$ *or* $240°$. ■

§ Problem 5.1.11. $\tan A = -1$ ◊

§§ Solution. Since $\tan A$ is negative, A must be in the second or fourth quadrant.

Now $\quad \tan A = -1 = -\tan 45^\circ = \tan(180^\circ - 45^\circ) = \tan 135^\circ$

also, $\quad \tan 135^\circ = \tan(180^\circ + 135^\circ) = \tan 315^\circ.$

$$\therefore A = 135^\circ \text{ or } 315^\circ. \qquad \blacksquare$$

§ Problem 5.1.12. $\cot A = -\sqrt{3}$ ◊

§§ Solution. Since $\cot A$ is negative, A must be in the second or fourth quadrant.

Now $\quad \cot A = -\sqrt{3} = -\cot 30^\circ = \cot(180^\circ - 30^\circ) = \cot 150^\circ$

also, $\quad \cot 150^\circ = \cot(180^\circ + 150^\circ) = \cot 330^\circ.$

$$\therefore A = 150^\circ \text{ or } 330^\circ. \qquad \blacksquare$$

§ Problem 5.1.13. $\sec A = -\dfrac{2}{\sqrt{3}}$ ◊

§§ Solution. Since $\sec A$ is negative, A must be in the second or third quadrant.

Now

$$\sec A = -\frac{2}{\sqrt{3}} = -\sec 30^\circ = \sec(180^\circ - 30^\circ) \text{ or } \sec(180^\circ + 30^\circ).$$

$$\therefore A = 150^\circ \text{ or } 210^\circ. \qquad \blacksquare$$

§ Problem 5.1.14. $\operatorname{cosec} A = -2$ ◊

§§ Solution. Since $\operatorname{cosec} A$ is negative, A must be in the third or fourth quadrant.

Now

$$\begin{aligned} \operatorname{cosec} A &= -2 = -\operatorname{cosec} 30^\circ \text{ or } -\operatorname{cosec} 150^\circ \\ &= \operatorname{cosec}(180^\circ + 30^\circ) \text{ or } \operatorname{cosec}(180^\circ + 150^\circ). \\ \therefore A &= 210^\circ \text{ or } 330^\circ. \end{aligned} \qquad \blacksquare$$

Express in terms of the ratios of a positive angle, which is less than $45°$, the quantities

§ Problem 5.1.15. $\sin(-65^\circ)$ ◊

§§ Solution. $\sin(-65^\circ) = -\sin 65^\circ = -\cos(90^\circ - 65^\circ) = -\cos 25^\circ.$ ■

§ Problem 5.1.16. $\cos(-84^\circ)$ ◊

§§ Solution. $\cos(-84^\circ) = \cos 84^\circ = \sin(90^\circ - 84^\circ) = \sin 6^\circ.$ ■

§ Problem 5.1.17. $\tan 137^\circ$ ◊

§§ Solution. $\tan 137^\circ = -\tan(180^\circ - 137^\circ) = -\tan 43^\circ.$ ■

§ Problem 5.1.18. $\sin 168^\circ$ ◊

§§ Solution. $\sin 168^\circ = \sin(180^\circ - 168^\circ) = \sin 12^\circ.$ ■

§ Problem 5.1.19. $\cos 287^\circ$ ◊

§§ Solution.

$$\begin{aligned} \cos 287^\circ &= \cos(180^\circ + 107^\circ) = -\cos 107^\circ \\ &= -\cos(90^\circ + 17^\circ) = \sin 17^\circ. \end{aligned} \qquad \blacksquare$$

§ Problem 5.1.20. $\tan(-246^\circ)$ ◊

§§ Solution.

$$\begin{aligned}\tan(-246^\circ) &= -\tan 246^\circ = -\tan(180^\circ + 66^\circ)\\ &= -\tan 66^\circ = -\cot(90^\circ - 66^\circ) = -\cot 24^\circ.\end{aligned}$$ ■

§ Problem 5.1.21. $\sin 843^\circ$ ◊

§§ Solution.

$$\begin{aligned}\sin 843^\circ &= \sin(2\times 360^\circ + 123^\circ) = \sin 123^\circ\\ &= \sin(180^\circ - 123^\circ) = \sin 57^\circ\\ &= \cos(90^\circ - 57^\circ) = \cos 33^\circ.\end{aligned}$$ ■

§ Problem 5.1.22. $\cos(-928^\circ)$ ◊

§§ Solution.

$$\begin{aligned}\cos(-928^\circ) &= \cos 928^\circ = \cos(2\times 360^\circ + 208^\circ)\\ &= \cos 208^\circ = \cos(180^\circ + 28^\circ) = -\cos 28^\circ.\end{aligned}$$ ■

§ Problem 5.1.23. $\tan 1145^\circ$ ◊

§§ Solution.

$$\begin{aligned}\tan 1145^\circ &= \tan(3\times 360^\circ + 65^\circ) = \tan 65^\circ\\ &= \cot(90^\circ - 65^\circ) = \cot 25^\circ.\end{aligned}$$ ■

§ Problem 5.1.24. $\cos 1410^\circ$ ◊

§§ Solution.

$$\begin{aligned}\cos 1410^\circ &= \cos(3\times 360^\circ + 330^\circ) = \cos 330^\circ\\ &= \cos(180^\circ + 150^\circ) = -\cos 150^\circ\\ &= -\cos(180^\circ - 30^\circ) = \cos 30^\circ.\end{aligned}$$ ■

§ Problem 5.1.25. $\cot(-1054^\circ)$ ◊

§§ Solution.

$$\begin{aligned}\cot(-1054^\circ) &= -\cot 1054^\circ = -\cot(2\times 360^\circ + 334^\circ) = -\cot 334^\circ\\ &= -\cot(180^\circ + 154^\circ) = -\cot 154^\circ\\ &= -\cot(180^\circ - 26^\circ) = \cot 26^\circ.\end{aligned}$$ ■

§ Problem 5.1.26. $\sec 1327^\circ$ ◊

§§ Solution.

$$\begin{aligned}\sec 1327^\circ &= \sec(3\times 360^\circ + 247^\circ) = \sec 247^\circ = \sec(180^\circ + 67^\circ)\\ &= -\sec 67^\circ = -\operatorname{cosec}(90^\circ - 67^\circ) = -\operatorname{cosec} 23^\circ.\end{aligned}$$ ■

§ Problem 5.1.27. $\operatorname{cosec}(-756^\circ)$ ◊

§§ Solution.

$$\begin{aligned}\operatorname{cosec}(-756^\circ) &= -\operatorname{cosec} 756^\circ\\ &= -\operatorname{cosec}(2\times 360^\circ + 36^\circ) = -\operatorname{cosec} 36^\circ.\end{aligned}$$ ■

What sign has $\sin A + \cos A$ for the following values of A ?

§ Problem 5.1.28. 140° ◊

§§ Solution.

$$\begin{aligned}\sin 140^\circ &= \sin(180^\circ - 140^\circ) = \sin 40^\circ\\ \cos 140^\circ &= -\cos(180^\circ - 140^\circ) = -\cos 40^\circ\end{aligned}$$

$\therefore \cos A$ is numerically greater than $\sin A$ and is negative
$\therefore \sin A + \cos A$ is negative. ■

§ Problem 5.1.29. 278° ◊

§§ Solution.

$$\begin{aligned}\sin 278^\circ &= -\sin 98^\circ = -\sin(180^\circ - 98^\circ) = -\sin 82^\circ\\ \cos 278^\circ &= \cos(180^\circ + 98^\circ) = -\cos 98^\circ = \cos(180^\circ - 98^\circ) = \cos 82^\circ.\end{aligned}$$

$\therefore \sin A$ is numerically greater than $\cos A$ and is negative
$\therefore \sin A + \cos A$ is negative. ■

§ Problem 5.1.30. -356° ◊

§§ Solution.

$$\begin{aligned}\sin(-356^\circ) &= -\sin 356^\circ = -\sin(180^\circ + 176^\circ)\\ &= \sin 176^\circ = \sin(180^\circ - 176^\circ) = \sin 4^\circ.\\ \cos(-356^\circ) &= \cos 356^\circ = \cos(180^\circ + 176^\circ)\\ &= -\cos 176^\circ = \cos(180^\circ - 176^\circ) = \cos 4^\circ.\end{aligned}$$

$\therefore$ $\sin A$ and $\cos A$ are both positive
$\therefore$ $\sin A + \cos A$ is positive. ■

§ Problem 5.1.31. -1125° ◊

§§ Solution.

$$-1125^\circ = -(3 \times 360 + 45^\circ)$$

$$\therefore \sin(-1125^\circ) = -\frac{1}{\sqrt{2}},\ \textit{and}\ \cos(-1125^\circ) = \frac{1}{\sqrt{2}}.$$

$$\therefore \sin A + \cos A = 0.$$ ■

What sign has $\sin A - \cos A$ for the following values of A ?

§ Problem 5.1.32. 215° ◊

§§ Solution.

$$\begin{aligned}\sin 215^\circ &= \sin(180^\circ + 35^\circ) = -\sin 35^\circ\\ \cos 215^\circ &= \cos(180^\circ + 35^\circ) = -\cos 35^\circ.\end{aligned}$$

$\therefore \sin A$ and $\cos A$ are both negative and $\cos A$ is numerically greater than $\sin A$;
$\therefore \sin A - \cos A$ is positive. ■

§ Problem 5.1.33. 825° ◊

§§ Solution.

$$\begin{aligned}825^\circ &= (2 \times 360^\circ + 105^\circ)\\ \sin 825^\circ &= \sin 105^\circ = \sin(180^\circ - 105^\circ) = \sin 75^\circ\\ \cos 825^\circ &= \cos 105^\circ = -\cos(180^\circ - 105^\circ) = -\cos 75^\circ.\end{aligned}$$

$\therefore \sin A$ is positive and $\cos A$ is negative ;
$\therefore \sin A - \cos A$ is positive. ■

§ Problem 5.1.34. -634° ◊

§§ Solution.

$$-634^\circ = -720^\circ + 86^\circ$$

$\therefore \sin A$ and $\cos A$ are both positive and $\sin A$ is numerically greater than $\cos A$;
$\therefore \sin A - \cos A$ is positive. ■

§ Problem 5.1.35. $-457°$ ◊

§§ Solution.

$$-457° = -(360° + 97°)$$
$$\therefore \sin(-457°) = \sin(-97°) = -\sin 97° = -\sin(180° - 97°) = -\sin 83°$$
$$\cos(-457°) = \cos 97° = -\cos(180° - 97°) = -\cos 83°$$

$\therefore \sin A$ and $\cos A$ are both negative and $\sin A$ is numerically greater than $\cos A$;
$\therefore \sin A - \cos A$ is negative. ■

§ Problem 5.1.36. *Find the sines and cosines of all angles in the first four quadrants whose tangents are equal to* $\cos 135°$*.* ◊

§§ Solution. If $\tan\theta = \cos 135° = -\frac{1}{\sqrt{2}}$, then $\sin\theta = \pm\frac{1}{\sqrt{3}}$ and $\cos\theta = \pm\sqrt{\frac{2}{3}}$.

Also, since $\tan\theta$ is negative, θ is in the second or fourth quadrant. In the second quadrant,

$$\sin\theta = \frac{1}{\sqrt{3}}, \text{ and } \cos\theta = -\sqrt{\frac{2}{3}}.$$

In the fourth quadrant,

$$\sin\theta = -\frac{1}{\sqrt{3}}, \text{ and } \cos\theta = \sqrt{\frac{2}{3}}.$$ ■

Prove that

§ Problem 5.1.37.

$$\sin(270° + A) = -\cos A \text{ and}$$
$$\tan(270° + A) = -\cot A.$$ ◊

§§ Solution.

$$\begin{aligned}\sin(270° + A) &= \sin[180° + (90° + A)] \\ &= -\sin(90° + A),\ [Art.\ 73] = -\cos A,\ [Art.\ 70]. \\ \tan(270° + A) &= \tan[180 + (90° + A)] \\ &= \tan(90° + A) = -\cot A.\end{aligned}$$

Otherwise thus: Cf. the figure of *Art.* 50. Let the revolving line, starting from OA, trace out any angle A in the first quadrant and let OP_1 be the position of the revolving line then, so that the $\angle AOP_1$ is A.

Let the revolving line then turn through three right angles $(270°)$ in the positive direction to the position OP_4, so that the $\angle AOP_4$ is $(270° + A)$.

Draw P_1M_1 and P_4M_4 perpendicular to OA.

By *Euc. I.* 26, the triangles P_1OM_1 and OP_4M_4 are geometrically equal and we have

$OP_1 = OP_4$ in magnitude and sign,
$P_1M_1 = OM_4$ in magnitude and sign,
and $P_4M_4 = OM_1$ in magnitude, but of opposite sign.

$$\therefore \sin(270° + A) = \frac{P_4M_4}{OP_4} = -\frac{OM_1}{OP_1} = -\cos A, \text{ and}$$
$$\tan(270° + A) = \frac{P_4M_4}{OM_4} = -\frac{OM_1}{P_1M_1} = -\cot A.$$ ■

§ Problem 5.1.38. $\cos(270^\circ - A) = -\sin A$ *and* $\cot(270^\circ - A) = \tan A.$ ◊

§§ Solution.

$$\begin{aligned}\cos(270^\circ - A) &= \cos[180^\circ + (90^\circ - A)] \\ &= -\cos(90^\circ - A) = -\sin A \\ \cot(270^\circ - A) &= \cot[180 + (90^\circ - A)] \\ &= \cot(90^\circ - A) = \tan A.\end{aligned}$$

Otherwise thus: Cf. the figure of *Art.* 50. Let the revolving line, starting from OA, trace out any angle A in the first quadrant and let OP_1 be the position of the revolving line then, so that the $\angle AOP_1$ is A.

To obtain the angle $(270^\circ - A)$, let the revolving line start from OA and after turning through three right angles, i.e. into the position OB', then turn back through an angle A into the position OP_3, so that the $\angle AOP_3$ is $(270^\circ - A)$.

Draw P_1M_1 and P_3M_3 perpendicular to AOA'.

By *Euc. I.* 26, the triangles P_1OM_1 and OP_3M_3 are geometrically equal and we have

$OP_1 = OP_3$ in magnitude and sign,

$OM_3 = P_1M_1$ in magnitude, but of opposite sign

and $P_3M_3 = OM_1$.

$$\therefore \cos(270^\circ - A) = \frac{OM_3}{OP_3} = -\frac{P_1M_1}{OP_1} = -\sin A, \textit{ and}$$

$$\cot(270^\circ - A) = \frac{OM_3}{P_3M_3} = \frac{-P_1M_1}{-OM_1} = \tan A. \qquad \blacksquare$$

§ Problem 5.1.39.

$$\cos A + \sin(270^\circ + A) - \sin(270^\circ - A) + \cos(180^\circ + A) = 0.$$ ◊

§§ Solution.

$$\begin{aligned}\sin(270^\circ + A) &= -\cos A \textit{ by } \S\textit{Problem } 5.1.37 \\ \sin(270^\circ - A) &= \sin[180^\circ + (90^\circ - A)] \\ &= -\sin(90^\circ - A) = -\cos A,\end{aligned}$$

and $\cos(180^\circ + A) = -\cos A$, *by Art.* 73.

Hence the given expression

$$= \cos A - \cos A + \cos A - \cos A = 0. \qquad \blacksquare$$

§ Problem 5.1.40.

$$\sec(270^\circ - A)\sec(90^\circ - A) - \tan(270^\circ - A)\tan(90^\circ + A) + 1 = 0.$$ ◊

§§ Solution.

$$\begin{aligned}\sec(270^\circ - A) &= \sec[180^\circ + (90^\circ - A)] \\ &= -\sec(90^\circ - A) = -\operatorname{cosec} A \\ \sec(90^\circ - A) &= \operatorname{cosec} A \\ \tan(270^\circ - A) &= \tan[180^\circ + (90^\circ - A)] \\ &= \tan(90^\circ - A) = \cot A \\ \tan(90^\circ + A) &= -\cot A.\end{aligned}$$

Hence the given expression

$$= -\operatorname{cosec}^2 A - \left(-\cot^2 A\right) + 1 = -\operatorname{cosec}^2 A + \operatorname{cosec}^2 A = 0. \qquad \blacksquare$$

§ Problem 5.1.41.

$$\cot A + \tan(180^\circ + A) + \tan(90^\circ + A) + \tan(360^\circ - A) = 0. \qquad \Diamond$$

§§ Solution.

The given expression

$$= \cot A + \tan A + (-\cot A) + (-\tan A) = 0. \qquad \blacksquare$$

Chapter 6

General Expressions for All Angles Having A Given Trigonometrical Ratio

6.1 Generic Values

What are the most general values of θ which satisfy the equations

§ Problem 6.1.1. $\sin\theta = \dfrac{1}{2}$. ◊

§§ Solution.

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$$
$$\therefore \theta = n\pi + (-1)^n\frac{\pi}{6}. \quad \blacksquare$$

§ Problem 6.1.2. $\sin\theta = -\dfrac{\sqrt{3}}{2}$. ◊

§§ Solution.

$$\sin\theta = -\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(-\frac{\pi}{3}\right)$$
$$\therefore \theta = n\pi + (-1)^n\left(-\frac{\pi}{3}\right) = n\pi - (-1)^n\frac{\pi}{3}. \quad \blacksquare$$

§ Problem 6.1.3. $\sin\theta = \dfrac{1}{\sqrt{2}}$. ◊

§§ Solution. $\sin\theta = \dfrac{1}{\sqrt{2}} = \sin\dfrac{\pi}{4}$.

$$\therefore \theta = n\pi + (-1)^n\left(\frac{\pi}{4}\right). \quad \blacksquare$$

§ Problem 6.1.4. $\cos\theta = -\dfrac{1}{2}$. ◊

§§ Solution.

$$\cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$$
$$\therefore \theta = 2n\pi \pm \frac{2\pi}{3}. \qquad \blacksquare$$

§ Problem 6.1.5. $\cos\theta = \dfrac{\sqrt{3}}{2}$. ◊

§§ Solution.

$$\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$
$$\therefore \theta = 2n\pi \pm \frac{\pi}{6}. \qquad \blacksquare$$

§ Problem 6.1.6. $\cos\theta = -\dfrac{1}{\sqrt{2}}$. ◊

§§ Solution.

$$\cos\theta = -\frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4}$$
$$\therefore \theta = 2n\pi \pm \frac{3\pi}{4}. \qquad \blacksquare$$

§ Problem 6.1.7. $\tan\theta = \sqrt{3}$. ◊

§§ Solution.

$$\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$$
$$\therefore \theta = n\pi + \frac{\pi}{3}. \qquad \blacksquare$$

§ Problem 6.1.8. $\tan\theta = -1$. ◊

§§ Solution.

$$\tan\theta = -1 = \tan\frac{3\pi}{4}$$
$$\therefore \theta = n\pi + \frac{3\pi}{4}. \qquad \blacksquare$$

§ Problem 6.1.9. $\cot\theta = 1$. ◊

§§ Solution.

$$\cot\theta = 1 = \cot\frac{\pi}{4}$$
$$\therefore \theta = n\pi + \frac{\pi}{4}. \qquad \blacksquare$$

§ Problem 6.1.10. $\sec\theta = 2$. ◊

§§ Solution.

$$\sec\theta = 2 = \sec\frac{\pi}{3}$$
$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}. \qquad \blacksquare$$

§ Problem 6.1.11. $\operatorname{cosec}\theta = \dfrac{2}{\sqrt{3}}$. ◊

§§ Solution.

$$\operatorname{cosec}\theta = \frac{2}{\sqrt{3}} = \operatorname{cosec}\frac{\pi}{3}$$
$$\therefore \theta = n\pi + (-1)^n\frac{\pi}{3}. \qquad \blacksquare$$

§ Problem 6.1.12. $\sin^2\theta = 1.$ ◊

§§ Solution.

$$\sin^2\theta = 1 = \sin^2\frac{\pi}{2}$$
$$\therefore \theta = n\pi \pm \frac{\pi}{2}.$$

Notes : $\theta = n\pi \pm \alpha$ is the general solution of any one of the following equations:

$$\sin^2\theta = \sin^2\alpha$$
$$\cos^2\theta = \cos^2\alpha$$
$$\tan^2\theta = \tan^2\alpha$$
$$\operatorname{cosec}^2\theta = \operatorname{cosec}^2\alpha$$
$$\sec^2\theta = \sec^2\alpha$$

and

$$\cot^2\theta = \cot^2\alpha.$$ ■

§ Problem 6.1.13. $\cos^2\theta = \frac{1}{4}.$ ◊

§§ Solution.

$$\cos^2\theta = \frac{1}{4} = \cos^2\frac{\pi}{3}$$
$$\therefore \theta = n\pi \pm \frac{\pi}{3}.$$ ■

§ Problem 6.1.14. $\tan^2\theta = \frac{1}{3}.$ ◊

§§ Solution.

$$\tan^2\theta = \frac{1}{3} = \tan^2\frac{\pi}{6}$$
$$\therefore \theta = n\pi \pm \frac{\pi}{6}.$$ ■

§ Problem 6.1.15. $4\sin^2\theta = 3.$ ◊

§§ Solution.

$$\sin^2\theta = \frac{3}{4} = \sin^2\frac{\pi}{3}$$
$$\therefore \theta = n\pi \pm \frac{\pi}{3}.$$ ■

§ Problem 6.1.16. $2\cot^2\theta = \operatorname{cosec}^2\theta.$ ◊

§§ Solution.

$$2\cot^2\theta = \operatorname{cosec}^2\theta = 1 + \cot^2\theta$$
$$\therefore \cot^2\theta = 1 = \cot^2\frac{\pi}{4}$$
$$\therefore \theta = n\pi \pm \frac{\pi}{4}.$$ ■

§ Problem 6.1.17. $\sec^2\theta = \frac{4}{3}.$ ◊

§§ Solution.

$$\sec^2\theta = \frac{4}{3} = \sec^2\frac{\pi}{6}$$
$$\therefore \theta = n\pi \pm \frac{\pi}{6}.$$ ■

§ Problem 6.1.18. *What is the most general value of θ that satisfies both of the equations*

$$\cos\theta = -\frac{1}{\sqrt{2}}, \textit{ and}$$
$$\tan\theta = 1. \qquad \Diamond$$

§§ Solution. The only value of θ between $0°$ and $360°$ satisfying both conditions is $225°$, i.e. $\frac{5\pi}{4}$.

Hence the general value of θ is $2n\pi + \frac{5\pi}{4}$. ∎

§ Problem 6.1.19. *What is the most general value of θ that satisfies both of the equations*

$$\cot\theta = -\sqrt{3}, \textit{ and}$$
$$\operatorname{cosec}\theta = -2. \qquad \Diamond$$

§§ Solution. θ must be either in the 4^{th} quadrant or in the 1^{st} negative quadrant and its value is $-\frac{\pi}{6}$.

Hence the general value of θ is $2n\pi - \frac{\pi}{6}$. ∎

§ Problem 6.1.20. *If $\cos(A-B) = \frac{1}{2}$, and $\sin(A+B) = \frac{1}{2}$, find the smallest positive values of A and B and also their most general values.* $\Diamond$

§§ Solution. $\cos(A-B) = \frac{1}{2} = \cos\frac{\pi}{3}$ and $\sin(A+B) = \frac{1}{2} = \sin\frac{\pi}{6}$.

$$\left.\begin{aligned} \text{Hence } A-B &= \frac{\pi}{3} \\ \text{and } A+B &= \frac{5\pi}{6} \end{aligned}\right\} \text{give the smallest positive values of } A \text{ and } B.$$

$\left[Not\ A+B = \frac{\pi}{6},\ \because A+B > A-B.\right]$

Hence, by addition, we have $2A = \frac{7\pi}{6}$, so that $A = \frac{7\pi}{12} = 105°$

and, by subtraction, we have $2B = \frac{\pi}{2}$, so that $B = \frac{\pi}{4} = 45°$.

The general values are given by

$$A - B = 2n\pi \pm \frac{\pi}{3}, \textit{ and}$$
$$A + B = m\pi \pm (-1)^m\frac{\pi}{6},$$

where m and n are any integers.

Hence we have $\quad A = \left(n + \frac{m}{2}\right)\pi \pm \frac{\pi}{6} + (-1)^m\frac{\pi}{12}$

and $\quad B = \left(\frac{m}{2} - n\right)\pi \mp \frac{\pi}{6} + (-1)^m\frac{\pi}{12}$. ∎

§ Problem 6.1.21. *If $\tan(A-B) = 1$, and $\sec(A+B) = \frac{2}{\sqrt{3}}$, find the smallest positive values of A and B and also their most general values.* $\Diamond$

§§ Solution. $\tan(A-B) = 1 = \tan\dfrac{\pi}{4}$ and $\sec(A+B) = \dfrac{2}{\sqrt{3}} = \sec\dfrac{\pi}{6}$.

$$\left.\begin{aligned} \text{Hence } A - B &= \frac{\pi}{4} \\ \text{and } A + B &= \frac{11\pi}{6} \end{aligned}\right\} \text{give the smallest positive values of } A \text{ and } B.$$

$\left[Not\ A+B = \dfrac{\pi}{6},\ \because A+B > A-B.\right]$

Hence, by addition, we have $2A = \dfrac{25\pi}{12}$, so that $A = \dfrac{25\pi}{24} = 187\dfrac{1}{2}^{\circ}$

and, by subtraction, we have $2B = \dfrac{19\pi}{12}$, so that $B = \dfrac{19\pi}{24} = 142\dfrac{1}{2}^{\circ}$.

The general values are given by

$$A - B = m\pi + \frac{\pi}{4},\ \textit{and}$$
$$A + B = 2n\pi \pm \frac{\pi}{6},$$

where m and n are any integers.

Hence we have $\qquad A = \left(n + \dfrac{m}{2}\right)\pi + \dfrac{\pi}{8} \pm \dfrac{\pi}{12}$

and $\qquad B = \left(n - \dfrac{m}{2}\right)\pi - \dfrac{\pi}{8} \pm \dfrac{\pi}{12}$. ■

§ Problem 6.1.22. *Find the angles between* 0° *and* 360° *which have respectively*

(1) their sines equal to $\dfrac{\sqrt{3}}{2}$,

(2) their cosines equal to $-\dfrac{1}{2}$, *and*

(3) their tangents equal to $\dfrac{1}{\sqrt{3}}$.

◊

§§ Solution. (1) If $\sin\theta = \dfrac{\sqrt{3}}{2}$, then θ is in the first or second quadrant and is equal to 60° or 120°.

(2) If $\cos\theta = -\dfrac{1}{2}$, then θ is in second or third quadrant and is equal to 120° or 240°.

(3) If $\tan\theta = \dfrac{1}{\sqrt{3}}$, then θ is in the first or third quadrant and is equal to 30° or 210°. ■

§ Problem 6.1.23. *Taking into consideration only angles between* 0° *and* 180° *how many values of* x *are there if*

(1) $\sin x = \dfrac{5}{7}$,

(2) $\cos x = \dfrac{1}{5}$,

(3) $\cos x = -\dfrac{4}{5}$,

(4) $\tan x = \frac{2}{3}$*, and*

(5) $\cot x = -7$ *?* ◊

§§ **Solution**. (1) Two values, supplementary.

(2) One value, between $0°$ and $90°$.

(3) One value, between $90°$ and $180°$.

(4) One value, acute.

(5) One value, obtuse. ■

§ **Problem 6.1.24.** *Given the angle x construct the angle y if*

(1) $\sin y = 2 \sin x$*,*

(2) $\tan y = 3 \tan x$*,*

(3) $\cos y = \frac{1}{2} \cos x$*,*

(4) $\sec y = \operatorname{cosec} x$*.* ◊

§§ **Solution**. (1) Describe a semicircle APB on AB as diameter. Make an angle $\angle PAB$ equal to x. Join BP. On the semicircle, take a point Q such that BQ is equal to $2 \cdot BP$. Join AQ. We then have

$$\sin \angle BAQ = \frac{BQ}{AB} = \frac{2 \cdot BP}{AB} = 2 \sin x.$$
$$\therefore \angle BAQ = y.$$

Notes : This construction is impossible if $\sin x > \frac{1}{2}$.

(2) Draw a straight line AB the length of which is unity and erect a perpendicular BP equal in length to x. Join AP. Produce BP to Q so that $BQ = 3 \cdot BP$. Join AQ. We then have

$$\tan \angle BAQ = \frac{BQ}{AB} = \frac{3 \cdot BP}{AB} = 3 \tan x.$$
$$\therefore \angle BAQ = y.$$

Notes : This construction is always possible.

(3) Describe a semicircle APB on AB as diameter. Make an angle $\angle PAB$ equal to x.

On the semicircle, take a point Q such that AQ is equal to $\frac{1}{2}AP$. Join BP and BQ. We then have

$$\cos \angle BAQ = \frac{AQ}{AB} = \frac{1}{2} \cdot \frac{AP}{AB} = \frac{1}{2} \cos x.$$
$$\therefore \angle BAQ = y.$$

Notes : This construction is always possible.

(4) In the right-angled triangle ΔPOM (figure of *Art.* 23), let the angle $\angle POM = x$. We then have

$$\sec \angle OPM = \frac{OP}{PM} = \operatorname{cosec} \angle POM = \operatorname{cosec} x.$$
$$\therefore \angle OPM = y.$$

Notes : y is obviously complement of x. ■

§ Problem 6.1.25. *Show that the same angles are indicated by the two following formulae :*

$$(2n-1)\frac{\pi}{2}+(-1)^n\frac{\pi}{3},\ \textit{and} \tag{6.1}$$

$$2n\pi \pm \frac{\pi}{6} \tag{6.2}$$

n being any integer. ♢

§§ Solution.

If $$\theta=(2n-1)\frac{\pi}{2}+(-1)^n\frac{\pi}{3}$$

then $$\theta+\frac{\pi}{2}=n\pi+(-1)^n\frac{\pi}{3}$$

this is the solution of the equation

$$\sin\left(\theta+\frac{\pi}{2}\right)=\sin\frac{\pi}{3}$$

$$\therefore \cos\theta=\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6}$$

the solution of which is $\theta=2n\pi\pm\dfrac{\pi}{6}$.

Otherwise thus : Putting $n=0,\ 1,\ 2,\ 3,\ \ldots$ in Eq. (6.1), we have the series of angles

$$-\frac{\pi}{6},\ \frac{\pi}{6},\ \frac{11\pi}{6},\ \frac{13\pi}{6},\ldots$$

Putting $n=0,\ 1,\ 2,\ 3,\ \ldots$ in Eq. (6.2), we have the series of angles

$$-\frac{\pi}{6},\ \frac{\pi}{6},\ \frac{11\pi}{6},\ \frac{13\pi}{6},\ldots$$

each value of n giving two angles. ■

§ Problem 6.1.26. *Prove that the two formulae*

$$\left(2n+\frac{1}{2}\right)\pi\pm\alpha,\ \textit{and} \tag{6.3}$$

$$n\pi+(-1)^n\left(\frac{\pi}{2}-\alpha\right) \tag{6.4}$$

denote the same angles, n being any integer.

Illustrate by a figure. ♢

§§ Solution. From the formula $\left(2n+\dfrac{1}{2}\right)\pi\pm\alpha$, we have, by putting $n=0,\ 1,\ 2,\ldots$ the series of angles

$$\frac{\pi}{2}\pm\alpha,\ \frac{5\pi}{2}\pm\alpha,\ \frac{9\pi}{2}\pm\alpha,\ \ldots. \tag{6.5}$$

From the second formula, we have the series of angles

$$\frac{\pi}{2}-\alpha,\ \pi-\left(\frac{\pi}{2}-\alpha\right),\ 2\pi+\frac{\pi}{2}-\alpha,\ 3\pi-\left(\frac{\pi}{2}-\alpha\right)$$

$$i.e.\ \frac{\pi}{2}-\alpha,\ \frac{\pi}{2}+\alpha,\ \frac{5\pi}{2}-\alpha,\ \frac{5\pi}{2}+\alpha,\ \ldots.$$

These two series are clearly the same.

As in the figure of $Art.$ 50, let OA be the initial line and OA' in the direction opposite to the initial line and OB the line bounding the first quadrant.

Let OP and OQ be lines, respectively between OB and OA and between OB and OA', such that $\angle POB=\angle BOQ=\alpha$.

Then, when the angle considered is one of those of Eq. (6.3), the bounding line is either in the position OP or in the position OQ.

For Eq. (6.4), firstly, let n be even and equal to $2m$; then the formula $= 2m\pi + \frac{\pi}{2} - \alpha$ and the corresponding bounding line is therefore in the position OP.

Secondly, let n be odd and equal to $2m+1$; then the formula $= (2m+1)\pi - \left(\frac{\pi}{2} - \alpha\right) = 2m\pi + \frac{\pi}{2} + \alpha$ and the corresponding bounding line is in the position OQ. Hence the proposition. ■

§ Problem 6.1.27. *If $\theta - \alpha = n\pi + (-1)^n\beta$, prove that*

$$\theta = 2m\pi + \alpha + \beta$$

or else that

$$\theta = (2m+1)\pi + \alpha - \beta$$

where m and n are any integers. ◊

§§ Solution. If n be even $(= 2m,\ say)$ we have

$$\theta - \alpha = 2m\pi + \beta$$

$$\therefore \theta = 2m\pi + \alpha + \beta$$

If n be odd $(= 2m+1,\ say)$ we have

$$\theta - \alpha = (2m+1)\pi - \beta$$

$$\therefore \theta = (2m+1)\pi + \alpha - \beta.$$ ■

§ Problem 6.1.28. *If $\cos p\theta + \cos q\theta = 0$, prove that the different values of θ form two arithmetical progressions in which the common differences are $\dfrac{2\pi}{p+q}$ and $\dfrac{2\pi}{p \sim q}$ respectively.* ◊

§§ Solution.

$$\cos p\theta + \cos q\theta = 0$$

$$\therefore \cos p\theta = -\cos q\theta = \cos(\pi - q\theta)$$

$$\therefore p\theta = 2n\pi \pm (\pi - q\theta).$$

Talking the upper sign, we have

$$(p+q)\theta = (2n+1)\pi$$

$$\therefore \theta = \frac{(2n+1)}{p+q}\pi.$$

By putting $n = 0,\ 1,\ 2, \ldots$ in succession, we have the series of angles

$$\frac{\pi}{p+q},\ \frac{3\pi}{p+q},\ \frac{5\pi}{p+q},\ \ldots$$

which are in $A.\ P.$, with common difference $\dfrac{2\pi}{p+q}$.

Taking the lower sign, we have

$$(p \sim q)\theta = (2n-1)\pi$$

$$\therefore \theta = \frac{(2n-1)}{p \sim q}\pi,$$

and we have the series of angles

$$-\frac{\pi}{p \sim q},\ \frac{\pi}{p \sim q},\ \frac{3\pi}{p+q},\ \ldots$$

which are in $A.\ P.$, with common difference $\dfrac{2\pi}{p \sim q}$. ■

§ Problem 6.1.29. *Construct the angle whose sine is $\dfrac{3}{2+\sqrt{5}}$.* ◊

§§ **Solution**. Take a straight line AB of length $2a$ and draw AC perpendicular to it of length a.

Join BC, then BC is equal to $a\sqrt{5}$.

Produce AB to O, making BO equal to BC.

On OA, describe a semicircle and in it draw AP, equal to $3a$ and meeting the semicircle in P.

Join OP. The angle $\angle APO$, being an angle in a semicircle, is right angle. We then have

$$\sin\angle AOP = \frac{AP}{OA} = \frac{AP}{AB+BO} = \frac{3a}{2a+a\sqrt{5}} = \frac{3}{2+\sqrt{5}}.$$

Hence $\angle AOP$ is the required angle. ■

6.2 Trigonometrical Equations

Solve the equations

§ Problem 6.2.1. $\cos^2\theta - \sin\theta - \dfrac{1}{4} = 0.$ ◊

§§ **Solution**.

$$\begin{aligned}
\cos^2\theta - \sin\theta - \frac{1}{4} &= 0\\
\therefore 1 - \sin^2\theta - \sin\theta - \frac{1}{4} &= 0\\
\therefore 4\sin^2\theta + 4\sin\theta - 3 &= 0\\
\therefore (2\sin\theta - 1)(2\sin\theta + 3) &= 0\\
\therefore 2\sin\theta - 1 = 0 \; or \; 2\sin\theta + 3 &= 0.
\end{aligned}$$

If $2\sin\theta - 1 = 0$, then

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$$
$$\therefore \theta = n\pi + (-1)^n\frac{\pi}{6}.$$

If $2\sin\theta + 3 = 0$, then

$$\sin\theta = -\frac{3}{2};$$

this value is inadmissible, since the sine of an angle cannot be numerically greater than unity. ■

§ Problem 6.2.2. $2\sin^2\theta + 3\cos\theta = 0.$ ◊

§§ **Solution**.

$$\begin{aligned}
2\sin^2\theta + 3\cos\theta &= 0\\
\therefore 2\left(1-\cos^2\theta\right) + 3\cos\theta &= 0\\
\therefore 2\cos^2\theta - 3\cos\theta - 2 &= 0\\
\therefore (2\cos\theta + 1)(\cos\theta - 2) &= 0\\
\therefore 2\cos\theta + 1 = 0, \; or \; \cos\theta - 2 &= 0.
\end{aligned}$$

If $2\cos\theta + 1 = 0$, then

$$\cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$$
$$\therefore \theta = 2n\pi \pm \frac{2\pi}{3}.$$

If $\cos\theta - 2 = 0$, then

$$\cos\theta = 2;$$

this value is inadmissible, since the cosine of an angle cannot be numerically greater than unity. ■

§ Problem 6.2.3. $2\sqrt{3}\cos^2\theta = \sin\theta$. ◊

§§ Solution.

$$\begin{aligned} 2\sqrt{3}\cos^2\theta &= \sin\theta \\ 2\sqrt{3}\left(1-\sin^2\theta\right) &= \sin\theta \\ \therefore 2\sqrt{3}\sin^2\theta + \sin\theta - 2\sqrt{3} &= 0 \\ \therefore \left(2\sin\theta - \sqrt{3}\right)\left(\sqrt{3}\sin\theta + 2\right) &= 0 \\ \therefore 2\sin\theta - \sqrt{3} = 0,\ \mathit{or}\ \sqrt{3}\sin\theta + 2 &= 0. \end{aligned}$$

If $2\sin\theta - \sqrt{3} = 0$, then

$$\begin{aligned} \sin\theta &= \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \\ \therefore \theta &= n\pi + (-1)^n\frac{\pi}{3}. \end{aligned}$$

If $\sqrt{3}\sin\theta + 2 = 0$, then

$$\sin\theta = -\frac{2}{\sqrt{3}};$$

this value is inadmissible. ■

§ Problem 6.2.4. $\cos\theta + \cos^2\theta = 1$. ◊

§§ Solution.

$$\begin{aligned} \cos\theta + \cos^2\theta &= 1 \\ \therefore \cos^2\theta + \cos\theta - 1 &= 0 \\ \therefore \cos\theta &= \frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}. \end{aligned}$$

The value $\dfrac{-\sqrt{5}-1}{2}$ is inadmissible.

$$\therefore \cos\theta = \frac{\sqrt{5}-1}{2}.$$ ■

§ Problem 6.2.5. $4\cos\theta - 3\sec\theta = 2\tan\theta$. ◊

§§ Solution.

$$\begin{aligned} 4\cos\theta - 3\sec\theta &= 2\tan\theta \\ \therefore 4\cos\theta - \frac{3}{\cos\theta} &= 2\frac{\sin\theta}{\cos\theta} \\ \therefore 4\cos^2\theta - 3 &= 2\sin\theta \qquad (6.6) \\ 4\left(1-\sin^2\theta\right) - 3 &= 2\sin\theta \\ \therefore 4\sin^2\theta + 2\sin\theta - 1 &= 0 \\ \therefore \sin\theta = \frac{-2\pm\sqrt{4+16}}{8} &= \frac{-2\pm2\sqrt{5}}{8} = \frac{\pm\sqrt{5}-1}{4}. \end{aligned}$$

Taking the upper sign, we have

$$\begin{aligned} \sin\theta &= \frac{\sqrt{5}-1}{4} = \sin\frac{\pi}{10} \\ \therefore \theta &= n\pi + (-1)^n\frac{\pi}{10}. \end{aligned}$$

Taking the lower sign, we have

$$\sin\theta = \frac{-\sqrt{5}-1}{4} = -\frac{\sqrt{5}+1}{4} = \sin\left(-\frac{3\pi}{10}\right)$$

$$\therefore \theta = n\pi - (-1)^n \frac{3\pi}{10}.$$

Otherwise thus : From Eq. (6.6), we have

$$4\cos^3\theta - 3\cos\theta = 2\sin\theta\cos\theta$$

$$\therefore \cos 3\theta = \sin 2\theta.$$

For the solution of this equation, see § Problem 6.2.20.

The result $\theta = 2n\pi - \dfrac{\pi}{2}$ (giving $\cos\theta = 0$) in that problem is not a solution here, but is due to the factor $\cos\theta$ introduced. ■

§ Problem 6.2.6. $\sin^2\theta - 2\cos\theta + \dfrac{1}{4} = 0.$ ◊

§§ Solution.

$$\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$$

$$\therefore 1 - \cos^2\theta - 2\cos\theta + \frac{1}{4} = 0$$

$$\therefore 4\cos^2\theta + 8\cos\theta - 5 = 0$$

$$\therefore (2\cos\theta - 1)(2\cos\theta + 5) = 0$$

$$\therefore 2\cos\theta - 1 = 0, \; or \; 2\cos\theta + 5 = 0.$$

If $2\cos\theta - 1 = 0$, then

$$\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

If $2\cos\theta + 5 = 0$, then

$$\cos\theta = -\frac{5}{2};$$

this value is inadmissible. ■

§ Problem 6.2.7. $\tan^2\theta - \left(1+\sqrt{3}\right)\tan\theta + \sqrt{3} = 0.$ ◊

§§ Solution.

$$\tan^2\theta - \left(1+\sqrt{3}\right)\tan\theta + \sqrt{3} = 0$$

$$\therefore (\tan\theta - 1)\left(\tan\theta - \sqrt{3}\right) = 0$$

$$\therefore \tan\theta = 1 = \tan\frac{\pi}{4} \; or \; \tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$$

$$\therefore \theta = n\pi + \frac{\pi}{4} \; or \; n\pi + \frac{\pi}{3}.$$ ■

§ Problem 6.2.8. $\cot^2\theta + \left(\sqrt{3} + \dfrac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0.$ ◊

§§ Solution.

$$\cot^2\theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0$$

$$\therefore \left(\cot\theta + \frac{1}{\sqrt{3}}\right)\left(\cot\theta + \sqrt{3}\right) = 0$$

$$\therefore \cot\theta = -\frac{1}{\sqrt{3}} = \cot\frac{2\pi}{3} \; or \; \cot\theta = -\sqrt{3} = \cot\frac{5\pi}{6}$$

$$\therefore \theta = n\pi + \frac{2\pi}{3} \; or \; n\pi + \frac{5\pi}{6}.$$ ■

§ Problem 6.2.9. $\cot\theta - ab\tan\theta = a - b.$ ◊

§§ Solution.

$$\begin{aligned}\cot\theta - ab\tan\theta &= a-b\\ \therefore \frac{1}{\tan\theta} - ab\tan\theta &= a-b\\ \therefore ab\tan^2\theta + (a-b)\tan\theta - 1 &= 0\\ \therefore (a\tan\theta - 1)(b\tan\theta+1) &= 0\\ \therefore \tan\theta &= \frac{1}{a} \ \mathit{or}\ -\frac{1}{b}.\end{aligned}$$ ■

§ Problem 6.2.10. $\tan^2\theta + \cot^2\theta = 2.$ ◊

§§ Solution.

$$\begin{aligned}\tan^2\theta + \cot^2\theta &= 2\\ \therefore \tan^2\theta + \frac{1}{\tan^2\theta} &= 2\\ \therefore \tan^4\theta - 2\tan^2\theta + 1 &= 0\\ \therefore \left(\tan^2\theta - 1\right)^2 &= 0\\ \therefore \tan\theta &= \pm 1\\ \therefore \theta &= n\pi \pm \frac{\pi}{4}.\end{aligned}$$ ■

§ Problem 6.2.11. $\sec\theta - 1 = \left(\sqrt{2}-1\right)\tan\theta.$ ◊

§§ Solution.

$$\begin{aligned}\sec\theta - 1 &= \left(\sqrt{2}-1\right)\tan\theta\\ \therefore \sec\theta - 1 &= \left(\sqrt{2}-1\right)\sqrt{\sec^2\theta - 1}\\ \therefore\ (i)\ \sqrt{\sec\theta - 1} &= 0,\ i.e.\ \sec\theta = 1,\ i.e.\ \theta = 2n\pi\end{aligned}$$

and

$$\begin{aligned}(ii)\ \sqrt{\sec\theta - 1} &= \left(\sqrt{2}-1\right)\sqrt{\sec\theta + 1}\\ \therefore \sec\theta - 1 &= \left(3 - 2\sqrt{2}\right)(\sec\theta + 1)\\ \therefore \left(\sqrt{2}-1\right)\sec\theta &= \sqrt{2}\left(\sqrt{2}-1\right)\\ \sec\theta &= \sqrt{2},\ i.e.\ \theta = 2n\pi + \frac{\pi}{4}.\end{aligned}$$

The value $\theta = 2n\pi - \dfrac{\pi}{4}$ is inadmissible ; it is introduced by the squaring and is the solution of the equation

$$\sec\theta - 1 = -\left(\sqrt{2}-1\right)\tan\theta.$$ ■

§ Problem 6.2.12. $3\left(\sec^2\theta + \tan^2\theta\right) = 5.$ ◊

§§ Solution.

$$\begin{aligned}3\left(\sec^2\theta + \tan^2\theta\right) &= 5\\ \therefore 3\left(1 + \tan^2\theta + \tan^2\theta\right) &= 5\\ \therefore 6\tan^2\theta &= 2\\ \therefore \tan^2\theta &= \frac{1}{3}\\ \therefore \tan\theta &= \pm\frac{1}{\sqrt{3}}\\ \therefore \theta &= n\pi \pm \frac{\pi}{6}.\end{aligned}$$ ■

§ Problem 6.2.13. $\cot\theta + \tan\theta = 2\,\mathrm{cosec}\ \theta.$ ◊

§§ Solution.

$$\cot\theta + \tan\theta = 2\,\mathrm{cosec}\ \theta$$
$$\therefore \frac{1+\tan^2\theta}{\tan\theta} = \frac{2}{\sin\theta} \tag{6.7}$$
$$\therefore \sec^2\theta = 2\sec\theta \tag{6.8}$$
$$\sec\theta = 2$$
$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

The value $\sec\theta = 0$ from Eq. (6.8) is inadmissible ; the original equation is also satisfied by the solution $\sin\theta = 0,$ *i.e.* $\theta = n\pi$. ∎

§ Problem 6.2.14. $4\cos^2\theta + \sqrt{3} = 2\left(\sqrt{3}+1\right)\cos\theta.$ ◊

§§ Solution.

$$4\cos^2\theta + \sqrt{3} = 2\left(\sqrt{3}+1\right)\cos\theta$$
$$\therefore \cos^2\theta - \left(\frac{\sqrt{3}+1}{2}\right)\cos\theta + \frac{\sqrt{3}}{4} = 0$$
$$\therefore \left(\cos\theta - \frac{1}{2}\right)\left(\cos\theta - \frac{\sqrt{3}}{2}\right) = 0$$
$$\therefore \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3} \ \ or\ \ \cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$
$$\therefore \theta = 2n\pi \pm \frac{\pi}{3} \ \ or\ \ 2n\pi \pm \frac{\pi}{6}.$$ ∎

§ Problem 6.2.15. $3\sin^2\theta - 2\sin\theta = 1.$ ◊

§§ Solution.

$$3\sin^2\theta - 2\sin\theta = 1$$
$$\therefore (3\sin\theta+1)(\sin\theta-1) = 0$$
$$\therefore \sin\theta = -\frac{1}{3} \ \ or\ \ \sin\theta = 1.$$ ∎

§ Problem 6.2.16. $\sin 5\theta = \frac{1}{\sqrt{2}}.$ ◊

§§ Solution.

$$\sin 5\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$
$$\therefore 5\theta = n\pi + (-1)^n\frac{\pi}{4}$$
$$\therefore \theta = \frac{n\pi}{5} + (-1)^n\frac{\pi}{20}.$$ ∎

§ Problem 6.2.17. $\sin 9\theta = \sin\theta.$ ◊

§§ Solution.

$$\sin 9\theta = \sin\theta$$
$$\therefore 9\theta = m\pi + (-1)^m\theta.$$

When m is even, $(= 2n$, say),

$$9\theta = 2n\pi + \theta$$
$$\therefore 8\theta = 2n\pi$$
$$\therefore \theta = \frac{n\pi}{4}.$$

When m is odd, ($= 2n+1$, say),

$$9\theta = (2n+1)\pi - \theta$$
$$\therefore 10\theta = (2n+1)\pi$$
$$\therefore \theta = \frac{(2n+1)\pi}{10}. \qquad \blacksquare$$

§ Problem 6.2.18. $\sin 3\theta = \sin 2\theta$. ◊

§§ Solution.

$$\sin 3\theta = \sin 2\theta$$
$$\therefore 3\theta = m\pi + (-1)^m 2\theta.$$

When m is even, ($= 2n$, say),

$$3\theta = 2n\pi + 2\theta$$
$$\therefore \theta = 2n\pi.$$

When m is odd, ($= 2n+1$, say),

$$3\theta = (2n+1)\pi - 2\theta$$
$$\therefore 5\theta = (2n+1)\pi$$
$$\therefore \theta = \frac{(2n+1)\pi}{5}. \qquad \blacksquare$$

§ Problem 6.2.19. $\cos m\theta = \cos n\theta$. ◊

§§ Solution.

$$\cos m\theta = \cos n\theta$$
$$\therefore m\theta = 2r\theta \pm n\theta, \text{ where } r \text{ is any integer.}$$

Taking the upper sign, we have

$$(m-n)\theta = 2r\pi$$
$$\therefore \theta = \frac{2r\pi}{m-n}.$$

Taking the lower sign, we have

$$(m+n)\theta = 2r\pi$$
$$\therefore \theta = \frac{2r\pi}{m+n}. \qquad \blacksquare$$

§ Problem 6.2.20. $\sin 2\theta = \cos 3\theta$. ◊

§§ Solution.

$$\sin 2\theta = \cos 3\theta$$
$$\therefore \cos 3\theta = \cos\left(\frac{\pi}{2} - 2\theta\right)$$
$$\therefore 3\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right).$$

Taking the upper sign, we have

$$5\theta = 2n\pi + \frac{\pi}{2}$$
$$\therefore \theta = \left(2n + \frac{1}{2}\right)\frac{\pi}{5}.$$

Taking the lower sign, we have

$$\theta = 2n\pi - \frac{\pi}{2}. \qquad \blacksquare$$

§ Problem 6.2.21. $\cos 5\theta = \cos 4\theta$. ◊

§§ Solution.

$$\cos 5\theta = \cos 4\theta$$
$$\therefore 5\theta = 2n\pi \pm 4\theta.$$

Taking the upper sign, we have

$$\theta = 2n\pi.$$

Taking the lower sign, we have

$$9\theta = 2n\pi$$
$$\therefore \theta = \frac{2n\pi}{9}.$$

■

§ Problem 6.2.22. $\cos m\theta = \sin n\theta$. ◊

§§ Solution.

$$\cos m\theta = \sin n\theta = \cos\left(\frac{\pi}{2} - n\theta\right)$$
$$\therefore m\theta = 2r\pi \pm \left(\frac{\pi}{2} - n\theta\right), \text{ where } r \text{ is any integer.}$$

Taking the upper sign, we have

$$(m+n)\theta = \left(2r + \frac{1}{2}\right)\pi$$
$$\therefore \theta = \left(2r + \frac{1}{2}\right)\frac{\pi}{m+n}.$$

Taking the lower sign, we have

$$(m-n)\theta = \left(2r - \frac{1}{2}\right)\pi$$
$$\therefore \theta = \left(2r - \frac{1}{2}\right)\frac{\pi}{m-n}.$$

■

§ Problem 6.2.23. $\cot\theta = \tan 8\theta$. ◊

§§ Solution.

$$\cot\theta = \tan 8\theta$$
$$\therefore \tan 8\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$
$$\therefore 8\theta = n\pi + \left(\frac{\pi}{2} - \theta\right)$$
$$\therefore 9\theta = \left(n + \frac{1}{2}\right)\pi$$
$$\therefore \theta = \left(n + \frac{1}{2}\right)\frac{\pi}{9}.$$

■

§ Problem 6.2.24. $\cot\theta = \tan n\theta$. ◊

§§ Solution.

$$\cot\theta = \tan n\theta$$
$$\therefore \tan n\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$
$$\therefore n\theta = m\pi + \left(\frac{\pi}{2} - \theta\right), \text{ where } m \text{ is any integer}$$
$$\therefore (n+1)\theta = \left(m + \frac{1}{2}\right)\pi$$
$$\therefore \theta = \left(m + \frac{1}{2}\right)\frac{\pi}{n+1}.$$

■

§ Problem 6.2.25. $\tan 2\theta = \tan\dfrac{2}{\theta}$. ◊

§§ Solution.

$$\tan 2\theta = \tan \frac{2}{\theta}$$
$$\therefore 2\theta = n\pi + \frac{2}{\theta}$$
$$\therefore 2\theta^2 - n\pi\theta - 2 = 0$$
$$\therefore \theta = \frac{n\pi \pm \sqrt{n^2\pi^2 + 16}}{4}. \qquad \blacksquare$$

§ Problem 6.2.26. $\tan 2\theta \tan \theta = 1.$ $\Diamond$

§§ Solution.

$$\tan 2\theta \tan \theta = 1$$
$$\therefore \tan 2\theta = \frac{1}{\tan \theta} = \cot \theta = \tan\left(\frac{\pi}{2} - \theta\right)$$
$$\therefore 2\theta = n\pi + \left(\frac{\pi}{2} - \theta\right)$$
$$\therefore 3\theta = \left(n + \frac{1}{2}\right)\pi$$
$$\therefore \theta = \left(n + \frac{1}{2}\right)\frac{\pi}{3}. \qquad \blacksquare$$

§ Problem 6.2.27. $\tan^2 3\theta = \cot^2 \alpha.$ $\Diamond$

§§ Solution.

$$\tan^2 3\theta = \cot^2 \alpha$$
$$\therefore \tan^2 3\theta = \tan^2\left(\frac{\pi}{2} - \alpha\right)$$
$$\therefore 3\theta = n\pi \pm \left(\frac{\pi}{2} - \alpha\right)$$
$$\therefore \theta = (2n+1)\frac{\pi}{6} \pm \frac{\alpha}{3} = \left(n + \frac{1}{2}\right)\frac{\pi}{3} \pm \frac{\alpha}{3}. \qquad \blacksquare$$

§ Problem 6.2.28. $\tan 3\theta = \cot \theta.$ $\Diamond$

§§ Solution.

$$\tan 3\theta = \cot \theta$$
$$\therefore \tan 3\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$
$$\therefore 3\theta = n\pi + \left(\frac{\pi}{2} - \theta\right)$$
$$\therefore 4\theta = \left(n + \frac{1}{2}\right)\pi$$
$$\therefore \theta = \left(n + \frac{1}{2}\right)\frac{\pi}{4}. \qquad \blacksquare$$

§ Problem 6.2.29. $\tan^2 3\theta = \tan^2 \alpha.$ $\Diamond$

§§ Solution.

$$\tan^2 3\theta = \tan^2 \alpha$$
$$\therefore 3\theta = n\pi \pm \alpha$$
$$\therefore \theta = \frac{n\pi}{3} \pm \frac{\alpha}{3}. \qquad \blacksquare$$

§ Problem 6.2.30. $3\tan^2 \theta = 1.$ $\Diamond$

§§ Solution.

$$\begin{aligned} 3\tan^2\theta &= 1 \\ \therefore \tan\theta &= \pm\frac{1}{\sqrt{3}} \\ \therefore \theta &= n\pi \pm \frac{\pi}{6}. \end{aligned}$$ ■

§ Problem 6.2.31. $\tan mx + \cot nx = 0.$ ◊

§§ Solution.

$$\begin{aligned} \tan mx + \cot nx &= 0 \\ \therefore \tan mx &= -\cot nx = \tan\left(\frac{\pi}{2} + nx\right) \\ \therefore mx &= r\pi + \left(\frac{\pi}{2} + nx\right), \text{ where } r \text{ is any integer} \\ \therefore (m-n)x &= \left(r + \frac{1}{2}\right)\pi \\ \therefore x &= \left(r + \frac{1}{2}\right)\frac{\pi}{m-n}. \end{aligned}$$ ■

§ Problem 6.2.32. $\tan(\pi\cot\theta) = \cot(\pi\tan\theta)$. ◊

§§ Solution.

$$\begin{aligned} \tan(\pi\cot\theta) &= \cot(\pi\tan\theta) \\ \therefore \tan(\pi\cot\theta) &= \tan\left(\frac{\pi}{2} - \pi\tan\theta\right) \\ \therefore \pi\cot\theta &= n\pi + \frac{\pi}{2} - \pi\tan\theta \\ \therefore \cot\theta + \tan\theta &= n + \frac{1}{2} \\ \therefore \frac{1}{\tan\theta} + \tan\theta &= n + \frac{1}{2} \\ \therefore 2\tan^2\theta - (2n+1)\tan\theta + 2 &= 0 \\ \therefore \tan\theta &= \frac{(2n+1) \pm \sqrt{(2n+1)^2 - 16}}{4} \\ &= \frac{1}{4}\left[2n + 1 \pm \sqrt{4n^2 + 4n - 15}\right]. \end{aligned}$$

Since $4n^2 + 4n - 15 = (2n-3)(2n+5)$, we see that unless $n > \frac{3}{2}$ or $< -\frac{5}{2}$, $4n^2 + 4n - 15$ will be negative and therefore $\tan\theta$ imaginary.

Hence in the above value, n is any integer except $1,\ 0,\ -1,\ -2.$ ■

§ Problem 6.2.33. $\sin(\theta - \phi) = \frac{1}{2}$ *and* $\cos(\theta + \phi) = \frac{1}{2}.$ ◊

§§ Solution.

$$\begin{aligned} \sin(\theta - \phi) &= \frac{1}{2} = \sin\frac{\pi}{6} \ \textit{and}\ \cos(\theta + \phi) = \frac{1}{2} = \cos\frac{\pi}{3} \\ \therefore \theta - \phi &= n\pi + (-1)^n\frac{\pi}{6},\ \textit{and}\ \theta + \phi = 2m\pi \pm \frac{\pi}{3}, \end{aligned}$$

where n and m are any integers.

$$\begin{aligned} \therefore \theta &= \left(m + \frac{n}{2}\right)\pi \pm \frac{\pi}{6} + (-1)^n\frac{\pi}{12},\ \textit{and} \\ \phi &= \left(m - \frac{n}{2}\right)\pi \pm \frac{\pi}{6} - (-1)^n\frac{\pi}{12}. \end{aligned}$$ ■

§ Problem 6.2.34. $\cos(2x+3y)=\dfrac{1}{2}$ *and* $\cos(3x+2y)=\dfrac{\sqrt{3}}{2}$. ◊

§§ Solution.

$$\cos(2x+3y)=\frac{1}{2}=\cos\frac{\pi}{3}\ \textit{and}\ \cos(3x+2y)=\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6}$$
$$\therefore 2x+3y=2n\pi\pm\frac{\pi}{3}\ \textit{and}\ 3x+2y=2m\pi\pm\frac{\pi}{6},$$
$$\text{where } n \text{ and } m \text{ are any integers}$$
$$\therefore 4x+6y=4n\pi\pm\frac{2\pi}{3}\ \textit{and}\ 9x+6y=6m\pi\pm\frac{\pi}{2}$$

hence, by subtraction,

$$5x=(6m-4n)\pi\pm\frac{\pi}{2}\mp\frac{2\pi}{3}$$
$$\therefore x=\frac{1}{5}\left[(6m-4n)\pi\pm\frac{\pi}{2}\mp\frac{2\pi}{3}\right].$$

Similarly, $$y=\frac{1}{5}\left[(6n-4m)\pi\pm\pi\mp\frac{\pi}{3}\right].$$ ■

§ Problem 6.2.35. *Find all the angles between* $0°$ *and* $90°$ *which satisfy the equation*

$$\sec^2\theta\operatorname{cosec}^2\theta+2\operatorname{cosec}^2\theta=8.$$ ◊

§§ Solution.

$$\sec^2\theta\operatorname{cosec}^2\theta+2\operatorname{cosec}^2\theta=8$$
$$\therefore \left(1+\tan^2\theta\right)\left(1+\cot^2\theta\right)+2\left(1+\cot^2\theta\right)=8$$
$$1+\tan^2\theta+\cot^2\theta+1+2+2\cot^2\theta=8$$
$$\therefore \tan^2\theta+3\cot^2\theta-4=0$$
$$\therefore \tan^2\theta+\frac{3}{\tan^2\theta}-4=0$$
$$\therefore \tan^4\theta-4\tan^2\theta+3=0$$
$$\therefore \left(\tan^2\theta-1\right)\left(\tan^2\theta-3\right)=0$$
$$\therefore \tan^2\theta-1=0\ \textit{or}\ \tan^2\theta-3=0$$
$$\therefore \tan\theta=\pm1\ \textit{or}\ \pm\sqrt{3}.$$

The required angles are therefore $45°$ and $60°$.

Otherwise thus :

$$\sec^2\theta\operatorname{cosec}^2\theta+2\operatorname{cosec}^2\theta=8$$
$$\therefore \frac{1}{\sin^2\theta\cos^2\theta}+\frac{2}{\sin^2\theta}=8$$
$$\therefore 1+2\cos^2\theta=8\sin^2\theta\cos^2\theta=8\left(1-\cos^2\theta\right)\cos^2\theta=8\cos^2\theta-8\cos^4\theta$$
$$8\cos^4\theta-6\cos^2\theta+1=0$$
$$\therefore \left(4\cos^2\theta-1\right)\left(2\cos^2\theta-1\right)=0$$
$$\therefore 4\cos^2\theta-1=0\ \textit{or}\ 2\cos^2\theta-1=0$$
$$\therefore \cos\theta=\pm\frac{1}{2}\ \textit{or}\ \pm\frac{1}{\sqrt{2}}.$$

The required angles are therefore $45°$ and $60°$. ■

§ Problem 6.2.36. *If* $\tan^2\theta=\dfrac{5}{4}$, *find* $versin\theta$ *and explain the double result.* ◊

§§ Solution.

If $$\tan^2\theta = \frac{5}{4}$$

then $$\sec^2\theta = 1 + \frac{5}{4} = \frac{9}{4}$$

$$\therefore \cos\theta = \pm\frac{2}{3},\ and$$

$$versin\theta = 1 - \cos\theta = 1 \mp \frac{2}{3} = \frac{1}{3}\ or\ \frac{5}{3}.$$

Since $\tan\theta = \pm\frac{\sqrt{5}}{2}$, there are four angles, one in each quadrant ; those in the first and fourth quadrants give $versin\theta = \frac{1}{3}$, and those in the second and third quadrants give $versin\theta = \frac{5}{3}$. ■

§ Problem 6.2.37. *If the* $coversin$ *of an angle be* $\frac{1}{3}$*, find its cosine and cotangent.* ◊

§§ Solution. If θ be the angle, we have $1-\sin\theta = \frac{1}{3}$, so that $\sin\theta = \frac{2}{3}$.

$$\therefore \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{4}{9}} = \sqrt{\frac{5}{9}} = \pm\frac{1}{3}\sqrt{5},\ and$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \pm\frac{1}{2}\sqrt{5}.$$ ■

Chapter 7

Trigonometrical Ratios of The Sum and Difference of Two Angles

7.1 Addition and Subtraction Theorems

§ **Problem 7.1.1.** *If* $\sin\alpha = \frac{3}{5}$ *and* $\cos\beta = \frac{9}{41}$*, find the values of* $\sin(\alpha-\beta)$ *and* $\cos(\alpha+\beta)$*. Verify by a graph and accurate measurement.* ◊

§§ **Solution**.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}.$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{9}{41}\right)^2} = \sqrt{\frac{1600}{1681}} = \frac{40}{41}.$$

$$\begin{aligned}\therefore \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= \frac{3}{5}\times\frac{9}{41} - \frac{4}{5}\times\frac{40}{41} = \frac{27-160}{205} = -\frac{133}{205}.\end{aligned}$$

Also,
$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{4}{5}\times\frac{9}{41} - \frac{3}{5}\times\frac{40}{41} = \frac{36-120}{205} = -\frac{84}{205}.\end{aligned}$$ ■

§ **Problem 7.1.2.** *If* $\sin\alpha = \frac{45}{53}$ *and* $\sin\beta = \frac{33}{65}$*, find the values of* $\sin(\alpha-\beta)$ *and* $\sin(\alpha+\beta)$. ◊

§§ **Solution**.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{45}{53}\right)^2}$$

$$= \sqrt{\frac{(53+45)(53-45)}{(53)^2}} = \sqrt{\frac{784}{(53)^2}} = \frac{28}{53}$$

and $$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\left(\frac{33}{65}\right)^2}$$

$$= \sqrt{\frac{(65+33)(65-33)}{(65)^2}} = \sqrt{\frac{3136}{(65)^2}} = \frac{56}{65}$$

$$\therefore \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{45}{53}\times\frac{56}{65} - \frac{28}{53}\times\frac{33}{65}$$

$$= \frac{28(90-33)}{53\times 65} = \frac{28\times 3(30-11)}{53\times 65} = \frac{84\times 19}{53\times 65} = \frac{1596}{3445}.$$

Also, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{28(90+33)}{53\times 65} = \frac{84\times 41}{53\times 65} = \frac{3444}{3445}.$$ ■

§ **Problem 7.1.3.** *If* $\sin\alpha = \frac{15}{17}$ *and* $\cos\beta = \frac{12}{13}$, *find the values of* $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$ *and* $\tan(\alpha+\beta)$. *Verify by a graph and accurate measurement.* ◊

§§ **Solution**.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{15}{17}\right)^2} = \sqrt{\frac{64}{(17)^2}} = \frac{8}{17}$$

and $$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{(13)^2}} = \frac{5}{13}$$

$$\therefore \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{15}{17}\div\frac{8}{17} = \frac{15}{8}$$

and $$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{5}{13}\div\frac{12}{13} = \frac{5}{12}.$$

$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{15}{17}\times\frac{12}{13} + \frac{8}{17}\times\frac{5}{13} = \frac{20(9+2)}{17\times 13} = \frac{220}{221}.$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \frac{8}{17}\times\frac{12}{13} + \frac{15}{17}\times\frac{5}{13} = \frac{3(32+25)}{17\times 13} = \frac{171}{221}.$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8}\times\frac{5}{12}} = \frac{220}{21}.$$ ■

Prove that

§ **Problem 7.1.4.**

$$\cos(45^\circ - A)\cos(45^\circ - B) - \sin(45^\circ - A)\sin(45^\circ - B) = \sin(A+B).$$ ◊

§§ **Solution**.

$$\begin{aligned}\cos(45^\circ - A)\cos(45^\circ - B) - \sin(45^\circ - A)\sin(45^\circ - B) &= \cos[(45^\circ - A) + (45^\circ - B)]\\ &= \cos[90^\circ - (A+B)]\\ &= \sin(A+B), \text{ by Art. } 69.\end{aligned}$$ ■

§ Problem 7.1.5.

$$\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A - B). \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}\sin(45^\circ + A)\cos(45^\circ - B) &+ \cos(45^\circ + A)\sin(45^\circ - B)\\ &= \sin[(45^\circ + A) + (45^\circ - B)]\\ &= \sin[90^\circ + (A - B)]\\ &= \cos(A - B),\ \textit{by Art. } 70. \qquad \blacksquare\end{aligned}$$

§ Problem 7.1.6. $\dfrac{\sin(A-B)}{\cos A\cos B} + \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} = 0.$ $\lozenge$

§§ Solution.

$$\begin{aligned}\frac{\sin(A-B)}{\cos A\cos B} &= \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B} = \tan A - \tan B\\ \frac{\sin(B-C)}{\cos B\cos C} &= \frac{\sin B\cos C - \cos B\sin C}{\cos B\cos C} = \tan B - \tan C\\ \frac{\sin(C-A)}{\cos C\cos A} &= \frac{\sin C\cos A - \cos C\sin A}{\cos C\cos A} = \tan C - \tan A\\ \therefore \frac{\sin(A-B)}{\cos A\cos B} &+ \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} = 0. \qquad \blacksquare\end{aligned}$$

§ Problem 7.1.7. $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ.$ $\lozenge$

§§ Solution.

$$\begin{aligned}\sin 105^\circ + \cos 105^\circ &= \sin(180^\circ - 105^\circ) - \cos(180^\circ - 105^\circ)\\ &= \sin 75^\circ - \cos 75^\circ = \sin 75^\circ - \sin(90^\circ - 75^\circ)\\ &= \sin 75^\circ - \sin 15^\circ = 2\cos 45^\circ \sin 30^\circ \qquad [\textit{Art. } 94.]\\ &= 2\cos 45^\circ \times \frac{1}{2} = \cos 45^\circ.\end{aligned}$$

Otherwise thus :

$$\begin{aligned}\sin 105^\circ + \cos 105^\circ &= \sin 75^\circ - \cos 75^\circ\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45^\circ. \qquad \blacksquare\end{aligned}$$

§ Problem 7.1.8. $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ.$ $\lozenge$

§§ Solution.

$$\begin{aligned}\sin 75^\circ - \sin 15^\circ &= \cos(90^\circ - 75^\circ) + \cos(90^\circ + 15^\circ),\ \textit{by Arts. } 69 \textit{ and } 70\\ &= \cos 15^\circ + \cos 105^\circ. \qquad \blacksquare\end{aligned}$$

§ Problem 7.1.9. $\cos\alpha\cos(\gamma - \alpha) - \sin\alpha\sin(\gamma - \alpha) = \cos\gamma.$ $\lozenge$

§§ Solution.

$$\begin{aligned}\cos\alpha\cos(\gamma - \alpha) &- \sin\alpha\sin(\gamma - \alpha)\\ &= \cos[\alpha + (\gamma - \alpha)] = \cos\gamma. \qquad \blacksquare\end{aligned}$$

§ Problem 7.1.10. $\cos(\alpha+\beta)\cos\gamma - \cos(\beta+\gamma)\cos\alpha = \sin\beta\sin(\gamma - \alpha).$ $\lozenge$

§§ Solution.

$$\begin{aligned}\cos(\alpha+\beta)\cos\gamma &- \cos(\beta+\gamma)\cos\alpha\\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\cos\gamma\\ &- (\cos\beta\cos\gamma - \sin\beta\sin\gamma)\cos\alpha\\ &= \sin\beta(\sin\gamma\cos\alpha - \cos\gamma\sin\alpha) = \sin\beta\sin(\gamma - \alpha). \qquad \blacksquare\end{aligned}$$

§ Problem 7.1.11. $\sin(n+1)A\sin(n-1)A+\cos(n+1)A\cos(n-1)A = \cos 2A.$ ◊

§§ Solution.

$$\begin{aligned}\sin(n+1)A\sin(n-1)A + \cos(n+1)A\cos(n-1)A \\ = \cos\left[(n+1)A-(n-1)A\right] = \cos 2A.\end{aligned}$$ ■

§ Problem 7.1.12. $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A = \cos A.$ ◊

§§ Solution.

$$\begin{aligned}\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A \\ = \cos\left[(n+1)A-(n+2)A\right] \\ = \cos(-A) = \cos A,\ by\ Art.\ 68.\end{aligned}$$ ■

7.2 Product Formulae

Prove that

§ Problem 7.2.1. $\dfrac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta.$ ◊

§§ Solution.

$$\begin{aligned}\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} &= \frac{2\cos\frac{1}{2}(7\theta+5\theta)\sin\frac{1}{2}(7\theta-5\theta)}{2\cos\frac{1}{2}(7\theta+5\theta)\cos\frac{1}{2}(7\theta-5\theta)} \\ &= \frac{\sin\theta}{\cos\theta} = \tan\theta.\end{aligned}$$ ■

§ Problem 7.2.2. $\dfrac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\tan\theta.$ ◊

§§ Solution.

$$\begin{aligned}\frac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} &= \frac{-2\sin\frac{1}{2}(6\theta+4\theta)\sin\frac{1}{2}(6\theta-4\theta)}{2\sin\frac{1}{2}(6\theta+4\theta)\cos\frac{1}{2}(6\theta-4\theta)} \\ &= -\frac{\sin\theta}{\cos\theta} = -\tan\theta.\end{aligned}$$ ■

§ Problem 7.2.3. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A.$ ◊

§§ Solution.

$$\begin{aligned}\frac{\sin A + \sin 3A}{\cos A + \cos 3A} &= \frac{2\sin\frac{1}{2}(A+3A)\cos\frac{1}{2}(A-3A)}{2\cos\frac{1}{2}(A+3A)\cos\frac{1}{2}(A-3A)} \\ &= \frac{\sin 2A}{\cos 2A} = \tan 2A.\end{aligned}$$ ■

§ Problem 7.2.4. $\dfrac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = \cos 4A\sec 5A.$ ◊

§§ Solution.

$$\frac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = \frac{2\cos\frac{1}{2}(7A + A)\sin\frac{1}{2}(7A - A)}{2\cos\frac{1}{2}(8A + 2A)\sin\frac{1}{2}(8A - 2A)}$$
$$= \frac{\cos 4A \sin 3A}{\cos 5A \sin 3A} = \cos 4A \cdot \frac{1}{\cos 5A}$$
$$= \cos 4A \sec 5A. \quad \blacksquare$$

§ Problem 7.2.5. $\dfrac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = \cot(A + B)\cot(A - B).$ ◊

§§ Solution.

$$\frac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = \frac{\cos 2A + \cos 2B}{\cos 2B - \cos 2A}$$
$$= \frac{2\cos\frac{1}{2}(2A + 2B)\cos\frac{1}{2}(2A - 2B)}{2\sin\frac{1}{2}(2A + 2B)\sin\frac{1}{2}(2A - 2B)}$$
$$= \frac{\cos(A + B)\cos(A - B)}{\sin(A + B)\sin(A - B)}$$
$$= \cot(A + B)\cot(A - B). \quad \blacksquare$$

§ Problem 7.2.6. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \dfrac{\tan(A + B)}{\tan(A - B)}.$ ◊

§§ Solution.

$$\frac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \frac{2\sin\frac{1}{2}(2A + 2B)\cos\frac{1}{2}(2A - 2B)}{2\cos\frac{1}{2}(2A + 2B)\sin\frac{1}{2}(2A - 2B)}$$
$$= \frac{\sin(A + B)\cos(A - B)}{\cos(A + B)\sin(A - B)}$$
$$= \frac{\tan(A + B)}{\tan(A - B)}. \quad \blacksquare$$

§ Problem 7.2.7. $\dfrac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot\dfrac{A}{2}.$ ◊

§§ Solution.

$$\frac{\sin A + \sin 2A}{\cos A - \cos 2A} = \frac{2\sin\frac{1}{2}(A + 2A)\cos\frac{1}{2}(A - 2A)}{2\sin\frac{1}{2}(2A + A)\sin\frac{1}{2}(2A - A)}$$
$$= \frac{\cos\left(-\frac{A}{2}\right)}{\sin\frac{A}{2}}$$
$$= \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}, \ \textit{by Art. } 68$$
$$= \cot\frac{A}{2}. \quad \blacksquare$$

§ Problem 7.2.8. $\dfrac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A.$ ◊

§§ Solution.

$$\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \frac{2\cos\frac{1}{2}(5A+3A)\sin\frac{1}{2}(5A-3A)}{2\cos\frac{1}{2}(3A+5A)\cos\frac{1}{2}(3A-5A)}$$
$$= \frac{\sin A}{\cos(-A)} = \frac{\sin A}{\cos A} = \tan A. \qquad \blacksquare$$

§ Problem 7.2.9. $\dfrac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \tan(A-B).$ ◊

§§ Solution.

$$\frac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \frac{2\sin\frac{1}{2}(2A+2B)\sin\frac{1}{2}(2A-2B)}{2\sin\frac{1}{2}(2A+2B)\cos\frac{1}{2}(2A-2B)}$$
$$= \frac{\sin(A-B)}{\cos(A-B)} = \tan(A-B). \qquad \blacksquare$$

§ Problem 7.2.10.

$$\cos(A+B) + \sin(A-B) = 2\sin(45^\circ + A)\cos(45^\circ + B). \qquad ◊$$

§§ Solution.

$$\cos(A+B) + \sin(A-B) = \sin[90^\circ + (A+B)] + \sin(A-B), \text{ by Art. } 70$$
$$= 2\sin\frac{1}{2}(90^\circ + A + B + A - B)\cos\frac{1}{2}(90^\circ + A + B - A + B)$$
$$= 2\sin(45^\circ + A)\cos(45^\circ + B). \qquad \blacksquare$$

§ Problem 7.2.11.

$$\frac{\cos 3A - \cos A}{\sin 3A - \sin A} + \frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \frac{\sin A}{\cos 2A \cos 3A}. \qquad ◊$$

§§ Solution.

$$\frac{\cos 3A - \cos A}{\sin 3A - \sin A} + \frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A}$$
$$= \frac{-2\sin\frac{1}{2}(3A+A)\sin\frac{1}{2}(3A-A)}{2\cos\frac{1}{2}(3A+A)\sin\frac{1}{2}(3A-A)} + \frac{2\sin\frac{1}{2}(4A+2A)\sin\frac{1}{2}(4A-2A)}{2\cos\frac{1}{2}(4A+2A)\sin\frac{1}{2}(4A-2A)}$$
$$= -\frac{\sin 2A}{\cos 2A} + \frac{\sin 3A}{\cos 3A} = \frac{\sin 3A\cos 2A - \cos 3A\sin 2A}{\cos 2A\cos 3A}$$
$$= \frac{\sin(3A-2A)}{\cos 2A\cos 3A} = \frac{\sin A}{\cos 2A\cos 3A}. \qquad \blacksquare$$

§ Problem 7.2.12. $\dfrac{\sin(4A-2B) + \sin(4B-2A)}{\cos(4A-2B) + \cos(4B-2A)} = \tan(A+B).$ ◊

§§ Solution.

$$\frac{\sin(4A-2B) + \sin(4B-2A)}{\cos(4A-2B) + \cos(4B-2A)}$$
$$= \frac{2\sin\frac{1}{2}(4A-2B+4B-2A)\cos\frac{1}{2}(4A-2B-4B+2A)}{2\cos\frac{1}{2}(4A-2B+4B-2A)\cos\frac{1}{2}(4A-2B-4B+2A)}$$

$$= \frac{\sin(A+B)}{\cos(A+B)} = \tan(A+B). \qquad \blacksquare$$

§ Problem 7.2.13. $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta \cos 4\theta.$ $\Diamond$

§§ Solution.

$$\begin{aligned}
\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} &= \frac{\dfrac{\sin 5\theta}{\cos 5\theta} + \dfrac{\sin 3\theta}{\cos 3\theta}}{\dfrac{\sin 5\theta}{\cos 5\theta} - \dfrac{\sin 3\theta}{\cos 3\theta}} \\
&= \frac{\sin 5\theta \cos 3\theta + \cos 5\theta \sin 3\theta}{\sin 5\theta \cos 3\theta - \cos 5\theta \sin 3\theta} \\
&= \frac{\sin(5\theta + 3\theta)}{\sin(5\theta - 3\theta)} = \frac{\sin 8\theta}{\sin 2\theta} \\
&= \frac{2\sin 4\theta \cos 4\theta}{\sin 2\theta} = \frac{4\sin 2\theta \cos 2\theta \cos 4\theta}{\sin 2\theta} \\
&= 4\cos 2\theta \cos 4\theta. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.2.14. $\dfrac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \cos 2\theta - \sin 2\theta \tan 3\theta.$ $\Diamond$

§§ Solution.

$$\begin{aligned}
\frac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} &= \frac{(\cos 7\theta + \cos 3\theta) + 2\cos 5\theta}{(\cos 5\theta + \cos\theta) + 2\cos 3\theta} \\
&= \frac{2\cos\frac{1}{2}(7\theta + 3\theta)\cos\frac{1}{2}(7\theta - 3\theta) + 2\cos 5\theta}{2\cos\frac{1}{2}(5\theta + \theta)\cos\frac{1}{2}(5\theta - \theta) + 2\cos 3\theta} \\
&= \frac{2\cos 5\theta \cos 2\theta + 2\cos 5\theta}{2\cos 3\theta \cos 2\theta + 2\cos 3\theta} \\
&= \frac{2\cos 5\theta\,(\cos 2\theta + 1)}{2\cos 3\theta\,(\cos 2\theta + 1)} = \frac{\cos 5\theta}{\cos 3\theta} = \frac{\cos(3\theta + 2\theta)}{\cos 3\theta} \\
&= \frac{\cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta}{\cos 3\theta} \\
&= \cos 2\theta - \sin 2\theta \tan 3\theta. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.2.15. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A.$ $\Diamond$

§§ Solution.

$$\begin{aligned}
&\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} \\
&= \frac{2\sin\frac{1}{2}(A + 3A)\cos\frac{1}{2}(A - 3A) + 2\sin\frac{1}{2}(5A + 7A)\cos\frac{1}{2}(5A - 7A)}{2\cos\frac{1}{2}(A + 3A)\cos\frac{1}{2}(A - 3A) + 2\cos\frac{1}{2}(5A + 7A)\cos\frac{1}{2}(5A - 7A)} \\
&= \frac{2\sin 2A\cos(-A) + 2\sin 6A\cos(-A)}{2\cos 2A\cos(-A) + 2\cos 6A\cos(-A)} \\
&= \frac{\sin 2A + \sin 6A}{\cos 2A + \cos 6A} \\
&= \frac{2\sin\frac{1}{2}(2A + 6A)\cos\frac{1}{2}(2A - 6A)}{2\cos\frac{1}{2}(2A + 6A)\cos\frac{1}{2}(2A - 6A)} = \frac{\sin 4A}{\cos 4A} = \tan 4A. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.2.16. $\dfrac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}=\tan\theta.$ ◊

§§ Solution.

$$\begin{aligned}
&\frac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}\\
&=\frac{\sin(\theta+\phi)+\sin(\theta-\phi)-2\sin\theta}{\cos(\theta+\phi)+\cos(\theta-\phi)-2\cos\theta}\\
&=\frac{2\sin\frac{1}{2}\left[(\theta+\phi)+(\theta-\phi)\right]\cos\frac{1}{2}\left[(\theta+\phi)-(\theta-\phi)\right]-2\sin\theta}{2\cos\frac{1}{2}\left[(\theta+\phi)+(\theta-\phi)\right]\cos\frac{1}{2}\left[(\theta+\phi)-(\theta-\phi)\right]-2\cos\theta}\\
&=\frac{2\sin\theta\cos\phi-2\sin\theta}{2\cos\theta\cos\phi-2\cos\theta}\\
&=\frac{2\sin\theta\,(\cos\phi-1)}{2\cos\theta\,(\cos\phi-1)}=\frac{\sin\theta}{\cos\theta}=\tan\theta.
\end{aligned}$$ ■

§ Problem 7.2.17. $\dfrac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}=\dfrac{\sin 3A}{\sin 5A}.$ ◊

§§ Solution.

$$\begin{aligned}
\frac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}&=\frac{\sin A+\sin 5A+2\sin 3A}{\sin 3A+\sin 7A+2\sin 5A}\\
&=\frac{2\sin\frac{1}{2}(A+5A)\cos\frac{1}{2}(A-5A)+2\sin 3A}{2\sin\frac{1}{2}(3A+7A)\cos\frac{1}{2}(3A-7A)+2\sin 5A}\\
&=\frac{2\sin 3A\cos(-2A)+2\sin 3A}{2\sin 5A\cos(-2A)+2\sin 5A}\\
&=\frac{2\sin 3A\,(\cos 2A+1)}{2\sin 5A\,(\cos 2A+1)}=\frac{\sin 3A}{\sin 5A}.
\end{aligned}$$ ■

§ Problem 7.2.18. $\dfrac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\dfrac{\sin A}{\sin B}.$ ◊

§§ Solution.

$$\begin{aligned}
&\frac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}\\
&=\frac{\sin(A-C)+\sin(A+C)+2\sin A}{\sin(B-C)+\sin(B+C)+2\sin B}\\
&=\frac{2\sin\frac{1}{2}\left[(A-C)+(A+C)\right]\cos\frac{1}{2}\left[(A-C)-(A+C)\right]+2\sin A}{2\sin\frac{1}{2}\left[(B-C)+(B+C)\right]\cos\frac{1}{2}\left[(B-C)-(B+C)\right]+2\sin B}\\
&=\frac{2\sin A\cos(-C)+2\sin A}{2\sin B\cos(-C)+2\sin B}=\frac{2\sin A\,(\cos C+1)}{2\sin B\,(\cos C+1)}=\frac{\sin A}{\sin B}.
\end{aligned}$$ ■

§ Problem 7.2.19. $\dfrac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}=\cot 4A.$ ◊

§§ Solution.

$$\begin{aligned}
&\frac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}\\
&=\frac{(\sin A-\sin 5A)+(\sin 9A-\sin 13A)}{(\cos A-\cos 5A)-(\cos 9A-\cos 13A)}
\end{aligned}$$

$$= \frac{2\cos\frac{1}{2}(A+5A)\sin\frac{1}{2}(A-5A) + 2\cos\frac{1}{2}(9A+13A)\sin\frac{1}{2}(9A-13A)}{2\sin\frac{1}{2}(A+5A)\sin\frac{1}{2}(5A-A) - 2\sin\frac{1}{2}(9A+13A)\sin\frac{1}{2}(13A-9A)}$$
$$= \frac{2\cos 3A\sin(-2A) + 2\cos 11A\sin(-2A)}{2\sin 3A\sin 2A - 2\sin 11A\sin 2A}$$
$$= \frac{-2\cos 3A\sin 2A - 2\cos 11A\sin 2A}{2\sin 3A\sin 2A - 2\sin 11A\sin 2A} = \frac{\cos 11A + \cos 3A}{\sin 11A - \sin 3A}$$
$$= \frac{2\cos\frac{1}{2}(11A+3A)\cos\frac{1}{2}(11A-3A)}{2\cos\frac{1}{2}(11A+3A)\sin\frac{1}{2}(11A-3A)}$$
$$= \frac{\cos 4A}{\sin 4A} = \cot 4A. \qquad \blacksquare$$

§ Problem 7.2.20. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$ $\Diamond$

§§ Solution.

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}$$
$$= \tan\frac{A+B}{2}\cot\frac{A-B}{2}. \qquad \blacksquare$$

§ Problem 7.2.21. $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \cot\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$ $\Diamond$

§§ Solution.

$$\frac{\cos A + \cos B}{\cos B - \cos A} = \frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}$$
$$= \cot\frac{A+B}{2}\cot\frac{A-B}{2}. \qquad \blacksquare$$

§ Problem 7.2.22. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\dfrac{A+B}{2}.$ $\Diamond$

§§ Solution.

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}$$
$$= \frac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}} = \tan\frac{A+B}{2}. \qquad \blacksquare$$

§ Problem 7.2.23. $\dfrac{\sin A - \sin B}{\cos B - \cos A} = \cot\dfrac{A+B}{2}.$ $\Diamond$

§§ Solution.

$$\frac{\sin A - \sin B}{\cos B - \cos A} = \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}} = \frac{\cos\frac{A+B}{2}}{\sin\frac{A+B}{2}} = \cot\frac{A+B}{2}. \qquad \blacksquare$$

§ Problem 7.2.24.

$$\frac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)-\sin(A-B+C)+\sin(A+B-C)}$$
$= \cot B.$ ◊

§§ Solution.

The numerator of the expression

$$\begin{aligned}
&= 2\cos\frac{1}{2}\left[(A+B+C)+(-A+B+C)\right] \\
&\times \cos\frac{1}{2}\left[(A+B+C)-(-A+B+C)\right] \\
&+ 2\cos\frac{1}{2}\left[(A-B+C)+(A+B-C)\right] \\
&\times \cos\frac{1}{2}\left[(A-B+C)-(A+B-C)\right] \\
&= 2\cos(B+C)\cos A + 2\cos A\cos(C-B) \\
&= 2\cos A\left[\cos(B+C)+\cos(C-B)\right] \\
&= 2\cos A\cdot 2\cos C\cos B.
\end{aligned}$$

The denominator

$$\begin{aligned}
&= 2\sin\frac{1}{2}\left[(A+B+C)+(-A+B+C)\right] \\
&\times \cos\frac{1}{2}\left[(A+B+C)-(-A+B+C)\right] \\
&- \{\sin(A-B+C)-\sin(A+B-C)\} \\
&= 2\sin\frac{1}{2}\left[(A+B+C)+(-A+B+C)\right] \\
&\times \cos\frac{1}{2}\left[(A+B+C)-(-A+B+C)\right] \\
&- \Big\{2\cos\frac{1}{2}\left[(A-B+C)+(A+B-C)\right] \\
&\times \sin\frac{1}{2}\left[(A-B+C)-(A+B-C)\right]\Big\} \\
&= 2\sin(B+C)\cos A - 2\cos A\sin(C-B) \\
&= 2\cos A\left[\sin(B+C)+\sin(B-C)\right] \\
&= 2\cos A\cdot\sin B\cos C.
\end{aligned}$$

Hence the expression $= \dfrac{4\cos A\cos B\cos C}{4\cos A\sin B\cos C} = \cot B.$ ■

§ Problem 7.2.25.

$$\begin{aligned}
&\cos 3A+\cos 5A+\cos 7A+\cos 15A \\
&\qquad = 4\cos 4A\cos 5A\cos 6A.
\end{aligned}$$ ◊

§§ Solution.

$$\begin{aligned}
&\cos 3A+\cos 5A+\cos 7A+\cos 15A \\
&= 2\cos\frac{1}{2}(3A+5A)\cos\frac{1}{2}(3A-5A)+2\cos\frac{1}{2}(7A+15A)\cos\frac{1}{2}(7A-15A) \\
&= 2\cos 4A\cos(-A)+2\cos 11A\cos(-4A) \\
&= 2\cos 4A\,(\cos A+\cos 11A) \\
&= 2\cos 4A\cdot 2\cos\frac{1}{2}(A+11A)\cos\frac{1}{2}(A-11A) \\
&= 2\cos 4A\cdot 2\cos 6A\cos(-5A) \\
&= 4\cos 4A\cos 5A\cos 6A.
\end{aligned}$$ ■

§ Problem 7.2.26.

$$\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)+\cos(A+B+C) = 4\cos A\cos B\cos C. \qquad \Diamond$$

§§ Solution.

The expression

$$\begin{aligned}
&= 2\cos\frac{1}{2}\left[(-A+B+C)+(A-B+C)\right] \\
&\times \cos\frac{1}{2}\left[(-A+B+C)-(A-B+C)\right] \\
&+ 2\cos\frac{1}{2}\left[(A+B-C)+(A+B+C)\right] \\
&\times \cos\frac{1}{2}\left[(A+B-C)-(A+B+C)\right] \\
&= 2\cos C\cos(B-A)+2\cos(A+B)\cos(-C) \\
&= 2\cos C\left[\cos(A-B)+\cos(A+B)\right] \\
&= 2\cos C\cdot 2\cos A\cdot\cos(-B) \\
&= 4\cos A\cos B\cos C. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.2.27.

$$\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0. \qquad \Diamond$$

§§ Solution.

$$\begin{aligned}
\sin 50^\circ - \sin 70^\circ + \sin 10^\circ &= 2\cos\frac{1}{2}(50^\circ+70^\circ)\sin\frac{1}{2}(50^\circ-70^\circ)+\sin 10^\circ \\
&= -2\cos 60^\circ\sin 10^\circ + \sin 10^\circ \\
&= -\sin 10^\circ + \sin 10^\circ = 0. \;\because \cos 60^\circ = \frac{1}{2}.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
\sin 50^\circ + \sin 10^\circ - \sin 70^\circ &= 2\sin\frac{1}{2}(50^\circ+10^\circ)\cos\frac{1}{2}(50^\circ-10^\circ)-\sin 70^\circ \\
&= 2\sin 30^\circ\cos 20^\circ - \sin 70^\circ \\
&= \cos 20^\circ - \cos(90^\circ-70^\circ),\; \because \sin 30^\circ = \frac{1}{2} \\
&= \cos 20^\circ - \cos 20^\circ = 0.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
\sin 10^\circ - \sin 70^\circ + \sin 50^\circ &= 2\cos\frac{1}{2}(10^\circ+70^\circ)\sin\frac{1}{2}(10^\circ-70^\circ)+\sin 50^\circ \\
&= -2\cos 40^\circ\sin 30^\circ + \sin 50^\circ \\
&= -\cos 40^\circ + \cos(90^\circ-50^\circ) \\
&= -\cos 40^\circ + \cos 40^\circ = 0. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.2.28. $\sin 10^\circ+\sin 20^\circ+\sin 40^\circ+\sin 50^\circ = \sin 70^\circ+\sin 80^\circ.$ $\Diamond$

§§ Solution.

$$\begin{aligned}
\sin 10^\circ + \sin 50^\circ &= 2\sin\frac{1}{2}(10^\circ+50^\circ)\cos\frac{1}{2}(10^\circ-50^\circ) \\
&= 2\sin 30^\circ\cos(-20^\circ) = \cos 20^\circ \\
\sin 20^\circ + \sin 40^\circ &= 2\sin\frac{1}{2}(20^\circ+40^\circ)\cos\frac{1}{2}(20^\circ-40^\circ) \\
&= 2\sin 30^\circ\cos(-10^\circ) = \cos 10^\circ \\
\text{and } \cos 20^\circ + \cos 10^\circ &= \sin(90^\circ-20^\circ)+\cos(90^\circ-10^\circ) \\
&= \sin 70^\circ + \sin 80^\circ. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.2.29.

$$\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha = 4\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
&\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha \\
&= 2\sin\frac{3\alpha}{2}\cos\frac{\alpha}{2} + 2\sin\frac{9\alpha}{2}\cos\frac{\alpha}{2} \\
&= 2\cos\frac{\alpha}{2}\left(\sin\frac{3\alpha}{2} + \sin\frac{9\alpha}{2}\right) \\
&= 2\cos\frac{\alpha}{2}\cdot 2\sin 3\alpha\cos\frac{3\alpha}{2} \\
&= 4\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha. \qquad \blacksquare
\end{aligned}$$

Simplify

§ Problem 7.2.30. $\cos\left\{\theta + \left(n - \frac{3}{2}\right)\phi\right\} - \cos\left\{\theta + \left(n + \frac{3}{2}\right)\phi\right\}.$ $\lozenge$

§§ Solution. The expression

$$\begin{aligned}
&= 2\sin\frac{1}{2}\left[\theta + \left(n + \frac{3}{2}\right)\phi + \left\{\theta + \left(n - \frac{3}{2}\right)\phi\right\}\right] \\
&\times \sin\frac{1}{2}\left[\theta + \left(n + \frac{3}{2}\right)\phi - \left\{\theta + \left(n - \frac{3}{2}\right)\phi\right\}\right] \\
&= 2\sin(\theta + n\phi)\sin\frac{3}{2}\phi. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.2.31. $\sin\left\{\theta + \left(n - \frac{1}{2}\right)\phi\right\} + \sin\left\{\theta + \left(n + \frac{1}{2}\right)\phi\right\}.$ $\lozenge$

§§ Solution. The expression

$$\begin{aligned}
&= 2\sin\frac{1}{2}\left[\theta + \left(n - \frac{1}{2}\right)\phi + \left\{\theta + \left(n + \frac{1}{2}\right)\phi\right\}\right] \\
&\times \cos\frac{1}{2}\left[\theta + \left(n - \frac{1}{2}\right)\phi - \left\{\theta + \left(n + \frac{1}{2}\right)\phi\right\}\right] \\
&= 2\sin(\theta + n\phi)\cos\left(-\frac{\phi}{2}\right) \\
&= 2\sin(\theta + n\phi)\cos\frac{\phi}{2}. \qquad \blacksquare
\end{aligned}$$

7.3 Converse Formulae

Express as a sum or difference the following :

§ Problem 7.3.1. $2\sin 5\theta\sin 7\theta.$ $\lozenge$

§§ Solution.

$$\begin{aligned}
2\sin 5\theta\sin 7\theta &= \cos(5\theta - 7\theta) - \cos(5\theta + 7\theta) \\
&= \cos(-2\theta) - \cos 12\theta \\
&= \cos 2\theta - \cos 12\theta, \ \textit{by Art. } 68. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.3.2. $2\cos 7\theta\sin 5\theta.$ $\lozenge$

§§ Solution.

$$\begin{aligned}
2\cos 7\theta\sin 5\theta &= \sin(7\theta + 5\theta) - \sin(7\theta - 5\theta) \\
&= \sin 12\theta - \sin 2\theta. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.3.3. $2\cos 11\theta \cos 3\theta$. ◊

§§ Solution.

$$\begin{aligned} 2\cos 11\theta \cos 3\theta &= \cos(11\theta + 3\theta) + \cos(11\theta - 3\theta) \\ &= \cos 14\theta + \cos 8\theta. \end{aligned}$$ ■

§ Problem 7.3.4. $2\sin 54^\circ \sin 66^\circ$. ◊

§§ Solution.

$$\begin{aligned} 2\sin 54^\circ \sin 66^\circ &= \cos\left(54^\circ - 66^\circ\right) - \cos\left(54^\circ + 66^\circ\right) \\ &= \cos\left(-12^\circ\right) - \cos 120^\circ = \cos 12^\circ - \cos 120^\circ. \end{aligned}$$ ■

Prove that

§ Problem 7.3.5. $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$. ◊

§§ Solution.

$$\begin{aligned} &\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} \\ &= \frac{1}{2}\left[\cos\left(\frac{\theta}{2} - \frac{7\theta}{2}\right) - \cos\left(\frac{\theta}{2} + \frac{7\theta}{2}\right)\right] \\ &+ \frac{1}{2}\left[\cos\left(\frac{3\theta}{2} - \frac{11\theta}{2}\right) - \cos\left(\frac{3\theta}{2} + \frac{11\theta}{2}\right)\right] \\ &= \frac{1}{2}\left[\cos(-3\theta) - \cos 4\theta + \cos(-4\theta) - \cos 7\theta\right] \\ &= \frac{1}{2}(\cos 3\theta - \cos 4\theta + \cos 4\theta - \cos 7\theta) \\ &= \frac{1}{2}(\cos 3\theta - \cos 7\theta) \\ &= \frac{1}{2} \times 2\sin\frac{7\theta + 3\theta}{2}\sin\frac{7\theta - 3\theta}{2} \\ &= \sin 5\theta \sin 2\theta. \end{aligned}$$ ■

§ Problem 7.3.6. $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$. ◊

§§ Solution.

$$\begin{aligned} &\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} \\ &= \frac{1}{2}\left[\cos\left(2\theta + \frac{\theta}{2}\right) + \cos\left(2\theta - \frac{\theta}{2}\right)\right] \\ &- \left[\cos\left(3\theta + \frac{9\theta}{2}\right) + \cos\left(3\theta - \frac{9\theta}{2}\right)\right] \\ &= \frac{1}{2}\left[\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\left(-\frac{3\theta}{2}\right)\right] \\ &= \frac{1}{2}\left(\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right) \\ &= \frac{1}{2}\left(\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right) \\ &= \frac{1}{2} \times 2\sin\frac{15\theta + 5\theta}{4}\sin\frac{15\theta - 5\theta}{4} \\ &= \sin 5\theta \sin\frac{5\theta}{2}. \end{aligned}$$ ■

§ Problem 7.3.7.

$$\begin{aligned} \sin A \sin(A + 2B) - \sin B \sin(B + 2A) \\ = \sin(A - B)\sin(A + B). \end{aligned}$$ ◊

§§ Solution.

$$\begin{aligned}
&\sin A\sin(A+2B)-\sin B\sin(B+2A)\\
&=\frac{1}{2}\left[\cos(A-A-2B)-\cos(A+A+2B)\right]\\
&-\frac{1}{2}\left[\cos(B-B-2A)-\cos(B+B+2A)\right]\\
&=\frac{1}{2}\left[\cos(-2B)-\cos 2(A+B)-\cos(-2A)-\cos 2(B+A)\right]\\
&=\frac{1}{2}(\cos 2B-\cos 2A)=\frac{1}{2}\times 2\sin\frac{2A+2B}{2}\sin\frac{2A-2B}{2}\\
&=\sin(A+B)\sin(A-B).
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
\sin A\sin(A+2B)&=\sin A\sin\left[(A+B)+B\right]\\
&=\sin A\left[\sin(A+B)\cos B+\cos(A+B)\sin B\right]\\
\sin B\sin(B+2A)&=\sin B\sin\left[(B+A)+A\right]\\
&=\sin B\left[\sin(B+A)\cos A+\cos(B+A)\sin A\right]\\
\therefore \sin A\sin(A+2B)&-\sin B\sin(B+2A)\\
&=\sin(A+B)\left(\sin A\cos B-\cos A\sin B\right)\\
&=\sin(A+B)\sin(A-B).
\end{aligned}$$

■

§ Problem 7.3.8. $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A=0.$ ◊

§§ Solution.

$$\begin{aligned}
&(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A\\
&=\left(2\sin\frac{3A+A}{2}\cos\frac{3A-A}{2}\right)\sin A\\
&+\left(2\sin\frac{A+3A}{2}\sin\frac{A-3A}{2}\right)\cos A\\
&=2\sin 2A\cos A\sin A-2\sin 2A\sin A\cos A=0.
\end{aligned}$$

■

§ Problem 7.3.9. $\dfrac{2\sin(A-C)\cos C-\sin(A-2C)}{2\sin(B-C)\cos C-\sin(B-2C)}=\dfrac{\sin A}{\sin B}.$ ◊

§§ Solution.

$$\begin{aligned}
&\frac{2\sin(A-C)\cos C-\sin(A-2C)}{2\sin(B-C)\cos C-\sin(B-2C)}\\
&=\frac{\sin(A-C+C)+\sin(A-C-C)-\sin(A-2C)}{\sin(B-C+C)+\sin(B-C-C)-\sin(B-2C)}\\
&=\frac{\sin A+\sin(A-2C)-\sin(A-2C)}{\sin B+\sin(B-2C)-\sin(B-2C)}=\frac{\sin A}{\sin B}.
\end{aligned}$$

■

§ Problem 7.3.10.

$$\begin{aligned}
&\frac{\sin A\sin 2A+\sin 3A\sin 6A+\sin 4A\sin 13A}{\sin A\cos 2A+\sin 3A\cos 6A+\sin 4A\cos 13A}\\
&=\tan 9A.
\end{aligned}$$

◊

§§ Solution. The expression

$$\begin{aligned}
&=\frac{\frac{1}{2}(\cos A-\cos 3A)+\frac{1}{2}(\cos 3A-\cos 9A)+\frac{1}{2}(\cos 9A-\cos 17A)}{\frac{1}{2}(\sin 3A-\sin A)+\frac{1}{2}(\sin 9A-\sin 3A)+\frac{1}{2}(\sin 17A-\sin 9A)}\\
&=\frac{\cos A-\cos 17A}{\sin 17A-\sin A}=\frac{2\sin 9A\sin 8A}{2\cos 9A\sin 8A}=\tan 9A.
\end{aligned}$$

■

§ Problem 7.3.11.

$$\frac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 6A \cot 5A. \qquad \Diamond$$

§§ Solution. The expression

$$= \frac{\frac{1}{2}(\cos A + \cos 5A) - \frac{1}{2}(\cos 5A + \cos 9A) + \frac{1}{2}(\cos 9A + \cos 11A)}{\frac{1}{2}(\cos A - \cos 7A) - \frac{1}{2}(\cos 3A - \cos 7A) + \frac{1}{2}(\cos 3A - \cos 11A)}$$

$$= \frac{\cos A + \cos 11A}{\cos A - \cos 11A} = \frac{2\cos 6A \cos 5A}{2 \sin 6A \sin 5A}$$

$$= \cot 6A \cot 5A. \qquad \blacksquare$$

§ Problem 7.3.12.

$$\cos(36^\circ - A)\cos(36^\circ + A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A. \qquad \Diamond$$

§§ Solution. We have

$$\cos(36^\circ - A) = \sin[90^\circ - (36^\circ - A)] = \sin(54^\circ + A)$$

and $\cos(36^\circ + A) = \sin[90^\circ - (36^\circ + A)] = \sin(54^\circ - A)$.

Hence the expression

$$= \sin(54^\circ + A)\sin(54^\circ - A) + \cos(54^\circ + A)\cos(54^\circ - A)$$

$$= \cos[(54^\circ + A) - (54^\circ - A)] = \cos 2A. \qquad \blacksquare$$

§ Problem 7.3.13.

$$\cos A \sin(B - C) + \cos B \sin(C - A) + \cos C \sin(A - B) = 0. \qquad \Diamond$$

§§ Solution. The expression

$$= \frac{1}{2}[\sin(A + B - C) - \sin(A - B + C)]$$
$$+ \frac{1}{2}[\sin(B + C - A) - \sin(B - C + A)]$$
$$+ \frac{1}{2}[\sin(C + A - B) - \sin(C - A + B)] = 0. \qquad \blacksquare$$

§ Problem 7.3.14. $\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\cos 2A.$ $\Diamond$

§§ Solution. $\sin(45^\circ + A)\sin(45^\circ - A)$

$$= \frac{1}{2}[\cos(45^\circ + A - 45^\circ + A) - \cos(45^\circ + A + 45^\circ - A)]$$

$$= \frac{1}{2}(\cos 2A - \cos 90^\circ) = \frac{1}{2}\cos 2A \quad \because \cos 90^\circ = 0. \qquad \blacksquare$$

§ Problem 7.3.15. $versin(A+B)versin(A-B) = (\cos A - \cos B)^2.$ $\Diamond$

§§ Solution.

$$versin(A + B)versin(A - B)$$
$$= [1 - \cos(A + B)][1 - \cos(A - B)]$$
$$= 1 - [\cos(A + B) + \cos(A - B)] + \cos(A + B)\cos(A - B)$$
$$= 1 - 2\cos A \cos B + \cos^2 A - \sin^2 B \, [Ex.\ 2,\ Art.\ 93]$$
$$= 1 - 2\cos A \cos B + \cos^2 A - \left(1 - \cos^2 B\right)$$
$$= \cos^2 A - 2\cos A \cos B + \cos^2 B = (\cos A - \cos B)^2. \qquad \blacksquare$$

§ Problem 7.3.16.
$$\sin(\beta-\gamma)\cos(\alpha-\delta)+\sin(\gamma-\alpha)\cos(\beta-\delta)$$
$$+\sin(\alpha-\beta)\cos(\gamma-\delta)=0. \qquad \lozenge$$

§§ Solution. The expression

$$\begin{aligned}
&=\frac{1}{2}\left[\sin(\beta-\gamma+\alpha-\delta)+\sin(\beta-\gamma-\alpha+\delta)\right]\\
&+\frac{1}{2}\left[\sin(\gamma-\alpha+\beta-\delta)+\sin(\gamma-\alpha-\beta+\delta)\right]\\
&+\frac{1}{2}\left[\sin(\alpha-\beta+\gamma-\delta)+\sin(\alpha-\beta-\gamma+\delta)\right]\\
&=\frac{1}{2}[\sin(\beta-\gamma+\alpha-\delta)+\sin(\beta-\gamma-\alpha+\delta)\\
&+\sin(\gamma-\alpha+\beta-\delta)-\sin(-\gamma+\alpha+\beta-\delta)\\
&-\sin(-\alpha+\beta-\gamma+\delta)-\sin(-\alpha+\beta+\gamma-\delta)]\ \textit{by Art. } 68\\
&=0.
\end{aligned}$$
■

§ Problem 7.3.17. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0.$ ◊

§§ Solution.

$$\begin{aligned}
2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}
&=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{\pi}{13}\cos\frac{4\pi}{13}\\
&=2\cos\frac{\pi}{13}\left(\cos\frac{9\pi}{13}+\cos\frac{4\pi}{13}\right)=0.\\
\because \cos\frac{9\pi}{13}&=-\cos\left(\pi-\frac{9\pi}{13}\right)=-\cos\frac{4\pi}{13}.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}
&=\cos\frac{8\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0.\\
\because \cos\frac{8\pi}{13}&=-\cos\left(\pi-\frac{8\pi}{13}\right)=-\cos\frac{5\pi}{13}\\
\text{and } \cos\frac{10\pi}{13}&=-\cos\left(\pi-\frac{10\pi}{13}\right)=-\cos\frac{3\pi}{13}.
\end{aligned}$$
■

7.4 Tangent of The Sum of Two Angles

§ Problem 7.4.1. *It* $\tan A=\frac{1}{2}$ *and* $\tan B=\frac{1}{3}$, *find the values of* $\tan(2A+B))$ *and* $\tan(2A-B)$. *Verify by a graph and accurate measurement.* ◊

§§ Solution.

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}=\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}=\frac{4}{3}.$$

$$\therefore \tan(2A+B)=\frac{\tan 2A+\tan B}{1-\tan 2A\tan B}=\frac{\frac{4}{3}+\frac{1}{3}}{1-\frac{4}{3}\times\frac{1}{3}}=\frac{5}{3}\div\frac{5}{9}=3.$$

$$\text{Also } \tan(2A-B) = \frac{\tan 2A - \tan B}{1+\tan 2A \tan B} = \frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\times\frac{1}{3}} = 1 \div \frac{13}{9} = \frac{9}{13}. \qquad \blacksquare$$

§ Problem 7.4.2. *If* $\tan A = \dfrac{\sqrt{3}}{4-\sqrt{3}}$ *and* $\tan B = \dfrac{\sqrt{3}}{4+\sqrt{3}}$*, prove that*
$$\tan(A-B) = \cdot 375. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
\tan(A-B) &= \frac{\tan A - \tan B}{1+\tan A \tan B} \\
&= \frac{\frac{\sqrt{3}}{4-\sqrt{3}} - \frac{\sqrt{3}}{4+\sqrt{3}}}{1+\frac{\sqrt{3}}{4-\sqrt{3}}\times\frac{\sqrt{3}}{4+\sqrt{3}}} \\
&= \frac{4\sqrt{3}+3-4\sqrt{3}+3}{\left(4-\sqrt{3}\right)\left(4+\sqrt{3}\right)+3} = \frac{6}{13+3} = \frac{6}{16} = \frac{3}{8} = \cdot 375. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.4.3. *If* $\tan A = \dfrac{n}{n+1}$ *and* $\tan B = \dfrac{1}{2n+1}$*, find* $\tan(A+B)$*.* $\lozenge$

§§ Solution.

$$\begin{aligned}
\tan(A+B) &= \frac{\tan A + \tan B}{1-\tan A \tan B} \\
&= \frac{\frac{n}{n+1}+\frac{1}{2n+1}}{1-\frac{n}{n+1}\times\frac{1}{2n+1}} \\
&= \frac{2n^2+n+n+1}{(n+1)(2n+1)-n} = \frac{2n^2+2n+1}{2n^2+2n+1} = 1. \qquad \blacksquare
\end{aligned}$$

§ Problem 7.4.4. *If* $\tan\alpha = \dfrac{5}{6}$ *and* $\tan\beta = \dfrac{1}{11}$*, prove that* $\alpha+\beta = \dfrac{\pi}{4}$*.*
Verify by a graph and accurate measurement. $\lozenge$

§§ Solution.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \\
&= \frac{\frac{5}{6}+\frac{1}{11}}{1-\frac{5}{6}\times\frac{1}{11}} = \frac{55+6}{66-5} = \frac{61}{61} = 1 = \tan\frac{\pi}{4}.
\end{aligned}$$

Hence one of the values of $\alpha+\beta$ is $\dfrac{\pi}{4}$. $\blacksquare$

Prove that

§ Problem 7.4.5. $\tan\left(\dfrac{\pi}{4}+\theta\right)\times\tan\left(\dfrac{3\pi}{4}+\theta\right) = -1.$ $\lozenge$

§§ Solution.

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta} = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\text{and } \tan\left(\frac{3\pi}{4}+\theta\right)=\frac{\tan\dfrac{3\pi}{4}+\tan\theta}{1-\tan\dfrac{3\pi}{4}\tan\theta}=\frac{-1+\tan\theta}{1+\tan\theta}$$

$$\therefore \tan\left(\frac{\pi}{4}+\theta\right)\times\tan\left(\frac{3\pi}{4}+\theta\right)=\frac{1+\tan\theta}{1-\tan\theta}\times\frac{-(1-\tan\theta)}{1+\tan\theta}=-1. \qquad \blacksquare$$

§ Problem 7.4.6. $\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)=1.$ $\Diamond$

§§ Solution.

$$\cot\left(\frac{\pi}{4}+\theta\right)=\frac{\cot\dfrac{\pi}{4}\cot\theta-1}{\cot\dfrac{\pi}{4}+\cot\theta}=\frac{\cot\theta-1}{1+\cot\theta}$$

$$\text{and } \cot\left(\frac{\pi}{4}-\theta\right)=\frac{\cot\dfrac{\pi}{4}\cot\theta+1}{\cot\theta-\cot\dfrac{\pi}{4}}=\frac{\cot\theta+1}{\cot\theta-1}$$

$$\therefore \cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)=1. \qquad \blacksquare$$

§ Problem 7.4.7. $1+\tan A\tan\frac{A}{2}=\tan A\cot\frac{A}{2}-1=\sec A.$ $\Diamond$

§§ Solution.

$$\begin{aligned}1+\tan A\tan\frac{A}{2}&=1+\frac{\sin A}{\cos A}\cdot\frac{\sin\dfrac{A}{2}}{\cos\dfrac{A}{2}}\\&=\frac{\cos A\cos\dfrac{A}{2}+\sin A\sin\dfrac{A}{2}}{\cos A\cos\dfrac{A}{2}}\\&=\frac{\cos\left(A-\dfrac{A}{2}\right)}{\cos A\cos\dfrac{A}{2}}=\frac{\cos\dfrac{A}{2}}{\cos A\cos\dfrac{A}{2}}\\&=\frac{1}{\cos A}=\sec A.\end{aligned}$$

Again,

$$\begin{aligned}\tan A\cot\frac{A}{2}-1&=\frac{\sin A}{\cos A}\cdot\frac{\cos\dfrac{A}{2}}{\sin\dfrac{A}{2}}-1\\&=\frac{\sin A\cos\dfrac{A}{2}-\cos A\sin\dfrac{A}{2}}{\cos A\sin\dfrac{A}{2}}\\&=\frac{\sin\left(A-\dfrac{A}{2}\right)}{\cos A\sin\dfrac{A}{2}}\\&=\frac{\sin\dfrac{A}{2}}{\cos A\sin\dfrac{A}{2}}=\frac{1}{\cos A}=\sec A.\end{aligned} \qquad \blacksquare$$

Chapter 8

The Trigonometrical Ratios of Multiple and Submultiple Angles

8.1 Multiple Angles

§ Problem 8.1.1. *Find the value of* $\sin 2\alpha$ *when*

(1) $\cos\alpha = \dfrac{3}{5}$

(2) $\sin\alpha = \dfrac{12}{13}$*, and*

(3) $\tan\alpha = \dfrac{16}{63}$. ◊

§§ Solution. (1)

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

$$\therefore \sin 2\alpha = 2\sin\alpha\cos\alpha = \pm 2\times\frac{4}{5}\times\frac{3}{5} = \pm\frac{24}{25}.$$

(2)

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

$$\therefore \sin 2\alpha = \pm 2\times\frac{12}{13}\times\frac{5}{13} = \pm\frac{120}{169}.$$

(3)

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}\ [Art.\ 109] = \frac{2\times\dfrac{16}{63}}{1+\left(\dfrac{16}{63}\right)^2}$$
$$= \frac{32}{63} \div \left[1+\frac{256}{3969}\right] = \frac{32}{63}\times\frac{3969}{4225} = \frac{2016}{4225}.$$ ■

§ Problem 8.1.2. *Find the value of* $\cos 2\alpha$ *when*

(1) $\cos\alpha = \dfrac{15}{17}$

(2) $\sin\alpha = \dfrac{4}{5}$*, and*

(3) $\tan\alpha = \dfrac{5}{12}$.

Verify by a graph and accurate measurement. ◊

§§ Solution. (1)

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 2\times\frac{225}{289} - 1 = \frac{450-289}{289} = \frac{161}{289}$$

(2)

$$\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2\times\frac{16}{25} = \frac{25-32}{25} = -\frac{7}{25}$$

(3)

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}\ [Art.\ 109]$$
$$= \frac{1-\dfrac{25}{144}}{1+\dfrac{25}{144}} = \frac{144-25}{144+25} = \frac{119}{169}.$$ ■

§ Problem 8.1.3. *If* $\tan\theta = \dfrac{b}{a}$*, find the value of* $a\cos 2\theta + b\sin 2\theta$. ◊

§§ Solution.

$$a\cos 2\theta + b\sin 2\theta = a\times\frac{1-\tan^2\theta}{1+\tan^2\theta} + b\times\frac{2\tan\theta}{1+\tan^2\theta}$$
$$= a\times\frac{a^2-b^2}{a^2+b^2} + b\times\frac{2ab^2}{a^2+b^2}$$
$$= a\left(\frac{a^2-b^2+2b^2}{a^2+b^2}\right) = a\left(\frac{a^2+b^2}{a^2+b^2}\right) = a.$$

Otherwise thus :

$$a\cos 2\theta + b\sin 2\theta = a\left(\cos 2\theta + \frac{b}{a}\sin 2\theta\right)$$
$$= a\left(\cos 2\theta + \tan\theta\sin 2\theta\right)$$
$$= \frac{a}{\cos\theta}\left(\cos 2\theta\cos\theta + \sin\theta\sin 2\theta\right)$$
$$= \frac{a}{\cos\theta}\cos(2\theta-\theta) = \frac{a}{\cos\theta}\times\cos\theta = a.$$ ■

Prove that

§ Problem 8.1.4. $\dfrac{\sin 2A}{1+\cos 2A} = \tan A.$ ◊

§§ Solution. $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{2\cos^2 A} = \dfrac{\sin A}{\cos A} = \tan A.$ ■

§ Problem 8.1.5. $\dfrac{\sin 2A}{1-\cos 2A} = \cot A.$ ◊

§§ Solution. $\dfrac{\sin 2A}{1-\cos 2A} = \dfrac{2\sin A\cos A}{2\sin^2 A} = \dfrac{\cos A}{\sin A} = \cot A.$ ■

§ Problem 8.1.6. $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A.$ ◊

§§ Solution. $\dfrac{1-\cos 2A}{1+\cos 2A} = \dfrac{2\sin^2 A}{2\cos^2 A} = \tan^2 A.$ ■

§ Problem 8.1.7. $\tan A + \cot A = 2\operatorname{cosec} 2A.$ ◊

§§ Solution.

$$\begin{aligned}\tan A + \cot A &= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A\cos A}\\ &= \frac{2}{2\sin A\cos A} = \frac{2}{\sin 2A} = 2\operatorname{cosec} 2A.\end{aligned}$$ ■

§ Problem 8.1.8. $\tan A - \cot A = -2\cot 2A.$ ◊

§§ Solution.

$$\begin{aligned}\tan A - \cot A &= \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A} = \frac{\sin^2 A - \cos^2 A}{\sin A\cos A}\\ &= \frac{-2\left(\cos^2 A - \sin^2 A\right)}{2\sin A\cos A} = \frac{-2\cos 2A}{\sin 2A} = -2\cot 2A.\end{aligned}$$ ■

§ Problem 8.1.9. $\operatorname{cosec} 2A + \cot 2A = \cot A.$ ◊

§§ Solution. $\operatorname{cosec} 2A + \cot 2A = \dfrac{1+\cos 2A}{\sin 2A} = \dfrac{2\cos^2 A}{2\sin A\cos A} = \cot A.$ ■

§ Problem 8.1.10. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\dfrac{A}{2}\cot\dfrac{B}{2}.$ ◊

§§ Solution.

$$\begin{aligned}&\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}\\ &= \frac{1-\cos(A+B)-(\cos A-\cos B)}{1-\cos(A+B)+(\cos A-\cos B)}\\ &= \frac{2\sin^2\dfrac{A+B}{2}+2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin^2\dfrac{A+B}{2}-2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}\\ &= \frac{2\sin\dfrac{A+B}{2}\left[\sin\dfrac{A+B}{2}+\sin\dfrac{A-B}{2}\right]}{2\sin\dfrac{A+B}{2}\left[\sin\dfrac{A+B}{2}-\sin\dfrac{A-B}{2}\right]}\\ &= \frac{\sin\dfrac{A+B}{2}+\sin\dfrac{A-B}{2}}{\sin\dfrac{A+B}{2}-\sin\dfrac{A-B}{2}}\end{aligned}$$

$$= \frac{2\sin\frac{A}{2}\cos\frac{B}{2}}{2\cos\frac{A}{2}\sin\frac{B}{2}} = \tan\frac{A}{2}\cot\frac{B}{2}.$$

■

§ Problem 8.1.11. $\dfrac{\cos A}{1 \mp \sin A} = \tan\left(45^\circ \pm \dfrac{A}{2}\right).$ ◊

§§ Solution.

$$\frac{\cos A}{1 \mp \sin A} = \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} \mp 2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2} \mp \sin\frac{A}{2}\right)^2}$$

$$= \frac{\cos\frac{A}{2} \pm \sin\frac{A}{2}}{\cos\frac{A}{2} \mp \sin\frac{A}{2}},$$

the upper or lower signs to be taken together,

$$= \frac{1 \pm \tan\frac{A}{2}}{1 \mp \tan\frac{A}{2}} = \tan\left(45^\circ \pm \frac{A}{2}\right).$$

■

§ Problem 8.1.12. $\dfrac{\sec 8A - 1}{\sec 4A - 1} = \dfrac{\tan 8A}{\tan 2A}.$ ◊

§§ Solution.

$$\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\frac{1}{\cos 8A} - 1}{\frac{1}{\cos 4A} - 1} = \frac{\cos 4A\,(1 - \cos 8A)}{\cos 8A\,(1 - \cos 4A)}$$

$$= \frac{\cos 4A}{\cos 8A} \cdot \frac{2\sin^2 4A}{2\sin^2 2A} = \frac{2\sin 4A\cos 4A}{\cos 8A} \cdot \frac{\sin 4A}{2\sin^2 2A}$$

$$= \frac{\sin 8A}{\cos 8A} \cdot \frac{2\sin 2A\cos 2A}{2\sin^2 2A} = \frac{\sin 8A}{\cos 8A} \cdot \frac{\cos 2A}{\sin 2A}$$

$$= \tan 8A\cot 2A = \frac{\tan 8A}{\tan 2A}.$$

■

§ Problem 8.1.13. $\dfrac{1 + \tan^2(45^\circ - A)}{1 - \tan^2(45^\circ - A)} = \operatorname{cosec}\, 2A.$ ◊

§§ Solution.

$$\frac{1 + \tan^2(45^\circ - A)}{1 - \tan^2(45^\circ - A)} = \frac{1 + \frac{\sin^2(45^\circ - A)}{\cos^2(45^\circ - A)}}{1 - \frac{\sin^2(45^\circ - A)}{\cos^2(45^\circ - A)}}$$

$$= \frac{\cos^2(45^\circ - A) + \sin^2(45^\circ - A)}{\cos^2(45^\circ - A) - \sin^2(45^\circ - A)}$$

$$= \frac{1}{\cos 2\,(45^\circ - A)}$$

$$= \frac{1}{\cos(90^\circ - 2A)} = \frac{1}{\sin 2A} = \operatorname{cosec}\, 2A.$$

■

§ Problem 8.1.14. $\dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.$ ◊

§§ Solution.

$$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\frac{2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}}{2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}}$$

$$=\tan\frac{\alpha+\beta}{2}\cot\frac{\alpha-\beta}{2}=\frac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.$$ ■

§ Problem 8.1.15. $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B).$ ◊

§§ Solution.

$$\frac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\frac{2\sin^2 A-2\sin^2 B}{2\sin A\cos A-2\sin B\cos B}$$

$$=\frac{1-\cos 2A-(1-\cos 2B)}{\sin 2A-\sin 2B}=\frac{\cos 2B-\cos 2A}{\sin 2A-\sin 2B}$$

$$=\frac{2\sin\frac{1}{2}(2A+2B)\sin\frac{1}{2}(2A-2B)}{2\cos\frac{1}{2}(2A+2B)\sin\frac{1}{2}(2A-2B)}$$

$$=\frac{\sin(A+B)}{\cos(A+B)}=\tan(A+B).$$ ■

§ Problem 8.1.16. $\tan\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right)=2\tan 2\theta.$ ◊

§§ Solution.

$$\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)$$

$$=\frac{1+\tan\theta}{1-\tan\theta}-\frac{1-\tan\theta}{1+\tan\theta},\ \textit{by Art. } 100,$$

$$=\frac{(1+\tan\theta)^2-(1-\tan\theta)^2}{1-\tan^2\theta}$$

$$=\frac{2\times 2\tan\theta}{1-\tan^2\theta}=2\tan 2\theta,\ \textit{by Art. } 105.$$ ■

§ Problem 8.1.17. $\dfrac{\cos A+\sin A}{\cos A-\sin A}-\dfrac{\cos A-\sin A}{\cos A+\sin A}=2\tan 2A.$ ◊

§§ Solution.

$$\frac{\cos A+\sin A}{\cos A-\sin A}-\frac{\cos A-\sin A}{\cos A+\sin A}$$

$$=\frac{(\cos A+\sin A)^2-(\cos A-\sin A)^2}{\cos^2 A-\sin^2 A}$$

$$=\frac{4\cos A\sin A}{\cos 2A}=\frac{2\sin 2A}{\cos 2A}=2\tan 2A.$$ ■

§ Problem 8.1.18. $\cot(A+15^\circ)-\tan(A-15^\circ)=\dfrac{4\cos 2A}{1+2\sin 2A}.$ ◊

§§ Solution.

$$\begin{aligned}
\cot(A+15^\circ)-\tan(A-15^\circ)&\\
&=\frac{\cos(A+15^\circ)}{\sin(A+15^\circ)}-\frac{\sin(A-15^\circ)}{\cos(A-15^\circ)}\\
&=\frac{\cos(A+15^\circ)\cos(A-15^\circ)-\sin(A+15^\circ)\sin(A-15^\circ)}{\sin(A+15^\circ)\cos(A-15^\circ)}\\
&=\frac{\cos[(A+15^\circ)+(A-15^\circ)]}{\frac{1}{2}(\sin 2A+\sin 30^\circ)}=\frac{2\cos 2A}{\sin 2A+\sin 30^\circ}\\
&=\frac{2\cos 2A}{\sin 2A+\frac{1}{2}}=\frac{4\cos 2A}{1+2\sin 2A}.
\end{aligned}$$

■

§ Problem 8.1.19. $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta.$ ◊

§§ Solution.

$$\begin{aligned}
\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}&=\frac{\sin\theta+2\sin\theta\cos\theta}{1+\cos\theta+2\cos^2\theta-1}\\
&=\frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)}=\frac{\sin\theta}{\cos\theta}=\tan\theta.
\end{aligned}$$

■

§ Problem 8.1.20. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\dfrac{\theta}{2}.$ ◊

§§ Solution.

$$\begin{aligned}
\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}&=\frac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}\\
&=\frac{2\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\\
&=\frac{\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}.
\end{aligned}$$

■

§ Problem 8.1.21.

$$\begin{aligned}
&\frac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos nA+\cos(n-1)A}\\
&=\tan\frac{A}{2}.
\end{aligned}$$

◊

§§ Solution.

$$\begin{aligned}
&\frac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos nA+\cos(n-1)A}\\
&=\frac{2\cos\frac{1}{2}(nA+A+nA-A)\sin\frac{1}{2}(nA+A-nA+A)}{2\cos\frac{1}{2}(nA+A+nA-A)\cos\frac{1}{2}(nA+A-nA+A)+2\cos nA}\\
&=\frac{2\cos nA\cdot\sin A}{2\cos nA\cos A+2\cos nA}\\
&=\frac{\sin A}{\cos A+1}=\tan\frac{A}{2}\ [by\ XVII.\ 4].
\end{aligned}$$

■

§ Problem 8.1.22.

$$\frac{\sin(n+1)A + 2\sin nA + \sin(n-1)A}{\cos(n-1)A - \cos(n+1)A} = \cot\frac{A}{2}. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
&\frac{\sin(n+1)A + 2\sin nA + \sin(n-1)A}{\cos(n-1)A - \cos(n+1)A} \\
&= \frac{2\sin\frac{1}{2}(nA + A + nA - A)\cos\frac{1}{2}(nA + A - nA + A) + 2\sin nA}{2\sin\frac{1}{2}(nA + A + nA - A)\sin\frac{1}{2}(nA + A - nA + A)} \\
&= \frac{2\sin nA\cos A + 2\sin nA}{2\sin nA\sin A} \\
&= \frac{\cos A + 1}{\sin A} = \cot\frac{A}{2}\ [by\ XVII.\ 4]. \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.23. $\sin(2n+1)A\sin A = \sin^2(n+1)A - \sin^2 nA.$ $\lozenge$

§§ Solution.

$$\begin{aligned}
\sin(2n+1)A\sin A &= \frac{1}{2}\left[\cos(2nA + A - A) - \cos(2nA + A + A)\right] \\
&= \frac{1}{2}\left[\cos 2nA - \cos 2(n+1)A\right] \\
&= \frac{1}{2}\left[1 - 2\sin^2 nA - \left\{1 - 2\sin^2(n+1)A\right\}\right] \\
&= \frac{1}{2}\left[2\sin^2(n+1)A - 2\sin^2 nA\right] \\
&= \sin^2(n+1)A - \sin^2 nA. \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.24. $\dfrac{\sin(A+3B) + \sin(3A+B)}{\sin 2A + \sin 2B} = 2\cos(A+B).$ $\lozenge$

§§ Solution.

$$\begin{aligned}
\frac{\sin(A+3B) + \sin(3A+B)}{\sin 2A + \sin 2B} &= \frac{2\sin\frac{1}{2}(4A+4B)\cos\frac{1}{2}(2A-2B)}{2\sin\frac{1}{2}(2A+2B)\cos\frac{1}{2}(2A-2B)} \\
&= \frac{2\sin 2(A+B)\cos(A-B)}{2\sin(A+B)\cos(A-B)} \\
&= \frac{\sin 2(A+B)}{\sin(A+B)} \\
&= \frac{2\sin(A+B)\cos(A+B)}{\sin(A+B)} \\
&= 2\cos(A+B). \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.25. $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}.$ $\lozenge$

§§ Solution.

$$\begin{aligned}
\sin 3A + \sin 2A - \sin A &= (\sin 3A - \sin A) + \sin 2A \\
&= 2\cos\frac{1}{2}(3A + A)\sin\frac{1}{2}(3A - A) + \sin 2A \\
&= 2\cos 2A\sin A + 2\sin A\cos A \\
&= 2\sin A\,(\cos 2A + \cos A) \\
&= 2\sin A\cdot 2\cos\frac{3A}{2}\cos\frac{A}{2}
\end{aligned}$$

$$= 4\sin A \cos\frac{A}{2}\cos\frac{3A}{2}. \qquad \blacksquare$$

§ Problem 8.1.26. $\tan 2A = (\sec 2A + 1)\sqrt{\sec^2 A - 1}$. ◊

§§ Solution.

$$\begin{aligned}
\tan 2A &= \frac{\sin 2A}{\cos 2A} = \frac{2\sin A\cos A}{\cos 2A} \\
&= \frac{2\cos^2 A}{\cos 2A}\cdot\frac{\sin A}{\cos A} = \frac{2\left(1-\sin^2 A\right)}{\cos 2A}\cdot\tan A \\
&= \frac{1-2\sin^2 A+1}{\cos 2A}\cdot\tan A = \frac{\cos 2A+1}{\cos 2A}\cdot\tan A \\
&= (\sec 2A+1)\sqrt{\sec^2 A-1}. \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.27. $\cos^3 2\theta + 3\cos 2\theta = 4\left(\cos^6\theta - \sin^6\theta\right)$. ◊

§§ Solution.

$$\begin{aligned}
\cos^3 2\theta + 3\cos 2\theta &= \cos 2\theta\left(\cos^2 2\theta + 3\right) \\
&= \cos 2\theta\left(4 - \sin^2 2\theta\right) \\
&= 4\cos 2\theta\left(1 - \sin^2\theta\cos^2\theta\right) \\
&= 4\cos 2\theta\left[\left(\cos^2\theta + \sin^2\theta\right)^2 - \sin^2\theta\cos^2\theta\right] \\
&= 4\left(\cos^2\theta - \sin^2\theta\right)\left(\cos^4\theta + \cos^2\theta\sin^2\theta + \sin^4\theta\right) \\
&= 4\left(\cos^6\theta - \sin^6\theta\right). \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.28. $1 + \cos^2 2\theta = 2\left(\cos^4\theta + \sin^4\theta\right)$. ◊

§§ Solution.

$$\begin{aligned}
1 + \cos^2 2\theta &= \left(\cos^2\theta + \sin^2\theta\right)^2 + \left(\cos^2\theta - \sin^2\theta\right)^2 \\
&= \cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta \\
&+ \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta \\
&= 2\left(\cos^4\theta + \sin^4\theta\right). \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.29. $\sec^2 A\,(1 + \sec 2A) = 2\sec 2A$. ◊

§§ Solution.

$$\begin{aligned}
\sec^2 A\,(1+\sec 2A) &= \frac{1}{\cos^2 A}\left(1 + \frac{1}{\cos 2A}\right) \\
&= \frac{1}{\cos^2 A}\left(\frac{\cos 2A + 1}{\cos 2A}\right) \\
&= \frac{1}{\cos^2 A}\cdot\frac{2\cos^2 A}{\cos 2A} \\
&= \frac{2}{\cos 2A} = 2\sec 2A. \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.30. $\operatorname{cosec} A - 2\cot 2A\cos A = 2\sin A$. ◊

§§ Solution.

$$\begin{aligned}
\operatorname{cosec} A - 2\cot 2A\cos A &= \frac{1}{\sin A} - \frac{2\cos 2A\cos A}{\sin 2A} \\
&= \frac{1}{\sin A} - \frac{2\cos 2A\cos A}{2\sin A\cos A} = \frac{1}{\sin A} - \frac{\cos 2A}{\sin A} \\
&= \frac{1-\cos 2A}{\sin A} = \frac{2\sin^2 A}{\sin A} = 2\sin A. \qquad \blacksquare
\end{aligned}$$

§ Problem 8.1.31. $\cot A = \frac{1}{2}\left(\cot\frac{A}{2} - \tan\frac{A}{2}\right).$ ◊

§§ Solution.

$$\begin{aligned}
\cot A = \frac{\cos A}{\sin A} &= \frac{\cos A}{2\sin\frac{A}{2}\cos\frac{A}{2}} \\
&= \frac{1}{2}\left(\frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right) \\
&= \frac{1}{2}\left(\frac{\cos^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}} - \frac{\sin^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right) \\
&= \frac{1}{2}\left(\cot\frac{A}{2} - \tan\frac{A}{2}\right).
\end{aligned}$$ ■

§ Problem 8.1.32. $\sin\alpha\sin(60^\circ - \alpha)\sin(60^\circ + \alpha) = \frac{1}{4}\sin 3\alpha.$ ◊

§§ Solution.

$$\begin{aligned}
&\sin\alpha\sin(60^\circ - \alpha)\sin(60^\circ + \alpha) \\
&\quad= \sin\alpha\left(\sin^2 60^\circ - \sin^2\alpha\right), \ \textit{by Ex. 2, Art. 93} \\
&\quad= \sin\alpha\left(\frac{3}{4} - \sin^2\alpha\right) = \frac{1}{4}\left(3\sin\alpha - 4\sin^3\alpha\right) = \frac{1}{4}\sin 3\alpha.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
&\sin\alpha\sin(60^\circ - \alpha)\sin(60^\circ + \alpha) \\
&\quad= \sin\alpha \times \frac{1}{2}\left(\cos 2\alpha - \cos 120^\circ\right) \\
&\quad= \frac{1}{2}\sin\alpha\left(\cos 2\alpha + \frac{1}{2}\right) \\
&\quad= \frac{1}{4}\left(2\sin\alpha\cos 2\alpha + \sin\alpha\right) \\
&\quad= \frac{1}{4}\left(\sin 3\alpha - \sin\alpha + \sin\alpha\right) = \frac{1}{4}\sin 3\alpha.
\end{aligned}$$ ■

§ Problem 8.1.33. $\cos\alpha\cos(60^\circ - \alpha)\cos(60^\circ + \alpha) = \frac{1}{4}\cos 3\alpha.$ ◊

§§ Solution.

$$\begin{aligned}
&\cos\alpha\cos(60^\circ - \alpha)\cos(60^\circ + \alpha) \\
&\quad= \cos\alpha\left(\cos^2 60^\circ - \sin^2\alpha\right), \ \textit{by Ex. 2, Art. 93} \\
&\quad= \cos\alpha\left(\cos^2\alpha - \sin^2 60^\circ\right) = \cos\alpha\left(\cos^2\alpha - \frac{3}{4}\right) \\
&\quad= \frac{1}{4}\left(4\cos^3\alpha - 3\cos\alpha\right) = \frac{1}{4}\cos 3\alpha.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
&\cos\alpha\cos(60^\circ - \alpha)\cos(60^\circ + \alpha) \\
&\quad= \cos\alpha \times \frac{1}{2}\left(\cos 120^\circ + \cos 2\alpha\right) = \frac{1}{2}\cos\alpha\left(\cos 2\alpha - \frac{1}{2}\right) \\
&\quad= \frac{1}{4}\left(2\cos\alpha\cos 2\alpha - \cos\alpha\right) \\
&\quad= \frac{1}{4}\left(\cos 3\alpha + \cos\alpha - \cos\alpha\right) = \frac{1}{4}\cos 3\alpha.
\end{aligned}$$ ■

§ Problem 8.1.34. $\cot\alpha + \cot(60^\circ + \alpha) - \cot(60^\circ - \alpha) = 3\cot 3\alpha.$ ◊

§§ Solution.

$$\begin{aligned}
&\cot\alpha + \cot(60^\circ + \alpha) - \cot(60^\circ - \alpha)\\
&= \frac{\cos\alpha}{\sin\alpha} + \frac{\cos(60^\circ + \alpha)}{\sin(60^\circ + \alpha)} - \frac{\cos(60^\circ - \alpha)}{\sin(60^\circ - \alpha)}\\
&= \frac{\cos\alpha}{\sin\alpha} + \frac{\sin(60^\circ - \alpha)\cos(60^\circ + \alpha) - \cos(60^\circ - \alpha)\sin(60^\circ + \alpha)}{\sin(60^\circ + \alpha)\sin(60^\circ - \alpha)}\\
&= \frac{\cos\alpha}{\sin\alpha} + \frac{\sin[(60^\circ - \alpha) - (60^\circ + \alpha)]}{\sin(60^\circ + \alpha)\sin(60^\circ - \alpha)}\\
&= \frac{\cos\alpha}{\sin\alpha} - \frac{2\sin 2\alpha}{2\sin(60^\circ + \alpha)\sin(60^\circ - \alpha)}\\
&= \frac{\cos\alpha}{\sin\alpha} - \frac{2\sin 2\alpha}{\cos 2\alpha - \cos 120^\circ}\\
&= \frac{\cos\alpha}{\sin\alpha} - \frac{2\times 4\sin\alpha\cos\alpha}{2\left(1 - 2\sin^2\alpha + \frac{1}{2}\right)}\\
&= \frac{\cos\alpha}{\sin\alpha} - \frac{8\sin^2\alpha\cos\alpha}{\sin\alpha\left(3 - 4\sin^2\alpha\right)}\\
&= \frac{3\cos\alpha - 4\sin^2\alpha\cos\alpha - 8\sin^2\alpha\cos\alpha}{3\sin\alpha - 4\sin^3\alpha}\\
&= \frac{3\left(\cos\alpha - 4\sin^2\alpha\cos\alpha\right)}{\sin 3\alpha}\\
&= \frac{3\left[\cos\alpha - 4\left(1 - \cos^2\alpha\right)\cos\alpha\right]}{\sin 3\alpha}\\
&= \frac{3\left(4\cos^3\alpha - 3\cos\alpha\right)}{\sin 3\alpha}\\
&= \frac{3\cos 3\alpha}{\sin 3\alpha} = 3\cot 3\alpha.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
&\cot\alpha + \cot(60^\circ + \alpha) - \cot(60^\circ - \alpha)\\
&= \frac{1}{\tan\alpha} + \frac{1}{\tan(60^\circ + \alpha)} - \frac{1}{\tan(60^\circ - \alpha)}\\
&= \frac{1}{\tan\alpha} + \frac{1 - \tan 60^\circ\tan\alpha}{\tan 60^\circ + \tan\alpha} - \frac{1 + \tan 60^\circ\tan\alpha}{\tan 60^\circ - \tan\alpha}\\
&= \frac{1}{\tan\alpha} + \frac{1 - \sqrt{3}\tan\alpha}{\sqrt{3} + \tan\alpha} - \frac{1 + \sqrt{3}\tan\alpha}{\sqrt{3} - \tan\alpha}\\
&= \frac{1}{\tan\alpha} - \frac{8\tan\alpha}{3 - \tan^2\alpha} = \frac{3 - \tan^2\alpha - 8\tan^2\alpha}{\tan\alpha(3 - \tan^2\alpha)}\\
&= \frac{3\left(1 - 3\tan^2\alpha\right)}{3\tan\alpha - \tan^3\alpha}\\
&= 3\times\frac{1}{\tan 3\alpha} = 3\cot 3\alpha.
\end{aligned}$$ ■

§ Problem 8.1.35. $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ = \frac{1}{16}.$ ◊

§§ Solution.

$$\cos 20^\circ\cos 40^\circ\cos 80^\circ = \frac{1}{2}\cos 20^\circ\left(\cos 120^\circ + \cos 40^\circ\right)$$

$$= \frac{1}{2}\cos 20^\circ \left[-\frac{1}{2} + \left(2\cos^2 20^\circ - 1\right)\right]$$
$$= \frac{1}{2}\cos 20^\circ \left(2\cos^2 20^\circ - \frac{3}{2}\right)$$
$$= \frac{1}{4}\left(4\cos^3 20^\circ - 3\cos 20^\circ\right) = \frac{1}{4}\cos 60^\circ.$$
$$\therefore \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{4}\cos^2 60^\circ = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}. \quad \blacksquare$$

§ Problem 8.1.36. $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \dfrac{3}{16}$. $\Diamond$

§§ Solution.

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$$
$$= \frac{1}{2}\sin 20^\circ \left(\cos 40^\circ - \cos 120^\circ\right)$$
$$= \frac{1}{2}\sin 20^\circ \left[1 - 2\sin^2 20^\circ - \left(-\frac{1}{2}\right)\right]$$
$$= \frac{1}{2}\sin 20^\circ \left(\frac{3}{2} - 2\sin^2 20^\circ\right)$$
$$= \frac{1}{4}\left(3\sin 20^\circ - 4\sin^3 20^\circ\right) = \frac{1}{4}\sin 60^\circ.$$
$$\therefore \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{1}{4}\sin^2 60^\circ = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}. \quad \blacksquare$$

§ Problem 8.1.37. $\cos 4\alpha = 1 - 8\cos^2\alpha + 8\cos^4\alpha.$ $\Diamond$

§§ Solution.

$$\cos 4\alpha = 2\cos^2 2\alpha - 1 = 2\left(2\cos^2\alpha - 1\right)^2 - 1$$
$$= 2\left(4\cos^4\alpha - 4\cos^2\alpha + 1\right) - 1$$
$$= 1 - 8\cos^2\alpha + 8\cos^4\alpha. \quad \blacksquare$$

§ Problem 8.1.38. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A.$ $\Diamond$

§§ Solution.

$$\sin 4A = 2\sin 2A\cos 2A = 4\sin A\cos A\left(\cos^2 A - \sin^2 A\right)$$
$$= 4\sin A\cos^3 A - 4\cos A\sin^3 A. \quad \blacksquare$$

§ Problem 8.1.39. $\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1.$ $\Diamond$

§§ Solution.

$$\cos 6\alpha = 2\cos^2 3\alpha - 1 = 2\left(4\cos^3\alpha - 3\cos\alpha\right)^2 - 1$$
$$= 2\left(16\cos^6\alpha - 24\cos^4\alpha + 9\cos^2\alpha\right) - 1$$
$$= 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1. \quad \blacksquare$$

§ Problem 8.1.40. $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A.$ $\Diamond$

§§ Solution. By *Art.* 107, we have

$$\tan 3A = \tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A\tan 2A}.$$
$$\therefore \tan 3A\left(1 - \tan A\tan 2A\right) = \tan A + \tan 2A$$
$$\therefore \tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A. \quad \blacksquare$$

§ Problem 8.1.41.

$$\frac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)$$
$$\left(2\cos 2^2\theta-1\right)\ldots\left(2\cos 2^{n-1}\theta-1\right).$$

◊

§§ Solution.
We have

$$\begin{aligned}(2\cos\theta+1)(2\cos\theta-1) &= 4\cos^2\theta-1\\ &= 2(1+\cos 2\theta)-1\\ &= 2\cos 2\theta+1.\end{aligned}$$

Similarly,

$$\begin{aligned}(2\cos 2\theta+1)(2\cos 2\theta-1) &= 2\cos 2^2\theta+1.\\ \left(2\cos 2^2\theta+1\right)\left(2\cos 2^2\theta-1\right) &= 2\cos 2^3\theta+1.\\ \ldots &= \ldots\\ \left(2\cos 2^{n-1}\theta+1\right)\left(2\cos 2^{n-1}\theta-1\right) &= 2\cos 2^n\theta+1.\end{aligned}$$

Hence,by multiplication and canceling, we have

$$(2\cos\theta+1)(2\cos\theta-1)(2\cos 2\theta-1)\left(2\cos 2^2\theta-1\right)$$
$$\ldots\left(2\cos 2^{n-1}\theta-1\right) = 2\cos 2^n\theta+1.$$
$$\therefore \frac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)$$
$$\left(2\cos 2^2\theta-1\right)\ldots\left(2\cos 2^{n-1}\theta-1\right).$$

■

8.2 Submultiple Angles

§ Problem 8.2.1. *If* $\sin\theta = \dfrac{1}{2}$ *and* $\sin\phi = \dfrac{1}{3}$, *find the values of* $\sin(\theta+\phi)$ *and* $\sin(2\theta+2\phi)$. ◊

§§ Solution.

$$\sin\theta = \frac{1}{2};\ \therefore \cos\theta = \sqrt{1-\left(\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}$$
$$\sin\phi = \frac{1}{3};\ \therefore \cos\phi = \sqrt{1-\left(\frac{1}{3}\right)^2} = \pm\frac{2\sqrt{2}}{3}$$
$$\begin{aligned}\therefore \sin(\theta+\phi) &= \sin\theta\cos\phi+\cos\theta\sin\phi\\ &= \frac{1}{2}\left(\pm\frac{2\sqrt{2}}{3}\right)+\left(\pm\frac{\sqrt{3}}{2}\right)\frac{1}{3}\\ &= \frac{\pm 2\sqrt{2}\pm\sqrt{3}}{6}.\end{aligned}$$

Again,

$$\cos 2\theta = 1-2\sin^2\theta = 1-2\left(\frac{1}{2}\right)^2 = 1-\frac{1}{2} = \frac{1}{2}$$
$$\therefore \sin 2\theta = \sqrt{1-\cos^2 2\theta} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}$$
$$\text{also, } \cos 2\phi = 1-2\sin^2\phi = 1-2\left(\frac{1}{3}\right)^2 = 1-\frac{2}{9} = \frac{7}{9}$$

$$\therefore \sin 2\phi = \sqrt{1 - \left(\frac{7}{9}\right)^2} = \pm\frac{4\sqrt{2}}{9}$$

$$\begin{aligned}\therefore \sin(2\theta + 2\phi) &= \sin 2\theta \cos 2\phi + \cos 2\theta \sin 2\phi \\ &= \pm\frac{\sqrt{3}}{2} \cdot \frac{7}{9} + \frac{1}{2}\left(\pm\frac{4\sqrt{2}}{9}\right) = \frac{\pm 7\sqrt{3} \pm 4\sqrt{2}}{18}.\end{aligned}$$ ■

§ **Problem 8.2.2.** *The tangent of an angle is* $2 \cdot 4$. *Find its cosecant, the cosecant of half the angle, and the cosecant of the supplement of double the angle.* ◊

§§ **Solution**. Cf. the figures of *Arts.* 31 and 50. Let θ be the angle and let the length OM be unity and let the corresponding length of MP be 2.4. Then

$$OP = \sqrt{OM^2 + MP^2} = \sqrt{1 + (2.4)^2} = \sqrt{6.76} = 2.6.$$

$$\therefore \text{cosec } \theta = \frac{OP}{MP} = \pm\frac{2.6}{2.4} = \pm\frac{13}{12}$$

$$\text{and } \cos\theta = \frac{OM}{MP} = \pm\frac{1}{2.6} = \pm\frac{5}{13}.$$

Otherwise thus :

Substitute in the formulae

$$\text{cosec } \theta = \frac{\sqrt{1 + \tan^2\theta}}{\tan\theta} \text{ and } \cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}}.$$

$$\begin{aligned}\therefore \sin\frac{\theta}{2} &= \pm\sqrt{\frac{1}{2}(1 - \cos\theta)} = \pm\sqrt{\frac{1}{2}\left(1 \mp \frac{5}{13}\right)} \\ &= \pm\frac{2}{\sqrt{13}} \text{ or } \pm\frac{3}{\sqrt{13}} \\ \therefore \text{cosec } \frac{\theta}{2} &= \pm\frac{\sqrt{13}}{2} \text{ or } \pm\frac{\sqrt{13}}{3}.\end{aligned}$$

Again,

$$\begin{aligned}\text{cosec } (180^\circ - 2\theta) &= \frac{1}{\sin(180^\circ - 2\theta)} = \frac{1}{\sin 2\theta} \\ &= \frac{1}{2\sin\theta\cos\theta} = \frac{1}{2\left(\pm\frac{12}{13}\right)\left(\pm\frac{5}{13}\right)} = \frac{169}{120}.\end{aligned}$$ ■

§ **Problem 8.2.3.** *If* $\cos\alpha = \frac{11}{61}$ *and* $\sin\beta = \frac{4}{5}$, *find the values of* $\sin^2\frac{\alpha - \beta}{2}$ *and* $\cos^2\frac{\alpha + \beta}{2}$, *the angles* α *and* β *being positive acute angles.* ◊

§§ **Solution**. If $\cos\alpha = \frac{11}{61}$, then

$$\begin{aligned}\sin\alpha &= \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \left(\frac{11}{61}\right)^2} \\ &= \sqrt{\frac{(61)^2 - (11)^2}{(61)^2}} = \sqrt{\frac{(61 + 11)(61 - 11)}{(61)^2}} \\ &= \sqrt{\frac{3600}{(61)^2}} = \frac{60}{61}.\end{aligned}$$

If $\sin\beta = \dfrac{4}{5}$, then

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$$

$$\begin{aligned}
\therefore \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&= \frac{11}{61}\cdot\frac{3}{5} + \frac{60}{61}\cdot\frac{4}{5} \\
&= \frac{33+240}{305} = \frac{273}{305}
\end{aligned}$$

$$\therefore \sin^2\frac{\alpha-\beta}{2} = \frac{1}{2}\left[1-\cos(\alpha-\beta)\right] = \frac{1}{2}\times\frac{32}{305} = \frac{16}{305}.$$

Also,

$$\begin{aligned}
\cos^2\frac{\alpha+\beta}{2} &= \frac{1}{2}\left[1+\cos(\alpha+\beta)\right] = \frac{1}{2}\left[1+\frac{33-240}{305}\right] \\
&= \frac{1}{2}\left[1-\frac{207}{305}\right] = \frac{1}{2}\times\frac{98}{305} = \frac{49}{305}.
\end{aligned}$$

■

§ Problem 8.2.4. *If* $\cos\alpha = \dfrac{3}{5}$ *and* $\cos\beta = \dfrac{4}{5}$*, find the value of* $\cos\dfrac{\alpha-\beta}{2}$*, the angles* α *and* β *being positive acute angles.* ◊

§§ Solution. If $\cos\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{4}{5}$, then

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+\frac{3}{5}}{2}} = \sqrt{\frac{8}{10}} = \frac{2}{\sqrt{5}}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{1+\cos\beta}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{2}{10}} = \frac{1}{\sqrt{5}}$$

$$\sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}}.$$

$$\begin{aligned}
\therefore \cos\frac{\alpha-\beta}{2} &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2} \\
&= \frac{2}{\sqrt{5}}\cdot\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{10}} = \frac{7}{\sqrt{50}} = \frac{7}{5\sqrt{2}}.
\end{aligned}$$

■

§ Problem 8.2.5. *Given* $\sec\theta = 1\frac{1}{4}$*, find* $\tan\dfrac{\theta}{2}$ *and* $\tan\theta$*. Verify by a graph.* ◊

§§ Solution. If $\sec\theta = 1\frac{1}{4} = \dfrac{5}{4}$, then $\cos\theta = \dfrac{4}{5}$.

$$\therefore \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \pm\sqrt{\frac{1-\frac{4}{5}}{1+\frac{4}{5}}} = \pm\frac{1}{3}.$$

Also,

$$\tan\theta = \pm\sqrt{\sec^2\theta - 1} = \pm\sqrt{\frac{25}{16} - 1} = \pm\frac{3}{4}. \qquad \blacksquare$$

§ Problem 8.2.6. *If* $\cos A = \cdot 28$, *find the value of* $\tan\frac{A}{2}$ *and explain the resulting ambiguity.* $\Diamond$

§§ Solution.

$$\begin{aligned}\tan\frac{A}{2} &= \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \pm\sqrt{\frac{1-.28}{1+.28}}\\ &= \pm\sqrt{\frac{.72}{1.28}} = \pm\sqrt{\frac{72}{128}} = \pm\sqrt{\frac{9}{16}} = \pm\frac{3}{4}.\end{aligned}$$

Since $\cos A = \cos(2n\pi \pm A)$, any equation giving $\tan\frac{A}{2}$ in terms of $\cos A$ will give also $\tan\left(n\pi \pm \frac{A}{2}\right)$, i.e. $\tan\left(\pm\frac{A}{2}\right)$, i.e. $\pm\tan\frac{A}{2}$. $\blacksquare$

§ Problem 8.2.7. *Find the values of*

(1) $\sin 7\frac{1}{2}^\circ$,

(2) $\cos 7\frac{1}{2}^\circ$,

(3) $\tan 22\frac{1}{2}^\circ$, *and*

(4) $\tan 11\frac{1}{4}^\circ$.

$\Diamond$

§§ Solution. (1)

$$\begin{aligned}\sin 7\frac{1}{2}^\circ &= \sqrt{\frac{1}{2}(1-\cos 15^\circ)} = \sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{3}+1}{2\sqrt{2}}\right)}\\ &= \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}} = \sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}}\\ &= \frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}\end{aligned}$$

(2) $\cos 7\frac{1}{2}^\circ = \sqrt{\frac{1}{2}(1+\cos 15^\circ)} = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$,

(3) $\tan 22\frac{1}{2}^\circ = \frac{\sqrt{1+\tan^2 45^\circ} - 1}{\tan 45^\circ} = \frac{\sqrt{1+1}-1}{1} = \sqrt{2} - 1.$

(4)

$$\begin{aligned}\tan 11\frac{1}{4}^\circ &= \frac{\sqrt{1+\tan^2 22\frac{1}{2}^\circ} - 1}{\tan 22\frac{1}{2}^\circ} = \frac{\sqrt{4-2\sqrt{2}}}{\sqrt{2}-1} - \frac{1}{\sqrt{2}-1}\\ &= \sqrt{\left(4-2\sqrt{2}\right)\left(\sqrt{2}+1\right)^2} - \left(\sqrt{2}+1\right)\end{aligned}$$

$$= \sqrt{\left(4 - 2\sqrt{2}\right)\left(3 + 2\sqrt{2}\right)} - \left(\sqrt{2} + 1\right)$$
$$= \sqrt{4 + 2\sqrt{2}} - \left(\sqrt{2} + 1\right). \qquad \blacksquare$$

§ Problem 8.2.8. *If* $\sin\theta + \sin\phi = a$ *and* $\cos\theta + \cos\phi = b$*, find the value of* $\tan\dfrac{\theta - \phi}{2}$. ◊

§§ Solution. Squaring and adding the given equations, we have

$$\left(\sin^2\theta + \cos^2\theta\right) + \left(\sin^2\phi + \cos^2\phi\right) + 2\left(\sin\theta\sin\phi + \cos\theta\cos\phi\right) = a^2 + b^2$$
$$\therefore 1 + 1 + 2\cos(\theta - \phi) = a^2 + b^2$$
$$\therefore 2\left[1 + \cos(\theta - \phi)\right] = a^2 + b^2$$
$$\therefore \cos(\theta - \phi) = \frac{a^2 + b^2 - 2}{2}.$$
$$\therefore \tan\frac{\theta - \phi}{2} = \pm\sqrt{\frac{1 - \cos(\theta - \phi)}{1 + \cos(\theta - \phi)}}, \ [\textit{by Art. } 110,\ (3)],$$
$$= \pm\sqrt{\frac{2 - \left(a^2 + b^2 - 2\right)}{2 + \left(a^2 + b^2 - 2\right)}} = \pm\sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}. \qquad \blacksquare$$

Prove that

§ Problem 8.2.9. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\dfrac{\alpha + \beta}{2}$. ◊

§§ Solution.

$$(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$
$$= \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$
$$= \left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) + 2\left(\cos\alpha\cos\beta - \sin\alpha\sin\beta\right)$$
$$= 1 + 1 + 2\cos(\alpha + \beta) = 2\left[1 + \cos(\alpha + \beta)\right] = 4\cos^2\frac{\alpha + \beta}{2}. \qquad \blacksquare$$

§ Problem 8.2.10. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\dfrac{\alpha - \beta}{2}$. ◊

§§ Solution.

$$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$$
$$= \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta$$
$$= 2\left[1 + \cos(\alpha - \beta)\right] = 4\cos^2\frac{\alpha - \beta}{2}. \qquad \blacksquare$$

§ Problem 8.2.11. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\dfrac{\alpha - \beta}{2}$. ◊

§§ Solution.

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$
$$= \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$
$$= 2 - 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right) = 2 - 2\cos(\alpha - \beta)$$
$$= 2\left[1 - \cos(\alpha - \beta)\right] = 4\sin^2\frac{\alpha - \beta}{2}. \qquad \blacksquare$$

§ Problem 8.2.12. $\sin A = \dfrac{2\tan\dfrac{A}{2}}{1 + \tan^2\dfrac{A}{2}}$. ◊

§§ Solution. See *Art.* 109. ■

§ Problem 8.2.13. $\cos A = \dfrac{1-\tan^2\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}}$. ◊

§§ Solution. See *Art*. 109. ■

§ Problem 8.2.14. $\sec\left(\dfrac{\pi}{4}+\theta\right)\sec\left(\dfrac{\pi}{4}-\theta\right) = 2\sec 2\theta$. ◊

§§ Solution.

$$\begin{aligned}
\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) &= \frac{1}{\cos\dfrac{\pi}{4}+\theta}\cdot\frac{1}{\cos\dfrac{\pi}{4}-\theta}\\
&= \frac{1}{\dfrac{1}{2}\left(\cos\dfrac{\pi}{2}+\cos 2\theta\right)} = \frac{2}{\cos 2\theta}\ \left[\because \cos\frac{\pi}{2}=0\right]\\
&= 2\sec 2\theta.
\end{aligned}$$

Otherwise thus :

$$\because \cos\left(\frac{\pi}{4}+\theta\right) = \sin\left[\frac{\pi}{2}-\left(\frac{\pi}{4}+\theta\right)\right] = \sin\left(\frac{\pi}{4}-\theta\right)$$

We have

$$\begin{aligned}
\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) &= \frac{2}{2\sin\left(\dfrac{\pi}{4}-\theta\right)\cos\left(\dfrac{\theta}{4}-\theta\right)}\\
&= \frac{2}{\sin\left(\dfrac{\pi}{2}-2\theta\right)} = \frac{2}{\cos 2\theta} = 2\sec 2\theta.
\end{aligned}$$

Otherwise thus :

By *Ex*. 2, *Art*. 93, we have

$$\cos\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right) = \cos^2\frac{\pi}{4}-\sin^2\theta$$

$$\begin{aligned}
\therefore \sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) &= \frac{1}{\cos^2\dfrac{\pi}{4}-\sin^2\theta} = \frac{1}{\dfrac{1}{2}-\sin^2\theta}\\
&= \frac{2}{1-2\sin^2\theta} = \frac{2}{\cos 2\theta} = 2\sec 2\theta.
\end{aligned}$$ ■

§ Problem 8.2.15. $\tan\left(45^\circ+\dfrac{A}{2}\right) = \sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A+\tan A$. ◊

§§ Solution.

$$\begin{aligned}
\tan\left(45^\circ+\frac{A}{2}\right) &= \frac{1+\tan\dfrac{A}{2}}{1-\tan\dfrac{A}{2}}\ [By\ Art.\ 100]\\
&= \frac{1+\dfrac{\sin\dfrac{A}{2}}{\cos\dfrac{A}{2}}}{1-\dfrac{\sin\dfrac{A}{2}}{\cos\dfrac{A}{2}}} = \sqrt{\frac{\left(\cos\dfrac{A}{2}+\sin\dfrac{A}{2}\right)^2}{\left(\cos\dfrac{A}{2}-\sin\dfrac{A}{2}\right)^2}}
\end{aligned}$$

$$= \sqrt{\frac{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} + 2\cos\frac{A}{2}\sin\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} - 2\cos\frac{A}{2}\sin\frac{A}{2}}}$$
$$= \sqrt{\frac{1+\sin A}{1-\sin A}} = \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}} = \frac{1+\sin A}{\cos A}$$
$$= \frac{1}{\cos A} + \frac{\sin A}{\cos A} = \sec A + \tan A.$$

Otherwise thus :

$$\tan\left(45^\circ + \frac{A}{2}\right) = \frac{\sin\left(45^\circ + \frac{A}{2}\right)}{\cos\left(45^\circ + \frac{A}{2}\right)}$$
$$= \sqrt{\frac{2\sin^2\left(45^\circ + \frac{A}{2}\right)}{2\cos^2\left(45^\circ + \frac{A}{2}\right)}} = \sqrt{\frac{1-\cos(90^\circ + A)}{1+\cos(90^\circ + A)}}$$
$$= \sqrt{\frac{1+\sin A}{1-\sin A}} = \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}} = \frac{1+\sin A}{\cos A}$$
$$= \frac{1}{\cos A} + \frac{\sin A}{\cos A} = \sec A + \tan A. \qquad \blacksquare$$

§ Problem 8.2.16. $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A.$ ♢

§§ Solution.

$$\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right)$$
$$= \frac{1}{2}\left[1 - \cos\left(\frac{\pi}{4} + A\right)\right] - \frac{1}{2}\left[1 - \cos\left(\frac{\pi}{4} - A\right)\right]$$
$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{4} - A\right) - \cos\left(\frac{\pi}{4} + A\right)\right]$$
$$= \frac{1}{2}\left[2\sin\frac{\pi}{4}\sin A\right] = \sin\frac{\pi}{4}\sin A = \frac{1}{\sqrt{2}}\sin A. \qquad \blacksquare$$

§ Problem 8.2.17. $\cos^2\alpha + \cos^2(\alpha + 120^\circ) + \cos^2(\alpha - 120^\circ) = \frac{3}{2}.$ ♢

§§ Solution.

$$\cos^2\alpha + \cos^2(\alpha + 120^\circ) + \cos^2(\alpha - 120^\circ)$$
$$= \frac{1}{2}\left[1 + \cos 2\alpha + 1 + \cos(2\alpha + 240^\circ) + 1 + \cos(2\alpha - 240^\circ)\right]$$
$$= \frac{1}{2}(3 + \cos 2\alpha + 2\cos 2\alpha\cos 240^\circ)$$
$$= \frac{1}{2}(3 + \cos 2\alpha - \cos 2\alpha) \left[\because \cos 240^\circ = -\frac{1}{2}\right]$$
$$= \frac{3}{2}. \qquad \blacksquare$$

§ Problem 8.2.18. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}.$ ♢

§§ Solution.

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$$
$$=\left(\frac{1+\cos\frac{\pi}{4}}{2}\right)^2+\left(\frac{1+\cos\frac{3\pi}{4}}{2}\right)^2+\left(\frac{1+\cos\frac{5\pi}{4}}{2}\right)^2$$
$$+\left(\frac{1+\cos\frac{7\pi}{4}}{2}\right)^2$$
$$=\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^2$$
$$=2\left[\frac{3+2\sqrt{2}}{8}+\frac{3-2\sqrt{2}}{8}\right]=2\times\frac{3}{4}=\frac{3}{2}.$$ ■

§ Problem 8.2.19. $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}.$ ◊

§§ Solution.

$$\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$$
$$=\left(\frac{1-\cos\frac{\pi}{4}}{2}\right)^2+\left(\frac{1-\cos\frac{3\pi}{4}}{2}\right)^2+\left(\frac{1-\cos\frac{5\pi}{4}}{2}\right)^2$$
$$+\left(\frac{1-\cos\frac{7\pi}{4}}{2}\right)^2$$
$$=\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^2$$
$$=2\left[\frac{3-2\sqrt{2}}{8}+\frac{3+2\sqrt{2}}{8}\right]=2\times\frac{3}{4}=\frac{3}{2}.$$ ■

§ Problem 8.2.20.

$$\cos 2\theta\cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)$$
$$=\cos(2\theta+2\phi).$$ ◊

§§ Solution.

$$\cos 2\theta\cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)$$
$$=\frac{1}{2}\left[\cos(2\theta+2\phi)+\cos(2\theta-2\phi)+1-\cos 2(\theta-\phi)-1+\cos 2(\theta+\phi)\right]$$
$$=\frac{1}{2}\times 2\cos(2\theta+2\phi)=\cos 2(\theta+\phi).$$ ■

§ Problem 8.2.21.

$$(\tan 4A+\tan 2A)\left(1-\tan^2 3A\tan^2 A\right)=2\tan 3A\sec^2 A.$$ ◊

§§ Solution.

$$(\tan 4A+\tan 2A)\left(1-\tan^2 3A\tan^2 A\right)$$
$$=\left(\frac{\sin 4A}{\cos 4A}+\frac{\sin 2A}{\cos 2A}\right)\left(1-\frac{\sin^2 3A\sin^2 A}{\cos^2 3A\cos^2 A}\right)$$

$$= \left(\frac{\sin 4A \cos 2A + \cos 4A \sin 2A}{\cos 4A \cos 2A}\right)\left(\frac{\cos^2 3A \cos^2 A - \sin^2 3A \sin^2 A}{\cos^2 3A \cos^2 A}\right)$$
$$= \left[\frac{\sin(4A+2A)}{\cos 4A \cos 2A}\right]$$
$$\left[\frac{(\cos 3A \cos A - \sin 3A \sin A)(\cos 3A \cos A + \sin 3A \sin A)}{\cos^2 3A \cos^2 A}\right]$$
$$= \left[\frac{\sin 6A}{\cos 4A \cos 2A}\right]\left[\frac{\cos(3A+A)\cos(3A-A)}{\cos^2 3A \cos^2 A}\right]$$
$$= \frac{\sin 6A}{\cos 4A \cos 2A} \cdot \frac{\cos 4A \cos 2A}{\cos^2 3A \cos^2 A}$$
$$= \frac{\sin 6A}{\cos^2 3A \cos^2 A} = \frac{2 \sin 3A \cos 3A}{\cos^2 3A \cos^2 A}$$
$$= \frac{2 \sin 3A}{\cos 3A} \cdot \frac{1}{\cos^2 A} = 2 \tan 3A \sec^2 A.$$ ■

§ Problem 8.2.22.
$$\left(1 + \tan\frac{\alpha}{2} - \sec\frac{\alpha}{2}\right)\left(1 + \tan\frac{\alpha}{2} + \sec\frac{\alpha}{2}\right)$$
$$= \sin\alpha \sec^2\frac{\alpha}{2}.$$ ◊

§§ Solution.
$$\left(1 + \tan\frac{\alpha}{2} - \sec\frac{\alpha}{2}\right)\left(1 + \tan\frac{\alpha}{2} + \sec\frac{\alpha}{2}\right)$$
$$= \left(1 + \tan\frac{\alpha}{2}\right)^2 - \sec^2\frac{\alpha}{2}$$
$$= 1 + \tan^2\frac{\alpha}{2} + 2\tan\frac{\alpha}{2} - \left(1 + \tan^2\frac{\alpha}{2}\right) = 2\tan\frac{\alpha}{2}$$
$$= \frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \cdot \frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\sin\alpha}{\cos^2\frac{\alpha}{2}}$$
$$= \sin\alpha \cdot \frac{1}{\cos^2\frac{\alpha}{2}} = \sin\alpha \sec^2\frac{\alpha}{2}.$$ ■

Find the proper signs to be applied to the radicals in the three following formulae.

§ Problem 8.2.23. $2\cos\frac{A}{2} = \pm\sqrt{1-\sin A} \pm \sqrt{1+\sin A}$, *when* $\frac{A}{2} = 278°$. ◊

§§ Solution. If $\frac{A}{2} = 278°$, then $\sin\frac{A}{2}$ is negative and numerically $> \cos\frac{A}{2}$.

$$\therefore \sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \text{ and}$$
$$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A}.$$

Hence, by subtraction, we have

$$2\cos\frac{A}{2} = +\sqrt{1-\sin A} - \sqrt{1+\sin A}.$$ ■

§ Problem 8.2.24. $2\sin\frac{A}{2} = \pm\sqrt{1-\sin A} \pm \sqrt{1+\sin A}$, *when* $\frac{A}{2} = \frac{19\pi}{11}$. ◊

§§ Solution. If $\dfrac{A}{2} = \dfrac{19\pi}{11}$, then, since $\dfrac{19\pi}{11}$ is slightly less than $\dfrac{7\pi}{4}$, $\sin\dfrac{A}{2}$ is negative and numerically $> \cos\dfrac{A}{2}$. Hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \textit{ and}$$
$$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A}.$$

Hence, by addition, we have

$$2\sin\frac{A}{2} = -\sqrt{1-\sin A} - \sqrt{1+\sin A}.$$ ■

§ Problem 8.2.25. $2\cos\dfrac{A}{2} = \pm\sqrt{1-\sin A} \pm \sqrt{1+\sin A}$, *when* $\dfrac{A}{2} = -140°$. ◊

§§ Solution. If $\dfrac{A}{2} = -140°$, then $\sin\dfrac{A}{2}$ and $\cos\dfrac{A}{2}$ are both negative, but $\cos\dfrac{A}{2}$ is numerically the greater. Hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \textit{ and}$$
$$\sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1-\sin A}.$$

Hence, by subtraction, we have

$$2\cos\frac{A}{2} = -\sqrt{1-\sin A} - \sqrt{1+\sin A}.$$ ■

§ Problem 8.2.26. *If* $A = 340°$, *prove that*

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}, \textit{ and}$$
$$2\cos\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$ ◊

§§ Solution. If $A = 340°$, then $\dfrac{A}{2} = 170°$ and $\cos\dfrac{A}{2}$ is negative and numerically $> \sin\dfrac{A}{2}$. Hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \textit{ and}$$
$$\sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1-\sin A}.$$

Hence, by addition, we have

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$$

and, by subtraction, we have

$$2\cos\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$ ■

§ Problem 8.2.27. *If* $A = 460°$, *prove that*

$$2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}.$$ ◊

§§ Solution. If $A = 460°$, then $\dfrac{A}{2} = 230°$ and $\sin\dfrac{A}{2}$ is negative and numerically $> \cos\dfrac{A}{2}$. Hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \textit{ and}$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A},$$

Hence, by subtraction, we have

$$2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}. \qquad \blacksquare$$

§ Problem 8.2.28. *If $A = 580°$, prove that*

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}. \qquad \Diamond$$

§§ Solution. If $A = 580°$ then $\frac{A}{2} = 290°$ and $\sin\frac{A}{2}$ is negative and numerically $> \cos\frac{A}{2}$. Hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \ \textit{and}$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A}.$$

Hence, by addition, we have

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}. \qquad \blacksquare$$

§ Problem 8.2.29. *Within what respective limits must $\frac{A}{2}$ lie when*

(1) $2\sin\frac{A}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}$

(2) $2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$

(3) $2\sin\frac{A}{2} = +\sqrt{1+\sin A} - \sqrt{1-\sin A}$

(4) $2\cos\frac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}.$

$\Diamond$

§§ Solution. (1) $2\sin\frac{A}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}$,

when $\sin\frac{A}{2} + \cos\frac{A}{2} = +\sqrt{1+\sin A}$

and $\sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1-\sin A}$

i.e. when $\sin\frac{A}{2}$ is positive and numerically $> \cos\frac{A}{2}$

i.e. when $\frac{A}{2}$ lies between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$.

(2) $2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$

when $\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}$

and $\sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1-\sin A}$

i.e. when $\cos\frac{A}{2}$ is positive and numerically $> \sin\frac{A}{2}$

i.e. when $\frac{A}{2}$ lies between $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$.

(3) $2\sin\dfrac{A}{2} = +\sqrt{1+\sin A} - \sqrt{1-\sin A}$

when $\qquad \sin\dfrac{A}{2} + \cos\dfrac{A}{2} = +\sqrt{1+\sin A}$

and $\qquad \sin\dfrac{A}{2} - \cos\dfrac{A}{2} = -\sqrt{1-\sin A}$

i.e. when $\cos\dfrac{A}{2}$ is positive and numerically $> \sin\dfrac{A}{2}$

i.e. when $\dfrac{A}{2}$ lies between $2n\pi - \dfrac{\pi}{4}$ and $2n\pi + \dfrac{\pi}{4}$.

(4) $2\cos\dfrac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}$.

when $\qquad \sin\dfrac{A}{2} + \cos\dfrac{A}{2} = +\sqrt{1+\sin A}$

and $\qquad \sin\dfrac{A}{2} - \cos\dfrac{A}{2} = +\sqrt{1-\sin A}$

i.e. when $\sin\dfrac{A}{2}$ is positive and numerically $> \cos\dfrac{A}{2}$

i.e. when $\dfrac{A}{2}$ lies between $2n\pi + \dfrac{\pi}{4}$ and $2n\pi + \dfrac{3\pi}{4}$. ■

§ Problem 8.2.30. *In the formula*

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$$

find within what limits $\dfrac{A}{2}$ *must lie when*

(1) the two positive signs are taken

(2) the two negative signs are taken, and

(3) the first sign is negative and the second positive. ◊

§§ Solution. (1) $2\cos\dfrac{A}{2} = +\sqrt{1+\sin A} + \sqrt{1-\sin A}$,

when $\qquad \sin\dfrac{A}{2} + \cos\dfrac{A}{2} = +\sqrt{1+\sin A}$

and $\qquad \sin\dfrac{A}{2} - \cos\dfrac{A}{2} = -\sqrt{1-\sin A}$

i.e. when $\cos\dfrac{A}{2}$ is positive and numerically $> \sin\dfrac{A}{2}$

i.e. when $\dfrac{A}{2}$ lies between $2n\pi - \dfrac{\pi}{4}$ and $2n\pi + \dfrac{\pi}{4}$.

(2) $2\cos\dfrac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$

when $\qquad \sin\dfrac{A}{2} + \cos\dfrac{A}{2} = -\sqrt{1+\sin A}$

and $\qquad \sin\dfrac{A}{2} - \cos\dfrac{A}{2} = +\sqrt{1-\sin A}$

i.e. when $\cos\dfrac{A}{2}$ is negative and numerically $> \sin\dfrac{A}{2}$

i.e. when $\dfrac{A}{2}$ lies between $2n\pi + \dfrac{3\pi}{4}$ and $2n\pi + \dfrac{5\pi}{4}$.

(3) $2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$

when $\qquad \sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}$

and $\qquad \sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A}$

i.e. when $\sin\frac{A}{2}$ is positive and numerically $> \cos\frac{A}{2}$

i.e. when $\frac{A}{2}$ lies between $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$. ■

§ Problem 8.2.31. *Prove that the sine is algebraically less than the cosine for any angle between* $2n\pi - \frac{3\pi}{4}$ *and* $2n\pi + \frac{\pi}{4}$ *where* n *is any integer.* ◊

§§ Solution. We have

$$\sin A - \cos A = \sqrt{2}\left(\sin A\frac{1}{\sqrt{2}} - \cos A\frac{1}{\sqrt{2}}\right)$$
$$= \sqrt{2}\left(\sin A\cos\frac{\pi}{4} - \cos A\sin\frac{\pi}{4}\right) = \sqrt{2}\sin\left(A - \frac{\pi}{4}\right).$$

Hence $\sin A - \cos A$ is negative, i.e. $\cos A > \sin A$ when $\sin\left(A - \frac{\pi}{4}\right)$ is negative, i.e. when $\left(A - \frac{\pi}{4}\right)$ lies between $2n\pi - \pi$ and $2n\pi$, in the third and fourth quadrants, i.e. when A lies between $2n\pi - \frac{3\pi}{4}$ and $2n\pi + \frac{\pi}{4}$.

Otherwise thus :

Since $2n\pi$ is equivalent to n complete revolutions of the revolving line, the angle $2n\pi - \frac{3\pi}{4}$ corresponds to the position of the revolving line at OR, [cf. figure of *Art.* 116], which bisects the third quadrant and the angle $2n\pi + \frac{\pi}{4}$ to the position of the revolving line at OP, which bisects the first quadrant.

By *Arts.* 53, 54 and 55, we see that from R to B' the sine and the cosine are both negative and the sine numerically $>$ the cosine; hence the sine is algebraically $<$ the cosine.

From B' to A the sine is negative and the cosine is positive; hence the sine is algebraically $<$ the cosine.

From A to P the sine and the cosine are both positive and the sine numerically $<$ the cosine; hence the sine is algebraically $<$ the cosine. ■

§ Problem 8.2.32. *If* $\sin\frac{A}{3}$ *be determined from the equation*

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$$

prove that we should expect to obtain also the values of

$$\sin\frac{\pi - A}{3} \textit{ and } -\sin\frac{\pi + A}{3}.$$

Give also a geometrical illustration. ◊

§§ Solution. $\because \sin A = \sin\,[n\pi + (-1)^n A]$, any equation giving $\sin\dfrac{A}{3}$ in terms of $\sin A$ should give also $\sin\dfrac{1}{3}\,[n\pi + (-1)^n A]$.

Now n is of the form $3m,\ or\ 3m \pm 1$.

If $n = 3m$, then

$$\begin{aligned}\sin\frac{1}{3}\,[n\pi + (-1)^n A] &= \sin\left[m\pi + (-1)^{3m}\frac{A}{3}\right]\\ &= \sin m\pi \cos(-1)^{3m}\frac{A}{3} + \cos m\pi \sin(-1)^{3m}\frac{A}{3}\\ &= \cos m\pi \sin(-1)^{3m}\frac{A}{3} = \sin\frac{A}{3}.\end{aligned}$$

whether m be even or odd, for $\sin m\pi = 0$ and $\cos m\pi = +1$ or -1 according as m is even or odd.

If $n = 3m+1$, then

$$\begin{aligned}\sin\frac{1}{3}\,[n\pi + (-1)^n A] &= \sin\left[m\pi + \frac{\pi + (-1)^{3m+1}A}{3}\right]\\ &= \cos m\pi \sin\frac{\pi + (-1)^{3m+1}A}{3}\\ &= \sin\frac{\pi - A}{3}\ \ or\ \ -\sin\frac{\pi + A}{3},\end{aligned}$$

according as m is even or odd.

If $n = 3m-1$, then

$$\begin{aligned}\sin\frac{1}{3}\,[n\pi + (-1)^n A] &= \sin\left[m\pi - \frac{\pi - (-1)^{3m-1}A}{3}\right]\\ &= -\cos m\pi \sin\frac{\pi - (-1)^{3m-1}A}{3}\\ &= \sin\frac{\pi - A}{3}\ \ or\ \ -\sin\frac{\pi + A}{3},\end{aligned}$$

according as m is odd or even.

Hence we have two values in addition to $\sin\dfrac{A}{3}$,

viz. $$\sin\frac{\pi - A}{3}\ \ and\ \ -\sin\frac{\pi + A}{3}.$$ ■

§ Problem 8.2.33. *If* $\cos\dfrac{A}{3}$ *be found from the equation*

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$$

prove that we should expect to obtain also the values of

$$\cos\frac{2\pi - A}{3}\ \ and\ \ \cos\frac{2\pi + A}{3}.$$

Give also a geometrical illustration. ◊

§§ Solution. $\because \cos A = \cos\,(2n\pi \pm A)$, any equation giving $\cos\dfrac{A}{3}$ in terms of $\cos A$ should give also $\cos\dfrac{1}{3}\,(2n\pi \pm A)$.

Now n is of the form $3m,\ or\ 3m \pm 1$.

If $n = 3m$, then

$$\cos\frac{1}{3}\,(2n\pi \pm A) = \cos\left(2m\pi \pm \frac{A}{3}\right) = \cos\frac{A}{3}.$$

If $n = 3m + 1$, then

$$\cos\frac{1}{3}(2n\pi \pm A) = \cos\left(2m\pi + \frac{2\pi}{3} \pm \frac{A}{3}\right) = \cos\frac{2\pi \pm A}{3}.$$

If $n = 3m - 1$, then

$$\begin{aligned}\cos\frac{1}{3}(2n\pi \pm A) &= \cos\left(2m\pi - \frac{2\pi}{3} \pm \frac{A}{3}\right)\\ &= \cos\left(2m\pi - \frac{2\pi \mp A}{3}\right) = \cos\frac{2\pi \mp A}{3}.\end{aligned}$$

Hence we have two values in addition to $\cos\frac{A}{3}$,

viz. $$\cos\frac{2\pi - A}{3} \; and \; -\cos\frac{2\pi + A}{3}.$$ ■

8.3 Angles of $9°$, $18°$, $36°$, $81°$

Prove that

§ Problem 8.3.1. $\sin^2 72° - \sin^2 60° = \dfrac{\sqrt{5}-1}{8}.$ ◊

§§ Solution.

$$\begin{aligned}\sin^2 72° - \sin^2 60° &= \left[\frac{\sqrt{10+2\sqrt{5}}}{4}\right]^2 - \left(\frac{\sqrt{3}}{2}\right)^2\\ &= \frac{10+2\sqrt{5}}{16} - \frac{3}{4} = \frac{5+\sqrt{5}-6}{8} = \frac{\sqrt{5}-1}{8}.\end{aligned}$$ ■

§ Problem 8.3.2. $\cos^2 48° - \sin^2 12° = \dfrac{\sqrt{5}+1}{8}.$ ◊

§§ Solution.

$$\begin{aligned}\cos^2 48° - \sin^2 12° &= \cos(48° + 12°)\cos(48° - 12°) \; [By\ Ex.\ 2,\ Art.\ 93]\\ &= \cos 60° \cos 36° = \frac{1}{2}\cdot\frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+1}{8}.\end{aligned}$$ ■

§ Problem 8.3.3. $\cos 12° + \cos 60° + \cos 84° = \cos 24° + \cos 48°$. *Verify by a graph.* ◊

§§ Solution.

$$\begin{aligned}\cos 12° + \cos 60° + \cos 84° &= \cos 12° + 2\cos 72° \cos 12°\\ &= \cos 12° (1 + 2\cos 72°)\\ &= \cos 12° \left(1 + \frac{\sqrt{5}-1}{2}\right)\\ &= \frac{\sqrt{5}+1}{2}\cdot\cos 12° = 2\cos 36° \cos 12°\\ &= \cos 24° + \cos 48°.\end{aligned}$$ ■

§ Problem 8.3.4. $\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}.$ ◊

§§ Solution.

$$\begin{aligned}\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} &= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{2\pi}{5}\sin\frac{\pi}{5}\\ &= \sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \sin^2 36° \sin^2 72°\\ &= \frac{10-2\sqrt{5}}{16}\cdot\frac{10+2\sqrt{5}}{16} = \frac{5}{16}.\end{aligned}$$ ■

§ Problem 8.3.5. $\sin\dfrac{\pi}{10}+\sin\dfrac{13\pi}{10}=-\dfrac{1}{2}.$ ◊

§§ Solution.

$$\begin{aligned}\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}&=2\sin\frac{7\pi}{10}\cos\frac{6\pi}{10}\\&=2\sin\frac{3\pi}{10}\left(-\cos\frac{4\pi}{10}\right)\\&=-2\sin 54°\cos 72°=-2\cdot\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{5}-1}{4}=-\frac{2}{4}=-\frac{1}{2}.\end{aligned}$$

Otherwise thus :

$$\begin{aligned}\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}&=\sin\frac{\pi}{10}-\sin\frac{3\pi}{10},\ \textit{by Art. }73,\\&=\sin 18°-\sin 54°=\frac{\sqrt{5}-1}{4}-\frac{\sqrt{5}+1}{4}\\&=-\frac{2}{4}=-\frac{1}{2}.\end{aligned}$$ ■

§ Problem 8.3.6. $\sin\dfrac{\pi}{10}\sin\dfrac{13\pi}{10}=-\dfrac{1}{4}.$ ◊

§§ Solution.

$$\begin{aligned}\sin\frac{\pi}{10}\sin\frac{13\pi}{10}&=\sin\frac{\pi}{10}\left(-\sin\frac{3\pi}{10}\right)\\&=-\frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{5}+1}{4}=-\frac{1}{4}.\end{aligned}$$ ■

§ Problem 8.3.7. $\tan 6°\tan 42°\tan 66°\tan 78°=1.$ ◊

§§ Solution.

$$\begin{aligned}\tan 6°\tan 42°\tan 66°\tan 78°&=\frac{\sin 6°\sin 42°\sin 66°\sin 78°}{\cos 6°\cos 42°\cos 66°\cos 78°}\\&=\frac{2\sin 6°\sin 66°\times 2\sin 42°\sin 78°}{2\cos 6°\cos 66°\times 2\cos 42°\cos 78°}\\&=\frac{(\cos 60°-\cos 72°)(\cos 36°-\cos 120°)}{(\cos 60°+\cos 72°)(\cos 36°+\cos 120°)}\\&=\frac{\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}+\frac{1}{2}\right)}{\left(\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}-\frac{1}{2}\right)}\\&=\frac{\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)}{\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)}=\frac{4}{4}=1.\end{aligned}$$ ■

§ Problem 8.3.8. $\cos\dfrac{\pi}{15}\cos\dfrac{2\pi}{15}\cos\dfrac{3\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{5\pi}{15}\cos\dfrac{6\pi}{15}\cos\dfrac{7\pi}{15}=\dfrac{1}{2^7}.$ ◊

§§ Solution.

$$\begin{aligned}\cos\frac{\pi}{15}\cos\frac{4\pi}{15}&=\frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{\pi}{5}\right)\\&=\frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right)=\frac{3+\sqrt{5}}{8}\\ \cos\frac{2\pi}{15}\cos\frac{7\pi}{15}&=\frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{3\pi}{5}\right)\end{aligned}$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right) = \frac{3-\sqrt{5}}{8}$$

$$\cos\frac{3\pi}{15}\cos\frac{6\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right) = \frac{1}{4},\ \mathit{and}$$

$$\cos\frac{5\pi}{15} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

$$\therefore \cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$= \frac{3+\sqrt{5}}{8} \times \frac{3-\sqrt{5}}{8} \times \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{2} \times \frac{1}{8} \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{2^7}.$$

Otherwise thus :

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$= \frac{2\sin\frac{\pi}{15}\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}}{2\sin\frac{\pi}{15}}$$

$$\times \frac{2\sin\frac{3\pi}{15}\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}}{2\sin\frac{3\pi}{15}} \times \cos\frac{5\pi}{15}$$

$$= \frac{\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}}{2\sin\frac{\pi}{15}} \times \frac{\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}} \times \frac{1}{2}$$

$$= \frac{\sin\frac{4\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}}{2\times 2\sin\frac{\pi}{15}} \times \frac{\sin\frac{4\pi}{5}}{2\times 2\sin\frac{\pi}{5}} \times \frac{1}{2}$$

$$= \frac{\sin\frac{8\pi}{15}\cos\frac{7\pi}{15}}{2\times 2\times 2\sin\frac{\pi}{15}} \times \frac{\sin\frac{\pi}{5}}{2\times 2\sin\frac{\pi}{5}} \times \frac{1}{2}$$

$$= \frac{\sin\frac{7\pi}{15}\cos\frac{7\pi}{15}}{2^3\sin\frac{\pi}{15}} \times \frac{1}{2^3} = \frac{\sin\frac{14\pi}{15}}{2^7\sin\frac{\pi}{15}} = \frac{1}{2^7}.$$ ■

§ Problem 8.3.9. $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = 1.$ ◊

§§ Solution.

$$16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}$$

$$= \frac{16\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}}{\sin\frac{2\pi}{15}}$$

$$= \frac{8\sin\dfrac{4\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15}}{\sin\dfrac{2\pi}{15}}$$

$$= \frac{4\sin\dfrac{8\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15}}{\sin\dfrac{2\pi}{15}} = \frac{2\sin\dfrac{16\pi}{15}\cos\dfrac{14\pi}{15}}{\sin\dfrac{2\pi}{15}}$$

$$= \frac{2\times\left(-\sin\dfrac{\pi}{15}\right)\times\left(-\cos\dfrac{\pi}{15}\right)}{\sin\dfrac{2\pi}{15}}$$

$$= \frac{2\sin\dfrac{\pi}{15}\cos\dfrac{\pi}{15}}{\sin\dfrac{2\pi}{15}} = \frac{\sin\dfrac{2\pi}{15}}{\sin\dfrac{2\pi}{15}} = 1.$$

■

§ Problem 8.3.10. *Two parallel chords of a circle, which are on the same side of the center, subtend angles of 72° and 144° respectively at the center. Prove that the perpendicular distance between the chords is half the radius of the circle.* ♢

§§ Solution. See figure of *Art.* 130. Let O be the center and r be the radius of the circle and let PP' and QQ' be the two chords, PP' being the one nearer to O.

Join OQ, OQ', OP and OP'. Draw OMN perpendicular to PP' and QQ', bisecting them in M and N respectively.

We have the $\angle QOQ' = 72^\circ$, and the $\angle POP' = 144^\circ$. Hence the required distance

$$\begin{aligned} &= MN = ON - OM = r\cos\angle QON - r\cos\angle POM \\ &= r\cos 36^\circ - r\cos 72^\circ = r\,(\cos 36^\circ - \cos 72^\circ) \\ &= r\left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right) = r\times\frac{2}{4} = \frac{r}{2}. \end{aligned}$$

■

§ Problem 8.3.11. *In any circle prove that the chord which subtends 108° at the center is equal to the sum of the two chords which subtend angles of 36° and 60°.* ♢

§§ Solution. If r be the radius of the circle, the lengths of the chords are $2r\sin 54^\circ$, $2r\sin 18^\circ$ and $2r\sin 30^\circ$ respectively.

Now $$\sin 54^\circ = \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \sin 18^\circ + \sin 30^\circ.$$

Hence we have the required result. ■

§ Problem 8.3.12. *Construct the angle whose cosine is equal to its tangent.* ♢

§§ Solution. If θ be the angle, we have

$$\cos\theta = \tan\theta;\ \therefore \cos^2\theta = \sin\theta;\ \therefore 1 - \sin^2\theta = \sin\theta$$

$$\therefore \sin^2\theta + \sin\theta - 1 = 0;$$

$$\therefore \sin\theta = \frac{-1+\sqrt{1+4}}{2} = \frac{\sqrt{5}-1}{2} = 2\sin 18^\circ.$$

Describe a semicircle APB on AB as diameter. Make an angle $\angle PAB$ equal to 18°. Join BP.

On this semicircle take a point Q, such that BQ equals $2BP$. Join AQ. We then have

$$\sin \angle BAQ = \frac{BQ}{AB} = \frac{2BP}{AB} = 2\sin 18^\circ.$$

Hence $\angle BAQ$ is the required angle. ■

§ Problem 8.3.13. *Solve the equation*

$$\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta.$$ ◊

§§ Solution.

$$\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta$$
$$\therefore 2\sin 5\theta \cos 3\theta = 2\sin 9\theta \cos 7\theta$$
$$\therefore \sin 8\theta + \sin 2\theta = \sin 16\theta + \sin 2\theta$$
$$\therefore \sin 8\theta = \sin 16\theta = 2\sin 8\theta \cos 8\theta$$
$$\therefore \sin 8\theta = 0 \ \textit{or} \ 2\cos 8\theta = 1.$$

If $$\sin 8\theta = 0, \ \textit{then} \ 8\theta = n\pi.$$

If $$2\cos 8\theta = 1, \ \textit{then} \ \cos 8\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$
$$\therefore 8\theta = 2n\pi \pm \frac{\pi}{3}.$$
$$\therefore \theta = \frac{n\pi}{8} \ \textit{or} \ \left(2n \pm \frac{1}{3}\right)\frac{\pi}{8}.$$ ■

Chapter 9

Identities And Trigonometrical Equations

9.1 Tangent of The Sum of Angles

§ Problem 9.1.1. *Prove that*
$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}. \qquad \Diamond$$

§§ Solution.
$$\begin{aligned}
\tan 5\theta &= \frac{s_1 - s_3 + s_5}{1 - s_2 + s_4} \\
&= \frac{{}^5C_1\tan\theta - {}^5C_3\tan^3\theta + {}^5C_5\tan^5\theta}{1 - {}^5C_2\tan^2\theta + {}^5C_4\tan^4\theta} \\
&= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}. \qquad \blacksquare
\end{aligned}$$

9.2 Identities

If $A + B + C = 180°$, prove that

§ Problem 9.2.1. $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C.$ $\Diamond$

§§ Solution.
$$\begin{aligned}
\sin 2A + \sin 2B - \sin 2C &= 2\sin(A+B)\cos(A-B) - 2\sin C\cos C \\
&= 2\sin C\cos(A-B) - 2\sin C\cos C \\
&= 2\sin C\,[\cos(A-B) - \cos C] \\
&= 2\sin C\,[\cos(A-B) + \cos(A+B)] \\
&= 2\sin C \cdot 2\cos A\cos B \\
&= 4\cos A\cos B\sin C. \qquad \blacksquare
\end{aligned}$$

§ Problem 9.2.2. $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C.$ ◊

§§ Solution.

$$\begin{aligned}\cos 2A+\cos 2B+\cos 2C &= 2\cos(A+B)\cos(A-B)+2\cos^2 C-1\\ &= 2\cos(A+B)\cos(A-B)+2\cos C\cdot\cos C-1\\ &= -2\cos C\cos(A-B)+2\cos C\,[-\cos(A+B)]-1\\ &= -2\cos C\,[\cos(A-B)+\cos(A+B)]-1\\ &= -2\cos C\cdot 2\cos A\cos B-1\\ &= -1-4\cos A\cos B\cos C.\end{aligned}$$ ■

§ Problem 9.2.3. $\cos 2A+\cos 2B-\cos 2C = 1-4\sin A\sin B\cos C.$ ◊

§§ Solution.

$$\begin{aligned}\cos 2A+\cos 2B-\cos 2C &= 2\cos(A+B)\cos(A-B)-\left(2\cos^2 C-1\right)\\ &= 2\cos(A+B)\cos(A-B)-2\cos C\cdot\cos C+1\\ &= -2\cos C\cos(A-B)-2\cos C\,[-\cos(A+B)]+1\\ &= -2\cos C\,[\cos(A-B)-\cos(A+B)]+1\\ &= -2\cos C\cdot 2\sin A\sin B+1\\ &= 1-4\sin A\sin B\cos C.\end{aligned}$$ ■

§ Problem 9.2.4. $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$ ◊

§§ Solution.

$$\begin{aligned}\sin A+\sin B+\sin C &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}\\ &= 2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\cos\frac{A+B}{2}\cos\frac{C}{2}\\ &= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right]\\ &= 2\cos\frac{C}{2}\cdot 2\cos\frac{A}{2}\cos\frac{B}{2}\\ &= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\end{aligned}$$ ■

§ Problem 9.2.5. $\sin A+\sin B-\sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$ ◊

§§ Solution.

$$\begin{aligned}\sin A+\sin B-\sin C &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{C}{2}\\ &= 2\cos\frac{C}{2}\cos\frac{A-B}{2}-2\cos\frac{A+B}{2}\cos\frac{C}{2}\\ &= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]\\ &= 2\cos\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}\\ &= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.\end{aligned}$$ ■

§ Problem 9.2.6. $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$ ◊

§§ Solution.

$$\begin{aligned}
\cos A + \cos B + \cos C &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} \\
&= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin\frac{C}{2}\sin\frac{C}{2} \\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\cos\frac{A+B}{2}\sin\frac{C}{2} \\
&= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right] + 1 \\
&= 2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2} + 1 \\
&= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.
\end{aligned}$$

■

§ Problem 9.2.7. $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C.$ ◊

§§ Solution.

$$\begin{aligned}
&\sin^2 A + \sin^2 B - \sin^2 C \\
&\quad = \frac{1}{2}(1-\cos 2A) + \frac{1}{2}(1-\cos 2B) - \frac{1}{2}(1-\cos 2C) \\
&\quad = \frac{1}{2}\left[1 - (\cos 2A + \cos 2B - \cos 2C)\right] \\
&\quad = \frac{1}{2}\left[1 - (1 - 4\sin A\sin B\cos C)\right] \{By\ \S Problem\ 9.2.3\} \\
&\quad = 2\sin A\sin B\cos C.
\end{aligned}$$

■

§ Problem 9.2.8. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C.$ ◊

§§ Solution.

$$\begin{aligned}
\cos^2 A + \cos^2 B + \cos^2 C &= \frac{1}{2}(1 + \cos 2A + 1 + \cos 2B + 1 + \cos 2C) \\
&= \frac{1}{2}\left[3 + (\cos 2A + \cos 2B + \cos 2C)\right] \\
&= \frac{1}{2}\left[3 - 1 - 4\cos A\cos B\cos C\right] \{By\ \S Problem\ 9.2.2\} \\
&= 1 - 2\cos A\cos B\cos C.
\end{aligned}$$

■

§ Problem 9.2.9. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A\sin B\cos C.$ ◊

§§ Solution.

$$\begin{aligned}
\cos^2 A + \cos^2 B - \cos^2 C &= \frac{1}{2}(1+\cos 2A) + \frac{1}{2}(1+\cos 2B) - \frac{1}{2}(1+\cos 2C) \\
&= \frac{1}{2}\left[1 + (\cos 2A + \cos 2B - \cos 2C)\right] \\
&= \frac{1}{2}\left[1 + 1 - 4\sin A\sin B\cos C\right] \{By\ \S Problem\ 9.2.3\} \\
&= 1 - 2\sin A\sin B\cos C.
\end{aligned}$$

■

§ Problem 9.2.10. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$ ◊

§§ Solution.

$$\begin{aligned}
&\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} \\
&\quad = \frac{1}{2}(1 - \cos A + 1 - \cos B + 1 - \cos C)
\end{aligned}$$

$$= \frac{1}{2}\left[3 - (\cos A + \cos B + \cos C)\right]$$
$$= \frac{1}{2}\left[3 - 1 - 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right] \{By\ \S Problem\ 9.2.6\}$$
$$= 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \qquad \blacksquare$$

§ Problem 9.2.11. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$. ♢

§§ Solution.

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2}$$
$$= \frac{1}{2}(1 - \cos A) + \frac{1}{2}(1 - \cos B) - \frac{1}{2}(1 - \cos C)$$
$$= \frac{1}{2}\left[1 - (\cos A + \cos B - \cos C)\right]$$
$$= \frac{1}{2}\left[1 + 1 - 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right] \{By\ Ex.\ 2,\ Art.\ 127\}$$
$$= 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}. \qquad \blacksquare$$

§ Problem 9.2.12. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$. ♢

§§ Solution. $\because \frac{A+B}{2} = 90^\circ - \frac{C}{2}$

$$\therefore \tan\frac{A+B}{2} = \tan\left(90^\circ - \frac{C}{2}\right) = \cot\frac{C}{2} = \frac{1}{\tan\frac{C}{2}}$$

$$\therefore \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}},\ by\ Art.\ 98$$
$$\therefore \tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$$
$$\therefore \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1. \qquad \blacksquare$$

§ Problem 9.2.13. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$. ♢

§§ Solution. $\because \frac{A+B}{2} = 90^\circ - \frac{C}{2}$

$$\therefore \cot\frac{A+B}{2} = \cot\left(90^\circ - \frac{C}{2}\right) = \tan\frac{C}{2} = \frac{1}{\cot\frac{C}{2}}$$

$$\therefore \frac{\cot\frac{A}{2}\cot\frac{B}{2} - 1}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \frac{1}{\cot\frac{C}{2}},\ by\ Art.\ 100$$
$$\therefore \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} - \cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2}$$
$$\therefore \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}. \qquad \blacksquare$$

§ Problem 9.2.14. $\cot B\cot C + \cot C\cot A + \cot A\cot B = 1$. ♢

§§ Solution. $\because A+B+C=180°$, $\therefore A+B=180°-C$

$$\therefore \cot(A+B)=\cot(180°-C)=-\cot C$$
$$\therefore \frac{\cot A\cot B-1}{\cot A+\cot B}=-\cot C$$
$$\therefore \cot A\cot B-1=-\cot C\,(\cot A+\cot B)$$
$$\therefore \cot B\cot C+\cot C\cot A+\cot A\cot B=1.$$ ■

§ Problem 9.2.15.

$$\sin(B+2C)+\sin(C+2A)+\sin(A+2B)$$
$$=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$$ ◊

§§ Solution.

$$\because B+2C=180°+C-A=180°-(A-C)$$
$$\therefore \sin(B+2C)=\sin(A-C)$$
$$\therefore \sin(B+2C)+\sin(C+2A)+\sin(A+2B)$$
$$=\sin(A-C)+\sin(B-A)+\sin(C-B)$$
$$=2\sin\frac{B-C}{2}\cos\frac{2A-B-C}{2}-2\sin\frac{B-C}{2}\cos\frac{B-C}{2}$$
$$=2\sin\frac{B-C}{2}\left(\cos\frac{2A-B-C}{2}-\cos\frac{B-C}{2}\right)$$
$$=2\sin\frac{B-C}{2}\times 2\sin\frac{A-C}{2}\sin\frac{B-A}{2}$$
$$=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$$ ■

§ Problem 9.2.16.

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1$$
$$=4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$$ ◊

§§ Solution.

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1$$
$$=\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)-1$$
$$=2\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\cos\frac{A-B}{4}+1-2\sin^2\frac{\pi-C}{4}-1$$
$$=2\sin\frac{\pi-C}{4}\left[\cos\frac{A-B}{4}-\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$
$$\left\{\because \cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)=\sin\frac{A+B}{4}=\sin\frac{\pi-C}{4}\right\}$$
$$=2\sin\frac{\pi-C}{4}\times 2\sin\left(\frac{\pi}{4}-\frac{B}{4}\right)\sin\left(\frac{\pi}{4}-\frac{A}{4}\right)$$
$$=4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$$ ■

§ Problem 9.2.17.

$$\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}$$
$$=4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}.$$ ◊

§§ Solution.

$$\cos\frac{A}{2}+\cos\frac{B}{2} = 2\cos\frac{A+B}{4}\cos\frac{A-B}{4}$$
$$= 2\cos\frac{\pi-C}{4}\cos\frac{A-B}{4}$$
$$\cos\frac{C}{2} = \sin\frac{A+B}{2} = 2\sin\frac{A+B}{4}\cos\frac{A+B}{4}$$
$$= 2\cos\frac{\pi-C}{4}\sin\frac{A+B}{4} = 2\cos\frac{\pi-C}{4}\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)$$
$$\therefore \cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}$$
$$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4}-\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$
$$= 2\cos\frac{\pi-C}{4}\times 2\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}$$
$$= 4\cos\frac{\pi-C}{4}\cos\left(\frac{\pi}{2}-\frac{\pi-A}{4}\right)\cos\left(\frac{\pi}{2}-\frac{\pi-B}{4}\right)$$
$$= 4\cos\frac{\pi-C}{4}\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}.$$ ■

§ Problem 9.2.18. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C} = 8\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}.$ ◊

§§ Solution.

$$\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}$$
$$= \frac{4\sin A\sin B\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \text{ [by Ex. 1, Art. 127 and §Problem 9.2.4]}$$
$$= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}\times 2\sin\frac{B}{2}\cos\frac{B}{2}\times 2\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$
$$= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$ ■

§ Problem 9.2.19.

$$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C) = 4\sin A\sin B\sin C.$$ ◊

§§ Solution.

$$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$$
$$= \sin(180^\circ-2A)+\sin(180^\circ-2B)+\sin(180^\circ-2C)$$
$$= \sin 2A+\sin 2B+\sin 2C$$
$$= 4\sin A\sin B\sin C \text{ [by Ex. 1, Art. 127]}.$$ ■

If $A+B+C=2S$, prove that

§ Problem 9.2.20. $\sin(S-A)\sin(S-B)+\sin S\sin(S-C) = \sin A\sin B.$ ◊

§§ Solution.

$$\sin(S-A)\sin(S-B)+\sin S\sin(S-C)$$
$$= \frac{1}{2}[\cos(B-A)-\cos(2S-A-B)+\cos C-\cos(2S-C)]$$
$$= \frac{1}{2}[\cos(A-B)-\cos C+\cos C-\cos(A+B)]$$

$$= \frac{1}{2}\left[\cos A \cos B + \sin A \sin B - (\cos A \cos B - \sin A \sin B)\right]$$
$$= \sin A \sin B. \qquad \blacksquare$$

§ Problem 9.2.21.

$$4 \sin S \sin(S-A) \sin(S-B) \sin(S-B) \sin(S-C)$$
$$= 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
&4 \sin S \sin(S-A) \sin(S-B) \sin(S-B) \sin(S-C)\\
&= [\cos A - \cos(2S-A)]\,[\cos(B-C) - \cos(2S-B-C)]\\
&= [\cos A - \cos(B+C)]\,[\cos(B-C) - \cos A]\\
&= \cos A\,[\cos(B-C) + \cos(B+C)]\\
&- \cos^2 A - \cos(B+C)\cos(B-C)\\
&= \cos A \cdot 2\cos B \cos C - \cos^2 A - \frac{1}{2}(\cos 2B + \cos 2C)\\
&= 2\cos A \cos B \cos C - \cos^2 A - \frac{1}{2}\left(2\cos^2 B - 1 + 2\cos^2 C - 1\right)\\
&= 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2\cos A \cos B \cos C. \qquad \blacksquare
\end{aligned}$$

§ Problem 9.2.22.

$$\sin(S-A) + \sin(S-B) + \sin(S-C) - \sin S$$
$$= 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
&\sin(S-A) + \sin(S-B) + \sin(S-C) - \sin S\\
&= 2\sin\frac{2S-A-B}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{2S-C}{2}\\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{A+B}{2}\\
&= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]\\
&= 2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}\\
&= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \qquad \blacksquare
\end{aligned}$$

§ Problem 9.2.23.

$$\cos^2 S + \cos^2(S-A) + \cos^2(S-B) + \cos^2(S-C)$$
$$= 2 + 2\cos A \cos B \cos C. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
&\cos^2 S + \cos^2(S-A) + \cos^2(S-B) + \cos^2(S-C)\\
&= \frac{1}{2}[1 + \cos 2S + 1 + \cos(2S-2A) + 1 + \cos(2S-2B) +\\
&\qquad 1 + \cos(2S-2C)]\\
&= 2 + \frac{1}{2}[\cos(A+B+C) + \cos(B+C-A)\\
&\qquad + \cos(A+C-B) + \cos(B+A-C)]\\
&= 2 + \frac{1}{2}[2\cos(B+C)\cos A + 2\cos A\cos(B-C)]\\
&= 2 + \cos A\,[\cos(B+C) + \cos(B-C)]\\
&= 2 + \cos A \cdot 2\cos B \cos C\\
&= 2 + 2\cos A \cos B \cos C. \qquad \blacksquare
\end{aligned}$$

§ Problem 9.2.24.

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C$$
$$= 1 + 4\cos S \cos(S-A)\cos(S-B)\cos(S-C). \quad \diamond$$

§§ Solution.

$$\begin{aligned}
&\cos^2 A + 2\cos A \cos B \cos C + \cos^2 B + \cos^2 C - 1\\
&= \cos^2 A + 2\cos A \cos B \cos C + \frac{1}{2}(\cos 2B + \cos 2C)\\
&= \cos^2 A + \cos A\,[\cos(B+C) + \cos(B-C)] + \cos(B+C)\cos(B-C)\\
&= [\cos A + \cos(B+C)]\,[\cos A + \cos(B-C)]\\
&= [\cos A + \cos(2S-A)]\,[\cos(2S-B-C) + \cos(B-C)]\\
&= 4\cos S \cos(S-A)\cos(S-B)\cos(S-C).
\end{aligned} \quad \blacksquare$$

§ Problem 9.2.25. *If $\alpha+\beta+\gamma+\delta = 2\pi$, prove that*

(1)

$$\cos\alpha + \cos\beta + \cos\gamma + \cos\delta + 4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2} = 0$$

(2)

$$\sin\alpha - \sin\beta + \sin\gamma - \sin\delta + 4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2} = 0$$

(3)

$$\tan\alpha + \tan\beta + \tan\gamma + \tan\delta = \tan\alpha\tan\beta\tan\gamma\tan\delta\left(\cot\alpha + \cot\beta + \cot\gamma + \cot\delta\right). \quad \diamond$$

§§ Solution. If $\alpha+\beta+\gamma+\delta = 2\pi$, then $\dfrac{\alpha+\beta}{2} = \pi - \dfrac{\gamma+\delta}{2}$.

(1)

$$\begin{aligned}
&\cos\alpha + \cos\beta + \cos\gamma + \cos\delta\\
&\quad= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos\frac{\gamma+\delta}{2}\cos\frac{\gamma-\delta}{2}\\
&\quad= 2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\gamma-\delta}{2}\right)\\
&\quad= 2\cos\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha-(2\pi-\alpha-\gamma-\delta)}{2} - \cos\frac{\gamma-\delta}{2}\right]\\
&\quad= 2\cos\frac{\alpha+\beta}{2}\times 2\sin\frac{\alpha+\gamma-\pi}{2}\sin\frac{\pi-\alpha-\delta}{2}\\
&\quad= 4\cos\frac{\alpha+\beta}{2}\left[\sin\left(\frac{\alpha+\gamma}{2} - \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2} - \frac{\alpha+\delta}{2}\right)\right]\\
&\quad= 4\cos\frac{\alpha+\beta}{2}\left(-\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}\right).
\end{aligned}$$

$$\therefore \cos\alpha + \cos\beta + \cos\gamma + \cos\delta + 4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2} = 0.$$

(2)

$$\sin\alpha - \sin\beta + \sin\gamma - \sin\delta + 4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}$$

$$
\begin{aligned}
&= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} + 2\cos\frac{\gamma+\delta}{2}\sin\frac{\gamma-\delta}{2} \\
&+ 4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2} \\
&= 2\cos\frac{\alpha+\beta}{2}\left(\sin\frac{\alpha-\beta}{2} - \sin\frac{\gamma-\delta}{2} + 2\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}\right) \\
&= 2\cos\frac{\alpha+\beta}{2}\left[\sin\frac{\alpha-\beta}{2} - \sin\frac{\gamma-\delta}{2} + \sin\left(\alpha+\frac{\gamma+\delta}{2}\right)\right. \\
&\qquad\left. + \sin\frac{\gamma-\delta}{2}\right] \\
&= 2\cos\frac{\alpha+\beta}{2}\left[\sin\frac{\alpha-\beta}{2} + \sin\left(\alpha+\frac{\gamma+\delta}{2}\right)\right] \\
&= 2\cos\frac{\alpha+\beta}{2}\left[\sin\frac{\alpha-\beta}{2} + \sin\left(\alpha+\pi-\frac{\alpha+\beta}{2}\right)\right] \\
&= 2\cos\frac{\alpha+\beta}{2}\left[\sin\frac{\alpha-\beta}{2} + \sin\left(\pi+\frac{\alpha-\beta}{2}\right)\right] \\
&= 2\cos\frac{\alpha+\beta}{2}\left[\sin\frac{\alpha-\beta}{2} - \sin\left(\frac{\alpha-\beta}{2}\right)\right] = 0.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\tan(\alpha+\beta) &= \tan\left[2\pi-(\gamma+\delta)\right] \\
\therefore \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} &= -\frac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}
\end{aligned}
$$

$$
\begin{aligned}
&\therefore \tan\alpha+\tan\beta+\tan\gamma+\tan\delta \\
&\quad = \tan\alpha\tan\gamma\tan\delta + \tan\beta\tan\gamma\tan\delta \\
&\quad + \tan\alpha\tan\beta\tan\gamma + \tan\alpha\tan\beta\tan\delta \\
&\quad = \tan\alpha\tan\beta\tan\gamma\tan\delta\left(\cot\alpha+\cot\beta+\cot\gamma+\cot\delta\right).
\end{aligned}
$$

■

§ Problem 9.2.26. *If the sum of four angles be* 180°*, prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.* ◊

§§ Solution. Let the four angles be A, B, C and D.

$$
\begin{aligned}
&\because A+B+C+D = 180^\circ \quad \therefore A+B = 180^\circ-(C+D) \\
&\therefore \cos(A+B) = -\cos(C+D) \\
&\therefore \cos A\cos B - \sin A\sin B = -\cos C\cos D + \sin C\sin D \\
&\therefore \cos A\cos B + \cos C\cos D = \sin A\sin B + \sin C\sin D.
\end{aligned}
$$

Similarly, $\cos A\cos C + \cos B\cos D = \sin A\sin C + \sin B\sin D$

and $\cos A\cos D + \cos B\cos C = \sin A\sin D + \sin B\sin C.$

Hence, by addition, we obtain the required result. ■

§ Problem 9.2.27. *Prove that*

$$
\begin{aligned}
&\sin 2\alpha + \sin 2\beta + \sin 2\gamma \\
&\quad = 2\left(\sin\alpha+\sin\beta+\sin\gamma\right)\left(1+\cos\alpha+\cos\beta+\cos\gamma\right)
\end{aligned}
$$

if $\alpha+\beta+\gamma=0.$ ◊

§§ Solution. Since $\alpha+\beta=-\gamma$, we have

$$
\begin{aligned}
\sin(\alpha+\beta) &= -\sin\gamma, \ \textit{and} \\
\cos(\alpha+\beta) &= \cos\gamma.
\end{aligned}
$$

$$
\begin{aligned}
&\sin 2\alpha + \sin 2\beta + \sin 2\gamma \\
&\quad = 2\sin(\alpha+\beta)\cos(\alpha-\beta) + 2\sin\gamma\cos\gamma \\
&\quad = -2\sin\gamma\cos(\alpha-\beta) + 2\sin\gamma\cos(\alpha+\beta)
\end{aligned}
$$

$$
\begin{aligned}
&= -2\sin\gamma\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]\\
&= -2\sin\gamma\cdot 2\sin\alpha\sin\beta = -4\sin\alpha\sin\beta\sin\beta\\
&= -32\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\\
&= -32\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cdot\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\\
&= -8\left[\left(\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\right)\sin\frac{\gamma}{2}\right]\\
&\qquad\left[\left(\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}\right)\cos\frac{\gamma}{2}\right]\\
&= 8\left[\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\right]\\
&\qquad\left[\cos^2\frac{\gamma}{2}+\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right]\\
&= 2\left(\sin\alpha+\sin\beta+\sin\gamma\right)\left(1+\cos\gamma+\cos\alpha+\cos\beta\right). \qquad \blacksquare
\end{aligned}
$$

§ Problem 9.2.28. *Verify that*

$$
\begin{aligned}
&\sin^3 a\sin(b-c)+\sin^3 b\sin(c-a)+\sin^3 c\sin(a-b)\\
&\qquad +\sin(a+b+c)\sin(b-c)\sin(c-a)\sin(a-b)=0. \qquad \diamondsuit
\end{aligned}
$$

§§ Solution. Since $\sin 3A = 3\sin A - 4\sin^3 A$, we have

$$
\begin{aligned}
&\sin^3 a\sin(b-c)+\sin^3 b\sin(c-a)+\sin^3 c\sin(a-b)\\
&=\left(\frac{3}{4}\sin a-\frac{1}{4}\sin 3a\right)\sin(b-c)+\left(\frac{3}{4}\sin b-\frac{1}{4}\sin 3b\right)\sin(c-a)\\
&\qquad +\left(\frac{3}{4}\sin c-\frac{1}{4}\sin 3c\right)\sin(a-b).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\sin a\sin(b-c)+\sin b\sin(c-a)+\sin c\sin(a-b)\\
&\qquad=\sin a\sin b\cos c-\sin a\cos b\sin c+\sin b\sin c\cos a\\
&\qquad-\sin b\cos c\sin a+\sin c\sin a\cos b-\sin c\cos a\sin b=0
\end{aligned}
$$

$$
\begin{aligned}
&\therefore \sin^3 a\sin(b-c)+\sin^3 b\sin(c-a)+\sin^3 c\sin(a-b)\\
&=-\frac{1}{4}\left[\sin 3a\sin(b-c)+\sin 3b\sin(c-a)+\sin 3c\sin(a-b)\right]\\
&=\frac{1}{8}\left[\cos(3a+b-c)-\cos(3a-b+c)+\cos(3b+c-a)\right.\\
&\qquad\left.-\cos(3b-c+a)+\cos(3c+a-b)-\cos(3c-a+b)\right].
\end{aligned}
$$

Hence taking the first and last terms together, the second and third terms together and the fourth and fifth terms together, within the square brackets, we have

$$
\begin{aligned}
&\sin^3 a\sin(b-c)+\sin^3 b\sin(c-a)+\sin^3 c\sin(a-b)\\
&=\frac{1}{4}\sin(a+b+c)\left[\sin 2(c-a)+\sin 2(a-b)+\sin 2(b-c)\right].
\end{aligned}
$$

Now

$$
\begin{aligned}
&\sin 2(c-a)+\sin 2(a-b)+\sin 2(b-c)\\
&=2\sin(c-b)\cos(c-2a+b)+2\sin(b-c)\cos(b-c)\\
&=2\sin(b-c)\left[\cos(b-c)-\cos(c-2a+b)\right]\\
&=2\sin(b-c)\cdot 2\sin(b-a)\sin(c-a)\\
&=-4\sin(b-c)\sin(c-a)\sin(a-b).
\end{aligned}
$$

Whence the required result follows. $\blacksquare$

If A, B, C **and** D **be any angles, prove that**

§ Problem 9.2.29.

$$\begin{aligned}\sin A\sin B\sin(A-B)+\sin B\sin C\sin(B-C)\\ +\sin C\sin A\sin(C-A)\\ +\sin(A-B)\sin(B-C)\sin(C-A)=0.\end{aligned} \quad \lozenge$$

§§ Solution.

$$\begin{aligned}&\sin A\sin B\sin(A-B)+\sin B\sin C\sin(B-C)+\sin C\sin A\sin(C-A)\\ &=\frac{1}{2}\sin B\left[\cos B-\cos(2A-B)+\cos(2C-B)-\cos B\right]\\ &\qquad+\sin C\sin A\sin(C-A)\\ &=-\sin B\sin(A-B+C)\sin(C-A)+\sin C\sin A\sin(C-A)\\ &=-\frac{1}{2}\sin(C-A)\left[\cos(A-2B+C)-\cos(A+C)-\cos(C-A)\right.\\ &\qquad\left.+\cos(A+C)\right]\\ &=-\sin(C-A)\sin(C-B)\sin(B-A)\\ &=-\sin(A-B)\sin(B-C)\sin(C-A)\end{aligned}$$

$$\begin{aligned}\therefore \sin A\sin B\sin(A-B)+\sin B\sin C\sin(B-C)\\ +\sin C\sin A\sin(C-A)\\ +\sin(A-B)\sin(B-C)\sin(C-A)=0.\end{aligned} \quad \blacksquare$$

§ Problem 9.2.30.

$$\begin{aligned}&\sin(A-B)\cos(A+B)+\sin(B-C)\cos(B+C)\\ &+\sin(C-D)\cos(C+D)+\sin(D-A)\cos(D+A)=0.\end{aligned} \quad \lozenge$$

§§ Solution. The expression

$$\begin{aligned}&=\frac{1}{2}(\sin 2A-\sin 2B+\sin 2B-\sin 2C+\sin 2C\\ &\qquad-\sin 2D+\sin 2D-\sin 2A)=0.\end{aligned} \quad \blacksquare$$

§ Problem 9.2.31.

$$\begin{aligned}&\sin(A+B-2C)\cos B-\sin(A+C-2B)\cos C\\ &=\sin(B-C)\left\{\cos(B+C-A)+\cos(C+A-B)+\cos(A+B-C)\right\}.\end{aligned} \lozenge$$

§§ Solution.

$$\begin{aligned}&\sin(A+B-2C)\cos B-\sin(A+C-2B)\cos C\\ &=\frac{1}{2}\left[\sin(A+2B-2C)+\sin(A-2C)-\sin(A+2C-2B)-\sin(A-2B)\right]\end{aligned}$$

by taking the first and third terms together and the second and fourth terms together,

$$\begin{aligned}&=\cos A\sin(2B-2C)+\cos(A-B-C)\sin(B-C)\\ &=\sin(B-C)\left\{2\cos A\cos(B-C)+\cos(A-B-C)\right\}\\ &=\sin(B-C)\left\{\cos(A+B-C)+\cos(A-B+C)+\cos(B+C-A)\right\}.\end{aligned} \quad \blacksquare$$

§ Problem 9.2.32.

$$\begin{aligned}\sin(A+B+C+D)+\sin(A+B-C-D)+\sin(A+B-C+D)\\ +\sin(A+B+C-D)=4\sin(A+B)\cos C\cos D.\end{aligned} \lozenge$$

§§ Solution. The expression

$$\begin{aligned}&=2\sin(A+B)\cos(C+D)+2\sin(A+B)\cos(C-D)\\ &=2\sin(A+B)\left[\cos(C+D)+\cos(C-D)\right]\\ &=2\sin(A+B)\cdot 2\cos C\cos D\\ &=4\sin(A+B)\cos C\cos D.\end{aligned} \quad \blacksquare$$

§ Problem 9.2.33. *If any theorem be true for values of* A*,* B *and* C *such that*

$$A + B + C = 180°$$

prove that the theorem is still true if we substitute for A*,* B *and* C *respectively the quantities*

(1) $90° - \dfrac{A}{2}$*,* $90° - \dfrac{B}{2}$ *and* $90° - \dfrac{C}{2}$*, or*

(2) $180° - 2A$*,* $180° - 2B$ *and* $180° - 2C$*.*

Hence deduce § Problem 9.2.16 and § Problem 9.2.17. ◊

§§ Solution. (1) If $A_1 = 90° - \dfrac{A}{2}$, $B_1 = 90° - \dfrac{B}{2}$ and $C_1 = 90° - \dfrac{C}{2}$, then

$$A_1 + B_1 + C_1 = 270° - \frac{A+B+C}{2} = 270° - 90° = 180°.$$

(2) If $A_2 = 180° - 2A$, $B_2 = 180° - 2B$ and $C_2 = 180° - 2C$, then

$$A_2 + B_2 + C_2 = 540° - 2(A + B + C) = 540° - 360° = 180°.$$

In § Problem 9.2.16, let

$$A = \frac{\pi}{2} - \frac{\alpha}{2},\ B = \frac{\pi}{2} - \frac{\beta}{2},\ C = \frac{\pi}{2} - \frac{\gamma}{2}.$$

Then $$\cos A = \sin\frac{\alpha}{2}\ \textit{and}\ \sin\frac{A}{2} = \sin\frac{\pi - \alpha}{4}.$$

Also, $$\alpha + \beta + \gamma = 3\pi - 2(A + B + C) = 3\pi - 2\pi = \pi.$$

$$\therefore \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} = 1 + 4\sin\frac{\pi-\alpha}{4}\sin\frac{\pi-\beta}{4}\sin\frac{\pi-\gamma}{4}.$$

In § Problem 9.2.17, let

$$A = \frac{\pi}{2} - \frac{\alpha}{2},\ B = \frac{\pi}{2} - \frac{\beta}{2},\ C = \frac{\pi}{2} - \frac{\gamma}{2}.$$

Then $$\sin A = \cos\frac{\alpha}{2},\ \sin\frac{A}{2} = \sin\frac{\pi-\alpha}{4},\ \cos\frac{C}{2} = \cos\frac{\pi-\gamma}{4}.$$

$$\begin{aligned}\therefore \cos\frac{\alpha}{2} + \cos\frac{\beta}{2} - \cos\frac{\gamma}{2} &= 4\sin\frac{\pi-\alpha}{4}\sin\frac{\pi-\beta}{4}\cos\frac{\pi-\gamma}{4} \\ &= 4\cos\frac{\pi+\alpha}{4}\cos\frac{\pi+\beta}{4}\cos\frac{\pi-\gamma}{4}.\end{aligned}$$ ■

If $x + y + z = xyz$, prove that

§ Problem 9.2.34.

$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2}\cdot\frac{3y - y^3}{1 - 3y^2}\cdot\frac{3z - z^3}{1 - 3z^2}.$$ ◊

§§ Solution. As in $Ex.$ 5, $Art.$ 127, if

$$x = \tan A,\ y = \tan B,\ \textit{and}\ z = \tan C,$$

we have $$A + B + C = n\pi + \pi$$

$$\begin{aligned}&\therefore \frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} \\ &= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} + \frac{3\tan B - \tan^3 B}{1 - 3\tan^2 B} + \frac{3\tan C - \tan^3 C}{1 - 3\tan^2 C} \\ &= \tan 3A + \tan 3B + \tan 3C\end{aligned}$$

By a proof similar to that of $Ex.$ 5, $Art.$ 127

$$= \tan 3A \tan 3B \tan 3C$$

$$= \frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}.$$

Proof :

$\because A+B+C=(n+1)\pi=m\pi$, say, where m is any integer,

$$\therefore 3A+3B+3C=3m\pi,\ \therefore 3A+3B=3m\pi-3C$$

$$\therefore \tan(3A+3B)=-\tan 3C,\ \therefore \frac{\tan 3A+\tan 3B}{1-\tan 3A\tan 3B}=-\tan 3C.$$

$$\therefore \tan 3A+\tan 3B+\tan 3C=\tan 3A\tan 3B\tan 3C.$$ ■

§ Problem 9.2.35.

$$x\left(1-y^2\right)\left(1-z^2\right)+y\left(1-z^2\right)\left(1-x^2\right) + z\left(1-x^2\right)\left(1-y^2\right)=4xyz.$$ ◊

§§ Solution. As in *Ex*. 5, *Art*. 127, we have

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}$$

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

$$x\left(1-y^2\right)\left(1-z^2\right)+y\left(1-z^2\right)\left(1-x^2\right) + z\left(1-x^2\right)\left(1-y^2\right)=4xyz.$$ ■

9.3 Trigonometrical Equations

Solve the equations

§ Problem 9.3.1. $\sin\theta+\sin 7\theta=\sin 4\theta.$ ◊

§§ Solution.

$$\sin\theta+\sin 7\theta=\sin 4\theta;\ \therefore 2\sin 4\theta\cos 3\theta=\sin 4\theta$$

$$\therefore \sin 4\theta=0 \ or\ 2\cos 3\theta=1.$$

If $$\sin 4\theta=0,\ then\ 4\theta=n\pi.$$

If $$2\cos 3\theta=1,\ then\ \cos 3\theta=\frac{1}{2}=\cos\frac{\pi}{3}$$

$$\therefore 3\theta=2n\pi\pm\frac{\pi}{3}=\left(2n\pm\frac{1}{3}\right)\pi.$$

$$\therefore \theta=\frac{n\pi}{4}\ or\ \left(2n\pm\frac{1}{3}\right)\frac{\pi}{3}.$$ ■

§ Problem 9.3.2. $\cos\theta+\cos 7\theta=\cos 4\theta.$ ◊

§§ Solution.

$$\cos\theta+\cos 7\theta=\cos 4\theta;\ \therefore 2\cos 4\theta\cos 3\theta=\cos 4\theta$$

$$\therefore \cos 4\theta=0\ or\ 2\cos 3\theta=1.$$

If $$\cos 4\theta=0,\ then\ 4\theta=2p\pi\pm\frac{\pi}{2}=4p\times\frac{\pi}{2}\pm\frac{\pi}{2}=(4p\pm 1)\frac{\pi}{2}.$$

$$=\text{ an odd multiple of }\frac{\pi}{2}=(2n+1)\frac{\pi}{2}=\left(n+\frac{1}{2}\right)\pi.$$

If $$2\cos 3\theta=1,\ then\ 3\theta=\left(2n\pm\frac{1}{3}\right)\pi[\text{ see last example}].$$

$$\therefore \theta=\left(n+\frac{1}{2}\right)\frac{\pi}{4}\ or\ \left(2n\pm\frac{1}{3}\right)\frac{\pi}{3}.$$ ■

§ Problem 9.3.3. $\cos\theta+\cos 3\theta=2\cos 2\theta.$ ◊

§§ Solution.

$$\cos\theta+\cos 3\theta=2\cos 2\theta;\ \therefore 2\cos 2\theta\cos\theta=2\cos 2\theta$$
$$\therefore \cos 2\theta=0\ or\ \ \cos\theta=1.$$

If $\quad \cos 2\theta=0,\ then\ 2\theta=\left(n+\frac{1}{2}\right)\pi.$[see last example].

If $\quad \cos\theta=1,\ then\ \theta=2n\pi.$

$$\therefore \theta=\left(n+\frac{1}{2}\right)\frac{\pi}{2}\ or\ 2n\pi.$$ ■

§ Problem 9.3.4. $\sin 4\theta-\sin 2\theta=\cos 3\theta.$ ◊

§§ Solution.

$$\sin 4\theta-\sin 2\theta=\cos 3\theta;\ \therefore 2\cos 3\theta\sin\theta=\cos 3\theta$$
$$\therefore \cos 3\theta=0\ or\ 2\sin\theta=1.$$

If $\quad \cos 3\theta=0\ then\ 3\theta=\left(n+\frac{1}{2}\right)\pi.$

If $\quad 2\sin\theta=1\ then\ \sin\theta=\frac{1}{2}=\sin\frac{\pi}{6}$

$$\therefore \theta=n\pi+(-1)^n\frac{\pi}{6}.$$
$$\therefore \theta=\left(n+\frac{1}{2}\right)\frac{\pi}{3}\ or\ n\pi+(-1)^n\frac{\pi}{6}.$$ ■

§ Problem 9.3.5. $\cos\theta-\sin 3\theta=\cos 2\theta.$ ◊

§§ Solution.

$$\cos\theta-\sin 3\theta=\cos 2\theta;\ \therefore \cos\theta-\cos 2\theta=\sin 3\theta$$
$$\therefore 2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}=2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2}$$
$$\therefore \sin\frac{3\theta}{2}=0\ or\ \sin\frac{\theta}{2}=\cos\frac{3\theta}{2}.$$

If $\quad \sin\frac{3\theta}{2}=0\ then\ \frac{3\theta}{2}=n\pi.$

If $\quad \cos\frac{3\theta}{2}=\sin\frac{\theta}{2}=\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right),$

we have $\quad \frac{3\theta}{2}=2n\pi\pm\left(\frac{\pi}{2}-\frac{\theta}{2}\right);$

Taking the upper sign,

$$2\theta=2n\pi+\frac{\pi}{2}$$

Taking the lower sign,

$$\theta=2n\pi-\frac{\pi}{2}.$$
$$\therefore \theta=\frac{2n\pi}{3}\ or\ \left(n+\frac{1}{4}\right)\pi\ or\ \left(2n-\frac{1}{2}\right)\pi.$$ ■

§ Problem 9.3.6. $\sin 7\theta=\sin\theta+\sin 3\theta.$ ◊

§§ Solution.

$$\sin 7\theta=\sin\theta+\sin 3\theta$$
$$\therefore \sin 7\theta-\sin\theta=\sin 3\theta$$
$$\therefore 2\cos 4\theta\sin 3\theta=\sin 3\theta$$
$$\therefore \sin 3\theta=0\ or\ 2\cos 4\theta=1.$$

If $\quad \sin 3\theta=0\ then\ 3\theta=n\pi.$

If $$2\cos 4\theta = 1,\ then\ \cos 4\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$
$$\therefore 4\theta = 2n\pi \pm \frac{\pi}{3}$$
$$\therefore \theta = \frac{n\pi}{3}\ or\ \left(2n \pm \frac{1}{3}\right)\frac{\pi}{4}.$$ ■

§ Problem 9.3.7. $\cos\theta + \cos 2\theta + \cos 3\theta = 0.$ ◊

§§ Solution.

$$\cos\theta + \cos 2\theta + \cos 3\theta = 0$$
$$\therefore \cos\theta + \cos 3\theta + \cos 2\theta = 0$$
$$\therefore 2\cos 2\theta\cos\theta + \cos 2\theta = 0$$
$$\therefore \cos 2\theta\,(2\cos\theta + 1) = 0$$
$$\therefore \cos 2\theta = 0\ or\ 2\cos\theta + 1 = 0.$$

If $$\cos 2\theta = 0\ then\ 2\theta = \left(n + \frac{1}{2}\right)\pi.$$

If $$2\cos\theta + 1 = 0,\ then\ \cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$$
$$\therefore \theta = 2n\pi \pm \frac{2\pi}{3}.$$
$$\therefore \theta = \left(n + \frac{1}{2}\right)\frac{\pi}{2}\ or\ 2n\pi \pm \frac{2\pi}{3}.$$ ■

§ Problem 9.3.8. $\sin\theta + \sin 3\theta + \sin 5\theta = 0.$ ◊

§§ Solution.

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$
$$\therefore \sin\theta + \sin 5\theta + \sin 3\theta = 0$$
$$\therefore 2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$$
$$\therefore \sin 3\theta\,(2\cos 2\theta + 1) = 0$$
$$\therefore \sin 3\theta = 0\ or\ 2\cos 2\theta + 1 = 0.$$

If $$\sin 3\theta = 0\ then\ 3\theta = n\pi.$$

If $$2\cos 2\theta + 1 = 0\ then\ \cos 2\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$$
$$\therefore 2\theta = 2n\pi \pm \frac{2\pi}{3}.$$
$$\therefore \theta = \frac{n\pi}{3}\ or\ \left(n \pm \frac{1}{3}\right)\pi.$$ ■

§ Problem 9.3.9. $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0.$ ◊

§§ Solution.

$$\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$$
$$\therefore \sin 2\theta - \sin\theta = \cos 2\theta - \cos\theta$$
$$\therefore 2\cos\frac{3\theta}{2}\sin\frac{\theta}{2} = -2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}$$
$$\therefore \sin\frac{\theta}{2} = 0\ or\ \cos\frac{3\theta}{2} = -\sin\frac{3\theta}{2}.$$

If $$\sin\frac{\theta}{2} = 0\ then\ \frac{\theta}{2} = n\pi.$$

If $$\cos\frac{3\theta}{2} = -\sin\frac{3\theta}{2}\ then\ \frac{3\theta}{2} = n\pi + \frac{3\pi}{4}.$$
$$\therefore \theta = 2n\pi\ or\ \left(\frac{2n}{3} + \frac{1}{2}\right)\pi.$$ ■

§ Problem 9.3.10. $\sin(3\theta+\alpha)+\sin(3\theta-\alpha)+\sin(\alpha-\theta)-\sin(\alpha+\theta)=\cos\alpha.$ $\lozenge$

§§ Solution.

$$\sin(3\theta+\alpha)+\sin(3\theta-\alpha)+\sin(\alpha-\theta)-\sin(\alpha+\theta)=\cos\alpha$$
$$\therefore 2\sin 3\theta\cos\alpha-2\cos\alpha\sin\theta=\cos\alpha$$
$$\therefore \sin 3\theta-\sin\theta=\frac{1}{2},\ \therefore 2\cos 2\theta\sin\theta=\frac{1}{2}$$
$$2\left(1-2\sin^2\theta\right)\sin\theta=\frac{1}{2}$$
$$\therefore 8\sin^3\theta-4\sin\theta+1=0$$
$$\therefore (2\sin\theta-1)\left(4\sin^2\theta+2\sin\theta-1\right)=0$$
$$\therefore 2\sin\theta-1=0 \textit{ or } 4\sin^2\theta+2\sin\theta-1=0.$$

If $$2\sin\theta-1=0 \textit{ then } \sin\theta=\frac{1}{2}=\sin\frac{\pi}{6}.$$

If $$4\sin^2\theta+2\sin\theta-1=0$$

then $$\sin\theta=\frac{\pm\sqrt{5}-1}{4}=\sin\frac{\pi}{10} \textit{ or } \sin\left(-\frac{3\pi}{10}\right)$$
$$\therefore \theta=n\pi+(-1)^n\frac{\pi}{6}$$

or $$n\pi+(-1)^n\frac{\pi}{10}$$

or $$n\pi-(-1)^n\frac{3\pi}{10}.$$ ■

§ Problem 9.3.11. $\cos(3\theta+\alpha)\cos(3\theta-\alpha)+\cos(5\theta+\alpha)\cos(5\theta-\alpha)=\cos 2\alpha.$ $\lozenge$

§§ Solution.

$$\cos(3\theta+\alpha)\cos(3\theta-\alpha)+\cos(5\theta+\alpha)\cos(5\theta-\alpha)=\cos 2\alpha$$
$$\therefore \cos 6\theta+\cos 2\alpha+\cos 10\theta+\cos 2\alpha=2\cos 2\alpha$$
$$\therefore \cos 6\theta+\cos 10\theta=0; \therefore 2\cos 8\theta\cos 2\theta=0$$
$$\therefore \cos 8\theta=0 \textit{ or } \cos 2\theta=0.$$

If $$\cos 8\theta=0,\ \textit{then } 8\theta=n\pi+\frac{\pi}{2}.$$

If $$\cos 2\theta=0,\ \textit{then } 2\theta=n\pi+\frac{\pi}{2}.$$
$$\therefore \theta=\left(n+\frac{1}{2}\right)\frac{\pi}{8} \textit{ or } \left(n+\frac{1}{2}\right)\frac{\pi}{2}.$$ ■

§ Problem 9.3.12. $\cos n\theta=\cos(n-2)\theta+\sin\theta.$ $\lozenge$

§§ Solution.

$$\cos n\theta=\cos(n-2)\theta+\sin\theta$$
$$\therefore \cos n\theta-\cos(n-2)\theta=\sin\theta$$
$$\therefore -2\sin(n-1)\theta\sin\theta=\sin\theta$$
$$\therefore \sin\theta=0 \textit{ or } 2\sin(n-1)\theta=-1.$$

If $\sin\theta=0$ *then* $\theta=m\pi,$ where m is any integer.

If $$2\sin(n-1)\theta=-1 \textit{ then } \sin(n-1)\theta=-\frac{1}{2}=\sin\left(-\frac{\pi}{6}\right)$$
$$\therefore (n-1)\theta=m\pi-(-1)^m\frac{\pi}{6}.$$
$$\therefore \theta=m\pi \textit{ or } \frac{1}{n-1}\left[m\pi-(-1)^m\frac{\pi}{6}\right].$$ ■

§ Problem 9.3.13. $\sin\dfrac{n+1}{2}\theta = \sin\dfrac{n-1}{2}\theta + \sin\theta.$ ◊

§§ Solution.

$$\sin\frac{n+1}{2}\theta = \sin\frac{n-1}{2}\theta + \sin\theta$$
$$\therefore \sin\frac{n+1}{2}\theta - \sin\frac{n-1}{2}\theta = \sin\theta$$
$$\therefore 2\cos\frac{n\theta}{2}\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$
$$\therefore \sin\frac{\theta}{2} = 0 \ or \ \cos\frac{n\theta}{2} = \cos\frac{\theta}{2}.$$

If $$\sin\frac{\theta}{2} = 0 \ then \ \frac{\theta}{2} = m\pi.$$

If $$\cos\frac{n\theta}{2} = \cos\frac{\theta}{2}, \ then \ \frac{n\theta}{2} = 2m\pi \pm \frac{\theta}{2}$$

taking the upper sign,

$$(n-1)\theta = 4m\pi$$

taking the lower sign,

$$(n+1)\theta = 4m\pi.$$
$$\therefore \theta = 2m\pi \ or \ \frac{4m\pi}{n\mp 1}.$$ ■

§ Problem 9.3.14. $\sin m\theta + \sin n\theta = 0.$ ◊

§§ Solution.

$$\sin m\theta + \sin n\theta = 0$$
$$\therefore 2\sin\frac{m+n}{2}\theta\cdot\cos\frac{m-n}{2}\theta = 0$$
$$\therefore \sin\frac{m+n}{2}\theta = 0 \ or \ \cos\frac{m-n}{2}\theta = 0.$$

If $$\sin\frac{m+n}{2}\theta = 0 \ then \ \frac{m+n}{2}\theta = r\pi,$$

where r is any integer.

If $$\cos\frac{m-n}{2}\theta = 0, \ then \ \frac{m-n}{2}\theta = r\pi + \frac{\pi}{2}.$$
$$\therefore \theta = \frac{2r\pi}{m+n} \ or \ (2r+1)\frac{\pi}{m-n}.$$ ■

§ Problem 9.3.15. $\cos m\theta + \cos n\theta = 0.$ ◊

§§ Solution.

$$\cos m\theta + \cos n\theta = 0\,[Cf.\ \S Problem\ 6.1.28.]$$
$$\therefore 2\cos\frac{m+n}{2}\theta\cdot\cos\frac{m-n}{2}\theta = 0$$
$$\therefore \cos\frac{m+n}{2}\theta = 0 \ or \ \cos\frac{m-n}{2}\theta = 0.$$

If $$\cos\frac{m+n}{2}\theta = 0 \ then \ \frac{m+n}{2}\theta = r\pi + \frac{\pi}{2}.$$

If $$\cos\frac{m-n}{2}\theta = 0, \ then \ \frac{m-n}{2}\theta = r\pi + \frac{\pi}{2}.$$
$$\therefore \theta = (2r+1)\frac{\pi}{m\pm n}.$$ ■

§ Problem 9.3.16. $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta.$ ◊

§§ Solution.

$$\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta$$

$$\therefore \sin\left[n\theta + (n-1)\theta\right]\sin\left[n\theta - (n-1)\theta\right] = \sin^2\theta, \ \textit{by Ex. 2, Art. 93}$$

$$\therefore \sin(2n-1)\theta\sin\theta = \sin^2\theta$$

$$\therefore \sin\theta = 0 \ or \ \sin(2n-1)\theta = \sin\theta.$$

If $$\sin\theta = 0 \ then \ \theta = m\pi.$$

If $$\sin(2n-1)\theta = \sin\theta,$$

then $$\sin(2n-1)\theta - \sin\theta = 0$$

$$\therefore 2\cos n\theta\sin(n-1)\theta = 0$$

$$\therefore \cos n\theta = 0 \ or \ \sin(n-1)\theta = 0$$

If $$\cos n\theta = 0 \ then \ n\theta = m\pi + \frac{\pi}{2}$$

If $$\sin(n-1)\theta = 0, \ then \ (n-1)\theta = m\pi.$$

$$\therefore \theta = m\pi \ or \ \left(m + \frac{1}{2}\right)\frac{\pi}{n} \ or \ \frac{m\pi}{n-1}.$$ ■

§ Problem 9.3.17. $\sin 3\theta + \cos 2\theta = 0.$ ◊

§§ Solution.

$$\sin 3\theta + \cos 2\theta = 0$$

$$\therefore \cos 2\theta - \cos\left(\frac{\pi}{2} + 3\theta\right) = 0$$

$$\therefore 2\sin\left(\frac{\pi}{4} + \frac{5\theta}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = 0$$

$$\therefore \sin\left(\frac{\pi}{4} + \frac{5\theta}{2}\right) = 0 \ or \ \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = 0.$$

If $$\sin\left(\frac{\pi}{4} + \frac{5\theta}{2}\right) = 0 \ then \ \frac{\pi}{4} + \frac{5\theta}{2} = n\pi.$$

If $$\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = 0 \ then \ \frac{\pi}{4} + \frac{\theta}{2} = n\pi.$$

$$\therefore \theta = \frac{1}{5}\left(2n\pi - \frac{\pi}{2}\right) \ or \ 2n\pi - \frac{\pi}{2}.$$ ■

§ Problem 9.3.18. $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}.$ ◊

§§ Solution. $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}.$

Dividing both sides of the equation by $\sqrt{3+1}$, i.e. 2, we have

$$\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}} \tag{9.1}$$

$$\therefore \sin\left(\frac{\pi}{3} + \theta\right) = \sin\frac{\pi}{4}$$

$$\therefore \left(\frac{\pi}{3} + \theta\right) = n\pi + (-1)^n\frac{\pi}{4}$$

$$\therefore \theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3} \tag{9.2}$$

Otherwise thus :

From Eq. (9.1), we have

$$\cos\left(\theta - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}; \ \therefore \theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$$

$$\therefore \theta = 2n\pi + \frac{5\pi}{12} \ or \ 2n\pi - \frac{\pi}{12} \tag{9.3}$$

To show that Eq. (9.2) and Eq. (9.3) give the same values for θ : in Eq. (9.2) let n be even ($= 2m$, say) and we have

$$\theta = 2m\pi + \frac{\pi}{4} - \frac{\pi}{3} = 2m\pi - \frac{\pi}{12}$$

let n be odd ($= 2m+1$, say) and we have

$$\theta = (2m+1)\pi - \frac{\pi}{4} - \frac{\pi}{3} = 2m\pi + \left(\pi - \frac{7\pi}{12}\right) = 2m\pi + \frac{5\pi}{12}. \qquad \blacksquare$$

§ Problem 9.3.19. $\sin\theta + \cos\theta = \sqrt{2}$. ◊

§§ Solution. $\sin\theta + \cos\theta = \sqrt{2}$.

Dividing both sides of the equation by $\sqrt{1+1}$, i.e. $\sqrt{2}$, we have

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = 1.$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = 1;\ \theta - \frac{\pi}{4} = 2n\pi \ i.e.\ \theta = \left(2n + \frac{1}{4}\right)\pi.$$

Otherwise thus :

we have
$$\sin\left(\theta + \frac{\pi}{4}\right) = 1.$$
$$\therefore \theta + \frac{\pi}{4} = 2n\pi + \frac{\pi}{2}$$
$$\therefore \theta = \left(2n + \frac{1}{4}\right)\pi. \qquad \blacksquare$$

§ Problem 9.3.20. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$. ◊

§§ Solution. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

Dividing both sides of the equation by $\sqrt{3+1}$, i.e. 2, we have

$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}.$$

$$\therefore \sin\left(\theta - \frac{\pi}{6}\right) = \sin\frac{\pi}{4};\ \theta - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{4}$$
$$\therefore \theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{6}. \qquad \blacksquare$$

§ Problem 9.3.21. $\sin x + \cos x = \sqrt{2}\cos A$. ◊

§§ Solution. $\sin x + \cos x = \sqrt{2}\cos A$.

Dividing both sides of the equation by $\sqrt{1+1}$, i.e. $\sqrt{2}$, we have

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \cos A.$$
$$\therefore \cos\left(x - \frac{\pi}{4}\right) = \cos A$$
$$\therefore x - \frac{\pi}{4} = 2n\pi \pm A$$
$$\therefore x = 2n\pi + \frac{\pi}{4} \pm A. \qquad \blacksquare$$

§ Problem 9.3.22. $5\sin\theta + 2\cos\theta = 5$ *(given* $\tan 21^\circ 48' = \cdot 4$*).* ◊

§§ Solution. $5\sin\theta + 2\cos\theta = 5$.

Dividing both sides of the equation by $\sqrt{5^2+2^2}$, i.e. $\sqrt{29}$, we have

$$\frac{5}{\sqrt{29}}\sin\theta + \frac{2}{\sqrt{29}}\cos\theta = \frac{5}{\sqrt{29}}.$$
$$\therefore \cos 21^\circ 48' \sin\theta + \sin 21^\circ 48' \cos\theta = \cos 21^\circ 48'$$
$$\therefore \sin\left(\theta + 21^\circ 48'\right) = \cos 21^\circ 48' = \sin\left(90^\circ - 21^\circ 48'\right) = \sin 68^\circ 12'$$
$$\therefore \theta + 21^\circ 48' = n \times 180^\circ + (-1)^n\left(68^\circ 12'\right)$$

$$\therefore \theta = -21^\circ 48' + n \times 180^\circ + (-1)^n \left(68^\circ 12'\right). \qquad \blacksquare$$

§ Problem 9.3.23. $6\cos x + 8\sin x = 9$ *(given* $\tan 53^\circ 8' = 1\frac{1}{3}$ *and* $\cos 25^\circ 50' = \cdot 9$*).* $\lozenge$

§§ Solution. $6\cos x + 8\sin x = 9$.

Dividing both sides of the equation by $\sqrt{6^2+8^2}$, i.e. 10, we have

$$\frac{3}{5}\cos x + \frac{4}{5}\sin x = \frac{9}{10}.$$
$$\therefore \cos 53^\circ 8' \cos x + \sin 53^\circ 8' \sin x = \cdot 9$$
$$\therefore \cos\left(x - 53^\circ 8'\right) = \cos 25^\circ 50'$$
$$\therefore x - 53^\circ 8' = 2n \times 180^\circ \pm 25^\circ 50'$$
$$\therefore x = 2n \times 180^\circ + 78^\circ 58' \text{ or } 2n \times 180^\circ + 27^\circ 18'. \qquad \blacksquare$$

§ Problem 9.3.24. $1 + \sin^2\theta = 3\sin\theta\cos\theta$ *(given* $\tan 71^\circ 34' = 3$*).* $\lozenge$

§§ Solution. $1 + \sin^2\theta = 3\sin\theta\cos\theta$.

$$\therefore 2 + 2\sin^2\theta = 6\sin\theta\cos\theta$$
$$\therefore 2 + 1 - \cos 2\theta = 3\sin 2\theta$$
$$\therefore \cos 2\theta + 3\sin 2\theta = 3.$$

Dividing both sides of this last equation by $\sqrt{1+3^2}$, i.e. $\sqrt{10}$, we have

$$\frac{1}{\sqrt{10}}\cos 2\theta + \frac{3}{\sqrt{10}}\sin 2\theta = \frac{3}{\sqrt{10}}.$$
$$\therefore \cos 71^\circ 34' \cos 2\theta + \sin 71^\circ 34' \sin 2\theta = \sin 71^\circ 34'$$
$$\therefore \cos\left(2\theta - 71^\circ 34'\right) = \cos\left(90^\circ - 71^\circ 34'\right) = \cos 18^\circ 26'$$
$$\therefore 2\theta - 71^\circ 34' = 2n \times 180^\circ \pm 18^\circ 26'$$

$$\therefore 2\theta = 2n \times 180^\circ + 90^\circ \text{ or } 2n \times 180^\circ + 52^\circ 8'$$
$$\therefore \theta = n \times 180^\circ + 45^\circ \text{ or } n \times 180^\circ + 26^\circ 34'. \qquad \blacksquare$$

§ Problem 9.3.25. $\operatorname{cosec}\theta = \cot\theta + \sqrt{3}$. $\lozenge$

§§ Solution. $\operatorname{cosec}\theta = \cot\theta + \sqrt{3}$.

$$\therefore \frac{1}{\sin\theta} = \frac{\cos\theta}{\sin\theta} + \sqrt{3}$$
$$\therefore \cos\theta + \sqrt{3}\sin\theta = 1.$$

Dividing both sides of this last equation by $\sqrt{1+3}$, i.e. 2, we have

$$\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = \frac{1}{2}.$$
$$\therefore \cos\left(\theta - \frac{\pi}{3}\right) = \cos\frac{\pi}{3}$$
$$\therefore \theta - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$$
$$\therefore \theta = 2n\pi \text{ or } 2n\pi + \frac{2\pi}{3}. \qquad \blacksquare$$

§ Problem 9.3.26. $\operatorname{cosec} x = 1 + \cot x$. $\lozenge$

§§ Solution. $\operatorname{cosec} x = 1 + \cot x$.

$$\therefore \frac{1}{\sin x} = 1 + \frac{\cos x}{\sin x}; \ \therefore \cos x + \sin x = 1.$$

Dividing both sides of this last equation by $\sqrt{1+1}$, i.e. $\sqrt{2}$, we have

$$\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}.$$

$$\therefore \cos\left(x-\frac{\pi}{4}\right)=\cos\frac{\pi}{4}$$
$$\therefore x-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$$
$$\therefore x=2n\pi \text{ or } 2n\pi+\frac{\pi}{2}. \qquad \blacksquare$$

§ Problem 9.3.27. $\left(2+\sqrt{3}\right)\cos\theta=1-\sin\theta.$ ◊

§§ Solution. $\left(2+\sqrt{3}\right)\cos\theta=1-\sin\theta.$

$$\therefore 2+\sqrt{3}=\frac{1-\sin\theta}{\cos\theta}=\frac{1+\cos\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right)}=\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \text{ [§Problem 8.1.4]}$$
$$\therefore \cot\frac{\pi}{12}=\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right);\ \therefore \frac{\pi}{4}+\frac{\theta}{2}=n\pi+\frac{\pi}{12}$$
$$\therefore \frac{\theta}{2}=n\pi-\frac{\pi}{6};\ \therefore \theta=2n\pi-\frac{\pi}{3}.$$

Also, both sides of the original equation become zero if

$$\cos\theta=0 \text{ and } 1-\sin\theta=0,\ \text{i.e. if } \theta=2n\pi+\frac{\pi}{2}. \qquad \blacksquare$$

§ Problem 9.3.28. $\tan\theta+\sec\theta=\sqrt{3}.$ ◊

§§ Solution. $\tan\theta+\sec\theta=\sqrt{3}.$

$$\therefore \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}=\sqrt{3}$$
$$\therefore \sqrt{3}\cos\theta-\sin\theta=1.$$

Dividing both sides of this last equation by $\sqrt{3+1}$, i.e. 2, we have

$$\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta=\frac{1}{2}$$
$$\therefore \cos\left(\theta+\frac{\pi}{6}\right)=\cos\frac{\pi}{3}$$
$$\therefore \theta+\frac{\pi}{6}=2n\pi\pm\frac{\pi}{3}$$
$$\therefore \theta=2n\pi+\frac{\pi}{6} \text{ or } 2n\pi-\frac{\pi}{2}. \qquad \blacksquare$$

§ Problem 9.3.29. $\cos 2\theta=\cos^2\theta.$ ◊

§§ Solution. $\cos 2\theta=\cos^2\theta.$

$$\therefore \cos^2\theta-\sin^2\theta=\cos^2\theta;\ \therefore \sin^2\theta=0;\ \therefore \theta=n\pi. \qquad \blacksquare$$

§ Problem 9.3.30. $4\cos\theta-3\sec\theta=\tan\theta.$ ◊

§§ Solution. $4\cos\theta-3\sec\theta=\tan\theta.$

$$\therefore 4\cos\theta-\frac{3}{\cos\theta}=\frac{\sin\theta}{\cos\theta}$$
$$\therefore 4\cos^2\theta-3=\sin\theta;\ \therefore 4\left(1-\sin^2\theta\right)-3=\sin\theta$$
$$\therefore 4\sin^2\theta+\sin\theta-1=0$$
$$\therefore \sin\theta=\frac{-1\pm\sqrt{1+16}}{8}=\frac{\pm\sqrt{17}-1}{8}. \qquad \blacksquare$$

§ Problem 9.3.31. $\cos 2\theta+3\cos\theta=0.$ ◊

§§ Solution. $\cos 2\theta + 3\cos\theta = 0.$

$$\therefore 2\cos^2\theta - 1 + 3\cos\theta = 0$$

$$\therefore \cos\theta = \frac{-3 \pm \sqrt{9+8}}{4} = \frac{\pm\sqrt{17}-3}{4}.$$

The value $\dfrac{-\sqrt{17}-3}{4}$ is inadmissible, $\therefore \cos\theta = \dfrac{\sqrt{17}-3}{4}.$ ■

§ Problem 9.3.32. $\cos 3\theta + 2\cos\theta = 0.$ ◊

§§ Solution. $\cos 3\theta + 2\cos\theta = 0.$

$$\therefore 4\cos^3\theta - 3\cos\theta + 2\cos\theta = 0\,[Art.\ 107]$$

$$\therefore 4\cos^3\theta - \cos\theta = 0;\ \therefore \cos\theta\left(4\cos^2\theta - 1\right) = 0$$

$$\therefore \cos\theta = 0\ or\ 4\cos^2\theta - 1 = 0.$$

If $$\cos\theta = 0\ then\ \theta = n\pi + \frac{\pi}{2}.$$

If $$4\cos^2\theta - 1 = 0\ then\ \cos\theta = \pm\frac{1}{2} = \pm\cos\frac{\pi}{3}.$$

$$\therefore \theta = \left(n + \frac{1}{2}\right)\pi\ or\ n\pi \pm \frac{\pi}{3}.$$ ■

§ Problem 9.3.33. $\cos 2\theta = \left(\sqrt{2}+1\right)\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right).$ ◊

§§ Solution. $\cos 2\theta = \left(\sqrt{2}+1\right)\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right).$

$$\therefore 2\cos^2\theta - 1 = \left(\sqrt{2}+1\right)\cos\theta - 1 - \frac{1}{\sqrt{2}}$$

$$\therefore 2\cos^2\theta - \left(\sqrt{2}+1\right)\cos\theta + \frac{1}{\sqrt{2}} = 0$$

$$\therefore (2\cos\theta - 1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right) = 0.$$

$$\therefore 2\cos\theta - 1 = 0\ or\ \cos\theta - \frac{1}{\sqrt{2}} = 0.$$

If $$2\cos\theta - 1 = 0,\ then\ \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}.$$

If $$\cos\theta - \frac{1}{\sqrt{2}} = 0,\ then\ \cos\theta = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}.$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}\ or\ 2n\pi \pm \frac{\pi}{4}.$$ ■

§ Problem 9.3.34. $\cot\theta - \tan\theta = 2.$ ◊

§§ Solution. $\cot\theta - \tan\theta = 2.$

$$\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = 2;\ \therefore \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$

$$\therefore \cos 2\theta = \sin 2\theta = \cos\left(\frac{\pi}{2} - 2\theta\right);\ \therefore 2\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right)$$

$$\therefore 4\theta = 2n\pi + \frac{\pi}{2};\ \therefore \theta = \left(n + \frac{1}{4}\right)\frac{\pi}{2}.$$

Otherwise thus :

By § Problem 8.1.5:

$$\cot\theta - \tan\theta = 2\cot 2\theta$$

$$\therefore 2\cot 2\theta = 2;\ \therefore \cot 2\theta = 1 = \cot\frac{\pi}{4}$$

$$\therefore 2\theta = n\pi + \frac{\pi}{4};\ \therefore \theta = \left(n + \frac{1}{4}\right)\frac{\pi}{2}.$$ ■

§ Problem 9.3.35. $4\cot 2\theta = \cot^2\theta - \tan^2\theta.$ ◊

§§ Solution. $4\cot 2\theta = \cot^2\theta - \tan^2\theta.$

$$\therefore 2(\cot\theta - \tan\theta) = \cot^2\theta - \tan^2\theta \text{ [see last example]}$$

$$\therefore \cot\theta - \tan\theta = 0 \text{ or } 2 = \cot\theta + \tan\theta.$$

If $$\cot\theta - \tan\theta = 0,\ then\ \frac{1}{\tan\theta} - \tan\theta = 0$$

$$\therefore 1 - \tan^2\theta = 0;\ \therefore \tan\theta = \pm 1,\ i.e.\ \theta = n\pi \pm \frac{\pi}{4}.$$

If $$\cot\theta + \tan\theta = 2,\ then\ \frac{1}{\tan\theta} + \tan\theta = 2$$

$$\therefore \tan^2\theta - 2\tan\theta + 1 = 0,\ i.e.\ (\tan\theta - 1)^2 = 0$$

$\therefore \tan\theta = 1,$ which is included in the former solution.

Otherwise thus :

The given equation may be written

$$\frac{4}{\tan 2\theta} = \frac{1}{\tan^2\theta} - \tan^2\theta$$

$$\therefore \frac{4\left(1 - \tan^2\theta\right)}{2\tan\theta} = \frac{1 - \tan^4\theta}{\tan^2\theta}$$

$$\therefore 1 - \tan^2\theta = 0 \text{ or } 2 = \frac{1 + \tan^2\theta}{\tan\theta},$$

and the solution follows as before. ■

§ Problem 9.3.36. $3\tan(\theta - 15°) = \tan(\theta + 15°).$ ◊

§§ Solution. $3\tan(\theta - 15°) = \tan(\theta + 15°).$

$$\therefore \frac{\tan(\theta - 15°)}{\tan(\theta + 15°)} = \frac{1}{3}.$$

Now $$\frac{\tan(\theta - 15°)}{\tan(\theta + 15°)} = \frac{\sin(\theta - 15°)}{\cos(\theta - 15°)} \div \frac{\sin(\theta + 15°)}{\cos(\theta + 15°)}$$

$$= \frac{2\cos(\theta + 15°)\sin(\theta - 15°)}{2\sin(\theta + 15°)\cos(\theta - 15°)}$$

$$= \frac{\sin 2\theta - \sin 30°}{\sin 2\theta + \sin 30°}$$

$$\therefore \frac{\sin 2\theta - \sin 30°}{\sin 2\theta + \sin 30°} = \frac{1}{3};\ \therefore \frac{2\sin 2\theta}{2\sin 30°} = \frac{4}{2}$$

$$\therefore \sin 2\theta = 2 \times \sin 30° = 2 \times \frac{1}{2} = 1$$

$$\therefore 2\theta = 2n\pi + \frac{\pi}{2};\ \therefore \theta = n\pi + \frac{\pi}{4}.$$ ■

§ Problem 9.3.37. $\tan\theta + \tan 2\theta + \tan 3\theta = 0.$ ◊

§§ Solution. $\tan\theta + \tan 2\theta + \tan 3\theta = 0.$

$$\therefore \frac{\sin\theta}{\cos\theta} + \frac{\sin 2\theta}{\cos 2\theta} + \frac{\sin 3\theta}{\cos 3\theta} = 0$$

$$\therefore \frac{\sin\theta\cos 3\theta + \sin 3\theta\cos\theta}{\cos\theta\cos 3\theta} + \frac{\sin 2\theta}{\cos 2\theta} = 0$$

$$\therefore \frac{\sin(\theta + 3\theta)}{\cos\theta\cos 3\theta} + \frac{\sin 2\theta}{\cos 2\theta} = 0$$

$$\therefore \frac{\sin 4\theta}{\cos\theta\cos 3\theta} + \frac{\sin 2\theta}{\cos 2\theta} = 0$$

$$\therefore \frac{2\sin 2\theta\cos 2\theta}{\cos\theta\cos 3\theta}+\frac{\sin 2\theta}{\cos 2\theta}=0$$

$$\therefore \textit{ either } \sin 2\theta=0,\ \therefore 2\theta=n\pi,\ \textit{i.e. } \theta=\frac{n\pi}{2};\ \textit{or}$$

$$2\cos^2 2\theta+\cos\theta\cos 3\theta=0.$$

$$\therefore 2\,(1+\cos 4\theta)+\cos 4\theta+\cos 2\theta=0$$

$$\therefore 3\cos 4\theta+\cos 2\theta+2=0$$

$$\therefore 3\left(2\cos^2 2\theta-1\right)+\cos 2\theta+2=0$$

$$\therefore 6\cos^2 2\theta+\cos 2\theta-1=0$$

$$\therefore (2\cos 2\theta+1)\,(3\cos 2\theta-1)=0$$

$$\therefore 2\cos 2\theta+1=0 \textit{ or } 3\cos 2\theta-1=0.$$

If $$2\cos 2\theta+1=0,\ \textit{then}\ \cos 2\theta=-\frac{1}{2}=\cos\frac{2\pi}{3}$$

$$\therefore 2\theta=2n\pi\pm\frac{2\pi}{3},\ \textit{i.e. } \theta=n\pi\pm\frac{\pi}{3}$$

If $$3\cos 2\theta-1=0,\ \textit{then}\ \cos 2\theta=\frac{1}{3}$$

$$\therefore 2\theta=2n\pi\pm\alpha,\ \textit{where}\ \cos\alpha=\frac{1}{3},\ \textit{i.e. } \theta=n\pi\pm\frac{\alpha}{2}.$$ ■

§ **Problem 9.3.38.** $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}$. ◊

§§ **Solution**. $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}$.

$$\therefore \tan\theta+\tan 2\theta=\sqrt{3}\,(1-\tan\theta\tan 2\theta)$$

$$\therefore \frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=\sqrt{3}=\tan\frac{\pi}{3}$$

$$\therefore \tan 3\theta=\tan\frac{\pi}{3}\quad [\textit{Art. } 107]$$

$$3\theta=n\pi+\frac{\pi}{3};\ \therefore \theta=\left(n+\frac{1}{3}\right)\frac{\pi}{3}.$$ ■

§ **Problem 9.3.39.** $\sin 3\alpha=4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$. ◊

§§ **Solution**. $\sin 3\alpha=4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$.

$$\therefore 3\sin\alpha-4\sin^3\alpha=4\sin\alpha\left(\sin^2 x-\sin^2\alpha\right)\quad [\textit{By Arts. } 107 \textit{ and } 93]$$

$$\therefore 3\sin\alpha=4\sin\alpha\sin^2 x;\ \therefore \sin^2 x=\frac{3}{4}$$

$$\therefore \sin x=\pm\frac{\sqrt{3}}{2}=\sin\left(\pm\frac{\pi}{3}\right);\ \therefore x=n\pi\pm\frac{\pi}{3}.$$ ■

§ **Problem 9.3.40.** *Prove that the equation $x^3-2x+1=0$ is satisfied by putting for x either of the values*

$$\sqrt{2}\sin 45^\circ,\ 2\sin 18^\circ \textit{ and } 2\sin 234^\circ.$$ ◊

§§ **Solution**. $x^3-2x+1=0$.

$$\therefore (x-1)\left(x^2+x-1\right)=0$$

$$\therefore x-1=0 \textit{ or } x^2+x-1=0.$$

If $$x-1=0 \textit{ then } x=1.$$

If $$x^2+x-1=0 \textit{ then } x=\frac{\pm\sqrt{5}-1}{2}.$$

Now $$\sqrt{2}\sin 45^\circ=\sqrt{2}\times\frac{1}{\sqrt{2}}=1$$

$$2\sin 18^\circ=2\times\frac{\sqrt{5}-1}{4}=\frac{\sqrt{5}-1}{2}\ \textit{and}$$

$$2\sin 234^\circ = -2\sin 54^\circ = -2\times\frac{\sqrt{5}+1}{4} = \frac{-\sqrt{5}-1}{2}.$$ ■

§ Problem 9.3.41. *If* $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, *prove that*

$$\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}.$$ ◊

§§ Solution. $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$.

$$\therefore \cos\left(\frac{\pi}{2} - \pi\cos\theta\right) = \cos(\pi\sin\theta)$$

$$\therefore \frac{\pi}{2} - \pi\cos\theta = 2n\pi \pm \pi\sin\theta,\ \text{where } n \text{ is any integer.}$$

$$\therefore \cos\theta \pm \sin\theta = \frac{1}{2} - 2n$$

$$\therefore \cos\theta\cdot\frac{1}{\sqrt{2}} \pm \sin\theta\cdot\frac{1}{\sqrt{2}} = \frac{1-4n}{2\sqrt{2}}$$

$$\therefore \cos\left(\theta \mp \frac{\pi}{4}\right) = \frac{1-4n}{2\sqrt{2}}.$$

Now n must be zero; for otherwise, since it is an integer, the right-hand member would be numerically greater than unity

$$\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}.$$ ■

§ Problem 9.3.42. *If* $\sin(\pi\cot\theta) = \cos(\pi\tan\theta)$, *prove that either* $\operatorname{cosec} 2\theta$ *or* $\cot 2\theta$ *is equal to* $n + \frac{1}{4}$ *where* n *is a positive or negative integer.* ◊

§§ Solution. $\sin(\pi\cot\theta) = \cos(\pi\tan\theta)$.

$$\therefore \cos(\pi\tan\theta) = \cos\left(\frac{\pi}{2} - \pi\cot\theta\right).$$

$$\therefore \pi\tan\theta = 2n\pi \pm \left(\frac{\pi}{2} - \pi\cot\theta\right),$$

where n is zero or some positive or negative integer.

Taking the upper sign,

$$2n + \frac{1}{2} = \tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta}$$

$$\therefore \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta = n + \frac{1}{4}.$$

Taking the lower sign,

$$2n - \frac{1}{2} = \tan\theta - \cot\theta = -\frac{2\cos 2\theta}{\sin 2\theta};\ \therefore \cot 2\theta = \frac{1}{4} - n.$$

Thus either $\operatorname{cosec} 2\theta$ or $\cot 2\theta$ is equal $n + \frac{1}{4}$. ■

Chapter 10

Logarithms

10.1 Characteristics, Mantissa and Properties

§ Problem 10.1.1. *Given* $\log 4 = .60206$ *and* $\log 3 = .4771213$*, find the logarithms of*

$$.8,\ .003,\ .0108\ \textit{and}\ (.00018)^{\frac{1}{7}}\ . \qquad \Diamond$$

§§ Solution. $.8 = \dfrac{8}{10} = \dfrac{2^3}{10}$

$$\therefore \log(.8) = 3\log 2 - \log 10 = 3\log\left(\sqrt{4}\right) - \log 10$$
$$= \frac{3}{2}\log 4 - \log 10 = .90309 - 1 = \bar{1}.90309$$
$$.003 = \frac{3}{1000} = \frac{3}{10^3}$$
$$\therefore \log(.003) = \log 3 - 3\log 10 = .4771213 - 3 = \bar{3}.4771213$$
$$.0108 = \frac{108}{10^4} = \frac{4\times 27}{10^4} = \frac{4\times 3^3}{10^4}$$
$$\therefore \log(.0108) = \log 4 + 3\log 3 - 4\log 10$$
$$= .60206 + 1.4313639 - 4 = \bar{2}.0334239.$$
$$.00018 = \frac{18}{10^5} = \frac{2\times 3^2}{10^5} = \frac{\sqrt{4}\times 3^2}{10^5}$$
$$\therefore \log(.00018)^{\frac{1}{7}} = \frac{1}{7}\left(\frac{1}{2}\log 4 + 2\log 3 - 5\log 10\right)$$
$$= \frac{1}{7}(.30103 + .9542426 - 5) = \bar{1}.4650389. \qquad \blacksquare$$

§ Problem 10.1.2. *Given* $\log 11 = 1.0413927$ *and* $\log 13 = 1.1139434$*, find the values of*

(1) $\log 1.43$

(2) $\log 133.1$

(3) $\log \sqrt[4]{143}$, *and*

(4) $\log \sqrt[3]{.00169}$. ◊

§§ Solution. (1) $1.43 = \dfrac{143}{100} = \dfrac{11 \times 13}{10^2}$

$$\begin{aligned}\therefore \log 1.43 &= \log 11 + \log 13 - 2\log 10\\ &= 1.0413927 + 1.1139434 - 2 = .1553361.\end{aligned}$$

(2) $133.1 = \dfrac{1331}{10} = \dfrac{11^3}{10}$

$$\therefore \log 133.1 = 3\log 11 - \log 10 = 3.1241781 - 1 = 2.1241781.$$

(3)

$$\begin{aligned}\log \sqrt[4]{143} = \log \sqrt[4]{11 \times 13} &= \frac{1}{4}(\log 11 + \log 13)\\ &= \frac{1}{4}(2.1553361) = .5288340.\end{aligned}$$

(4) $.00169 = \dfrac{169}{10^5} = \dfrac{13^2}{10^5}$

$$\begin{aligned}\therefore \log \sqrt[3]{.00169} &= \frac{1}{3}(2\log 13 - 5\log 10) = \frac{1}{3}(2.2278868 - 5)\\ &= \frac{1}{3}\left(\bar{3}.2278868\right) = \bar{1}.0759623.\end{aligned}$$ ■

§ Problem 10.1.3. *What are the characteristics of the logarithms of* 243.7, $.0153$, 2.8713, $.00057$, $.023$, $\sqrt[5]{24615}$ *and* $(24589)^{\frac{3}{4}}$ *?* ◊

§§ Solution.

$$\begin{aligned}\log 243.7 &= 2. \ \ldots [Art.\ 142,\ (i)].\\ \log(.0153) &= \bar{2}. \ \ldots [Art.\ 142,\ (ii)].\\ \log 2.8713 &= 0. \ \ldots [Art.\ 142,\ (i)].\\ \log(.00057) &= \bar{4}. \ \ldots [Art.\ 142,\ (ii)].\\ \log(.023) &= \bar{2}. \ \ldots [Art.\ 142,\ (ii)].\\ \log 24615 &= 4. \ \ldots [Art.\ 142,\ (i)].\\ \therefore \log \sqrt[5]{24615} &= \frac{1}{5}\log 24615 = 0. \ \ldots\\ \log 24589 &= 4. \ \ldots [Art.\ 142,\ (i)].\\ \therefore \log (24589)^{\frac{3}{4}} &= \frac{3}{4}\log 24589 = 3. \ \ldots\end{aligned}$$ ■

§ Problem 10.1.4. *Find the* 5^{th} *root of* $.003$, *having given* $\log 3 = .4771213$ *and*

$$\log 312936 = 5.4954243.$$ ◊

§§ Solution.

$$\begin{aligned}\log(.003)^{\frac{1}{5}} &= \frac{1}{5}(3.4771213), \ \textit{by} \ \S\textit{Problem}\ 10.1.1\\ &= \frac{1}{5}\left(\bar{5} + 2.4771213\right) = \bar{1}.4954243 = \log(.312936)\\ \therefore (.003)^{\frac{1}{5}} &= .312936.\end{aligned}$$ ■

§ Problem 10.1.5. *Find the value of*

(1) $7^{\frac{1}{7}}$

(2) $(84)^{\frac{2}{5}}$*, and*

(3) $(.021)^{\frac{1}{5}}$

having given

$$\log 2 = .30103,\ \log 3 = .4771213$$
$$\log 7 = .8450980,\ \log 132057 = 5.1207283$$
$$\log 588453 = 5.7697117 \textit{ and } \log 461791 = 5.6644438.$$

◊

§§ Solution. (1)

$$\begin{aligned}\log 7^{\frac{1}{7}} &= \frac{1}{7}\log 7 = \frac{1}{7}(.8450980)\\ &= .1207283 = \log 1.32057\\ \therefore 7^{\frac{1}{7}} &= 1.32057.\end{aligned}$$

(2)

$$\begin{aligned}84 = 12 \times 7 &= 2^2 \times 3 \times 7\\ \therefore \log(84)^{\frac{2}{5}} &= \frac{2}{5}\left(2\log 2 + \log 3 + \log 7\right)\\ &= \frac{2}{5}\left(.60206 + .4771213 + .8450980\right)\\ &= \frac{2}{5}(1.9242793) = .7697117 = \log 5.88453\\ \therefore (84)^{\frac{2}{5}} &= 5.88453.\end{aligned}$$

(3) $(.021)^{\frac{1}{5}} = \frac{21}{10^3} = \frac{3 \times 7}{10^3}$

$$\begin{aligned}\therefore \log(.021)^{\frac{1}{5}} &= \frac{1}{5}\left(\log 3 + \log 7 - 3\log 10\right) = \frac{1}{5}\left(\bar{2}.3222193\right)\\ &= \frac{1}{5}\left(\bar{5} + 3.3222193\right) = \bar{1}.6644439 = \log(.461791)\\ \therefore (.021)^{\frac{1}{5}} &= .461791.\end{aligned}$$

■

§ Problem 10.1.6. *Having given* $\log 3 = .4771213$*, find the number of digits in*

(1) 3^{43}

(2) 3^{27} *and*

(3) 3^{62}

and the position of the first significant figure in

(1) 3^{-13}

(2) 3^{-43} *and*

(3) 3^{-65}.

◊

§§ Solution. (1) $\log 3^{43} = 43\log 3 = 43 \times .4771213 = 20.5162159.$

Hence there are 21 digits in 3^{43}.

(2) $\log 3^{27} = 27 \log 3 = 27 \times .4771213 = 12.8822751.$
Hence there are 13 digits in 3^{27}.

(3) $\log 3^{62} = 62 \log 3 = 62 \times .4771213 = 29.5815206.$
Hence there are 30 digits in 3^{62}.

(4) $\log 3^{-13} = -13 \log 3 = -13 \times .4771213 = -6.2025769 = \bar{7}.7974231.$
Hence the first significant figure is in the seventh place of decimals.

(5) $\log 3^{-43} = -43 \log 3 = -43 \times .4771213 = -20.5162159 = \bar{21}.4837841.$
Hence the first significant figure is in the twenty-first place of decimals.

(6) $\log 3^{-65} = -65 \log 3 = -65 \times .4771213 = -31.0128845 = \bar{32}.9871155.$
Hence the first significant figure is in the thirty-second place of decimals. ■

§ **Problem 10.1.7.** *Given* $\log 2 = .30103,\ \log 3 = .4771213$ *and* $\log 7 = .8450980$*, solve the equations*

(1) $2^x \cdot 3^{x+4} = 7^x$

(2) $2^{2x+1} \cdot 3^{3x+2} = 7^{4x}$

(3) $7^{2x} \div 2^{x-4} = 3^{3x-7}$*, and*

(4)
$$\left.\begin{aligned} 7^{x+y} \times 3^{2x+y} &= 9 \\ 3^{x-y} \div 2^{x-2y} &= 3^x \end{aligned}\right\}$$
◊

§§ **Solution.** (1) $2^x \cdot 3^{x+4} = 7^x$.

Taking logarithms of both sides, we have
$$x \log 2 + (x+4) \log 3 = x \log 7$$
$$\therefore x (\log 2 + \log 3 - \log 7) = -4 \log 3$$
$$\begin{aligned} \therefore x &= \frac{4 \log 3}{\log 7 - (\log 2 + \log 3)} \\ &= \frac{1.9084852}{.8450980 - .7781513} = \frac{1.9084852}{.0669467} = 28.5\ldots \end{aligned}$$

(2) $2^{2x+1} \cdot 3^{3x+2} = 7^{4x}$

Taking logarithms of both sides, we have
$$(2x+1) \log 2 + (3x+2) \log 3 = 4x \log 7$$
$$\begin{aligned} \therefore x &= \frac{\log 2 + 2 \log 3}{4 \log 7 - (2 \log 2 + 3 \log 3)} \\ &= \frac{1.2552726}{3.3803920 - 2.0334239} = \frac{1.2552726}{1.3469681} = .93\ldots \end{aligned}$$

(3) $7^{2x} \div 2^{x-4} = 3^{3x-7}$

Taking logarithms of both sides, we have
$$2x \log 7 - (x-4) \log 2 = (3x-7) \log 3$$
$$\begin{aligned} \therefore x &= \frac{7 \log 3 + 4 \log 2}{3 \log 3 + \log 2 - 2 \log 7} \\ &= \frac{3.3398491 + 1.20412}{1.7323939 - 1.6901960} = \frac{4.5439691}{.0421979} = 107.68\ldots \end{aligned}$$

(4) We have

$$\left.\begin{array}{r}(x+y)\log 7+(2x+y)\log 3=2\log 3\\(x-y)\log 3+(2y-x)\log 2=x\log 3\end{array}\right\}$$

or

$$\left.\begin{array}{r}(x+y)c+(2x+y)b=2b\\(x-y)b+(2y-x)a=xb\end{array}\right\}$$

where $a=\log 2,\ b=\log 3\ \textit{and}\ c=\log 7$

$$\left.\begin{array}{r}\therefore (c+2b)x+(c+b)y=2b\\ax+(b-2a)y=0\end{array}\right\}$$

Solving these equations, we have

$$x=\frac{2b(2a-b)}{5ab+3ac-2b^2-bc}\ \textit{and}\ y=\frac{2ab}{5ab+3ac-2b^2-bc}. \qquad \blacksquare$$

§ Problem 10.1.8. *From the tables, find the seventh root of* $.000026751$. ◊

§§ Solution.

$$\begin{aligned}\log(.000026751)&=\bar{5}.4273400\\ \therefore \log(.000026751)^{\frac{1}{7}}&=\frac{1}{7}\left(\bar{5}.4273400\right)\\ &=\frac{1}{7}\left(\bar{7}+2.4273400\right)\\ &=\bar{1}.3467629\approx\log(.22221)\end{aligned}$$

$$\therefore (000026751)^{\frac{1}{7}}\approx .22221. \qquad \blacksquare$$

Making use of the tables, find the approximate values of

§ Problem 10.1.9. $\sqrt[3]{645.3}$. ◊

§§ Solution.

$$\begin{aligned}\log 645.3&=2.8097617\\ \therefore \log\sqrt[3]{645.3}&=\frac{1}{3}(2.8097617)\\ &=.9365872\approx\log 8.6415\end{aligned}$$

$$\therefore \sqrt[3]{645.3}\approx 8.6415. \qquad \blacksquare$$

§ Problem 10.1.10. $\sqrt[5]{82357}$. ◊

§§ Solution.

$$\begin{aligned}\log 82357&=4.9157005\\ \therefore \log\sqrt[5]{82357}&=\frac{1}{5}(4.9157005)\\ &=.9831401\approx\log 9.6192\end{aligned}$$

$$\therefore \sqrt[5]{82357}\approx 9.6192. \qquad \blacksquare$$

§ Problem 10.1.11. $\dfrac{\sqrt{5}\times\sqrt[3]{7}}{\sqrt[4]{8}\times\sqrt[5]{9}}$. ◊

§§ Solution. Let $x=\dfrac{\sqrt{5}\times\sqrt[3]{7}}{\sqrt[4]{8}\times\sqrt[5]{9}}$.

$$\begin{aligned}\therefore \log x&=\frac{1}{2}\log 5+\frac{1}{3}\log 7-\left(\frac{1}{4}\log 8+\frac{1}{5}\log 9\right)\\ &=.6311843-.4166210=.2145633\approx\log 1.6389\end{aligned}$$

$$\therefore x\approx 1.6389. \qquad \blacksquare$$

§ Problem 10.1.12. $\sqrt[3]{\dfrac{7.2 \times 8.3}{9.4 \div 16.5}}$. ◊

§§ Solution. Let $x = \sqrt[3]{\dfrac{7.2 \times 8.3}{9.4 \div 16.5}}$.

$$\begin{aligned}
\therefore \log x &= \frac{1}{3}\left[\log 7.2 + \log 8.3 - (\log 9.4 - \log 16.5)\right] \\
&= \frac{1}{3}\left[(\log 7.2 + \log 8.3 + \log 16.5) - \log 9.4\right] \\
&= \frac{1}{3}\left[2.9938945 - .9731279\right] \\
&= \frac{1}{3}(2.0207666) = .6735889 \approx \log 4.7162 \\
\therefore x &\approx 4.7162.
\end{aligned}$$ ■

§ Problem 10.1.13. $\sqrt{\dfrac{8^{\frac{1}{5}} \times 11^{\frac{1}{3}}}{\sqrt{74} \times \sqrt[5]{62}}}$. ◊

§§ Solution. Let $x = \sqrt{\dfrac{8^{\frac{1}{5}} \times 11^{\frac{1}{3}}}{\sqrt{74} \times \sqrt[5]{62}}}$.

$$\begin{aligned}
\therefore \log x &= \frac{1}{2}\left[\frac{1}{5}\log 8 + \frac{1}{3}\log 11 - \left(\frac{1}{2}\log 74 + \frac{1}{5}\log 62\right)\right] \\
&= \frac{1}{2}\left[.5277489 - 1.2930942\right] = \frac{1}{2}\left(\bar{1}.2346547\right) \\
&= \frac{1}{2}\left(\bar{2} + 1.2346547\right) = \bar{1}.6173274 \approx \log(.41431) \\
\therefore x &\approx .41431.
\end{aligned}$$ ■

Chapter 11

Tables of Logarithms And Trigonometrical Ratios, Principle of Proportional Parts

11.1 Proportional Parts

§ Problem 11.1.1.

Given $\log 35705 = 4.5527290$

and $\log 35706 = 4.5527412$

find the values of $\log 35705.7$ *and* $\log 35.70585$. ◊

§§ Solution.

$$\text{The difference for } 1 = 4.5527412 - 4.5527290 = .0000122.$$

$$\therefore \text{ the difference for } .7 = .0000085$$

$$\therefore \log 35705.7 = 4.5527290 + .0000085 = 4.5527375.$$

Again, we have $\log 35.705 = 1.5527290$

and $\log 35.706 = 1.5527412.$

Hence the difference for $.001 = .0000122$

$$\therefore \text{ the difference for } .00085 = .0000104$$

$$\therefore \log 35.70585 = 1.5527290 + .0000104 = 1.5527394.$$ ■

§ Problem 11.1.2.

Given $\log 5.8743 = .7689487$

and $\log 587.44 = 2.7689561$

find the values of $\log 58743.57$ *and* $\log .00587432$. ◊

§§ Solution.

Given $\log 58743 = 4.7689487$

and $\log 58744 = 4.7689561.$

$$\text{The difference for } 1 = .0000074.$$
$$\therefore \text{ the difference for } .57 = .0000042$$
$$\therefore \log 58743.57 = 4.7689487 + .0000042 = 4.7689529.$$
Again the difference for $\quad .2 = .0000015$
$$\therefore \log 58743.2 = 4.7689487 + .0000015 = 4.7689502$$
$$\therefore \log .00587432 = \bar{3}.7689502.$$ ■

§ Problem 11.1.3.
Given $\log 47847 = 4.6798547$
and $\log 47848 = 4.6798638$
find the numbers whose logarithms are respectively
$$2.6798593 \textit{ and } \bar{3}.6798617.$$ ◊

§§ Solution. We have
$$\log 478.47 = 2.6798547 \tag{11.1}$$
and
$$\log 478.48 = 2.6798638 \tag{11.2}$$
Let
$$\log(478.47 + x) = 2.6798593 \tag{11.3}$$
From Eq. (11.1) and Eq. (11.2), we have the difference for
$$.01 = .0000091.$$
From Eq. (11.1) and Eq. (11.3), we have the difference for
$$x = .0000046.$$
$$\therefore \frac{x}{.01} = \frac{.0000046}{.0000091}.$$
$$\therefore x = \frac{46}{91} \times .01 = \frac{.46}{91} \approx .005.$$
$$\therefore \text{ the required number } = 478.47 + .005 = 478.475.$$
Again, let
$$\log(478.47 + y) = 2.6798617 \tag{11.4}$$
From Eq. (11.1) and Eq. (11.4), we have the difference for $y = .0000070$.
$$\therefore y = \frac{70}{91} \times .01 = \frac{.7}{91} \approx .0077.$$
$$\therefore \log(478.4777) = 2.6798617; \ \therefore \log .004784777 = \bar{3}.6798617.$$
$$\therefore \text{ the required number } = .004784777.$$ ■

§ Problem 11.1.4.
Given $\log 258.36 = 2.4122253$
and $\log 2.5837 = .4122421,$
find the numbers whose logarithms are
$$.4122378 \textit{ and } \bar{2}.4122287.$$ ◊

§§ Solution. We have
$$\log 2.5836 = .4122253 \tag{11.5}$$
and
$$\log 2.5837 = .4122421 \tag{11.6}$$
Let
$$\log(2.5836 + x) = .4122378 \tag{11.7}$$
From Eq. (11.5) and Eq. (11.6), we have the difference for
$$.0001 = .0000168.$$
From Eq. (11.5) and Eq. (11.7), we have the difference for
$$x = .0000125.$$
Hence we have $\quad x = \dfrac{125}{168} \times .0001 = \dfrac{.0125}{168} \approx .000074.$

Hence the required number

$$= 2.5836 + .000074 = 2.583674.$$

Again, we have $$\log .025836 = \bar{2}.4122253 \qquad (11.8)$$

and $$\log .025837 = \bar{2}.4122421 \qquad (11.9)$$

Let $$\log (.025836 + y) = \bar{2}.4122287 \qquad (11.10)$$

From Eq. (11.8) and Eq. (11.9), we have the difference for

$$.000001 = .0000168.$$

From Eq. (11.8) and Eq. (11.10), we have the difference for

$$y = .000034.$$

Hence we have

$$y = \frac{34}{168} \times .000001 = \frac{.000034}{168} = .0000002.$$

Hence the required number

$$= .025836 + .0000002 = .0258362.$$ ■

§ **Problem 11.1.5.** *From the table in* Art. 144*, find the logarithms of*

(1) 52538.97

(2) 527.286

(3) $.000529673$*,*

and the numbers whose logarithms are

(4) 3.7221098

(5) $\bar{2}.7210075$*, and*

(6) $.7210386$. ◊

§§ **Solution.** (1)

$$\begin{aligned} \log 52538 &= 4.7204735 \\ \text{diff. for } .9 &= .0000074 \\ \text{diff. for } .07 &= .0000006 \\ \therefore \log 52538.97 &= 4.7204815 \end{aligned}$$

(2)

$$\begin{aligned} \log 52728 &= 4.7220413 \\ \text{diff. for } .6 &= .0000049 \\ \therefore \log 52728.6 &= 4.7220462 \\ \therefore \log 527.286 &= 2.7220462 \end{aligned}$$

(3)

$$\begin{aligned} \log 52967 &= 4.7240054 \\ \text{diff. for } .3 &= .0000025 \\ \therefore \log 52967.3 &= 4.7240079 \\ \therefore \log .000529673 &= \bar{4}.7240079 \end{aligned}$$

(4)

Let $$\log x = 4.7221098$$

We have $$\log 52736 = 4.7221072$$

$$\text{diff.} \quad = .0000026$$

$$\begin{aligned}
\text{diff. for } .3 &= .0000025\\
\therefore \log 52736.3 &= \log x\\
\therefore x &= 52736.3.
\end{aligned}$$

Hence the required number is 5273.63.

(5)

$$\begin{aligned}
\text{Let} \qquad \log x &= 4.7240075\\
\text{We have} \qquad \log 52967 &= 4.7240054\\
\text{diff.} &= .0000021\\
\text{diff. for } .2 &= .0000016\\
\text{diff. for } .06 &= .00000049\\
\therefore \log 52967.26 &= \log x\\
\therefore x &= 52967.26.
\end{aligned}$$

Hence the required number is $.05296726$.

(6)

$$\begin{aligned}
\text{Let} \qquad \log x &= 4.7210386\\
\text{We have} \qquad \log 52606 &= 4.7210353\\
\text{diff.} &= .0000033\\
\text{diff. for } .4 &= .0000033\\
\therefore \log 52606.4 &= \log x\\
\therefore x &= 52606.4.
\end{aligned}$$

Hence the required number is 5.26064. ■

§ Problem 11.1.6.

Given $\sin 43°23' = .6868761$

and $\sin 43°24' = .6870875,$

find the value of $\sin 43°23'47''$. ◊

§§ Solution. For an increase of $60''$ in the angle, there is an increase of

$$.6870875 - .6868761, \; i.e. \; .0002114,$$

in the logarithm.

Hence for an increase of $47''$ in the angle, the corresponding increase in the logarithm

$$= \frac{47}{60} \times .0002114 = .0001656.$$

$$\therefore \sin 43°23'47'' = .6868761 + .0001656 = .6870417.$$ ■

§ Problem 11.1.7. *Find also the angle whose sine is* $.6870349$. ◊

§§ Solution. Let the required angle be $43°23' + x''$, so that

$$\sin\left(43°23' + x''\right) = .6870349.$$

$$\therefore \frac{x''}{60''} = \frac{.6870349 - .6868761}{.6870875 - .6868761} = \frac{.0001588}{.0002114}$$

$$\therefore x = 60'' \times \frac{1588}{2114} \approx 45''.$$

Hence the required angle is $43°23'45''$. ■

§ Problem 11.1.8.

Given $\cos 32°16' = .8455726$

and $\cos 32°17' = .8454172$

find the values of $\cos 32°16'24''$ *and of* $\cos 32°16'47''$. ◊

§§ Solution. For an increase of $60''$ in the angle, there is a decrease of

$$.8455726 - .8454172 = .0001554 \text{ in the logarithm.}$$

Hence, for an increase of $24''$ in the angle, the corresponding decrease in the logarithm

$$= \frac{24}{60} \times .0001554 = .0000622.$$

And, for an increase of $47''$ in the angle, the corresponding decrease in the logarithm

$$= \frac{47}{60} \times .0001554 = .0001217.$$

$$\therefore \cos 32^\circ 16' 24'' = .8455726 - .0000622 = .8455104$$

and $$\cos 32^\circ 16' 47'' = .8455726 - .0001217 = .8454509.$$ ■

§ Problem 11.1.9. *Find also the angles whose cosines are*

$$.8454832 \textit{ and } .8455176.$$ ◊

§§ Solution. (1) Let the required angle be $32^\circ 16' + x''$.

$$\therefore \cos\left(32^\circ 16' + x''\right) = .8454832.$$

$$\therefore \frac{x''}{60''} = \frac{.8455726 - .8454832}{.8455726 - .8454172} = \frac{.0000894}{.0001554}$$

$$\therefore x = 60'' \times \frac{894}{1554} \approx 35''.$$

Hence the required angle is $32^\circ 16' 35''$.

(2) Let the required angle be $32^\circ 16' + y''$.

$$\therefore \cos\left(32^\circ 16' + y''\right) = .8455176.$$

$$\therefore \frac{y''}{60''} = \frac{.8455726 - .8455176}{.8455726 - .8454172} = \frac{.0000550}{.0001554}$$

$$\therefore x = 60'' \times \frac{550}{1554} \approx 21''.$$

Hence the required angle is $32^\circ 16' 21''$. ■

§ Problem 11.1.10.

Given $$\tan 76^\circ 21' = 4.1177784$$

and $$\tan 76^\circ 22' = 4.1230079,$$

find the values of $\tan 76^\circ 21' 29''$ *and* $\tan 76^\circ 21' 47''$. ◊

§§ Solution.

$$\tan 76^\circ 22' = 4.1230079$$

$$\tan 76^\circ 21' = 4.1177784$$

$$\therefore \text{ diff. for } 60'' = .0052295.$$

$$\therefore \text{ diff. for } 29'' = \frac{29}{60} \times .0052295 = .0025276$$

$$\therefore \text{ diff. for } 47'' = \frac{47}{60} \times .0052295 = .0040964.$$

$$\therefore \tan 76^\circ 21' 29'' = 4.1177784 + .0025276 = 4.1203060$$

$$\therefore \tan 76^\circ 21' 47'' = 4.1177784 + .0040964 = 4.1218748.$$ ■

§ Problem 11.1.11.

Given $$\operatorname{cosec} 13^\circ 8' = 4.4010616$$

and $$\operatorname{cosec} 13^\circ 9' = 4.3955817,$$

find the values of $\operatorname{cosec} 13^\circ 8' 19''$ *and* $\operatorname{cosec} 13^\circ 8' 37''$. ◊

§§ Solution.

$$\begin{aligned}
\text{cosec } 13°9' &= 4.3955817\\
\text{cosec } 13°8' &= 4.4010616\\
\therefore \text{ diff. for } 60'' &= -.0054799.\\
\therefore \text{ diff. for } 19'' &= \frac{19}{60} \times (-.0054799) = -.0017353\\
\therefore \text{ diff. for } 37'' &= \frac{37}{60} \times (-.0054799) = -.0033793.\\
\text{cosec } 13°8'19'' &= 4.4010616 - .0017353 = 4.3993263\\
\text{cosec } 13°8'37'' &= 4.4010616 - .0033793 = 4.3976823.
\end{aligned}$$ ■

§ Problem 11.1.12. *Find also the angle whose cosecant is* 4.396789. ◊

§§ Solution.

Let
$$\begin{aligned}
\text{cosec } 13°8'x'' &= 4.3967890\\
\text{cosec } 13°8' &= 4.4010616\\
\therefore \text{ diff. for } x'' &= -.0042726.\\
\therefore \frac{x''}{60''} &= \frac{.0042726}{.0054799}, \ \therefore x \approx 47''\\
\therefore \text{ required angle } &= 13°8'47''.
\end{aligned}$$ ■

§ Problem 11.1.13.

Given $$L\cos 34°44'' = 9.9147729$$
and $$L\cos 34°45'' = 9.9146852,$$
find the value of $L\cos 34°44°27''$. ◊

§§ Solution.

$$\begin{aligned}
L\cos 34°45'' &= 9.9146852\\
L\cos 34°44'' &= 9.9147729\\
\therefore \text{ diff. for } 60'' &= -.0000877.\\
\therefore \text{ diff. for } 27'' &= \frac{27}{60} \times (-.0000877) = -.0000395\\
\therefore L\cos 34°44°27'' &= 9.9147729 - .0000395 = 9.9147334.
\end{aligned}$$ ■

§ Problem 11.1.14. *Find also the angle* θ*, where*
$$L\cos\theta = 9.9147328.$$ ◊

§§ Solution.

Let
$$\begin{aligned}
L\cos 34°44'x'' &= 9.9147328\\
L\cos 34°44' &= 9.9147729\\
\therefore \text{ diff. for } x'' &= -.0000401.
\end{aligned}$$
But
$$\begin{aligned}
\text{diff. for } 60'' &= -.0000877\\
\therefore x &= \frac{401}{877} \times 60 \approx 27\\
\therefore \theta &= 34°44'27''.
\end{aligned}$$ ■

§ Problem 11.1.15.

Given $$L\cot 71°27' = 9.5257779$$
and $$L\cot 71°28' = 9.5253589,$$
find the value of $L\cot 71°27'47''$,
and solve the equation $L\cot\theta = 9.5254782$. ◊

§§ Solution.

$$\begin{aligned} L\cot 71^\circ 28' &= 9.5253589 \\ L\cot 71^\circ 27' &= 9.5257779 \\ \therefore \text{ diff.for } 60'' &= -.0004190. \\ \therefore \text{ diff. for } 47'' &= \frac{47}{60} \times (-.0004190) = -.0003282. \\ \therefore L\cot 71^\circ 27' 47'' &= 9.5257779 - .0003282 = 9.5254497. \end{aligned}$$

Again, let $\theta = 71^\circ 27' x''$. Then

$$\begin{aligned} L\cot 71^\circ 27' x'' &= 9.5254782 \\ L\cot 71^\circ 27' &= 9.5257779 \\ \therefore \text{ diff. for } x'' &= -.0002997. \end{aligned}$$

But
$$\text{diff. for } 60'' = -.0004190$$

$$\therefore x = \frac{2997}{4190} \times 60 = 43; \ \therefore \theta = 71^\circ 27' 43''.$$ ■

§ Problem 11.1.16.
Given $L\sec 18^\circ 27' = 10.0229168$
and $L\sec 18^\circ 28' = 10.0229590,$
find the value of $L\sec 18^\circ 27' 35''$. ◊

§§ Solution.

$$\begin{aligned} L\sec 18^\circ 28' &= 10.0229590 \\ L\sec 18^\circ 27' &= 10.0229168 \\ \therefore \text{ diff. for } 60'' &= .0000422. \\ \therefore \text{ diff. for } 35'' &= \frac{35}{60} \times (.0000422) = .0000246 \\ \therefore L\sec 18^\circ 27' 35'' &= 10.0229168 + .0000246 = 10.0229414. \end{aligned}$$ ■

§ Problem 11.1.17. *Find also the angle whose* $L\sec$ *is* 10.0229285. ◊

§§ Solution.

Let
$$\begin{aligned} L\sec 18^\circ 27' x'' &= 10.0229285 \\ L\sec 18^\circ 27' &= 10.0229168 \\ \therefore \text{ diff. for } x'' &= .0000117. \end{aligned}$$

But
$$\text{diff. for } 60'' = .0000422$$

$$\therefore x = \frac{117}{422} \times 60 = 17$$

$$\therefore \text{ required angle } = 18^\circ 27' 17''.$$ ■

§ Problem 11.1.18. *Find in degrees, minutes and seconds the angle whose sine is* $.6$, *given that*

$$\log 6 = 7781513, \ L\sin 36^\circ 52' = 9.7781186$$

and $$L\sin 36^\circ 53' = 9.7782870.$$ ◊

§§ Solution.

Let
$$\sin\theta = .6 = \frac{6}{10}$$

$$\therefore L\sin\theta = 10 + \log 6 - \log 10 = 9.7781513.$$

Let
$$\theta = 36^\circ 52' x''.$$

Then
$$\begin{aligned} L\sin 36^\circ 52' x'' &= 9.7781513 \\ L\sin 36^\circ 52' &= 9.7781186 \\ \therefore \text{ diff. for } x'' &= .0000327 \\ L\sin 36^\circ 53' &= 9.7782870 \end{aligned}$$

$$\begin{aligned} L\sin 36°52' &= 9.7781186 \\ \therefore\ \text{diff. for } 60'' &= .0001684 \\ \therefore x &= \frac{327}{1684} \times 60 \approx 12. \\ \therefore\ \text{the angle } &= 36°52'12''. \end{aligned}$$ ■

11.2 Logarithmic Sines, Tangents And Secants

§ Problem 11.2.1. *Find θ, given that $\cos\theta = .9725382$,*

$$\cos 13°27' = .9725733, \quad \textit{diff. for } 1' = 677.$$ ◊

§§ Solution.

Since $\qquad \cos\theta < \cos 13°27',\ \therefore \theta > 13°27'.$

Let then $\qquad \cos\left(13°27'x''\right) = .9725382.$

Since $\qquad .9725733 - .9725382 = .0000351$

we have $\qquad x = 60'' \times \dfrac{351}{677} \approx 31''.$

Hence $\qquad \theta = 13°27'31''.$ ■

§ Problem 11.2.2. *Find the angle whose sine is $\frac{3}{8}$, given*

$$\sin 22°1' = .3748763, \ \textit{diff. for } 1' = 2696.$$ ◊

§§ Solution. Since $\frac{3}{8} = .375$, let the required angle be $22°1' + x''$, so that

$$\sin\left(22°1' + x''\right) = .375.$$

$$\because .375 - .3748763 = .0001237,$$

we have $\qquad x = 60'' \times \dfrac{1237}{2696} \approx 28''.$

Hence the required angle is $22°1'28''$. ■

§ Problem 11.2.3.

Given $\qquad \operatorname{cosec} 65°24' = 1.0998243$

$\qquad\qquad$ *diff. for* $1' = 1464,$

find the value of $\qquad \operatorname{cosec} 65°24'37''.$

and the angle whose cosec *is* 1.0997938. ◊

§§ Solution. Since $\frac{37}{60} \times 1464 \approx 903$, we have

$$\operatorname{cosec} 65°24'37'' = 1.0998243 - .0000903 = 1.0997340.$$

Again, let θ be the required angle.

$$\because \operatorname{cosec}\ \theta < \operatorname{cosec}\ 65°24',\ \therefore \theta > 65°24'.$$

Let then $\qquad \theta = 65°24' + x''$

$$\therefore \operatorname{cosec}\ \left(65°24' + x''\right) = 1.0997938.$$

$$\because 1.0998243 - 1.0997938 = .0000305,$$

$$\therefore x = 60'' \times \frac{305}{1464} = 12.5''.$$

$$\therefore \theta = 65°24'12.5''.$$ ■

§ Problem 11.2.4.
Given $L\tan 22^\circ 37' = 9.6197205$
diff. for $1'' = 3557$,
find the value of $L\tan 22^\circ 37' 22''$,
and the angle whose $L\tan$ *is* 9.6195283. ◊

§§ Solution. Since $\dfrac{22}{60} \times 3557 \approx 1304$, we have

$$L\tan 22^\circ 37' 22'' = 9.6197205 + .0001304 = 9.6198509.$$

Again, let θ be the required angle.

$$\because L\tan\theta < L\tan 22^\circ 37', \ \therefore \theta < 22^\circ 37'.$$

Let then $\theta = 22^\circ 37' - x''$, $\therefore L\tan\left(22^\circ 37' - x''\right) = 9.6195283.$

$$\because 9.6197205 - 9.6195283 = .0001922$$

$$\therefore x = 60'' \times \frac{1922}{3557} \approx 32''.$$

$$\therefore \theta = 22^\circ 37' - 32'' = 22^\circ 36' 28''.$$ ■

§ Problem 11.2.5. *Find the angle whose* $L\cos$ *is* 9.993, *given*
$L\cos 10^\circ 15' = 9.9930131$, *diff. for* $1' = 229$. ◊

§§ Solution. Let θ be the required angle.

$$\because L\cos\theta < L\cos 10^\circ 15'', \ \therefore \theta > 10^\circ 15'.$$

Let then $\theta = 10^\circ 15' + x''$, $\therefore L\cos\theta = 9.993.$

$$\because 9.9930131 - 9.993 = .0000131$$

$$\therefore x = 60'' \times \frac{131}{229} \approx 34''.$$

$$\therefore \theta = 10^\circ 15' 34''.$$ ■

§ Problem 11.2.6. *Find the angle whose* $L\sec$ *is* 10.15, *given*
$L\sec 44^\circ 55' = 10.1498843$, *diff. for* $1' = 1260$. ◊

§§ Solution. Let θ be the required angle.

$$\because L\sec\theta > L\sec 44^\circ 55', \ \therefore \theta > 44^\circ 55'.$$

Let then $\theta = 44^\circ 55' + x''$, $\therefore L\sec\left(44^\circ 55' + x''\right) = 10.15.$

$$\because 10.15 - 10.1498843 = .0001157,$$

$$\therefore x = 60 \times \frac{1157}{1260} \approx 55.$$

$$\therefore \theta = 44^\circ 55' 55''.$$ ■

§ Problem 11.2.7. *From the table in* $Art.\ 159$, *find the values of*

(1) $L\sin 32^\circ 18' 23''$

(2) $L\cos 32^\circ 16' 49''$

(3) $L\cot 32^\circ 29' 43''$

(4) $L\sec 32^\circ 52' 27''$

(5) $L\tan 57^\circ 45' 28''$

(6) $L\,\mathrm{cosec}\ 57^\circ 48' 21''$, *and*

(7) $L\cos 57^\circ 58' 29''$.

◊

§§ Solution. (1) Since $\dfrac{23}{60} \times 1998 \approx 766$, we have

$$L \sin 32°18'23'' = 9.7278277 + .0000766 = 9.7279043.$$

(2) Since $\dfrac{49}{60} \times 798 \approx 652$, we have

$$L \cos 32°16'49'' = 9.9271509 - .0000652 = 9.9270857.$$

(3) Since $\dfrac{43}{60} \times 2788 \approx 1998$, we have

$$L \cot 32°29'43'' = 10.1960915 - .0001998 = 10.1958917.$$

(4) Since $\dfrac{27}{60} \times 817 \approx 368$, we have

$$L \sec 32°52'27'' = 10.0757539 + .0000368 = 10.0757907.$$

(5) Since $\dfrac{28}{60} \times 2800 \approx 1307$, we have

$$L \tan 57°45'28'' = 10.2000030 + .0001307 = 10.2001337.$$

(6) Since $\dfrac{21}{60} \times 795 \approx 278$, we have

$$L \operatorname{cosec} 57°48'21'' = 10.0725305 - .0000278 = 10.0725027.$$

(7) Since $\dfrac{29}{60} \times 2020 \approx 976$, we have

$$L \cos 57°58'29'' = 9.7246138 - .0000976 = 9.7245162.$$ ■

§ Problem 11.2.8. *With the help of the table in* Art. 159*, solve the equations*

(1) $L \tan \theta = 10.1959261$

(2) $L \operatorname{cosec} \theta = 10.0738125$

(3) $L \cos \theta = 9.9259283$*, and*

(4) $L \sin \theta = 9.9241352.$

◊

§§ Solution. (1)

$$\because L \tan 57°30' = 10.1958127$$
$$\text{diff. for } 1' = 2788$$

Let $$\theta = 57°30' + x''$$

$$\therefore L \tan \left(57°30' + x''\right) = 10.1959261.$$

Since $$10.1959261 - 10.1958127 = .0001134$$

$$\therefore x = 60'' \times \frac{1134}{2788} \approx 24''.$$
$$\therefore \theta = 57°30'24''.$$

(2)

$$\because L \operatorname{cosec} 57°32' = 10.0738099$$
$$\text{diff. for } 1' = 805$$

Let $$\theta = 57°32' - x''$$

$$\therefore L \operatorname{cosec} \left(57°32' - x''\right) = 10.0738125.$$

Since $$10.0738125 - 10.0738099 = .0000026$$

$$\therefore x = 60'' \times \frac{26}{805} \approx 2''.$$
$$\therefore \theta = 57^\circ 32' - x'' = 57^\circ 31' 58''.$$

(3)

$$\because L\cos 32^\circ 32' = 9.9258681$$
$$\text{diff. for } 1' = 806$$

Let $\theta = 32^\circ 32' - x''$

$$\therefore L\cos\left(32^\circ 32' - x''\right) = 9.9259283.$$

Since $9.9259283 - 9.9258681 = .0000602$

$$\therefore x = 60'' \times \frac{602}{805} \approx 45''.$$
$$\therefore \theta = 32^\circ 32' - 45'' = 32^\circ 31' 15''.$$

(4)

$$\because L\sin 57^\circ 6' = 9.9240827$$
$$\text{diff. for } 1' = 817$$

Let $\theta = 57^\circ 6' + x''$

$$\therefore L\sin\left(57^\circ 6' + x''\right) = 9.9241352.$$

Since $9.9241352 - 9.9240827 = .0000525$

$$\therefore x = 60'' \times \frac{525}{817} \approx 39''.$$
$$\therefore \theta = 57^\circ 6' 39''.$$

■

§ Problem 11.2.9. *Take out of the tables $L\tan 16^\circ 6' 23''$ and calculate the value of the square root of the tangent.* ◊

§§ Solution.

$$L\tan 16^\circ 6' = 9.4603492$$
$$\text{diff. for } 23'' = \frac{23}{60} \times 4740 = 1817$$
$$\therefore L\tan 16^\circ 6' 23'' = 9.4603492 + .0001817 = 9.4605309$$
$$\therefore \log\tan 16^\circ 6' 23'' = L\tan 16^\circ 6' 23'' - 10 = \bar{1}.4605309.$$

Let $x = \sqrt{\tan 16^\circ 6' 23''}$.
Taking logarithms of both sides, we have

$$\log x = \frac{1}{2}\left(\log\tan 16^\circ 6' 23''\right) = \frac{1}{2}\left(\bar{1}.4605309\right)$$
$$= \frac{1}{2}\left(\bar{2} + 1.4605309\right) = \bar{1}.7302655.$$

Now $\log 53736 = 4.7302653, \text{ diff. for } 1 = 81.$

Let $\log(53736 + y) = 4.7302655.$

We then have $y = \frac{2}{81} = .02$

$$\therefore \log 53736.02 = 4.7302655.$$
$$\therefore x = .5373602.$$

■

§ Problem 11.2.10. *Change into a form more convenient for logarithmic computation (i.e. express in the form of products of quantities) the quantities*

(1) $1 + \tan x \tan y$

(2) $1 - \tan x \tan y$

(3) $\cot x + \tan y$

(4) $\cot x - \tan y$

(5) $\dfrac{1 - \cos 2x}{1 + \cos 2x}$, *and*

(6) $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

◊

§§ Solution. (1)

$$\begin{aligned} 1 + \tan x \tan y &= 1 + \frac{\sin x \sin y}{\cos x \cos y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y} \\ &= \cos(x - y) \sec x \sec y. \end{aligned}$$

(2)

$$\begin{aligned} 1 - \tan x \tan y &= 1 - \frac{\sin x \sin y}{\cos x \cos y} \\ &= \frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y} \\ &= \cos(x + y) \sec x \sec y. \end{aligned}$$

(3)

$$\begin{aligned} \cot x + \tan y &= \frac{\cos x}{\sin x} + \frac{\sin y}{\cos y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y} \\ &= \cos(x - y) \operatorname{cosec}\, x \sec y. \end{aligned}$$

(4)

$$\begin{aligned} \cot x - \tan y &= \frac{\cos x}{\sin x} - \frac{\sin y}{\cos y} \\ &= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y} \\ &= \cos(x + y) \operatorname{cosec}\, x \sec y. \end{aligned}$$

(5) $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \dfrac{2 \sin^2 x}{2 \cos^2 x} = \tan^2 x.$

(6)

$$\begin{aligned} \frac{\tan x + \tan y}{\cot x + \cot y} &= \left[\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}\right] \div \left[\frac{\cos x}{\sin x} + \frac{\cos y}{\sin y}\right] \\ &= \left[\frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}\right] \div \left[\frac{\cos x \sin y + \sin x \cos y}{\sin x \sin y}\right] \\ &= \frac{\sin x \sin y}{\cos x \cos y} = \tan x \tan y. \end{aligned}$$

■

Chapter 12

Relations Between The Sides and The Trigonometrical Ratios of The Angles of Any Triangle

12.1 Basics

In a triangle

§ Problem 12.1.1. *Given*

$$a = 25,\ b = 52 \ and\ c = 63;$$

find
$$\tan\frac{A}{2},\ \tan\frac{B}{2}\ and\ \tan\frac{C}{2}. \qquad \Diamond$$

§§ Solution. We have

$$s = \frac{1}{2}(25+52+63) = 70,\ s-a = 45,\ s-b = 18\ and\ s-c = 7.$$

$$\therefore \tan\frac{A}{2} = \sqrt{\frac{18\times 7}{70\times 45}} = \frac{1}{5},\ \tan\frac{B}{2} = \sqrt{\frac{7\times 45}{70\times 18}} = \frac{1}{2},$$

and
$$\tan\frac{C}{2} = \sqrt{\frac{45\times 18}{70\times 7}} = \frac{9}{7}. \qquad \blacksquare$$

§ Problem 12.1.2. *Given*

$$a = 125,\ b = 123\ and\ c = 62;$$

find the sines of half the angles and the sines of the angles. $\Diamond$

§§ Solution.

$$s = \frac{1}{2}(125+123+62) = 155,\ s-a = 30,\ s-b = 32\ and\ s-c = 93.$$

$$\begin{aligned}\therefore \sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{155\times 30\times 32\times 93}\\ &= \sqrt{31\times 5\times 5\times 3\times 64\times 31\times 3}\end{aligned}$$

$$= 31 \times 5 \times 3 \times 8.$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{32 \times 93}{123 \times 62}} = \frac{4}{\sqrt{41}},\ \sin\frac{B}{2} = \sqrt{\frac{93 \times 30}{62 \times 125}} = \frac{3}{5}$$

$$\sin\frac{C}{2} = \sqrt{\frac{30 \times 32}{125 \times 123}} = \frac{8}{5\sqrt{41}}.$$

$$\sin A = \frac{2}{123 \times 62} \times 31 \times 5 \times 3 \times 8 = \frac{40}{41}$$

$$\sin B = \frac{2}{125 \times 62} \times 31 \times 5 \times 3 \times 8 = \frac{24}{25}$$

and $$\sin C = \frac{2}{125 \times 123} \times 31 \times 5 \times 3 \times 8 = \frac{496}{1025}.$$ ■

§ Problem 12.1.3. *Given*

$$a = 18,\ b = 24 \textit{ and } c = 30,$$

find $$\sin A,\ \sin B \textit{ and } \sin C.$$

Verify by a graph. ◊

§§ Solution.

$$s = \frac{1}{2}(18 + 24 + 30) = 36,\ s - a = 18,\ s - b = 12 \textit{ and } s - c = 6.$$

$$\therefore \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{36 \times 18 \times 12 \times 6}$$

$$= \sqrt{36 \times 36 \times 6 \times 6} = 36 \times 6.$$

$$\therefore \sin A = \frac{2}{24 \times 30} \times 36 \times 6 = \frac{3}{5}$$

$$\sin B = \frac{2}{18 \times 30} \times 36 \times 6 = \frac{4}{5}$$

and $$\sin C = \frac{2}{18 \times 24} \times 36 \times 6 = 1.$$ ■

§ Problem 12.1.4. *Given*

$$a = 35,\ b = 84 \textit{ and } c = 91,$$

find $$\tan A,\ \tan B \textit{ and } \tan C.$$ ◊

§§ Solution.

$$s = \frac{1}{2}(35 + 84 + 91) = 105,\ s - a = 70,\ s - b = 21 \textit{ and } s - c = 14.$$

$$\therefore \tan\frac{A}{2} = \sqrt{\frac{21 \times 14}{105 \times 70}} = \frac{1}{5},$$

and $$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}} = \frac{2}{5} \div \left(1 - \frac{1}{25}\right) = \frac{2}{5} \times \frac{25}{24} = \frac{5}{12}$$

$$\tan\frac{B}{2} = \sqrt{\frac{14 \times 70}{105 \times 21}} = \frac{2}{3}$$

and $$\tan B = \frac{2\tan\frac{B}{2}}{1 - \tan^2\frac{B}{2}} = \frac{4}{3} \div \left(1 - \frac{4}{9}\right) = \frac{4}{3} \times \frac{9}{5} = \frac{12}{5}$$

$$\tan\frac{C}{2} = \sqrt{\frac{70 \times 21}{105 \times 14}} = 1$$

and
$$\tan C = \frac{2\tan\frac{C}{2}}{1-\tan^2\frac{C}{2}} = \frac{2}{1-1} = \frac{2}{0} = \infty.$$
■

§ Problem 12.1.5. *Given*
$$a = 13,\ b = 14 \textit{ and } c = 15,$$
find the sines of the angles. Verify by a graph. ◊

§§ Solution.
$$s = \frac{1}{2}(13+14+15) = 21,\ s-a = 8,\ s-b = 7 \textit{ and } s-c = 6$$
$$\therefore \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21\times 8\times 7\times 6} = \sqrt{42\times 4\times 42} = 42\times 2.$$
$$\therefore \sin A = \frac{2}{14\times 15}\times 42\times 2 = \frac{4}{5},\ \sin B = \frac{2}{13\times 15}\times 42\times 2 = \frac{56}{65}$$
and
$$\sin C = \frac{2}{13\times 14}\times 42\times 2 = \frac{12}{13}.$$
■

§ Problem 12.1.6. *Given*
$$a = 287,\ b = 816 \textit{ and } c = 865,$$
find the values of $\tan\frac{A}{2}$ *and* $\tan A$. ◊

§§ Solution.
$$s = \frac{1}{2}(287+816+865) = 984,\ s-a = 697,\ s-b = 168,\ s-c = 119.$$
$$\therefore \tan\frac{A}{2} = \sqrt{\frac{168\times 119}{984\times 697}} = \sqrt{\frac{7\times 24\times 17\times 7}{41\times 24\times 17\times 41}} = \frac{7}{41}$$
$$\therefore \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \frac{14}{41}\div\left(1-\frac{49}{41\times 41}\right) = \frac{14}{41}\times\frac{41\times 41}{1632} = \frac{287}{816}.$$
■

§ Problem 12.1.7. *Given*
$$a = \sqrt{3},\ b = \sqrt{2} \textit{ and } c = \frac{\sqrt{6}+\sqrt{2}}{2},$$
find the angles. ◊

§§ Solution. We have
$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{\left(\sqrt{2}\right)^2 + \left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 - \left(\sqrt{3}\right)^2}{2\sqrt{2}\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)}$$
$$= \frac{2+2+\sqrt{3}-3}{2\sqrt{3}+2} = \frac{\sqrt{3}+1}{2\left(\sqrt{3}+1\right)} = \frac{1}{2} = \cos 60^\circ$$
$$\therefore A = 60^\circ.$$
$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{2+\sqrt{3}+3-2}{3\sqrt{2}+\sqrt{6}}$$
$$= \frac{3+\sqrt{3}}{\sqrt{2}\left(3+\sqrt{3}\right)} = \frac{1}{\sqrt{2}} = \cos 45^\circ$$
$$\therefore B = 45^\circ$$
$$\therefore C = 180^\circ - (A+B) = 180^\circ - 105^\circ = 75^\circ.$$
■

12.2 Sides And Angles of A Triangle

In any triangle ABC, prove that

§ Problem 12.2.1. $\sin\dfrac{B-C}{2}=\dfrac{b-c}{a}\cos\dfrac{A}{2}$. ◊

§§ Solution. $\because \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k\ (say)$, we have

$$\begin{aligned}
\frac{b-c}{a}&=\frac{k\sin B-k\sin C}{k\sin A}=\frac{\sin B-\sin C}{\sin A}\\
&=\frac{2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}\\
&=\frac{\sin\dfrac{A}{2}\sin\dfrac{B-C}{2}}{\sin\dfrac{A}{2}\cos\dfrac{A}{2}}=\frac{\sin\dfrac{B-C}{2}}{\cos\dfrac{A}{2}}.\\
\therefore \sin\frac{B-C}{2}&=\frac{b-c}{a}\cos\frac{A}{2}.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
\sin\frac{B-C}{2}&=\sin\frac{B}{2}\cos\frac{C}{2}-\cos\frac{B}{2}\sin\frac{C}{2}\\
&=\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{s(s-c)}{ab}}-\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}\\
&=\frac{s-c}{a}\sqrt{\frac{s(s-a)}{bc}}-\frac{s-b}{a}\sqrt{\frac{s(s-a)}{bc}}\\
&=\left(\frac{s-c}{a}-\frac{s-b}{a}\right)\sqrt{\frac{s(s-a)}{bc}}\\
&=\frac{b-c}{a}\sqrt{\frac{s(s-a)}{bc}}=\frac{b-c}{a}\cos\frac{A}{2}.
\end{aligned}$$ ■

§ Problem 12.2.2. $b^2\sin 2C+c^2\sin 2B=2bc\sin A$. ◊

§§ Solution.

$$\begin{aligned}
b^2\sin 2C+c^2\sin 2B&=2b^2\sin C\cos C+2c^2\sin B\cos B\\
&=2b\sin C\,(b\cos C+c\cos B)\\
&=2ab\sin C=2bc\sin A.
\end{aligned}$$ ■

§ Problem 12.2.3. $a\,(b\cos C-c\cos B)=b^2-c^2$. ◊

§§ Solution.

$$\begin{aligned}
a\,(b\cos C-c\cos B)&=\frac{a^2+b^2-c^2}{2}-\frac{a^2+c^2-b^2}{2}\\
&=b^2-c^2.
\end{aligned}$$ ■

§ Problem 12.2.4. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$. ◊

§§ Solution.

$$\begin{aligned}
&(b+c)\cos A+(c+a)\cos B+(a+b)\cos C\\
&=(b\cos A+a\cos B)+(c\cos B+b\cos C)+(c\cos A+a\cos C)\\
&=c+a+b,\ by\ Art.\ 170.
\end{aligned}$$ ■

§ Problem 12.2.5. $a\,(\cos B+\cos C)=2(b+c)\sin^2\dfrac{A}{2}$. ◊

§§ Solution. From the equations

$$b = c\cos A + a\cos C,\ \textit{and}$$
$$c = a\cos B + b\cos A$$

we have, by addition,

$$c\cos A + a\cos C + a\cos B + b\cos A = b + c$$
$$\therefore a\,(\cos B + \cos C) = (b+c)\,(1 - \cos A) = 2(b+c)\sin^2\frac{A}{2}.$$

Otherwise thus :

$$\begin{aligned}\cos B + \cos C &= 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}\\ &= 2\sin\frac{A}{2}\cos\frac{B-C}{2}\\ &= \frac{2\sin\frac{A}{2}\cos\frac{B-C}{2}\sin A}{\sin A}\\ &= \frac{2\sin^2\frac{A}{2}\times 2\cos\frac{A}{2}\cos\frac{B-C}{2}}{\sin A}\\ &= \frac{2\sin^2\frac{A}{2}\times 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\sin A}\\ &= \frac{2\sin^2\frac{A}{2}\,(\sin B + \sin C)}{\sin A}\\ &= \frac{2\sin^2\frac{A}{2}(b+c)}{a}\end{aligned}$$

$$\therefore a\,(\cos B + \cos C) = 2(b+c)\sin^2\frac{A}{2}.$$ ■

§ Problem 12.2.6. $a\,(\cos C - \cos B) = 2(b-c)\cos^2\dfrac{A}{2}$. ◊

§§ Solution. From the equations

$$b = c\cos A + a\cos C,\ \textit{and}$$
$$c = a\cos B + b\cos A$$

we have, by subtraction,

$$a\,(\cos C - \cos B) + (c-b)\cos A = b - c$$
$$\therefore a\,(\cos C - \cos B) = (b-c)\,(1 + \cos A) = 2(b-c)\cos^2\frac{A}{2}.$$

Otherwise thus :

$$\begin{aligned}\cos C - \cos B &= 2\sin\frac{B+C}{2}\sin\frac{B-C}{2}\\ &= 2\cos\frac{A}{2}\sin\frac{B-C}{2}\\ &= \frac{2\cos\frac{A}{2}\sin\frac{B-C}{2}\sin A}{\sin A}\\ &= \frac{2\cos^2\frac{A}{2}\times 2\sin\frac{A}{2}\sin\frac{B-C}{2}}{\sin A}\end{aligned}$$

$$= \frac{2\cos^2\frac{A}{2} \times 2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{\sin A}$$
$$= \frac{2\cos^2\frac{A}{2}(\sin B - \sin C)}{\sin A}$$
$$= \frac{2\cos^2\frac{A}{2}(b-c)}{a}$$
$$\therefore a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}. \qquad \blacksquare$$

§ Problem 12.2.7. $\dfrac{\sin(B-C)}{\sin(B+C)} = \dfrac{b^2-c^2}{a^2}$. ◊

§§ Solution.
$$\frac{\sin(B-C)}{\sin(B+C)} = \frac{\sin(B-C)\sin(B+C)}{\sin^2(B+C)}$$
$$= \frac{\sin^2 B - \sin^2 C}{\sin^2 A} = \left(\frac{\sin B}{\sin A}\right)^2 - \left(\frac{\sin C}{\sin A}\right)^2$$
$$= \left(\frac{b}{a}\right)^2 - \left(\frac{c}{a}\right)^2 = \frac{b^2-c^2}{a^2}. \qquad \blacksquare$$

§ Problem 12.2.8. $\dfrac{a+b}{a-b} = \tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}$. ◊

§§ Solution. We have $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$.
$$\therefore \frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$
$$= \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}$$
$$= \tan\frac{A+B}{2}\cot\frac{A-B}{2}. \qquad \blacksquare$$

§ Problem 12.2.9. $a\sin\left(\dfrac{A}{2}+B\right) = (b+c)\sin\dfrac{A}{2}$. ◊

§§ Solution.
$$\frac{b+c}{a} = \frac{k\sin B + k\sin C}{k\sin A} = \frac{\sin B + \sin C}{\sin A}$$
$$= \frac{\sin B + \sin(A+B)}{\sin A} = \frac{2\sin\left(B+\frac{A}{2}\right)\cos\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$
$$\therefore a\sin\left(\frac{A}{2}+B\right) = (b+c)\sin\frac{A}{2}. \qquad \blacksquare$$

§ Problem 12.2.10. $\dfrac{a^2\sin(B-C)}{\sin B + \sin C} + \dfrac{b^2\sin(C-A)}{\sin C + \sin A} + \dfrac{c^2\sin(A-B)}{\sin A + \sin B} =$
0. ◊

§§ Solution. Since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (*say*), the given expression

$$= k^2 \left[\frac{\sin A \sin(B+C)\sin(B-C)}{\sin B + \sin C} + \frac{\sin B \sin(C+A)\sin(C-A)}{\sin C + \sin A} + \frac{\sin C \sin(A+B)\sin(A-B)}{\sin A + \sin B}\right]$$

$$= k^2 \left[\frac{\sin A\left(\sin^2 B - \sin^2 C\right)}{\sin B + \sin C} + \frac{\sin B\left(\sin^2 C - \sin^2 A\right)}{\sin C + \sin A} + \frac{\sin C\left(\sin^2 A - \sin^2 B\right)}{\sin A + \sin B}\right]$$

$$= k^2 \left[\sin A\left(\sin B - \sin C\right) + \sin B\left(\sin C - \sin A\right) + \sin C\left(\sin A - \sin B\right)\right]$$
$$= 0.$$ ■

§ Problem 12.2.11. $(b+c-a)\left(\cot\dfrac{B}{2} + \cot\dfrac{C}{2}\right) = 2a\cot\dfrac{A}{2}.$ ◊

§§ Solution.

$$\begin{aligned}
&(b+c-a)\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right)\\
&\qquad = 2(s-a)\left[\sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}\right]\\
&\qquad = 2\sqrt{s}\cdot(s-a)\left[\frac{s-b+s-c}{\sqrt{(s-a)(s-b)(s-c)}}\right]\\
&\qquad = 2\sqrt{s}\cdot(s-a)\left[\frac{2s-(b+c)}{\sqrt{(s-a)(s-b)(s-c)}}\right]\\
&\qquad = 2a\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = 2a\cot\frac{A}{2}.
\end{aligned}$$ ■

§ Problem 12.2.12. $a^2+b^2+c^2 = 2\left(bc\cos A + ca\cos B + ab\cos C\right).$ ◊

§§ Solution. By *Art*. 164, we have

$$\begin{aligned}
b^2 + c^2 - a^2 &= 2bc\cos A\\
c^2 + a^2 - b^2 &= 2ca\cos B, \ \textit{and}\\
a^2 + b^2 - c^2 &= 2ab\cos C.
\end{aligned}$$

Hence, by addition,

$$a^2 + b^2 + c^2 = 2\left(bc\cos A + ca\cos B + ab\cos C\right).$$ ■

§ Problem 12.2.13. $\left(a^2 - b^2 + c^2\right)\tan B = \left(a^2 + b^2 - c^2\right)\tan C.$ ◊

§§ Solution.

$$\begin{aligned}
\frac{\tan B}{\tan C} &= \frac{\sin B\cos C}{\cos B\sin C} = \frac{\sin B\cos C}{\sin C\cos B}\\
&= \frac{b\cos C}{c\cos B} = \frac{a^2+b^2-c^2}{2a} \div \frac{c^2+a^2-b^2}{2a}.\\
\therefore \left(a^2 - b^2 + c^2\right)\tan B &= \left(a^2 + b^2 - c^2\right)\tan C.
\end{aligned}$$ ■

§ Problem 12.2.14. $c^2 = (a-b)^2\cos^2\dfrac{C}{2} + (a+b)^2\sin^2\dfrac{C}{2}.$ ◊

§§ Solution.

$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
&\left(a^2+b^2\right)\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right) - 2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right) \\
&= \left(a^2-2ab+b^2\right)\cos^2\frac{C}{2} + \left(a^2+2ab+b^2\right)\sin^2\frac{C}{2} \\
&= (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2}.
\end{aligned}$$

■

§ Problem 12.2.15. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$ ◊

§§ Solution. Since $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k\ (say)$, we have

$$\begin{aligned}
&a\sin(B-C)+b\sin(C-A)+c\sin(A-B) \\
&\quad= k\sin A\sin(B-C)+k\sin B\sin(C-A)+k\sin C\sin(A-B) \\
&\quad= k\left[\sin(B+C)\sin(B-C)+\sin(C+A)\sin(C-A) + \sin(A+B)\sin(A-B)\right] \\
&\quad= k\left[\sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A+\sin^2 A-\sin^2 B\right] \\
&\quad[by\ Ex.\ 2,\ Art.\ 93] \\
&\quad= 0.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
&a\sin(B-C)+b\sin(C-A)+c\sin(A-B) \\
&\quad= a\left(\sin B\cos C-\cos B\sin C\right)+b\left(\sin C\cos A-\cos C\sin A\right) \\
&\quad+ c\left(\sin A\cos B-\cos A\sin B\right) \\
&\quad= \cos C\left(a\sin B-b\sin A\right)+\cos B\left(c\sin A-a\sin C\right) \\
&\quad+ \cos A\left(b\sin C-c\sin B\right) \\
&\quad= 0, \\
&\quad\because a\sin B=b\sin A,\ c\sin A=a\sin C\ and\ b\sin C=c\sin B.
\end{aligned}$$

■

§ Problem 12.2.16. $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{c\sin(A-B)}{a^2-b^2}.$ ◊

§§ Solution. Since $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k\ (say)$, we have

$$\begin{aligned}
\frac{a\sin(B-C)}{b^2-c^2} &= \frac{k\sin A\sin(B-C)}{k^2\sin^2 B-k^2\sin^2 C} \\
&= \frac{1}{k}\cdot\frac{\sin(B+C)\sin(B-C)}{\sin^2 B-\sin^2 C} \\
&= \frac{1}{k}\cdot\frac{\sin^2 B-\sin^2 C}{\sin^2 B-\sin^2 C},\ by\ Ex.\ 2,\ Art.\ 93 \\
&= \frac{1}{k}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2},\ similarly.
\end{aligned}$$

■

§ Problem 12.2.17.

$$a\sin\frac{A}{2}\sin\frac{B-C}{2}+b\sin\frac{B}{2}\sin\frac{C-A}{2} + c\sin\frac{C}{2}\sin\frac{A-B}{2}=0.$$

◊

§§ Solution.

$$\begin{aligned}
&a\sin\frac{A}{2}\sin\frac{B-C}{2}+b\sin\frac{B}{2}\sin\frac{C-A}{2}+c\sin\frac{C}{2}\sin\frac{A-B}{2} \\
&= a\cos\frac{B+C}{2}\sin\frac{B-C}{2}+b\cos\frac{C+A}{2}\sin\frac{C-A}{2}+c\cos\frac{A+B}{2}\sin\frac{A-B}{2}
\end{aligned}$$

$$= \frac{a}{2}(\sin B - \sin C) + \frac{b}{2}(\sin C - \sin A) + \frac{c}{2}(\sin A - \sin B)$$
$$= 0, \because a \sin B = b \sin A,\ c \sin A = a \sin C \textit{ and } b \sin C = c \sin B. \qquad \blacksquare$$

§ Problem 12.2.18.
$$a^2\left(\cos^2 B - \cos^2 C\right) + b^2\left(\cos^2 C - \cos^2 A\right) + c^2\left(\cos^2 A - \cos^2 B\right) = 0. \qquad \lozenge$$

§§ Solution.
$$\begin{aligned} & a^2\left(\cos^2 B - \cos^2 C\right) + b^2\left(\cos^2 C - \cos^2 A\right) + c^2\left(\cos^2 A - \cos^2 B\right) \\ &= k^2\Big[\sin^2 A\left(\cos^2 B - \cos^2 C\right) + \sin^2 B\left(\cos^2 C - \cos^2 A\right) \\ &\qquad + \sin^2 C\left(\cos^2 A - \cos^2 B\right)\Big] \\ &= k^2\Big[\sin^2 A\left(\sin^2 C - \sin^2 B\right) + \sin^2 B\left(\sin^2 A - \sin^2 C\right) \\ &\qquad + \sin^2 C\left(\sin^2 B - \sin^2 A\right)\Big] \\ &= k^2[0] = 0. \end{aligned} \qquad \blacksquare$$

§ Problem 12.2.19.
$$\frac{b^2 - c^2}{a^2}\sin 2A + \frac{c^2 - a^2}{b^2}\sin 2B + \frac{a^2 - b^2}{c^2}\sin 2C = 0. \qquad \lozenge$$

§§ Solution.
$$\begin{aligned} \frac{b^2 - c^2}{a^2}\sin 2A &= \frac{k^2\sin^2 B - k^2\sin^2 C}{k^2\sin^2 A}\cdot 2\sin A\cos A \\ &= \frac{2\cos A\sin(B+C)\sin(B-C)}{\sin A}, \textit{ by Ex. } 2,\ \textit{Art. } 93 \\ &= 2\cos A\sin(B-C), \because \sin(B+C) = \sin A \\ &= -2\cos(B+C)\sin(B-C) = \sin 2C - \sin 2B. \end{aligned}$$

Similarly, $$\frac{c^2 - a^2}{b^2}\sin 2B = \sin 2A - \sin 2C$$

and $$\frac{a^2 - b^2}{c^2}\sin 2C = \sin 2B - \sin 2A.$$

Hence the given expression
$$= \sin 2C - \sin 2B + \sin 2A - \sin 2C + \sin 2B - \sin 2A = 0. \qquad \blacksquare$$

§ Problem 12.2.20. $$\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C}. \qquad \lozenge$$

§§ Solution.
$$\begin{aligned} \frac{(a+b+c)^2}{a^2+b^2+c^2} &= \frac{(\sin A + \sin B + \sin C)^2}{\sin^2 A + \sin^2 B + \sin^2 C} \\ &= \frac{16\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{\sin^2 A + \sin^2 B + \sin^2 C}, \textit{ by } \S\textit{Problem } 9.2.4 \\ &= 2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \times \frac{\sin A\sin B\sin C}{\sin^2 A + \sin^2 B + \sin^2 C} \\ &= 2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \div \frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin A\sin B\sin C} \\ &= 2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \div \left[\frac{\sin A}{\sin B\sin C} + \frac{\sin B}{\sin A\sin C} + \frac{\sin C}{\sin A\sin B}\right] \end{aligned}$$

$$= 2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \div \left[\frac{\sin(B+C)}{\sin B\sin C} + \frac{\sin(A+C)}{\sin A\sin C} + \frac{\sin(A+B)}{\sin A\sin B}\right]$$
$$= 2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \div [\cot C + \cot B + \cot C + \cot A + \cot B + \cot A]$$
$$= \frac{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{\cot A + \cot B + \cot C}$$
$$= \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C},$$ *by* §*Problem* 9.2.13.

Otherwise thus :

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{(\sin A + \sin B + \sin C)^2}{\sin^2 A + \sin^2 B + \sin^2 C}$$
$$= \frac{16\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{2\,(1+\cos A\cos B\cos C)},$$ *by* §*Problem* 9.2.4 *and Ex.* 3 *of Art.* 127
$$= \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \div \left[\frac{\cos A\cos B\cos C - \cos(A+B+C)}{\sin A\sin B\sin C}\right],$$

by dividing numerator and denominator by $2\sin A\sin B\sin C$, and putting $\cos(A+B+C) = -1$.

Now

$$\frac{\cos A\cos B\cos C - \cos(A+B+C)}{\sin A\sin B\sin C}$$
$$= \cot A\cot B\cot C - \frac{\cos(A+B+C)}{\sin A\sin B\sin C}$$
$$= \cot A\cot B\cot C - (\cot A\cot B\cot C - \cot A - \cot B - \cot C),$$

by expanding $\cos(A+B+C)$, as in *Art.* 124;

also $$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$$
$$\therefore \frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C}. \qquad \blacksquare$$

§ Problem 12.2.21.

$$a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc. \qquad \lozenge$$

§§ Solution.

$$a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B)$$
$$= k^3\left[\sin^3 A\cos(B-C) + \sin^3 B\cos(C-A) + \sin^3 C\cos(A-B)\right]$$
$$= k^3\left[\sin^2 A\sin(B+C)\cos(B-C) + \sin^2 B\sin(C+A)\cos(C-A) + \sin^2 C\sin(A+B)\cos(A-B)\right]$$
$$= \frac{k^3}{2}\left[\sin^2 A\,(\sin 2B + \sin 2C) + \sin^2 B\,(\sin 2C + \sin 2A) + \sin^2 C\,(\sin 2A + \sin 2B)\right]$$
$$= k^3\left(\sin^2 A\sin B\cos B + \sin^2 A\sin C\cos C + \sin^2 B\sin C\cos C + \sin^2 B\sin A\cos A\right.$$

$$
\begin{aligned}
&\qquad + \sin^2 C \sin A \cos A + \sin^2 C \sin B \cos B\Big) \\
&= k^3 \left[\sin A \sin B \left(\sin A \cos B + \cos A \sin B\right)\right. \\
&\qquad\qquad + \sin B \sin C \left(\sin B \cos C + \cos B \sin C\right) \\
&\qquad\qquad \left. + \sin C \sin A \left(\sin A \cos C + \cos A \sin C\right)\right] \\
&= k^3 \left[\sin A \sin B \sin(A+B) + \sin B \sin C \sin(B+C)\right. \\
&\qquad\qquad\qquad \left. + \sin C \sin A \sin(A+C)\right] \\
&= k^3 \left(\sin A \sin B \sin C + \sin B \sin C \sin A + \sin C \sin A \sin B\right) \\
&= 3k^3 \sin A \sin B \sin C = 3abc. \qquad \blacksquare
\end{aligned}
$$

§ Problem 12.2.22. *In a triangle whose sides are* $3,\ 4$ *and* $\sqrt{38}$ *feet respectively, prove that the largest angle is greater than* $120°$*.* $\lozenge$

§§ Solution. If $a = 3,\ b = 4$ and $c = \sqrt{38}$, then C is the largest angle. We have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 16 - 38}{2 \times 3 \times 4} = -\frac{13}{24}.$$

Now $$\cos 120° = -\frac{1}{2} = -\frac{12}{24}.$$

$\because \cos C$ is less than $\cos 120°$,

$$\therefore \angle C > 120°. \qquad \blacksquare$$

§ Problem 12.2.23. *The sides of a right-angled triangle are* 21 *and* 28 *feet; find the length of the perpendicular drawn to the hypotenuse from the right angle.* $\lozenge$

§§ Solution. Let ABC be the right-angled triangle and CD be the perpendicular drawn to the hypotenuse from the right angle $\angle C$.

Let $BC = 21\ feet\ and\ AC = 28\ feet.$

In the triangle ABC, we have

$$\tan A = \frac{21}{28} = \frac{3}{4},\ so\ that\ \sin A = \frac{3}{5}.$$

In the triangle ACD, we have

$$CD = AC \sin A = 28 \times \frac{3}{5} = \frac{84}{5} = 16\frac{4}{5}\ feet. \qquad \blacksquare$$

§ Problem 12.2.24. *If in any triangle the angles be to one another as* $1 : 2 : 3$*, prove that the corresponding sides are as* $1 : \sqrt{3} : 2$*.* $\lozenge$

§§ Solution. Let A, $2A$ and $3A$ be the angles, so that $A + 2A + 3A = 6A = 180°$ and $A = 30°$; i.e. the angles are $30°$, $60°$ and $90°$. If a, b and c denote the sides opposite these angles respectively, we have, by $Art.$ 163,

$$\frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{c}{\sin 90°}$$

$$\therefore a : b : c = \sin 30° : \sin 60° : \sin 90° = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2. \qquad \blacksquare$$

§ Problem 12.2.25. *In any triangle, if*

$$\tan\frac{A}{2} = \frac{5}{6}\ and\ \tan\frac{B}{2} = \frac{20}{37},$$

find $\tan\dfrac{C}{2}$ *and prove that in this triangle* $a + c = 2b$*.* $\lozenge$

§§ Solution. We have

$$\begin{aligned}
\tan\frac{C}{2} &= \tan\left(90^\circ - \frac{A+B}{2}\right) = \cot\frac{A+B}{2}\\
&= \frac{1}{\tan\dfrac{A+B}{2}} = \frac{1-\tan\dfrac{A}{2}\tan\dfrac{B}{2}}{\tan\dfrac{A}{2}+\tan\dfrac{B}{2}}\\
&= \frac{1-\dfrac{5}{6}\cdot\dfrac{20}{37}}{\dfrac{5}{6}+\dfrac{20}{37}} = \frac{222-100}{185+120} = \frac{122}{305} = \frac{2\times 61}{5\times 61} = \frac{2}{5}.\\
\because \tan\frac{A}{2}\tan\frac{C}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\\
\therefore \frac{5}{6}\cdot\frac{2}{5} &= \frac{s-b}{s};\ \therefore \frac{1}{3} = \frac{a+c-b}{a+c+b}\\
\therefore \frac{3-1}{3+1} &= \frac{b}{a+c};\ \therefore a+c = 2b.
\end{aligned}$$

■

§ Problem 12.2.26. *In an isosceles right-angled triangle, a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle into parts whose cotangents are* 2 *and* 3. ♢

§§ Solution. Let ACB be the triangle, right-angled at C and D be the middle point of AC. Join DB.

Let $AC = a = CB$. We then have

$$\cot\angle DBC = \frac{BC}{CD} = \frac{a}{\left(\dfrac{a}{2}\right)} = 2.$$

Also,

$$\begin{aligned}
\cot\angle ABD &= \cot\left(\angle ABC - \angle DBC\right)\\
&= \frac{\cot\angle ABC\cot\angle DBC + 1}{\cot\angle DBC - \cot\angle ABC}\\
&= \frac{2+1}{2-1} = 3,\\
\because \cot\angle ABC &= \frac{BC}{AC} = 1.
\end{aligned}$$

■

§ Problem 12.2.27. *The perpendicular* AD *to the base of a triangle* ABC *divides it into segments such that* BD, CD *and* AD *are in the ratio of* $2,\ 3$ *and* 6*; prove that the vertical angle of the triangle is* 45°. ♢

§§ Solution. Denote the $\angle BAD$ by α and the $\angle DAC$ by β. In the triangle ABD,

we have $$\tan\alpha = \frac{2}{6} = \frac{1}{3}.$$

In the ΔACD, we have $$\tan\beta = \frac{3}{6} = \frac{1}{2}.$$

In the ΔABC, we have

$$\tan A = \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = \frac{5}{5} = 1 = \tan 45^\circ.$$

The angle A is therefore 45°. ■

§ Problem 12.2.28. *A ring, ten inches in diameter, is suspended from a point one foot above its center by* 6 *equal strings attached to its circumference at equal intervals. Find the cosine of the angle between consecutive strings.* ◊

§§ Solution. Let O be the center of the ring and P be the point of suspension, so that $OP = 12''$.

Let $A,\ B,\ C,\ D,\ E$ and F be the points of attachment of the strings to the circumference. Then $A,\ B,\ C,\ D,\ E$ and F are the angular points of a regular hexagon inscribed in the ring.

Consider the points B and C. Join BC.

We have, $BC =$ radius of ring $= 5''$. Also

$$PB = PC = \sqrt{OP^2 + OC^2} = \sqrt{(12)^2 + 5^2} = \sqrt{144+25} = 13''.$$

$$\therefore \cos\angle BPC = \frac{(13)^2 + (13)^2 - 5^2}{2\times 13\times 13}$$

$$= \frac{313}{338} = \text{ the cosine of the angle required.}$$ ■

§ Problem 12.2.29. *If* a^2, b^2 *and* c^2 *be in* $A.\ P.$, *prove that* $\cot A$, $\cot B$ *and* $\cot C$ *are in* $A.\ P.$ *also.* ◊

§§ Solution. Given $a^2 - b^2 = b^2 - c^2$, i.e. $\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C$;

$$\therefore \sin(A+B)\sin(A-B) = \sin(B+C)\sin(B-C).$$

Dividing both sides of this last equation by $\sin A \sin B \sin C$, we have

$$\frac{\sin(A-B)}{\sin A\sin B} = \frac{\sin(B-C)}{\sin B\sin C}$$

$$\therefore \frac{\sin A\cos B - \cos A\sin B}{\sin A\sin B} = \frac{\sin B\cos C - \cos B\sin C}{\sin B\sin C}$$

$$\therefore \cot B - \cot A = \cot C - \cot B$$

$$\therefore \cot A - \cot B = \cot B - \cot C$$

$$\therefore\ \cot A,\ \cot B \textit{ and } \cot C \textit{ are in } A.\ P.$$

Otherwise thus :

We have $$2b^2 = a^2 + c^2$$

$$\therefore \frac{b^2}{ac} = \frac{a^2+c^2-b^2}{ac};\ \therefore \frac{\sin^2 B}{\sin A\sin C} = 2\cos B$$

$$\therefore \frac{\sin B}{\sin A\sin C} = \frac{2\cos B}{\sin B};\ \therefore \frac{\sin(A+C)}{\sin A\sin C} = \frac{2\cos B}{\sin B}$$

$$\therefore \frac{\cos A}{\sin A} + \frac{\cos C}{\sin C} = \frac{2\cos B}{\sin B};\ \therefore \cot A + \cot C = 2\cot B$$

$$\therefore\ \cot A,\ \cot B \textit{ and } \cot C \textit{ are in } A.\ P.$$ ■

§ Problem 12.2.30. *If* $a,\ b$ *and* c *be in* $A.\ P.$, *prove that* $\cos A\cot\frac{A}{2}$, $\cos B\cot\frac{B}{2}$ *and* $\cos C\cot\frac{C}{2}$ *are in* $A.\ P.$ ◊

§§ Solution.

Given $$a-b=b-c, \ \therefore \sin A-\sin B=\sin B-\sin C$$

$$\therefore 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}=2\cos\frac{B+C}{2}\sin\frac{B-C}{2}.$$

Dividing both sides of this last equation by $2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, we have

$$\frac{\sin\frac{A-B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}=\frac{\sin\frac{B-C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\therefore \frac{\sin\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}=\frac{\sin\frac{B}{2}\cos\frac{C}{2}-\cos\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\therefore \cot\frac{B}{2}-\cot\frac{A}{2}=\cot\frac{C}{2}-\cot\frac{B}{2}$$

$$\therefore \cot\frac{A}{2}-\cot\frac{B}{2}=\cot\frac{B}{2}-\cot\frac{C}{2}$$

$$\therefore \cot\frac{A}{2},\ \cot\frac{B}{2} \ \textit{and} \ \cot\frac{C}{2} \ \textit{are in} \ A.\ P.$$

Also $\sin A$, $\sin B$ and $\sin C$ are in $A.\ P.$:

$$\therefore \cot\frac{A}{2}-\sin A,\ \cot\frac{B}{2}-\sin B \ \textit{and} \ \cot\frac{C}{2}-\sin C \ \textit{are in} \ A.\ P.$$

Now $$\cot\frac{A}{2}-\sin A=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}-2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\left(1-2\sin^2\frac{A}{2}\right)=\cot\frac{A}{2}\cos A.$$

Similarly,

$$\cot\frac{B}{2}-\sin B=\cot\frac{B}{2}\cot B$$

and $$\cot\frac{C}{2}-\sin C=\cot\frac{C}{2}\cot C.$$

Hence $\cos A\cot\frac{A}{2}$, $\cos B\cot\frac{B}{2}$ and $\cos C\cot\frac{C}{2}$ are in $A.\ P.$ ■

§ Problem 12.2.31. *If* $a,\ b$ *and* c *are in* $H.\ P.$*, prove that* $\sin^2\frac{A}{2}$*,* $\sin^2\frac{B}{2}$ *and* $\sin^2\frac{C}{2}$ *are also in* $H.\ P.$ ◊

§§ Solution.

Given $$a:c=a-b:b-c$$

$$\therefore \sin A\,(\sin B-\sin C)=\sin C\,(\sin A-\sin B)$$

$$\therefore 2\sin\frac{A}{2}\cos\frac{A}{2}\cdot 2\cos\frac{B+C}{2}\sin\frac{B-C}{2}$$

$$=2\sin\frac{C}{2}\cos\frac{C}{2}\cdot 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\therefore \sin^2\frac{A}{2}\sin\frac{B+C}{2}\sin\frac{B-C}{2}=\sin^2\frac{C}{2}\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\therefore \sin^2\frac{A}{2}\left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)=\sin^2\frac{C}{2}\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)$$
$$\therefore \sin^2\frac{A}{2},\ \sin^2\frac{B}{2}\ \textit{and}\ \sin^2\frac{C}{2}\ \textit{are in H. P.}$$ ■

§ Problem 12.2.32. *The sides of a triangle are in A. P. and the greatest and least angles are θ and ϕ; prove that*
$$4\,(1-\cos\theta)\,(1-\cos\theta)=\cos\theta+\cos\phi.$$ ◊

§§ Solution. The third angle of the triangle is $\pi-(\theta+\phi)$. We are given

$$\sin\theta+\sin\phi=2\sin\left[\pi-(\theta+\phi)\right]=2\sin(\theta+\phi)$$
$$\therefore 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=4\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2}$$
$$\therefore \cos\frac{\theta-\phi}{2}=2\cos\frac{\theta+\phi}{2}$$
$$\therefore \cos\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\theta}{2}\sin\frac{\phi}{2}=2\cos\frac{\theta}{2}\cos\frac{\phi}{2}-2\sin\frac{\theta}{2}\sin\frac{\phi}{2}$$
$$\therefore 3\sin\frac{\theta}{2}\sin\frac{\phi}{2}=\cos\frac{\theta}{2}\cos\frac{\phi}{2} \tag{12.1}$$

by adding $\sin\frac{\theta}{2}\sin\frac{\phi}{2}$ to each side of Eq. (12.1):

$$4\sin\frac{\theta}{2}\sin\frac{\phi}{2}=\cos\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\theta}{2}\sin\frac{\phi}{2}=\cos\frac{\theta-\phi}{2} \tag{12.2}$$

by subtracting $\sin\frac{\theta}{2}\sin\frac{\phi}{2}$ from each side of Eq. (12.1):

$$2\sin\frac{\theta}{2}\sin\frac{\phi}{2}=\cos\frac{\theta}{2}\cos\frac{\phi}{2}-\sin\frac{\theta}{2}\sin\frac{\phi}{2}=\cos\frac{\theta+\phi}{2} \tag{12.3}$$

Hence, from Eq. (12.2) and Eq. (12.3), by multiplication, we have

$$16\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$
$$\therefore 4\,(1-\cos\theta)\,(1-\cos\phi)=\cos\theta+\cos\phi.$$

Otherwise thus :

Let a be the least side. Since $b=\frac{1}{2}(a+c)$, we have

$$\cos\theta=\frac{a^2+b^2-c^2}{2ab}=\frac{a^2-c^2}{2ab}+\frac{b^2}{2ab}=\frac{a-c}{a}+\frac{a+c}{4a}=\frac{5a-3c}{4a},\ \textit{and}$$
$$\cos\phi=\frac{b^2+c^2-a^2}{2bc}=\frac{(c+a)^2+4\left(c^2-a^2\right)}{4c(c+a)}=\frac{c+a+4(c-a)}{4c}=\frac{5c-3a}{4c}.$$
$$\therefore 4\,(1-\cos\theta)\,(1-\cos\phi)=\frac{(3c-a)(3a-c)}{4ac}=\frac{10ac-3a^2-3c^2}{4ac}$$
$$\therefore \cos\theta+\cos\phi=\frac{(5a-3c)c+(5c-3a)a}{4ac}=\frac{10ac-3a^2-3c^2}{4ac}$$
$$\therefore 4\,(1-\cos\theta)\,(1-\cos\phi)=\cos\theta+\cos\phi.$$ ■

§ Problem 12.2.33. *The sides of a triangle are in A. P. and the greatest angle exceeds the least by $90°$; prove that the sides are proportional to $\sqrt{7}+1$, $\sqrt{7}$ and $\sqrt{7}-1$.* ◊

§§ Solution. Let A be the greatest and C be the least angle of the triangle ABC.

Given, $$a+c=2b \tag{12.4}$$

and $$A-C=90^\circ \tag{12.5}$$

From Eq. (12.4), we have

$$\sin A+\sin C=2\sin B=2\sin(A+C)$$
$$\therefore 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}=4\sin\frac{A+C}{2}\cos\frac{A+C}{2}$$
$$\therefore \cos\frac{A-C}{2}=2\cos\frac{A+C}{2}=2\sin\frac{B}{2}$$
$$\therefore \cos 45^\circ=\frac{1}{\sqrt{2}}=2\sin\frac{B}{2}$$
$$\therefore \sin\frac{B}{2}=\frac{1}{2\sqrt{2}} \textit{ and } \cos\frac{B}{2}=\sqrt{1-\sin^2\frac{B}{2}}=\frac{\sqrt{7}}{2\sqrt{2}}$$
$$\therefore \sin A+\sin C=2\sin B=4\sin\frac{B}{2}\cos\frac{B}{2}=\frac{\sqrt{7}}{2}.$$

Also

$$\sin A-\sin C=2\cos\frac{A+C}{2}\sin\frac{A-C}{2}=2\sin\frac{B}{2}\sin 45^\circ=\frac{1}{2}.$$

Hence, by addition, we have

$$2\sin A=\frac{\sqrt{7}+1}{2},\ \therefore \sin A=\frac{\sqrt{7}+1}{4}$$

and, by subtraction, we have

$$2\sin C=\frac{\sqrt{7}-1}{2},\therefore \sin C=\frac{\sqrt{7}-1}{4}.$$
$$\therefore a:b:c=\sin A:\sin B:\sin C$$
$$=\sqrt{7}+1:\sqrt{7}:\sqrt{7}-1.$$

■

§ Problem 12.2.34. *If $C=60^\circ$, then prove that*

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}.$$

◊

§§ Solution. We have

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$$

if $$(2c+a+b)(c+a+b)=3(c+a)(c+b)$$

i.e. if $$a^2-ab+b^2=c^2$$

i.e. if $$a^2-ab+b^2=a^2-2ab\cos C+b^2$$

i.e. if $$2\cos C=1$$

i.e. if $$C=60^\circ.$$

■

§ Problem 12.2.35. *In any triangle ABC, if D be any point of the base BC, such that $BD:DC::m:n$ and if $\angle BAD=\alpha$, $\angle DAC=\beta$, $\angle CDA=\theta$ and $AD=x$, prove that*

$$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$$
$$=n\cot B-m\cot C$$

and $$(m+n)^2\cdot x^2=(m+n)\left(mb^2+nc^2\right)-mna^2.$$ ◊

§§ Solution. Take the figure of *Art.* 218, with m and n for x and y respectively.

We have

$$\frac{m}{n} = \frac{BD}{DC} = \frac{BD}{AD} \cdot \frac{AD}{DC} = \frac{\sin \angle BAD}{\sin \angle ABD} \cdot \frac{\sin \angle ACD}{\sin \angle DAC}$$
$$= \frac{\sin(\theta - B)}{\sin B} \cdot \frac{\sin C}{\sin(\theta + C)}$$
$$= \frac{\sin\theta \cot B - \cos\theta}{\sin\theta \cot C + \cos\theta} = \frac{\cot B - \cot\theta}{\cot C + \cot\theta}$$
$$\therefore (m+n)\cot\theta = n\cot B - m\cot C.$$

Again, we have $$BD = BD \times \frac{BC}{BC}$$

and $$DC = DC \times \frac{BC}{BC};\ i.\ e.\ BD = \frac{ma}{m+n}$$

and $$DC = \frac{na}{m+n}.$$

Hence in the triangle ADC, we have

$$AD^2 + \left(\frac{na}{m+n}\right)^2 - 2AD \cdot \frac{na}{m+n} \cdot \cos\theta = b^2 \tag{12.6}$$

Hence in the triangle ADB, we have

$$AD^2 + \left(\frac{ma}{m+n}\right)^2 + 2AD \cdot \frac{ma}{m+n} \cdot \cos\theta = c^2 \tag{12.7}$$

Multiplying Eq. (12.6) by m and Eq. (12.7) by n, by addition,

$$(m+n)AD^2 + \frac{mna^2}{m+n} = mb^2 + nc^2$$
$$\therefore (m+n)^2 AD^2 = (m+n)\left(mb^2 + nc^2\right) - mna^2. \qquad \blacksquare$$

§ **Problem 12.2.36.** *If in a triangle, the bisector of the side c be perpendicular to the side b, prove that*

$$2\tan A + \tan C = 0. \qquad \Diamond$$

§§ **Solution**. Let D be the middle point of c. Then the $\angle ACD = 90°$. Through B, draw BE parallel to DC, meeting AC produced in E. Then,

$$\because AD = DB,\ \therefore AC = CE.$$

We have $$\tan A = \frac{DC}{AC}$$

and $$\tan C = -\frac{BE}{CE} = -\frac{2DC}{AC} = -2\tan A.$$

$$\therefore 2\tan A + \tan C = 0. \qquad \blacksquare$$

§ **Problem 12.2.37.** *In any triangle, prove that, if θ be any angle, then*

$$b\cos\theta = c\cos(A-\theta) + a\cos(C+\theta). \qquad \Diamond$$

§§ **Solution**. By *Art*. 170 and $\because c\sin A = a\sin C$; we have

$$b\cos\theta = (c\cos A + a\cos C)\cos\theta + (c\sin A - a\sin C)\sin\theta$$
$$\therefore b\cos\theta = c(\cos A\cos\theta + \sin A\sin\theta) + a(\cos C\cos\theta - \sin C\sin\theta)$$
$$= c\cos(A-\theta) + a\cos(C+\theta). \qquad \blacksquare$$

§ **Problem 12.2.38.** *If p and q be the perpendiculars from the angular points A and B on any line passing through the vertex C of the triangle ABC, then prove that*

$$a^2p^2 + b^2q^2 - 2abpq\cos C = a^2b^2\sin^2 C. \qquad \Diamond$$

§§ Solution. Let p and q meet the line through C in M and N respectively. We then have

$$\sin\angle ACM = \frac{p}{b},\ \textit{and}\ \sin\angle BCN = \frac{q}{a}.$$

Also, $$\angle ACM + \angle BCN = 180^\circ - C$$

$$\therefore \cos\left(\angle ACM + \angle BCN\right) = \cos\left(180^\circ - C\right)$$

$$\therefore \cos\angle ACM \cos\angle BCN - \sin\angle ACM \sin\angle BCN = -\cos C$$

$$\therefore \sqrt{1-\frac{p^2}{b^2}}\sqrt{1-\frac{q^2}{a^2}} - \frac{p}{b}\cdot\frac{q}{a} = -\cos C$$

$$\therefore \left(1-\frac{p^2}{b^2}\right)\left(1-\frac{q^2}{a^2}\right) = \left(\frac{pq}{ab}-\cos C\right)^2$$

$$\therefore 1 - \frac{p^2}{b^2} - \frac{q^2}{a^2} + \frac{p^2q^2}{a^2b^2} = \frac{p^2q^2}{a^2b^2} - 2\frac{pq}{ab}\cos C + \cos^2 C$$

$$\therefore a^2p^2 + b^2q^2 - 2abpq\cos C = a^2b^2\left(1-\cos^2 C\right) = a^2b^2\sin^2 C. \qquad \blacksquare$$

§ Problem 12.2.39. *In the triangle ABC, lines OA, OB and OC are drawn so that the angles $\angle OAB$, $\angle OBC$ and $\angle OCA$ are each equal to ω; prove that*

$$\cot\omega = \cot A + \cot B + \cot C$$

and $$\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C. \qquad \Diamond$$

§§ Solution. Since the $\angle OBC = \angle OCA$

$$\therefore \angle BOC = 180^\circ - \angle OCB - \angle OCA = 180^\circ - C.$$

Similarly, $$\angle COA = 180^\circ - A,\ \textit{and}\ \angle AOB = 180^\circ - B.$$

In the triangle OAB, we have

$$\frac{OA}{c} = \frac{\sin\left(B-\omega\right)}{\sin B}.$$

In the triangle OAC, we have

$$\frac{b}{OA} = \frac{\sin A}{\sin\omega}.$$

Hence, by multiplication, we have

$$\frac{b}{c} = \frac{\sin\left(B-\omega\right)}{\sin\omega}\cdot\frac{\sin A}{\sin B}$$

$$\therefore \frac{\sin B\cdot\sin B}{\sin C\cdot\sin A} = \frac{\sin\left(B-\omega\right)}{\sin\omega} = \frac{\sin B\cos\omega - \cos B\sin\omega}{\sin\omega}$$

$$\therefore \frac{\sin^2 B}{\sin C\sin A} = \sin B\cot\omega - \cos B$$

$$\therefore \cot\omega = \cot B + \frac{\sin B}{\sin C\sin A} = \cot B + \frac{\sin(A+C)}{\sin C\sin A}$$

$$\therefore \cot\omega = \cot A + \cot B + \cot C.$$

Again,

$$\begin{aligned}
\operatorname{cosec}^2\omega &= 1 + \cot^2\omega \\
&= 1 + \cot^2 A + \cot^2 B + \cot^2 C \\
&\qquad + 2\left(\cot A\cot B + \cot B\cot C + \cot C\cot A\right) \\
&= 3 + \cot^2 A + \cot^2 B + \cot^2 C,\ \textit{by}\ \S\textit{Problem}\ 9.2.14 \\
&= 1 + \cot^2 A + 1 + \cot^2 B + 1 + \cot^2 C \\
&= \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C. \qquad \blacksquare
\end{aligned}$$

Chapter 13

Solution of Triangles

13.1 Solution of Right-Angled Triangles

§ **Problem 13.1.1.** *In a right-angled triangle ABC, where C is the right angle, if $a = 50$ and $B = 75°$, find the sides.*

$$\left[\tan 75° = 2 + \sqrt{3}.\right]$$ ◊

§§ **Solution**. By *Art.* 177, we have $\dfrac{b}{50} = \tan 75° = 2 + \sqrt{3}$

$$\therefore b = 50\left(2 + \sqrt{3}\right) = 50 \times 3.73205 = 186.60\ldots$$

Also, $$c = \frac{a}{\cos B} = \frac{50}{\cos 75°}.$$

Now by *Art.* 93, $$\cos 75° = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\therefore c = \frac{100\sqrt{2}}{\sqrt{3} - 1} = \frac{100\sqrt{2}\left(\sqrt{3} + 1\right)}{2} = 50\left(\sqrt{6} + \sqrt{2}\right)$$

$$= 50\,[2.4494 + 1.4142] = 193.18.$$ ∎

§ **Problem 13.1.2.** *Solve the triangle of which two-sides are equal to 10 and 20 feet and of which the included angle is 90° ; given that* $\log 20 = 1.30103$, *and*

$$L\tan 26°33' = 9.6986847,\ diff.\ for\ 1' = 3160.$$ ◊

§§ **Solution**. In *Art.* 176, putting $a = 20$ and $b = 10$, we have

$$\tan B = \frac{10}{20} = \frac{1}{2}$$

$$\therefore L\tan B = 10 - \log 2 = 10 - 1.30103 = 9.69897.$$

$$\text{Let } B = 26°33' + x'',\ \therefore L\tan\left(26°33' + x''\right) = 9.69897.$$

$$\text{The diff. for } x'' = 9.69897 - 9.6986847 = .0002853.$$
$$\text{Hence } x = 60'' \times \frac{2853}{3160} \approx 54''$$
$$\therefore B = 26^\circ 33' 54''; \therefore A = 90^\circ - B = 63^\circ 26' 6''.$$
$$\text{Also, } c = \sqrt{20^2 + 10^2} = 10\sqrt{5}\ feet.$$ ■

§ Problem 13.1.3. *The length of the perpendicular from one angle of a triangle upon the base is* 3 *inches and the lengths of the sides containing this angle are* 4 *and* 5 *inches. Find the angles, having given*

$$\log 2 = .30103, \ \log 3 = .4771213$$
$$L\sin 36^\circ 52' = 9.7781186, \ diff. \ for \ 1' = 1684$$
$$\text{and } L\sin 48^\circ 35'' = 9.8750142, \ diff. \ for \ 1' = 1115.$$ ◊

§§ Solution. Let $AD (= 3\ inches)$ be the perpendicular from A upon BC the base of the ΔABC and let $AB = 4\ inches$ and $AC = 5\ inches$.

We then have $\sin C = \frac{3}{5} = \frac{3 \times 2}{10}$ and $\sin B = \frac{3}{4} = \frac{3}{2^2}$.

$$\therefore L\sin C = 10 + \log 3 + \log 2 - \log 10$$
$$= 10 + .4771213 + .30103 - 1 = 9.7781513.$$

Let $\quad C = 36^\circ 52' + x'', \ \therefore L\sin\left(36^\circ 52' + x''\right) = 9.7781513.$

The diff. for $\quad x'' = 9.7781513 - 9.7781186 = .0000327.$

$$\therefore x = 60'' \times \frac{327}{1684} \approx 12''; \ \therefore C = 36^\circ 52' 12''.$$

Again, $\quad L\sin B = 10 + \log 3 - 2\log 2$

$$= 10 + .4771213 - .60206 = 9.8750613.$$

Let $\quad B = 48^\circ 35' + x''$

$$\therefore L\sin\left(48^\circ 35' + x''\right) = 9.8750613.$$

The diff. for $\quad x'' = 9.8750613 - 9.8750142 = .0000471.$

$$\therefore x = 60'' \times \frac{471}{1115} \approx 25''$$
$$\therefore B = 48^\circ 35' 25''.$$

Also, $\quad A = 180^\circ - (B + C) = 94^\circ 32' 23''.$ ■

§ Problem 13.1.4. *Find the acute angles of a right-angled triangle whose hypotenuse is four times as long as the perpendicular drawn to it from the opposite angle.* ◊

§§ Solution. Let ABC be the triangle, C the right angle and CD be the perpendicular from C upon AB. We are given $AB = 4CD$.

$$\therefore AC\sec A = 4AC\sin A$$
$$\therefore 4\sin A\cos A = 1$$
$$\therefore 2\sin 2A = 1$$
$$\therefore \sin 2A = \frac{1}{2} = \sin 30^\circ \text{ or } \sin 150^\circ$$
$$\therefore 2A = 30^\circ \text{ or } 150^\circ$$
$$\therefore A = 15^\circ \text{ or } 75^\circ.$$
$$If \ A = 15^\circ, \ then \ B = 90^\circ - 15^\circ = 75^\circ.$$
$$If \ A = 75^\circ, \ then \ B = 90^\circ - 75^\circ = 15^\circ.$$ ■

13.2 Solution of Non-Right-Angled Triangles

§ Problem 13.2.1. *If the sides of a triangle be* $56,\ 65$*, and* 33 *feet, find the greatest angle.* $\Diamond$

§§ Solution. The greatest angle is opposite the greatest side, so that if $a=56$ feet, $b=65$ feet and $c=33$ feet,then B is the required angle. We have

$$s=\frac{1}{2}(56+65+33)=77,\ s-a=21,\ s-b=12,\ and\ s-c=44.$$

$$\therefore \tan\frac{B}{2}=\sqrt{\frac{44\times 21}{77\times 12}}=1=\tan 45^\circ$$

$$\therefore \frac{B}{2}=45^\circ,\ and\ \therefore B=90^\circ.$$

Alternative Sol. Since $65^2=56^2+33^2$, the angle opposite the side 65 is a right angle. ∎

§ Problem 13.2.2. *The sides of a triangle are* $7,\ 4\sqrt{3}$*, and* $\sqrt{13}$ *yards respectively. Find the number of degrees in its smallest angle.* $\Diamond$

§§ Solution. If $a=7$ yards, $b=4\sqrt{3}$ yards, and $c=\sqrt{13}$ yards, then C is the required angle.

We have $$\cos C=\frac{7^2+\left(4\sqrt{3}\right)^2-\left(\sqrt{13}\right)^2}{2\times 7\times 4\sqrt{3}}=\frac{49+48-13}{56\sqrt{3}}$$

$$=\frac{84}{56\sqrt{3}}=\frac{3}{2\sqrt{3}}=\frac{\sqrt{3}}{2}=\cos 30^\circ.$$

$$\therefore C=30^\circ.$$ ∎

§ Problem 13.2.3. *The sides of a triangle are* x^2+x+1*,* $2x+1$ *and* x^2-1 *; prove that the greatest angle is* 120°*.* $\Diamond$

§§ Solution. If θ be the required angle, we have

$$\begin{aligned}\cos\theta&=\frac{\left(x^2-1\right)^2+(2x+1)^2-\left(x^2+x+1\right)^2}{2\left(x^2-1\right)(2x+1)}\\&=\frac{x^4-2x^2+1+4x^2+4x+1-\left(x^4+2x^3+4x^2+2x+1\right)}{2\left(x^2-1\right)(2x+1)}\\&=\frac{-\left(2x^3+x^2-2x-1\right)}{2\left(2x^3+x^2-2x-1\right)}=-\frac{1}{2}=\cos 120^\circ.\end{aligned}$$

$$\therefore \theta=120^\circ.$$ ∎

§ Problem 13.2.4. *The sides of a triangle are* a*,* b *and* $\sqrt{a^2+ab+b^2}$ *feet; find the greatest angle.* $\Diamond$

§§ Solution. If θ be the required angle, we have

$$\cos\theta=\frac{a^2+b^2-\left(a^2+ab+b^2\right)}{2ab}=\frac{-ab}{2ab}=-\frac{1}{2}=\cos 120^\circ.$$

$$\therefore \theta=120^\circ.$$ ∎

§ Problem 13.2.5. *If* $a=2$*,* $b=\sqrt{6}$ *and* $c=\sqrt{3}-1$*, solve the triangle.* $\Diamond$

§§ Solution. We have

$$\cos A = \frac{\left(\sqrt{6}\right)^2 + \left(\sqrt{3}-1\right)^2 - 2^2}{2\sqrt{6}\left(\sqrt{3}-1\right)} = \frac{6+4-2\sqrt{3}-4}{2\sqrt{6}\left(\sqrt{3}-1\right)}$$
$$= \frac{3-\sqrt{3}}{\sqrt{6}\left(\sqrt{3}-1\right)} = \frac{\sqrt{3}\left(\sqrt{3}-1\right)}{\sqrt{6}\left(\sqrt{3}-1\right)} = \frac{1}{\sqrt{2}} = \cos 45^\circ.$$
$$\therefore A = 45^\circ.$$
$$\cos B = \frac{\left(\sqrt{3}-1\right)^2 + 2^2 - \left(\sqrt{6}\right)^2}{4\left(\sqrt{3}-1\right)} = \frac{4-2\sqrt{3}+4-6}{4\left(\sqrt{3}-1\right)}$$
$$= \frac{1-\sqrt{3}}{2\left(\sqrt{3}-1\right)} = -\frac{1}{2} = \cos 120^\circ.$$
$$\therefore B = 120^\circ.$$
$$\therefore C = 180^\circ - (A+B) = 180^\circ - 165^\circ = 15^\circ.$$
■

§ Problem 13.2.6. *If $a = 2$, $b = \sqrt{6}$ and $c = \sqrt{3}+1$, solve the triangle.* ◊

§§ Solution. We have

$$\cos A = \frac{\left(\sqrt{6}\right)^2 + \left(\sqrt{3}+1\right)^2 - 2^2}{2\sqrt{6}\left(\sqrt{3}+1\right)} = \frac{6+4+2\sqrt{3}-4}{2\sqrt{6}\left(\sqrt{3}+1\right)}$$
$$= \frac{\sqrt{3}\left(\sqrt{3}+1\right)}{\sqrt{6}\left(\sqrt{3}+1\right)} = \frac{1}{\sqrt{2}} = \cos 45^\circ.$$
$$\therefore A = 45^\circ.$$
$$\cos B = \frac{\left(\sqrt{3}+1\right)^2 + 2^2 - \left(\sqrt{6}\right)^2}{4\left(\sqrt{3}+1\right)} = \frac{4+2\sqrt{3}+4-6}{4\left(\sqrt{3}+1\right)}$$
$$= \frac{\sqrt{3}+1}{2\left(\sqrt{3}+1\right)} = \frac{1}{2} = \cos 60^\circ.$$
$$\therefore B = 60^\circ.$$
$$\therefore C = 180^\circ - (A+B) = 180^\circ - 105^\circ = 75^\circ.$$
■

§ Problem 13.2.7. *If $a = 9$, $b = 10$ and $c = 11$, find B, given*

$$\log 2 = .30103,\ L\tan 29^\circ 29' = 9.7523472$$

and $$L\tan 29^\circ 30' = 9.7526420.$$

◊

§§ Solution. We have

$$s = \frac{1}{2}(9+10+11) = 15,\ s-a = 6,\ s-b = 5 \textit{ and } s-c = 4$$
$$\therefore \tan\frac{B}{2} = \sqrt{\frac{4\times 6}{15\times 5}} = \sqrt{\frac{8}{25}} = \sqrt{\frac{2^5}{100}}$$
$$\therefore L\tan\frac{B}{2} = 10 + \frac{1}{2}\left(5\log 2 - 2\right) = 9.7525750.$$

Let $\dfrac{B}{2} = 29^\circ 29' + x''$, $\therefore L\tan\left(29^\circ 29' + x''\right) = 9.7525750.$

The diff. for $x'' = 9.7525750 - 9.7523472 = .0002278.$

The diff. for $60'' = 9.7526420 - 9.7523472 = .0002948.$

$$\therefore x = 60'' \times \frac{2278}{2948} \approx 46.3''.$$

$$\therefore \frac{B}{2} = 29°29'46.3'' \textit{ and } B = 58°59'33''. \qquad \blacksquare$$

§ Problem 13.2.8. *The sides of a triangle are* 130, 123 *and* 77 *feet. Find the greatest angle,having given*

$$\log 2 = .30103, \ L\tan 38°39' = 9.9029376$$

and
$$L\tan 38°40' = 9.9031966. \qquad \lozenge$$

§§ Solution. If $a = 130$ *feet*, $b = 123$ *feet* and $c = 77$ *feet*, then A is the required angle. We have

$$s = \frac{1}{2}(130 + 123 + 77) = 165, \ s - a = 35, \ s - b = 42 \textit{ and } s - c = 88$$

$$\therefore \tan\frac{A}{2} = \sqrt{\frac{42 \times 88}{165 \times 35}} = \frac{4}{5} = \frac{2^3}{10}$$

$$\therefore L\tan\frac{A}{2} = 10 + 3\log 2 - \log 10 = 9.90309.$$

Let $\frac{A}{2} = 38°39' + x''$, so that $L\tan(38°39' + x'') = 9.90309$.

The diff. for $x'' = 9.90309 - 9.9029376 = .0001524.$

The diff. for $60'' = 9.9031966 - 9.9029376 = .0002590.$

$$\therefore x = 60'' \times \frac{1524}{2590} \approx 35.3''.$$

$$\therefore \frac{A}{2} = 38°39'35.3'' \textit{ and } A = 77°19'11''. \qquad \blacksquare$$

§ Problem 13.2.9. *Find the greatest angle of a triangle whose sides are* 212, 188 *and* 270 *feet, having given*

$$\log 2 = .30103, \ \log 3 = .4771213, \ \log 7 = .8450980$$

$$L\tan 38°20' = 9.8980104 \textit{ and } L\tan 38°19' = 9.8977507. \qquad \lozenge$$

§§ Solution. If $a = 242$ *feet*, $b = 188$ *feet* and $c = 270$ *feet*, then C is the required angle.

We have

$$s = \frac{1}{2}(242 + 188 + 270) = 350, \ s - a = 108$$

$$s - b = 162 \textit{ and } s - c = 80.$$

$$\therefore \tan\frac{C}{2} = \sqrt{\frac{108 \times 162}{350 \times 80}} = \sqrt{\frac{2 \times 3^7}{7 \times 10^3}}$$

$$\begin{aligned}\therefore L\tan\frac{C}{2} &= 10 + \frac{1}{2}(7\log 3 + \log 2 - \log 7 - 3\log 10) \\ &= 10 + \frac{1}{2}(3.3398491 + .3010300 - .8450980 - 3) = 9.8978905.\end{aligned}$$

Let $\frac{C}{2} = 38°19' + x''$, so that $L\tan(38°19' + x'') = 9.8978905$.

The diff. for $x'' = 9.8978905 - 9.8977507 = .0001398.$

The diff. for $60'' = 9.8980104 - 9.8977507 = .0002597.$

Hence
$$x = 60'' \times \frac{1398}{2597} \approx 32.3''$$

$$\therefore \frac{C}{2} = 38°19'32.3'' \textit{ and } C = 76°39'5''. \qquad \blacksquare$$

§ Problem 13.2.10. *The sides of a triangle are* 2, 3 *and* 4*; find the greatest angle, having given*

$$\log 2 = .30103,\ \log 3 = .4771213$$
$$L\tan 52°14' = 10.1108395$$

and $$L\tan 52°15' = 10.1111004.$$ ◊

§§ Solution. If $a = 2$, $b = 3$ and $c = 4$, then C is the required angle. We have

$$s = \frac{1}{2}(2+3+4) = \frac{9}{2},\ s-a = \frac{5}{2},\ s-b = \frac{3}{2}\ \textit{and}\ s-c = \frac{1}{2}$$

$$\therefore \tan\frac{C}{2} = \sqrt{\frac{\frac{5}{2}\times\frac{3}{2}}{\frac{9}{2}\times\frac{1}{2}}} = \sqrt{\frac{5}{3}} = \sqrt{\frac{10}{3\times 2}}.$$

$$\therefore L\tan\frac{C}{2} = 10 + \frac{1}{2}(\log 10 - \log 2 - \log 3) = 10.1109243.$$

Let $\dfrac{C}{2} = 52°14' + x''$, so that $L\tan(52°14' + x'') = 10.1109243.$

The diff. for $x'' = 10.1109243 - 10.1108395 = .0000848.$

The diff. for $60'' = 10.1111004 - 10.1108395 = .0002609.$

$$\therefore x = 60''\times\frac{848}{2609} \approx 19.5''$$

$$\therefore \frac{C}{2} = 52°14'19.5''\ \textit{and}\ C = 104°28'39''.$$ ■

Making use of the tables, find all the angles when

§ Problem 13.2.11. $a = 25,\ b = 26\ \textit{and}\ c = 27.$ ◊

§§ Solution. We have

$$s = \frac{1}{2}(25+26+27) = 39,\ s-a = 14,\ s-b = 13\ \textit{and}\ s-c = 12$$

$$\therefore \tan\frac{A}{2} = \sqrt{\frac{13\times 12}{39\times 14}} = \sqrt{\frac{2}{7}}$$

$$\therefore L\tan\frac{A}{2} = 10 + \frac{1}{2}(\log 2 - \log 7) = 9.7279660.$$

Now $L\tan 28°7' = 9.7278048$, diff. for $1' = 3039$.

Let $\dfrac{A}{2} = 28°7' + x''$, so that $L\tan(28°7' + x'') = 9.7279660.$

The diff. for $x'' = 9.7279660 - 9.7278048 = .0001612.$

$$\therefore x = 60''\times\frac{1612}{3039} \approx 32''$$

$$\therefore \frac{A}{2} = 28°7'32''\ \textit{and}\ A = 56°15'4''.$$

Again, $$\tan\frac{B}{2} = \sqrt{\frac{12\times 14}{39\times 13}} = \frac{2\sqrt{14}}{13}$$

$$\therefore L\tan\frac{B}{2} = 10 + \log 2 + \frac{1}{2}\log 14 - \log 13 = 9.7601506.$$

Now $L\tan 29°55' = 9.7599794,$ diff. for $1' = 2922.$

Let $\dfrac{B}{2} = 29°55' + x''$, so that $L\tan(29°55' + x'') = 9.7601506.$

The diff. for $x'' = 9.7601506 - 9.7599794 = .0001712.$

Hence $$x = 60'' \times \frac{1712}{2922} \approx 35''$$
$$\therefore \frac{B}{2} = 29°55'35'' \textit{ and } B = 59°51'10''.$$
Also, $$C = 180° - (A + B) = 180° - 116°6'14'' = 63°53'46''.$$ ■

§ Problem 13.2.12. $a = 17,\ b = 20$ *and* $c = 27.$ ◊

§§ Solution. We have
$$s = \frac{1}{2}(17 + 20 + 27) = 32,\ s - a = 15,\ s - b = 12 \textit{ and } s - c = 5$$
$$\therefore \tan\frac{A}{2} = \sqrt{\frac{12 \times 5}{32 \times 15}} = \sqrt{\frac{1}{2^3}}$$
$$\therefore L\tan\frac{A}{2} = 10 + \frac{1}{2}(\log 1 - 3\log 2) = 9.5484550.$$
Now $L\tan 19°28' = 9.5483452$, diff. for $1' = 4019.$

Let $\frac{A}{2} = 19°28' + x''$, so that $L\tan(19°28' + x'') = 9.5484550.$

The diff. for $x'' = 9.5484550 - 9.5483452 = .0001098.$
$$\therefore x = 60'' \times \frac{1098}{4019} \approx 16.4''$$
$$\therefore \frac{A}{2} = 19°28'16.4'' \textit{ and } A = 38°56'33''.$$
Again, $$\tan\frac{B}{2} = \sqrt{\frac{5 \times 15}{32 \times 12}} = \frac{5}{\sqrt{2^7}} = \frac{10}{\sqrt{2^9}}$$
$$\therefore L\tan\frac{B}{2} = 10 + \log 10 - \frac{1}{2}(9\log 2) = 9.6453650.$$
Now $L\tan 23°50' = 9.6451743$, diff. for $1' = 3417.$

Let $\frac{B}{2} = 23°50' + x''$, so that $L\tan(23°50' + x'') = 9.6453650.$

The diff. for $x'' = 9.6453650 - 9.6451743 = .0001907.$

Hence $$x = 60'' \times \frac{1907}{3417} \approx 33.5''$$
$$\therefore \frac{B}{2} = 23°50'33.5'' \textit{ and } B = 47°41'7''.$$
Also, $$C = 180° - (A + B) = 180° - 86°37'40'' = 93°22'20''.$$ ■

§ Problem 13.2.13. $a = 2000,\ b = 1050$ *and* $c = 1150.$ ◊

§§ Solution. We have
$$s = \frac{1}{2}(2000 + 1050 + 1150) = 2100,\ s - a = 100,\ s - b = 1050$$
$$\textit{and } s - c = 950.$$
$$\therefore \tan\frac{A}{2} = \sqrt{\frac{1050 \times 950}{2100 \times 100}} = \frac{\sqrt{19}}{2}$$
$$\therefore L\tan\frac{A}{2} = 10 + \frac{1}{2}\log 19 - \log 2 = 10.3383468.$$
Now $L\tan 65°21' = 10.3382897$, diff. for $1' = 3334.$

Let $\frac{A}{2} = 65°21' + x''$, so that $L\tan(65°21' + x'') = 10.3383468.$

The diff. for $x'' = 10.3383468 - 10.3382897 = .0000571.$
$$\therefore x = 60'' \times \frac{571}{3334} \approx 10.276''$$

$$\therefore \frac{A}{2} = 65°21'10.276'' \text{ and } A = 130°42'20.5''.$$

Again, $$\tan\frac{B}{2} = \sqrt{\frac{950 \times 100}{2100 \times 1050}} = \frac{\sqrt{19}}{21}$$

$$\therefore L\tan\frac{B}{2} = 10 + \frac{1}{2}\log 19 - \log 21 = 9.3171575.$$

Now $$L\tan 11°43' = 9.3167950, \text{ diff. for } 1' = 6349.$$

Let $\frac{B}{2} = 11°43' + x''$, so that $L\tan(11°43' + x'') = 9.3171575$.

The diff. for $x'' = 9.3171575 - 9.3167950 = .0003625$.

Hence $$x = 60'' \times \frac{3625}{6349} \approx 34.25''$$

$$\therefore \frac{B}{2} = 11°43'34.25'' \text{ and } B = 23°27'8.5''.$$

Also, $C = 180° - (A + B) = 180° - 154°9'29'' = 25°50'31''$. ■

13.3 Two Sides and Included Angle

§ Problem 13.3.1. *If $b = 90$, $c = 70$ and $A = 72°48'30''$, find B and C, given*

$$\log 2 = .30103, \; L\cot 36°24'15'' = 10.1323111$$
$$L\tan 9°37' = 9.2290071$$

and $$L\tan 9°38' = 9.2297735.$$ ◊

§§ Solution. We have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{8}\cot\frac{A}{2} = \frac{1}{2^3}\cot 36°24'15''$$

$$\begin{aligned}\therefore L\tan\frac{B-C}{2} &= \log 1 - 3\log 2 + L\cot 36°24'15'' \\ &= 0 - .90309 + 10.1323111 = 9.2292211.\end{aligned}$$

Let $\frac{B-C}{2} = 9°37' + x''$, so that $L\tan(9°37' + x'') = 9.2292211$.

The diff. for $x'' = 9.2292211 - 9.2290071 = .0002140$.

The diff. for $60'' = 9.2297735 - 9.2290071 = .0007664$.

$$\therefore x = 60'' \times \frac{2140}{7664} \approx 17''$$

$$\therefore \frac{B-C}{2} = 9°37'17'' \tag{13.1}$$

But $$\frac{B+C}{2} = 90° - \frac{A}{2} = 90° - 36°24'15'' = 53°35'45'' \tag{13.2}$$

By adding (13.1) and (13.2), we have $B = 63°13'2''$.

By subtracting (13.1) from (13.2), we have $C = 43°58'28''$. ■

§ Problem 13.3.2. *If $a = 21$, $b = 11$ and $C = 34°42'30''$, find A and B, given*

$$\log 2 = .30103$$

and $$L\tan 72°38'45'' = 10.50515.$$ ◊

§§ Solution. We have

$$\tan\frac{A-b}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{10}{32}\cot 17°21'15''$$

$$\therefore L\tan\frac{A-B}{2} = \log 10 - 5\log 2 + L\tan 72°38'45''$$
$$= 1 - 1.50515 + 10.50515 = 10.$$
$$\therefore 10 + \log\tan\frac{A-B}{2} = 10$$
$$\therefore \log\tan\frac{A-B}{2} = 0, \ \therefore \tan\frac{A-B}{2} = 1 = \tan 45°$$
$$\therefore \frac{A-B}{2} = 45° \tag{13.3}$$
but $$\frac{A+B}{2} = 90° - \frac{C}{2} = 90° - 17°21'15'' = 72°38'45'' \tag{13.4}$$
By adding (13.3) and (13.4), we have $A = 117°38'45''$.
By subtracting (13.3) from (13.4), we have $B = 27°38'45''$. ∎

§ Problem 13.3.3. *If the angles of a triangle be in* A. P. *and the lengths of the greatest and least sides be* 24 *and* 16 *feet respectively, find the length of the third side and the angles, given*
$$\log 2 = .30103, \ \log 3 = .4771213$$
and $$L\tan 19°6' = 9.5394287, \textit{ diff. for } 1' = 4084.$$ ◊

§§ Solution. Let A, B and C be the angles. Since they are in A. P., we have
$$2B = A + C$$
$$\therefore 3B = A + B + C = 180°; \ \therefore B = 60° \textit{ and } A + C = 120°.$$
Let $a = 24$ *feet* and $c = 16$ *feet*. We then have
$$\tan\frac{A-C}{2} = \frac{a-c}{a+c}\tan\frac{A+C}{2} = \frac{2}{10}\tan 60° = \frac{2\sqrt{3}}{10}$$
$$\therefore L\tan\frac{A-C}{2} = 10 + \log 2 + \frac{1}{2}\log 3 - \log 10$$
$$= 10 + .30103 + .2385607 - 1 = 9.5395907.$$
Let $\dfrac{A-C}{2} = 19°6' + x''$, so that $L\tan\dfrac{A-C}{2} = 9.5395907$.
The diff. for $x'' = 9.5395907 - 9.5394287 = .0001620$.
The diff. for $60'' = .0004084$.
$$\therefore x = 60'' \times \frac{1620}{4084} \approx 24''$$
$$\therefore \frac{A-C}{2} = 19°6'24'' \tag{13.5}$$
But $$\frac{A+C}{2} = 60° \tag{13.6}$$
By adding (13.5) and (13.6), we have $A = 79°6'24''$.
By subtracting (13.5) from (13.6), we have $C = 40°53'36''$.
Also, $b^2 = a^2 + c^2 - 2ac\cos B = 576 + 256 - 384 = 448 = 64 \times 7$
$$\therefore b = 8\sqrt{7} \textit{ feet}.$$ ∎

§ Problem 13.3.4. *If* $a = 13$, $b = 7$ *and* $C = 60°$, *find* A *and* B, *given that*
$$\log 3 = .4771213$$
and $$L\tan 27°27' = 9.7155508, \textit{ diff. for } 1' = 3087.$$ ◊

§§ Solution. We have
$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{3}{10}\cot 30° = \frac{3\sqrt{3}}{10} = \frac{3^{\frac{3}{2}}}{10}$$

$$\therefore L\tan\frac{A-B}{2} = 10 + \frac{3}{2}\log 3 - \log 10 = 10 + .7156819 - 1 = 9.7156819.$$

Let $\dfrac{A-B}{2} = 27°27' + x''$, so that $L\tan(27°27' + x'') = 9.7156819$.

The diff. for $x'' = 9.7156819 - 9.7155508 = .0001311$.

The diff. for $60'' = .0003087$.

$$\therefore x = 60'' \times \frac{1311}{3087} \approx 25.5''$$

$$\therefore \frac{A-B}{2} = 27°27'25.5'' \tag{13.7}$$

But $$\frac{A+B}{2} = 90° - \frac{C}{2} = 90° - 30° = 60° \tag{13.8}$$

By adding (13.7) and (13.8), we have $A = 87°27'25.5''$.

By subtracting (13.7) from (13.8), we have $B = 32°32'34.5''$. ■

§ Problem 13.3.5. *If $a = 2b$ and $C = 120°$, find the values of A, B and the ratio of c to a, given that*

$$\log 3 = .4771213$$

and $$L\tan 10°53' = 9.2839070, \text{ diff. for } 1' = 6808.$$ ◊

§§ Solution. We have

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{2b-b}{2b+b}\cot 60° = \frac{1}{3}\cdot\frac{1}{\sqrt{3}} = \frac{1}{3^{\frac{3}{2}}}$$

$$\therefore L\tan\frac{A-B}{2} = 10 - \frac{3}{2}\log 3 = 10.7156819 = 9.2843181.$$

Let $\dfrac{A-B}{2} = 10°53' + x''$, so that $L\tan(10°53' + x'') = 9.2843181$.

The diff. for $x'' = 9.2843181 - 9.283907 = .0004111$.

The diff. for $60'' = .0006808$.

$$\therefore x = 60'' \times \frac{4111}{6808} \approx 36''$$

$$\therefore \frac{A-B}{2} = 10°53'36'' \tag{13.9}$$

But $$\frac{A+B}{2} = 90° - \frac{C}{2} = 90° - 60° = 30° \tag{13.10}$$

By adding (13.9) and (13.10), we have $A = 40°53'36''$.

By subtracting (13.9) from (13.10), we have $B = 19°6'24''$.

Also, $$c^2 = a^2 + b^2 - 2ab\cos C = a^2 + \frac{a^2}{4} - 2a \times \frac{a}{2}\left(-\frac{1}{2}\right)$$

$$= a^2 + \frac{a^2}{4} + \frac{a^2}{2} = \frac{7a^2}{4}$$

$$\therefore \frac{c^2}{a^2} = \frac{7}{4},\ \therefore \frac{c}{a} = \frac{\sqrt{7}}{2}$$

$$\therefore c : a = \sqrt{7} : 2.$$ ■

§ Problem 13.3.6. *If $b = 14$, $c = 11$ and $A = 60°$, find B and C, given that*

$$\log 2 = .30103,\ \log 3 = .4771213$$

$$L\tan 11°44' = 9.3174299$$

and $$L\tan 11°45' = 9.3180640.$$ ◊

§§ **Solution**. We have

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{3}{25}\cot 30^\circ=\frac{3\sqrt{3}}{25}=\frac{3^{\frac{3}{2}}\times 2^2}{100}$$

$$\therefore L\tan\frac{B-C}{2}=10+\frac{3}{2}\log 3+2\log 2-\log 100$$

$$=10+.7156819+.60206-2=9.3177419.$$

Let $\dfrac{B-C}{2}=11^\circ 44'+x''$, so that $L\tan(11^\circ 44'+x'')=9.3177419$.

The diff. for $x''=9.3177419-9.3174299=.0003120$.

The diff. for $60''=9.3180640-9.3174299=.0006341$.

$$\therefore x=60''\times\frac{3120}{6341}\approx 30''$$

$$\therefore\frac{B-C}{2}=11^\circ 44' 30'' \tag{13.11}$$

But $$\frac{B+C}{2}=90^\circ-\frac{A}{2}=90^\circ-30^\circ=60^\circ \tag{13.12}$$

By adding (13.11) and (13.12), we have $B=71^\circ 44' 30''$.

By subtracting (13.11) from (13.12), we have $C=48^\circ 15' 30''$. ■

§ **Problem 13.3.7.** *The two sides of a triangle are* 540 *and* 420 *yards long respectively and include an angle of* $52^\circ 6'$. *Find the remaining angles, given that*

$$\log 2=.30103,\ L\tan 26^\circ 3'=9.6891430$$

$$L\tan 14^\circ 20'=9.4074189 \textit{ and } L\tan 14^\circ 21'=9.4079453.$$ ◊

§§ **Solution**. If $b=540$ *yards*, $c=420$ *yards*, and $A=52^\circ 6'$, we have, since

$$\tan\frac{B-C}{2}\tan\frac{A}{2}=\frac{b-c}{b+c},\ \tan\frac{B-C}{2}\tan 26^\circ 6'=\frac{540-420}{540+420}=\frac{1}{8}=\frac{1}{2^3}$$

$$\therefore L\tan\frac{B-C}{2}+L\tan 26^\circ 3'=20+\log 1-3\log 2$$

$$\therefore L\tan\frac{B-C}{2}=20+0-.90309-9.6891430=9.4077670.$$

Let $\dfrac{B-C}{2}=14^\circ 20'+x''$, so that $L\tan(14^\circ 20'+x'')=9.4077670$.

The diff. for $x''=9.4077670-9.4074189=.0003481$.

The diff. for $60''=9.4079453-9.4074189=.0005264$.

$$\therefore x=60''\times\frac{3481}{5264}\approx 40''$$

$$\therefore\frac{B-C}{2}=14^\circ 20' 40'' \tag{13.13}$$

But $$\frac{B+C}{2}=90^\circ-\frac{A}{2}=90^\circ-26^\circ 3'=63^\circ 57' \tag{13.14}$$

By adding (13.13) and (13.14), we have $B=78^\circ 17' 40''$.

By subtracting (13.13) from (13.14), we have $C=49^\circ 36' 20''$. ■

§ **Problem 13.3.8.** *If* $b=2\frac{1}{2}$ *ft., $c=2$ ft. and $A=22^\circ 20'$, find the other angles and show that the third side is nearly one foot, given*

$$\log 2=.30103,\ \log 3=.47712$$

$$L\cot 11^\circ 10'=10.70465,\ L\sin 22^\circ 20'=9.57977$$

$$L\tan 29^\circ 22' 20''=9.75038,\ L\tan 29^\circ 22' 30''=9.75013$$

and $$L\sin 49°27'34'' = 9.88079.$$ ◊

§§ Solution. We have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{\left(\frac{1}{2}\right)}{4\frac{1}{2}}\cot 11°10' = \frac{1}{3^2}\cot 11°10'$$

$$\therefore L\tan\frac{B-C}{2} = \log 1 - 2\log 3 + L\cot 11°10'$$
$$= 0 - .95424 + 10.70465 = 9.75041.$$

Let $\dfrac{B-C}{2} = 29°22'20'' + x''$, so that $L\tan(29°22'20'' + x'') = 9.75041$.

The diff. for $\qquad x'' = 9.75041 - 9.75038 = .00003.$

The diff. for $\qquad 10'' = 9.75043 - 9.75038 = .00005.$

$$\therefore x = 10'' \times \frac{3}{5} = 6''$$

$$\therefore \frac{B-C}{2} = 29°22'26'' \tag{13.15}$$

But $$\frac{B+C}{2} = 90° - \frac{A}{2} = 90° - 11°10' = 78°50' \tag{13.16}$$

By adding (13.15) and (13.16), we have $B = 108°12'26''$.

By subtracting (13.15) from (13.16), we have $C = 49°27'34''$.

To find a, we have $$a = \frac{c\sin A}{\sin C}$$

$$\therefore \log a = \log c + L\sin A - L\sin C = .30103 + 9.57977 - 9.88079$$
$$= .00001 \approx 0$$

$$\therefore a \approx 1\ foot.$$ ■

§ Problem 13.3.9. *If* $a = 2$, $b = 1 + \sqrt{3}$ *and* $C = 60°$, *solve the triangle.* ◊

§§ Solution. We have

$$c^2 = a^2 + b^2 - 2ab\cos C = 4 + 4 + 2\sqrt{3} - 4\left(1+\sqrt{3}\right)\frac{1}{2} = 6$$

$$\therefore c = \sqrt{6}.$$

Also, $$\sin A = \frac{a}{c}\sin C = \frac{2}{\sqrt{6}}\sin 60° = \frac{2}{\sqrt{6}}\cdot\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore A = 45°$$

$$\therefore B = 180° - (A + C) = 180° - 105° = 75°.$$ ■

§ Problem 13.3.10. *Two sides of a triangle are* $\sqrt{3}+1$ *and* $\sqrt{3}-1$ *and the included angle is* $60°$*; find the other side and angles.* ◊

§§ Solution. If $b = \sqrt{3}+1$, $c = \sqrt{3}-1$ and $A = 60°$, we have

$$a^2 = b^2 + c^2 - 2bc\cos A = 4 + 2\sqrt{3} + 4 - 2\sqrt{3} - 2\times 2\times\frac{1}{2} = 6$$

$$\therefore a = \sqrt{6}.$$

Also, $$\sin C = \frac{c}{a}\sin A = \frac{\sqrt{3}-1}{\sqrt{6}}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\therefore C = 15°$$

$$\therefore B = 180° - (A + C) = 180° - 75° = 105°.$$ ■

§ Problem 13.3.11. *If $b = 1$, $c = \sqrt{3} - 1$ and $A = 60°$, find the length of the side a.* $\Diamond$

§§ Solution.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 1 + 4 - 2\sqrt{3} - 2\left(\sqrt{3} - 1\right)\frac{1}{2} = 6 - 3\sqrt{3} \\ &= 6 - 3\,(1.73205) = .80385 \\ \therefore a &= \sqrt{.80385} = .8965. \end{aligned}$$

$\blacksquare$

§ Problem 13.3.12. *If $b = 91$, $c = 125$ and $\tan\dfrac{A}{2} = \dfrac{17}{6}$, prove that $a = 204$.* $\Diamond$

§§ Solution. We have

$$\cos A = \frac{1 - \tan^2\dfrac{A}{2}}{1 + \tan^2\dfrac{A}{2}} = \frac{1 - \dfrac{289}{36}}{1 + \dfrac{289}{36}} = -\frac{253}{325}$$

$$\begin{aligned} \therefore a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 8281 + 15625 + 2 \times 91 \times 125 \times \frac{253}{325} \\ &= 8281 + 15625 + 17710 = 41616 \\ \therefore a &= 204. \end{aligned}$$

$\blacksquare$

§ Problem 13.3.13. *If $a = 5$, $b = 4$ and $\cos(A - B) = \dfrac{31}{32}$, prove that the third side c will be 6.* $\Diamond$

§§ Solution. We have

$$\tan\frac{A - B}{2} = \sqrt{\frac{1 - \cos(A - B)}{1 + \cos(A - B)}} = \sqrt{\frac{1 - \dfrac{31}{32}}{1 + \dfrac{31}{32}}} = \sqrt{\frac{1}{63}} = \frac{1}{3\sqrt{7}}.$$

Also, $$\tan\frac{A + B}{2} = \frac{a + b}{a - b}\tan\frac{A - B}{2} = \frac{9}{1}\cdot\frac{1}{3\sqrt{7}} = \frac{3}{\sqrt{7}} = \cot\frac{C}{2}$$

$$\therefore \cos C = \frac{\cot^2\dfrac{C}{2} - 1}{\cot^2\dfrac{C}{2} + 1} = \frac{\dfrac{9}{7} - 1}{\dfrac{9}{7} + 1} = \frac{2}{16} = \frac{1}{8}$$

$$\therefore c^2 = a^2 + b^2 - 2ab\cos C = 25 + 16 - 2 \times 5 \times 4 \times \frac{1}{8} = 36$$

$$\therefore c = 6.$$

$\blacksquare$

§ Problem 13.3.14. *One angle of a triangle is $30°$ and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$ yards. Find the length of the third side and the number of degrees in the other angles.* $\Diamond$

§§ Solution. If $b = 40$ *yards*, $c = 40\sqrt{3}$ *yards* and $A = 30°$, we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 1600 + 4800 - 2 \times 40 \times 40\sqrt{3} \times \frac{\sqrt{3}}{2} = 1600 \\ \therefore a &= 40 \text{ yards } = b. \\ \therefore B &= A = 30°. \\ \therefore C &= 180° - (A + B) = 180° - 60° = 120°. \end{aligned}$$

$\blacksquare$

§ Problem 13.3.15. *The sides of a triangle are* 9 *and* 3 *and the difference of the angles opposite to them is* 90. *Find the base and the angles, having given*

$$\log 2 = .30103,\ \log 3 = .4771213$$
$$\log 75894 = 4.8802074,\ \log 75895 = 4.8802132$$
$$L\tan 26^\circ 33' = 9.6986847$$

and $$L\tan 26^\circ 34' = 9.6990006.$$ ◊

§§ Solution. If $b = 9$, $c = 3$ and $B - C = 90^\circ$, we have, since

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$
$$\tan 45^\circ = \frac{1}{2}\cot\frac{A}{2}$$
$$\therefore 1 = \frac{1}{2}\cot\frac{A}{2},\ \therefore \tan\frac{A}{2} = \frac{1}{2}$$
$$\therefore L\tan\frac{A}{2} = \log 1 - \log 2 + 10 = 0 - .30103 + 10 = 9.69897.$$

Let $$\frac{A}{2} = 26^\circ 33' + x'',\ \therefore L\tan\left(26^\circ 33' + x''\right) = 9.69897.$$

The diff. for $x'' = 9.69897 - 9.6986847 = .0002853.$

The diff. for $60'' = 9.6990006 - 9.6986847 = .0003159.$

$$\therefore x = \frac{2853}{3159} \times 60'' = 54.2''$$
$$\therefore \frac{A}{2} = 26^\circ 33' 54.2'',\ \therefore A = 53^\circ 7' 48''$$
$$\therefore \frac{B+C}{2} = 90^\circ - \frac{A}{2} = 90^\circ - 26^\circ 33' 54'' = 63^\circ 26' 6'' \tag{13.17}$$

also, $$\frac{B-C}{2} = 45^\circ \tag{13.18}$$

By adding (13.17) and (13.18), we have $B = 108^\circ 26' 6''$.
By subtracting (13.18) from (13.17), we have $C = 18^\circ 26' 6''$.

Again, $$\cos A = \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}} = \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} = \frac{3}{5}$$

$$\therefore a^2 = b^2 + c^2 - 2bc\cos A = 81 + 9 - 2 \times 9 \times 3 \times \frac{3}{5}$$
$$= 90 - \frac{162}{5} = 90 - 32.4 = 57.6 = \frac{(24)^2}{10} = \frac{3^2 \times 2^6}{10}$$
$$\therefore 2\log a = 2\log 3 + 6\log 2 - \log 10$$
$$\therefore \log a = \log 3 + 3\log 2 - \frac{1}{2}\log 10$$
$$= .4771213 + .90309 - .5 = .8802113.$$

We have $$\log 7.5894 = .8802074 \tag{13.19}$$

and $$\log 7.5895 = .8802132 \tag{13.20}$$

Let $$\log(7.5894 + x) = .8802113 \tag{13.21}$$

From (13.19) and (13.20), the diff. for $.0001 = .0000058$.
From (13.19)and (13.21), the diff. for $x = .0000039$.

$$\therefore x = \frac{39}{58} \times .0001 = \frac{.0039}{58} \approx .000067.$$
$$\therefore a = 7.589467.$$ ■

§ **Problem 13.3.16.**

If $$\tan\phi = \frac{a-b}{a+b}\cot\frac{C}{2}$$

prove that $$c = (a+b)\frac{\sin\dfrac{C}{2}}{\cos\phi}.$$

If $a = 3$, $b = 1$ *and* $C = 53°7°48''$, *find* c *without getting* A *and* B, *given*

$$\log 2 = .30103,\ \log 25298 = 4.4030862$$

$$\log 25299 = 4.4031034,\ L\cos 26°33'54'' = 9.9515452$$

and $$L\tan 26°33'54'' = 9.6989700.$$ ◊

§§ **Solution**. We have

$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C\\
&= \left(a^2+b^2\right)\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right) - 2ab\left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right)\\
&= (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2}\\
&= (a+b)^2\sin^2\frac{C}{2}\left[\left(\frac{a-b}{a+b}\right)^2\cot^2\frac{C}{2} + 1\right]\\
&= (a+b)^2\sin^2\frac{C}{2}\left(\tan^2\phi + 1\right)\\
&= (a+b)^2\sin^2\frac{C}{2}\sec^2\phi
\end{aligned}$$

$$\therefore c = (a+b)\frac{\sin\dfrac{C}{2}}{\cos\phi}.$$

If $a = 3$, $b = 1$ and $C = 53°7'48''$, we have

$$\tan\phi = \frac{1}{2}\cot\frac{C}{2} = \frac{1}{2\tan\dfrac{C}{2}}.$$

$$\therefore L\tan\phi - 10 = \log 1 - \log 2 - L\tan\frac{C}{2} + 10$$

$$\therefore L\tan\phi = 20 - .30103 - 9.69897 = 10$$

$$\therefore 10 + \log\tan\phi = 10,\ \therefore \log\tan\phi = 0,\ \therefore \tan\phi = 1$$

$$\therefore \phi = 45°.$$

Also, $$\begin{aligned}L\sin\frac{C}{2} &= L\tan\frac{C}{2} + L\cos\frac{C}{2} - 10\\ &= 9.69897 + 9.9515452 - 10 = 9.6505152.\end{aligned}$$

Now $$c = (a+b)\sin\frac{C}{2}\sec 45° = 4\sin\frac{C}{2}\times\sqrt{2}$$

$$\begin{aligned}\therefore \log c &= 2\log 2 + L\sin\frac{C}{2} - 10 + \frac{1}{2}\log 2\\ &= \frac{5}{2}(.30103) + 9.6505152 - 10 = .4030902.\end{aligned}$$

Now $$\log 2.5298 = .4030862 \tag{13.22}$$

and $$\log 2.5299 = .4031034 \tag{13.23}$$

Let $$\log(2.5298 + x) = .4030902 \tag{13.24}$$

From (13.22)and (13.23), the diff. for $.0001 = .0000172$.
From (13.23) and (13.24), the diff. for $x = .0000040$.

$$\therefore x = \frac{40}{172}\times .0001 = \frac{.004}{172} = \frac{.001}{43} \approx .000023.$$

$$\therefore c = 2.529823. \qquad \blacksquare$$

§ Problem 13.3.17. *Two sides of a triangle are* 237 *and* 158 *feet and the contained angle is* $66°40'$; *find the base and the other angles, having given*

$$\log 2 = .30103, \ \log 79 = 1.89763$$
$$\log 22687 = 4.35578, \ L\cot 33°20' = 10.18197$$
$$L\sin 33°20' = 9.73998, \ L\tan 16°54' = 9.48262$$
$$L\tan 16°55' = 9.48308, \ L\sec 16°54' = 10.01917$$

and $$L\sec 16°55' = 10.01921.$$

Note : *Use either the formula* $\cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2}$ *or the formula of the preceding problem.* ♢

§§ Solution. If $b = 237$, $c = 158$ and $A = 66°40'$, we have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{79}{395}\cot\frac{A}{2} = \frac{1}{5}\cot\frac{A}{2} = \frac{2}{10}\cot 33°20'$$

$$\begin{aligned}\therefore L\tan\frac{B-C}{2} &= \log 2 - \log 10 + L\cot 33°20' \\ &= .30103 - 1 + 10.18197 = 9.48300.\end{aligned}$$

Let $\frac{B-C}{2} = 16°54' + x''$, so that $L\tan(16°54' + x'') = 9.48300$.

The diff. for $\quad x'' = 9.48300 - 9.48262 = .00038.$

The diff. for $\quad 60'' = 9.48308 - 9.48262 = .00046.$

$$\therefore x = 60'' \times \frac{38}{46} \approx 50''.$$

$$\therefore \frac{B-C}{2} = 16°54'50'' \tag{13.25}$$

But $$\frac{B+C}{2} = 90° - \frac{A}{2} = 90° - 33°20' = 56°40' \tag{13.26}$$

By adding (13.25) and (13.26), we have $B = 73°34'50''$.
subtracting (13.25) from (13.26), we have $C = 39°45'10''$.

Again, we have $$a = (b+c)\sin\frac{A}{2}\sec\frac{B-C}{2}$$

$$\begin{aligned}\therefore \log a &= \log 395 + L\sin 33°20' + L\sec 16°54'50'' - 20 \\ &= \log 790 - \log 2 + L\sin 33°20' + L\sec 16°54'50'' - 20.\end{aligned}$$

Now

$$\begin{aligned}L\sec 16°54'50'' &= 10.01917 + \frac{50}{60}(10.01921 - 10.01917) \\ &= 10.01917 + \frac{5}{6}(.00004) = 10.01917 + .00003 = 10.01920\end{aligned}$$

$$\therefore \log a = 2.89763 - .30103 + 9.73998 + 10.01920 - 20 = 2.35578$$

$$\therefore a = 226.87 \ feet. \qquad \blacksquare$$

In the following four examples, the required logarithms must be taken from the tables :

§ Problem 13.3.18. *If* $a = 242.5$, $b = 164.3$ *and* $C = 54°36'$, *solve the triangle.* ♢

§§ Solution. We have

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{39.1}{203.4}\cot 27°18'$$

$$\therefore L\tan\frac{A-B}{2} = \log 39.1 - \log 203.4 + L\cot 27^\circ 18'$$
$$= 1.5921768 - 2.3083509 + 10.2872338 = 9.5710597.$$

Now $\qquad L\tan 20^\circ 25' = 9.5708088,$ diff. for $1' = 3863.$

Let $\qquad \dfrac{A-B}{2} = 20^\circ 25' + x'',\ \therefore L\tan\left(20^\circ 25'\right) = 9.5710597.$

The diff. for $\quad x'' = 9.5710597 - 9.5708088 = .0002509.$

$$\therefore x = 60'' \times \frac{2509}{3863} \approx 39''$$

$$\therefore \frac{A-B}{2} = 20^\circ 25' 39'' \tag{13.27}$$

But $$\frac{A+B}{2} = 90^\circ - \frac{C}{2} = 90^\circ - 27^\circ 18' = 62^\circ 42' \tag{13.28}$$

By adding (13.27) and (13.28), we have $A = 83^\circ 7' 39''$.
By subtracting (13.27) from (13.28), we have $B = 42^\circ 16' 21''$.

To find c, we have $\qquad c = \dfrac{a\sin C}{\sin A}$

$$\therefore \log c = \log a + L\sin C - L\sin A$$
$$= \log 242.5 + L\sin 54^\circ 36' - L\sin 83^\circ 7' 39''.$$

Now $\qquad L\sin 83^\circ 7' 39'' = 9.9968584 + \dfrac{39}{60} \times .0000152$
$$= 9.9968584 + .0000099 = 9.9968683.$$

$$\therefore \log c = 2.3847117 + 9.9112257 - 9.9968683 = 2.2990691.$$

Now $\qquad \log 199.09 = 2.2980494,$ diff. for $.01 = .0000219.$

Let $\qquad \log(199.09 + x) = 2.2990691.$

The diff. for $\quad x = 2.2990691 - 2.2980494 = .0000197.$

$$\therefore x = \frac{197}{219} \times .01 = \frac{1.97}{219} \approx .009$$
$$\therefore c = 199.099.$$

■

§ **Problem 13.3.19.** *If* $b = 130$*,* $c = 63$ *and* $A = 42^\circ 15' 30''$*, solve the triangle.* ◊

§§ **Solution**. We have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{67}{193}\cot 21^\circ 7' 45''.$$

$$\therefore L\tan\frac{B-C}{2} = \log 67 - \log 193 + L\cot 21^\circ 7' 45''.$$

Now $\qquad L\cot 21^\circ 7' 45'' = 10.4131853 - \dfrac{45}{60} \times .0003757$
$$= 10.4131853 - .0002818 = 10.4129035.$$

$$\therefore L\tan\frac{B-C}{2} = 1.8260748 - 2.2855573 + 10.4129035$$
$$= 9.9534210 = L\tan 41^\circ 56'.$$

$$\therefore \frac{B-C}{2} = 41^\circ 56' \tag{13.29}$$

But $$\frac{B+C}{2} = 90^\circ - \frac{A}{2} = 90^\circ - 21^\circ 7' 45'' = 68^\circ 52' 15'' \tag{13.30}$$

By adding (13.29) and (13.30), we have $B = 110^\circ 48' 15''$.
By subtracting (13.29) from (13.30), we have $C = 26^\circ 56' 15''$.

To find a, we have $\qquad a = \dfrac{c\sin A}{\sin C}$

$$\begin{aligned}\therefore \log a &= \log c + L\sin A - L\sin C\\ &= \log 63 + L\sin 42^\circ 15' 30'' - L\sin 26^\circ 56' 15''.\end{aligned}$$

Now $$L\sin 42^\circ 15'30'' = 9.8276063 + \frac{30}{60} \times .0001390$$
$$= 9.8276063 + \frac{1}{2} \times .0001390$$
$$= 9.8276758$$

and $$L\sin 26^\circ 56'15'' = 9.6560536 + \frac{15}{60} \times .0002485$$
$$= 9.6560536 + \frac{1}{4} \times .0002485$$
$$= 9.6561157$$

$$\therefore \log a = 1.7993405 + 9.8276758 - 9.6561157 = 1.9709006.$$

Now $\log 93.519 = 1.9708999,$ diff. for $.001 = .0000046.$

Let $\log(93.519 + x) = 1.9709006.$

The diff. for $x = 1.9709006 - 1.9708999 = .0000007.$

$$\therefore x = \frac{7}{46} \times .001 = \frac{.007}{46} \approx .0002, \ \therefore a = 93.5192.$$ ■

§ Problem 13.3.20. *Two sides of a triangle being* 2265.4 *and* 1779 *feet and the included angle* $58^\circ 17'$, *find the remaining angles.* ◊

§§ Solution. If $b = 2265.4$ *feet*, $c = 1779$ *feet* and $A = 58^\circ 17'$, we have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{486.4}{4044.4}\cot 29^\circ 8'30''$$
$$\therefore L\tan\frac{B-C}{2} = \log 486.4 - \log 4044.4 + L\cot 29^\circ 8'30''$$

Now $$L\cot 29^\circ 8'30'' = 10.2538680 - \frac{30}{60} \times .0002970$$
$$= 10.2538680 - \frac{1}{2} \times .0002970 = 10.2537195$$

$$\therefore L\tan\frac{B-C}{2} = 2.6869936 - 3.6068541 + 10.2537195 = 9.3338590.$$

Now $L\tan 12^\circ 10' = 9.3336463,$ diff. for $1' = 6128.$

Let $\dfrac{B-C}{2} = 12^\circ 10' + x'', \ \therefore L\tan\left(12^\circ 10' + x''\right) = 9.3338590.$

The diff. for $x'' = 9.3338590 - 9.3336463 = .0002127.$

$$\therefore x = 60'' \times \frac{2127}{6128} \approx 21''$$
$$\therefore \frac{B-C}{2} = 12^\circ 10' 21'' \tag{13.31}$$

But $$\frac{B+C}{2} = 90^\circ - \frac{A}{2} = 90^\circ - 29^\circ 8'30'' = 60^\circ 51'30'' \tag{13.32}$$

By adding (13.31) and (13.32), we have $B = 73^\circ 1'51''$.
By subtracting (13.31) from (13.31), we have $C = 48^\circ 41'9''$. ■

§ Problem 13.3.21. *Two sides of a triangle being* 237.09 *and* 130.96 *feet and the included angle* $57^\circ 59'$, *find the remaining angles.* ◊

§§ Solution. If $b = 237.09, c = 130.96$ and $A = 57^\circ 59'$, we have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{106.13}{368.05}\cot 28^\circ 59'30''$$

$$\therefore L\tan\frac{B-C}{2} = \log 106.13 - \log 368.05 + L\cot 28^\circ 59'30''.$$

Now $\quad L\cot 28^\circ 59'30'' = 10.2565460 - \dfrac{30}{60} \times .0002980$

$$= 10.2565460 - \frac{1}{2} \times .0002980 = 10.2563970.$$

$$\therefore \tan\frac{B-C}{2} = 2.0258382 - 2.5659068 + 10.2563970 = 9.7163284.$$

Now $\quad L\tan 27^\circ 29' = 9.7161682, \text{ diff. for } 1' = 3085.$

Let $\quad \dfrac{B-C}{2} = 27^\circ 29' + x'', \therefore L\tan\left(27^\circ 29' + x''\right) = 9.7163284.$

The diff. for $\quad x'' = 9.7163284 - 9.7161682 = .0001602.$

$$\therefore x = 60'' \times \frac{1602}{3085} \approx 31''$$

$$\therefore \frac{B-C}{2} = 27^\circ 29' 31'' \tag{13.33}$$

But $\quad \dfrac{B+C}{2} = 90^\circ - \dfrac{A}{2} = 90^\circ - 28^\circ 59'30'' = 61^\circ 0'30'' \qquad (13.34)$

By adding (13.33) and (13.34), we have $B = 88^\circ 30'1''$.
By subtracting (13.33) from (13.34), we have $C = 33^\circ 30'59''$. ■

13.4 Ambiguous Case

§ **Problem 13.4.1.** *If* $a = 5$, $b = 7$ *and* $\sin A = \dfrac{3}{4}$, *is there any ambiguity ?* ◊

§§ **Solution.** We have

$$b\sin A = 7 \times \frac{3}{4} = \frac{21}{4} = 5\frac{1}{4}$$
$$\therefore a < b\sin A.$$

Hence there is no triangle. ■

§ **Problem 13.4.2.** *If* $a = 2$, $c = \sqrt{3}+1$ *and* $A = 45^\circ$, *solve the triangle.* ◊

§§ **Solution.**

$$\because c\sin A = \frac{\sqrt{3}+1}{\sqrt{2}} = \frac{1}{2}\left(\sqrt{6}+\sqrt{2}\right)$$
$$= \frac{1}{2}(2.4495 + 1.4142) = 1.9318,$$

we have $a > c\sin A$; also $a < c$ and A is acute.

Hence there are two triangles.

We have

$$\sin C = \frac{c}{a}\sin A = \frac{\sqrt{3}+1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$\therefore C_1 = 75^\circ \textit{ and } C_2 = 105^\circ$$
$$\therefore B_1 = 60^\circ \textit{ and } B_2 = 30^\circ$$
$$\therefore b_1 = \frac{a\sin B_1}{\sin A} = \left(2 \times \frac{\sqrt{3}}{2}\right) \div \frac{1}{\sqrt{2}} = \sqrt{6}$$

and $\quad b_2 = \dfrac{a\sin B_2}{\sin A} = \left(2 \times \dfrac{1}{2}\right) \div \dfrac{1}{\sqrt{2}} = \sqrt{2}.$ ■

§ **Problem 13.4.3.** *If* $a = 100$, $c = 100\sqrt{3}$ *and* $A = 30^\circ$, *solve the triangle.* ◊

§§ **Solution**.

$$\because c\sin A = 100\sqrt{3} \times \frac{1}{2} = 50\sqrt{3}$$

we have $a > c\sin A$; also $a < c$ and A is acute.

Hence there are two triangles.

We have

$$\sin C = \frac{c}{a}\sin A = \frac{100\sqrt{3}}{100} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$\therefore C_1 = 60^\circ \textit{ and } C_2 = 120^\circ$$

$$\therefore B_1 = 90^\circ \textit{ and } B_2 = 30^\circ$$

$$\therefore b_1 = \frac{a\sin B_1}{\sin A} = 100 \div \frac{1}{2} = 200$$

and $$b_2 = \frac{a\sin B_2}{\sin A} = \left(100 \times \frac{1}{2}\right) \div \frac{1}{2} = 100.$$ ■

§ **Problem 13.4.4.** *If* $2b = 3a$ *and* $\tan^2 A = \frac{3}{5}$, *prove that there are two values to the third side, one of which is double the other.* ◊

§§ **Solution**. Since $\tan A = \sqrt{\frac{3}{5}}$, $\therefore \sin A = \frac{\sqrt{3}}{2\sqrt{2}}$ (the positive sign being taken, as A must be $< 180^\circ$ in a triangle) and $\cos A = \frac{\sqrt{5}}{2\sqrt{2}}$.

$$\therefore b\sin A = \frac{3a}{2} \cdot \frac{\sqrt{3}}{2\sqrt{2}} = \sqrt{\frac{27}{32}} \cdot a$$

$$\therefore a > b\sin A \textit{ and } < b \left(\because a = \frac{2}{3}b\right).$$

Also A is acute; for since $b > a$, $\therefore B > A$, and if A were obtuse, there would be two obtuse angles in the triangle, which is impossible. The two values of c are given by

$$c^2 = a^2 - b^2 + 2bc\cos A$$

$$\therefore c^2 - 2bc\cos A + \left(b^2 - a^2\right) = 0.$$

$$\therefore c^2 - 3ac \cdot \frac{\sqrt{5}}{2\sqrt{2}} + \frac{9a^2}{4} - a^2 = 0$$

$$\therefore 4c^2 - 3a\sqrt{10} \cdot c + 5a^2 = 0$$

$$\therefore c = \frac{3\sqrt{10} \pm \sqrt{90-80}}{8} \cdot a = \frac{3 \pm 1}{8} \cdot a\sqrt{10}$$

$$\therefore c = \frac{a\sqrt{10}}{2} \textit{ and } \frac{a\sqrt{10}}{4}.$$

Otherwise thus :

Take the figure 3 of *Art.* 186, putting C for A, A for B and B_1 and B_2 for C_1 and C_2 respectively. We then have

$$AD = b\cos A = \frac{3a}{2} \cdot \frac{\sqrt{5}}{2\sqrt{2}} = \frac{3\sqrt{5}}{4\sqrt{2}} \cdot a$$

and $$CD = b\sin A = \frac{3a}{2} \cdot \frac{\sqrt{3}}{2\sqrt{2}} = \frac{3\sqrt{3}}{4\sqrt{2}} \cdot a.$$

$$\therefore B_1D = \sqrt{B_1C^2 - CD^2} = \sqrt{a^2 - \frac{27}{32}a^2} = \sqrt{\frac{5a^2}{32}} = \frac{a\sqrt{5}}{4\sqrt{2}}$$

$$\therefore AB_1 = AD + DB_1 = \left(\frac{3\sqrt{5}}{4\sqrt{2}} + \frac{\sqrt{5}}{4\sqrt{2}}\right)a = \frac{a\sqrt{10}}{2}$$

and $$AB_2 = AD - DB_1 = \left(\frac{3\sqrt{5}}{4\sqrt{2}} - \frac{\sqrt{5}}{4\sqrt{2}}\right)a = \frac{a\sqrt{10}}{4}.$$ ■

§ Problem 13.4.5. *If $A = 30°$, $b = 8$ and $a = 6$, find c.* ◊

§§ Solution. Since $b \sin A = 8 \times \frac{1}{2} = 4$, we have $a > b \sin A$; also $a < b$ and A is acute; hence there are two triangles. The two values of c are given by

$$c^2 - 2bc \cos A + b^2 - a^2 = 0$$

$$\therefore c^2 - 2 \cdot 8 \cdot c \cdot \frac{\sqrt{3}}{2} + 64 - 36 = 0$$

$$\therefore c^2 - 8\sqrt{3} \cdot c + 28 = 0$$

$$\therefore c = \frac{8\sqrt{3} \pm \sqrt{64 \times 3 - 4 \times 28}}{2}$$

$$= 4\sqrt{3} \pm \sqrt{48 - 28} = 4\sqrt{3} \pm \sqrt{20} = 4\sqrt{3} \pm 2\sqrt{5}.$$ ■

§ Problem 13.4.6. *Given $B = 30°$, $c = 150$ and $b = 50\sqrt{3}$, prove that of the two triangles which satisfy the data, one will be isosceles and the other right-angled. Find the greater value of the third side.*

Would the solution have been ambiguous had

$$B = 30°, \ c = 150 \ \textit{and} \ b = 75?$$ ◊

§§ Solution. We have

$$\sin C = \frac{c}{b} \sin B = \frac{150 \times \frac{1}{2}}{50\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\therefore C_1 = 60° \ \textit{and} \ C_2 = 120°$$

$$\therefore A_1 = 90° \ \textit{and} \ A_2 = 30°.$$

Hence one triangle is isosceles and the other is right-angled. The greater value of a is when $A = 90°$ and we have

$$a = \sqrt{b^2 + c^2} = \sqrt{(50)^2(3 + 9)} = 50\sqrt{12} = 100\sqrt{3}.$$

$$\left[\textit{Or thus}: \ a = c \sec B = 150 \times \frac{2}{\sqrt{3}} = 100\sqrt{3}.\right]$$

If $B = 30°, \ c = 150$ and $b = 75$, we have $c \sin B = 150 \times \frac{1}{2} = 75 = b$.

Hence there is one triangle right-angled. ■

§ Problem 13.4.7. *In the ambiguous case given a, b and A, prove that the difference between the two values of c is $2\sqrt{a^2 - b^2 \sin^2 A}$.* ◊

§§ Solution. The two values of c (c_1 *and* c_2) are given by

$$c^2 - 2bc \cos A + b^2 - a^2 = 0$$

and we have $$c_1 + c_2 = 2b \cos A \ \textit{and} \ c_1c_2 = b^2 - a^2.$$

$$\therefore (c_1 \sim c_2)^2 = (c_1 + c_2)^2 - 4c_1c_2 = 4b^2 \cos^2 A - 4\left(b^2 - a^2\right)$$

$$= 4b^2\left(1 - \sin^2 A\right) - 4\left(b^2 - a^2\right) = 4\left(a^2 - b^2 \sin^2 A\right)$$

$$c_1 \sim c_2 = 2\sqrt{a^2 - b^2 \sin^2 A}.$$

Otherwise thus : Take *fig.* 3 of *Art.* 186, with C for A, A for B and B_2 and B_1 for C_2 and C_1 respectively. We then have

$$c_1 \sim c_2 = B_1B_2 = 2B_2D = 2\sqrt{BC^2 - CD^2} = 2\sqrt{a^2 - (b\sin A)^2}. \quad \blacksquare$$

§ Problem 13.4.8. *If $a = 5,\ b = 4$ and $A = 45°$, find the other angles, having given*

$$\log 2 = .30103,\ L\sin 33°29' = 9.7520507$$

and $$L\sin 33°30' = 9.7530993. \quad \Diamond$$

§§ Solution. Since $a > b$, there is only one triangle.

We have

$$\sin B = \frac{b}{a}\sin A = \frac{4}{5} \times \frac{1}{\sqrt{2}} = \frac{4\sqrt{2}}{10} = \frac{2^{\frac{5}{2}}}{10}$$

$$\therefore L\sin B = 10 + \frac{5}{2}\log 2 - \log 10 = 10 + .7525750 - 1 = 9.7525750.$$

Let $\quad B = 33°29' + x'',\ \therefore L\sin\left(33°29' + x''\right) = 9.7525750.$

The diff. for $\quad 60'' = 9.7530993 - 9.7520507 = .0010486.$

The diff. for $\quad x'' = 9.7525750 - 9.7520507 = .0005243.$

$$\therefore x = 60'' \times \frac{5243}{10486} = 60'' \times \frac{1}{2} = 30''$$

$$\therefore B = 33°29'30''$$

and $\quad C = 180° - (A + B) = 180° - 78°29'30'' = 101°30'30''. \quad \blacksquare$

§ Problem 13.4.9. *If $a = 9,\ b = 12$ and $A = 30°$, find c, having given*

$$\log 2 = .30103, \qquad \log 3 = .47712$$

$$\log 171 = 2.23301, \qquad \log 368 = 2.56635$$

$$L\sin 11°48'39'' = 9.31108,\ L\sin 41°48'39'' = 9.82391$$

and $$L\sin 108°11'21'' = 9.97774. \quad \Diamond$$

§§ Solution. Since $b\sin A = 12 \times \dfrac{1}{2} = 6$, we have $a > b\sin A$; also $a < b$ and A is acute. Hence there are two triangles.

We have

$$\sin B = \frac{b}{a}\sin A = \frac{6}{9} = \frac{2}{3}$$

$$\therefore L\sin B = 10 + \log 2 - \log 3 = 9.82391$$

$$\therefore B_1 = 41°48'39''$$

and $$B_2 = 180° - 41°48'39'' = 138°11'21''$$

$$\therefore C_1 = 180° - (A + B_1) = 180° - 71°48'39'' = 108°11'21''$$

and $$C_2 = 180° - (A + B_2) = 180° - 168°11'21'' = 11°48'39''.$$

$$\therefore c_1 = \frac{a\sin C_1}{\sin A} = \frac{9\sin C_1}{\frac{1}{2}} = 2 \times 3^2 \times \sin 108°11'21''$$

$$\begin{aligned}\therefore \log c_1 &= \log 2 + 2\log 3 + L\sin 108°11'21'' - 10\\ &= .30103 + .95424 + 9.7774 - 10 = 1.23301\end{aligned}$$

$$\therefore c_1 = 17.1.$$

Again, $$c_2 = \frac{a\sin C_2}{\sin A} = \frac{9\sin C_2}{\left(\frac{1}{2}\right)} = 2 \times 3^2 \times \sin 11°48'39''$$

$$\begin{aligned}\therefore \log c_2 &= \log 2 + 2\log 3 + L\sin 11°48'39'' - 10\\ &= .30103 + .95424 + 9.31108 - 10 = .56635\\ \therefore c_2 &= 3.68.\end{aligned}$$ ■

§ Problem 13.4.10. *Point out whether or no the solutions of the following triangles are ambiguous.*

Find the smaller value of the third side in the ambiguous case and the other angles in both cases :

(1) $A = 30°,\ c = 250\ feet\ and\ a = 125\ feet$

(2) $A = 30°,\ c = 250\ feet\ and\ a = 200\ feet.$

Given $\log 2 = .30103,\ \log 6.03893 = .7809601$
$L\sin 38°41' = 9.7958800$
and $L\sin 8°41' = 9.1789001.$ ◊

§§ Solution. (1) We have $\sin C = \dfrac{c}{a}\sin A = \dfrac{250}{125}\cdot\dfrac{1}{2} = 1; \therefore C = 90°.$

Hence the triangle is not ambiguous.

(2) We have $\sin C = \dfrac{c}{a}\sin A = \dfrac{250}{200}\cdot\dfrac{1}{2} = \dfrac{5}{8} = \dfrac{10}{16} = \dfrac{10}{2^4}.$

Since $a > c\sin A,\ a < c$, and A is acute, the triangle is ambiguous.

We have $L\sin C = 10 + \log 10 - 4\log 2$
$$\begin{aligned}&= 10 + 1 - 1.2041200 = 9.7958800\\ \therefore C_1 &= 38°41'\end{aligned}$$
and $C_2 = 180° - 38°41' = 141°19'$
$$\therefore B_1 = 180° - (A + C_1) = 180° - 68°41' = 111°19'$$
and $B_2 = 180° - (A + C_2) = 180° - 171°19' = 8°41'.$

For the smaller value of b, we have
$$b = \frac{a\sin 8°41'}{\sin A} = \frac{200\sin 8°41'}{\left(\frac{1}{2}\right)} = 2^2 \times 100 \times \sin 8°41'$$
$$\begin{aligned}\therefore \log b &= 2\log 2 + \log 100 + L\sin 8°41' - 10\\ &= .60206 + 2 + 9.1789001 - 10 = 1.7809601\\ \therefore b &= 60.3893.\end{aligned}$$ ■

§ Problem 13.4.11. *Given* $a = 250,\ b = 240$ *and* $A = 72°4'48''$, *find the angles* B *and* C *and state whether they can have more than one value, given*

$\log 2.5 = .3979400,\qquad \log 2.4 = .3802112$
$L\sin 72°4' = 9.9783702,\qquad L\sin 72°5' = 9.9784111$
and $L\sin 65°59' = 9.9606739.$ ◊

§§ Solution. Since $a > b$, there is only one triangle.

We have $\sin B = \dfrac{b}{a}\sin A = \dfrac{24}{25}\sin 72°4'48''$
$$\therefore L\sin B = \log 24 + L\sin 72°4'48'' - \log 25.$$

To find $L\sin 72°4'48''$, we have

diff. for $60'' = 9.9784111 - 9.9783702 = .0000409$

$\therefore$ diff. for $48' = \dfrac{48}{60} \times .0000409 = \dfrac{8}{10} \times .0000409 = .0000327$

$$\therefore L\sin 72°4'48'' = 9.9783702 + .0000327 = 9.9784029$$
$$\therefore L\sin B = 1.3802112 + 9.9784029 - 1.3979400 = 9.9606741$$
$$\therefore B \approx 65°59'$$
$$\therefore C = 180° - (A+B) = 180° - 138°3'48'' = 41°56'12''.$$ ∎

§ Problem 13.4.12. *Two straight roads intersect at an angle of $30°$; from the point of junction two pedestrians A and B start at the same time, A walking along one road at the rate of 5 miles per hour and B walking uniformly along the other road. At the end of 3 hours, they are 9 miles apart. Show that there are two rates at which B may walk to fulfill this condition and find them.* ♢

§§ Solution. In the triangle ABC, let B represent the point of intersection of the roads. At the end of 3 hours, let the pedestrians A and B be at A and C respectively. We then have the $\angle ABC = 30°$, $BA(c) = 15$ miles and $AC(b) = 9$ miles. The distance walked by B in 3 hours will be given by the value of a.

Since $c\sin B = 15\sin 30° = 7\frac{1}{2}$, we have $b > c\sin B$; also $b < c$ and the $\angle B$ is acute. Hence there are two values of a, i.e. two rates at which B may walk. These values of a are given by
$$a^2 - 2ca\cos B + c^2 - b^2 = 0.$$
We have
$$a^2 - 2 \times 15 \times a \times \frac{\sqrt{3}}{2} + 225 - 81 = 0$$
$$\therefore a^2 - 15\sqrt{3}\cdot a + 144 = 0$$
$$\therefore a = \frac{15\sqrt{3} \pm \sqrt{675 - 576}}{2} = \frac{15\sqrt{3} \pm \sqrt{99}}{2}$$
$$= \frac{15\sqrt{3} \pm 3\sqrt{11}}{2}.$$
Hence the require rates
$$= \frac{a}{3} = \frac{5\sqrt{3} \pm \sqrt{11}}{2} = \frac{5 \times 1.73205 \pm 3.31661}{2}$$
$$= \frac{8.66025 \pm 3.31662}{2} = 5.9884\ldots \textit{ or } 2.6718\ldots \textit{ miles per hour}$$
In $fig.$ 3 of $Art.$ 186, we see that when the pedestrian A is at A, the pedestrian B may be either at C_2 or C_1 to be 9 miles (b) apart. ∎

For the following three examples, a book of tables will be required.

§ Problem 13.4.13. *Two sides of a triangle are 1015 feet and 732 feet and the angle opposite the latter side is $40°$; find the angle opposite the former and prove that more than one value is admissible.* ♢

§§ Solution. Let $b = 732\ feet,\ c = 1015\ feet\ and\ B = 40°$. Since, by the table of natural sines,
$$c\sin B = 1015 \times .6427876 \approx 625,$$
we have $b > c\sin B$; also $b < c$ and B is acute.

Hence there are two triangles.

We have
$$\sin C = \frac{c}{b}\sin B = \frac{1015}{732}\sin 40°.$$
$$\therefore L\sin C = \log 1015 + L\sin 40° - \log 732$$
$$= 3.0064660 + 9.8080675 - 2.8645111 = 9.9500224.$$

Now $\quad L\sin 63°2' = 9.9500095,$ diff. for $1' = 643.$

Let $\quad C = 63°2' + x'', \therefore L\sin\left(63°2' + x''\right) = 9.9500224.$

The diff. for $\quad x'' = 9.9500224 - 9.9500095 = .0000129.$

$$\therefore x = 60'' \times \frac{129}{643} \approx 12''.$$
$$\therefore L\sin C = L\sin 63°2'12''.$$
$$\therefore C = 63°2'12'' \textit{ or } 180° - 63°2'12'' = 116°57'48''.$$

■

§ Problem 13.4.14. *Two sides of a triangle being* 5374.5 *and* 1586.6 *feet and the angle opposite the latter being* $15°11'$, *calculate the other angles of the triangle or triangles.* ♢

§§ Solution. Let $b = 1586.6,\ c = 5374.5$ and $B = 15°11'$. Since, by the table of natural sines,

$$c\sin B = 5374.5 \times .2619085 \approx 1408,$$

we have $b > c\sin B$; also $b < c$ and B is acute.

Hence there are two triangles.

We have $\quad \sin C = \frac{c}{b}\sin B = \frac{5374.5}{1586.6}\sin 15°11'.$

$$\therefore L\sin C = \log 5374.5 + L\sin 15°11' - \log 1586.6$$
$$= 3.7303381 + 9.4181495 - 3.2004674 = 9.9480202.$$

Now $\quad L\sin 62°31' = 9.9479947,$ diff. for $1' = 657.$

Let $\quad C = 62°31' + x'', \therefore L\sin\left(62°31' + x''\right) = 9.9480202.$

The diff. for $\quad x'' = 9.9480202 - 9.9479947 = .0000255.$

Hence $\quad x = 60'' \times \frac{255}{657} \approx 23.$

$$\therefore L\sin C = L\sin 62°31'23''.$$
$$\therefore C = 62°31'23'' \textit{ or } 180° - 62°31'23'' = 117°28'37''$$
$$\therefore C_1 = 62°31'23'' \textit{ and } C_2 = 117°28'37''.$$
$$\therefore A_1 = 180° - \left(15°11' + 62°31'23''\right) = 102°17'37''$$

and $\quad A_2 = 180° - \left(15°11' + 117°28'37''\right) = 47°20'23''.$ ■

§ Problem 13.4.15. *Given* $A = 10°,\ a = 2308.7$ *and* $b = 7903.2$, *find the smaller value of* c. ♢

§§ Solution. We have $\sin B = \frac{b}{a}\sin A = \frac{7903.2}{2308.7}\sin 10°.$

$$\therefore L\sin B = \log 7903.2 + L\sin 10° - \log 2308.7$$
$$= 3.8978030 + 9.2396702 - 3.3633675 = 9.7741057.$$

Now $\quad L\sin 36°28'' = 9.7740459,$ diff. for $1' = 1709.$

Let $\quad B = 36°28' + x'', \therefore L\sin\left(36°28' + x''\right) = 9.7741057.$

The difference for

$$x'' = 9.7741057 - 9.7740459 = .0000598.$$
$$\therefore x = 60'' \times \frac{598}{1709} \approx 21''.$$
$$\therefore L\sin B = L\sin 36°28'21''.$$
$$\therefore B = 36°28'21'' \textit{ or } 180° - 36°28'21'' = 143°31'39'',$$

and, by the question, we must take $B = 143°31'39''$.

$$\therefore C = 180° - \left(10° + 143°31'39''\right) = 26°28'21''.$$

To find c, we have

$$c = \frac{a \sin C}{\sin A} = \frac{2308.7 \sin 26°28'21''}{\sin 10°}.$$

$$\therefore \log c = \log 2308.7 + L \sin 26°28'21'' - L \sin 10°.$$

Now $\quad L \sin 26°28' = 9.6490203,$ diff. for $1' = 2537.$

$$\therefore\ diff.\ for\ 21'' = \frac{21}{60} \times 2537 \approx 888$$

$$\therefore L \sin 26°28'21'' = 9.6490203 + .0000888 = 9.6491091$$

$$\therefore \log c = 3.3633675 + 9.6491091 - 9.2396702 = 3.7728064.$$

Now $\quad \log 5926.6 = 3.7728056,$ diff. for $.1 = 73.$

Let $\quad \log(5926.6 + x) = 3.7728064.$

The diff. for $\quad x = 3.7728064 - 3.7728056 = .0000008.$

$$\therefore x = \frac{8}{73} \times .1 = \frac{.8}{73} \approx .011$$

$$\therefore \log c = \log 5926.611.$$

Hence the smaller value of $c = 5926.611$. ■

13.5 One Side Two Angles vs. Three Angles

§ Problem 13.5.1. *If* $\cos A = \dfrac{17}{22}$ *and* $\cos C = \dfrac{1}{14}$*, find the ratio of* $a : b : c$*.* ◊

§§ Solution.

We have $\quad \sin A = \sqrt{1 - \left(\dfrac{17}{22}\right)^2} = \dfrac{\sqrt{195}}{22}$

and $\quad \sin C = \sqrt{1 - \left(\dfrac{1}{14}\right)^2} = \dfrac{\sqrt{195}}{14}$

$$\begin{aligned}\therefore \sin B = \sin(A + C) &= \sin A \cos C + \cos A \sin C \\ &= \frac{\sqrt{195}}{22} \cdot \frac{1}{14} + \frac{17}{22} \cdot \frac{\sqrt{195}}{14} \\ &= \frac{9\sqrt{195}}{11 \times 14}.\end{aligned}$$

$$\therefore \text{ the required ratio } = \frac{1}{22} : \frac{9}{11 \times 14} : \frac{1}{14} = 7 : 9 : 11.$$ ■

§ Problem 13.5.2. *The angles of a triangle are as* $1 : 2 : 7$*; prove that the ratio of the greatest side to the least side is* $\sqrt{5} + 1 : \sqrt{5} - 1$*.* ◊

§§ Solution. If $x°$, $2x°$ and $7x°$ be the angles, we have

$$x° + 2x° + 7x° = 10x° = 180°$$

$$\therefore x° = 18° \textit{ and } 7x° = 126°.$$

Hence the greatest side : the least side

$$\begin{aligned}&= \sin 126° : \sin 18° \\ &= \sin 54° : \sin 18° \\ &= \frac{\sqrt{5} + 1}{4} : \frac{\sqrt{5} - 1}{4} \\ &= \sqrt{5} + 1 : \sqrt{5} - 1.\end{aligned}$$ ■

§ Problem 13.5.3. *If $A = 45°$, $B = 75°$ and $C = 60°$, prove that $a + c\sqrt{2} = 2b$.* ◊

§§ Solution. We have

$$\frac{a}{\sin 45°} = \frac{b}{\sin 75°}$$

$$\therefore a = \frac{b}{\sqrt{2}} \div \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{2b}{\sqrt{3}+1}.$$

Also,

$$\frac{c}{\sin 60°} = \frac{b}{\sin 75°}$$

$$\therefore c = \frac{b\sqrt{3}}{2} \div \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{b\sqrt{6}}{\sqrt{3}+1}.$$

$$\therefore a + c\sqrt{2} = \frac{2b + b\sqrt{12}}{\sqrt{3}+1} = \frac{2b\left(1+\sqrt{3}\right)}{1+\sqrt{3}} = 2b.$$ ■

§ Problem 13.5.4. *Two angles of a triangle are $41°13'22''$ and $71°19'5''$ and the side opposite the first angle is 55; find the side opposite the latter angle, given*

$$\log 55 = 1.7403627,\ \log 79063 = 4.8979775$$

$$L\sin 41°13'22'' = 9.8188779$$

and

$$L\sin 71°19'5'' = 9.9764927.$$ ◊

§§ Solution. Let $A = 41°13'22''$, $B = 71°19'5''$ and $a = 55$.

We have

$$b = \frac{a\sin B}{\sin A}$$

$$\begin{aligned} \therefore \log b &= \log a + L\sin B - L\sin A \\ &= 1.7403627 + 9.9764927 - 9.8188779 \\ &= 1.8979775 = \log 79.063. \end{aligned}$$

$$\therefore b = 79.063.$$ ■

§ Problem 13.5.5. *From each of two ships, one mile apart, the angle is observed which is subtended by the other ship and a beacon on shore; these angles are found to be $52°25'15''$ and $75°9'30''$ respectively. Given*

$$L\sin 75°9'30'' = 9.9852635$$

$$L\sin 52°25'15'' = 9.8990055,\ \log 1.2197 = .0862530$$

and

$$\log 1.2198 = .0862886,$$

find the distance of the beacon from each of the ships. ◊

§§ Solution. The third angle of the triangle formed by the ships and the beacon

$$= 180° - \left(52°25'15'' + 75°9'30''\right) = 52°25'15''.$$

Thus the triangle is isosceles, so that if a and b be the required distances respectively, we have $b = 1\ mile$.

Also,

$$a = \frac{1 \times \sin 75°9'30''}{\sin 52°25'15''}.$$

$$\begin{aligned} \therefore \log a &= \log 1 + L\sin 75°9'30'' - L\sin 52°25'15'' \\ &= 0 + 9.9852635 - 9.8990055 = .0862580. \end{aligned}$$

We have

$$\log 1.2197 = .0862530 \tag{13.35}$$

and

$$\log 1.2198 = .0862886 \tag{13.36}$$

Let

$$\log(1.2197 + x) = .0862580 \tag{13.37}$$

From (13.35) and (13.36), the diff. for $.0001 = .0000356$.

From (13.35) and (13.37), the diff. for $x = .0000050$.

$$\therefore x = \frac{50}{356} \times .0001 = \frac{.005}{356} \approx .000014.$$

Hence the required distance $= 1.219714\ mile$. ∎

§ Problem 13.5.6. *The base angles of a triangle are* $22\frac{1}{2}^\circ$ *and* $112\frac{1}{2}^\circ$*; prove that the base is equal to twice the height.* ◊

§§ Solution. In the triangle ABC, if $A = 22\frac{1}{2}^\circ$ and $B = 112\frac{1}{2}^\circ$, then $C = 45^\circ$.

Let h denote the height of the triangle. We then have

$$h = b \sin A = \frac{c \sin B \sin A}{\sin C} = \frac{c \sin 112\frac{1}{2}^\circ \sin 22\frac{1}{2}^\circ}{\sin 45^\circ}$$

$$= \frac{c \sin 67\frac{1}{2}^\circ \sin 22\frac{1}{2}^\circ}{\sin 45^\circ} = \frac{c \cos 22\frac{1}{2}^\circ \sin 22\frac{1}{2}^\circ}{2 \cos 22\frac{1}{2}^\circ \sin 22\frac{1}{2}^\circ} = \frac{c}{2}.$$ ∎

For the following five problems, a book of tables is required.

§ Problem 13.5.7. *The base of a triangle being seven feet and the base angles* $129^\circ 23'$ *and* $38^\circ 36'$*, find the length of its shorter side.* ◊

§§ Solution. The third angle of the triangle

$$= 180^\circ - \left(129^\circ 33' + 38^\circ 36'\right) = 12^\circ 1'.$$

To find the side opposite the angle $38^\circ 36'$ ($a\ feet,\ say$), we have

$$a = \frac{7 \sin 38^\circ 36'}{\sin 12^\circ 1'}$$

$$\therefore \log a = \log 7 + L \sin 38^\circ 36' - L \sin 12^\circ 1'$$
$$= .8450980 + 9.7951008 - 9.3184728 = 1.3217260.$$

Now $\log 20.976 = 1.3217227,$ diff. for $.001 = .0000207$.

Let $\log (20.976 + x) = 1.3217260.$

The diff. for $x = 1.3217260 - 1.3217227 = .0000033.$

$$\therefore x = \frac{33}{207} \times .001 = \frac{.033}{207} = \frac{.011}{69} \approx .00016$$

$$\therefore a = 20.97616\ feet.$$ ∎

§ Problem 13.5.8. *If the angles of triangle be as* $5 : 10 : 21$ *and the side opposite the smaller angle be* 3 *feet, find the other sides.* ◊

§§ Solution. If $5x^\circ$, $10x^\circ$ and $21x^\circ$ be the angles, we have

$$5x^\circ + 10x^\circ + 21x^\circ = 36x^\circ = 180^\circ,$$

so that $x^\circ = 5^\circ$ and the angles are 25°, 50° and 105°.

We have $$\frac{\sin 25^\circ}{3} = \frac{\sin 50^\circ}{b} = \frac{\sin 105^\circ}{c},$$

where b feet and c feet are the sides opposite the angles 50° and 105° respectively.

$$\therefore b = \frac{3 \sin 50^\circ}{\sin 25^\circ} = \frac{6 \sin 25^\circ \cos 25^\circ}{\sin 25^\circ} = 6 \cos 25^\circ$$
$$= 6 \times .9063078 = 5.4378468\ feet,$$

by the table of natural cosines.

Also,
$$c = \frac{3\sin 105^\circ}{\sin 25^\circ} = \frac{3\sin 75^\circ}{\sin 25^\circ}$$
$$\therefore \log c = \log 3 + L\sin 75^\circ - L\sin 25^\circ$$
$$= .4771213 + 9.9849438 - 9.6259483 = .8361168.$$

Now $\log 6.8567 = .8361151$, diff. for $.0001 = .0000064$.

Let $\log(6.8567 + x) = .8361168$.

The diff. for $x = .8361168 - .8361151 = .0000017$.
$$\therefore x = \frac{17}{64} \times .0001 = \frac{.0017}{64} \approx .00003$$
$$\therefore c = 6.85673\ feet.$$
■

§ Problem 13.5.9. *The angles of a triangle being* 150°,$18^\circ 20'$ *and* $11^\circ 40'$ *and the longest side being* 1000 *feet, find the length of the shortest side.* ◊

§§ Solution. Let $A = 150^\circ$, $B = 18^\circ 20'$, $C = 11^\circ 40'$ and $a = 1000\ feet$.
To find c, we have
$$c = \frac{a\sin C}{\sin A} = \frac{1000\sin 11^\circ 40'}{\sin 150^\circ} = \frac{1000\sin 11^\circ 40'}{\sin 30^\circ}$$
$$= [1000 \times .2022176] \div \frac{1}{2} = 404.4352\ feet.$$
■

§ Problem 13.5.10. *To get the distance of a point* A *from a point* B, *a line* BC *and the angles* $\angle ABC$ *and* $\angle BCA$ *are measured and are found to be* 287 *yards and* $55^\circ 32' 10''$ *and* $51^\circ 8' 20''$ *respectively. Find the distance* AB. ◊

§§ Solution. We have
$$A = 180^\circ - (B + C) = 180^\circ - 106^\circ 40' 30'' = 73^\circ 19' 30''.$$

Also,
$$AB = \frac{BC\sin C}{\sin A}.$$
$$\therefore \log AB = \log BC + L\sin C - L\sin A$$
$$= \log 287 + L\sin 51^\circ 8' 20'' - L\sin 73^\circ 19' 30''.$$

Now
$$L\sin 51^\circ 8' 20'' = 9.8913191 + \frac{20}{60} \times .0001017 = 9.8913530$$
and
$$L\sin 73^\circ 19' 30'' = 9.9813229 + \frac{30}{60} \times .0000379 = 9.9813419$$
$$\therefore \log AB = 2.4578819 + 9.8913530 - 9.9813419 = 2.3678930.$$

Now $\log 233.28 = 2.3678775$, diff. for $.01 = .0000186$.

Let $\log(233.28 + x) = 2.3678930$.

The diff. for $x = 2.3678930 - 2.3678775 = .0000155$.
$$\therefore x = \frac{155}{186} \times .01 = \frac{1.55}{186} \approx .0083$$
$$\therefore AB = 233.2883\ yards.$$
■

§ Problem 13.5.11. *To find the distance from* A *to* P *a distance,* AB *of* 1000 *yards is measured in a convenient direction. At* A *the angle* $\angle PAB$ *is found to be* $41^\circ 18'$ *and at* B *the angle* $\angle PBA$ *is found to be* $114^\circ 38'$. *What is the required distance to the nearest yard ?* ◊

§§ **Solution**. The angle $\angle APB = 180° - (41°18' + 114°38') = 24°4'$.

We have
$$AP = \frac{AB \sin 114°38'}{\sin 24°4'} = \frac{1000 \sin 65°22'}{\sin 24°4'}$$
$$\begin{aligned}\therefore \log AP &= \log 1000 + L \sin 65°22' - L \sin 24°4' \\ &= 3 + 9.9585609 - 9.6104465 = 3.3481144.\end{aligned}$$

Hence $AP = 2229\ yards$, to the nearest yard. ■

Chapter 14

Heights and Distances

14.1 Inaccessible Object and Distant Points

§ **Problem 14.1.1.** *A flagstaff stands on the middle of a square tower. A man on the ground, opposite the middle of one face and distant from it* 100 *feet, just sees the flag ; on his receding another* 100 *feet, the tangents of elevation of the top of the tower and the top of the flagstaff are found to be* $\frac{1}{2}$ *and* $\frac{5}{9}$*. Find the dimensions of the tower and the height of the flagstaff, the ground being horizontal.* ◊

§§ **Solution**. Let $CHKL$ represent the vertical section of the tower through its center, HK being the base. Let A and B be the two positions of the man respectively, opposite H. Let DF (k feet, say) be the flagstaff, F being the middle point of CL.

Produce DF to meet HK in E. Let h feet be the height, and b feet be the breadth of the tower, so that $CH = FE = LK = h$, and $HE = EK = \frac{b}{2}$.

We are given $BA = AH = 100$ feet, $\tan\angle CBE = \frac{1}{2}$, and $\tan\angle DBE = \frac{5}{9}$. Also ACD is a straight line.

In the ΔCBH, we have $CH = BH\tan\angle CBE$,

i.e. $$h = 200 \times \frac{1}{2} = 100';$$

$$\therefore \angle CAH = 45^\circ = \angle DCF;$$

$$\therefore k = \frac{b}{2}.$$

In the ΔDBE, we have

$$DE = BE\tan\angle DBE,$$

i.e.
$$h+k=\left(200+\frac{b}{2}\right)\frac{5}{9};$$
$$\therefore 9(100+k)=5(200+k);$$
$$\therefore 4k=100,\ \therefore \boldsymbol{k=25'};$$
and
$$\frac{b}{2}=k=25,\ \therefore \boldsymbol{b=50'}.$$
■

§ **Problem 14.1.2.** *A man, walking on a level plane towards a tower, observes that at a certain point the angular height of the tower is* 10°*, and, after going* 50 *yards nearer the tower, the elevation is found to be* 15°*. Having given*
$$L\sin 15^\circ=9.4129962,\qquad L\cos 5^\circ=9.9983442,$$
$$\log 25.783=1.4113334,\ \textit{and}\ \log 25.784=1.4113503,$$
find, to 4 *places of decimals, the height of the tower in yards.* ◊

§§ **Solution**. Take the figure of *Art*. 192. Let PQ (x yards, say) represent the tower and A and B be the points at which the angles of elevation are taken.

We are given $AB=50$ yards, $\angle PAQ=10^\circ$ and $\angle PBQ=15^\circ$, so that $\angle APB=15^\circ-10^\circ=5^\circ$.

We then have
$$\frac{x}{BP}=\sin 15^\circ,\ \text{and}\ \frac{BP}{50}=\frac{\sin 10^\circ}{\sin 5^\circ}=\frac{2\sin 5^\circ\cos 5^\circ}{\sin 5^\circ}=2\cos 5^\circ.$$
Hence by multiplication, we have
$$\frac{x}{50}=2\sin 15^\circ\cos 5^\circ$$
$$\therefore x=100\sin 15^\circ\cos 5^\circ$$
$$\therefore \log x=\log 100+L\sin 15^\circ-10+L\cos 5^\circ-10$$
$$=2+9.4129962+9.9983442-20=1.4113404.$$

We have
$$\log 25.783=1.4113334 \tag{14.1}$$
$$\log 25.784=1.4113503 \tag{14.2}$$
Let
$$\log(25.783+y)=1.4113404 \tag{14.3}$$
From (14.1) and (14.2), we have the difference for $.001=.0000169$.
From (14.1) and (14.3), we have the difference for $y=.0000070$.
$$\therefore y=\frac{70}{169}\times .001=\frac{.07}{169}\approx .0004$$
$$\therefore \boldsymbol{x=25.7834}\ \textbf{\textit{yards}}.$$
■

§ **Problem 14.1.3.** *DE is a tower standing on a horizontal plane and $ABCD$ is a straight line in the plane. The height of the tower subtends an angle θ at A, 2θ at B, and 3θ at C. If AB and BC be respectively* 50 *and* 20 *feet, find the height of the tower and the distance CD.* ◊

§§ **Solution**. Let the height of the tower be h feet and the distance CD be x feet.

We then have $\cot\theta=\dfrac{70+x}{h}$, $\cot 2\theta=\dfrac{20+x}{h}$ and $\cot 3\theta=\dfrac{x}{h}$.

Now $\cot 2\theta=\dfrac{\cot^2\theta-1}{2\cot\theta}$, so that we have
$$\frac{20+x}{h}=\left[\left(\frac{70+x}{h}\right)^2-1\right]\div\left[\frac{2(70+x)}{h}\right]=\frac{(70+x)^2-h^2}{2h(70+x)}$$

$$\therefore 2(20+x)(70+x) = (70+x)^2 - h^2 \tag{14.4}$$

Again, $$\cot 3\theta = \cot(2\theta+\theta) = \frac{\cot\theta\cot 2\theta - 1}{\cot\theta + \cot 2\theta}.$$

$$\begin{aligned}\therefore \frac{x}{h} &= \left[\frac{(70+x)(20+x)}{h^2} - 1\right] \div \left[\frac{70+x}{h} + \frac{20+x}{h}\right]\\ &= \frac{(70+x)(20+x) - h^2}{h(90+2x)}\end{aligned}$$

$$\therefore x(90+2x) = (70+x)(20+x) - h^2 \tag{14.5}$$

Subtracting (14.5) from (14.4), we have

$$3(20+x)(70+x) - x(90+2x) = (70+x)^2$$

$$\therefore 40x = 700, \text{ and } \boldsymbol{x = \frac{70}{4} = 17\frac{1}{2}\ feet.}$$

Also, from (14.4), we have

$$\begin{aligned}h^2 &= (70+x)\left[(70+x) - (40+2x)\right] = (70+x)(30-x)\\ &= \left(70+\frac{70}{4}\right)\left(30-\frac{70}{4}\right) = 70 \times \frac{5}{4} \times \frac{50}{4} = 25 \times 25 \times \frac{7}{4}.\end{aligned}$$

$$\therefore \boldsymbol{h = \frac{25\sqrt{7}}{2} = 33.07189\ feet.}$$ ■

§ Problem 14.1.4. *A tower,* 50 *feet high, stands on the top of a mound ; from a point on the ground the angles of elevation of the top and bottom of the lower are found to be* 75° *and* 45° *respectively ; find the height of tho mound.* ◊

§§ Solution. If h feet be the height of the mound, and x feet be the distance of the point from the base of the mound, we have

$$\frac{h}{x} = \tan 45^\circ = 1, \ \therefore h = x.$$

Also, $$\frac{h+50}{x} = \tan 75^\circ = 2+\sqrt{3}$$

$$\therefore h + 50 = h(2+\sqrt{3})$$

$$\therefore h = \frac{50}{\sqrt{3}+1} = \frac{50(\sqrt{3}-1)}{2} = \frac{100}{4} \times .732\ldots$$

$$\therefore \boldsymbol{h = 18.3\ feet}\ .$$ ■

§ Problem 14.1.5. *A vertical pole (more than* 100 *feet high) consists of two parts, the lower being* $\frac{1}{3}$*rd of the whole. From a point in a horizontal plane through the foot of the pole and* 40 *feet from it, the upper part subtends an angle whose tangent is* $\frac{1}{2}$*. Find the height of the pole.* ◊

§§ Solution. Let the required height be $3h$ feet, so that the lower part of the pole is h feet and the upper part is $2h$ feet. Then, if α and β be the angles subtended at the point in the horizontal plane by the pole and the lower part of it respectively, we have

$$\tan\alpha = \frac{3h}{40} \text{ and } \tan\beta = \frac{h}{40}.$$

Also, $$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{1}{2}.$$

$$\therefore \left(\frac{3h-h}{40}\right) \div \left[1+\frac{3h^2}{(40)^2}\right] = \frac{1}{2}$$
$$\therefore 3h^2 - 160h + 1600 = 0$$
$$\therefore (3h-40)(h-40) = 0.$$

Since the pole is more than 100 feet high, we must take the solution

$$h - 40 = 0, \ \therefore h = 40.$$

Hence the required height $= \boldsymbol{3h = 120}$ ***feet***. ■

§ **Problem 14.1.6.** *A tower subtends an angle α at a point on the same level as the foot of the tower, and at a second point, h feet above the first, the depression of the foot of the tower is β. Find the height of the tower.* ◊

§§ **Solution**. If x be the required height and y be the distance of the point from the foot of the tower, we have

$$y = x\cot\alpha \text{ and } y = h\cot\beta.$$

Hence
$$x\cot\alpha = h\cot\beta$$
$$\therefore x = h\tan\alpha\cot\beta.$$ ■

§ **Problem 14.1.7.** *A person in a balloon, which has ascended vertically from flat land at the sea level, observes the angle of depression of a ship at anchor to be 30° ; after descending vertically for 600 feet, he finds the angle of depression to be 15° ; find the horizontal distance of the ship from the point of ascent.* ◊

§§ **Solution**. Let B and C be the positions of the balloon respectively when the observations are taken and S be the position of the ship whose horizontal distance SA (x feet, say) is required, A being the point of ascent.

Draw BN and CM parallel to AS, so that BN and CM are the horizontal lines passing through B and C respectively.

We are given

$$\angle NBS = 30^\circ = \angle BSA, \text{ and}$$
$$\angle MCS = 15^\circ = \angle CSA.$$
$$\therefore \frac{x}{CS} = \sin\angle ACS = \cos 15^\circ, \text{ and}$$
$$\frac{CS}{600} = \frac{\sin\angle CBS}{\sin\angle CSB} = \frac{\sin 60^\circ}{\sin 15^\circ}.$$

Hence, by multiplication, we have

$$\frac{x}{600} = \frac{\cos 15^\circ \sin 60^\circ}{\sin 15^\circ} = \sin 60^\circ \cot 15^\circ = \frac{\sqrt{3}}{2}(2+\sqrt{3}).$$
$$\therefore x = 300(2\sqrt{3}+3) = 300 \times 6.464\ldots$$
$$\therefore \boldsymbol{x = 1939.2\ldots} \ \textbf{\textit{feet}}.$$

Otherwise thus : If $CA = h$ feet, we have

$$\frac{h+600}{x} = \tan 30^\circ = \frac{1}{\sqrt{3}}, \text{ and } \frac{h}{x} = \tan 15^\circ = 2-\sqrt{3},$$
$$\therefore \frac{600}{x} = \frac{1}{\sqrt{3}} - 2 + \sqrt{3} = \frac{4-2\sqrt{3}}{\sqrt{3}}$$
$$\therefore \boldsymbol{x = 300(2\sqrt{3}+3) = 1939.2} \ \textbf{\textit{feet}}.$$ ■

§ Problem 14.1.8. *PQ is a tower standing on a horizontal plane, Q being its foot ; A and B are two points on the plane such that the $\angle QAB$ is $90°$, and AB is 40 feet. It is found that*

$$\cot PAQ = \frac{3}{10} \text{ and } \cot PBQ = \frac{1}{2}.$$

Find the height of the tower. ◊

§§ Solution. If h feet be the height of the tower, we have

$$AQ = PQ \cot \angle PAQ = \frac{3}{10}h, \text{ and } BQ = PQ \cot \angle PBQ = \frac{h}{2}.$$

Also,
$$BQ^2 = BA^2 + AQ^2.$$
$$\therefore \frac{h^2}{4} = (40)^2 + \frac{9h^2}{100}$$
$$\therefore 16h^2 = (400)^2, \ \therefore \boldsymbol{h = 100\ feet}\ .$$ ■

§ Problem 14.1.9. *A column is E.S.E. of an observer, and at noon the end of the shadow is North-East of him. The shadow is 80 feet long and the elevation of the column at the observer's station is $45°$. Find the height of the column.* ◊

§§ Solution. Let CB, perpendicular to the plane of the paper, be the column and O be the position of the observer, so that the $\angle COB = 45°$, and the $\angle EOB = 22\frac{1}{2}^\circ$ (the column being E.S.E of O) where E is the extremity of the horizontal line due east from O. Let BD represent the shadow, which lies due N, since at noon the sun is due S. D, the extremity of the shadow, is N.E, of O, so that the $\angle DOE = 45°$. We have $OD = 80$ feet, and the $\angle DOB = 67\frac{1}{2}^\circ$.

Also
$$\angle DBO = 180° - 45° - 67\frac{1}{2}^\circ = 67\frac{1}{2}^\circ = \angle DOB.$$
$$\therefore OB = BC \cot COB = BC \cot 45° = BC;$$
$$\therefore OB = 2OD \cos DOB = 2 \times 80 \sin 22\frac{1}{2}^\circ$$
$$= 80\sqrt{2 - 2\cos 45°} = 80\sqrt{2 - \sqrt{2}}$$
$$= 80\sqrt{.5857864} = 80 \times .7653 = 61.224\ feet.$$
$$= BC, \text{ the required height.}$$ ■

§ Problem 14.1.10. *A tower is observed from two stations A and B, It is found to be due north of A and north-west of B. B is due east of A and distant from it 100 feet. The elevation of the tower as seen from A is the complement of the elevation as seen from B. Find the height of the tower.* ◊

§§ Solution. Let T and R be the top and foot of the tower respectively, and let h feet be its height. Let α and β be the angles of elevation at A and B respectively.

We have

$$\alpha + \beta = 90°,$$
$$RA = h \cot \alpha, \text{ and}$$
$$RB = h \cot \beta = h \tan \alpha.$$

Also,

$$\because \angle ABR = 45°, \ \therefore RA = AB = 100\ feet$$

$$\therefore RB = 100\sqrt{2}\ feet.$$

$$\therefore h \cot\alpha = 100, \text{ and } h\tan\alpha = 100\sqrt{2}.$$

Hence, by multiplication, we have

$$h^2 = (100)^2 \times \sqrt{2}$$

$$\therefore h = 100\sqrt[4]{2}\ feet.$$

■

§ Problem 14.1.11. *The elevation of a steeple at a place due south of it is* $45°$ *and at another place due west of the former place the elevation is* $15°$. *If the distance between the two places be* a, *prove that the height of the steeple is*

$$\frac{a\left(\sqrt{3}-1\right)}{2\sqrt[4]{3}}.$$

◊

§§ Solution. Take the figure of *Art.* 193, with $PQ(= x)$ representing the steeple,

$$\angle PBQ = 45°,\ \angle PAQ = 15°,\ \angle QBA = 90°, \text{ and } AB = a.$$

We then have

$$BQ = x\cot 45° = x, \text{ and}$$

$$AQ = x\cot 15° = x\frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

Also,

$$AQ^2 = BQ^2 + AB^2$$

$$\therefore \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 x^2 = x^2 + a^2$$

$$\therefore \left[\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 - 1\right] x^2 = a^2$$

$$\therefore \left[\frac{4\sqrt{3}}{(\sqrt{3}-1)^2}\right] x^2 = a^2$$

$$\therefore x^2 = \frac{a^2}{4}\left[\frac{(\sqrt{3}-1)^2}{\sqrt{3}}\right]$$

$$\therefore x = \frac{a}{2}\left[\frac{\sqrt{3}-1}{\sqrt[4]{3}}\right].$$

■

§ Problem 14.1.12. *A person stands in the diagonal produced of the square base of a church tower, at a distance* $2a$ *from it, and observes the angles of elevation of each of the two outer corners of the top of the tower to be* $30°$, *whilst that of the nearest corner is* $45°$. *Prove that the breadth of the tower is*

$$a\left(\sqrt{10}-\sqrt{2}\right).$$

◊

§§ Solution. Let A and C be the two upper corners and B and D the two lower corners, B being the corner nearest to the observer at E.

Let x be the required breadth.

We have

$$BE = 2a,\ \angle AEB = 45° \text{ and } \angle CED = 30°.$$

Then

$$AB = EB\tan 45° = 2a = CD$$

and

$$ED = CD\cot 30° = 2a\sqrt{3}$$

also, $$\angle EBD = 135^\circ.$$

$$\therefore ED^2 = EB^2 + BD^2 - 2EB.BD\cos 135^\circ$$
$$12a^2 = 4a^2 + x^2 + 2 \times 2a \times x \times \frac{1}{\sqrt{2}}$$
$$\therefore x^2 + 2a\sqrt{2}x - 8a^2 = 0$$
$$\therefore x = \frac{-2a\sqrt{2} \pm \sqrt{8a^2 + 32a^2}}{2}$$
$$= -a\sqrt{2} + \sqrt{10a^2} = a\left(\sqrt{10} - \sqrt{2}\right).$$ ∎

§ Problem 14.1.13. *A person standing at a point A due south of a tower built on a horizontal plane observes the altitude of the tower to be 60°. He then walks to B due west of A and observes the altitude to be 45°, and again at C in AB produced he observes it to be 30°. Prove that B is midway between A and C.* ◊

§§ Solution. If T and R be the top and foot of the tower respectively and h be its height.
We have
$$AR = h\cot 60^\circ = \frac{h}{\sqrt{3}}$$
$$BR = h\cot 45^\circ = h, \text{ and}$$
$$CR = h\cot 30^\circ = h\sqrt{3}.$$
$$\therefore AB^2 = BR^2 - AR^2 = h^2 - \frac{h^2}{3} = \frac{2h^2}{3}, \text{ and}$$
$$AC^2 = CR^2 - AR^2 = 3h^2 - \frac{h^2}{3} = \frac{8h^2}{3}.$$
$$\therefore AC^2 = 4AB^2$$
$$\therefore AC = 2AB.$$ ∎

§ Problem 14.1.14. *At each end of a horizontal base of length $2a$ it is found that the angular height of a certain peak is θ and that at the middle point it is ϕ. Prove that the vertical height of the peak is*
$$\frac{a\sin\theta\sin\phi}{\sqrt{\sin(\phi+\theta)\sin(\phi-\theta)}}.$$ ◊

§§ Solution. Let AB be the base of length $2a$, C be its middle point and P and Q be the top and foot of the peak respectively.

Let the required height, PQ, be h.

We the have
$$CQ = h\cot\phi, \text{ and}$$
$$AQ = BQ = h\cot\theta.$$
$\therefore \Delta ABQ$ is isosceles
$$\therefore \angle ACQ = 90^\circ$$
$$\therefore AQ^2 = AC^2 + CQ^2$$
$$\therefore h^2\cot^2\theta = a^2 + h^2\cot^2\phi$$
$$\therefore h^2\left(\cot^2\theta - \cot^2\phi\right) = a^2$$
$$\therefore h^2\left(\frac{\cos^2\theta}{\sin^2\theta} - \frac{\cos^2\phi}{\sin^2\phi}\right) = a^2$$

$$\begin{aligned}
\therefore h^2 &= \frac{a^2\sin^2\theta\sin^2\phi}{\sin^2\phi\cos^2\theta - \sin^2\theta\cos^2\phi} \\
&= \frac{a^2\sin^2\theta\sin^2\phi}{(\sin\phi\cos\theta + \cos\phi\sin\theta)(\sin\phi\cos\theta - \cos\phi\sin\theta)} \\
&= \frac{a^2\sin^2\theta\sin^2\phi}{\sin(\phi+\theta)\sin(\phi-\theta)} \\
\therefore h &= \frac{a\sin\theta\sin\phi}{\sqrt{\sin(\phi+\theta)\sin(\phi-\theta)}}.
\end{aligned}$$

■

§ Problem 14.1.15. *A and B are two stations 1000 feet apart ; P and Q are two stations in the same plane as AB and on the same side of it ; the angles $\angle PAB$, $\angle PBA$, $\angle QAB$ and $\angle QBA$ are respectively $75°$, $30°$, $45°$, and $90°$; find how far P is from Q and how far each is from A and B.* ♢

§§ Solution. Draw a figure as in *Art.* 194, with the angles and AB as given.

Since the $\angle QAB = 45°$, we have

$$BQ = AB = 1000\ feet;\ AQ = 1000\sqrt{2}\ feet.$$

Also,
$$\angle APB = 180° - (75° + 30°) = 75°$$
$$\therefore PB = AB = 1000\ feet.$$
$$\because \angle PBQ = 60°$$
$$\therefore \Delta PBQ \text{ is equilateral.}$$
$$\therefore PQ = 1000\ feet.$$

Also, in the isosceles ΔAPB, we have

$$AP = 2\times 1000\cos 75° = 1000\times\frac{\sqrt{3}-1}{\sqrt{2}} = 500\left(\sqrt{6}-\sqrt{2}\right)\ feet.$$

■

For the following seven examples, a book of tables will be required.

§ Problem 14.1.16. *At a point on a horizontal plane the elevation of the summit of a mountain is found to be $22°15'$, and at another point on the plane, a mile further away in a direct line, its elevation is $10°12'$; find the height of the mountain.* ♢

§§ Solution. Take the figure of *Art.* 192. Let PQ (x miles, say) be the mountain and B and A be the two points at which the angles of elevation are taken.

We are given

$$BA = 1\ mile,\ \angle PBQ = 22°15' \text{ and } \angle PAQ = 10°12'.$$

Also,
$$\angle APB = 22°15' - 10°12' = 12°3'.$$

From the ΔPBQ, we have

$$\frac{x}{BP} = \sin 22°15' \tag{14.6}$$

From the ΔPAB, we have

$$\frac{PB}{1\ mile} = \frac{\sin 10°12'}{\sin 12°3'} \tag{14.7}$$

From (14.6) and (14.7), by multiplication, we have

$$\begin{aligned}
\frac{x}{1} &= \frac{\sin 10°12'\ \sin 22°15'}{\sin 12°3'} = \sin 10°12'\ \sin 22°15'\ \operatorname{cosec} 12°3' \\
\log x &= L\sin 10°12' + L\sin 22°15' + L\operatorname{cosec} 12°3' - 30 \\
&= 9.2481811 + 9.5782364 + 10.6809341 - 30 = \bar{1}.5067594.
\end{aligned}$$

Now

$$\log .32118 = \bar{1}.5067485, \text{ and}$$
$$\bar{1}.5067594 - \bar{1}.5067485 = .0000109,$$

for which difference the proportional part is 8.

$$\therefore x = .321188 \approx .32119 \; mile.$$

■

§ Problem 14.1.17. *From the top of a hill the angles of depression of two successive milestones, on level ground and in the same vertical plane with the observer, are found to be* $5°$ *and* $10°$ *respectively. Find the height of the hill and the horizontal distance to the nearest milestone.* ♢

§§ Solution. Take the figure of *Art.* 192. Draw PN parallel to QA, so that PN is the horizontal line passing through P. Let P be the position of the observer on the hill PQ and A and B be the positions of the milestones.

We are given

$$AB = 1 \; mile,$$
$$\angle PAQ = \angle NPA = 5° \text{ and}$$
$$\angle PBQ = \angle NPB = 10°.$$

Also,

$$\angle APB = 10° - 5° = 5° = \angle PAB,$$
$$\therefore AP = AB = 1 \; mile.$$
$$\therefore PQ = AP \sin 10° = \sin 10° = .1736482 \; mile, \text{ and}$$
$$BQ = AP \cos 10° = \cos 10° = .9848078 \; mile,$$

by the table of natural sines and cosines. ■

§ Problem 14.1.18. *A castle and a monument stand on the same horizontal plane. The height of the castle is* 140 *feet, and the angles of depression of the top and bottom of the monument as seen from the top of the castle are* $40°$ *and* $80°$ *respectively. Find the height of the monument.* ♢

§§ Solution. Take the figure of *Ex.* 3, *Art.* 45. Let AB be the castle and CD be the monument.

We are given

$$AB = 140 \; feet, \; \angle EAC = 40°, \text{ and}$$
$$\angle EAD = 80° = \angle ADB.$$

We then have

$$BD = 140 \cot \angle ADB = 140 \cot 80° = AE, \text{ and}$$
$$CE = AE \tan 40° = 140 \cot 80° \tan 40°$$
$$\therefore \log CE = \log 140 + L \cot 80° + L \tan 40° - 20$$
$$= 2.1461280 + 9.2463188 + 9.9238135 - 20 = 1.3162603.$$

Now $\log 20.713 = 1.3162430, \text{ diff. for } .001 = .0000210.$

Let $\log(20.713 + x) = 1.3162603.$

The diff. for x $= 1.3162603 - 1.3162430 = .0000173.$

$$\therefore x = \frac{173}{210} \times .001 = \frac{.173}{210} \approx .0008238$$
$$\therefore CE = 20.7138238 \; feet, \text{ and}$$
$$DE = 140 - CE \approx 119.2862 \; feet.$$

■

§ **Problem 14.1.19.** *A flagstaff PN stands on level ground. A base AB is measured at right angles to AN, the points A,B and N being in the same horizontal plane, and the angles $\angle PAN$ and $\angle PBN$ are found to be α and β respectively.*

Prove that the height of the flagstaff is

$$AB\frac{\sin\alpha\sin\beta}{\sqrt{\sin(\alpha-\beta)\sin(\alpha+\beta)}}.$$

If $AB = 100$ $feet$, $\alpha = 70°$ and $\beta = 50°$, calculate the height. ◊

§§ **Solution**. Take the figure of $Art.$ 193, with N for Q, A for B and B for A.

We then have

$$AN = x\cot\alpha, \text{ and } BN = x\cot\beta.$$

Also, since the $\angle BAN$ is a right angle,

$$\therefore BN^2 = AN^2 + AB^2$$
$$\therefore x^2\cot^2\beta = x^2\cot^2\alpha + AB^2$$
$$\therefore x^2\left(\cot^2\beta - \cot^2\alpha\right) = AB^2$$
$$\therefore x = AB\frac{\sin\alpha\sin\beta}{\sqrt{\sin(\alpha-\beta)\sin(\alpha+\beta)}}$$

as in (14.1.14).

If $AB = 100$ $feet$, $\alpha = 70°$ and $\beta = 50°$, we have

$$x = \frac{100\sin 70°\sin 50°}{\sqrt{\sin 20°\sin 120°}}$$

$$\begin{aligned}\therefore \log x &= \log 100 + L\sin 70° - 10 + L\sin 50° - 10\\ &\qquad - \frac{1}{2}\left[L\sin 20° - 10 + L\sin 60° - 10\right]\\ &= 2 + 9.9729858 + 9.8842540 - 20\\ &\qquad - \frac{1}{2}[9.5340517 + 9.9375306 - 20]\\ &= 1.8572398 - \frac{1}{2}[-.5284177] = 2.1214487.\end{aligned}$$

Now $\log 132.26 = 2.1214285$, diff. for $.01 = .0000329$.

Let $\log(132.26 + y) = 2.1214487$.

The diff. for $y = 2.1214487 - 2.1214285 = .0000202$.

Hence we have $y = \dfrac{202}{329} \times .01 = \dfrac{2.02}{329} \approx .006$.

$$\therefore \log x = \log(132.26 + .006) = \log 132.266$$
$$\therefore x = 132.266\ feet.$$

■

§ **Problem 14.1.20.** *A man, standing due south of a tower on a horizontal plane through its foot, finds the elevation of the top of the tower to be $54°16'$; he goes east 100 yards and finds the elevation to be then $50°8'$. Find the height of the tower.* ◊

§§ **Solution**. As in the last example, if h yards be the required height, we have

$$h = \frac{100\sin 54°16'\sin 50°8'}{\sqrt{\sin(54°16' + 50°8')\sin(54°16' - 50°8')}}$$

$$= \frac{100 \sin 54°16' \sin 50°8'}{\sqrt{\sin 104°24' \sin 4°8'}}$$

$$\therefore \log h = \log 100 + L\sin 54°16' - 10 + L\sin 50°8' - 10$$
$$- \frac{1}{2}\left[L\sin 75°36' - 10 + L\sin 4°8' - 10\right]$$
$$= 2 + 9.9094190 + 9.8851000 - 20$$
$$- \frac{1}{2}[9.9861369 + 8.8578010 - 20] = 2.3725500.$$

Now $\log 235.80 = 2.3725438$, diff. for $.01 = .0000184$.
Let $\log(235.80 + x) = 2.3725500$.
The diff. for $x = 2.3725500 - 2.3725438 = .0000062$.

$$\therefore x = \frac{62}{184} \times .01 = \frac{.62}{184} \approx .00337$$
$$\therefore h \approx 235.8034 \; yards.$$

■

§ Problem 14.1.21. *A man in a balloon observes that the angle of depression of an object on the ground bearing due north is* $33°$ *; the balloon drifts* 3 *miles due west and the angle of depression is now found to be* $21°$. *Find the height of the balloon.* ◊

§§ Solution. Let C be the object, A be the first position of the balloon, AD (h miles) its height and B be the second position of the balloon, BE (h miles) its height.

We then have

$$CD = h\cot 33° \text{ and } CE = h\cot 21°$$

also,
$$CE^2 = CD^2 + DE^2$$
$$\therefore h^2\cot^2 21° = h^2\cot^2 33° + 3^2.$$

Hence, as in § Problem 14.1.14, we have

$$h = \frac{3\sin 33° \sin 21°}{\sqrt{\sin 54° \sin 12°}}$$

$$\therefore \log h = \log 3 + L\sin 33° - 10 + L\sin 21° - 10$$
$$- \frac{1}{2}[L\sin 54° - 10 + L\sin 12° - 10]$$
$$= .4771213 + 9.7361088 + 9.5543292 - 20$$
$$- \frac{1}{2}[9.9079576 + 9.3178789 - 20] = .1546410.$$

Now $\log 1.4277 = .1546370$, diff. for $.0001 = .0000304$.
Let $\log(1.4277 + x) = .1546410$.
The diff. for $x = .1546410 - .1546370 = .0000040$.

$$\therefore x = \frac{40}{304} \times .0001 = \frac{.001}{76} \approx .000013$$
$$\therefore h = 1.427713 \; mile.$$

■

§ Problem 14.1.22. *From the extremities of a horizontal base-line* AB, *whose length is* 1000 *feet, the bearings of the foot* C *of a tower are observed and it is found that* $\angle CAB = 56°23'$, $\angle CBA = 47°15'$, *and that the elevation of the tower from A is* $9°25'$ *; find the height of the tower.* ◊

§§ Solution. Let D be the top of the tower.

We have

$$\angle DAC = 9°25', \text{ and } \angle ACB = 180° - \left(56°23' + 47°15'\right) = 76°22',$$

$$\text{and } \frac{AC}{AB} = \frac{\sin\angle ABC}{\sin\angle ACB}$$

also,
$$\frac{CD}{AC} = \tan\angle DAC.$$

Hence, by multiplication, we have

$$\frac{CD}{AB} = \tan\angle DAC \sin\angle ABC \operatorname{cosec}\angle ACB$$
$$\therefore CD = 1000\tan 9°25' \sin 47°15' \operatorname{cosec} 76°22'$$
$$\therefore \log CD = \log 1000 + L\tan 9°25' + L\sin 47°15' + L\operatorname{cosec} 76°22' - 30$$
$$= 3 + 9.2197097 + 9.8658868 + 10.0124124 - 30 = 2.0980089.$$

Now $\log 125.31 = 2.0979857,$ diff. for $.01 = .0000346.$

Let $\log(125.31 + x) = 2.0980089.$

The diff. for $x = 2.0980089 - 2.0979857 = .0000232.$

$$\therefore x = \frac{232}{346} \times .01 = \frac{2.32}{346} \approx .0067$$
$$\therefore CD = 125.3167\ feet.$$ ■

14.2 Angle Subtended at Two Points

§ Problem 14.2.1. *A bridge has* 5 *equal spans, each of* 100 *feet measured from the center of the piers, and a boat is moored in a line with one of the middle piers. The whole length of the bridge subtends a right angle as seen from the boat. Prove that the distance of the boat from the bridge is* $100\sqrt{6}\ feet.$ ♢

§§ Solution. Let AB be the bridge, C be the pier and D be the position of the boat.

We are given

$$AC = 200\ ft.,\ CB = 300 ft.,\ \text{and } \angle ADB = 90°.$$

Let the required distance CD be x ft. and $\angle ADC = \alpha.$

We then have

$$x = AC\cot\alpha = 200\cot\alpha$$

also
$$x = BC\cot(90° - \alpha) = 300\tan\alpha.$$

Hence, by multiplication, we have

$$x^2 = 60000$$
$$\therefore x = 100\sqrt{6}\ feet.$$ ■

§ Problem 14.2.2. *A ladder placed at an angle of* $75°$ *with the ground just reaches the sill of a window at a height of* $27\ feet$ *above the ground on one side of a street. On turning the ladder over without moving its foot, it is found that when it rests against a wall on the other side of the street it is at an angle of* $15°$ *with the ground. Prove that the breadth of the street and the length of the ladder are respectively*

$$27\left(3 - \sqrt{3}\right) \text{ and } 27\left(\sqrt{6} - \sqrt{2}\right)\ feet.$$ ♢

§§ Solution. Let AC and CE be the two positions of the ladder (length $l\ feet$) respectively and DB be the breadth of the street, B being vertically below A and D below E.

We are given

$$AB = 27\ ft.,\ \angle ACB = 75° \text{ and } \angle ECD = 15°.$$

The triangles ABC and CDE are identically equal.

$$\therefore CD = AB = 27\ ft.$$

Also, $$CB = 27\cot 75^\circ = 27\left(2-\sqrt{3}\right)\ ft.$$

$\therefore$ the breadth of the street

$$= DC + CB = 27\left(3-\sqrt{3}\right)\ ft.$$

Again, $$l = 27\,\text{cosec}\ 75^\circ = 27 \times \frac{2\sqrt{2}}{\sqrt{3}+1} = 27\left(\sqrt{6}-\sqrt{2}\right)\ ft.$$

Otherwise thus :

$$\begin{aligned}\text{The breadth of the street } &= l\cos 75^\circ + l\cos 15^\circ \\ &= 2l\cos 45^\circ \cos 30^\circ \\ &= 54\left(\sqrt{6}-\sqrt{2}\right)\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} \\ &= 27(\sqrt{3}-1)\sqrt{3} = 27\left(3-\sqrt{3}\right)\ ft.\end{aligned}$$

■

§ Problem 14.2.3. *From a house on one side of a street observations are made of the angle subtended by the height of the opposite house; from the level of the street the angle subtended is the angle whose tangent is* 3 *; from two windows one above the other the angle subtended is found to be the angle whose tangent is* -3 *; the height of the opposite house being* $60\ feet$*, find the height above the street of each of the two windows.* ♢

§§ Solution. Let A be the top and B be the bottom of the house of height 60 feet and C be the bottom of the other house, i.e. C is at the level of the street.

We then have

$$CB = AB\cot\angle ACB = 60 \times \frac{1}{\tan\angle ACB} = 60 \times \frac{1}{3} = 20\ feet.$$

Let W' and W be the upper and lower windows respectively and let $x\ feet$ be the height of either window above the street.

Draw $W'D$ and WE perpendicular to AB, so that $W'D = WE = CB = 20\ feet$.

We have

$$\angle AWE + \angle BWE = \angle AWB.$$

$$\therefore \frac{\tan\angle AWE + \tan\angle BWE}{1-\tan\angle AWE.\tan\angle BWE} = \tan\angle AWB$$

$$\therefore \frac{\dfrac{x}{20} + \dfrac{60-x}{20}}{1-\dfrac{x}{20}.\dfrac{60-x}{20}} = -3$$

$$\therefore x^2 - 60x + 800 = 0$$

$$\therefore (x-20)(x-40) = 0$$

$$\therefore x = 20 \text{ or } 40.$$

Thus the windows are $20\ ft.$ and $40\ ft.$ above the street. ■

§ Problem 14.2.4. *A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground ; find the longest shadow it can cast on the ground.*

Calculate the altitude of the sun when the longest shadow it can cast is $3\frac{1}{2}$ *times the length of the rod.* ♢

§§ **Solution**. Let BA be the rod of length l, AC be its shadow, γ be the sun's altitude and θ be the inclination of the rod to the ground. We then have

$$AC = \frac{l\sin(\theta+\gamma)}{\sin\gamma},$$

and this is greatest when $\theta+\gamma = 90°$, i.e. when $\theta = 90° - \gamma$, i.e. when ABC is a right angle.

$$\therefore AC = l \operatorname{cosec} \gamma.$$

Again, if $AC = 3\frac{1}{2} \times l$, then

$$\operatorname{cosec} \gamma = \frac{7}{2}, \ \therefore \sin\gamma = \frac{2}{7}.$$ ■

§ **Problem 14.2.5.** *A person on a ship A observes another ship B leaving a harbour, whose bearing is then N.W. After 10 minutes A, having sailed one mile N.E., sees B due west and the harbour then bears 60° West of North. After another 10 minutes B is observed to bear S.W. Find the distances between A and B at the first observation and also the direction and rate of B.* ♢

§§ **Solution**. Let A, A_1, A_2, B, B_1 and B_2 be the positions of A and B respectively as given in the question.

$A_2A_1B_2$ is a straight line since both A_1 and B_2 are in a direction S.E. of A_2.

We have

$$\angle BAA_1 = 90° \text{ and } \angle BA_1A = 75°$$

$$\therefore AB = AA_1 \tan 75° = 1 \times (2+\sqrt{3}) = 3.732\ldots \ miles.$$

Also, $$BB_1 = B_1B_2$$

$$\therefore \frac{A_1B_1}{BB_1} = \frac{A_1B_1}{B_1B_2}$$

$$\therefore \frac{\sin(\theta-30°)}{\sin 30°} = \frac{\sin(\theta+45°)}{\sin 45°} \text{ where } \theta \equiv \angle A_1B_1B_2$$

$$\therefore \sin\theta\cot 30° - \cos\theta = \sin\theta + \cos\theta$$

$$\therefore \sin\theta(\cot 30° - 1) = 2\cos\theta$$

$$\therefore \tan\theta = \frac{2}{\sqrt{3}-1} = \sqrt{3}+1.$$

$\therefore B's$ course is at an angle whose tangent is $\sqrt{3}+1$, S. of E.

Again, $$BB_2 = AB \operatorname{cosec} \angle BB_2A = (2+\sqrt{3}) \operatorname{cosec}(\theta+45°)$$

$$= \frac{2+\sqrt{3}}{\sin(\theta+45°)} = \frac{\sqrt{2}(2+\sqrt{3})}{\sin\theta+\cos\theta}$$

$$= \sqrt{2}(2+\sqrt{3}) \div \frac{\sqrt{3}+1+1}{\sqrt{5+2\sqrt{3}}} = \sqrt{2}\sqrt{5+2\sqrt{3}}$$

$$= \sqrt{10+4\sqrt{3}} = \sqrt{16.9282} = 4.114\ldots \ miles.$$

Hence in 1 hour the distance $= 4.114\ldots \times 3 = 12.342\ldots \ miles.$ ■

§ **Problem 14.2.6.** *A person on a ship sailing north sees two light-houses, which are 6 miles apart, in a line due west ; after an hour's sailing one of them bears S.W. and the other S.S.W. Find the ship's rate.* ♢

§§ Solution. Let S and N be the two positions of the ship respectively, A be the nearer lighthouse and B be the further lighthouse. We are given

$$\angle BNS = 45^\circ \text{ and } \angle ANS = 22\frac{1}{2}^\circ$$
$$\therefore \angle NAS = 67\frac{1}{2}^\circ.$$

We then have

$$\frac{BN}{BA} = \frac{\sin\angle BAN}{\sin\angle BNA} = \frac{\sin\left(180^\circ - 67\frac{1}{2}^\circ\right)}{\sin\left(45^\circ - 22\frac{1}{2}^\circ\right)}$$
$$\therefore \frac{BN}{6} = \frac{\sin 67\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \frac{\cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \cot 22\frac{1}{2}^\circ = \sqrt{2}+1$$
$$\therefore BN = 6\left(\sqrt{2}+1\right).$$

Also, $$SN = BN\sin 45^\circ = \frac{6\left(\sqrt{2}+1\right)}{\sqrt{2}} = 3\sqrt{2}\left(\sqrt{2}+1\right)$$
$$= 3\left(2+\sqrt{2}\right) = 3\times 3.4142 = 10.2426\ miles$$

i.e. the required rate is $10.2426\ miles$ per hour. ■

§ Problem 14.2.7. *A person on a ship sees a lighthouse N.W. of himself. After sailing for 12 miles in a direction 15° south of W. the lighthouse is seen due N. Find the distance of the lighthouse from the ship in each position.* ◊

§§ Solution. Let S and H be the two positions of the ship and L be the light house.
We then have

$$\frac{SL}{SH} = \frac{\sin\angle SHL}{\sin\angle SLH}$$
$$\therefore SL = \frac{12\sin 75^\circ}{\sin 45^\circ} = 12\times\frac{\sqrt{3}+1}{2}$$
$$= 6\left(\sqrt{3}+1\right) = 6\times 2.73205 = 16.3923\ldots\ miles$$

Also, $$\frac{HL}{SH} = \frac{\sin\angle HSL}{\sin\angle SLH}$$
$$\therefore HL = \frac{12\sin 60^\circ}{\sin 45^\circ} = 6\sqrt{3}\sqrt{2} = 6\sqrt{6}$$
$$= 6\times 2.4494897 \approx 14.697\ miles$$ ■

§ Problem 14.2.8. *A man, traveling west along a straight road, observes that when he is due south of a certain windmill the straight line drawn to a distant tower makes an angle of 30° with the road. A mile further on the bearings of the windmill and tower are respectively N.E. and N.W. Find the distances of the tower from the windmill and from the nearest point of the road.* ◊

§§ Solution. Let W be the windmill, T be the tower, M and A be the two points at which the directions are taken and let TN be perpendicular to the road.

We are given

$$MA = 1\ mile,\ \angle TMA = 30^\circ \text{ and } \angle WAM = 45^\circ = \angle TAN.$$

We then have

$$\frac{AT}{AM} = \frac{\sin\angle AMT}{\sin\angle ATM} = \frac{\sin 30^\circ}{\sin 15^\circ} = 2\cos 15^\circ$$

$$\therefore AT = \frac{\sqrt{3}+1}{\sqrt{2}}.$$

Also, $$AW = AM\sec 45^\circ = \sqrt{2}.$$

$\because \angle TAW = 90^\circ$, we have

$$TW^2 = AT^2 + AW^2 = 2 + \sqrt{3} + 2 = 4 + \sqrt{3}.$$

$$\therefore TW = \sqrt{4+\sqrt{3}} = \sqrt{5.73205} = 2.39\ miles.$$

Also, $$TN = AT\sin 45^\circ = \frac{\sqrt{3}+1}{\sqrt{2}}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2}$$

$$= \frac{2.73205}{2} = 1.366\ miles.$$ ■

§ Problem 14.2.9. *An observer on a headland sees a ship due north of him ; after a quarter of an hour he sees it due east and after another half-hour he sees it due south-east; find the direction that the ship's course makes with the meridian and the time after the ship is first seen until it is nearest the observer, supposing that it sails uniformly in a straight line.* ♢

§§ Solution. Let O be the position of the observer and S, H and I be the three positions of the ship respectively.

Draw OP perpendicular to the straight line SHI ; join OH ; and draw IA perpendicular to SO produced.

Then P is the nearest position of the ship to the observer after it is first seen.

Also, $\angle HOI = 45^\circ$. We have $SH =$ the distance the ship sails in $\frac{1}{4}$ hour and $HI =$ the distance it sails in $\frac{1}{2}$ hour.

$$\therefore HI = 2SH$$

$$\therefore OA = 2OS = AI$$

$$\therefore \tan\angle ASI = \frac{AI}{AS} = \frac{2}{3},$$

so that the ship's course makes with the meridian an angle whose tangent is $\frac{2}{3}$.

Also,

$$SP = OS\cos\angle OSP = SH\cos^2\angle OSP = SH\cos^2\angle ASI = SH \times \frac{9}{13}.$$

Hence the required time $= \left(\frac{9}{13} \times \frac{1}{4}\right)\ hour = \frac{9}{52}\ hour.$ ■

§ Problem 14.2.10. *A man walking along a straight road, which runs in a direction* 30° *east of north, notes when he is due south of a certain house ; when he has walked a mile further, he observes that the house lies due west and that a windmill on the opposite side of the road is N.E. of him ; three miles further on he finds that he is due* *north of the windmill; prove that the line joining the house and the windmill makes with the road the angle whose tangent is*

$$\frac{48-25\sqrt{3}}{11}.$$

◊

§§ Solution. Let A, and B and C be the three positions of the man, H be the house and W be the windmill.

Let HW meet AC in K and CW meet HB produced in D.

We then have

$$HB = AB\sin 30^\circ = \frac{1}{2}\ mile, \text{ and}$$

$$BD = BC\sin 30^\circ = \frac{3}{2}\ mile = DW, \because \angle WBD = 45^\circ$$

$$\therefore HD = 2\ miles.$$

Let ϕ be the required angle, so that

$$\phi = \angle CKW = \angle CBD - \angle WHD = 60^\circ - \angle WHD.$$

We have
$$\tan\angle WHD = \frac{WD}{HD} = \frac{3}{2} \div 2 = \frac{3}{4}$$

$$\therefore \tan(60^\circ - \phi) = \frac{3}{4}$$

$$\therefore \frac{\tan 60^\circ - \tan\phi}{1+\tan 60^\circ\tan\phi} = \frac{3}{4}$$

$$\therefore 4\left(\sqrt{3}-\tan\phi\right) = 3\left(1+\sqrt{3}\tan\phi\right)$$

$$\therefore \tan\phi = \frac{4\sqrt{3}-3}{3\sqrt{3}+4} = \frac{\left(4\sqrt{3}-3\right)\left(3\sqrt{3}-4\right)}{27-16} = \frac{48-25\sqrt{3}}{11}.$$

■

§ Problem 14.2.11. *A, B, and C are three consecutive milestones on a straight road from each of which a distant spire is visible. The spire is observed to bear north-east at A, east at B, and 60° east of south at C. Prove that the shortest distance of the spire from the road is*

$$\frac{7+5\sqrt{3}}{13}\ miles.$$

◊

§§ Solution. Let S be the spire and SD (x miles, say) be the shortest distance of the spire from the road.

Let $\angle BSD = \theta$ and $\tan\theta = t$.

We have

$$\begin{aligned} 1 = BC = CD - BD &= x\tan(30^\circ+\theta) - x\tan\theta \\ &= x\left(\frac{1+t\sqrt{3}}{\sqrt{3}-t} - t\right) = x\left(\frac{1+t^2}{\sqrt{3}-t}\right). \end{aligned}$$

Also,

$$\begin{aligned} 1 = AB = BD + AD &= x\tan\theta + x\tan(45^\circ-\theta) \\ &= x\left(t + \frac{1-t}{1+t}\right) = x\left(\frac{1+t^2}{1+t}\right). \end{aligned}$$

Hence, equating these values, we have

$$x\left(\frac{1+t^2}{\sqrt{3}-t}\right) = x\left(\frac{1+t^2}{1+t}\right)$$

$$\therefore \sqrt{3}-t = 1+t,\ i.e.\ 2t = \sqrt{3}-1,\ i.e.\ t = \frac{\sqrt{3}-1}{2}.$$

$$\therefore x = \frac{1+t}{1+t^2} = \frac{1+\dfrac{\sqrt{3}-1}{2}}{1+\dfrac{2-\sqrt{3}}{2}}$$
$$= \frac{1+\sqrt{3}}{4-\sqrt{3}} = \frac{(1+\sqrt{3})(4+\sqrt{3})}{16-3} = \frac{7+5\sqrt{3}}{13}. \qquad \blacksquare$$

§ Problem 14.2.12. *Two stations due south of a tower, which leans towards the north, are at distances a and b from its foot; if α and β be the elevations of the top of the tower from these stations, prove that its inclination to the horizontal is*
$$\cot^{-1}\left(\frac{b\cot\alpha - a\cot\beta}{b-a}\right). \qquad \Diamond$$

§§ Solution. Let T be the top and R be the foot of the tower and let A and B be the two stations respectively.

Then, if θ be the required inclination, we have
$$\frac{b}{a} = \frac{BR}{AR} = \frac{BR}{TR}.\frac{TR}{AR} = \frac{\sin(\theta-\beta)}{\sin\beta}.\frac{\sin\alpha}{\sin(\theta-\alpha)}$$
$$= \frac{\sin\theta\cot\beta - \cos\theta}{\sin\theta\cot\alpha - \cos\theta} = \frac{\cot\beta - \cot\theta}{\cot\alpha - \cot\theta}$$
$$\therefore b\cot\alpha - a\cot\beta = (b-a)\cot\theta$$
$$\therefore \theta = \cot^{-1}\left(\frac{b\cot\alpha - a\cot\beta}{b-a}\right).$$

Otherwise thus :

From T draw TN perpendicular to the ground. Let $TN = h$ and $RN = x$. Then

In the ΔTBN, we have $b + x = h\cot\beta$

In the ΔTAN, we have $a + x = h\cot\alpha$, and

In the ΔTRN, we have $x = h\cot\theta$.

$$\therefore b = h(\cot\beta - \cot\theta) \text{ and } a = h(\cot\alpha - \cot\theta)$$
$$\therefore \frac{b}{a} = \frac{\cot\beta - \cot\theta}{\cot\alpha - \cot\theta}$$
$$\therefore b\cot\alpha - a\cot\beta = (b-a)\cot\theta$$
$$\therefore \theta = \cot^{-1}\left(\frac{b\cot\alpha - a\cot\beta}{b-a}\right). \qquad \blacksquare$$

§ Problem 14.2.13. *From a point A on a level plane the angle of elevation of a balloon is α, the balloon being south of A ; from a point B, which is at a distance c south of A, the balloon is seen northwards at an elevation of β; find the distance of the balloon from A and its height above the ground.* $\Diamond$

§§ Solution. If C be the position of the balloon and CD be its height above the ground,we have
$$\frac{AC}{AB} = \frac{\sin\beta}{\sin(\alpha+\beta)}$$
$$\therefore AC = c\sin\beta\,\mathrm{cosec}\,(\alpha+\beta).$$

Also, $$CD = AC\sin\alpha = c\sin\alpha\sin\beta\,\mathrm{cosec}\,(\alpha+\beta). \qquad \blacksquare$$

§ Problem 14.2.14. *A statue on the top of a pillar subtends the same angle α at distances of 9 and 11 yards from the pillar ; if $\tan\alpha = \dfrac{1}{10}$, find the height of the pillar and of the statue.* ◊

§§ Solution. Take the figure of *Art.* 196.

Let RP (x yards, say) be the statue and PQ (y yards, say) be the pillar and let B and A be the points of observation, so that $BQ = 9\ yards$ and $AQ = 11\ yards$.

Since the $\angle PAR = \angle PBR = \alpha$, a circle will go through the four points A, B, P and R.

We have

$$\frac{y}{11} = \tan\beta \tag{14.8}$$

$$\frac{x+y}{11} = \tan(\alpha+\beta), \text{ and} \tag{14.9}$$

$$\frac{9}{x+y} = \tan\beta \tag{14.10}$$

From (14.9) and (14.10), by multiplication, we have

$$\frac{9}{11} = \tan\beta\tan(\alpha+\beta) = \tan\beta\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \tan\beta\frac{\frac{1}{10}+\tan\beta}{1-\frac{1}{10}\tan\beta} = \frac{\tan\beta+10\tan^2\beta}{10-\tan\beta}$$

$$\therefore 11\tan^2\beta + 2\tan\beta - 9 = 0$$

$$\therefore (11\tan\beta - 9)(\tan\beta + 1) = 0$$

$$\therefore 11\tan\beta - 9 = 0$$

$$\therefore \tan\beta = \frac{9}{11}.$$

Hence, from (14.8), we have $y = 9\ yards$, and, from (14.10), $x+y = 11$.

$$\therefore x = 11 - y = 2\ yards.$$

Otherwise thus :

Let O be the center of the circle passing though the four points A, B, P and R and from O draw OM and ON perpendicular to RP and AB, bisecting them in M and N respectively.

Join OR and OP.

$\angle ROM = \frac{1}{2}\angle ROP = \angle RAP$ by *Euclid. III.* 20, and we have

$$\tan\angle RAP = \frac{1}{10} = \tan\angle ROM = \frac{RM}{MO}$$

$$\therefore RM = \frac{1}{10}MO = \frac{1}{10}\left(BQ + \frac{1}{2}AB\right) = \frac{1}{10}\times\left(9+\frac{1}{2}\times 2\right) = 1\ yard$$

$$\therefore RP = 2\ yards.$$

Also, by *Euclid. III.* 36, *Cor.*, we have

$$QB.QA = QR.QP$$

$$\therefore 9\times 11 = QP(QP+2)$$

$$\therefore QP^2 + 2QP - 99 = 0$$

$$\therefore QP = \frac{-2\pm\sqrt{4+396}}{2} = -1\pm 10 = 9\ yards.$$ ■

§ Problem 14.2.15. *A flagstaff on the top of a tower is observed to subtend the same angle α at two points on a horizontal plane, which he on a line passing through the center of the base of the tower and whose distance from one another is $2a$, and an angle β at a point halfway between them. Prove that the height of the flagstaff is*

$$a\sin\alpha\sqrt{\frac{2\sin\beta}{\cos\alpha\sin(\beta-\alpha)}}.$$

◊

§§ Solution. Take the figure of $Art.$ 196, with N the middle point of AB. Let P and Q be the top and foot of the tower respectively, and let PR be the flagstaff.

Let A, B and N be the points of observation, so that

$$\angle RBP = \alpha = \angle RAP,\ \angle RNP = \beta \text{ and } BN = a = NA.$$

$$\because \angle RBP = \angle RAP,$$

$\therefore$ a circle will go through the four points A, B, P and R.

Let O be the center of this circle and from O draw OM and ON perpendicular to RP and AB, bisecting them in M and N respectively.

Join OR, OP.

Let $\angle PNQ = \phi$. We then have the $\angle ROP = 2\alpha$, so that

$$\angle ROM = \alpha = \angle MOP$$

$$\therefore OM = \frac{x}{2}\cot\alpha = QN, \text{ where } RP = x, \text{ and}$$

$$OR = \frac{x}{2}\operatorname{cosec}\alpha.$$

Also,

$$ON = \sqrt{OB^2 - BN^2} = \sqrt{OR^2 - BN^2} = \sqrt{\frac{x^2}{4}\operatorname{cosec}^2\alpha - a^2} = MQ$$

$$\therefore \tan(\beta+\phi) = \frac{\sqrt{\frac{x^2}{4}\operatorname{cosec}^2\alpha - a^2} + \frac{x}{2}}{\frac{x}{2}\cot\alpha}, \text{ and}$$

$$\tan\phi = \frac{\sqrt{\frac{x^2}{4}\operatorname{cosec}^2\alpha - a^2} - \frac{x}{2}}{\frac{x}{2}\cot\alpha}.$$

$$\therefore \tan(\beta+\phi) - \tan\phi = \frac{x}{\frac{x}{2}\cot\alpha} = \frac{2}{\cot\alpha}, \text{ and}$$

$$1 + \tan(\beta+\phi)\tan\phi = 1 + \frac{\frac{x^2}{4}\operatorname{cosec}^2\alpha - a^2 - \frac{x^2}{4}}{\frac{x^2}{4}\cot^2\alpha} = \frac{\frac{x^2}{2}\cot^2\alpha - a^2}{\frac{x^2}{4}\cot^2\alpha}$$

$$\therefore \tan\beta = \tan(\beta+\phi-\phi) = \frac{\tan(\beta+\phi)-\tan\phi}{1+\tan(\beta+\phi)\tan\phi} = \frac{\frac{x^2}{2}\cot^2\alpha}{\frac{x^2}{2}\cot^2\alpha - a^2}$$

$$\therefore \cot\beta = \cot\alpha - \frac{2a^2}{x^2\cot\alpha}$$

$$\therefore \frac{2a^2}{x^2}\cdot\frac{1}{\cot\alpha} = \cot\alpha - \cot\beta.$$

$$\therefore x^2 = \frac{2a^2}{\cot\alpha(\cot\alpha - \cot\beta)} = \frac{2a^2 \sin^2\alpha \sin\beta}{\cos\alpha \sin(\beta - \alpha)}$$

$$\therefore x = a\sin\alpha\sqrt{\frac{2\sin\beta}{\cos\alpha\sin(\beta-\alpha)}}.$$

■

§ **Problem 14.2.16.** *An observer in the first place stations himself at a distance a feet from a column standing upon a mound. He finds that the column subtends an angle, whose tangent is $\frac{1}{2}$, at his eye which may be supposed to be on the horizontal plane through the base of the mound. On moving $\frac{2}{3}a$ feet nearer the column, he finds that the angle subtended is unchanged. Find the height of the mound and of the column.* ◊

§§ **Solution**. Take the figure of *Art.* 196.

Let P and Q be the top and foot of the mound respectively and let PR be the column.

Let A and B be the points of observation, so that $AQ = a\ ft.$ and $AB = \frac{2}{3}a\ ft.$, and therefore $BQ = \frac{a}{3}\ ft.$

$\because \angle PAR = \angle PBR = \alpha$, a circle will go through the four points A, B, P and R.

Let $y\ ft.$ and $x\ ft.$ be the heights of the mound and column respectively and β be as marked in the figure.

We then have

$$\tan\alpha = \frac{1}{2}.$$

Also, $\quad x + y = a\tan(\alpha+\beta)$ and $y = a\tan\beta$.

Also, in the ΔRBQ,

$$x + y = \frac{a}{3}\cot\beta.$$

$$\therefore \frac{1}{3}\cot\beta = \tan(\alpha+\beta) = \frac{\frac{1}{2} + \tan\beta}{1 - \frac{1}{2}\tan\beta} = \frac{1 + 2\tan\beta}{2 - \tan\beta}$$

$$\therefore \frac{1}{3\tan\beta} = \frac{1+2\tan\beta}{2-\tan\beta}$$

$$\therefore 3\tan^2\beta + 2\tan\beta - 1 = 0$$

$$\therefore (3\tan\beta - 1)(\tan\beta + 1) = 0$$

$$\therefore \tan\beta = \frac{1}{3}.$$

$$\therefore y = a\tan\beta = \frac{a}{3}\ ft., \text{ and}$$

$$x = \frac{a}{3}\cot\beta - y = a - \frac{a}{3} = \frac{2a}{3}\ ft.$$

Otherwise thus:

Let O be the center of the circle passing through the points A, B, P and R.

From O draw OM and ON perpendicular to RP and AB, bisecting them in M and N respectively.

Join OR, OP. The $\angle ROP = 2\alpha$, and $\angle ROM = \alpha = \angle MOP$.

$$\therefore OM = \frac{x}{2}\cot\alpha = \frac{x}{2} \times 2 \qquad \left[\because \tan\alpha = \frac{1}{2}\right]$$
$$= x = NQ = \frac{2a}{3}.$$

Also, $ON = OM$, since the chords RP and AB are equal ;

$$\therefore ON = \frac{2a}{3} = MQ$$
$$\therefore PQ = \frac{a}{3}. \qquad \blacksquare$$

§ Problem 14.2.17. *A church tower stands on the bank of a river, which is* $150\ feet$ *wide, and on the top of the tower is a spire* $30\ feet$ *high. To an observer on the opposite bank of the river, the spire subtends the same angle that a pole six feet high subtends when placed upright on the ground at the foot of the tower. Prove that the height of the tower is nearly* $285\ feet$*.* ◊

§§ Solution. Let AB be the spire, BD be the tower, CD be the pole and O be the position of the observer, so that

$$AB = 30\ feet,\ CD = 6\ feet \text{ and } OD = 150\ feet.$$

Let $\angle AOB = \alpha = \angle COD$, $\angle BOC = \beta$ and x feet be the required height.

We then have

$$\frac{6}{150} = \tan\alpha,\ \frac{x}{150} = \tan(\alpha+\beta) \text{ and}$$

$$\frac{x+30}{150} = \tan(\alpha+\beta+\alpha) = \frac{\tan(\alpha+\beta)+\tan\alpha}{1-\tan(\alpha+\beta)\tan\alpha} = \frac{\dfrac{x}{150}+\dfrac{6}{150}}{1-\dfrac{6x}{(150)^2}}$$

$$\therefore x^2 + 30x - 90000 = 0$$
$$\therefore x \approx 285\ feet. \qquad \blacksquare$$

§ Problem 14.2.18. *A person, wishing to ascertain the height of a tower, stations himself on a horizontal plane through its foot at a point at which the elevation of the top is* $30°$*. On walking a distance* a *in a certain direction he finds that the elevation of the top is the same as before, and on then walking a distance* $\frac{5}{3}a$ *at right angles to his former direction he finds the elevation of the top to be* $60°$*. Prove that the height of the tower is either* $\sqrt{\frac{5}{6}}a$ *or* $\sqrt{\frac{85}{48}}a$*.* ◊

§§ Solution. Let T and R be the top and the foot of the tower, whose height h is required and let A, B and C be the points at which the angles of elevation are taken respectively, so that

$$\angle TAR = 30°,\ \angle TBR = 30°,\ \angle TCR = 60°,\ AB = a \text{ and } BC = \frac{5}{3}a.$$

We then have

$$AR = h\cot 30° = h\sqrt{3} = BR \text{ and } CR = h\cot 60° = \frac{h}{\sqrt{3}}.$$

Also,
$$\angle CBR = 90° - \angle ABR$$
$$\therefore \cos\angle CBR = \sin\angle ABR.$$

Now
$$\cos\angle ABR = \frac{a}{2} \div BR = \frac{a}{2h\sqrt{3}}$$

$$\therefore \cos\angle CBR = \sqrt{1-\cos^2\angle ABR} = \sqrt{1-\frac{a^2}{12h^2}}.$$

Again, $$RC^2 = BC^2 + BR^2 - 2BC.BR\cos\angle CBR$$

$$\therefore \frac{h^2}{3} = \frac{25a^2}{9} + 3h^2 - 2.\frac{5a}{3}.h\sqrt{3}\sqrt{1-\frac{a^2}{12h^2}}$$

$$\therefore \frac{5a}{3}\sqrt{12h^2-a^2} = \frac{8}{3}h^2 + \frac{25a^2}{9}$$

$$\therefore 15a\sqrt{12h^2-a^2} = 24h^2 + 25a^2$$

$$\therefore 225a^2\left(12h^2-a^2\right) = 576h^4 + 1200a^2h^2 + 625a^4$$

$$\therefore 288h^4 - 750a^2h^2 + 425a^4 = 0$$

$$\therefore h^2 = a^2.\frac{750\pm\sqrt{72900}}{576} = a^2.\frac{750\pm 270}{576}$$

$$= a^2.\frac{1020}{576} \text{ or } a^2.\frac{480}{576}$$

$$= a^2.\frac{85}{48} \text{ or } a^2.\frac{5}{6}$$

$$\therefore h = a\sqrt{\frac{85}{48}} \text{ or } a\sqrt{\frac{5}{6}}.$$

■

§ **Problem 14.2.19.** *The angles of elevation of the top of a tower, standing on horizontal plane, from two points distant a and b from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ feet, and, if θ be the angle subtended at the top of the tower by the line joining the two points, then*

$$\sin\theta = \frac{a\sim b}{a+b}.$$

◊

§§ **Solution**. Let h denote the height of the tower and α the greater angle of elevation.

We then have

$$a = h\cot\alpha \text{ and } b = h\cot(90^\circ-\alpha) = h\tan\alpha, \text{ where } a<b.$$

Hence, by multiplication,

$$h^2 = ab, \therefore h = \sqrt{ab}.$$

Again, $$\theta = \alpha - (90^\circ-\alpha) = 2\alpha - 90^\circ.$$

$$\therefore \sin\theta = -\cos 2\alpha = \frac{\sin^2\alpha-\cos^2\alpha}{\sin^2\alpha+\cos^2\alpha} = \frac{\tan\alpha-\cot\alpha}{\tan\alpha+\cot\alpha}$$

$$= \left(\frac{b}{h}-\frac{a}{h}\right)\div\left(\frac{b}{h}+\frac{a}{h}\right) = \frac{b-a}{b+a}.$$

If $a>b$, we have

$$b = h\cot\alpha \text{ and } a = h\tan\alpha$$

$$\therefore \sin\theta = \left(\frac{a}{h}-\frac{b}{h}\right)\div\left(\frac{a}{h}+\frac{b}{h}\right) = \frac{a-b}{a+b}$$

$$\therefore \sin\theta = \frac{a\sim b}{a+b}.$$

■

§ **Problem 14.2.20.** *A tower 150 feet high stands on the top of a cliff 80 feet high. At what point on the plane passing through the foot of the cliff must an observer place himself so that the tower and* *the cliff may subtend equal angles, the height of his eye being* 5 *feet ?* ♢

§§ Solution. Let A be the top of the tower, B and C be the top and the foot of the cliff respectively, O be the position of the observer's eye and x feet be the required distance.

From O, draw OD perpendicular to BC, so that $OD = x\ feet$.

Then, since OB bisects the $\angle AOC$, we have, by $Euclid\ VI.\ 3$,

$$\frac{OA}{OC} = \frac{AB}{BC}$$

$$\therefore \frac{\sqrt{OD^2 + AD^2}}{\sqrt{x^2 + 5^2}} = \frac{150}{80} = \frac{15}{8}$$

$$\therefore \frac{x^2 + (150 + 80 - 5)^2}{x^2 + 5^2} = \frac{225}{64}$$

$$\therefore \frac{x^2 + (225)^2}{x^2 + 5^2} = \frac{225}{64}$$

$$\therefore 161x^2 = 225 \times 25(64 \times 9 - 1) = 225 \times 25 \times 575$$

$$\therefore (7 \times 23)x^2 = 225 \times (25)^2 \times 23$$

$$\therefore x = \frac{15 \times 25}{\sqrt{7}} = \frac{375}{\sqrt{7}}\ feet.$$

Otherwise thus :

Let E be the point on the ground vertically below O.

Let the $\angle AOB = \alpha = \angle BOC$ and the $\angle ECO = \theta = \angle COD$.

We then have

$$\frac{5}{x} = \tan\theta,\ \frac{75}{x} = \tan(\alpha - \theta) \text{ and } \frac{225}{x} = \tan(2\alpha - \theta).$$

Now
$$2\alpha - \theta - \theta = 2(\alpha - \theta)$$

$$\therefore \tan(2\alpha - \theta - \theta) = \tan 2(\alpha - \theta)$$

$$\therefore \frac{\tan(2\alpha - \theta) - \tan\theta}{1 + \tan(2\alpha - \theta)\tan\theta} = \frac{2\tan(\alpha - \theta)}{1 - \tan^2(\alpha - \theta)}$$

$$\therefore \frac{\frac{225}{x} - \frac{5}{x}}{1 + \frac{225}{x}.\frac{5}{x}} = \frac{\frac{150}{x}}{1 - \left(\frac{75}{x}\right)^2}$$

$$\therefore \frac{220}{x^2 + 1125} = \frac{150}{x^2 - 5625}$$

$$\therefore x = \frac{375}{\sqrt{7}}, \text{ as before.}$$ ■

§ Problem 14.2.21. *A statue on the top of a pillar, standing on level ground, is found to subtend the greatest angle* α *at the eye of an observer when his distance from the pillar is* $c\ feet$ *; prove that the height of the statue is* $2c\tan\alpha$ *feet, and find the height of the pillar.* ♢

§§ Solution. Let P and Q be the top and the foot of the pillar respectively and let PR be the statue.

If D be the point at which the greatest angle is subtended by PR, then D must be the point where a circle drawn through P and R touches the ground.

Also, O the center of this circle (to which QD is a tangent) is in the vertical line through D.

Draw OM perpendicular to RP, bisecting it in M. Join OR and OP.

We have $OM = DQ = c$; also the $\angle ROP = 2 \times \angle RDP = 2\alpha$, so that the $\angle ROM = \alpha = \angle MOP$.

$$\therefore PR = PM + MR = 2.OM \tan\alpha = 2c\tan\alpha.$$

Also, $\quad \because OD = OR = c\sec\alpha$ and $PM = c\tan\alpha$

$$\therefore PQ = c(\sec\alpha - \tan\alpha) = c\left(\frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha}\right) = c(1-\sin\alpha)\sec\alpha.$$

Otherwise thus :

We may obtain this last result thus :

By *Euclid III*. 36,

$$QP.QR = QD^2$$
$$\therefore QP(QP + PR) = QD^2$$
$$\therefore QP^2 + QP.2c\tan\alpha = c^2$$
$$\therefore QP^2 + QP.2c\tan\alpha + c^2\tan^2\alpha = c^2\left(1+\tan^2\alpha\right) = c^2\sec^2\alpha$$
$$\therefore QP = c(\sec\alpha - \tan\alpha) = c(1-\sin\alpha)\sec\alpha.$$

■

§ Problem 14.2.22. *A tower stood at the foot of an inclined plane whose inclination to the horizon was* $9°$*. A line* 100 *feet in length was measured straight up the incline from the foot of the tower, and at the end of this line the tower subtended an angle of* $54°$*. Find the height of the tower, having given*

$$\log 2 = .30103, \qquad \log 114.4123 = 2.0584726,$$

and $\quad L\sin 54° = 9.9079576.$ ◊

§§ Solution. Let AB (h feet, say) be the tower at the foot of the inclined plane AC.

$$\angle BAC = 90° - 9° = 81°.$$

Hence, if AC be the line 100 feet in length, so that

$$\angle BCA = 54°$$
$$\therefore \angle ABC = 180° - (\angle BAC + \angle BCA) = 180° - (81° + 54°) = 45°.$$

We then have

$$\frac{h}{\sin 54°} = \frac{AC}{\sin 45°}$$
$$\therefore h = 100\sqrt{2}\sin 54°$$
$$\therefore \log h = \log 100 + \frac{1}{2}\log 2 + L\sin 54° - 10$$
$$= 2 + .150515 + 9.9079576 - 10$$
$$= 2.0584726 = \log 114.4123.$$

Hence the required height $= 114.4123$ $feet$. ■

§ Problem 14.2.23. *A vertical tower stands on a declivity which is inclined at* $15°$ *to the horizon. From the foot of the tower a man ascends the declivity for* 80 *feet, and then finds that the tower subtends an angle of* $30°$*. Prove that the height of the tower is*

$$40(\sqrt{6} - \sqrt{2})\ feet.$$ ◊

§§ Solution. Let AB (h feet, say) be the tower and AC be the distance 80 feet.

We have $\quad \angle BAC = 90° - 15° = 75°$ and $\angle BCA = 30°$

Also, $\quad \angle ABC = 180° - (75° + 30°) = 75°.$

We then have
$$\frac{h}{\sin 30^\circ} = \frac{80}{\sin 75^\circ}$$
$$\therefore 2h = 80 \div \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{160\sqrt{2}}{\sqrt{3}+1}$$
$$\therefore h = \frac{80\sqrt{2}(\sqrt{3}-1)}{3-1} = 40(\sqrt{6}-\sqrt{2})\ feet. \qquad \blacksquare$$

§ **Problem 14.2.24.** *The altitude of a certain rock is 47°, and after walking towards it 1000 feet up a slope inclined at 30° to the horizon an observer finds its altitude to be 77°. Find the vertical height of the rook above the first point of observation, given that*
$$\sin 47^\circ = .73135. \qquad \lozenge$$
§§ **Solution**. Let B and C be the top and the foot of the rock respectively and A be the first point of observation, so that $\angle BAC = 47^\circ$.

Let D be the second point of observation, so that
$$AD = 1000\ feet,\ \angle DAC = 30^\circ \text{ and } \angle BDE = 77^\circ,$$
where DE is parallel to AC, meeting BC in E.

The $\angle BAD = 47^\circ - 30^\circ = 17^\circ$.

In the ΔBAC, C is a right angle and $\angle BAC = 47^\circ$
$$\therefore \angle ABC = 90^\circ - 47^\circ = 43^\circ.$$
In the ΔBDE, E is a right angle and $\angle BDE = 77^\circ$
$$\therefore \angle DBE = 90^\circ - 77^\circ = 13^\circ.$$
$$\therefore \angle ABD = 43^\circ - 13^\circ = 30^\circ, \text{ and}$$
$$\angle BDA = 180^\circ - (17^\circ + 30^\circ) = (180^\circ - 47^\circ)$$
From the ΔABD, we have
$$\frac{AB}{\sin \angle BDA} = \frac{AD}{\sin \angle ABD}$$
$$\therefore AB = \frac{AD \sin 47^\circ}{\sin 30^\circ} = 2000 \sin 47^\circ.$$
From the ΔABC, we have
$$BC = AB \sin 47^\circ = 2000 \sin^2 47^\circ = 2000 \times (.73135)^2 = 1069.745645\ ft. \blacksquare$$

§ **Problem 14.2.25.** *A man observes that when he has walked c feet up an inclined plane the angular depression of an object in a horizontal plane through the foot of the slope is α, and that, when he has walked a further distance of c feet, the depression is β. Prove that the inclination of the slope to the horizon is the angle whose cotangent is*
$$(2\cot\beta - \cot\alpha). \qquad \lozenge$$
§§ **Solution**. Let O be the object, M be the foot of the inclined plane and A and N be the two points of observation.

From A and N draw AP and NQ perpendicular to the horizontal plane which contains O and M.

Draw AR and NS parallel to MO, so that AR and NS are the horizontal lines passing through A and N.

We are given
$$\angle RAO = \alpha = \angle AOP,\ \angle SNO = \beta = \angle NOQ, \text{ and}$$
$$MA = c\ feet = AN.$$
Let the distance OM be x feet and the required $\angle NMQ$ be $\theta(= \angle AMP)$.

We then have
$$MP = PQ = c\cos\theta,\ AP = c\sin\theta \text{ and } NQ = 2c\sin\theta.$$

In the ΔOAP, we have $OP = AP \cot\alpha$,

$$\therefore x + c\cos\theta = c\sin\theta\cot\alpha \tag{14.11}$$

In the ΔONQ, we have $OQ = NQ\cot\beta$,

$$\therefore x + 2c\cos\theta = c\sin\theta\cot\beta \tag{14.12}$$

(14.13)

From (14.11) and (14.12), by subtraction, we have

$$c\cos\theta = c\sin\theta\,(2\cot\beta - \cot\alpha)\,.$$

$$\therefore \cot\theta = 2\cot\beta - \cot\alpha$$

$$\therefore \theta = \cot^{-1}(2\cot\beta - \cot\alpha)\,.$$

■

§ Problem 14.2.26. *A regular pyramid on a square base has an* 150 *feet long, and the length of the side of its base is* 200 *feet. Find the inclination of its face to the base.* ◊

§§ Solution. Let V be the vertex and F be the center of the base $ABCD$ of the pyramid.

Let E be the middle point of AB. We then have

$$AV = 150\ ft.,\ AE = 100\ ft.,\ AF = \frac{200}{\sqrt{2}} = 100\sqrt{2}\ ft.,$$

and
$$VF = \sqrt{VA^2 - FA^2} = 50\sqrt{9-8} = 50\ ft.$$

$$\therefore \tan\angle VEF = \frac{VF}{EF} = \frac{50}{100} = \frac{1}{2},$$

i.e. the required inclination is the angle whose tangent is $\frac{1}{2}$. ■

§ Problem 14.2.27. *A pyramid has for base a square of side* a *; its vertex lies on a line through the middle point of the base and perpendicular to it, and at a distance* h *from it ; prove that the angle* α *between the two lateral faces is given by the equation*

$$\sin\alpha = \frac{2h\sqrt{2a^2+4h^2}}{a^2+4h^2}.$$

◊

§§ Solution. Let V be the vertex and F be the center of the base $ABCD$ of the pyramid.

Draw BK and DK perpendicular to VA, so that $\alpha = \angle BKD$; and let the $\angle VAF = \phi$.

$$\tan\frac{\alpha}{2} = \frac{BF}{FK} = \frac{\left(\frac{a}{\sqrt{2}}\right)}{AF\sin\phi} = \frac{1}{\sin\phi} = \frac{VA}{VF}$$

$$= \frac{\sqrt{FA^2+VF^2}}{VF} = \frac{\sqrt{\frac{a^2}{2}+h^2}}{h} = \frac{\sqrt{a^2+2h^2}}{h\sqrt{2}}$$

$$\therefore \sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

$$= \frac{\sqrt{2a^2+4h^2}}{h} \div \left(1 + \frac{a^2+2h^2}{2h^2}\right)$$

$$= \frac{2h\sqrt{2a^2+4h^2}}{a^2+4h^2}.$$

■

§ **Problem 14.2.28.** *A flagstaff,* 100 *feet high, stands in the center of an equilateral triangle which is horizontal. From the top of the flagstaff each side subtends an angle of* 60° *; prove that the length of the side of the triangle is* $50\sqrt{6}\ feet$. ◊

§§ **Solution**. Let ABC be the triangle, D be its center and ED be the flagstaff.

We have $EA = EB = EC$ and the angles at E each 60°.

Therefore the triangles EBC, ECA and EAB are equilateral.

Let $x\ feet = BC = CA = AB = EA = EB = EC$. We then have

$$AD = \frac{2}{3}.x \sin 60^\circ = \frac{2}{3}.\frac{x\sqrt{3}}{2} = \frac{x}{\sqrt{3}}\ feet.$$

Also,

$$AD^2 + DE^2 = AE^2$$

$$\therefore \frac{x^2}{3} + (100)^2 = x^2$$

$$\therefore 2x^2 = 3 \times (100)^2$$

$$\therefore x = 100\sqrt{\frac{3}{2}} = 50\sqrt{6}\ feet.$$

■

§ **Problem 14.2.29.** *The extremity of the shadow of a flagstaff, which is* 6 *feet high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant* 56 *and* 8 *feet respectively from the extremities of that side. Find the sun's altitude if the height of the pyramid be* 34 *feet.* ◊

§§ **Solution**. Let $ABCD$ be the square base and O be its center.

Let E be the top of the pyramid and EF be the flag staff and let α be the sun's altitude, so that

$$\tan\alpha = \frac{FO}{GO} = \frac{6+34}{GO} = \frac{40}{GO},$$

where G is on AB, such that $GA = 8\ feet$ and $GB = 56\ feet$.

To find GO, we have AOB a right-angled isosceles triangle,

$$AB = (56+8)\ feet = 64\ feet \text{ and } AO = OB = \frac{64}{\sqrt{2}} = 32\sqrt{2}\ feet.$$

Also,

$$\angle GAO = 45^\circ.$$

$$\therefore GO^2 = OA^2 + AG^2 - 2OA.AG\cos 45^\circ$$

$$= \left(32\sqrt{2}\right)^2 + 8^2 - 2\times 32\sqrt{2}\times 8\times\frac{1}{\sqrt{2}}$$

$$= 2048 + 64 - 512 = 1600$$

$$\therefore GO = 40\ feet.$$

Hence $\tan\alpha = \frac{40}{40} = 1$, so that the sun's altitude $= 45^\circ$. ■

§ **Problem 14.2.30.** *The extremity of the shadow of a flagstaff, which is* 6 *feet high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant* x *feet and* y *feet respectively from the ends of that side; prove that the height of the pyramid is*

$$\sqrt{\frac{x^2+y^2}{2}}\tan\alpha - 6,$$

where α *is the elevation of the sun.* ◊

§§ **Solution**. Let $ABCD$ be the square base and O be its center.

Let E be the top of the pyramid, EF be flagstaff and h be the required height EO.

We have $\tan\alpha = \dfrac{FO}{GO}$, where G is on AB, such that $AG = x$ feet and $GB = y$ feet.

Thus $GO = FO\cot\alpha = (6+h)\cot\alpha$.

Also, $\angle GAO = 45°$, and

$$AO = (x+y)\cos 45° = \frac{x+y}{\sqrt{2}}.$$

$$\because GO^2 = OA^2 + AG^2 - 2OA.AG\cos 45°$$

$$\therefore (6+h)\cot\alpha = \sqrt{\frac{(x+y)^2}{2} + x^2 - \frac{2(x+y)x}{\sqrt{2}}.\frac{1}{\sqrt{2}}}$$

$$= \sqrt{\frac{x^2+2xy+y^2}{2} + x^2 - x^2 - xy}$$

$$= \sqrt{\frac{x^2+y^2}{2}}$$

$$\therefore h = \sqrt{\frac{x^2+y^2}{2}}\tan\alpha - 6.$$

■

§ **Problem 14.2.31.** *The angle of elevation of a cloud from a point h feet above a lake is α, and the angle of depression of its reflexion in the lake is β ; prove that its height is*

$$h\frac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

◊

§§ **Solution**. Let C be the cloud and CD the perpendicular upon the surface of the water.

Produce CD to L so that CD and DL are equal.

Then, by the laws of Optics, L is the image of C in the water.

Let O be the observer and draw OP perpendicular to CD.

Then $\angle POC = \alpha$ and $\angle POL = \beta$.

If x be the height of the cloud, we have $CD = DL = x$.

$$\therefore \frac{x-h}{x+h} = \frac{CP}{PD} = \frac{OP\tan\alpha}{OP\tan\beta} = \frac{\tan\alpha}{\tan\beta}$$

$$\therefore \frac{x}{h} = \frac{\tan\alpha+\tan\beta}{\tan\beta-\tan\alpha} = \frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}.$$

■

§ **Problem 14.2.32.** *The shadow of a tower is observed to be half the known height of the tower and sometime afterwards it is equal to the known height ; how much will the sun have gone down in the interval, given*

$$\log 2 = .30103,\ L\tan 63°26' = 10.3009994,$$

and $$diff.\ for\ 1' = 3159?$$

◊

§§ **Solution**. At the first observation, the elevation of the sun $= \tan^{-1} 2 = \alpha$, say, so that $\tan\alpha = 2$.

$$\therefore L\tan\alpha = 10 + \log 2 = 10.3010300.$$

Let $\alpha = 63°26' + x''$, so that $L\tan\big(63°26' + x''\big) = 10.3010300.$

The diff. for $x'' = 10.3010300 - 10.3009994 = .0000306.$

$$\therefore x = 60'' \times \frac{306}{3159} \approx 6''$$
$$\therefore \alpha = 63^\circ 26' 6''.$$

At the second observation, the elevation of the sun $= 45^\circ$.
Hence the required difference $= 63^\circ 26' 6'' - 45^\circ = 18^\circ 26' 6''$. ■

§ Problem 14.2.33. *An isosceles triangle of wood is placed in a vertical plane, vertex upwards, and faces the sun. If $2a$ be the base of the triangle, h its height, and 30° the altitude of the sun, prove that the tangent of the angle at the apex of the shadow is*

$$\frac{2ah\sqrt{3}}{3h^2 - a^2}.$$

◊

§§ Solution. Let C be the vertex of the triangle and D be the middle point of its base, so that $AD = DB = a$ and $CD = h$.

Let E be the apex of the shadow, so that

$$DE = h \cot 30^\circ = h\sqrt{3}.$$

We then have, if 2θ be the $\angle BEA$,

$$\tan \angle BED = \tan\theta = \frac{BD}{DE} = \frac{a}{h\sqrt{3}}$$
$$\therefore \tan \angle BEA = \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$= \frac{2a}{h\sqrt{3}} \div \left(1 - \frac{a^2}{3h^2}\right)$$
$$= \frac{2ah\sqrt{3}}{3h^2 - a^2}.$$

■

§ Problem 14.2.34. *A rectangular target faces due south, being vertical and standing on a horizontal plane. Compare the area of the target with that of its shadow on the ground when the sun is β° from the south at an altitude of α°.* ◊

§§ Solution. Let $ACBD$ be the target and $ACB'D'$ be its shadow.

$$\angle DD'A = \text{ the sun's altitude } = \alpha \text{ and } \angle D'AC = 90^\circ - \beta.$$

We then have

$$\frac{\text{area of target}}{\text{area of shadow}} = \frac{AC.AD}{AC.AD' \sin(90^\circ - \beta)} = \frac{AC.AD}{AC.AD' \cos\beta}$$
$$= \frac{AC.AD}{AC.AD \cot\alpha \cos\beta} = \frac{1}{\cot\alpha\cos\beta} = \tan\alpha\sec\beta$$
$$\therefore \text{ area of target : area of shadow} = \tan\alpha\sec\beta : 1.$$

■

§ Problem 14.2.35. *A spherical ball, of diameter δ, subtends an angle α at a man's eye when the elevation of its center is β ; prove that the height of the center of the ball is*

$$\frac{1}{2}\delta \sin\beta \operatorname{cosec} \frac{\alpha}{2}.$$

◊

§§ Solution. Let O be the man's eye and A be the center of the ball.

The angle subtended by $\frac{\delta}{2}$ at $O = \frac{\alpha}{2}$, and the angle subtended by h, the height of the center of the ball, at $A = \beta$.

$$\therefore h = OA \sin\beta = \frac{\delta}{2} \operatorname{cosec} \frac{\alpha}{2} \sin\beta.$$

■

§ Problem 14.2.36. *A man standing on a plane observes a row of equal and equidistant pillars, the 10^{th} and 17^{th} of which subtend the same angle that they would do if they were in the position of the first and were respectively $\frac{1}{2}$ and $\frac{1}{3}$ of their height. Prove that, neglecting the height of the man's eye, the line of pillars is inclined to the line drawn from his eye to the first at an angle whose secant is nearly* 2.6. ◊

§§ Solution. Let A, B and C be the positions of the feet of the first, the tenth and the seventeenth pillars respectively, and let d be the distance between any two consecutive pillars, so that

$$AB = 9d \text{ and } AC = 16d.$$

Let O be the position of the observer, h be the height of a pillar and α and β be the angles subtended by the pillars at B and C respectively. We then have

$$OB = h\cot\alpha,\ OA = \frac{h}{2}\cot\alpha,\ OC = h\cot\beta \text{ and } OA = \frac{h}{3}\cot\beta$$

$$\therefore OB = 2.OA = 2a, \text{ say, and } OC = 3.OA = 3a, \text{ say.}$$

Let θ be the angle OA makes with CBA produced. We have

$$\text{in the } \Delta OAB,\ (2a)^2 = a^2 + (9d)^2 + 2a.9d\cos\theta,$$

$$\text{in the } \Delta OAC,\ (3a)^2 = a^2 + (16d)^2 + 2a.16d\cos\theta$$

$$\therefore a^2 = 27d^2 + 6ad\cos\theta \tag{14.14}$$

and

$$a^2 = 32d^2 + 4ad\cos\theta \tag{14.15}$$

From (14.14) and (14.15), by subtraction, we have

$$0 = -5d^2 + 2ad\cos\theta, \text{ so that } d = \frac{2a\cos\theta}{5}.$$

Substituting this value of d in (14.15), we have

$$a^2 = \frac{128a^2\cos^2\theta}{25} + \frac{8a^2\cos^2\theta}{5}$$

$$\therefore \cos^2\theta = \frac{25}{168}$$

$$\therefore \sec^2\theta = \frac{168}{25}$$

$$\therefore \sec\theta = \frac{2\sqrt{42}}{5} = \frac{4\times 6.4809}{10} \approx 2.6.$$ ■

For the following four examples a book of tables will be required.

§ Problem 14.2.37. *A and B are two points, which are on the banks of a river and opposite to one another, and between them is the mast, PN, of a ship ; the breadth of the river is 1000 feet, and the angular elevation of P at A is $14°20'$ and at B it is $8°10'$. What is the height of P above AB ?* ◊

§§ Solution. Let h be the required height of PN.

Then $$\frac{AN}{h} = \cot 14°20',\ \therefore AN = h\cot 14°20',$$

and $$\frac{BN}{h} = \cot 8°10',\ \therefore BN = h\cot 8°10'.$$

Hence, by addition, we have

$$AN + BN = AB = 1000 = h\left(\cot 14°20' + \cot 8°10'\right)$$

$$\begin{aligned}\therefore h &= \frac{1000}{\cot 14^\circ 20' + \cot 8^\circ 10'} = \frac{1000}{3.9136420 + 6.9682335} \\ &= \frac{1000}{10.8818755} \approx 91.896\ feet\end{aligned}$$

Otherwise thus :

We have

$$\angle APB = 180^\circ - \left(14^\circ 20' + 8^\circ 10'\right) = 157^\circ 30'.$$

Now $\qquad \dfrac{PB}{AB} = \dfrac{\sin \angle PAB}{\sin \angle APB}$ and $\dfrac{PN}{PB} = \sin \angle PBN.$

Hence, by multiplication, we have

$$\frac{PN}{AB} = \frac{\sin \angle PAB \sin \angle PBN}{\sin \angle APB}$$

$$\begin{aligned}\therefore h &= 1000 \sin 14^\circ 20' \sin 8^\circ 10' \operatorname{cosec} 22^\circ 30' \\ \therefore \log h &= \log 1000 + L \sin 14^\circ 20' + L \sin 8^\circ 10' + L \operatorname{cosec} 22^\circ 30' - 30 \\ &= 3 + 9.3936852 + 9.1524507 + 10.4171603 - 30 = 1.9632962. \\ \therefore h &\approx 91.896\ feet.\end{aligned}$$

■

§ Problem 14.2.38. *AB is a line 1000 yards long; B is due north of A and from B a distant point P bears 70° east of north ; at A it bears $41^\circ 22'$ east of north ; find the distance from A to P.* ◊

§§ Solution. We have

$$\frac{AP}{AB} = \frac{\sin \angle ABP}{\sin \angle APB} = \frac{\sin (180^\circ - 70^\circ)}{\sin (70^\circ - 41^\circ 22')} = \frac{\sin 70^\circ}{\sin 28^\circ 38'}$$

$$\therefore AP = \frac{1000 \sin 70^\circ}{\sin 28^\circ 38'}$$

$$\begin{aligned}\therefore \log AP &= \log 1000 + L \sin 70^\circ - L \sin 28^\circ 38' \\ &= 3 + 9.9729858 - 9.6805191 = 3.2924667.\end{aligned}$$

Now $\qquad \log 1960.9 = 3.2924554,\ diff.\ for\ .1 = .0000222.$

Let $\qquad \log(1960.9 + x) = 3.2924667.$

The diff. for $\quad x = 3.2924667 - 3.2924554 = .0000113.$

$$\therefore x = \frac{113}{222} \times .1 = \frac{11.3}{222} \approx .05$$

$$\therefore \log AP = \log(1960.9 + .05) = \log 1960.95$$

$$\therefore AP = 1960.95\ yards.$$

■

§ Problem 14.2.39. *A is a station exactly 10 miles west of B, The bearing of a particular rock from A is $74^\circ 19'$ east of north, and its bearing from B is $20^\circ 51'$ west of north. How far is it north of the line AB ?* ◊

§§ Solution. Let R denote the rock and let RQ be drawn perpendicular to AB.

We have

$$\angle RAQ = 90^\circ - 74^\circ 19' = 15^\circ 41',\ \angle RBA = 90^\circ - 26^\circ 51' = 63^\circ 9'$$

$$\angle ARB = 180^\circ - \left(15^\circ 41' + 63^\circ 9'\right) = 180^\circ - 78^\circ 50' = 101^\circ 10'.$$

Now $\qquad \dfrac{RQ}{AR} = \sin \angle RAQ = \sin 15^\circ 41'$

and $\qquad \dfrac{AR}{AB} = \dfrac{\sin \angle ABR}{\sin \angle ARB} = \dfrac{\sin 63^\circ 9'}{\sin 101^\circ 10'} = \dfrac{\sin 63^\circ 9'}{\sin 78^\circ 50'}.$

Hence by multiplication, we have

$$\frac{RQ}{AB} = \frac{\sin 15°41' \sin 63°9'}{\sin 78°50'}$$

$$\therefore RQ = \frac{10 \sin 15°41' \sin 63°9'}{\sin 78°50'}$$

$$\log RQ = \log 10 + L \sin 15°41' + L \sin 63°9' - 10 - L \sin 78°50'$$
$$= 1 + 9.4318788 + 9.9504583 - 10 - 9.9916991 = .3906380.$$

Now $\quad \log 2.4583 = .3906349,\ diff.\ for\ .0001 = .0000176.$

Let $\quad \log(2.4583 + x) = .3906380.$

The diff. for $\quad x = .3906380 - .3906349 = .0000031.$

$$\therefore x = \frac{31}{176} \times .0001 = \frac{.0031}{176} \approx .00002,$$

Hence, the required distance $= 2.45832\ miles$.

Otherwise thus :

We have

$$AQ = RQ \cot \angle RAQ = RQ \cot 15°41' \text{ and}$$
$$BQ = RQ \cot \angle RBQ = RQ \cot 63°9'.$$

Hence, by addition,

$$AQ + BQ = RQ \left(\cot 15°41' + \cot 63°9'\right)$$

and, by the table of natural cotangents, we have

$$10 = RQ(3.5615900 + .5062322) = RQ(4.0678222)$$

$$\therefore RQ = \frac{10}{4.0678222} = 2.45832\ miles.$$

Notes :

$$\because \cot A + \cot B = \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B}$$
$$= \frac{\sin B \cos A + \cos B \sin A}{\sin A \sin B} = \frac{\sin(B + A)}{\sin A \sin B},$$

We have $RQ = \dfrac{10 \sin 15°41' \sin 63°9'}{\sin 78°50'}$, as in the first solution. ■

§ Problem 14.2.40. *The summit of a spire is vertically over the middle point of a horizontal square enclosure whose side is of length a feet ; the height of the spire is h feet above the level of the square. If the shadow of the spire just reach a corner of the square when the sun has an altitude θ, prove that*

$$h\sqrt{2} = a \tan \theta.$$

Calculate h, having given $a = 1000\ feet$ and $\theta = 25°15'$. ◊

§§ Solution. We have

$$\frac{h}{a \cos 45°} = \tan \theta$$

$$\therefore h = a \tan \theta \cos 45° = \frac{a \tan \theta}{\sqrt{2}}$$

$$\therefore h\sqrt{2} = a \tan \theta.$$

If $a = 1000\ feet$, and $\theta = 25°15'$, we have

$$\log h + \frac{1}{2} \log 2 = \log a + L \tan \theta - 10.$$

$$\therefore \log h = \log 1000 + L \tan 25°15' - 10 - \frac{1}{2} \log 2$$

$$= 3 + 9.6736020 - 10 - .1505150 = 2.5230870.$$

Now $\log 333.49 = 2.5230828,\ diff.\ for\ .01 = .0000130.$

Let $\log(333.49 + x) = 2.5230870.$

The diff. for $x = 2.5230870 - 2.5230828 = .0000042.$

$$\therefore x = \frac{42}{130} \times .01 = \frac{.42}{130} \approx .0032$$

$$\therefore h = 333.4932\ feet.$$

■

Chapter 15

Properties of A Triangle

15.1 Area of a Given Triangle

Find the area of the triangle ABC **when**

§ Problem 15.1.1. *a = 13, b = 14 and c = 15.* ◊

§§ Solution. $s = \dfrac{13+14+15}{2} = 21,\ s-a = 8,\ s-b = 7$ and $s-c = 6$

$$\therefore \Delta = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{42 \times 4 \times 42} = 42 \times 2 = 84.$$ ■

§ Problem 15.1.2. *a = 18, b = 24 and c = 30.* ◊

§§ Solution. $s = \dfrac{18+24+30}{2} = 36,\ s-a = 18,\ s-b = 12$ and $s-c = 6$

$$\therefore \Delta = \sqrt{36 \times 18 \times 12 \times 6} = \sqrt{36 \times 36 \times 36} = 6 \times 6 \times 6 = 216.$$ ■

§ Problem 15.1.3. *a = 25, b = 52 and c = 63.* ◊

§§ Solution. $s = \dfrac{25+52+63}{2} = 70,\ s-a = 45,\ s-b = 18$ and $s-c = 7$

$$\begin{aligned}\therefore \Delta &= \sqrt{70 \times 45 \times 18 \times 7} = \sqrt{7 \times 2 \times 5 \times 45 \times 18 \times 7}\\ &= \sqrt{49 \times 36 \times 225} = 7 \times 6 \times 15 = 630.\end{aligned}$$ ■

§ Problem 15.1.4. *a = 125, b = 123 and c = 62.* ◊

§§ Solution. $s = \dfrac{125+123+62}{2} = 155,\ s-a = 30,\ s-b = 32$ and $s-c = 93$

$$\begin{aligned}\therefore \Delta &= \sqrt{155 \times 30 \times 32 \times 93} = \sqrt{31 \times 5 \times 15 \times 2 \times 32 \times 31 \times 3}\\ &= \sqrt{31 \times 31 \times 15 \times 15 \times 64} = 31 \times 15 \times 8 = 3720.\end{aligned}$$ ■

§ Problem 15.1.5. *a = 15, b = 36 and c = 39.* ◊

§§ Solution. $s = \dfrac{15+36+39}{2} = 45,\ s-a = 30,\ s-b = 9$ and $s-c = 6$

$$\therefore \Delta = \sqrt{45 \times 30 \times 9 \times 6} = \sqrt{270 \times 270} = 270. \qquad \blacksquare$$

§ Problem 15.1.6. *a = 287, b = 816 and c = 865.* ◊

§§ Solution. $s = \dfrac{287+816+865}{2} = 984,\ s{-}a = 697,\ s{-}b = 168$ and $s-c = 119$

$$\begin{aligned}\therefore \Delta &= \sqrt{984 \times 697 \times 168 \times 119}\\ &= \sqrt{24 \times 41 \times 17 \times 41 \times 24 \times 7 \times 17 \times 7}\\ &= 24 \times 41 \times 17 \times 7 = 117096.\end{aligned} \qquad \blacksquare$$

§ Problem 15.1.7. *a = 35, b = 84 and c = 91.* ◊

§§ Solution. $s = \dfrac{35+84+91}{2} = 105,\ s-a = 70,\ s-b = 21$ and $s-c = 14$

$$\begin{aligned}\therefore \Delta &= \sqrt{105 \times 70 \times 21 \times 14}\\ &= \sqrt{15 \times 7 \times 7 \times 10 \times 7 \times 3 \times 7 \times 2}\\ &= \sqrt{7 \times 7 \times 7 \times 7 \times 900} = 7 \times 7 \times 30 = 1470.\end{aligned} \qquad \blacksquare$$

§ Problem 15.1.8. $a = \sqrt{3},\ b = \sqrt{2}$ *and* $c = \dfrac{\sqrt{6}+\sqrt{2}}{2}$. ◊

§§ Solution. $\therefore 2s = a+b+c = \dfrac{2\sqrt{3}+3\sqrt{2}+\sqrt{6}}{2}$

$$\therefore s = \frac{2\sqrt{3}+3\sqrt{2}+\sqrt{6}}{4}$$

$$s-a = \frac{3\sqrt{2}+\sqrt{6}-2\sqrt{3}}{4}$$

$$s-b = \frac{2\sqrt{3}-\sqrt{2}+\sqrt{6}}{4} \text{ and}$$

$$s-c = \frac{2\sqrt{3}+\sqrt{2}-\sqrt{6}}{4}$$

$$\begin{aligned}\therefore \Delta &= \frac{1}{16}\sqrt{\left[\left(3\sqrt{2}+\sqrt{6}\right)+2\sqrt{3}\right]\left[\left(3\sqrt{2}+\sqrt{6}\right)-2\sqrt{3}\right]} \times\\ &\quad \sqrt{\left[2\sqrt{3}+\left(\sqrt{6}-\sqrt{2}\right)\right]\left[2\sqrt{3}-\left(\sqrt{6}-\sqrt{2}\right)\right]}\\ &= \frac{1}{16}\sqrt{\left(24+12\sqrt{3}-12\right)\left(12-8+4\sqrt{3}\right)}\\ &= \frac{1}{16}\sqrt{4(3+3\sqrt{3}).4(1+\sqrt{3})} = \frac{1}{4}\sqrt{(1+\sqrt{3}3.(1+\sqrt{3})}\\ &= \frac{1}{4}(1+\sqrt{3})\sqrt{3} = \frac{1}{4}(3+\sqrt{3}) = \frac{1}{4} \times 4.732\ldots = 1.183\ldots.\end{aligned} \qquad \blacksquare$$

§ Problem 15.1.9. *If* $B = 45°$, $C = 60°$ *and* $a = 2(\sqrt{3}+1)$ *inches, prove that the area of the triangle is* $6+2\sqrt{3}$ *sq. inches.* ◊

§§ Solution. $B = 45°$, $C = 60°$, $a = 2(\sqrt{3}+1)$ ins., $A = 180° - (B + C) = 75°$

$$\begin{aligned}\therefore \Delta &= \frac{1}{2}ab\sin C = \frac{1}{2}a.\frac{a\sin B}{\sin A}.\sin C\\ &= \frac{a^2}{2}.\frac{\sin B \sin C}{\sin A}\end{aligned}$$

$$= 2(4+2\sqrt{3})\frac{\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}$$
$$= 2(\sqrt{3}+1)\sqrt{3} = (6+2\sqrt{3})\ sq.\ ins.$$ ■

§ Problem 15.1.10. *The sides of a triangle are* 119*,* 111*, and* 92 *yards; prove that its area is* 10 *sq. yards less than an acre.* ◊

§§ Solution. $s = \dfrac{119+111+92}{2} = 161$ yards, $s-a = 42$ yards, $s-b = 50$ yards, and $s-c = 69$ yards

$$\therefore \Delta = \sqrt{161\times 42\times 50\times 69}\ sq.\ yds.$$
$$= \sqrt{23\times 7\times 14\times 3\times 25\times 2\times 23\times 3}\ sq.\ yds.$$
$$= \sqrt{23\times 23\times 14\times 14\times 3\times 3\times 25}\ sq.\ yds.$$
$$= (23\times 14\times 3\times 5)\ sq.\ yds. = 4830\ sq.\ yds.$$
$$= 10\ sq.\ yds.\text{ less than an acre.}$$ ■

§ Problem 15.1.11. *The sides of a triangular field are* 242*,* 1212 *and* 1450 *yards; prove that the area of the field is* 6 *acres.* ◊

§§ Solution. $s = \dfrac{242+1212+1450}{2} = 1452$ yards

$s-a = 1210$ yards, $s-b = 240$ yards and $s-c = 2$ yards

$$\therefore \Delta = \sqrt{1452\times 1210\times 240\times 2}\ sq.\ yds.$$
$$= \sqrt{11\times 11\times 12\times 11\times 11\times 10\times 8\times 3\times 10\times 2}\ sq.\ yds.$$
$$= \sqrt{11\times 11\times 11\times 11\times 36\times 10\times 10\times 16}\ sq.\ yds.$$
$$= (11\times 11\times 6\times 10\times 4)\ sq.\ yds.$$
$$= (4840\times 6)\ sq.\ yds.$$
$$= 6\ acres.$$ ■

§ Problem 15.1.12. *A workman is told to make a triangular enclosure of sides* 50*,* 41 *and* 21 *yards respectively ; having made the first side one yard too long, what length must he make the other two sides in order to enclose the prescribed area with the prescribed length of fencing ?* ◊

§§ Solution. We have $s = \dfrac{50+41+21}{2} = 56$ yards.

Also,
$$51+b+c = 112$$
$$\therefore b+c = 61\ yards \tag{15.1}$$
$$\sqrt{s(s-51)(s-b)(s-c)} = \sqrt{s\times 6\times 15\times 35}$$
$$\therefore 5\left[s^2-(b+c)s+bc\right] = 6\times 15\times 35$$
$$\therefore s^2-(b+c)s+bc = 6\times 3\times 35$$
$$\therefore s(s-61)+bc = 6\times 3\times 35$$
$$\therefore 56(-5)+bc = 630$$
$$\therefore bc = 630+280 = 910$$
$$\therefore (b-c)^2 = (b+c)^2-4bc = 3721-3640 = 81.$$
$$\therefore b-c = 9\ yards \tag{15.2}$$

From (15.1) and (15.2), we have $b = 35$ yds. and $c = 26$ yds. ■

§ **Problem 15.1.13.** *Find, correct to* $.0001$ *of an inch, the length of one of the equal sides of an isosceles triangle on a base of* 14 *inches having the same area as a triangle whose sides are* 13.6, 15 *and* 15.4 *inches.* ♢

§§ **Solution**. If a be the length of one of the equal sides of the isosceles triangle, we have

$$s = \frac{1}{2}(13.6 + 15 + 15.4) = 22\ ins., \text{ and}$$
$$s' = \frac{1}{2}(a + a + 14) = a + 7.$$

Equating areas, we have

$$\sqrt{(a+7) \times 7 \times 7 \times (a-7)} = \sqrt{22 \times 8.4 \times 7 \times 6.6}$$
$$\therefore a^2 - 7 = 22 \times 1.2 \times 6.6$$
$$\therefore a^2 = 49 + 174.24 = 223.24$$
$$\therefore a = 14.941 \ldots ins.$$

■

§ **Problem 15.1.14.** *Prove that the area of a triangle is*

$$\frac{1}{2}a^2\frac{\sin B \sin C}{\sin A}.$$

If one angle of a triangle be $60°$, *the area* $10\sqrt{3}$ *square feet, and the perimeter* 20 *feet, find the lengths of the sides.* ♢

§§ **Solution**.

$$\begin{aligned}\Delta &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}a.\frac{a\sin B}{\sin A}.\sin C \\ &= \frac{a^2}{2}.\frac{\sin B \sin C}{\sin A}.\end{aligned}$$

Given $A = 60°$, $\Delta = 10\sqrt{3}\ sq.\ ft.$ and $a + b + c = 20\ ft.$, we have

$$\frac{1}{2}bc\sin A = 10\sqrt{3}$$
$$\therefore \frac{1}{2}bc.\frac{\sqrt{3}}{2} = 10\sqrt{3}$$
$$\therefore bc = 40 \tag{15.3}$$

Also,

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc\cos 60° \\ &= b^2 + c^2 - bc = (b+c)^2 - 3bc \\ &= (20-a)^2 - 3 \times 40 = 280 - 40a + a^2\end{aligned}$$
$$\therefore 40a = 280,\ \therefore a = 7\ feet.$$
$$\therefore b + c = 20 - 7 = 13\ feet \tag{15.4}$$

From (15.3) and (15.4), we have $b = 8\ feet$ and $c = 5\ feet$. ■

§ **Problem 15.1.15.** *The sides of a triangle are in* $A.\ P.$ *and its area is* $\frac{3}{5}th$ *of an equilateral triangle of the same perimeter ; prove that its sides are in the ratio* $3 : 5 : 7$, *and find the greatest angle of the triangle.* ♢

§§ **Solution**. Let $x - y$, x and $x + y$ denote the sides of the triangle. Then

$$2s = x - y + x + x + y = 3x,\ \therefore s = \frac{3x}{2},$$

and $$\Delta = \sqrt{\frac{3x}{2}\left(\frac{x}{2}+y\right)x\left(\frac{x}{2}-y\right)} = \frac{x}{4}\sqrt{3\left(x^2-4y^2\right)}.$$

Also, the area of an equilateral triangle of the same perimeter, i.e. with side x is

$$= \frac{1}{2}x^2\sin 60^\circ = \frac{\sqrt{3}.x^2}{4}.$$

$$\therefore \frac{x}{4}\sqrt{3\left(x^2-4y^2\right)} = \frac{3}{5}.\frac{\sqrt{3}.x^2}{4}$$

$$\therefore 25\left(x^2-4y^2\right) = 9x^2$$

$$\therefore 16x^2 = 100y^2$$

$$\therefore x = \frac{5y}{2}.$$

Thus the sides are in the ratio

$$\frac{3y}{2},\ \frac{5y}{2},\ \frac{7y}{2},\ i.e.\ 3:5:7.$$

Also, if θ be the greatest angle, we have

$$\cos\theta = \frac{3^2+5^2-7^2}{2.3.5} = \frac{9+25-49}{30} = -\frac{15}{30} = -\frac{1}{2}$$

$$\therefore \theta = 120^\circ.$$

■

§ Problem 15.1.16. *In a triangle the least angle is 45° and the tangents of the angles are in A. P. If its area be 3 square yards, prove that the lengths of the sides are $3\sqrt{5}$, $6\sqrt{2}$ and 9 feet, and that the tangents of the other angles are respectively 2 and 3.* ◊

§§ Solution. In the triangle ABC, let the $\angle = 45^\circ$ and let

$$\tan A = 1,\ \tan B = 1+d,\ \tan C = 1+2d.$$

We have

$$B+C = 180^\circ - A = 135^\circ,\ \therefore C = 135^\circ - B$$

$$\therefore \tan C = \frac{-1-\tan B}{1-\tan B}$$

$$\therefore 1+2d = \frac{-1-1-d}{1-1-d} = \frac{2+d}{d}$$

$$\therefore d+2d^2 = 2+d,\ \therefore 2d^2 = 2,\ \therefore d = 1$$

$$\therefore \tan B = 2 \text{ and } \tan C = 3$$

$$\therefore \sin A = \frac{1}{\sqrt{2}},\ \sin B = \frac{2}{\sqrt{5}} \text{ and } \sin C = \frac{3}{\sqrt{10}}.$$

$$\because \Delta = \frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}$$

$$\therefore a^2 = \frac{2\times 3\times 9\times\dfrac{1}{\sqrt{2}}}{\dfrac{2}{\sqrt{5}}\times\dfrac{3}{\sqrt{10}}} = 9\times 5;\ \therefore a = 3\sqrt{5}\ feet.$$

Also, $$b = \frac{a\sin B}{\sin A} = 3\sqrt{5}\times\frac{2}{\sqrt{5}}\times\sqrt{2} = 6\sqrt{2}\ feet$$

and $$c = \frac{a\sin C}{\sin A} = 3\sqrt{5}\times\frac{3}{\sqrt{10}}\times\sqrt{2} = 9\ feet.$$

■

§ Problem 15.1.17. *The lengths of two sides of a triangle are one foot and $\sqrt{2}$ feet respectively, and the angle opposite side the shorter* *side is* 30° *; prove that there are two triangles satisfying these conditions, find their angles, and show that their areas are in the ratio*

$$\sqrt{3}+1 : \sqrt{3}-1. \qquad \Diamond$$

§§ Solution. Given $a = 1\ ft.$, $b = \sqrt{2}\ ft.$, $A = 30^\circ$, we have

$$\sin B = \frac{b \sin A}{a} = \frac{\sqrt{2} \times \frac{1}{2}}{1} = \frac{1}{\sqrt{2}}$$

$$\therefore B_1 = 45^\circ,\ C_1 = 105^\circ;\ B_2 = 135^\circ,\ C_2 = 15^\circ.$$

$$\therefore \frac{\Delta_1}{\Delta_2} = \frac{\frac{1}{2} ab \sin C_1}{\frac{1}{2} ab \sin C_2} = \frac{\sin C_1}{\sin C_2} = \frac{\sin 105^\circ}{\sin 15^\circ} = \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

■

§ Problem 15.1.18. *Find by the aid of the tables the area of the larger of the two triangles given by the data*

$$A = 31^\circ 15',\ a = 5\ ins. \text{ and } b = 7\ ins. \qquad \Diamond$$

§§ Solution. We have

$$\sin B = \frac{b}{a} \sin A = \frac{7}{5} \sin A = \frac{14}{10} \sin 13^\circ 15'$$

$$\therefore L \sin B = \log 14 + L \sin 31^\circ 15' - \log 10$$

$$= 1.1461280 + 9.7149776 - 1 = 9.8611056.$$

Now $\quad L \sin 46^\circ 34' = 9.8610412,\ diff.\ for\ 1' = 1196.$

Let $\quad B = 46^\circ 34' + x''$

$$\therefore L \sin \left(46^\circ 34' + x''\right) = 9.8611056.$$

The diff.for $\quad x'' = 9.8611056 - 9.8610412 = .0000644.$

$$\therefore x = 60'' \times \frac{644}{1196} \approx 32'',\ \therefore B = 46^\circ 34' 32''$$

and $\quad C = 180^\circ - (A + B) = 180^\circ - 77^\circ 49' 32'' = 102^\circ 10' 28''.$

$$\therefore \text{ the required area } = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 5 \times 7 \times \sin 77^\circ 49' 32'' = \frac{35}{2} \sin 77^\circ 49' 32''$$

$$\therefore \log(area) = \log 35 + L \sin 77^\circ 49' 32'' - 10 - \log 2.$$

Now $\quad L \sin 77^\circ 49' 32'' = 9.9901067 + \frac{32}{60} \times .0000272 = 9.9901212$

$$\therefore \log(area) = 1.5440680 + 9.9901212 - 10 - .3010300 = 1.2331592.$$

Now $\quad \log 17.106 = 1.2331485,\ diff.\ for\ .001 = .0000254.$

Let $\quad \log(17.106 + x) = 1.2331592.$

The diff. for $\quad x = 1.2331592 - 1.2331485 = .0000107.$

$$\therefore x = \frac{107}{254} \times .001 = \frac{.107}{254} \approx .0004.$$

$$\therefore \log(area) = \log(17.106 + .0004)$$

$$\therefore \text{ the required area } = 17.1064\ sq.\ ins.$$

■

15.2 The Circles Connected With A Triangle

§ Problem 15.2.1. *In a triangle whose sides are* 18, 24, *and* 30 *inches respectively, prove that the circumradius, the inradius and the radii* *of the three escribed circles are respectively* 15*,* 6*,* 12*,* 18 *and* 36 *inches.* ◊

§§ Solution.

$$s = \frac{18+24+30}{2} = 36\ ins.,\ s-a = 18\ ins.,$$
$$s-b = 12\ ins.,\ \text{and } s-c = 6\ ins.$$
$$S = \sqrt{36 \times 18 \times 12 \times 6} = \sqrt{36 \times 36 \times 36} = 6 \times 6 \times 6 = 216\ sq.\ ins.$$
$$\therefore R = \frac{18 \times 24 \times 30}{4 \times 6 \times 6 \times 6}\ ins = 15\ ins.$$
$$r = \frac{6 \times 6 \times 6}{36}\ ins. = 6\ ins.$$
$$r_1 = \frac{6 \times 6 \times 6}{18}\ ins. = 12\ ins.$$
$$r_2 = \frac{6 \times 6 \times 6}{12}\ ins. = 18\ ins.$$
$$r_3 = \frac{6 \times 6 \times 6}{6}\ ins. = 36\ ins.$$

■

§ Problem 15.2.2. *The sides of a triangle are* 13*,* 14 *and* 15 *feet ; prove that*

(1) $R = 8\frac{1}{8}\ ft.$

(2) $r = 4\ ft.$

(3) $r_1 = 10\frac{1}{2}\ ft.$

(4) $r_2 = 12\ ft.$ *and*

(5) $r_3 = 14\ ft.$

◊

§§ Solution.

$$s = \frac{13+14+15}{2} = 21\ ins.,\ s-a = 8\ ins.,$$
$$s-b = 7\ ins.,\ \text{and } s-c = 6\ ins.$$
$$S = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{42 \times 4 \times 42} = 42 \times 2 = 84\ sq.\ ft.$$

(1) $R = \frac{13 \times 14 \times 15}{4 \times 42 \times 2}\ ft. = \frac{65}{8}\ ft. = 8\frac{1}{8}\ ft.$

(2) $r = \frac{42 \times 2}{21}\ ft. = 4\ ft.$

(3) $r_1 = \frac{42 \times 2}{8}\ ft. = \frac{21}{2} = 10\frac{1}{2}\ ft.$

(4) $r_2 = \frac{42 \times 2}{7}\ ft. = 12\ ft.$

(5) $r_3 = \frac{42 \times 2}{6}\ ft. = 14\ ft.$

■

§ Problem 15.2.3. *In a* ΔABC *if* $a = 13$*,* $b = 4$ *and* $\cos C = -\frac{5}{13}$*, find* $R,\ r,\ r_1,\ r_2$ *and* r_3. ◊

§§ Solution. $a = 13,\ b = 4$ and $\cos C = -\dfrac{5}{13}$

$$\begin{aligned}\therefore c^2 &= a^2 + b^2 - 2ab\cos C\\ &= 169 + 16 + 2 \times 13 \times 4 \times \frac{5}{13} = 185 + 40 = 225\\ &\therefore c = 15\end{aligned}$$

$$\therefore s = \frac{13+4+15}{2} = 16,\ s-a = 3,\ s-b = 12 \text{ and } s-c = 1$$

$$\therefore S = \sqrt{16 \times 3 \times 12 \times 1} = \sqrt{16 \times 36} = 4 \times 6 = 24$$

$$\begin{aligned}\therefore R &= \frac{13 \times 4 \times 15}{4 \times 4 \times 6} = \frac{65}{8} = 8\frac{1}{8}\\ r &= \frac{24}{16} = \frac{3}{2} = 1\frac{1}{2}\\ r_1 &= \frac{24}{3} = 8\\ r_2 &= \frac{24}{12} = 2 \text{ and}\\ r_3 &= \frac{24}{1} = 24.\end{aligned}$$

■

§ Problem 15.2.4. *In the ambiguous case of the solution of triangles prove that the circumcircles of the two triangles are equal.* ◊

§§ Solution. Given a, b and A, we have $R = \dfrac{a}{2\sin A}$, which is the same for both triangles.

For the two triangles have the same value for a and the same value of A. ■

Prove that

§ Problem 15.2.5. $r_1(s-a) = r_2(s-b) = r_3(s-c) = rs = S.$ ◊

§§ Solution. By *Arts.* 202 and 205, we have

$$r_1 = \frac{S}{s-a},\ r_2 = \frac{S}{s-b},\ r_3 = \frac{S}{s-c} \text{ and } r = \frac{S}{s}$$

$$\therefore r_1(s-a) = S = r_2(s-b) = r_3(s-c) = rs.$$

■

§ Problem 15.2.6. $\dfrac{rr_1}{r_2r_3} = \tan^2\dfrac{A}{2}.$ ◊

§§ Solution.

$$\begin{aligned}\frac{rr_1}{r_2r_3} &= \left[\left(\frac{S}{s}.\frac{S}{s-a}\right) \div \left(\frac{S}{s-b}.\frac{S}{s-c}\right)\right]\\ &= \frac{S^2}{s(s-a)} \times \frac{(s-b)(s-c)}{S^2} = \frac{(s-b)(s-c)}{s(s-a)} = \tan^2\frac{A}{2}.\end{aligned}$$

■

§ Problem 15.2.7. $rr_1r_2r_3 = S^2.$ ◊

§§ Solution.

$$\begin{aligned}rr_1r_2r_3 &= \frac{S^4}{s(s-a)(s-b)(s-c)}\\ &= \frac{S^4}{S^2} = S^2.\end{aligned}$$

■

§ Problem 15.2.8. $r_1r_2r_3 = r^3\cot^2\dfrac{A}{2}\cot^2\dfrac{B}{2}\cot^2\dfrac{C}{2}.$ ◊

§§ Solution. We have $\dfrac{r_1}{r} = \dfrac{s}{s-a}$, $\dfrac{r_2}{r} = \dfrac{s}{s-b}$ and $\dfrac{r_3}{r} = \dfrac{s}{s-c}$.

$$\begin{aligned}
\therefore \frac{r_1 r_2 r_3}{r^3} &= \frac{s^3}{(s-a)(s-b)(s-c)} \\
&= \frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)} \cdot \frac{s(s-c)}{(s-a)(s-b)} \\
&= \cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}. \\
\therefore r_1 r_2 r_3 &= r^3 \cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}.
\end{aligned}$$ ■

§ Problem 15.2.9. $rr_1 \cot\dfrac{A}{2} = S.$ ◊

§§ Solution.

$$\begin{aligned}
\therefore rr_1 \cot\frac{A}{2} &= \frac{S}{s} \cdot \frac{S}{s-a}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \\
&= \frac{s(s-a)(s-b)(s-c)}{s(s-a)}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \\
&= \sqrt{s(s-a)(s-b)(s-c)} = S.
\end{aligned}$$ ■

§ Problem 15.2.10. $r_2r_3 + r_3r_1 + r_1r_2 = s^2.$ ◊

§§ Solution.

$$\begin{aligned}
r_1r_2 + r_2r_3 + r_3r_1 &= S^2\left(\frac{1}{s-a}\cdot\frac{1}{s-b} + \frac{1}{s-b}\cdot\frac{1}{s-c} + \frac{1}{s-c}\cdot\frac{1}{s-a}\right) \\
&= S^2\left[\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}\right] \\
&= S^2\left[\frac{3s-(a+b+c)}{(s-a)(s-b)(s-c)}\right] \\
&= s(3s-2s) = s^2.
\end{aligned}$$

Otherwise thus :

$$r_1 = s\tan\frac{A}{2},\ r_2 = s\tan\frac{B}{2} \text{ and } r_3 = s\tan\frac{C}{2}.$$

By § Problem 9.2.12, we have

$$\begin{aligned}
\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} &= 1. \\
\therefore \frac{r_1}{s}\cdot\frac{r_2}{s} + \frac{r_2}{s}\cdot\frac{r_3}{s} + \frac{r_3}{s}\cdot\frac{r_1}{s} &= 1 \\
\therefore r_1r_2 + r_2r_3 + r_3r_1 &= s^2.
\end{aligned}$$ ■

§ Problem 15.2.11. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} - \dfrac{1}{r} = 0.$ ◊

§§ Solution.

$$\begin{aligned}
\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r} &= \frac{s-a}{S} + \frac{s-b}{S} + \frac{s-c}{S} - \frac{s}{S} \\
&= \frac{1}{S}\left[3s-(a+b+c)-s\right] = \frac{1}{S}(3s-2s-s) = 0.
\end{aligned}$$ ■

§ Problem 15.2.12. $a\,(rr_1 + r_2r_3) = b\,(rr_2 + r_3r_1) = c\,(rr_3 + r_1r_2).$ ◊

§§ Solution.

$$\begin{aligned}
a\,(rr_1 + r_2r_3) &= a\left[\frac{S^2}{s(s-a)} + \frac{S^2}{(s-b)(s-c)}\right] \\
&= a[(s-b)(s-c) + s(s-a)] \\
&= a\left[2s^2 - (a+b+c)s + bc\right] \\
&= a\left(2s^2 - 2s^2 + bc\right) = abc.
\end{aligned}$$

■

§ Problem 15.2.13. $(r_1 + r_2)\tan\dfrac{C}{2} = (r_3 - r)\cot\dfrac{C}{2} = c.$ ◊

§§ Solution.

$$\begin{aligned}
(r_1 + r_2)\tan\frac{C}{2} &= \left(s\tan\frac{A}{2} + s\tan\frac{B}{2}\right)\tan\frac{C}{2} \\
&= s\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} + \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\right)\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}} \\
&= s\left(\frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}} \\
&= \frac{s\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{(a+b+c)\sin\frac{C}{2}}{2\cos\frac{A}{2}\cos\frac{B}{2}} \\
&= c\cdot\frac{\sin A + \sin B + \sin C}{\sin C}\cdot\frac{\sin\frac{C}{2}}{2\cos\frac{A}{2}\cos\frac{B}{2}} \\
&= c\cdot\frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}\cdot\frac{\sin\frac{C}{2}}{2\cos\frac{A}{2}\cos\frac{B}{2}} = c.
\end{aligned}$$

Similarly, $$(r_3 - r)\cot\frac{C}{2} = c.$$

Otherwise thus :

$$\begin{aligned}
r_1 + r_2 &= \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} + \frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}} \\
&= 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} \quad (Art.\ 207,\ Cor.) \\
&= 4R\cos\frac{C}{2}\cdot\sin\frac{A+B}{2} = 4R\cos^2\frac{C}{2}
\end{aligned}$$

$$\therefore (r_1 + r_2)\tan\frac{C}{2} = 4R\sin\frac{C}{2}\cos\frac{C}{2} = 2R\sin C = c$$

Again, $$r_3 - r = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}} - \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

$$= \frac{c}{\cos\dfrac{C}{2}} \cdot \cos\frac{A+B}{2} = \frac{c\sin\dfrac{C}{2}}{\cos\dfrac{C}{2}} = c\tan\frac{C}{2}$$

$$\therefore (r_3 - r)\cot\frac{C}{2} = c\tan\frac{C}{2}\cdot\cot\frac{C}{2} = c.$$

Otherwise thus :

$$\begin{aligned}(r_1+r_2)\tan\frac{C}{2} &= \left[\frac{S}{s-a}+\frac{S}{s-b}\right]\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\\ &= S\left[\frac{2s-(a+b)}{(s-a)(s-b)}\right]\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\\ &= \frac{Sc}{\sqrt{s(s-a)(s-b)(s-c)}} = c.\end{aligned}$$

Similarly, $$\begin{aligned}(r_3-r)\cot\frac{C}{2} &= \left[\frac{S}{s-c}-\frac{S}{s}\right]\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}\\ &= \frac{Sc}{s(s-c)}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = c.\end{aligned}$$ ■

§ Problem 15.2.14. $S = 2R^2\sin A\sin B\sin C.$ ◊

§§ Solution.

$$\begin{aligned}S &= \frac{1}{2}a\cdot b\sin C = \frac{1}{2}\left(2R\sin A\cdot 2R\sin B\cdot\sin C\right)\\ &= 2R^2\sin A\sin B\sin C.\end{aligned}$$ ■

§ Problem 15.2.15. $4R\sin A\sin B\sin C = a\cos A + b\cos B + c\cos C.$ ◊

§§ Solution.

$$\begin{aligned}4R\sin A\sin B\sin C &= R\left(\sin 2A+\sin 2B+\sin 2C\right),\ \textit{by Art. } 127,\ \textit{Ex. } 1\\ &= 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C\\ &= a\cos A + b\cos B + c\cos C,\end{aligned}$$

$\because a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C,$ *by Art.* 200. ■

§ Problem 15.2.16. $S = 4Rr\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}.$ ◊

§§ Solution.

$$\begin{aligned}S = rs &= r\cdot\frac{1}{2}(a+b+c) = r\cdot\frac{1}{2}\left(2R\sin A + 2R\sin B + 2R\sin C\right)\\ &= Rr\left(\sin A + \sin B + \sin C\right)\\ &= 4Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},\ \textit{by §Problem 9.2.4}.\end{aligned}$$

Otherwise thus :

$$\begin{aligned}4Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} &= 4\cdot\frac{abc}{4S}\cdot\frac{S}{s}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}\\ &= \sqrt{s(s-a)(s-b)(s-c)} = S.\end{aligned}$$ ■

§ Problem 15.2.17. $\dfrac{1}{r^2}+\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2} = \dfrac{a^2+b^2+c^2}{S^2}.$ ◊

§§ Solution.

$$\begin{aligned}\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} &= \frac{s^2}{S^2}+\frac{(s-a)^2}{S^2}+\frac{(s-b)^2}{S^2}+\frac{(s-c)^2}{S^2}\\ &= \frac{4s^2-2(a+b+c)s+a^2+b^2+c^2}{S^2}\\ &= \frac{4s^2-4s^2+a^2+b^2+c^2}{S^2}=\frac{a^2+b^2+c^2}{S^2}.\end{aligned}$$ ■

§ Problem 15.2.18. $r_1+r_2+r_3-r=4R.$ ◊

§§ Solution.

$$\begin{aligned}r_1+r_2+r_3-r &= S\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right]\\ &= S\left[\frac{s-b+s-a}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}\right]\\ &= S\left[\frac{2s-(a+b)}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right]\\ &= cS\left[\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right]\\ &= \frac{cS}{S^2}[s(s-c)+(s-a)(s-b)]\\ &= \frac{c}{S}\left[2s^2-(a+b+c)s+ab\right]\\ &= \frac{c}{S}\left[2s^2-2s^2+ab\right]=\frac{abc}{S}=4R.\end{aligned}$$

Otherwise thus :

$$\begin{aligned}r_1+r_2+r_3-r=4R\Big[&\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}\\ &+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\Big]\\ =4R\sin\frac{1}{2}&(A+B+C),\ by\ Art.\ 124\\ =4R\sin 90^\circ&=4R.\end{aligned}$$ ■

§ Problem 15.2.19. $(r_1-r)(r_2-r)(r_3-r)=4Rr^2.$ ◊

§§ Solution.

$$\begin{aligned}&(r_1-r)(r_2-r)(r_3-r)\\ &=\left[s\tan\frac{A}{2}-(s-a)\tan\frac{A}{2}\right]\left[s\tan\frac{B}{2}-(s-b)\tan\frac{B}{2}\right]\times\\ &\qquad\left[s\tan\frac{C}{2}-(s-c)\tan\frac{C}{2}\right]\\ &=a\tan\frac{A}{2}\cdot b\tan\frac{B}{2}\cdot c\tan\frac{C}{2}\\ &=abc\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\\ &=\frac{abc}{s^2}\sqrt{s(s-a)(s-b)(s-c)}=\frac{abc}{s^2}\cdot S=4\cdot\frac{abc}{4S}\cdot\frac{S^2}{s^2}=4Rr^2.\end{aligned}$$

Otherwise thus :

$$(r_1-r)(r_2-r)(r_3-r)=\left(\frac{S}{s-a}-\frac{S}{s}\right)\left(\frac{S}{s-b}-\frac{S}{s}\right)\left(\frac{S}{s-c}-\frac{S}{s}\right)$$

$$\begin{aligned}
&= \frac{aS}{s(s-a)} \cdot \frac{bS}{s(s-b)} \cdot \frac{cS}{s(s-c)} \\
&= \frac{abcS}{s^2} = \frac{abcr^2}{S} = 4Rr^2. \qquad \blacksquare
\end{aligned}$$

§ Problem 15.2.20. $\dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab} = \dfrac{1}{2Rr}$. $\Diamond$

§§ Solution.

$$\begin{aligned}
\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} &= \frac{c+a+b}{abc} \\
&= \frac{2s}{abc} \\
&= \frac{2S}{r} \div 4RS \\
&= \frac{1}{2Rr}. \qquad \blacksquare
\end{aligned}$$

§ Problem 15.2.21. $\dfrac{r_1}{bc} + \dfrac{r_2}{ca} + \dfrac{r_3}{ab} = \dfrac{1}{r} - \dfrac{1}{2R}$. $\Diamond$

§§ Solution.

$$\begin{aligned}
&\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{ar_1 + br_2 + cr_3}{abc} \\
&= \frac{1}{abc}\left[\frac{aS}{s-a} + \frac{bS}{s-b} + \frac{cS}{s-c}\right] \\
&= \frac{S}{abc}\left[\frac{a(s-b)(s-c) + b(s-a)(s-c) + c(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right] \\
&= \frac{s}{abcS}\left[(a+b+c)s^2 - \{a(b+c) + b(a+c) + c(a+b)\}\, s + 3abc\right] \\
&= \frac{s}{abcS}\left[2s^3 - 2s(bc+ca+ab) + 3abc\right] \\
&= \frac{s}{S} - \frac{2s\left[-s^3 + s(bc+ca+ab) - abc\right]}{abcS} \\
&= \frac{1}{r} - \frac{2s\left[-s^3 + (s-a)(s-b)(s-c) - s^3 + (a+b+c)s^2\right]}{abcS}, \\
&\left\{\because (s-a)(s-b)(s-c) = s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc\right\} \\
&= \frac{1}{r} - \frac{2s(s-a)(s-b)(s-c)}{abcS} \\
&= \frac{1}{r} - \frac{2S}{abc} \\
&= \frac{1}{r} - \frac{1}{2R}.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
\frac{r_1}{bc} &= \frac{a\cos\dfrac{B}{2}\cos\dfrac{C}{2}}{bc\cos\dfrac{A}{2}} \\
&= \frac{2R\sin A\cos\dfrac{B}{2}\cos\dfrac{C}{2}}{2R\sin B\cdot 2R\sin C\cos\dfrac{A}{2}}
\end{aligned}$$

$$= \frac{1}{4R} \cdot \frac{\sin\dfrac{A}{2}}{\sin\dfrac{B}{2}\sin\dfrac{C}{2}}.$$

$$\therefore \frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{4R}\left[\frac{\sin\dfrac{A}{2}}{\sin\dfrac{B}{2}\sin\dfrac{C}{2}} + \frac{\sin\dfrac{B}{2}}{\sin\dfrac{C}{2}\sin\dfrac{A}{2}} + \frac{\sin\dfrac{C}{2}}{\sin\dfrac{A}{2}\sin\dfrac{B}{2}}\right]$$

$$= \frac{1}{4R}\left[\frac{\sin^2\dfrac{A}{2} + \sin^2\dfrac{B}{2} + \sin^2\dfrac{C}{2}}{\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}}\right]$$

$$= \frac{1 - 2\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}}{4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}}, \text{ by } \S\textit{Problem } 9.2.10$$

$$= \frac{1}{4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}} - \frac{1}{2R}$$

$$= \frac{1}{r} - \frac{1}{2R}, \text{ by } \textit{Art}.\ 204,\ \textit{Cor}. \qquad \blacksquare$$

§ Problem 15.2.22. $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$. ◊

§§ Solution.

$$r^2 + r_1^2 + r_2^2 + r_3^2$$
$$= (r_1 + r_2 + r_3 - r)^2 + 2r(r_1 + r_2 + r_3) - 2(r_2r_3 + r_3r_1 + r_1r_2).$$

Now $r_1 + r_2 + r_3 - r = 4R$, by *§Problem 15.2.18*,

and $r_2r_3 + r_3r_1 + r_1r_2 = s^2$, by *§Problem 15.2.10*.

Also,

$$2r(r_1 + r_2 + r_3) = 2\left[\frac{S^2}{s(s-a)} + \frac{S^2}{s(s-b)} + \frac{S^2}{s(s-c)}\right]$$
$$= 2[(s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b)]$$
$$= 2\left[3s^2 - 2(a+b+c)s + bc + ca + ab\right]$$
$$= -2s^2 + 2(bc + ca + ab).$$

$$\therefore r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - 2s^2 + 2(bc + ca + ab) - 2s^2$$
$$= 16R^2 - \left[(a+b+c)^2 - 2(bc + ca + ab)\right]$$
$$= 16R^2 - a^2 - b^2 - c^2. \qquad \blacksquare$$

15.3 Orthocenter and Pedal Triangle

If I, I_1, I_2, and I_3 be respectively the centers of the incircle and the three escribed circles of a triangle ABC, prove that

§ Problem 15.3.1. $AI = r\operatorname{cosec}\dfrac{A}{2}$. ◊

§§ Solution. Taking the figure of *Art*. 202, we have

$$\frac{IE}{AI} = \sin\frac{A}{2}$$

$$\therefore AI = r\operatorname{cosec}\frac{A}{2}. \qquad \blacksquare$$

§ Problem 15.3.2. $IA \cdot IB \cdot IC = abc \tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}$. ◊

§§ Solution. With the same figure, we have

$$\begin{aligned} IA \cdot IB \cdot IC &= r \operatorname{cosec}\frac{A}{2} \cdot r \operatorname{cosec}\frac{B}{2} \cdot r \operatorname{cosec}\frac{C}{2} \\ &= \frac{a \sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\sin\frac{A}{2}} \cdot \frac{b \sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}\sin\frac{B}{2}} \cdot \frac{c \sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}\sin\frac{C}{2}} \\ &= abc \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}. \end{aligned}$$

■

§ Problem 15.3.3. $AI_1 = r_1 \operatorname{cosec}\dfrac{A}{2}$. ◊

§§ Solution. Taking the figure of *Art.* 205, we have

$$\frac{I_1E_1}{AI_1} = \sin\frac{A}{2}$$

$$\therefore AI_1 = r_1 \operatorname{cosec}\frac{A}{2}.$$

■

§ Problem 15.3.4. $II_1 = a \sec\dfrac{A}{2}$. ◊

§§ Solution.

$$\begin{aligned} II_1 = AI_1 - AI &= r_1 \operatorname{cosec}\frac{A}{2} - r \operatorname{cosec}\frac{A}{2} \\ &= \left[s \tan\frac{A}{2} - (s-a)\tan\frac{A}{2}\right] \operatorname{cosec}\frac{A}{2} \\ &= a \sec\frac{A}{2}. \end{aligned}$$

■

§ Problem 15.3.5. $I_2I_3 = a \operatorname{cosec}\dfrac{A}{2}$. ◊

§§ Solution. If E_2 be the point of contact of the circle whose center is I_2 with the side AC of the triangle ABC, we have

$$\begin{aligned} AI_2 &= AE_2 \sec I_2AE_2 = AE_2 \sec\left(90^\circ - \frac{A}{2}\right) \\ &= (s-c)\operatorname{cosec}\frac{A}{2}. \end{aligned}$$

Similarly,
$$AI_3 = (s-b)\operatorname{cosec}\frac{A}{2}$$

$$\begin{aligned} \therefore I_2I_3 &= AI_2 + AI_3 \\ &= (2s-b-c)\operatorname{cosec}\frac{A}{2} \\ &= a \operatorname{cosec}\frac{A}{2}. \end{aligned}$$

Otherwise thus :

We have

$$\frac{r_2}{AI_2} = \sin\left(90^\circ - \frac{A}{2}\right) = \cos\frac{A}{2} = \frac{r_3}{AI_3}$$

$$\begin{aligned} \therefore I_2I_3 = AI_2 + AI_3 &= (r_2 + r_3)\sec\frac{A}{2} \\ &= S\left(\frac{1}{s-b} + \frac{1}{s-c}\right)\sqrt{\frac{bc}{s(s-a)}} \end{aligned}$$

$$= \frac{aS\sqrt{bc}}{(s-b)(s-c)\sqrt{s(s-a)}}$$
$$= \frac{a\sqrt{bc}}{\sqrt{(s-b)(s-c)}} = a\,\mathrm{cosec}\,\frac{A}{2}. \qquad \blacksquare$$

§ Problem 15.3.6. $II_1 \cdot II_2 \cdot II_3 = 16R^2r.$ ◊

§§ Solution.

$$\begin{aligned}
\therefore II_1 \cdot II_2 \cdot II_3 &= a\sec\frac{A}{2}\cdot b\sec\frac{B}{2}\cdot c\sec\frac{C}{2}\\
&= abc\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}\\
&= 8R^3\sin A\sin B\sin C\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}\\
&= \frac{8R^3\cdot 8\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\\
&= 64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\
&= 16R^2\times 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\
&= 16R^2r,\ \textit{by Art. } 204,\ \textit{Cor.} \qquad \blacksquare
\end{aligned}$$

§ Problem 15.3.7. $(I_2I_3)^2 = 4R(r_2+r_3).$ ◊

§§ Solution.

$$\begin{aligned}
(I_2I_3)^2 &= \left(a\,\mathrm{cosec}\,\frac{A}{2}\right)^2, \textit{by } \S\textit{Problem } 15.3.5\\
&= \frac{a^2\cdot bc}{(s-b)(s-c)} = abc\left[\frac{s-b+s-c}{(s-b)(s-c)}\right]\\
&= abc\left[\frac{1}{s-b}+\frac{1}{s-c}\right] = \frac{abc}{S}\left[\frac{S}{s-b}+\frac{S}{s-c}\right]\\
&= 4R(r_2+r_3). \qquad \blacksquare
\end{aligned}$$

§ Problem 15.3.8. $\angle I_3I_1I_2 = \frac{B+C}{2}.$ ◊

§§ Solution.

$$\begin{aligned}
\angle I_3I_1I_2 &= \angle BI_1C = 180^\circ - (I_1BC + I_1CB)\\
&= 180^\circ - \left(90^\circ - \frac{B}{2} + 90^\circ - \frac{C}{2}\right) = \frac{B+C}{2}. \qquad \blacksquare
\end{aligned}$$

§ Problem 15.3.9. $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2.$ ◊

§§ Solution. By § Problem 15.3.4 and § Problem 15.3.5, we have

$$\begin{aligned}
II_1^2 + I_2I_3^2 &= a^2\left(\sec^2\frac{A}{2} + \mathrm{cosec}^2\frac{A}{2}\right)\\
&= a^2\left(\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}\right) = \left(\frac{a}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)^2\\
&= \left(\frac{2a}{\sin A}\right)^2\\
&= (4R)^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2,\ \textit{similarly.} \qquad \blacksquare
\end{aligned}$$

§ **Problem 15.3.10.** *Area of* $\Delta I_1I_2I_3 = 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{abc}{2r}$. ♢

§§ **Solution.**

$$\begin{aligned}
\Delta I_1I_2I_3 &= \frac{1}{2}I_2I_3 \times AI_1\\
&= \frac{1}{2}\cdot a\,\text{cosec}\,\frac{A}{2}\cdot r_1\,\text{cosec}\,\frac{A}{2},\ \textit{by §Problem 15.3.5 and §Problem 15.3.3}\\
&= \frac{1}{2}\cdot\frac{2R\sin A}{\sin\frac{A}{2}}\cdot\frac{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}}\\
&= 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\\
&= 2R^2\left(\sin A+\sin B+\sin C\right),\ \textit{by §Problem 9.2.4}\\
&= R(a+b+c) = R\cdot 2s = \frac{abc}{4S}\cdot 2s\\
&= abc \div \frac{2S}{s} = \frac{abc}{2r}.
\end{aligned}$$

Otherwise thus :

$$\begin{aligned}
8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} &= 8R^2\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}\\
&= 8R^2\times\frac{sS}{abc}\\
&= \frac{abcs}{2S} = \frac{abc}{2r}.
\end{aligned}$$

■

§ **Problem 15.3.11.** $\dfrac{II_1\cdot I_2I_3}{\sin A} = \dfrac{II_2\cdot I_3I_1}{\sin B} = \dfrac{II_3\cdot I_1I_2}{\sin C}$. ♢

§§ **Solution.**

$$\begin{aligned}
\frac{II_1\cdot I_2I_3}{\sin A} &= \frac{a\sec\frac{A}{2}\cdot a\,\text{cosec}\,\frac{A}{2}}{\sin A}\\
&= \frac{2a^2}{2\cos\frac{A}{2}\sin\frac{A}{2}\sin A}\\
&= \frac{2a^2}{\sin^2 A}\\
&= 2\left(2R\right)^2 = 8R^2\\
&= \frac{II_2\cdot I_3I_1}{\sin B} = \frac{II_3\cdot I_1I_2}{\sin C},\ \textit{similarly.}
\end{aligned}$$

■

If I, O, and P be respectively the incenter, circumcenter and orthocenter, and G the centroid of the triangle ABC, prove that

§ **Problem 15.3.12.** $IO^2 = R^2\left(3 - 2\cos A - 2\cos B - 2\cos C\right)$. ♢

§§ **Solution.** By *Art.* 217, we have

$$\begin{aligned}
IO^2 &= R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\\
&= R^2\left[1 - 4\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\sin\frac{C}{2}\right]\\
&= R^2\left(1 - 4\cos\frac{A-B}{2}\cos\frac{A+B}{2} + 4\sin^2\frac{C}{2}\right)
\end{aligned}$$

$$= R^2 \left[1 - 2(\cos A + \cos B) + 2(1 - \cos C)\right]$$
$$= R^2 \left(3 - 2\cos A - 2\cos B - 2\cos C\right). \qquad \blacksquare$$

§ Problem 15.3.13. $IP^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$ $\Diamond$

§§ Solution. We have

$$\begin{aligned}
IP^2 &= AP^2 + AI^2 - 2AP \cdot AI \cdot \cos\angle IAP \\
&= 4R^2\cos^2 A + 16R^2 \sin^2\frac{B}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} \\
&\qquad - 16R^2 \cos A \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2} \\
&= 4R^2\left(\cos^2 A + 4\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 4\cos A\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}\cos\frac{B}{2}\right. \\
&\qquad \left. -4\cos A\sin^2\frac{B}{2}\sin^2\frac{C}{2}\right) \\
&= 4R^2\left[\cos^2 A + 4\sin^2\frac{B}{2}\sin^2\frac{C}{2}(1-\cos A) + \cos A\sin B\sin C\right] \\
&= 4R^2\left(8\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} + \cos^2 A - \cos A\sin B\sin C\right) \\
&= 2r^2 + 4R^2\cos A\left(\cos A - \sin B\sin C\right) \\
&= 2r^2 - 4R^2\cos A\left[\cos(B+C) + \sin B\sin C\right] \\
&= 2r^2 - 4R^2\cos A\cos B\cos C.
\end{aligned}$$

Notes : In the above,

$$\angle IAP = \frac{A}{2} - \angle PAC = \frac{A}{2} - (90^\circ - C) = \frac{1}{2}(C - B). \qquad \blacksquare$$

§ Problem 15.3.14. $OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right).$ $\Diamond$

§§ Solution. By the figure of *Art.* 215, we have $OG = \frac{1}{3}OP$

$$\begin{aligned}
\therefore OG^2 &= \frac{1}{9}\cdot OP^2 = \frac{1}{9}\left(R^2 - 8R^2\cos A\cos B\cos C\right) \\
&= \frac{1}{9}R^2\left[1 - 4\left\{\cos(A+B) + \cos(A-B)\right\}\cos C\right] \\
&= \frac{1}{9}R^2\left[1 + 4\cos^2 C + 4\cos(A-B)\cos(A+B)\right] \\
&= \frac{1}{9}R^2\left[1 + 2(1+\cos 2C) + 2(\cos 2A + \cos 2B)\right] \\
&= \frac{1}{9}R^2\left(3 + 2\cos 2A + 2\cos 2B + 2\cos 2C\right) \\
&= \frac{1}{9}R^2\left[9 - 2(1-\cos 2A) - 2(1-\cos 2B) - 2(1-\cos 2C)\right] \\
&= R^2 - \frac{4R^2}{9}\left(\sin^2 A + \sin^2 B + \sin^2 C\right) \\
&= R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right). \qquad \blacksquare
\end{aligned}$$

§ Problem 15.3.15. *Area of* $\Delta IOP = 2R^2 \sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$ $\Diamond$

§§ Solution.

$$\begin{aligned}
\Delta IOP &= \Delta API + \Delta AIO - \Delta APO \\
&= \frac{1}{2}AP\cdot AI\cdot\sin\angle API + \frac{1}{2}AI\cdot AO\cdot\sin\angle AIO
\end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{2} AP \cdot AO \sin \angle APO \\
&= \frac{1}{2} \cdot 8R^2 \cos A \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{B-C}{2} \\
&\quad + \frac{1}{2} \cdot 4R^2 \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{B-C}{2} - \frac{1}{2} \cdot 2R^2 \cos A \sin(B-C) \\
&\left[\because AP = 2R\cos A,\ AI = 4R \sin \frac{B}{2} \sin \frac{C}{2} \textit{ and } AO = R \right] \\
&= 2R^2 \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{B-C}{2} (2\cos A + 1) \\
&\quad - 2R^2 \cos A \sin \frac{B-C}{2} \cos \frac{B-C}{2} \\
&= 2R^2 \sin \frac{B-C}{2} \left[(2\cos A + 1) \sin \frac{B}{2} \sin \frac{C}{2} \right. \\
&\qquad \left. - \cos A \left(\cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right) \right] \\
&= 2R^2 \sin \frac{B-C}{2} \left[\cos A \left(\sin \frac{B}{2} \sin \frac{C}{2} - \cos \frac{B}{2} \cos \frac{C}{2} \right) \right. \\
&\qquad \left. + \sin \frac{B}{2} \sin \frac{C}{2} \right] \\
&= 2R^2 \sin \frac{B-C}{2} \left(- \cos A \cos \frac{B+C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right) \\
&= R^2 \sin \frac{B-C}{2} \left[- \cos \left(A + \frac{B+C}{2} \right) - \cos \left(A - \frac{B+C}{2} \right) + \right. \\
&\qquad \left. \cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right] \\
&= R^2 \sin \frac{B-C}{2} \left[\sin \frac{A}{2} - \cos \left(A - \frac{B+C}{2} \right) \right. \\
&\qquad \left. + \cos \frac{B-C}{2} - \sin \frac{A}{2} \right] \\
&= 2R^2 \sin \frac{B-C}{2} \sin \frac{A-C}{2} \sin \frac{B-A}{2} \\
&= 2R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}. \qquad \blacksquare
\end{aligned}
$$

§ Problem 15.3.16. *Area of* $\Delta IPG = \dfrac{4}{3} R^2 \sin \dfrac{B-C}{2} \sin \dfrac{C-A}{2} \sin \dfrac{A-B}{2}$. ◊

§§ Solution. Cf. figure of *Art.* 215. We have $PG = \dfrac{2}{3} OP$.

$$
\begin{aligned}
\therefore \Delta IPG &= \frac{2}{3} \Delta IOP = \frac{2}{3} \textit{ result of } \S\textit{Problem } 15.3.15 \\
&= \frac{4}{3} R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}.
\end{aligned}
$$

Also, $$\Delta IOG = \frac{2}{3} R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}. \qquad \blacksquare$$

§ Problem 15.3.17. *Prove that the distance of the center of the nine-point circle from the angle* A *is* $\dfrac{R}{2}\sqrt{1 + 8\cos A \sin B \sin C}$. ◊

§§ Solution. Let N be the middle point of OP, where O and P are respectively the circumcenter and orthocenter of the ΔABC.

The distance AN is required.

We have

$$2\left(AN^2 + PN^2\right) = AP^2 + AO^2$$
$$\therefore 2AN^2 + \frac{OP^2}{2} = AP^2 + AO^2.$$

$$\begin{aligned}
\therefore AN^2 &= 2R^2\cos^2 A + \frac{R^2}{2} - \frac{1}{4}\left(R^2 - 8R^2\cos A\cos B\cos C\right)\\
&= \frac{R^2}{4}\left(8\cos^2 A + 2 - 1 + 8\cos A\cos B\cos C\right)\\
&= \frac{R^2}{4}\left[1 + 8\cos A\left(\cos A + \cos B\cos C\right)\right]\\
&= \frac{R^2}{4}\left[1 + 8\cos A\left\{\cos B\cos C - \cos(B+C)\right\}\right]\\
&= \frac{R^2}{4}\left[1 + 8\cos A(\sin B\sin C)\right]\\
\therefore AN &= \frac{R}{2}\sqrt{1 + 8\cos A\sin B\sin C}.
\end{aligned}$$

■

§ **Problem 15.3.18.** *DEF is the pedal triangle of ABC ; prove that*

(1) its area is $2S\cos A\cos B\cos C$

(2) the radius of its circumcircle is $\frac{R}{2}$ and

(3) the radius of its incircle is $2R\cos A\cos B\cos C$.

◊

§§ **Solution**. For the pedal triangle, if a', b', c', A', B' and C' denote the sides and angles respectively, we have, by $Art.\ 210$:

$$a' = a\cos A, \qquad A' = 180^\circ - 2A,$$
$$b' = b\cos B, \qquad B' = 180^\circ - 2B, \text{ and}$$
$$c' = c\cos C, \text{ and } C' = 180^\circ - 2C.$$

so that, if Δ', R' and r' denote its area, radius of its circumcircle and radius of its incircle respectively, we have

(1)

$$\begin{aligned}
\Delta &= \frac{1}{2}a'b'\sin C' = \frac{1}{2}ab\cos A\cos B\sin\left(180^\circ - 2C\right)\\
&= \frac{1}{2}ab\cos A\cos B\sin 2C = ab\cos A\cos B\sin C\cos C\\
&= 2S\cos A\cos B\cos C, \ \because S = \frac{1}{2}ab\sin C
\end{aligned}$$

(2) $R' = \dfrac{a'}{2\sin A'} = \dfrac{a\cos A}{2\sin 2A} = \dfrac{a}{4\sin A} = \dfrac{R}{2}.$

(3)

$$\begin{aligned}
r' &= 4R'\sin\frac{A'}{2}\sin\frac{B'}{2}\sin\frac{C'}{2}\\
&= 4\cdot\frac{R}{2}\cdot\cos A\cos B\cos C = 2R\cos A\cos B\cos C.
\end{aligned}$$

■

§ **Problem 15.3.19.** *$O_1O_2O_3$ is the triangle formed by the centers of the escribed circles of the triangle ABC; prove that*

(1) its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$,

(2) *its angles are* $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ *and* $\frac{\pi}{2} - \frac{C}{2}$, *and*

(3) *its area is* $2Rs$. ♢

§§ Solution. ABC is the pedal triangle of the triangle $O_1O_2O_3$ [*Art.* 211], so that, if a', b', c', A', B' and C', denote the sides and angles of the $\Delta O_1O_2O_3$, we have

$$a = a' \cos A', \qquad A = 180^\circ - 2A',$$
$$b = b' \cos B', \qquad B = 180^\circ - 2B', \text{ and}$$
$$c = c' \cos C', \text{ and } C = 180^\circ - 2C'.$$

Hence

(1) $A' = 90^\circ - \frac{A}{2}$, $B' = 90^\circ - \frac{B}{2}$ and $C' = 90^\circ - \frac{C}{2}$

(2)

$$\begin{aligned} a' &= a \sec A' = a \operatorname{cosec} \frac{A}{2} \\ &= 2R \sin A \operatorname{cosec} \frac{A}{2} = \frac{2R \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{A}{2}} = 4R \cos \frac{A}{2}. \end{aligned}$$

Similarly, $b' = 4R \cos \frac{B}{2}$ and $c' = 4R \cos \frac{C}{2}$.

(3) $\Delta O_1O_2O_3 = \frac{1}{2} O_2O_3 \cdot AO_1$.

Now $$O_2O_3 = a \operatorname{cosec} \frac{A}{2}$$

and $$AO_1 = r_1 \operatorname{cosec} \frac{A}{2} = s \tan \frac{A}{2} \operatorname{cosec} \frac{A}{2} = s \sec \frac{A}{2}.$$

$$\therefore \Delta O_1O_2O_3 = \frac{1}{2} \cdot \frac{as}{\sin \frac{A}{2} \cos \frac{A}{2}} = \frac{as}{\sin A} = 2Rs.$$

■

§ Problem 15.3.20. *DEF is the triangle formed by joining the points of contact of the incircle with the sides of the triangle ABC, prove that*

(1) *its sides are* $2r \cos \frac{A}{2}$, $2r \cos \frac{B}{2}$ *and* $2r \cos \frac{C}{2}$,

(2) *its angles are* $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ *and* $\frac{\pi}{2} - \frac{C}{2}$, *and*

(3) *its area is* $\frac{2S^3}{abcs}$, *i.e.* $\frac{1}{2} \frac{r}{R} S$. ♢

§§ Solution. If I be the center of the incircle and r its radius, we have

(1) $EF = 2r \sin \frac{\angle EIF}{2} = 2r \sin \frac{180^\circ - A}{2} = 2r \cos \frac{A}{2}$.

So $$FD = 2r \cos \frac{B}{2}, \text{ and } DE = 2r \cos \frac{C}{2}.$$

(2) The $\angle FDE = \frac{1}{2}\angle FIE = 90^\circ - \frac{A}{2}$, as in (1).

So $\angle DEF = 90^\circ - \frac{B}{2}$, and the $\angle DFE = 90^\circ - \frac{C}{2}$.

(3)

$$\begin{aligned} Area &= \Delta EIF + \Delta FID + \Delta DIE \\ &= \frac{1}{2}r^2 \sin \angle EIF + \frac{1}{2}r^2 \sin \angle FID + \frac{1}{2}r^2 \sin \angle DIE \\ &= \frac{1}{2}r^2 (\sin A + \sin B + \sin C) \\ &= \frac{1}{2}r^2 \left(\frac{a+b+c}{2R}\right) = \frac{r^2 s}{2R} = \frac{rS}{2R} \\ &= \frac{rS}{\frac{2abc}{4S}} = \frac{2rS^2}{abc} = \frac{2S^3}{abcs}. \end{aligned}$$

■

§ Problem 15.3.21. *D, E and F are the middle points of the sides of the triangle ABC ; prove that the centroid of the triangle DEF is the same as that of ABC and that its orthocenter is the circumcenter of ABC.* ♢

§§ Solution. AD bisects EF, at K, say, and $DK = \frac{1}{2}AD$.

If G be the centroid of the ΔDEF, then $DG = \frac{2}{3}DK = \frac{1}{3}AD$;

$\therefore G$ is the centroid of the ΔABC.

Again, draw DL perpendicular to EF and therefore perpendicular to BC ; also draw EM perpendicular to FD, and therefore perpendicular to AC.

The orhtocenter of the triangle DEF lies in DL, also in EM ; but DL and EM meet at the circumcenter of the ΔABC ; hence the orthocenter of the ΔDEF is the circumcenter of the ΔABC. ■

In any ΔABC prove that

§ Problem 15.3.22. *The perpendicular from A divides BC into portions which are proportional to the cotangents of the adjacent angles, and that it divides the angle A into portions whose cosines are inversely proportional to the adjacent sides.* ♢

§§ Solution. Let the perpendicular from A meet BC in D and let the $\angle BAD$ and $\angle DAC$ be denoted by β and γ respectively. We have

$$BD = DA \cot B \text{ and } CD = DA \cot C$$
$$\therefore BD : DC = \cot B : \cot C.$$

Also,
$$\cos\beta = \frac{AD}{AB}$$
$$\therefore AB\cos\beta = AD = AC\cos\gamma$$
$$\therefore \cos\beta : \cos\gamma = AC : AB = \frac{1}{AB} : \frac{1}{AC}.$$
■

§ Problem 15.3.23. *The median through A divides it into angles whose cotangents are $2\cot A + \cot C$ and $2\cot A + \cot B$, and makes with the base an angle whose cotangent is $\frac{1}{2}(\cot C \sim \cot B)$.* ♢

§§ **Solution**. Let β and γ be the angles which the median AD makes with CA and AB respectively.

We have
$$\frac{BD}{DA} = \frac{CD}{DA}$$
$$\therefore \frac{\sin\gamma}{\sin B} = \frac{\sin\beta}{\sin C}$$
$$\therefore \frac{\sin(A-\beta)}{\sin\beta} = \frac{\sin B}{\sin C} = \frac{\sin(C+A)}{\sin C}$$
$$[\because \gamma = A-\beta \text{ and } B = 180^\circ - (C+A)]$$
$$\therefore \sin A\cot\beta - \cos A = \cos A + \cot C\sin A$$
$$\therefore \cot\beta - \cot A = \cot A + \cot C$$
$$\therefore \cot\beta = 2\cot A + \cot C.$$
Similarly,
$$\cot\gamma = 2\cot A + \cot B.$$

Otherwise thus :

Draw BH and DG perpendicular to AC, so that $CG = GH$ and $BH = 2DG$.
$$\begin{aligned}\cot\beta &= \frac{AG}{DG} = \frac{2AG}{BH} = \frac{2AH+HC}{BH}\\ &= \frac{2AH}{BH} + \frac{HC}{BH} = 2\cot A + \cot C.\end{aligned}$$
Similarly,
$$\cot\gamma = 2\cot A + \cot B.$$
Again, if AM be the perpendicular from A on BC, we have
$$\cot\angle ADB = \frac{DM}{AM} = \frac{1}{2}\cdot\frac{CM\sim BM}{AM} = \frac{1}{2}\left(\cot C\sim\cot B\right).$$ ■

§ **Problem 15.3.24.** *The distance between the middle point of BC and the foot of the perpendicular from A is*
$$\frac{b^2\sim c^2}{2a}.$$ ◊

§§ **Solution**. With the figure of § Problem 15.3.23, we have
$$\begin{aligned}DM &= \frac{1}{2}\left(CM\sim BM\right) = \frac{1}{2}\cdot\frac{\left(CM\sim BM\right)\left(CM+BM\right)}{CM+BM}\\ &= \frac{1}{2}\cdot\frac{CM^2\sim BM^2}{BC} = \frac{\left(b^2-AM^2\right)\sim\left(c^2-AM^2\right)}{2a}\\ &= \frac{b^2\sim c^2}{2a}.\end{aligned}$$

Otherwise thus :
$$DM = BM\sim BD = \frac{a}{2}\sim c\cos B = \frac{a}{2}\sim\frac{c^2+a^2-b^2}{2a} = \frac{b^2\sim c^2}{2a}.$$ ■

§ **Problem 15.3.25.** *O is the orthocenter of a triangle ABC ; prove that the radii of the circles circumscribing the triangles BOC, COA, AOB, and ABC are all equal.* ◊

§§ **Solution**. The angles at O are known.

The radius of the circle circumscribing the ΔBOC
$$= \frac{BC}{2\sin\angle BOC} = \frac{a}{2\sin(B+C)} = \frac{a}{2\sin A} = R.$$
Similarly, the radii of the circles circumscribing the triangles COA and AOB each $= R$. ■

§ Problem 15.3.26. *AD, BE, and CF are the perpendiculars from the angular points of a triangle ABC upon the opposite sides; prove that the diameters of the circumcircles of the triangles AEF, BDF and CDE are respectively $a\cot A$, $b\cot B$ and $c\cot C$ and that the perimeters of the ΔDEF and ΔABC are in the ratio $r : R$.* ♢

§§ Solution. We have $AE = c\cos A$, $AF = b\cos A$ and $\angle FAE = A$.

Thus the ΔAEF is similar to the ΔABC and has its sides equal to $a\cos A$, $b\cos A$ and $c\cos A$.

Hence the radius of the circumcircle of the ΔAEF

$$= R\cos A = \frac{a}{2\sin A}\cdot\cos A = \frac{a}{2}\cot A,$$

and therefore the diameter $= a\cot A$.

Similarly, for the diameters of the other two triangles.

Again, the perimeter of the ΔDEF

$$\begin{aligned}
&= a\cos A + b\cos B + c\cos C\\
&= 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C\\
&= R(\sin 2A + \sin 2B + \sin 2C) = 4R\sin A\sin B\sin C\\
&= 4R\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R} = \frac{abc}{2R^2} = \frac{4S}{2R} = \frac{2S}{R}.
\end{aligned}$$

Hence the perimeter of the ΔDEF : the perimeter of the ΔABC

$$= \frac{2S}{R} : 2s = \frac{S}{s} : R = r : R.$$ ■

§ Problem 15.3.27. *Prove that the product of the distances of the incenter from the angular points of a triangle is $4Rr^2$.* ♢

§§ Solution. We have

$$\begin{aligned}
AI\cdot BI\cdot CI &= r\operatorname{cosec}\frac{A}{2}\cdot r\operatorname{cosec}\frac{B}{2}\cdot\operatorname{cosec}\frac{C}{2}\\
&= r^2\cdot 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cdot\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}\\
&\left[\because r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]\\
&= 4Rr^2.
\end{aligned}$$ ■

§ Problem 15.3.28. *The triangle DEF circumscribes the three escribed circles of the ΔABC ; prove that*

$$\frac{EF}{a\cos A} = \frac{FD}{b\cos B} = \frac{DE}{c\cos C}.$$ ♢

§§ Solution. Let P, Q and R be the centers of the three escribed circles which are respectively opposite to A, B and C, and to D, E and F.

From P, draw PL, PK, PH, PN and PM respectively, perpendicular to DF, AB, BC, AC and ED.

Since RP, AC and FD meet, if produced, at a center of similitude of circles centers R and P, we have

$$\angle LPB = \angle BPN;\ \text{but } \angle KPB = \angle HPB$$

$$\therefore \text{ the remainder },\angle LPK = \text{ the remainder}, \angle NPH = C.$$

Similarly, $\quad \angle KPC = \angle MPC;\ \text{but } \angle HPC = \angle NPC$

$$\therefore \text{ the remainder },\angle KPH (= B) = \text{ the remainder}, \angle MPN$$

$$\therefore \angle LPM = 360^\circ - 2B - 2C = 2A$$

$$\therefore D = 180^\circ - 2A;\ \text{so } E = 180^\circ - 2B,\ \text{and } F = 180^\circ - 2C.$$

Hence the sides are as

$$\begin{aligned}\sin(180^\circ - 2A) &: \sin(180^\circ - 2B) : \sin(180^\circ - 2C)\\ &\equiv \sin 2A : \sin 2B : \sin 2C\\ &\equiv 2\sin A\cos A : 2\sin B\cos B : 2\sin C\cos C\\ &\equiv a\cos A : b\cos B : c\cos C.\end{aligned}$$

■

§ Problem 15.3.29. *If a circle be drawn touching the inscribed and circumscribed circles of a triangle and the side BC externally, prove that its radius is*

$$\frac{\Delta}{a}\tan^2\frac{A}{2}.$$

◊

§§ Solution. Let K be the center and x be the radius of the circle, and I and O be the centers of the incircle and circumcircle respectively.

Let D be the point of contact of the circle, center K, the inscribed circle and the side BC.

Then IDK is perpendicular to BC.

Draw $OA'L$ parallel to IDK, meeting BC in A' and draw KL perpendicular to IK, meeting $OA'L$ in L.

We then have

$$OK = R - x,\ OL = OA' + x = R\cos A + x, \text{ and}$$

$$KL = A'D = BA' - BD = \frac{a}{2} - (s-b) = \frac{a-2s+2b}{2} = \frac{b-c}{2}.$$

Now

$$OL^2 + LK^2 = OK^2$$

$$\therefore (R\cos A + x)^2 + \left(\frac{b-c}{2}\right)^2 = (R-x)^2$$

$$\therefore R^2\cos^2 A + 2Rx\cos A + x^2 + \left(\frac{b-c}{2}\right)^2 = R^2 - 2Rx + x^2$$

$$\therefore 2Rx(1+\cos A) = R^2\sin^2 A - \left(\frac{b-c}{2}\right)^2$$

$$\begin{aligned}\therefore 4Rx\cos^2\frac{A}{2} &= \left(\frac{a}{2}\right)^2 - \left(\frac{b-c}{2}\right)^2 = \frac{a^2-b^2-c^2+2bc}{4}\\ &= \frac{2bc - 2bc\cos A}{4} = bc\sin^2\frac{A}{2}\end{aligned}$$

$$\therefore x = \frac{bc}{4R}\tan^2\frac{A}{2} = \frac{\Delta}{a}\tan^2\frac{A}{2}.$$

Otherwise thus :

We have

$$OK^2 = IK^2 + OI^2 - 2IK\cdot OI\cos\angle OIK$$

$$\therefore (R-x)^2 = (r+x)^2 + \left(R^2 - 2Rr\right) - 2(r+x)(r - R\cos A)$$

For, draw OQ perpendicular to IK and we have

$$\begin{aligned}\cos\angle OIK = \frac{OQ}{OI} \text{ and } OQ &= IK - QK\\ &= (r+x) - OL = (r+x) - (R\cos A + x)\end{aligned}$$

$$\begin{aligned}\therefore R^2 - 2Rx + x^2 = r^2 &+ 2rx + x^2 + R^2 - 2Rr - 2r^2 - 2rx\\ &+ 2Rr\cos A + 2Rx\cos A\end{aligned}$$

$$\therefore 2Rx(1+\cos A) = r^2 + 2Rr(1-\cos A)$$
$$\therefore \frac{ax(1+\cos A)}{\sin A} = r\left[r + \frac{a(1-\cos A)}{\sin A}\right]$$
$$\therefore ax\cot\frac{A}{2} = r\left[\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} + \frac{a\sin\frac{A}{2}}{\cos\frac{A}{2}}\right]$$
$$= \frac{ar}{\cos\frac{A}{2}}\left(\sin\frac{B}{2}\sin\frac{C}{2} + \cos\frac{B+C}{2}\right) = \frac{ar}{\cos\frac{A}{2}}\cdot\cos\frac{B}{2}\cos\frac{C}{2}$$
$$= \frac{ar}{4\cos^2\frac{A}{2}}(\sin A + \sin B + \sin C)$$
$$= \frac{ar\sin\frac{A}{2}}{2\cos\frac{A}{2}}\cdot\left(\frac{\sin A + \sin B + \sin C}{\sin A}\right)$$
$$= \frac{ar}{2}\tan\frac{A}{2}\left(\frac{a+b+c}{a}\right) = rs\tan\frac{A}{2} = \Delta\tan\frac{A}{2}.$$
$$\therefore x = \frac{\Delta}{a}\tan^2\frac{A}{2}.$$ ■

§ Problem 15.3.30. *If a, b and c be the radii of three circles which touch one another externally, and r_1 and r_2 be the radii of the two circles that can be drawn to touch these three, prove that*
$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$ ◊

§§ Solution. Let A, B and C be the centers of the three circles respectively and O be the center of a circle touching them.

Join AB, BC and CA.

Let $\angle BOC = \alpha$, $\angle AOC = \beta$ and $\angle AOB = \gamma$.

Now if O be in the plane of the ΔABC, whether within or without ΔABC, we have
$$\beta + \gamma = 180^\circ - \alpha \text{ or } \beta + \gamma = \alpha.$$
Hence, in all cases,
$$\cos(\beta+\gamma) = \cos\alpha$$
$$\therefore \cos\beta\cos\gamma - \cos\alpha = \sin\beta\sin\gamma.$$
Squaring, we have
$$\cos^2\beta\cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha$$
$$= \left(1-\cos^2\beta\right)\left(1-\cos^2\gamma\right)$$
$$= 1 - \cos^2\beta - \cos^2\gamma + \cos^2\beta\cos^2\gamma$$
$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 1 \tag{15.5}$$
The symmetry of this expression shows that it is true for all positions of O.

Now
$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - x, \text{ say}$$
$$\cos\beta = 1 - 2\sin^2\frac{\beta}{2} = 1 - y, \text{ say}$$
$$\cos\gamma = 1 - 2\sin^2\frac{\gamma}{2} = 1 - z, \text{ say}$$

$\therefore$ from (15.5), we have

$$(1-x)^2 + (1-y)^2 + (1-z)^2 - 2(1-x)(1-y)(1-z) = 1$$

$$\therefore x^2 + y^2 + z^2 - 2(yz + zx + xy) + 2xyz = 0 \tag{15.6}$$

Now $\qquad AB = a + b,\ BC = b + c$ and $CA = c + a$

For the smaller circle (O within ΔABC), in the ΔOBC the sides are

$$b + c,\ c + r_1,\ r_1 + b$$

$$\therefore s = r_1 + b + c$$

$$\therefore \sin\frac{\alpha}{2} = \sqrt{\frac{bc}{(r_1 + b)(r_1 + c)}} \tag{15.7}$$

For the larger circle (O not necessarily within ΔABC, in the ΔOBC the side are

$$b + c,\ r_2 - c,\ r_2 - b$$

$$\therefore s = r_2$$

$$\therefore \sin\frac{\alpha}{2} = \sqrt{\frac{bc}{(r_2 - b)(r_2 - c)}} \tag{15.8}$$

From (15.7) and (15.8) :

$$\therefore 2\sin^2\frac{\alpha}{2} = \frac{\frac{2}{r_1^2}}{\left(\frac{1}{b} + \frac{1}{r_1}\right)\left(\frac{1}{c} + \frac{1}{r_1}\right)} \text{ or } \frac{\frac{2}{r_2^2}}{\left(\frac{1}{b} - \frac{1}{r_2}\right)\left(\frac{1}{c} - \frac{1}{r_2}\right)} = x.$$

Hence, if r denote r_1 or $-r_2$, from (15.6), we have

$$\left[\frac{\frac{2}{r^2}}{\left(\frac{1}{r} + \frac{1}{b}\right)\left(\frac{1}{r} + \frac{1}{c}\right)}\right]^2 + \ldots + \ldots$$

$$- 2\left[\frac{\frac{4}{r^4}}{\left(\frac{1}{r} + \frac{1}{a}\right)^2\left(\frac{1}{r} + \frac{1}{b}\right)\left(\frac{1}{r} + \frac{1}{c}\right)} + \ldots + \ldots\right]$$

$$+ 2 \cdot \frac{\frac{8}{r^6}}{\left(\frac{1}{r} + \frac{1}{a}\right)^2\left(\frac{1}{r} + \frac{1}{b}\right)^2\left(\frac{1}{r} + \frac{1}{c}\right)^2} = 0$$

$$\therefore \frac{4}{r^4}\left[\left(\frac{1}{r} + \frac{1}{a}\right)^2 + \ldots + \ldots\right]$$

$$- \frac{8}{r^4}\left[\left(\frac{1}{r} + \frac{1}{b}\right)\left(\frac{1}{r} + \frac{1}{c}\right) + \cdots + \ldots\right] + \frac{16}{r^6} = 0.$$

$$\therefore \frac{3}{r^2} + \frac{2}{r}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$- 2\left[\frac{3}{r^2} + \frac{2}{r}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right] + \frac{4}{r^2} = 0,$$

so that $\frac{1}{r_1}$ and $-\frac{1}{r_2}$ are roots of the equation

$$\frac{1}{r^2} - \frac{2}{r}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 2\left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right) = 0.$$

Hence, by the theory of quadratic equations, we have
$$\frac{1}{r_1} - \frac{1}{r_2} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \qquad \blacksquare$$

§ Problem 15.3.31. *If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle, whose area is Δ and Δ_1, Δ_2 and Δ_3 the corresponding areas for the escribed circles, prove that*
$$\Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 = 2\Delta. \qquad \lozenge$$

§§ Solution. Take the figure of *Art.* 205. Join F_1D_1, D_1E_1 and F_1E_1. We then have
$$\begin{aligned}\Delta_1 &= \Delta F_1D_1I_1 + \Delta D_1I_1E_1 - \Delta F_1I_1E_1\\ &= \frac{1}{2}r_1^2(\sin B + \sin C - \sin A)\\ &= \frac{1}{2}r_1^2\left(\frac{b+c-a}{2R}\right)\\ &= \frac{1}{2}r_1 \cdot \frac{\Delta}{s-a} \cdot \frac{2(s-a)}{2R}\\ &= \frac{r_1\Delta}{2R}.\end{aligned}$$
Similarly, for Δ_2 and Δ_3.

Also, $\Delta_0 = \frac{r\Delta}{2R}$, by § Problem 15.3.20.
$$\begin{aligned}\therefore \Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 &= \frac{\Delta}{2R}(r_1 + r_2 + r_3 - r)\\ &= \frac{\Delta}{2R} \cdot 4R, \ \textit{by } \S\textit{Problem } 15.2.18\\ &= 2\Delta. \qquad \blacksquare\end{aligned}$$

§ Problem 15.3.32. *If the bisectors of the angles of a ΔABC meet the opposite sides in A', B' and C', prove that the ratio of the areas of the triangles $A'B'C'$ and ABC is*
$$2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : \cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}. \qquad \lozenge$$

§§ Solution. By *Euclid VI.* 3, [*cf. Art.* 218], we have
$$\begin{aligned}AB' &= \frac{bc}{c+a},\\ AC' &= \frac{bc}{a+b},\\ BC' &= \frac{ca}{a+b},\\ BA' &= \frac{ca}{b+c}\\ CB' &= \frac{ab}{c+a}\\ \text{and } CA' &= \frac{ab}{b+c};\\ \therefore \Delta AB'C' &= \frac{1}{2} \cdot \frac{bc}{c+a} \cdot \frac{bc}{a+b}\sin A\\ &= \frac{bc}{(c+a)(a+b)} \cdot \Delta ABC,\end{aligned}$$

$$\therefore \Delta A'BC' = \frac{ca}{(a+b)(b+c)} \cdot \Delta ABC, \text{ and}$$
$$\Delta A'CB' = \frac{ab}{(b+c)(c+a)} \cdot \Delta ABC;$$

$$\begin{aligned}
&\therefore \Delta A'B'C' \\
&= \Delta ABC \left[1 - \frac{bc}{(c+a)(a+b)} - \frac{ca}{(a+b)(b+c)} - \frac{ab}{(b+c)(c+a)}\right] \\
&= \Delta ABC \left[1 - \frac{bc(b+c)+ca(c+a)+ab(a+b)}{(b+c)(c+a)(a+b)}\right] \\
&= \Delta ABC \left[\frac{2abc}{(b+c)(c+a)(a+b)}\right]
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{\Delta A'B'C'}{\Delta ABC} &= \frac{2\sin A \sin B \sin C}{(\sin B + \sin C)(\sin C + \sin A)(\sin A + \sin B)} \\
&= \frac{16 \sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{8\sin\frac{B+C}{2}\cos\frac{B-C}{2}\sin\frac{C+A}{2}\cos\frac{C-A}{2}\sin\frac{A+B}{2}\cos\frac{A-B}{2}} \\
&= \frac{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2}}
\end{aligned}$$

$$\begin{aligned}
&\therefore \Delta A'B'C' : \Delta ABC \\
&= 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : \cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}. \qquad \blacksquare
\end{aligned}$$

§ **Problem 15.3.33.** *Through the angular points of a triangle are drawn straight lines which make the same angle α with the opposite sides of the triangle; prove that the area of the triangle formed by them is to the area of the original triangle as*

$$4\cos^2\alpha : 1. \qquad \Diamond$$

§§ **Solution**. Let ABC be the original triangle and $A'B'C'$ be the new triangle whose sides are a', b', c'.

$$\begin{aligned}
\because \angle A'B'C' &= \angle B'AC + \angle B'CA \\
&= \alpha - C + \pi - A + \alpha = \pi - A - C = B.
\end{aligned}$$

$\therefore \Delta A'B'C'$ is similar to the ΔABC.

We then have

$$\frac{AB'}{AC} = \frac{\sin(\alpha + A)}{\sin B} \text{ and } \frac{AC'}{AB} = \frac{\sin(\alpha - A)}{\sin C}.$$

$$\because AB' = \frac{b}{\sin B} \cdot \sin(\alpha + A) = 2R\sin(\alpha + A), \text{ and}$$
$$AC' = \frac{c}{\sin C} \cdot \sin(\alpha - A) = 2R\sin(\alpha - A).$$
$$\therefore a' = AB' - AC' = 2R \cdot 2\cos\alpha \sin A = 2a\cos\alpha.$$

Also, since the triangles are similar, we have, by $Euclid\ VI.$ 19,

$$\frac{\Delta'}{\Delta} = \frac{a'^2}{a^2} = \frac{4a^2\cos^2\alpha}{a^2} = 4\cos^2\alpha$$
$$\therefore \Delta' : \Delta = 4\cos^2\alpha : 1. \qquad \blacksquare$$

§ Problem 15.3.34. *Two circles, of radii a and b, cut each other at an angle θ. Prove that the length of the common chord is*

$$\frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}.$$

◊

§§ Solution. Let A and B be the centers of the two circles respectively and C be a point of intersection.

The $\angle\theta$ is the angle between the tangents at C.

Also, the $\angle ACB = \frac{\pi}{2}+\frac{\pi}{2}-\theta = \pi-\theta$.

If the length of the common chord be d, we then have

$$\begin{aligned}\frac{1}{2}\cdot AB\cdot\frac{d}{2} &= \Delta ACB = \frac{1}{2}ab\sin\angle ACB.\\ \therefore d &= \frac{2ab\sin\theta}{AB}\\ &= \frac{2ab\sin\theta}{\sqrt{a^2+b^2-2ab\cos\angle ACB}}\\ &= \frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}.\end{aligned}$$

■

§ Problem 15.3.35. *Three equal circles touch one another; find the radius of the circle which touches all three.* ◊

§§ Solution. Let A, B and C be the centers of the three equal circles of radius r, D be the point of contact of circles whose centers are B and C, and O be the center of the required circles.

Join AB, BC and CA.

ABC is an equilateral Δ.

Also, $AD = r\tan 60^\circ = r\sqrt{3}$ and $AO = \frac{2}{3}r\sqrt{3}$.

$\therefore$ the required radii

$$\begin{aligned}&= \frac{2}{3}r\sqrt{3}\pm r\\ &= r(1.1547\pm 1)\\ &= 2.1547r,\ \textit{or}\ .1547r.\end{aligned}$$

■

§ Problem 15.3.36. *Three circles, whose radii are a, b and c, touch one another externally and the tangents at their points of contact meet in a point ; prove that the distance of this point from either of their points of contact is*

$$\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$

◊

§§ Solution. Let A, B and C be the centers of three circles respectively, and ID, IE and IF be the three common tangents.

Join AB, BC and CA.

Since ID, IE and IF are perpendicular to BC, CA and AB respectively, and are equal to one another.

$\therefore I$ is the incenter of the ΔABC.

Now the sides of the ΔABC are

$$b+c(=a',\ say),\ c+a(=b')\text{ and }a+b(=c')$$
$$\therefore 2s = 2(a+b+c)\text{ and }s = a+b+c.$$

Hence the required distance

$$= IE = ID = IF = \frac{\Delta ABC}{s}$$

$$= \sqrt{\frac{(s-a')(s-b')(s-c')}{s}}$$

$$= \sqrt{\frac{abc}{a+b+c}}. \qquad \blacksquare$$

§ Problem 15.3.37. *On the sides BC, CA, AB are taken three points A', B', C' such that*

$$BA' : A'C = CB' : B'A = AC' : C'B = m : n;$$

prove that if AA', BB' and CC' be joined they will form by their intersections a triangle whose area is to that of the triangle ABC as

$$(m-n)^2 : m^2 + mn + n^2. \qquad \Diamond$$

§§ Solution. Let BB' and CC', CC' and AA', and AA' and BB', meet respectively in the points D, E and F.

Let the lengths of the perpendiculars from F upon the sides of the triangle be called α, β and γ, and let the area of the ΔABC be Δ.

$$\begin{aligned}
\therefore \frac{\beta}{\gamma} &= \frac{AF \sin \angle A'AC}{AF \sin \angle A'AB} \\
&= \frac{AD \sin \angle A'AC}{AD \sin A'AB} \\
&= \text{ ratio of perpendiculars from } A' \text{ on the sides} \\
&= \frac{n \sin C}{m \sin B} = \frac{nc}{mb}.
\end{aligned}$$

$$\therefore \frac{\gamma}{\alpha} = \frac{na}{mc}$$

$$\begin{aligned}
\therefore \frac{c\gamma}{1} &= \frac{a\alpha}{\frac{m}{n}} = \frac{b\beta}{\frac{n}{m}} = \frac{a\alpha + b\beta + c\gamma}{\frac{m}{n} + \frac{n}{m} + 1} \\
&= \frac{\text{twice sum of the areas of } \Delta's\ AFB,\ BDC,\ CEA}{\frac{m}{n} + \frac{n}{m} + 1} \\
&= 2\Delta \times \frac{mn}{m^2 + mn + n^2}
\end{aligned}$$

$$\therefore \textit{ area } \Delta AFB = \frac{1}{2} c\gamma = \Delta \times \frac{mn}{m^2 + mn + n^2}$$

So each of the areas BDC and CEA is equal to the same quantity;

$$\begin{aligned}
\therefore \textit{ area } \Delta DEF &= \Delta - \text{ sum of the areas of } \Delta AFB,\ BDC \text{ and } CEA \\
&= \Delta \left[1 - \frac{3mn}{m^2 + mn + n^2} \right] \\
&= \Delta \times \frac{(m-n)^2}{m^2 + mn + n^2}. \qquad \blacksquare
\end{aligned}$$

§ Problem 15.3.38. *The circle inscribed in the triangle ABC touches the sides BC, CA and AB in the points A_1, B_1 and C_1 respectively; similarly the circle inscribed in the triangle $A_1B_1C_1$ touches the sides in A_2, B_2, C_2 respectively, and so on ; if $A_nB_nC_n$ be the n^{th} triangle so formed, prove that its angles are*

$$\frac{\pi}{3} + (-2)^{-n}\left(A - \frac{\pi}{3}\right), \ \frac{\pi}{3} + (-2)^{-n}\left(B - \frac{\pi}{3}\right) \textit{ and } \frac{\pi}{3} + (-2)^{-n}\left(C - \frac{\pi}{3}\right).$$

Hence prove that the triangle so formed is ultimately equilateral. $\Diamond$

§§ Solution. If I be the center of the incircle of the ΔABC, we have

$$A_1 = \frac{1}{2} B_1 I C_1 = \frac{1}{2}(\pi - A) = \frac{\pi}{2} - \frac{A}{2} = \frac{\pi}{3} - \frac{1}{2}\left(A - \frac{\pi}{3}\right).$$

$$\therefore A_2 = \frac{\pi}{2} - \frac{A_1}{2} = \frac{\pi}{3} - \frac{1}{2}\left(A_1 - \frac{\pi}{3}\right) = \frac{\pi}{3} + \frac{1}{4}\left(A - \frac{\pi}{3}\right).$$

If
$$A_n = \frac{\pi}{3} + (-2)^{-n}\left(A - \frac{\pi}{3}\right)$$

then
$$\begin{aligned} A_{n+1} &= \frac{\pi}{2} - \frac{A_n}{2} \\ &= \frac{\pi}{2} - \frac{1}{2}\left[\frac{\pi}{3} + (-2)^{-n}\left(A - \frac{\pi}{3}\right)\right] \\ &= \frac{\pi}{3} + (-2)^{-n-1}\left(A - \frac{\pi}{3}\right). \end{aligned}$$

Hence the expression is true for $n+1$, if true for n.

But it is true when $n = 1$ and 2 ; $\therefore$ if $n = 3,\ 4\ \ldots$.

Similarly for the other angles.

If n be very large, $2^{-n} = 0$, and the angles are $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$, i.e. ultimately the triangle becomes equilateral. ■

§ Problem 15.3.39. *$A_1B_1C_1$ is the triangle formed by joining the feet of the perpendiculars drawn from ABC upon the opposite sides ; in like manner $A_2B_2C_2$ is the triangle obtained by joining the feet of the perpendiculars from A_1, B_1 and C_1 on the opposite sides, and so on. Find the values of the angles A_n, B_n and C_n in the n^{th} of these triangles.* ♢

§§ Solution. Here we have

$$A_1 = \pi - 2A = \frac{\pi}{3} - 2\left(A - \frac{\pi}{3}\right), \text{ and}$$

$$A_2 = \pi - 2A_1 = \pi - \frac{2\pi}{3} + 4\left(A - \frac{\pi}{3}\right) = \frac{\pi}{3} + 4\left(A - \frac{\pi}{3}\right).$$

If
$$A_n = \frac{\pi}{3} + (-2)^n\left(A - \frac{\pi}{3}\right),$$

then
$$\begin{aligned} A_{n+1} &= \pi - 2A_n = \pi - \frac{2\pi}{3} + (-2)^{n+1}\left(A - \frac{\pi}{3}\right) \\ &= \frac{\pi}{3} + (-2)^{n+1}\left(A - \frac{\pi}{3}\right). \end{aligned}$$

Hence the expression is true for $n+1$, if true for n.

But it is true when $n = 1$ and 2 ; $\therefore$ if $n = 3,\ 4\ \ldots$.

Similarly for the other angles. ■

Chapter 16

On Quadrilaterals And Regular Polygons

16.1 Area of a Quadrilateral

§ Problem 16.1.1. *Find the area of a quadrilateral, which can be inscribed in a circle, whose sides are*

(1) 3, 5, 7, and 9 feet ; and

(2) 7, 10, 5, and 2 feet. ◊

§§ Solution. (1) If $a = 3\ ft.$, $b = 5\ ft.$, $c = 7\ ft.$, and $d = 9\ ft.$, then

$$s = 12\ ft.,\ s - a = 9\ ft.,\ s - b = 7\ ft.,$$
$$s - c = 5\ ft.,\ \text{and}\ s - d = 3\ ft.$$
$$\therefore\ \text{the area}\ = \sqrt{9 \times 7 \times 5 \times 3} = 3\sqrt{105}\ sq.\ ft.$$

(2) If $a = 7\ ft.$, $b = 10\ ft.$, $c = 5\ ft.$, and $d = 2\ ft.$, then

$$s = 12\ ft.,\ s - a = 5\ ft.,\ s - b = 2\ ft.,$$
$$s - c = 7\ ft.,\ \text{and}\ s - d = 10\ ft.$$
$$\therefore\ \text{the area}\ = \sqrt{5 \times 2 \times 7 \times 10} = 10\sqrt{7}\ sq.\ ft.$$ ■

§ Problem 16.1.2. *The sides of a quadrilateral are respectively* 3, 4, 5, *and* 6 $feet$, *and the sum of a pair of opposite angles is* $120°$ *; prove that the area of the quadrilateral is* $3\sqrt{30}$ *square feet.* ◊

§§ Solution. If $a = 3\ ft.$, $b = 4\ ft.$, $c = 5\ ft.$, and $d = 6\ ft.$, then

$$s = 9\ ft.,\ s - a = 6\ ft.,\ s - b = 5\ ft.,$$
$$s - c = 4\ ft.,\ \text{and}\ s - d = 3\ ft.$$

$$\begin{aligned}
\therefore\ \text{the area} &= \sqrt{6 \times 5 \times 4 \times 3 - 3 \times 4 \times 5 \times 6 \times \cos^2 60^\circ} \\
&= \sqrt{6 \times 5 \times 4 \times 3\,(1 - \cos^2 60^\circ)} \\
&= \sqrt{6 \times 5 \times 4 \times 3} \times \sin 60^\circ \\
&= 6\sqrt{10} \times \frac{\sqrt{3}}{2} \\
&= 3\sqrt{30}\ sq.\ ft.
\end{aligned}$$

■

§ Problem 16.1.3. *Tho sides of a quadrilateral which can he inscribed in a circle are 3, 3, 4 and 4 feet ; find the radii of the incircle and circumcircle.* ♢

§§ Solution. Let $ABCD$ be the quadrilateral. Take

$$BA = 3\ ft.,\ DA = 4\ ft.,\ BC = 3\ ft.\ and\ DC = 4\ ft.$$

By symmetry, BD is diameter and is equal to $\sqrt{4^2 + 3^2} = 5\ ft.$

Hence the radius of the circumcircle $= \frac{5}{2}\ ft. = 2\frac{1}{2}\ ft.$

Also, since $AB + CD = BC + DA$, a circle can be inscribed in the quadrilateral and if r be the radius of this circle, we have (as in $Art.$ 202),

the area of the quadrilateral

$$= \frac{r}{2}(3 + 4 + 4 + 3).$$
$$\therefore 7r = \sqrt{3 \times 4 \times 4 \times 3} = 3 \times 4$$
$$\therefore r = \frac{12}{7} = 1\frac{5}{7}\ ft.$$

■

§ Problem 16.1.4. *Prove that the area of any quadrilateral is one-half the product of the two diagonals and the sine of the angle between them.* ♢

§§ Solution. Take the figure of $Art.$ 221. Let the diagonals AC and BD intersect in the point O and let the $\angle AOD = \alpha = \angle BOC$.

We then have

$$\begin{aligned}
\Delta ADC &= \Delta AOD + \Delta COD \\
&= \frac{1}{2} DO \cdot AO\ \sin\alpha + \frac{1}{2} DO \cdot CO\ \sin(\pi - \alpha) \\
&= \frac{1}{2} DO(AO + CO) \sin\alpha \\
&= \frac{1}{2} DO \cdot AC\ \sin\alpha.
\end{aligned}$$

Similarly, $$\Delta ABC = \frac{1}{2} BO \cdot AC \sin\alpha.$$

Hence the area of the quadrilateral

$$= \frac{1}{2}(DO + BO)AC\ \sin\alpha = \frac{1}{2} BD \cdot AC\ \sin\alpha.$$

■

§ Problem 16.1.5. *If a quadrilateral can be inscribed in one circle and circumscribed about another circle, prove that its area is* $\sqrt{abcd}$ *and that the radius of the latter circle is*

$$\frac{2\sqrt{abcd}}{a+b+c+d}.$$ ◊

§§ Solution. See *Art.* 222.

If r be the radius of the incircle, we have

$$\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc+\frac{1}{2}rd=\sqrt{abcd}$$
$$\therefore r=\frac{2\sqrt{abcd}}{a+b+c+d}.$$ ■

§ Problem 16.1.6. *A quadrilateral $ABCD$ is described about a circle; prove that*

$$AB\sin\frac{A}{2}\sin\frac{B}{2}=CD\sin\frac{C}{2}\sin\frac{D}{2}.$$ ◊

§§ Solution. If r be the radius of the circle, we have

$$AB=r\cot\frac{A}{2}+r\cot\frac{B}{2}=r\frac{\sin\dfrac{A+B}{2}}{\sin\dfrac{A}{2}\sin\dfrac{B}{2}}$$
$$\therefore AB\sin\frac{A}{2}\sin\frac{B}{2}=r\sin\frac{A+B}{2}.$$

Similarly,
$$CD\sin\frac{C}{2}\sin\frac{D}{2}=r\sin\frac{C+D}{2}.$$
$$\because A+B=360^\circ-(C+D)$$
$$\therefore \frac{A+B}{2}=180^\circ-\frac{C+D}{2}$$
$$\therefore \sin\frac{A+B}{2}=\sin\frac{C+D}{2}.$$
$$\therefore AB\sin\frac{A}{2}\sin\frac{B}{2}=CD\sin\frac{C}{2}\sin\frac{D}{2}.$$ ■

§ Problem 16.1.7. *a, b, c and d are the sides of a quadrilateral taken in order, and α is the angle between the diagonals opposite to b or d; prove that the area of the quadrilateral is*

$$\frac{1}{4}\left(a^2-b^2+c^2-d^2\right)\tan\alpha.$$ ◊

§§ Solution. Take the figure of *Art.* 221. Let the diagonals AC and BD intersect in the point O and let the $\angle AOD=\alpha=\angle BOC$.

We then have

$$a^2=AO^2+BO^2+2AO\cdot BO\cos\alpha$$
$$b^2=BO^2+CO^2-2BO\cdot CO\cos\alpha$$
$$c^2=CO^2+DO^2+2CO\cdot DO\cos\alpha,\ \text{and}$$
$$d^2=DO^2+AO^2-2DO\cdot AO\cos\alpha$$

$$\begin{aligned}\therefore a^2-b^2+c^2-d^2 \\ &=\left(a^2+c^2\right)-\left(b^2+d^2\right)\\ &=2\cos\alpha\left[(AO\cdot BO+CO\cdot DO)+(BO\cdot CO+DO\cdot AO)\right]\\ &=2\cos\alpha\left[AO(BO+DO)+CO(DO+BO)\right]\\ &=2AC\cdot BD\cos\alpha.\end{aligned}$$

Also, by § Problem 16.1.4,
the area of the quadrilateral

$$= \frac{1}{2} AC \cdot BD \sin\alpha.$$

Hence, by substitution, we have

$$area \ = \frac{1}{4}\left(a^2 - b^2 + c^2 - d^2\right)\tan\alpha.$$

■

§ Problem 16.1.8. *If a, b, c and d be the sides and x and y the diagonals of a quadrilateral, prove that its area is*

$$\frac{1}{4}\left[4x^2y^2 - \left(b^2 + d^2 - a^2 - c^2\right)^2\right]^{\frac{1}{2}}.$$

◊

§§ Solution. By § Problem 16.1.4,

$$\begin{aligned} area \ &= \frac{1}{2} xy \sin\alpha = \sqrt{\frac{1}{4} x^2y^2 \sin^2\alpha} \\ &= \sqrt{\frac{1}{4}x^2y^2 - \frac{1}{4}x^2y^2\cos^2\alpha} \\ &= \frac{1}{4}\sqrt{4x^2y^2 - 4x^2y^2\cos^2\alpha} \\ &= \frac{1}{4}\sqrt{4x^2y^2 - (a^2 - b^2 + c^2 - d^2)^2}, by\ \S Problem\ 16.1.7. \end{aligned}$$

■

§ Problem 16.1.9. *If a quadrilateral can be inscribed in a circle, prove that the angle between its diagonals is*

$$\sin^{-1}\left[2\sqrt{(s-a)(s-b)(s-c)(s-d)} \div (ac+bd)\right].$$

If the same quadrilateral can also be circumscribed about a circle, prove that this angle is then

$$\cos^{-1}\frac{ac-bd}{ac+bd}.$$

◊

§§ Solution. If x and y be the diagonals and α be the angle between them, we have

the area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}, [Art.\ 219]$,

and also $= \frac{1}{2}xy\sin\alpha.$ [§*Problem* 16.1.4.]

But $xy = ac + bd\,[Euclid.\ VI.\ D.]$

$$\therefore \sin\alpha = \frac{2\ area}{xy} = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ac+bd}$$

$$\therefore \alpha = \sin^{-1}\left[2\sqrt{(s-a)(s-b)(s-c)(s-d)} \div (ac+bd)\right].$$

In the second case, we have the area

$$= \sqrt{abcd}$$

$$\therefore \sin\alpha = \frac{2\sqrt{abcd}}{ac+bd}$$

$$\therefore \cos^2\alpha = 1 - \frac{4abcd}{(ac+bd)^2} = \left(\frac{ac-bd}{ac+bd}\right)^2$$

$$\therefore \cos\alpha = \frac{ac-bd}{ac+bd}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{ac-bd}{ac+bd}\right). \qquad \blacksquare$$

§ Problem 16.1.10. *The sides of a quadrilateral are divided in order in tho ratio $m : n$, and a new quadrilateral is formed by joining the points of division ; prove that its area is to the area of the original figure as*

$$m^2 + n^2 \ to \ (m+n)^2. \qquad \lozenge$$

§§ Solution. Let the sides AB, BC, CD and DA of the quadrilateral $ABCD$ be divided at the points A', B', C' and D' respectively, so that

$$\frac{AA'}{A'B} = \frac{BB'}{B'C} = \frac{CC'}{C'D} = \frac{DD'}{D'A} = \frac{m}{n}.$$

We then have

$$\frac{A'B}{AB} = \frac{n}{m+n}, \ and \ \frac{BB'}{BC} = \frac{m}{m+n}$$

$$\begin{aligned}
\therefore \Delta A'BB' &= \frac{1}{2}A'B \cdot BB' \sin B \\
&= \frac{1}{2} \cdot \frac{mn}{(m+n)^2} \cdot AB \cdot BC \sin B \\
&= \frac{mn}{(m+n)^2} \cdot \Delta ABC.
\end{aligned}$$

Similarly, $$\Delta C'DD' = \frac{mn}{(m+n)^2} \cdot \Delta ADC$$

$$\therefore \Delta A'BB' + \Delta C'DD' = \frac{mn}{(m+n)^2} \cdot \ area \ of \ ABCD.$$

$$\therefore \Delta A'AD' + \Delta C'CB' = \frac{mn}{(m+n)^2} \cdot \ area \ of \ ABCD.$$

Thus the area of the four triangles at the corners

$$= \frac{2mn}{(m+n)^2} \cdot \ area \ of \ ABCD.$$

Hence the area of $A'B'C'D'$

$$\begin{aligned}
&= area \ of \ ABCD \left[1 - \frac{2mn}{(m+n)^2}\right] \\
&= area \ of \ ABCD \times \frac{m^2+n^2}{(m+n)^2}
\end{aligned}$$

$$\therefore \ area \ of \ A'B'C'D' : area \ of \ ABCD = m^2 + n^2 : (m+n)^2. \qquad \blacksquare$$

§ Problem 16.1.11. *If $ABCD$ be a quadrilateral inscribed in a circle, prove that*

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}}$$

and that the product of the segments into which one diagonal is divided by the other diagonal is

$$\frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}. \qquad \lozenge$$

§§ Solution. By $Art.$ 219, we have

$$\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

$$\therefore \tan^2\frac{B}{2} = \frac{1-\cos B}{1+\cos B} = \frac{(c+d)^2-(a-b)^2}{(a+b)^2-(c-d)^2}$$

$$= \frac{(c+d+a-b)(c+d-a+b)}{(a+b+c-d)(a+b-c+d)}$$
$$= \frac{2(s-b)2(s-a)}{2(s-d)2(s-c)}$$
$$\therefore \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}}.$$

Again, take the figure of *Art.* 221 and let O be the intersection of the diagonals. We then have

$$\frac{AO}{OC} = \frac{\Delta AOB}{\Delta COB} = \frac{\Delta AOD}{\Delta COD} = \frac{\Delta BAD}{\Delta BCD} = \frac{ad}{bc}$$
$$\therefore AO = \frac{ad}{ad+bc}\cdot AC \text{ and } OC = \frac{bc}{ad+bc}\cdot AC$$
$$\therefore AO\cdot OC = \frac{abcd}{(ad+bc)^2}\cdot AC^2$$
$$= \frac{abcd}{(ad+bc)^2}\cdot\frac{(ac+bd)(ad+bc)}{(ab+cd)}$$
$$= \frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}. \qquad \blacksquare$$

§ Problem 16.1.12. *If a, b, c and d be the sides of a quadrilateral taken in order, prove that*

$$d^2 = a^2 + b^2 + c^2 - 2ab\cos\alpha - 2bc\cos\beta - 2ca\cos\gamma$$

where α, β and γ denote the angles between the sides a and b, b and c, and c and a respectively. ♢

§§ Solution. Take the figure of *Art.* 221. Produce BA and CD to meet in E, so that the $\angle AED = \gamma$.

Let $AE = m$ and $DE = n$. We then have

$$d^2 = m^2 + n^2 - 2mn\cos\gamma \qquad (16.1)$$

$$b^2 + c^2 - 2bc\cos\beta = BD^2 = (a+m)^2 + n^2 - 2(a+m)n\cos\gamma, \text{ and} \qquad (16.2)$$

$$a^2 + b^2 - 2ab\cos\alpha = AC^2 = (c+n)^2 + m^2 - 2(c+n)m\cos\gamma. \qquad (16.3)$$

From (16.2) and (16.3), by addition, we have

$$a^2 + 2b^2 + c^2 - 2bc\cos\beta - 2ab\cos\alpha$$
$$= (a+m)^2 + (c+n)^2 + m^2 + n^2 - 2(an+cm+2mn)\cos\gamma \qquad (16.4)$$

Also, $\quad b^2 = (a+m)^2 + (c+n)^2 - 2(a+m)(c+n)\cos\gamma \qquad (16.5)$

From (16.4) and (16.5), by subtraction, we have

$$a^2 + b^2 + c^2 - 2bc\cos\beta - 2ab\cos\alpha$$
$$= m^2 + n^2 + 2ac\cos\gamma - 2mn\cos\gamma = d^2 + 2ac\cos\gamma$$
$$\therefore d^2 = a^2 + b^2 + c^2 - 2ab\cos\alpha - 2bc\cos\beta - 2ca\cos\gamma. \qquad \blacksquare$$

16.2 Regular Polygons

§ Problem 16.2.1. *Find, correct to .01 of an inch, the length of the perimeter of a regular decagon which surrounds a circle of radius one foot.* [$\tan 18^\circ = \cdot 32492.$] ♢

§§ Solution. If a be a side of the decagon, we have

$$a = 2r\tan\frac{\pi}{n} = (2\times 12\tan 18^\circ) \text{ inches}$$

$$\therefore \text{ the length of the perimeter } = (240 \tan 18^\circ) \ inches.$$

Now

$$\begin{aligned}
\tan 18^\circ &= \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}} \, [Art.\ 120] \\
&= \sqrt{\frac{6-2\sqrt{5}}{\sqrt{10+2\sqrt{5}}}} = \sqrt{\frac{3-\sqrt{5}}{5+\sqrt{5}}} \\
&= \sqrt{\frac{\left(3-\sqrt{5}\right)\left(5-\sqrt{5}\right)}{\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)}} \\
&= \sqrt{\frac{20-8\sqrt{5}}{25-5}} = \sqrt{\frac{5-2\sqrt{5}}{5}} \\
&= \sqrt{\frac{5-2\times 2.2360680}{5}} = \sqrt{\frac{\cdot 5278640}{5}} = \sqrt{\cdot 1055728} \\
&= \cdot 3249\ldots
\end{aligned}$$

[this may be obtained at once from the table of natural tangents].

Hence the length of the perimeter

$$= (240 \times \cdot 3249) \ inches \approx 77.98 \ inches.$$

■

§ Problem 16.2.2. *Find to* 3 *places of decimals the length of the side of a regular polygon of* 12 *sides which is circumscribed to a circle of unit radius.* ♢

§§ Solution.

Here

$$\begin{aligned}
a &= 2r \tan \frac{\pi}{n} = 2 \tan \frac{\pi}{12} = 2 \tan 15^\circ \\
&= 2\left(2-\sqrt{3}\right) = 2(2-1.73205) = 2(.26795) = .5359.
\end{aligned}$$

■

§ Problem 16.2.3. *Find the area of*

(1) a pentagon,

(2) a hexagon,

(3) an octagon,

(4) a decagon and

(5) a dodecagon,

each being a regular figure of side 1 *foot.*

$[\cot 18^\circ = 3.07768\ldots; \ \cot 36^\circ = 1.37638\ldots]$ ♢

§§ Solution. (1)

$$\begin{aligned}
Area &= n.\frac{a^2}{4} \cot \frac{\pi}{n} = 5 \times \frac{1}{4} \times \cot \frac{\pi}{5} = \frac{5}{4} \cot 36^\circ \\
&= \frac{5}{4} \times 1.3763819 = 1.720\ldots \ sq.ft.
\end{aligned}$$

(2) $Area = 6 \times \frac{1}{4} \times \cot \frac{\pi}{6} = \frac{3}{2} \cot 30^\circ = \frac{3}{2} \times 1.7320508 = 2.598\ldots \ sq.ft.$

(3) $Area = 8\times\frac{1}{4}\times\cot\frac{\pi}{8} = 2\cot 22\frac{1}{2}^{\circ} = 2\times 2.4142136 = 4.8284\ldots\ sq.ft.$

(4) $Area = 10\times\frac{1}{4}\times\cot\frac{\pi}{10} = \frac{5}{2}\times\cot 18^{\circ} = \frac{5}{2}\times 3.0776835 = 7.694\ldots\ sq.ft.$

(5) $Area = 12\times\frac{1}{4}\times\cot\frac{\pi}{12} = 3\cot 15^{\circ} = 3\times 3.7320508 = 11.196\ldots\ sq.ft.$

■

§ Problem 16.2.4. *Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each be* 24 *feet.* ◊

§§ Solution. For the octagon, the length of a side $(a) = \frac{24}{8}\ feet = 3\ feet$

$\therefore$ the area $= 8\times\frac{9}{4}\times\cot\frac{\pi}{8} = 18\cot 22\frac{1}{2}^{\circ} = 18\times 2.4142136 = 43.4558448\ sq.ft.$

For the hexagon, the length of a side $(a) = \frac{24}{6}\ feet = 4\ feet$

$\therefore$ the area $= 6\times\frac{16}{4}\times\cot\frac{\pi}{6} = 24\cot 30^{\circ} = 24\times 1.7320508 = 41.5692192\ sq.ft.$

Hence the required difference

$$= (43.4558448 - 41.5692192)\ sq.ft. = 1.8866\ldots\ sq.ft.$$

Otherwise thus :

Area of a polygon

$$= \frac{na^2}{4}\cot\frac{\pi}{n} = \frac{(na)^2}{4n}\cot\frac{\pi}{n} = \frac{p^2}{4n}\cot\frac{\pi}{n},$$ where p is the perimeter.

Hence the required difference

$$\begin{aligned}
&= \frac{(24)^2}{4\times 8}\cot\frac{\pi}{8} - \frac{(24)^2}{4\times 6}\cot\frac{\pi}{6}\\
&= 24\times 6\left[\frac{1}{8}\cot\frac{\pi}{8} - \frac{1}{6}\cot\frac{\pi}{6}\right]\\
&= 24\times 6\left[\frac{\sqrt{2}+1}{8} - \frac{\sqrt{3}}{6}\right]\\
&= 6\left[3\left(\sqrt{2}+1\right) - 4\sqrt{3}\right] = 6\,[7.2426408 - 6.9282032]\\
&= 6\times .3144376 = 1.8866\ldots\ sq.ft.
\end{aligned}$$

■

§ Problem 16.2.5. *A square, whose side is* 2 *feet, has its corners cut away so as to form a regular octagon ; find its area.* ◊

§§ Solution. If a be a side of the octagon, we have

$$a + 2a\cos 45^{\circ} = 2$$

$$\therefore a + 2a\times\frac{1}{\sqrt{2}} = 2$$

$$\therefore a = \frac{2}{\sqrt{2}+1} = 2\left(\sqrt{2}-1\right).$$

Hence the required area

$$= 8\times\frac{a^2}{4}\times\cot\frac{\pi}{8} = 8\times\left(\sqrt{2}-1\right)^2\times\left(\sqrt{2}+1\right)$$

$$= 8\left(\sqrt{2}-1\right) = 8 \times .4142136 = 3.3137\ldots \ sq.ft. \qquad \blacksquare$$

§ Problem 16.2.6. *Compare the areas and perimeters of octagons which are respectively inscribed in and circumscribed to a given circle, and show that the areas of the inscribed hexagon and octagon are as $\sqrt{27}$ to $\sqrt{32}$.* $\Diamond$

§§ Solution. We have, if ρ be the radius of the circle,

$$\frac{area\ oct.\ insc.}{area\ oct.\ circ.} = \frac{\frac{8}{2}\rho^2 \sin\frac{2\pi}{8}}{8\rho^2 \tan\frac{\pi}{8}}$$

$$= \frac{2\sin\frac{\pi}{8}\cos\frac{\pi}{8}}{2\dfrac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}}}$$

$$= \frac{2\cos^2\frac{\pi}{8}}{2} = \frac{1+\cos\frac{\pi}{4}}{2} = \frac{2+\sqrt{2}}{4}.$$

Again,

$$\frac{side\ oct.\ insc.}{side\ oct.\ circ.} = \frac{2\rho\sin\frac{\pi}{8}}{2\rho\tan\frac{\pi}{8}}$$

$$= \frac{2\cos\frac{\pi}{8}}{2} = \frac{\sqrt{2+2\cos\frac{\pi}{4}}}{2}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Also,

$$\frac{area\ hex.\ insc.}{area\ oct.\ insc.} = \frac{\frac{6}{2}R^2\sin\frac{2\pi}{6}}{\frac{8}{2}R^2\sin\frac{2\pi}{8}} = \frac{3\sin\frac{\pi}{3}}{4\sin\frac{\pi}{4}}$$

$$= \frac{\left(\frac{3\sqrt{3}}{2}\right)}{\left(\frac{4}{\sqrt{2}}\right)} = \frac{3\sqrt{6}}{8} = \sqrt{\frac{54}{64}} = \sqrt{\frac{27}{32}}. \qquad \blacksquare$$

§ Problem 16.2.7. *Prove that the radius of the circle described about a regular pentagon is nearly $\frac{17}{20}ths$ of the side of the pentagon.* $\Diamond$

§§ Solution. We have $R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{5}$.

$$\therefore \frac{R}{a} = \frac{1}{2\sin\frac{\pi}{5}} = 1 \div \frac{2\sqrt{10-2\sqrt{5}}}{4}$$

$$= \frac{2}{\sqrt{10-2\sqrt{5}}} = \frac{2\sqrt{10+2\sqrt{5}}}{100-20}$$

$$= \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{20}} = \frac{\sqrt{50+10\sqrt{5}}}{10}$$
$$= \frac{\sqrt{72.36}}{10} \approx \frac{8.5}{10}.$$
$$\therefore R = a \times \frac{17}{20}.$$
■

§ Problem 16.2.8. *If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as $2:3$.* ◊

§§ Solution. If $6a =$ the perimeter of the triangle and of the hexagon, then a side of the triangle $= 2a$, and a side of the hexagon $= a$.

$$\text{Area} = \frac{na^2}{4}\cot\frac{\pi}{n} = \frac{(na)^2}{4n}\cot\frac{\pi}{n}, \text{ where } na = \text{the perimeter.}$$

Hence we have

$$\frac{\textit{area of triangle}}{\textit{area of hexagon}} = \frac{\frac{1}{12}\cot\frac{\pi}{3}}{\frac{1}{24}\cot\frac{\pi}{6}} = \frac{2\cot\frac{\pi}{3}}{\cot\frac{\pi}{6}} = \frac{\left(\frac{2}{\sqrt{3}}\right)}{\sqrt{3}} = \frac{2}{3}.$$
■

§ Problem 16.2.9. *If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2:\sqrt{5}$.* ◊

§§ Solution. If $10a =$ the perimeter of the pentagon and of the decagon, then a side of the pentagon $= 2a$, and a side of the decagon $= a$.

Hence we have

$$\frac{\textit{area of pentagon}}{\textit{area of decagon}} = \frac{\frac{1}{20}\cot\frac{\pi}{5}}{\frac{1}{40}\cot\frac{\pi}{10}} = \frac{2\cot\frac{\pi}{5}}{\cot\frac{\pi}{10}}$$
$$= 2\frac{\cos\frac{\pi}{5}}{\sin\frac{\pi}{5}} \cdot \frac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}}$$
$$= \frac{2\cos\frac{\pi}{5}}{2\cos^2\frac{\pi}{10}}$$
$$= \frac{2\cos\frac{\pi}{5}}{1+\cos\frac{\pi}{5}} = \frac{2\left(\sqrt{5}+1\right)}{4} \div \left[1+\frac{\sqrt{5}+1}{4}\right]$$
$$= \frac{2\left(\sqrt{5}+1\right)}{\sqrt{5}\left(\sqrt{5}+1\right)} = \frac{2}{\sqrt{5}}.$$
■

§ Problem 16.2.10. *Prove that the sum of the radii of the circles, which are respectively inscribed in and circumscribed about a regular polygon of n sides, is*

$$\frac{a}{2}\cot\frac{\pi}{2n},$$

where a is a side of the polygon. ◊

§§ Solution. We have

$$\begin{aligned} r+R &= \frac{a}{2}\cot\frac{\pi}{n}+\frac{a}{2}\operatorname{cosec}\frac{\pi}{n} \\ &= \frac{a}{2}\left[\frac{\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{n}}\right] \\ &= \frac{a}{2}\left[\frac{2\cos^2\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\right] \\ &= \frac{a}{2}\cot\frac{\pi}{2n}. \end{aligned}$$

■

§ Problem 16.2.11. *Of two regular polygons of n sides, one circumscribes and the other is inscribed in a given circle. Prove that the perimeters of the circumscribing polygon, the circle, and the inscribed polygon are in the ratio*

$$\sec\frac{\pi}{n} : \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} : 1,$$

and that the areas of the polygons are in the ratio $\cos^2\frac{\pi}{n} : 1$. ♢

§§ Solution. If ρ be the radius of the circle, the perimeters are as

$$\begin{aligned} & 2n\rho\tan\frac{\pi}{n} : 2n\rho\sin\frac{\pi}{n} : 2\pi\rho \\ \therefore\ & \tan\frac{\pi}{n} : \sin\frac{\pi}{n} : \frac{\pi}{n} \\ \therefore\ & \sec\frac{\pi}{n} : 1 : \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} \end{aligned}$$

or, in descending order of magnitude,

$$\therefore \sec\frac{\pi}{n} : \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} : 1.$$

$$\text{Also, } \frac{\textit{area pol. insc.}}{\textit{area pol. circ.}} = \left[\frac{n}{2}\rho^2\sin\frac{2\pi}{n}\right] \div \left[n\rho^2\tan\frac{\pi}{n}\right] = \frac{\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{\left(\dfrac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}\right)} = \cos^2\frac{\pi}{n} : 1.$$

■

§ Problem 16.2.12. *Given that the area of a polygon of n sides circumscribed about a circle is to the area of the circumscribed polygon of $2n$ sides as $3:2$, find n.* ♢

§§ Solution. If ρ be the radius of the circle, we have

$$n\rho^2\tan\frac{\pi}{n} : 2n\rho^2\tan\frac{\pi}{2n} = 3:2$$

$$\therefore 3\tan\frac{\pi}{2n} = \tan\frac{\pi}{n} = \frac{2\tan\frac{\pi}{2n}}{1-\tan^2\frac{\pi}{2n}}$$

where $\tan\frac{\pi}{2n}=0$, or $2 = 3-3\tan^2\frac{\pi}{2n}$, so that $\tan\frac{\pi}{2n} = \pm\frac{1}{\sqrt{3}}$.

The only value admissible is $\tan\dfrac{\pi}{2n} = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$.
$\therefore 2n = 6$ and $n = 3$. ■

§ Problem 16.2.13. *Prove that the area of a regular polygon of* $2n$ *sides inscribed in a circle is a mean proportional between the areas of the regular inscribed and circumscribed polygons of* n *sides.* ♢
§§ Solution. If ρ be the radius of the circle, the areas of the polygons are

$$n\rho^2\sin\frac{\pi}{n} : \frac{n}{2}\rho^2\sin\frac{2\pi}{n} \text{ and } n\rho^2\tan\frac{\pi}{n}$$

respectively. Now

$$\begin{aligned} n\rho^2\sin\frac{\pi}{n} &= \sqrt{n^2\rho^4\sin^2\frac{\pi}{n}} \\ &= \sqrt{n\rho^2\tan\frac{\pi}{n}\times n\rho^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}} \\ &= \sqrt{n\rho^2\tan\frac{\pi}{n}\times\frac{n}{2}\rho^2\sin\frac{2\pi}{n}}. \end{aligned}$$

■

§ Problem 16.2.14. *The area of a regular polygon of* n *sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as* 3 *is to* 4*. Find the value of* n*.* ♢
§§ Solution. If ρ be the radius of the circle, we have

$$\frac{n}{2}\rho^2\sin\frac{2\pi}{n} : n\rho^2\tan\frac{\pi}{n} :: 3 : 4$$

$$\therefore \frac{\sin\frac{2\pi}{n}}{2\tan\frac{\pi}{n}} = \frac{2\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{\dfrac{2\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}} = \cos^2\frac{\pi}{n} = \frac{3}{4}$$

$$\therefore \cos\frac{\pi}{n} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$

$$\therefore n = 6.$$

■

§ Problem 16.2.15. *The interior angles of a polygon are in* $A.\ P.$ *; the least angle is* $120°$ *and the common difference is* $5°$ *; find the number of sides.* ♢
§§ Solution. If n be the number of sides of the polygon, the angles of the polygon are $120°, 125°, 130°, \ldots$ to n terms.

Their sum $= \dfrac{n}{2}\left[2\times 120° + (n-1)5°\right] = (2n-4)90°$
$[By\ Euc.\ I.\ 32,\ Cor.\ 1]$.

$$\therefore 120n + \left(n^2 - n\right)\frac{5}{2} = 180n - 360$$

$$\therefore n^2 - 25n + 144 = 0$$

$$\therefore n = 9 \text{ or } 16.$$

Now the greatest angle of the polygon must be less than $180°$ and with $n = 9$, this would be $120° + 40°$, i.e. $160°$ and with $n = 16$, the greatest angle would be $120° + 75°$, i.e. $195°$.

Hence the value 9 of n is applicable only, i.e. the polygon has 9 sides. ■

§ **Problem 16.2.16.** *There are two regular polygons the number of sides in one being double the number in the other, and an angle of one polygon is to an angle of the other as* 9 *to* 8 *; find the number of sides of each polygon.* $\Diamond$

§§ **Solution**. Let $2n$ and n be the number of sides in the two polygons respectively. By $Art.$ 223, we then have

$$\frac{(4n-4)90}{2n} : \frac{(2n-4)90}{n} = 9 : 8$$
$$\therefore 16n - 16 = 18n - 36$$
$$\therefore n = 10.$$

So that the polygons have 20 sides and 10 sides respectively. ■

§ **Problem 16.2.17.** *Show that there are eleven pairs of regular polygons such that the number of degrees in the angle of one is to the number in the angle of the other as* $10 : 9$*. Find the number of sides in each.* $\Diamond$

§§ **Solution**. Let m and n be the number of sides in the polygons, so that

$$\frac{(2m-4)90}{m} : \frac{(2n-4)90}{n} = 10 : 9$$
$$\therefore 18 - \frac{36}{m} = 20 - \frac{40}{n}$$
$$\therefore mn = 20m - 18n$$
$$\therefore n = 20\left(\frac{m+18-18}{m+18}\right) = 20 - \frac{360}{m+18}.$$

Now n must be positive; also m and n must be > 2, since there must be more than 2 sides in a polygon. Hence,

$$\begin{aligned} \because 360 &= 360 \times 1 = 180 \times 2 = 120 \times 3 = 90 \times 4 \\ &= 72 \times 5 = 60 \times 6 = 45 \times 8 = 40 \times 9 \\ &= 36 \times 10 = 30 \times 12 = 24 \times 15 = 20 \times 18 \end{aligned}$$
$$\therefore m + 18 = 360,\ 180,\ 120,\ 90,\ 72,\ 60,\ 45,\ 40,\ 36,\ 30,\ or\ 24$$
$$\therefore m = 342,\ 162,\ 102,\ 72,\ 54,\ 42,\ 27,\ 22,\ 18,\ 12,\ or\ 6.$$

Also, the corresponding values of

$$n = \frac{20m}{m+18} = 19,\ 18,\ 17,\ 16,\ 15,\ 14,\ 12,\ 11,\ 10,\ 8,\ or\ 5.$$

Thus there are eleven pairs of polygons. ■

§ **Problem 16.2.18.** *The side of a base of a square pyramid is* a *feet and its vertex is at a height of* h *feet above the center of the base ; if* θ *and* ϕ *be respectively the inclinations of any face to the base, and of any two faces to one another, prove that*

$$\tan\theta = \frac{2h}{a}\ and\ \tan\frac{\phi}{2} = \sqrt{1 + \frac{a^2}{2h^2}}.$$ $\Diamond$

§§ **Solution**. Let V be the vertex and E be the center of the base $ABCD$ of the pyramid. Let H be the middle point of AD. We then have

$$\tan\theta = \tan\angle VHE = \frac{h}{\left(\frac{a}{2}\right)} = \frac{2h}{a}.$$

Again, draw BF and DF perpendicular to VA, so that $\phi = \angle BFD$ and let the $\angle VAE = \alpha$.

$$\therefore \tan\frac{\phi}{2} = \frac{BE}{EF} = \frac{\left(\frac{a}{\sqrt{2}}\right)}{AE\sin\alpha} = \frac{1}{\sin\alpha} = \frac{VA}{VE}$$
$$= \frac{\sqrt{VE^2+AE^2}}{VE} = \frac{\sqrt{h^2+\frac{a^2}{2}}}{h} = \sqrt{1+\frac{a^2}{2h^2}}. \qquad \blacksquare$$

§ Problem 16.2.19. *A pyramid stands on a regular hexagon as base. The perpendicular from the vertex of the pyramid on the base passes through the center of the hexagon, and its length is equal to that of a side of the base. Find the tangent of the angle between the base and any face of the pyramid, and also of half the angle between any two side faces.* ◊

§§ Solution. Let V be the vertex and E be the center of the base of the pyramid.

Let AB and BC be the two sides of the base and D be the middle point of AB.

Then, if θ be the angle between the base and any face of the pyramid and a be a side of the base, we have

$$\tan\theta = \frac{VE}{ED} = \frac{a}{a\sin 60°} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$

Again, since $VE = EB$, $\therefore \angle VBE = 45°$.

Draw AH perpendicular to VB and AK perpendicular to EB ; then HK is perpendicular to VB and to AK and if ϕ be the angle between any two side faces, we have

$$\tan\frac{\phi}{2} = \tan\angle AHK = \frac{AK}{HK} = \frac{a\sin 60°}{BK\sin 45°}$$
$$= \frac{a\sqrt{3}}{2} \div \left(\frac{a}{2}\cdot\frac{1}{\sqrt{2}}\right) = \sqrt{6}. \qquad \blacksquare$$

§ Problem 16.2.20. *A. regular pyramid has for its base a polygon of n sides, each of length a and the length of each slant side is l ; prove that the cosine of the angle between two adjacent lateral faces is*

$$\frac{4l^2\cos\frac{2\pi}{n} + a^2}{4l^2 - a^2}.$$ ◊

§§ Solution. Let V be the vertex and E be the center of the base of the polygon.

Let AB and BC be two sides of the base.

We have the $\angle AEB = \frac{2\pi}{n}$ and the $\angle ABE = \frac{1}{2}\left(\pi - \frac{2\pi}{n}\right) = \frac{\pi}{2} - \frac{\pi}{n}$.

Draw AH perpendicular to VB and AK perpendicular to EB ; then HK is perpendicular to VB and to AK and if ϕ be the angle between two adjacent lateral faces, we have

$$\tan\frac{\phi}{2} = \frac{AK}{HK} = \frac{a\sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right)}{BK\sin\angle VBE} = \frac{a\cos\frac{\pi}{n}}{a\sin\frac{\pi}{n}\cdot\frac{VE}{l}}$$

$$= \frac{l \cot \dfrac{\pi}{n}}{\sqrt{l^2 - BE^2}} = \frac{l \cot \dfrac{\pi}{n}}{\sqrt{l^2 - \left(\dfrac{a}{2} \operatorname{cosec} \dfrac{\pi}{n}\right)^2}}$$

$$\therefore \frac{\tan^2 \dfrac{\phi}{2}}{1} = \frac{l^2 \cot^2 \dfrac{\pi}{n}}{l^2 - \dfrac{a^2}{4} \operatorname{cosec}^2 \dfrac{\pi}{n}} = \frac{4l^2 \cos^2 \dfrac{\pi}{n}}{4l^2 \sin^2 \dfrac{\pi}{n} - a^2}$$

$$\therefore \cos\phi = \frac{1 - \tan^2 \dfrac{\phi}{2}}{1 + \tan^2 \dfrac{\phi}{2}}$$

$$= \frac{-a^2 - 4l^2 \left(\cos^2 \dfrac{\pi}{n} - \sin^2 \dfrac{\pi}{n}\right)}{4l^2 \left(\sin^2 \dfrac{\pi}{n} + \cos^2 \dfrac{\pi}{n}\right) - a^2}$$

$$= -\frac{a^2 + 4l^2 \cos \dfrac{2\pi}{n}}{4l^2 - a^2}.$$

The cosine of the acute angle between two faces

$$= \frac{4l^2 \cos \dfrac{2\pi}{n} + a^2}{4l^2 - a^2}. \qquad \blacksquare$$

Chapter 17

Trigonometrical Ratios of Small Angles, Area of A Circle, DIP of The Horizon.

17.1 Ratios of Small Angles

$$\left[\pi = 3.14159265;\ \frac{1}{\pi} = .31831\ldots\right]$$

Find, to 5 places of decimals, the value of

§ Problem 17.1.1. $\sin 7'$. ◊

§§ Solution. Since $7' = \dfrac{7^\circ}{60} = \dfrac{7\pi^c}{180 \times 60}$, we have

$$\sin 7' = \sin\left(\frac{7\pi}{180 \times 60}\right)^c = \frac{7\pi}{180 \times 60} = \frac{7 \times 3.14159265}{180 \times 60} = .00204. \quad \blacksquare$$

§ Problem 17.1.2. $\sin 15'$. ◊

§§ Solution. Since $15'' = \dfrac{1'}{4} = \dfrac{1^\circ}{4 \times 60} = \dfrac{\pi^c}{180 \times 4 \times 60}$, we have

$$\begin{aligned}\sin 15'' &= \sin\left(\frac{\pi}{180 \times 4 \times 60}\right)^c \\ &= \frac{\pi}{180 \times 4 \times 60} = \frac{3.14159265}{180 \times 4 \times 60} = .00007. \quad \blacksquare\end{aligned}$$

§ Problem 17.1.3. $\sin 1'$. ◊

§§ Solution. Since $1' = \dfrac{1^\circ}{60} = \dfrac{\pi^c}{180 \times 60}$, we have

$$\sin 1' = \sin\left(\frac{\pi}{180 \times 60}\right)^c$$

$$= \frac{\pi}{180 \times 60} = .00029. \qquad \blacksquare$$

§ Problem 17.1.4. $\cos 15'$. $\Diamond$

§§ Solution. Since $15' = \dfrac{15^\circ}{60} = \dfrac{1^\circ}{4} = \dfrac{\pi^c}{180 \times 4}$, we have

$$\sin 15' = \sin\left(\frac{\pi}{180 \times 4}\right)^c$$
$$= \frac{\pi}{180 \times 4} \approx .0043633.$$

Also, $\cos 15' = \sqrt{1 - \sin^2 15'} = [1 - .00001904]^{\frac{1}{2}}$

By the Binomial Theorem,

$$\approx 1 - \frac{1}{2}[.00001904]$$
$$= 1 - .00000952 = .99999 \text{ to } 5 \text{ places of decimals.} \qquad \blacksquare$$

§ Problem 17.1.5. $\operatorname{cosec} 8''$. $\Diamond$

§§ Solution. $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta} \approx \dfrac{1}{\theta}$.

Here $\qquad \theta = \dfrac{8}{60 \times 60} \times \dfrac{\pi}{180}$

$$\therefore \frac{1}{\theta} = \frac{900}{2} \times \frac{180}{\pi} = \frac{900}{2} \times 57.2957795 = 25783.10077. \qquad \blacksquare$$

§ Problem 17.1.6. $\sec 5'$. $\Diamond$

§§ Solution. Since $5' = \dfrac{1^\circ}{12} = \dfrac{\pi^c}{180 \times 12}$.

We have

$$\sin 5' = \sin\left(\frac{\pi}{180 \times 12}\right)^c = \frac{\pi}{180 \times 12} \approx .001454.$$

Also, $\sec 5' = \dfrac{1}{\cos 5'} = \left[1 - \sin^2 5'\right]^{-\frac{1}{2}}$

$$= [1 - .000002114]^{-\frac{1}{2}}$$

By the Binomial Theorem,

$$\approx 1 + \frac{1}{2}[.000002114] = 1 + .000001057 \approx 1.0000011. \qquad \blacksquare$$

Solve approximately the equations

§ Problem 17.1.7. $\sin \theta = .01$. $\Diamond$

§§ Solution. Here θ is very small.

$$\therefore \theta \approx .01^c = .01 \times \frac{180^\circ}{\pi}$$
$$= .01 \times 206265'' \approx 2063'' = 34'23''. \qquad \blacksquare$$

§ Problem 17.1.8. $\sin \theta = .48$. $\Diamond$

§§ Solution. Since $\sin \theta$ is nearly equal to $\dfrac{1}{2}$, θ must be nearly equal to $\dfrac{\pi}{6}$.

Let then $\theta = \dfrac{\pi}{6} - x$, where x is very small;

$$\therefore .48 = \sin\left(\frac{\pi}{6} - x\right) = \sin\frac{\pi}{6}\cos x - \cos\frac{\pi}{6}\sin x = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x.$$

Since x is very small, we have

$$\cos x = 1 \; and \; \sin x \approx x; \; \therefore .48 = \frac{1}{2} - \frac{\sqrt{3}}{2}x$$

$$\therefore x = .02 \times \frac{2}{\sqrt{3}} \; radians \; = \left(\frac{.04 \times \sqrt{3}}{3}\right)^c = .023094^c \approx 1°19'23''.$$

$$\therefore \theta = 30° - 1°29'23'' = 28°40'37''. \qquad \blacksquare$$

§ Problem 17.1.9. $\cos\left(\frac{\pi}{3} + \theta\right) = .49.$ ◊

§§ Solution. $\cos\left(\frac{\pi}{3} + \theta\right) = .49.$

Since $.49$ is very nearly equal to $\frac{1}{2}$, which is the value of $\cos\frac{\pi}{3}$, it follows that θ must be very small.

The equation may be written

$$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = .49.$$

Also, since θ is very small, we have

$$\cos\theta = 1 \; and \; \sin\theta \approx \theta$$

$$\therefore \frac{1}{2} - \frac{\sqrt{3}}{2}\theta = .49$$

$$\therefore \theta = .01 \times \frac{2}{\sqrt{3}} \; radians \; = \left(\frac{.02 \times \sqrt{3}}{3}\right)^c = .011547^c$$

$$= (.011547 \times 206265)'' \approx 2382'' \approx 39'42''. \qquad \blacksquare$$

§ Problem 17.1.10. $\cos\theta = .999.$ ◊

§§ Solution. $\cos\theta = .999.$ Hence θ is very small.

We have
$$1 - \frac{\theta^2}{2} = 1 - .001;$$

$$\therefore \frac{\theta^2}{2} = .001; \; \therefore \theta^2 = .002$$

$$\therefore \theta \approx .0447213^c$$

$$= (.0447213 \times 57.2957795)° = 2°33'44''. \qquad \blacksquare$$

§ Problem 17.1.11. *Find approximately the distance at which a halfpenny, which is an inch in diameter, must be placed so as to just hide the moon, the angular diameter of the moon, that is the angle its diameter subtends at the observer's eye, being taken to be $30'$.* ◊

§§ Solution. If x be the required distance in inches, we have

$$\frac{1}{x} = \text{ the number of radians in } 30' = \frac{30}{60} \times \frac{\pi}{180}$$

$$\therefore x = 2 \times \frac{180}{\pi} = 2 \times 57.2957795 = 114.59\ldots \; inches. \qquad \blacksquare$$

§ Problem 17.1.12. *A person walks in a straight line toward a very distant object and observes that at three points A, B and C the angles of elevation of the top of the object are α, 2α and 3α respectively; prove that*

$$AB \approx 3BC.$$ ◊

§§ Solution. Let E be the top of the object.

In the ΔABE, we have the $\angle AEB = \angle BAE = \alpha$, so that $BE = AB$.

In the ΔBEC, the $\angle BEC = \alpha$ and we have

$$\frac{BE}{BC} = \frac{\sin 3\alpha}{\sin \alpha}$$

$$\therefore \frac{AB}{BC} = \frac{\sin 3\alpha}{\sin \alpha} = 3 - 4\sin^2 \alpha.$$

Since the object is very distant, α is very small.

$$\therefore \frac{AB}{BC} \approx 3, \therefore AB \approx 3BC. \qquad \blacksquare$$

§ Problem 17.1.13. *If θ be the number of radians in an angle which is less than a right angle, prove that*

$$\cos\theta \ is \ < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}. \qquad \Diamond$$

§§ Solution. We have

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \ and \ \sin\frac{\theta}{2} > \frac{\theta}{2} - \frac{\theta^3}{32}$$

$$\therefore \cos\theta < 1 - 2\left(\frac{\theta}{2} - \frac{\theta^3}{32}\right)^2$$

$$< 1 - 2\left[\frac{\theta^2}{4} - \frac{\theta^4}{32} + \frac{\theta^6}{(32)^2}\right]$$

$$< 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16} - \frac{2\theta^6}{(32)^2}.$$

$$\therefore \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}. \qquad \blacksquare$$

§ Problem 17.1.14. *Prove the theorem of Euler, viz. that*

$$\sin\theta = \theta\cdot\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\ ad.\ inf. \qquad \Diamond$$

§§ Solution. We have

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2^2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}$$

$$= 2^3\sin\frac{\theta}{2^3}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2} = \ldots\ldots$$

$$= 2^n\sin\frac{\theta}{2^n}\times\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\ldots\cos\frac{\theta}{2^n}.$$

Make n indefinitely great so that, by *Art.* 228 *Cor.*,

$$2^n\sin\frac{\theta}{2^n} = \theta.$$

$$\therefore \sin\theta = \theta\cdot\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\ ad.\ inf. \qquad \blacksquare$$

§ Problem 17.1.15. *Prove that*

$$\left(1 - \tan^2\frac{\theta}{2}\right)\left(1 - \tan^2\frac{\theta}{2^2}\right)\left(1 - \tan^2\frac{\theta}{2^3}\right)\ldots\ ad.\ inf. = \theta\cdot\cot\theta. \qquad \Diamond$$

§§ Solution. We have

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$$

$$\tan\frac{\theta}{2} = \frac{2\tan\dfrac{\theta}{2^2}}{1-\tan^2\dfrac{\theta}{2^2}}$$
$$\tan\frac{\theta}{2^2} = \frac{2\tan\dfrac{\theta}{2^3}}{1-\tan^2\dfrac{\theta}{2^3}}$$
$$\ldots = \ldots$$

Hence, by multiplication, we have

$$\tan\theta = \frac{2^n\tan\dfrac{\theta}{2^n}}{\left(1-\tan^2\dfrac{\theta}{2}\right)\left(1-\tan^2\dfrac{\theta}{2^2}\right)\ldots}$$

Now $$2^n\tan\frac{\theta}{2^n} = \theta\times\frac{\tan\dfrac{\theta}{2^n}}{\dfrac{\theta}{2^n}}$$
$$= \theta, \text{ when } n \text{ is indefinitely increased;}$$
$$\therefore \tan\theta = \frac{\theta}{\left(1-\tan^2\dfrac{\theta}{2}\right)\left(1-\tan^2\dfrac{\theta}{2^2}\right)\ldots}$$
$$\therefore \left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)\ldots = \theta\cot\theta.$$ ■

17.2 Area of A Circle

$$\left[\textit{Assume that } \pi = 3.14159\ldots,\ \frac{1}{\pi} = .31831 \textit{ and } \log\pi = .49715.\right]$$

§ Problem 17.2.1. *Find the area of a circle whose circumference is* 74 *feet.* ◊

§§ Solution. We have $2\pi R = 74\ feet$, $\therefore R = \dfrac{37}{\pi}\ feet$.

$$\therefore \textit{ the area } = \pi R^2 = \pi\times\frac{(37)^2}{\pi^2} = \frac{(37)^2}{\pi}$$
$$= 1369\times .31831 \approx 435.77\ sq.\ ft.$$ ■

§ Problem 17.2.2. *The diameter of a circle is* 10 *feet; find the area of a sector whose arc is* $22\frac{1}{2}^\circ$. ◊

§§ Solution. The required area

$$= \frac{22\frac{1}{2}}{360}\times \text{ area of whole circle}$$
$$= \frac{1}{16}\times\left(\pi\times 5^2\right) = \frac{100\pi}{64}$$
$$= \frac{314.1592}{64} = 4.9087\ sq.\ ft.$$ ■

§ Problem 17.2.3. *The area of a certain sector of a circle is* 10 *square feet; if the radius of the circle be* 3 *feet, find the angle of the sector.* ◊

§§ Solution. We have $\frac{1}{2} \times 3^2 \times \alpha = 10$;

$$\therefore \alpha = \frac{20}{9}\ radians\ = \frac{20}{9} \times \frac{180^\circ}{\pi} = \frac{20}{9} \times 57.2957795^\circ \approx 127^\circ 19' 26''. \blacksquare$$

§ Problem 17.2.4. *The perimeter of a certain sector of a circle is* 10 *feet; if the radius of the circle be* 3 *feet, find the area of the sector.* ♢

§§ Solution. We have

$$2R + R\alpha = 10\ and\ R = 3\ feet;\ \therefore 3\alpha = 4,\ \therefore \alpha = \frac{4}{3}.$$

Hence the required area

$$= \frac{1}{2}R^2 \cdot \alpha = \frac{1}{2} \times 9 \times \frac{4}{3} = 6\ sq.\ ft. \qquad \blacksquare$$

§ Problem 17.2.5. *A strip of paper, two miles long and* .003 *of an inch thick, is rolled up into a solid cylinder; find approximately the radius of the circular ends of the cylinder.* ♢

§§ Solution. If d be the width of the paper and r be the radius of the cylinder, we have

$$\pi r^2 d = 2 \times 1760 \times 36 \times d \times .003$$
$$\therefore r^2 = \frac{6 \times 176 \times 36}{100\pi}$$
$$\therefore r = 6 \times 4 \times \sqrt{\frac{6 \times 11}{100\pi}} = 6 \times 4 \times \sqrt{6 \times 11 \times .0031831}$$
$$= 6 \times 4 \times .45835 = 11.0004\ inches. \qquad \blacksquare$$

§ Problem 17.2.6. *A strip of paper, one mile long, is rolled tightly up into a solid cylinder, the diameter of whose circular ends is* 6 *inches; find the thickness of the paper.* ♢

§§ Solution. If x inch be the required thickness, we have

$$\pi \times 3^2 = 1760 \times 36 \times x$$
$$\therefore x = \frac{\pi}{1760 \times 4} = \frac{3.14159265}{11 \times 80 \times 8} = .00044625\ inch. \qquad \blacksquare$$

§ Problem 17.2.7. *Given two concentric circles of radii* r *and* $2r$*; two parallel tangents to the inner circle cut off an arc from the outer circle; find its length.* ♢

§§ Solution. Let O be the center of the circles and AB be the arc cut off and let the tangents through A and B meet the inner circle in C and D respectively.

We the have
$$\cos \angle AOC = \frac{r}{2r} = \frac{1}{2},$$
$$\therefore \angle AOC = \frac{\pi}{3} = \angle BOD;\ \therefore \angle AOB = \frac{\pi}{3}.$$
$$\therefore\ the\ arc\ AB = 2r \times \frac{\pi}{3} = \frac{2}{3}\pi r. \qquad \blacksquare$$

§ Problem 17.2.8. *The circumference of a semi-circle is divided into two arcs such that the chord of one is double that of the other. Prove that the sum of the areas of the two segments cut off by these chords is to the area of the semicircle as* 27 *is to* 55.

$$\left[\pi = \frac{22}{7}.\right]$$ ♢

§§ Solution. Take the figure of *Art.* 106. Let the circumference OPQ be divided into the two arcs OP and PQ, such that the chord OP ($2x$, say) is double the chord $PQ(x)$. The $\angle OPQ$ is a right angle, being an angle in a semicircle and the other angles are as marked in the figure.

We then have $\tan A = \frac{x}{2x} = \frac{1}{2}$ *and* $\therefore \sin A = \frac{1}{\sqrt{5}}$

Also, if r be the radius, then

$$x = 2r\sin A = \frac{2r}{\sqrt{5}}.$$

The sum of the areas of the two segments OP and OQ

$$\begin{aligned}
&= \text{the area of the semicircle } - \Delta OPQ \\
&= \frac{\pi r^2}{2} - \frac{1}{2}\cdot 2x\cdot x = \frac{\pi r^2}{2} - x^2 \\
&= \frac{\pi r^2}{2} - \frac{4r^2}{5} = r^2\left(\frac{\pi}{2} - \frac{4}{5}\right) \\
&= r^2\left(\frac{11}{7} - \frac{4}{5}\right) = \frac{27}{35}r^2.
\end{aligned}$$

Hence the required ratio

$$= \frac{27}{35}r^2 : \frac{\pi r^2}{2} = \frac{27}{35} : \frac{11}{7} = 27 : 55.$$ ■

§ Problem 17.2.9. *If each of three circles, of radius a, touch the other two, prove that the area included between them is nearly equal to $\frac{4}{25}a^2$.* ◊

§§ Solution. Let A, B and C be the centers of the three equal circles and D, E and F be the points of contact of circles A and B, B and C and C and A respectively.

Join AB, BC and CA. ABC is an equilateral triangle of side $2a$.

The required area = the area of the triangle ABC - the areas of the three equal sectors DAF, FCE and DBE

$$\begin{aligned}
\therefore \text{ the required area } &= \frac{1}{2}(2a)^2\sin\frac{\pi}{3} - 3\left(\frac{a^2}{2}\times\frac{\pi}{3}\right) \\
&= a^2\sqrt{3} - a^2\frac{\pi}{2} = a^2\left(\sqrt{3} - \frac{\pi}{2}\right) \\
&= a^2\left(1.732 - 1.5708\right) \approx a^2 \times .16 \\
&= \frac{16a^2}{100} = \frac{4}{25}a^2.
\end{aligned}$$ ■

§ Problem 17.2.10. *Six equal circles, each of radius a, are placed so that each touches two others, their centers being all on the circumference of another circle; prove that the area which they enclose is*

$$2a^2\left(3\sqrt{3} - \pi\right).$$ ◊

§§ Solution. Let A, B, C, D, E and F be the centers of the six equal circles and let G, H, K, L, M and N be the points of contact of circles A and B, B and C, C and D, D and E, E and F and F and A respectively.

Join AB, BC, CD, DE, EF and FA.

Then $ABCDEF$ is a regular hexagon of side $2a$.

The required area = the area of the hexagon $ABCDEF$ - the areas of the six equal sectors GAN, HBG, KCH, LDK, MEL and NFM

$$\begin{aligned}\therefore \text{ the required area } &= 6 \times \frac{4a^2}{4} \cot\frac{\pi}{6} - 6\left(a^2 \times \frac{\pi}{3}\right)\\ &= 6a^2\sqrt{3} - 2a^2\pi = 2a^2\left(3\sqrt{3} - \pi\right).\end{aligned}$$

■

§ Problem 17.2.11. *From the vertex A of a triangle, a straight line AD is drawn making an angle θ with the base and meeting it at D. Prove that the area common to the circumscribing circles of the triangles ABD and ACD is*

$$\frac{1}{4}\left(b^2\gamma + c^2\beta - bc\sin A\right)\operatorname{cosec}^2\theta,$$

where β and γ are the number of radians in the angles B and C respectively. ♢

§§ Solution. Let O and O' be the centers and R and R' be the radii of the circles respectively. We then have

$$R = \frac{AB}{2\sin\angle ADB} = \frac{c}{2\sin\theta} \text{ and } R' = \frac{b}{2\sin\theta}.$$

Also, $\quad \angle AOD = 2\angle ABD = 2\beta, \text{ and } \angle AO'D = 2\angle ACD = 2\gamma.$

The required area

$$\begin{aligned}&= \text{ area of sector } AOD - \text{ area of triangle } AOD\\ &+ \text{ area of sector } AO'D - \text{ area of triangle } AO'D\\ &= R^2\beta - \frac{1}{2}R^2\sin 2B + R'^2\gamma - \frac{1}{2}R'^2\sin 2C\\ &= \frac{1}{4\sin^2\theta}\left(c^2\beta - \frac{1}{2}c^2\sin 2B + b^2\gamma - \frac{1}{2}b^2\sin 2C\right)\\ &= \frac{1}{4}\left(b^2\gamma + c^2\beta - c^2\sin B\cos B - b^2\sin C\cos C\right)\operatorname{cosec}^2\theta\\ &= \frac{1}{4}\left(b^2\gamma + c^2\beta - bc\sin C\cos B - bc\sin B\cos C\right)\operatorname{cosec}^2\theta\\ &= \frac{1}{4}\left[b^2\gamma + c^2\beta - bc\sin(B+C)\right]\operatorname{cosec}^2\theta\\ &= \frac{1}{4}\left(b^2\gamma + c^2\beta - bc\sin A\right)\operatorname{cosec}^2\theta.\end{aligned}$$

■

17.3 Dip of The Horizon

[Unless otherwise stated, the earth's radius may be taken to be 4000 **miles.]**

§ Problem 17.3.1. *Find in degrees, minutes and seconds, the dip of the horizon from the top of a mountain* 4200 *feet high, the earth's radius being* 21×10^6 *feet.* ♢

§§ Solution. The dip of the horizon

$$\begin{aligned}&= \sqrt{\frac{2h}{r}} \text{ radians } = \sqrt{\frac{2 \times 4400}{21 \times 10^6}}\\ &= \left(\frac{2}{10^2}\sqrt{\frac{22}{21}} \cdot \frac{180}{\pi}\right) \text{ degrees}\\ &= \frac{2 \times 1.02}{100} \times 57.2957795^\circ = 1^\circ 10' 8''.\end{aligned}$$

■

§ **Problem 17.3.2.** *The lamp of a lighthouse is* 196 *feet high; how far off can it be seen ?* ◊

§§ **Solution**. The required distance

$$= \sqrt{2hr} = \sqrt{2 \times 196 \times 21 \times 10^6} \; feet \; = 14000\sqrt{42} \; feet$$
$$= \frac{14000 \times 6.4807}{1760 \times 3} \; miles \approx 17.2 \; miles.$$ ■

§ **Problem 17.3.3.** *If the radius of the earth be* 4000 *miles, find the height of a balloon when the dip is* 1°.

Find also the dip when the balloon is 2 *miles high.* ◊

§§ **Solution**. Since the dip $= \sqrt{\frac{2h}{r}}$ radians, we have $\frac{\pi}{180} = \sqrt{\frac{2h}{4000}}$, so that

$$h = 2000\left(\frac{\pi}{180}\right)^2 \; miles \; \approx .61 \; mile.$$

Again, the dip

$$= \sqrt{\frac{2h}{r}} \; radians \; = \sqrt{\frac{2 \times 2}{4000}} = \sqrt{\frac{1}{1000}}$$
$$= \frac{\sqrt{10}}{100} \; radian = \left(\frac{\sqrt{10}}{100} \times 57.2957795\right)^{\circ} \approx 1^{\circ}48'.$$
$$\left[N.\ B.\ \sqrt{10} = 3.1622777.\right]$$ ■

§ **Problem 17.3.4.** *From the top of the mast of a ship, which is* 66 *feet above the sea, the light of a lighthouse which is known to be* 132 *feet high can just be seen; prove that its distance is* 24 *miles nearly.* ◊

§§ **Solution**. If x and y be the distances in miles of the horizon seen from the top of the mast and the lighthouse respectively, we have

$$x = \sqrt{\frac{2 \times 66 \times 4000}{1760 \times 3}} \; miles$$

and
$$y = \sqrt{\frac{2 \times 132 \times 4000}{1760 \times 3}} \; miles.$$

Hence the required distance

$$= (x + y) \; miles \; = \sqrt{\frac{2 \times 66 \times 4000}{1760 \times 3}}\left(1 + \sqrt{2}\right)$$
$$= 10\left(1 + \sqrt{2}\right) \approx 24 \; miles.$$ ■

§ **Problem 17.3.5.** *From the top of the mast,* 66 *feet above the sea, the top of the mast of another ship can just be seen at a distance of* 20 *miles; prove that the heights of the masts are the same.* ◊

§§ **Solution**. If x and y be the distances in miles of the horizon seen from the tops of the masts respectively, we have

$$x = \sqrt{\frac{2 \times 66 \times 4000}{1760 \times 3}} \; miles \; = 10 \; miles$$

and
$$x + y = 20 \; miles$$
$$\therefore y = 10 \; miles.$$

Hence the height of the second mast

$$= \frac{y^2}{2r} = \frac{100 \times 1760 \times 3}{2 \times 4000} = 66\ feet. \qquad \blacksquare$$

§ Problem 17.3.6. *From the top of the mast of a ship which is* 44 *feet above the sea-level, the light of a lighthouse can just be seen; after sailing for* 15 *minutes the light can just be seen from the deck which is* 11 *feet above the sea-level; prove that the rate of sailing of the ship is nearly* 16.33 *miles per hour.* ◊

§§ Solution. Let S and S' be the positions on the sea of the vessel at the two instants spoken of.

Let P be the top of the mast at the first instant; if we draw a tangent PA to the earth, meeting it in A, then PA produced will pass through the top L of the lighthouse.

At the second instant, let P' be the point on the deck at which L can just be seen and hence P' lies on PA between P and A.

Then $$SP = 44\ feet\ and\ S'P' = 11\ feet.$$

$$\therefore PA = \sqrt{\frac{2 \times 44 \times 4000}{1760 \times 3}}\ miles\ = 10\sqrt{\frac{2}{3}}\ miles$$

and $$P'A = \sqrt{\frac{2 \times 11 \times 4000}{1760 \times 3}}\ miles\ = 10\sqrt{\frac{1}{6}}\ miles$$

$$\therefore SS' = SA - S'A = PA - P'A = 10\left(\frac{2-1}{\sqrt{6}}\right)\ miles$$

$$= \frac{10\sqrt{6}}{6} = \frac{24.494897}{6} = 4.082483\ miles,$$

i.e. the ship sails $\approx$ 4.0825 miles in 15 minutes or 16.33 miles per hour. ■

§ Problem 17.3.7. *Prove that, if the height of the place of observation be n feet, the distance that the observer can see is $\sqrt{\frac{3n}{2}}$ miles nearly.* ◊

§§ Solution.

The required distance

$$= \sqrt{\frac{2n \times 4000}{1760 \times 3}}\ miles = \sqrt{\frac{100n}{66}}\ miles$$

$$\approx \sqrt{\frac{99n}{66}}\ miles\ \approx \sqrt{\frac{3n}{2}}\ miles. \qquad \blacksquare$$

§ Problem 17.3.8. *There are* 10 *million meters in a quadrant of the earth's circumference. Find approximately the distance at which the top of the Eiffel tower should be visible, its height being* 300 *meters.* ◊

§§ Solution. If r be the radius of the earth, we have

$$\frac{1}{4} \times 2\pi r = 10^7\ meters,\ \therefore r = \frac{2 \times 10^7}{\pi}\ meters.$$

The required distance

$$= \sqrt{2hr} = \sqrt{2 \times 300 \times \frac{2 \times 10^7}{\pi}}\ meters$$

$$= 2 \times 10^4 \times \sqrt{\frac{30}{\pi}}\ meters\ = 2 \times 10^4 \times \sqrt{30 \times .31831}\ meters$$

$$\approx 2 \times 10^4 \times 3.09\ meters\ \approx 61800\ meters\ \approx 38\frac{1}{2}\ miles. \qquad \blacksquare$$

§ Problem 17.3.9. *Three vertical posts are placed at intervals of a mile along a straight canal, each rising to the same height above the surface of the water. The visual line joining the tops of the two extreme posts cuts the middle post at a point* 8 *inches below its top. Find the radius of the earth to the nearest mile.* ◊

§§ Solution. Take the figure of $Art.\ 235$, with T, O and T' representing the tops of the posts. Let TT' meet CA in N.

We then have, very approximately.

$$1\ mile\ =\ line\ AT = \sqrt{2r \cdot AN} = \sqrt{2 \times \frac{8}{12 \times 3 \times 1760} \times r}$$

$$\therefore r = (6 \times 3 \times 220)\ \ miles\ = 3960\ miles. \qquad \blacksquare$$

Chapter 18

Inverse Circular Functions

18.1 Identities and Equations

Prove that

§ Problem 18.1.1. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$. ◊

§§ Solution.

$$\begin{aligned}
&\sin\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}\right)\\
&\quad=\sin\left(\sin^{-1}\frac{3}{5}\right)\cos\left(\sin^{-1}\frac{8}{17}\right)+\cos\left(\sin^{-1}\frac{3}{5}\right)\sin\left(\sin^{-1}\frac{8}{17}\right)\\
&\quad=\frac{3}{5}\times\sqrt{1-\frac{64}{289}}+\sqrt{1-\frac{9}{25}}\times\frac{8}{17}=\frac{3}{5}\times\frac{15}{17}+\frac{4}{5}\times\frac{8}{17}=\frac{77}{85}
\end{aligned}$$

$$\therefore\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}.$$ ■

§ Problem 18.1.2. $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\left(\frac{253}{325}\right)$. ◊

§§ Solution.

$$\begin{aligned}
&\cos\left(\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}\right)\\
&\quad=\cos\left(\sin^{-1}\frac{5}{13}\right)\cos\left(\sin^{-1}\frac{7}{25}\right)-\sin\left(\sin^{-1}\frac{5}{13}\right)\sin\left(\sin^{-1}\frac{7}{25}\right)\\
&\quad=\sqrt{1-\frac{25}{169}}\times\sqrt{1-\frac{49}{625}}-\frac{5}{13}\times\frac{7}{25}\\
&\quad=\frac{12}{13}\times\frac{24}{25}-\frac{5}{13}\times\frac{7}{25}=\frac{253}{325}
\end{aligned}$$

$$\therefore \sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\left(\frac{253}{325}\right).$$ ■

§ Problem 18.1.3. $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$. ◊

§§ Solution.

$$\begin{aligned}\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}&=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\\&=\tan^{-1}\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}=\tan^{-1}\frac{15+12}{20-9}=\tan^{-1}\frac{27}{11}.\end{aligned}$$

N. B. Such relations as $\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}$ can be seen at once by drawing a figure as in *Art.* 240. ■

§ Problem 18.1.4. $\cos^{-1}\frac{4}{5}+\cos 6{-}1\frac{12}{13}=\cos^{-1}\frac{33}{65}$. ◊

§§ Solution.

$$\begin{aligned}&\cos\left(\cos^{-1}\frac{4}{5}+\cos 6{-}1\frac{12}{13}\right)\\&=\cos\left(\cos^{-1}\frac{4}{5}\right)\cos\left(\cos^{-1}\frac{12}{13}\right)-\sin\left(\cos^{-1}\frac{4}{5}\right)\sin\left(\cos^{-1}\frac{12}{13}\right)\\&=\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\frac{16}{25}}\times\sqrt{1-\frac{144}{169}}=\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}=\frac{33}{65}\end{aligned}$$

$$\therefore \cos^{-1}\frac{4}{5}+\cos 6{-}1\frac{12}{13}=\cos^{-1}\frac{33}{65}.$$ ■

§ Problem 18.1.5. $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$. ◊

§§ Solution.

Let $\quad \sin^{-1}\sqrt{\frac{1-x}{2}}=\alpha,\ \therefore \sin\alpha=\sqrt{\frac{1-x}{2}}.$

Then $\quad \cos 2\alpha=1-2\sin^2\alpha=1-2\left(\frac{1-x}{2}\right)=x$

$$\therefore 2\alpha=\cos^{-1}x,\ \therefore 2\sin^{-1}\sqrt{\frac{1-x}{2}}=\cos^{-1}x.$$

Again, let $\quad \cos^{-1}\sqrt{\frac{1+x}{2}}=\alpha,\ \therefore \cos\alpha=\sqrt{\frac{1+x}{2}}.$

Then $\quad \cos 2\alpha=2\cos^2\alpha-1=2\left(\frac{1+x}{2}\right)-1=x$

$$\therefore 2\alpha=\cos^{-1}x,\ \therefore 2\cos^{-1}\sqrt{\frac{1+x}{2}}=\cos^{-1}x.$$ ■

§ Problem 18.1.6. $2\cos^{-1}\frac{3}{\sqrt{13}}+\cot^{-1}\frac{16}{63}+\frac{1}{2}\cos^{-1}\frac{7}{25}=\pi$. ◊

§§ Solution.

$$2\cos^{-1}\frac{3}{\sqrt{13}}+\cot^{-1}\frac{16}{63}+\frac{1}{2}\cos^{-1}\frac{7}{25}$$

For, as in the previous problem, $\frac{1}{2}\cos^{-1}x=\tan^{-1}\sqrt{\frac{1-x}{1+x}}$.

$$=2\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{63}{16}+\tan^{-1}\sqrt{\frac{1-\frac{7}{25}}{1+\frac{7}{25}}}$$

$$=\tan^{-1}\frac{\frac{4}{3}}{1-\frac{4}{9}}+\tan^{-1}\frac{63}{16}+\tan^{-1}\sqrt{\frac{18}{32}}$$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{3}{4}=\tan^{-1}\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}+\tan^{-1}\frac{63}{16}$$

$$=\tan^{-1}\left(-\frac{63}{16}\right)+\tan^{-1}\frac{63}{16}=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16}=\pi.$$ ■

§ Problem 18.1.7. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=45°.$ ◊

§§ Solution. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=\tan^{-1}1=45°.$

From a figure, we see at once that

$$\tan^{-1}\frac{1}{2}=\sin^{-1}\frac{1}{\sqrt{5}},\ \textit{and}\ \cot^{-1}3=\tan^{-1}\frac{1}{3}.$$ ■

§ Problem 18.1.8. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}.$ ◊

§§ Solution.

$$\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}$$

$$=\tan^{-1}\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}=\tan^{-1}\frac{20}{90}=\tan^{-1}\frac{2}{9}.$$ ■

§ Problem 18.1.9. $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}.$ ◊

§§ Solution. $2\tan^{-1}\frac{2}{3}=\tan^{-1}\frac{\frac{4}{3}}{1-\frac{4}{9}}=\tan^{-1}\frac{12}{5}.$

$$\therefore\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}.$$ ■

§ Problem 18.1.10. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}.$ ◊

§§ Solution.

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\tan^{-1}\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}$$
$$=\tan^{-1}\frac{17}{34}=\tan^{-1}\frac{1}{2}=\alpha\ (say),\ \therefore \tan\alpha=\frac{1}{2}.$$
$$\therefore \cos 2\alpha=\frac{1-\tan^2\alpha}{1+\tan^2\alpha}=\frac{1-\frac{1}{4}}{1+\frac{1}{4}}=\frac{3}{5}$$
$$\therefore 2\alpha=\cos^{-1}\frac{3}{5},\ and\ \alpha=\frac{1}{2}\cos^{-1}\frac{3}{5}$$
$$\therefore \tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}.$$ ■

§ Problem 18.1.11. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}.$ ◊

§§ Solution.

$$2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}$$
$$=2\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right)+\tan^{-1}\frac{1}{7}$$
$$=2\tan^{-1}\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}+\tan^{-1}\frac{1}{7}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$
$$=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$
$$=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}=\tan^{-1}1=\frac{\pi}{4}.$$ ■

§ Problem 18.1.12. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}.$ ◊

§§ Solution.

$$\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}$$
$$=\tan^{-1}\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}-\tan^{-1}\frac{8}{19}$$
$$=\tan^{-1}\frac{27}{11}-\tan^{-1}\frac{8}{19}$$
$$=\tan^{-1}\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}=\tan^{-1}1=\frac{\pi}{4}.$$ ■

§ Problem 18.1.13. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}.$ ◊

§§ Solution.

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}$$

$$=\tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}}+\tan^{-1}\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}=\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{3}{11}$$

$$=\tan^{-1}\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}=\tan^{-1}1=\frac{\pi}{4}.$$

■

§ Problem 18.1.14. $3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{20}=\frac{\pi}{4}-\tan^{-1}\frac{1}{1985}.$ ◊

§§ Solution. Let $\tan^{-1}\frac{1}{4}=\alpha,\ \tan^{-1}\frac{1}{20}=\beta$, and $\tan^{-1}\frac{1}{1985}=\gamma$,

$$\therefore \tan\alpha=\frac{1}{4},\ \tan\beta=\frac{1}{20},\ \textit{and}\ \tan\gamma=\frac{1}{1985}.$$

$$\therefore \tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\frac{\frac{3}{4}-\frac{1}{64}}{1-\frac{3}{16}}=\frac{47}{52}$$

$$\therefore \tan(3\alpha+\beta)=\frac{\tan 3\alpha+\tan\beta}{1-\tan 3\alpha\tan\beta}=\frac{\frac{47}{52}+\frac{1}{20}}{1-\frac{47}{52}\cdot\frac{1}{20}}=\frac{992}{993}.$$

Again, $$\tan\left(\frac{\pi}{4}-\gamma\right)=\frac{1-\tan\gamma}{1+\tan\gamma}=\frac{1-\frac{1}{1985}}{1+\frac{1}{1985}}=\frac{992}{993}.$$

$$\therefore 3\alpha+\beta=\frac{\pi}{4}-\gamma$$

$$\therefore 3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{20}=\frac{\pi}{4}-\tan^{-1}\frac{1}{1985}.$$

■

§ Problem 18.1.15. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}.$ ◊

§§ Solution. If $\tan^{-1}\frac{1}{5}=\alpha$, then $\tan 4\alpha=\frac{120}{119}$.
$[Ex.\ 3,\ Art.\ 240]$.

Let $$\tan^{-1}\frac{1}{70}=\beta,\ \textit{and}\ \tan^{-1}\frac{1}{99}=\gamma$$

$$\therefore \tan\beta=\frac{1}{70},\ \textit{and}\ \tan\gamma=\frac{1}{99}.$$

Then $$\tan(\beta-\gamma)=\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\cdot\frac{1}{99}}=\frac{29}{6931}=\frac{1}{239}.$$

Again, $$\tan[4\alpha-(\beta-\gamma)]=\frac{\tan 4\alpha-\tan(\beta-\gamma)}{1+\tan 4\alpha\tan(\beta-\gamma)}$$

$$=\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}=\frac{28561}{28561}=1=\tan\frac{\pi}{4}.$$

$$\therefore 4\alpha-\beta+\gamma=\frac{\pi}{4}$$

$$\therefore 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}. \qquad \blacksquare$$

§ Problem 18.1.16. $\tan^{-1}\dfrac{120}{119} = 2\sin^{-1}\dfrac{5}{13}$. ◊

§§ Solution. Let $\tan^{-1}\dfrac{120}{119} = \alpha$, so that $\tan\alpha = \dfrac{120}{119}$.

Let $$\sin^{-1}\frac{5}{13} = \beta,\ \therefore \sin\beta = \frac{5}{13},\ \textit{and}\ \therefore \tan\beta = \frac{5}{12}.$$

$$\therefore \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{\left(\frac{10}{12}\right)}{1-\frac{25}{144}} = \frac{120}{119} = \tan\alpha$$

$$\therefore \alpha = 2\beta,\ \textit{i.e.}\ \tan^{-1}\frac{120}{119} = 2\sin^{-1}\frac{5}{13}. \qquad \blacksquare$$

§ Problem 18.1.17. $\tan^{-1}\dfrac{m}{n} - \tan^{-1}\dfrac{m-n}{m+n} = \dfrac{\pi}{4}$. ◊

§§ Solution. Let $\tan^{-1}\dfrac{m}{n} = \alpha$, so that $\tan\alpha = \dfrac{m}{n}$.

Then $$\tan\left(\alpha - \frac{\pi}{4}\right) = \frac{\tan\alpha - 1}{\tan\alpha + 1} = \frac{m-n}{m+n}$$

$$\therefore \alpha - \frac{\pi}{4} = \tan^{-1}\frac{m-n}{m+n}$$

$$\therefore \tan^{-1}\frac{m}{n} - \tan^{-1}\frac{m-n}{m+n} = \frac{\pi}{4}. \qquad \blacksquare$$

§ Problem 18.1.18.

$$\tan^{-1} t + \tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{3t-t^3}{1-3t^2},\ t>0,\ \textit{if}\ t < \frac{1}{\sqrt{3}}\ \textit{or}\ > \sqrt{3}$$

$$= \pi + \tan^{-1}\frac{3t-t^3}{1-3t^2},\ \textit{if}\ t > \frac{1}{\sqrt{3}}\ \textit{and}\ < \sqrt{3}.$$

◊

§§ Solution. $$\tan^{-1} t + \tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{t + \frac{2t}{1-t^2}}{1 - \frac{2t^2}{1-t^2}} = \tan^{-1}\frac{3t-t^3}{1-3t^2}.$$

If $t < \dfrac{1}{\sqrt{3}}$, then $\dfrac{3t-t^3}{1-3t^2}$ is positive and $\tan^{-1}\dfrac{3t-t^3}{1-3t^2}$ lies between $0°$ and $90°$.

If $t > \dfrac{1}{\sqrt{3}}$, then $\dfrac{3t-t^3}{1-3t^2}$ is negative and $\tan^{-1}\dfrac{3t-t^3}{1-3t^2}$ is a negative angle; also $\pi + \tan^{-1}\dfrac{3t-t^3}{1-3t^2}$ is a positive angle with the same tangent, so that

$$\tan^{-1} t + \tan^{-1}\frac{2t}{1-t^2} = \pi + \tan^{-1}\frac{3t-t^3}{1-3t^2}. \qquad \blacksquare$$

§ Problem 18.1.19.

$$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}$$
$$+ \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}} = \pi.$$ ◊

§§ Solution.

$$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}$$

$$= \tan^{-1}\frac{\sqrt{\dfrac{a(a+b+c)}{bc}}+\sqrt{\dfrac{b(a+b+c)}{ca}}}{1-\sqrt{\dfrac{ab(a+b+c)^2}{abc^2}}}$$

$$= \tan^{-1}\frac{a\sqrt{c(a+b+c)}+b\sqrt{c(a+b+c)}}{c\sqrt{ab}-\sqrt{ab}(a+b+c)}$$

$$= \tan^{-1}\left(-\frac{(a+b)\sqrt{c(a+b+c)}}{\sqrt{ab}(a+b)}\right)$$

$$= \pi-\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$\therefore \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=\pi. \qquad \blacksquare$$

§ Problem 18.1.20. $\cot^{-1}\dfrac{ab+1}{a-b}+\cot^{-1}\dfrac{bc+1}{b-c}+\cot^{-1}\dfrac{ca+1}{c-a}=0.$ ◊

§§ Solution.

$$\cot^{-1}\frac{ab+1}{a-b}=\tan^{-1}\frac{a-b}{1+ab}=\tan^{-1}a-\tan^{-1}b$$

$$\cot^{-1}\frac{bc+1}{b-c}=\tan^{-1}\frac{b-c}{1+bc}=\tan^{-1}b-\tan^{-1}c$$

and $$\cot^{-1}\frac{ca+1}{c-a}=\tan^{-1}\frac{c-a}{1+ca}=\tan^{-1}c-\tan^{-1}a.$$

Hence, by addition, we obtain the required result. ■

§ Problem 18.1.21. $\tan^{-1}n+\cot^{-1}(n+1)=\tan^{-1}\left(n^2+n+1\right).$ ◊

§§ Solution.

$$\tan^{-1}n+\cot^{-1}(n+1)=\tan^{-1}n+\tan^{-1}\frac{1}{n+1}$$

$$=\tan^{-1}\frac{n+\dfrac{1}{n+1}}{1-\dfrac{n}{n+1}}$$

$$=\tan^{-1}\frac{n^2+n+1}{n+1-n}=\tan^{-1}\left(n^2+n+1\right). \qquad \blacksquare$$

§ Problem 18.1.22. $\cos\left(2\tan^{-1}\dfrac{1}{7}\right)=\sin\left(4\tan^{-1}\dfrac{1}{3}\right).$ ◊

§§ Solution. Let $\tan^{-1}\dfrac{1}{7}=\alpha$, so that $\tan\alpha=\dfrac{1}{7}$.

Then $$\cos 2\alpha=\frac{1-\tan^2\alpha}{1+\tan^2\alpha}=\frac{1-\dfrac{1}{49}}{1+\dfrac{1}{49}}=\frac{48}{50}=\frac{24}{25}.$$

Let $\tan^{-1}\frac{1}{3}=\beta$, so that $\tan\beta=\frac{1}{3}$.

Then $$\tan 2\beta=\frac{\left(\frac{2}{3}\right)}{1-\frac{1}{9}}=\frac{3}{4}$$

and $$\sin 4\beta=\frac{2\tan 2\beta}{1+\tan^2 2\beta}=\frac{\left(\frac{3}{2}\right)}{1+\frac{9}{16}}=\frac{24}{25}$$

$$\therefore \cos 2\alpha=\sin 4\beta,\ i.e.\ \cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right). \qquad \blacksquare$$

§ Problem 18.1.23.

$$2\tan^{-1}\left[\tan\left(45^\circ-\alpha\right)\tan\frac{\beta}{2}\right]$$
$$=\cos^{-1}\left[\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right]. \qquad \Diamond$$

§§ Solution. Let $\tan^{-1}\left[\tan\left(45^\circ-\alpha\right)\tan\frac{\beta}{2}\right]=\theta$,

$$\therefore \tan\theta=\tan\left(45^\circ-\alpha\right)\tan\frac{\beta}{2}=\frac{1-\tan\alpha}{1+\tan\alpha}\cdot\tan\frac{\beta}{2}$$
$$=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\cdot\sqrt{\frac{1-\cos\beta}{1+\cos\beta}}$$
$$\therefore \tan^2\theta=\frac{1-\sin 2\alpha}{1+\sin 2\alpha}\cdot\frac{1-\cos\beta}{1+\cos\beta}$$
$$\therefore \frac{\tan^2\theta}{1}=\frac{1-\cos\beta-\sin 2\alpha+\sin 2\alpha\cos\beta}{1+\cos\beta+\sin 2\alpha+\sin 2\alpha\cos\beta}$$
$$\therefore \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}$$
$$\therefore 2\theta=\cos^{-1}\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}$$
$$\therefore 2\tan^{-1}\left[\tan\left(45^\circ-\alpha\right)\tan\frac{\beta}{2}\right]=\cos^{-1}\left[\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right]. \qquad \blacksquare$$

§ Problem 18.1.24. $\tan^{-1}x=2\tan^{-1}\left[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x\right]$. $\Diamond$

§§ Solution. Let $\tan^{-1}x=\theta$, so that $\tan\theta=x$.

$$\therefore 2\tan^{-1}\left[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x\right]$$
$$=2\tan^{-1}\left[\operatorname{cosec}\theta-\tan\left(\frac{\pi}{2}-\theta\right)\right]$$
$$=2\tan^{-1}\left[\operatorname{cosec}\theta-\cot\theta\right]=2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$
$$=2\tan^{-1}\left[\tan\frac{\theta}{2}\right]=2\left[\frac{\theta}{2}\right]=\theta=\tan^{-1}x. \qquad \blacksquare$$

§ Problem 18.1.25.

$$2\tan^{-1}\left[\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right]$$
$$=\tan^{-1}\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}. \qquad \Diamond$$

§§ Solution. Let $\tan^{-1}\left[\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)\right]=\theta$,

$$\therefore \tan\theta = \tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$$

$$= \frac{1-\cos\alpha}{\sin\alpha}\cdot\frac{1-\cos\left(\frac{\pi}{2}-\beta\right)}{\sin\left(\frac{\pi}{2}-\beta\right)} = \frac{1-\cos\alpha}{\sin\alpha}\cdot\frac{1-\sin\beta}{\cos\beta}.$$

$$\therefore \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{1-\cos\alpha}{\sin\alpha}\right)\left(\frac{1-\sin\beta}{\cos\beta}\right)}{1-\left(\frac{1-\cos\alpha}{\sin\alpha}\right)^2\left(\frac{1-\sin\beta}{\cos\beta}\right)^2}$$

$$= \frac{2\sin\alpha\cos\beta\,(1-\cos\alpha)\,(1-\sin\beta)}{(1-\cos^2\alpha)\left(1-\sin^2\beta\right)-(1-\cos\alpha)^2\,(1-\sin\beta)^2}$$

$$= \frac{2\sin\alpha\cos\beta}{(1+\cos\alpha)\,(1+\sin\beta)-(1-\cos\alpha)\,(1-\sin\beta)}$$

$$= \frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$$

$$\therefore 2\theta = \tan^{-1}\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$$

$$\therefore 2\tan^{-1}\left[\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right] = \tan^{-1}\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}.$$ ■

§ Problem 18.1.26. *Show that*

$$\cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cot^{-1}\sqrt{\frac{a-x}{x-b}}$$

$$= \frac{1}{2}\sin^{-1}\frac{2\sqrt{(a-x)(x-b)}}{a-b}.$$ ◊

§§ Solution. Let $\cos^{-1}\sqrt{\dfrac{a-x}{a-b}}=\alpha$, so that $\cos\alpha=\sqrt{\dfrac{a-x}{a-b}}$

$$\therefore \sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{a-x}{a-b}} = \sqrt{\frac{x-b}{a-b}}.$$

$$\therefore \alpha = \sin^{-1}\sqrt{\frac{x-b}{a-b}},\ i.e\ \cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}}.$$

Again, $$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \sqrt{\frac{a-x}{a-b}}\div\sqrt{\frac{x-b}{a-b}} = \sqrt{\frac{a-x}{x-b}}$$

$$\therefore \alpha = \cot^{-1}\sqrt{\frac{a-x}{x-b}},$$

$$i.e.\ \cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cot^{-1}\sqrt{\frac{a-x}{x-b}}.$$

$$\therefore \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\frac{\sqrt{(x-b)(a-x)}}{a-b}$$

$$\therefore 2\alpha = \sin^{-1}\frac{2\sqrt{(x-b)(a-x)}}{a-b}$$

$$\therefore \alpha = \frac{1}{2}\sin^{-1}\frac{2\sqrt{(x-b)(a-x)}}{a-b}$$

$$\therefore \cos^{-1}\sqrt{\frac{a-x}{x-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cot^{-1}\sqrt{\frac{a-x}{x-b}}$$
$$= \frac{1}{2}\sin^{-1}\frac{2\sqrt{(a-x)(x-b)}}{a-b}. \qquad \blacksquare$$

§ Problem 18.1.27. *If* $\cos^{-1}\dfrac{x}{a} + \cos^{-1}\dfrac{y}{b} = \alpha$, *prove that*
$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha. \qquad \Diamond$$

§§ Solution. Given $\cos^{-1}\dfrac{x}{a} = \alpha - \cos^{-1}\dfrac{y}{b}$

$$\therefore \frac{x}{a} = \cos\left(\alpha - \cos^{-1}\frac{y}{b}\right) = \cos\alpha \cdot \frac{y}{b} + \sin\alpha\sqrt{1-\frac{y^2}{b^2}}$$
$$\therefore \frac{x}{a} - \frac{y}{b}\cos\alpha = \sin\alpha\sqrt{1-\frac{y^2}{b^2}}.$$

Squaring both sides, we have
$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2}\cos^2\alpha = \sin^2\alpha - \frac{y^2}{b^2}\sin^2\alpha.$$
$$\therefore \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha. \qquad \blacksquare$$

Solve the equations :

§ Problem 18.1.28. $\tan^{-1}\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \beta.$ $\Diamond$

§§ Solution. We have
$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \tan\beta$$
$$\therefore \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} = \frac{1+\tan\beta}{1-\tan\beta} = \frac{\cos\beta+\sin\beta}{\cos\beta-\sin\beta}$$
$$\therefore \frac{1+x^2}{1-x^2} = \left(\frac{\cos\beta+\sin\beta}{\cos\beta-\sin\beta}\right)^2 = \frac{1+\sin 2\beta}{1-\sin 2\beta}$$
$$\therefore x^2 = \sin 2\beta,\ i.e.\ x = \pm\sqrt{\sin 2\beta}. \qquad \blacksquare$$

§ Problem 18.1.29. $\tan^{-1}2x + \tan^{-1}3x = \dfrac{\pi}{4}.$ $\Diamond$

§§ Solution.
$$\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$$
$$\therefore \tan^{-1}\frac{2x+3x}{1-2x\times 3x} = \tan^{-1}1$$
$$\therefore \frac{5x}{1-6x^2} = 1;\ \therefore\ 6x^2+5x-1=0$$
$$\therefore (6x-1)(x+1) = 0;\ \therefore x = \frac{1}{6}\ or\ -1.$$

The latter value of x is inadmissible, by *Art.* 240, *Ex.* 4, since in this case we should have $ab > 1$. $\blacksquare$

§ Problem 18.1.30. $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}.$ $\Diamond$

§§ Solution. We have

$$\tan^{-1}\frac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{x-1}{x-2}\times\dfrac{x+1}{x+2}}=\tan^{-1}1$$
$$\therefore \frac{(x-1)(x+2)+(x+1)(x-2)}{x^2-4-(x^2-1)}=1$$
$$\therefore \frac{2x^2-4}{-3}=1;\ \therefore 2x^2=1$$
$$\therefore x^2=\frac{1}{2},\ i.e.\ x=\pm\frac{1}{\sqrt{2}}.$$ ■

§ Problem 18.1.31. $\tan^{-1}(x+1)+\cot^{-1}(x-1)=\sin^{-1}\dfrac{4}{5}+\cos^{-1}\dfrac{3}{5}.$ ◊

§§ Solution.

$$\tan^{-1}(x+1)+\cot^{-1}(x-1)=\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{3}{5}$$
$$\therefore \tan^{-1}(x+1)+\tan^{-1}\left(\frac{1}{x-1}\right)=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{4}{3}=2\tan^{-1}\frac{4}{3}$$
$$\therefore \tan^{-1}\frac{x+1+\dfrac{1}{x-1}}{1-\dfrac{x+1}{x-1}}=\tan^{-1}\frac{\left(\dfrac{8}{3}\right)}{1-\dfrac{16}{9}}$$
$$\therefore \frac{x^2-1+1}{x-1-(x+1)}=-\frac{24}{7}$$
$$\therefore \frac{x^2}{2}=\frac{24}{7},\ i.e.\ x^2=\frac{48}{7},\ i.e.\ x=\pm 4\sqrt{\frac{3}{7}}.$$ ■

§ Problem 18.1.32. $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\dfrac{8}{31}.$ ◊

§§ Solution.

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$$
$$\therefore \tan^{-1}\frac{x+1+x-1}{1-(x+1)(x-1)}=\tan^{-1}\frac{8}{31}$$
$$\therefore \frac{2x}{2-x^2}=\frac{8}{31},\ i.e.\ 4x^2+31x-8=0$$
$$\therefore x=\frac{-31\pm\sqrt{961+128}}{8}=\frac{-31\pm 33}{8}=-8\ or\ \frac{1}{4}.$$ ■

§ Problem 18.1.33. $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x).$ ◊

§§ Solution.

Let $\tan^{-1}(\cos x)=\alpha\ and\ \tan^{-1}(2\operatorname{cosec} x)=\beta$

$$\therefore \tan\alpha=\cos x\ and\ \tan\beta=2\operatorname{cosec} x.$$

We are given $2\alpha=\beta$

$$\therefore \tan 2\alpha=\tan\beta$$
$$\therefore \frac{2\cos x}{1-\cos^2 x}=\frac{2}{\sin x};\ \therefore \frac{\cos x}{\sin^2 x}=\frac{1}{\sin x}$$
$$\therefore \sin x=0\ or\ \cos x=\sin x.$$

If $\sin x=0\ then\ x=n\pi.$

If $\cos x=\sin x,\ then\ \tan x=1=\tan\dfrac{\pi}{4}$

$$\therefore x = n\pi + \frac{\pi}{4}.$$ ■

§ Problem 18.1.34. $\tan^{-1} x + 2\cot^{-1} x = \frac{2}{3}\pi.$ ◊

§§ Solution.

$$\tan^{-1} x + 2\cot^{-1} x = \frac{2}{3}\pi$$

Let $$\tan^{-1} x = \theta;\ then\ \cot^{-1} x = \left(\frac{\pi}{2} - \theta\right).$$

The equation becomes

$$\theta + \pi - 2\theta = \frac{2\pi}{3},\ i.e.\ \theta = \frac{\pi}{3}$$
$$\therefore x = \tan\theta = \tan\frac{\pi}{3} = \sqrt{3}.$$ ■

§ Problem 18.1.35. $\tan\cos^{-1} x = \sin\cot^{-1}\frac{1}{2}.$ ◊

§§ Solution. Let $\cos^{-1} x = \alpha$, so that $\cos\alpha = x$,

$$\therefore \tan\alpha = \frac{\sqrt{1-x^2}}{x}.$$

Let $$\cot^{-1}\frac{1}{2} = \beta,\ \therefore \cot\alpha = \frac{1}{2}$$
$$\therefore \sin\alpha = \frac{2}{\sqrt{5}}.$$

The equation becomes

$$\frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}}.$$
$$\therefore 5 - 5x^2 = 4x^2,\ i.e.\ 9x^2 = 5,\ i.e.\ x = \frac{\sqrt{5}}{3}.$$

The value $x = -\frac{\sqrt{5}}{3}$ is inadmissible; for then $\cos^{-1} x$ would be between $\frac{\pi}{2}$ and π and therefore $\tan\cos^{-1} x$ would be negative. ■

§ Problem 18.1.36. $\cot^{-1} x - \cot^{-1}(x+2) = 15°.$ ◊

§§ Solution.

$$\cot^{-1} x - \cot^{-1}(x+2) = 15°$$
$$\therefore \tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2} = 15°$$
$$\therefore \tan^{-1}\frac{\frac{1}{x} - \frac{1}{x+2}}{1 + \frac{1}{x(x+2)}} = 15°$$
$$\therefore \frac{x+2-x}{x^2+2x+1} = \tan 15° = 2 - \sqrt{3}$$
$$\therefore \frac{2}{(x+1)^2} = 2 - \sqrt{3}$$
$$\therefore (x+1)^2 = \frac{2}{2-\sqrt{3}} = 2\left(2+\sqrt{3}\right) = 4 + 2\sqrt{3} = \left(\sqrt{3}+1\right)^2$$
$$\therefore x = \sqrt{3}\ or\ -\left(\sqrt{3}+2\right).$$ ■

§ Problem 18.1.37. $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}.$ ◊

§§ Solution.

$$\cos^{-1}\frac{x^2-1}{x^2+1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}.$$

If $x=\cot\theta$, we have

$$\frac{x^2-1}{x^2+1}=\frac{\cot^2\theta-1}{\cot^2\theta+1}=\cos 2\theta$$

and
$$\frac{2x}{x^2-1}=\frac{2\cot\theta}{\cot^2\theta-1}=\frac{1}{\cot 2\theta}=\tan 2\theta.$$

Thus the equation becomes

$$2\theta+2\theta=\frac{2\pi}{3};\ \therefore 4\theta=\frac{2\pi}{3};\ \therefore \theta=\frac{\pi}{6}$$

$$\therefore x=\cot\theta=\cot\frac{\pi}{6}=\sqrt{3}.$$ ■

§ Problem 18.1.38. $\cot^{-1}x+\cot^{-1}\left(n^2-x+1\right)=\cot^{-1}(n-1).$ ◊

§§ Solution.

$$\cot^{-1}x+\cot^{-1}\left(n^2-x+1\right)=\cot^{-1}(n-1)$$

$$\therefore \tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{n^2-x+1}=\tan^{-1}\frac{1}{n-1}$$

$$\therefore \tan^{-1}\frac{\frac{1}{x}+\frac{1}{n^2-x+1}}{1-\frac{1}{x}\cdot\frac{1}{n^2-x+1}}=\tan^{-1}\frac{1}{n-1}$$

$$\therefore \frac{n^2+1}{n^2x-x^2+x-1}=\frac{1}{n-1}$$

$$\therefore n^2x-x^2+x=n^3-n^2+n$$

$$\therefore x^2-\left(n^2+1\right)x+n\left(n^2-n+1\right)=0$$

$$\therefore (x-n)\left[x-\left(n^2-n+1\right)\right]=0$$

$$\therefore x=n \text{ or } n^2-n+1.$$ ■

§ Problem 18.1.39. $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}.$ ◊

§§ Solution.

$$\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$$

$$\therefore \cos\left(\sin^{-1}x+\sin^{-1}2x\right)=\frac{1}{2}$$

$$\therefore \sqrt{1-x^2}\times\sqrt{1-4x^2}-x\times 2x=\frac{1}{2}$$

$$\therefore \left(1-x^2\right)\left(1-4x^2\right)=4x^4+2x^2+\frac{1}{4}$$

$$\therefore 1-5x^2+4x^4=4x^4+2x^2+\frac{1}{4}$$

$$\therefore 7x^2=\frac{3}{4};\ \therefore x^2=\frac{1}{4}\times\frac{3}{7}$$

$$\therefore x=\pm\frac{1}{2}\sqrt{\frac{3}{7}}.$$

The negative value is inadmissible, since x is necessarily positive.

Otherwise thus :

$$\sin^{-1} 2x = \frac{\pi}{3} - \sin^{-1} x$$
$$\therefore 2x = \sin\left(\frac{\pi}{3} - \sin^{-1} x\right) = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{1}{2} \times x$$
$$\therefore 5x = \sqrt{3}\left(1-x^2\right);\ \therefore 25x^2 = 3 - 3x^2$$
$$\therefore 28x^2 = 3,\ i.e.\ x^2 = \frac{3}{28},\ i.e.\ x = \frac{1}{2}\sqrt{\frac{3}{7}}.$$ ■

§ Problem 18.1.40. $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$. ◊

§§ Solution.
$$\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$$
Let
$$\sin^{-1}\frac{5}{x} = \alpha\ and\ \sin^{-1}\frac{12}{x} = \beta$$
$$\therefore \sin\alpha = \frac{5}{x}\ and\ \sin\beta = \frac{12}{x}.$$
Now
$$\alpha + \beta = \frac{\pi}{2};\ \therefore \sin\beta = \cos\alpha$$
$$\therefore \sin^2\beta = 1 - \sin^2\alpha;\ \therefore \sin^2\alpha + \sin^2\beta = 1$$
$$\therefore \frac{25}{x^2} + \frac{144}{x^2} = 1;\ \therefore x^2 = 169;\ \therefore x = 13.$$ ■

§ Problem 18.1.41. $\tan^{-1}\frac{a}{x} + \tan^{-1}\frac{b}{x} + \tan^{-1}\frac{c}{x} + \tan^{-1}\frac{d}{x} = \frac{\pi}{2}$. ◊

§§ Solution. We have
$$\tan^{-1}\frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{x^2}} + \tan^{-1}\frac{\frac{c}{x}+\frac{d}{x}}{1-\frac{cd}{x^2}} = \frac{\pi}{2}$$
$$\therefore \tan^{-1}\frac{(a+b)x}{x^2-ab} = \frac{\pi}{2} - \tan^{-1}\frac{(c+d)x}{x^2-cd}$$
$$\therefore \frac{(a+b)x}{x^2-ab} = \frac{x^2-cd}{(c+d)x}$$
$$\therefore (a+b)(c+d)x^2 = \left(x^2-ab\right)\left(x^2-cd\right)$$
$$\therefore x^4 - (ab+ac+ad+bc+bd+cd)x^2 + abcd = 0.$$ ■

§ Problem 18.1.42. $\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1} b - \sec^{-1} a$. ◊

§§ Solution. The given equation may be written
$$\cos^{-1}\frac{a}{x} + \cos^{-1}\frac{1}{a} = \cos^{-1}\frac{b}{x} + \cos^{-1}\frac{1}{b}$$
$$\therefore \cos\left(\cos^{-1}\frac{a}{x} + \cos^{-1}\frac{1}{a}\right) = \cos\left(\cos^{-1}\frac{b}{x} + \cos^{-1}\frac{1}{b}\right)$$
$$\therefore \frac{a}{x} \times \frac{1}{a} - \sqrt{1-\frac{a^2}{x^2}} \times \sqrt{1-\frac{1}{a^2}} = \frac{b}{x} \times \frac{1}{b} - \sqrt{1-\frac{b^2}{x^2}} \times \sqrt{1-\frac{1}{b^2}}$$
$$\therefore \frac{\left(x^2-a^2\right)\left(a^2-1\right)}{a^2x^2} = \frac{\left(x^2-b^2\right)\left(b^2-1\right)}{b^2x^2}$$
$$\therefore \left(a^2-b^2\right)x^2 = a^2b^2\left(a^2-b^2\right)$$
$$\therefore x^2 = a^2b^2,\ i.e.\ x = ab.$$ ■

§ Problem 18.1.43. $\operatorname{cosec}^{-1} x = \operatorname{cosec}^{-1} a + \operatorname{cosec}^{-1} b.$ ◊

§§ Solution. The given equation may be written

$$\sin^{-1}\frac{1}{x} = \sin^{-1}\frac{1}{a} + \sin^{-1}\frac{1}{b}$$

$$\therefore \sin\left(\sin^{-1}\frac{1}{x}\right) = \sin\left(\sin^{-1}\frac{1}{a} + \sin^{-1}\frac{1}{b}\right)$$

$$\therefore \frac{1}{x} = \frac{1}{a}\sqrt{1-\frac{1}{b^2}} + \frac{1}{b}\sqrt{1-\frac{1}{a^2}} = \frac{1}{ab}\left(\sqrt{b^2-1} + \sqrt{a^2-1}\right)$$

$$\therefore x = \frac{ab}{\sqrt{b^2-1}+\sqrt{a^2-1}}.$$ ■

§ Problem 18.1.44. $2\tan^{-1} x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}.$ ◊

§§ Solution.

$$\because \cos^{-1}\frac{1-a^2}{1+a^2} = 2\tan^{-1} a \textit{ and } \cos^{-1}\frac{1-b^2}{1+b^2} = 2\tan^{-1} b,$$

the equation becomes

$$\tan^{-1} x = \tan^{-1} a - \tan^{-1} b = \tan^{-1}\frac{a-b}{1+ab}$$

$$\therefore x = \frac{a-b}{1+ab}.$$ ■

Chapter 19

On Some Simple Trigonometrical Series

19.1 Simple Series

Sum the series :

§ Problem 19.1.1. $\cos\theta + \cos 3\theta + \cos 5\theta + \ldots$ *to n terms.* ◊

§§ Solution. By *Art.* 242, we have

$$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots \textit{ to n terms}$$
$$= \frac{\cos\left[\theta + \dfrac{n-1}{2}\cdot 2\theta\right]\sin\dfrac{n\cdot 2\theta}{2}}{\sin\dfrac{2\theta}{2}}$$
$$= \frac{\cos n\theta \sin n\theta}{\sin\theta} = \frac{1}{2}\sin 2n\theta \operatorname{cosec}\theta.$$ ■

§ Problem 19.1.2. $\cos\dfrac{A}{2} + \cos 2A + \cos\dfrac{7A}{2} + \ldots$ *to n terms.* ◊

§§ Solution.

$$\cos\frac{A}{2} + \cos 2A + \cos\frac{7A}{2} + \ldots \textit{ to n terms}$$
$$= \frac{\cos\left[\dfrac{A}{2} + \dfrac{n-1}{2}\cdot\dfrac{3A}{2}\right]\sin\dfrac{n}{2}\cdot\dfrac{3A}{2}}{\sin\dfrac{1}{2}\cdot\dfrac{3A}{2}}$$
$$= \cos\frac{3n-1}{4}A \sin\frac{3n}{4}A \operatorname{cosec}\frac{3}{4}A.$$ ■

Prove that :

§ Problem 19.1.3. $\dfrac{\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha}{\cos\alpha + \cos 2\alpha + \ldots + \cos n\alpha} = \tan\dfrac{n+1}{2}\alpha.$ ♢

§§ Solution. The given expression

$$= \left\{\frac{\sin\left[\alpha + \dfrac{n-1}{2}\alpha\right]\sin\dfrac{n\alpha}{2}}{\sin\dfrac{\alpha}{2}}\right\} \div \left\{\frac{\cos\left[\alpha + \dfrac{n-1}{2}\alpha\right]\sin\dfrac{n\alpha}{2}}{\sin\dfrac{\alpha}{2}}\right\}$$

$$= \tan\left[\alpha + \frac{n-1}{2}\alpha\right] = \tan\frac{n+1}{2}\alpha.$$ ■

§ Problem 19.1.4.

$$\frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots + \cos(2n-1)\alpha} = \tan n\alpha.$$ ♢

§§ Solution. The given expression

$$= \left\{\frac{\sin\left[\alpha + \dfrac{n-1}{2}\cdot 2\alpha\right]\sin\dfrac{n\cdot 2\alpha}{2}}{\sin\dfrac{2\alpha}{2}}\right\}$$

$$\div \left\{\frac{\cos\left[\alpha + \dfrac{n-1}{2}\cdot 2\alpha\right]\sin\dfrac{n\cdot 2\alpha}{2}}{\sin\dfrac{2\alpha}{2}}\right\}$$

$$= \tan\left[\alpha + (n-1)\alpha\right] = \tan n\alpha.$$ ■

§ Problem 19.1.5.

$$\frac{\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \textit{ to } n \textit{ terms}}{\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots \textit{ to } n \textit{ terms}} = \tan\left[\alpha + \frac{n-1}{2}(\beta+\pi)\right].$$ ♢

§§ Solution. The given expression

$$= \frac{\sin\alpha + \sin(\alpha+\beta+\pi) + \sin(\alpha+2\beta+2\pi) + \ldots \textit{ to } n \textit{ terms}}{\cos\alpha + \cos(\alpha+\beta+\pi) + \cos(\alpha+2\beta+2\pi) + \ldots \textit{ to } n \textit{ terms}}$$

$$= \frac{\sin\left[\alpha + \dfrac{n-1}{2}(\beta+\pi)\right]\sin\dfrac{n}{2}(\beta+\pi)\operatorname{cosec}\dfrac{1}{2}(\beta+\pi)}{\cos\left[\alpha + \dfrac{n-1}{2}(\beta+\pi)\right]\sin\dfrac{n}{2}(\beta+\pi)\operatorname{cosec}\dfrac{1}{2}(\beta+\pi)}$$

$$= \tan\left[\alpha + \frac{n-1}{2}(\beta+\pi)\right].$$ ■

Sum the following series :

§ Problem 19.1.6. $\cos\dfrac{\pi}{2n+1} + \cos\dfrac{3\pi}{2n+1} + \cos\dfrac{5\pi}{2n+1} + \ldots$ *to n terms.* ♢

§§ Solution.

$$\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \ldots \textit{ to } n \textit{ terms}$$

$$
\begin{aligned}
&= \frac{\left\{\cos\left[\left(\dfrac{\pi}{2n+1}\right) + \dfrac{n-1}{2}\cdot\left(\dfrac{2\pi}{2n+1}\right)\right]\sin\dfrac{n}{2}\cdot\dfrac{2\pi}{2n+1}\right\}}{\sin\dfrac{\pi}{2n+1}} \\
&= \cos\frac{n\pi}{2n+1}\sin\frac{n\pi}{2n+1}\operatorname{cosec}\frac{\pi}{2n+1} \\
&= \frac{1}{2}\sin\frac{2n\pi}{2n+1}\operatorname{cosec}\frac{\pi}{2n+1} \\
&= \frac{1}{2}\sin\frac{\pi}{2n+1}\operatorname{cosec}\frac{\pi}{2n+1} = \frac{1}{2}.
\end{aligned}
$$

■

§ Problem 19.1.7. $\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \ldots$ *to* $2n$ *terms.* ♢

§§ Solution.

$$
\begin{aligned}
&\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \ldots \textit{ to } 2n \textit{ terms} \\
&= \cos\alpha + \cos(\alpha+\beta+\pi) + \cos(\alpha+2\beta+\pi) + \ldots \textit{ to } 2n \textit{ terms} \\
&= \cos\left[\alpha + \frac{2n-1}{2}(\beta+\pi)\right]\sin\frac{2n}{2}(\beta+\pi)\operatorname{cosec}\frac{1}{2}(\beta+\pi) \\
&= \cos\left[\alpha + \left(n-\frac{1}{2}\right)\beta + n\pi - \frac{\pi}{2}\right]\sin(n\beta+n\pi)\operatorname{cosec}\left(\frac{\beta}{2}+\frac{\pi}{2}\right) \\
&= \sin\left[\alpha + \left(n-\frac{1}{2}\right)\beta + n\pi\right]\sin(n\beta+n\pi)\sec\frac{\beta}{2} \\
&= \sin\left[\alpha + \left(n-\frac{1}{2}\right)\beta\right]\cos n\pi\cdot\sin n\beta\cos n\pi\sec\frac{\beta}{2} \\
&= \sin\left[\alpha + \left(n-\frac{1}{2}\right)\beta\right]\sin n\beta\sec\frac{\beta}{2},\ \because \cos^2 n\pi = +1.
\end{aligned}
$$

■

§ Problem 19.1.8. $\sin\theta + \sin\dfrac{n-4}{n-2}\theta + \sin\dfrac{n-6}{n-2}\theta + \ldots$ *to* n *terms.* ♢

§§ Solution.

$$
\begin{aligned}
&\sin\theta + \sin\frac{n-4}{n-2}\theta + \sin\frac{n-6}{n-2}\theta + \ldots \textit{ to } n \textit{ terms} \\
&= \sin\left[\theta + \frac{n-1}{2}\left(\frac{-2\theta}{n-2}\right)\right]\sin\frac{n}{2}\left(\frac{-2\theta}{n-2}\right)\operatorname{cosec}\left(\frac{-\theta}{n-2}\right) \\
&= \sin\left[\theta - \frac{n-1}{n-2}\theta\right]\sin\frac{n\theta}{n-2}\operatorname{cosec}\frac{\theta}{n-2} \\
&= \sin\left(\frac{-\theta}{n-2}\right)\sin\frac{n\theta}{n-2}\operatorname{cosec}\frac{\theta}{n-2} = -\sin\frac{n\theta}{n-2}.
\end{aligned}
$$

■

§ Problem 19.1.9. $\cos x + \sin 3x + \cos 5x + \sin 7x + \ldots + \sin(4n-1)x.$ ♢

§§ Solution.

$$
\begin{aligned}
&\cos x + \sin 3x + \cos 5x + \sin 7x + \ldots + \sin(4n-1)x \\
&\quad = (\cos x + \cos 5x + \ldots \textit{ to } n \textit{ terms}) \\
&\qquad\quad + (\sin 3x + \sin 7x + \ldots \textit{ to } n \textit{ terms}) \\
&\quad = \cos\left[x + \frac{n-1}{2}\cdot 4x\right]\sin\left(\frac{n}{2}\cdot 4x\right)\operatorname{cosec}\frac{4x}{2} \\
&\qquad\quad + \sin\left[3x + \frac{n-1}{2}\cdot 4x\right]\sin\left(\frac{n}{2}\cdot 4x\right)\operatorname{cosec}\frac{4x}{2} \\
&\quad = \{\cos(2n-1)x + \sin(2n+1)x\}\sin 2nx\operatorname{cosec} 2x \\
&\quad = (\cos 2nx\cos x + \sin 2nx\sin x \\
&\qquad\quad + \sin 2nx\cos x + \cos 2nx\sin x)\sin 2nx\operatorname{cosec} 2x \\
&\quad = (\cos 2nx + \sin 2nx)(\cos x + \sin x)\sin 2nx\operatorname{cosec} 2x.
\end{aligned}
$$

■

§ Problem 19.1.10.

$$\sin\alpha\sin 2\alpha + \sin 2\alpha\sin 3\alpha + \sin 3\alpha\sin 4\alpha + \dots \textit{ to } n \textit{ terms}. \qquad \Diamond$$

§§ Solution.

If

$$S = \sin\alpha\sin 2\alpha + \sin 2\alpha\sin 3\alpha + \sin 3\alpha\sin 4\alpha + \dots \textit{ to } n \textit{ terms}$$

then

$$\begin{aligned}
2S &= \cos\alpha - \cos 3\alpha + \cos\alpha - \cos 5\alpha + \cos\alpha - \cos 7\alpha + \dots \textit{ to } n \textit{ terms} \\
&= n\cos\alpha - (\cos 3\alpha + \cos 5\alpha + \cos 7\alpha + \dots \textit{ to } n \textit{ terms}) \\
&= n\cos\alpha - \cos\left[3\alpha + (n-1)\alpha\right]\sin n\alpha \operatorname{cosec}\,\alpha \\
&= \frac{n}{2}\sin 2\alpha \operatorname{cosec}\,\alpha - \cos(n+2)\alpha\sin n\alpha\operatorname{cosec}\,\alpha \\
&= \frac{1}{2}\left[n\sin 2\alpha - \sin(2n+2)\alpha + \sin 2\alpha\right]\operatorname{cosec}\,\alpha \\
\therefore S &= \frac{1}{4}\left[(n+1)\sin 2\alpha - \sin(2n+2)\alpha\right]\operatorname{cosec}\,\alpha.
\end{aligned}$$

■

§ Problem 19.1.11.

$$\cos\alpha\sin 2\alpha + \sin 2\alpha\cos 3\alpha + \cos 3\alpha\sin 4\alpha + \sin 4\alpha\cos 5\alpha + \dots \textit{ to } 2n \textit{ terms}. \qquad \Diamond$$

§§ Solution.

If

$$S = \cos\alpha\sin 2\alpha + \sin 2\alpha\cos 3\alpha + \cos 3\alpha\sin 4\alpha + \sin 4\alpha\cos 5\alpha + \dots \textit{ to } 2n \textit{ terms}$$

then

$$\begin{aligned}
2S &= \sin 3\alpha + \sin\alpha + \sin 5\alpha - \sin\alpha + \sin 7\alpha + \sin\alpha \\
&\quad + \sin 9\alpha - \sin\alpha + \dots \\
&= \sin 3\alpha + \sin 5\alpha + \sin 7\alpha + \sin 9\alpha + \dots \textit{ to } 2n \textit{ terms} \\
&= \sin\left[3\alpha + (2n-1)\alpha\right]\sin 2n\alpha\operatorname{cosec}\,\alpha \\
\therefore S &= \frac{1}{2}\sin(2n+2)\alpha\sin 2n\alpha\operatorname{cosec}\,\alpha.
\end{aligned}$$

■

§ Problem 19.1.12.

$$\sin\alpha\sin 3\alpha + \sin 2\alpha\sin 4\alpha + \sin 3\alpha\sin 5\alpha + \dots \textit{ to } n \textit{ terms}. \qquad \Diamond$$

§§ Solution.

$$\begin{aligned}
S &= \sin\alpha\sin 3\alpha + \sin 2\alpha\sin 4\alpha + \sin 3\alpha\sin 5\alpha + \dots \textit{ to } n \textit{ terms} \\
\therefore 2S &= \cos 2\alpha - \cos 4\alpha + \cos 2\alpha - \cos 6\alpha + \cos 2\alpha - \cos 8\alpha + \dots \\
&= n\cos 2\alpha - (\cos 4\alpha + \cos 6\alpha + \cos 8\alpha + \dots \textit{ to } n \textit{ terms}) \\
&= n\cos 2\alpha - \cos\left[4\alpha + (n-1)\alpha\right]\sin n\alpha\operatorname{cosec}\,\alpha. \\
\therefore S &= \frac{n}{2}\cos 2\alpha - \frac{1}{2}\cos(n+3)\alpha\sin n\alpha\operatorname{cosec}\,\alpha.
\end{aligned}$$

■

§ Problem 19.1.13.

$$\cos\alpha\cos\beta + \cos 3\alpha\cos 2\beta + \cos 5\alpha\cos 3\beta + \dots \textit{ to } n \textit{ terms}. \qquad \Diamond$$

§§ Solution.

$$
\begin{aligned}
S &= \cos\alpha\cos\beta + \cos 3\alpha\cos 2\beta + \cos 5\alpha\cos 3\beta + \ldots \textit{ to } n \textit{ terms}\\
\therefore 2S &= \cos(\alpha-\beta) + \cos(\alpha+\beta) + \cos(3\alpha-2\beta) + \cos(3\alpha+2\beta)\\
&\qquad + \cos(5\alpha-3\beta) + \cos(5\alpha+3\beta) + \ldots\\
&= \{\cos(\alpha-\beta) + \cos(3\alpha-2\beta) + \cos(5\alpha-3\beta) + \ldots \textit{ to } n \textit{ terms}\}\\
&+ \{\cos(\alpha+\beta) + \cos(3\alpha+2\beta) + \cos(5\alpha+3\beta) + \ldots \textit{ to } n \textit{ terms}\}\\
&= \cos\left[(\alpha-\beta) + \frac{n-1}{2}(2\alpha-\beta)\right]\sin\frac{n}{2}(2\alpha-\beta)\operatorname{cosec}\frac{1}{2}(2\alpha-\beta)\\
&+ \cos\left[(\alpha+\beta) + \frac{n-1}{2}(2\alpha+\beta)\right]\sin\frac{n}{2}(2\alpha+\beta)\operatorname{cosec}\frac{1}{2}(2\alpha+\beta)\\
&= \frac{\begin{array}{l}\cos\left[n\alpha - \frac{n+1}{2}\beta\right]\sin\left(n\alpha - \frac{n\beta}{2}\right)\sin\left(\alpha+\frac{\beta}{2}\right)\\ + \cos\left[n\alpha + \frac{n+1}{2}\beta\right]\sin\left(n\alpha + \frac{n\beta}{2}\right)\sin\left(\alpha-\frac{\beta}{2}\right)\end{array}}{\sin\frac{2\alpha-\beta}{2}\sin\frac{2\alpha+\beta}{2}}\\
&= \frac{\begin{array}{l}\cos\left(n\alpha - \frac{n+1}{2}\beta\right)\\ \left[\cos\left(n\alpha-\alpha-\frac{n+1}{2}\beta\right) - \cos\left(n\alpha+\alpha-\frac{n-1}{2}\beta\right)\right]\\ + \mathit{cos}\left(n\alpha+\frac{n+1}{2}\beta\right)\\ \left[\cos\left(n\alpha-\alpha+\frac{n+1}{2}\beta\right) - \cos\left(n\alpha+\alpha+\frac{n-1}{2}\beta\right)\right]\end{array}}{(\cos\beta - \cos 2\alpha)}\\
\therefore 4S &= \frac{\begin{array}{l}\cos[2n\alpha - \alpha - (n+1)\beta]\\ + \cos\alpha - \cos(2n\alpha + \alpha - n\beta) - \cos(\alpha-\beta)\\ + \cos[2n\alpha - \alpha + (n+1)\beta]\\ + \cos\alpha - \cos(2n\alpha+\alpha+n\beta) - \cos(\alpha+\beta)\end{array}}{(\cos\beta - \cos 2\alpha)}\\
\therefore S &= \frac{\begin{array}{l}\cos(2n\alpha-\alpha)\cos(n+1)\beta - \cos(2n\alpha+\alpha)\cos n\beta\\ \qquad + \cos\alpha(1-\cos\beta)\end{array}}{2(\cos\beta - \cos 2\alpha)}.
\end{aligned}
$$

■

§ Problem 19.1.14. $\sin^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \ldots$ *to n terms.* ◊

§§ Solution.

$$
\begin{aligned}
S &= \sin^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \ldots \textit{ to } n \textit{ terms}\\
\therefore 2S &= 1 - \cos 2\alpha + 1 - \cos 4\alpha + 1 - \cos 6\alpha + \ldots\\
&= n - (\cos 2\alpha + \cos 4\alpha + \cos 6\alpha + \ldots \textit{ to } n \textit{ terms})\\
&= n - \cos[2\alpha + (n-1)\alpha]\sin n\alpha\operatorname{cosec}\alpha\\
&= \{n\sin\alpha - \cos(n+1)\alpha\sin n\alpha\}\operatorname{cosec}\alpha\\
\therefore 4S &= \{2n\sin\alpha - \sin(2n+1)\alpha + \sin\alpha\}\operatorname{cosec}\alpha\\
\therefore S &= \frac{1}{4}\{(2n+1)\sin\alpha - \sin(2n+1)\alpha\}\operatorname{cosec}\alpha.
\end{aligned}
$$

■

§ Problem 19.1.15. $\sin^2\theta + \sin^2(\theta+\alpha) + \sin^2(\theta+2\alpha) + \ldots$ *to n terms.* ◊

§§ Solution.

$$S = \sin^2\theta + \sin^2(\theta+\alpha) + \sin^2(\theta+2\alpha) + \ldots \textit{ to } n \textit{ terms}$$

$$\begin{aligned}\therefore 2S &= 1-\cos 2\theta + 1 - \cos(2\theta+2\alpha) + 1 - \cos(2\theta+4\alpha) + \ldots \\ &= n - \{\cos 2\theta + \cos(2\theta+2\alpha) + \cos(2\theta+4\alpha) + \ldots \textit{ to } n \textit{ terms}\} \\ &= n - \cos[2\theta+(n-1)\alpha]\sin n\alpha \operatorname{cosec}\alpha \\ \therefore S &= \frac{n}{2} - \frac{1}{2}\cos[2\theta+(n-1)\alpha]\sin n\alpha\operatorname{cosec}\alpha. \end{aligned}$$ ■

§ Problem 19.1.16. $\sin^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \ldots$ *to n terms.* ◊

§§ Solution. $\because \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$, i.e. $4\sin^3\alpha = 3\sin\alpha - \sin 3\alpha$,

$$\begin{aligned} S &= \sin^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \ldots \textit{ to } n \textit{ terms} \\ \therefore 4S &= 3(\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots \textit{ to } n \textit{ terms}) \\ &\quad - (\sin 3\alpha + \sin 6\alpha + \sin 9\alpha + \ldots \textit{ to } n \textit{ terms}) \\ &= 3\sin\left[\alpha + (n-1)\frac{\alpha}{2}\right]\sin\frac{n\alpha}{2}\operatorname{cosec}\frac{\alpha}{2} \\ &\quad - \sin\left[3\alpha + \frac{n-1}{2}\cdot 3\alpha\right]\sin\frac{3n\alpha}{2}\operatorname{cosec}\frac{3\alpha}{2} \\ \therefore S &= \frac{3}{4}\sin\frac{(n+1)\alpha}{2}\sin\frac{n\alpha}{2}\operatorname{cosec}\frac{\alpha}{2} \\ &\quad - \frac{1}{4}\sin\frac{3(n+1)\alpha}{2}\sin\frac{3n\alpha}{2}\operatorname{cosec}\frac{3\alpha}{2}. \end{aligned}$$ ■

§ Problem 19.1.17. $\sin^4\alpha + \sin^4 2\alpha + \sin^4 3\alpha + \ldots$ *to n terms.* ◊

§§ Solution.

$$\begin{aligned} S &= \sin^4\alpha + \sin^4 2\alpha + \sin^4 3\alpha + \ldots \textit{ to } n \textit{ terms} \\ \because 4\sin^4\alpha &= \left(2\sin^2\alpha\right)^2 = (1-\cos 2\alpha)^2 \\ &= 1 - 2\cos 2\alpha + \cos^2 2\alpha \\ \therefore 8\sin^4\alpha &= 2 - 4\cos 2\alpha + (1+\cos 4\alpha) \\ &= 3 - 4\cos 2\alpha + \cos 4\alpha \\ \therefore 8S &= 3n - 4(\cos 2\alpha + \cos 4\alpha + \cos 6\alpha + \ldots \textit{ to } n \textit{ terms}) \\ &\quad + (\cos 4\alpha + \cos 8\alpha + \cos 12\alpha + \ldots \textit{ to } n \textit{ terms}) \\ &= 3n - 4\cos\left[2\alpha + \frac{n-1}{2}\cdot 2\alpha\right]\sin n\alpha\operatorname{cosec}\alpha \\ &\quad + \cos\left[4\alpha + \frac{n-1}{2}\cdot 4\alpha\right]\sin 2n\alpha\operatorname{cosec}2\alpha \\ \therefore S &= \frac{1}{8}\{3n - 4\cos(n+1)\alpha\sin n\alpha\operatorname{cosec}\alpha \\ &\quad + \cos(2n+2)\alpha\sin 2n\alpha\operatorname{cosec}2\alpha\}. \end{aligned}$$ ■

§ Problem 19.1.18. $\cos^4\alpha + \cos^4 2\alpha + \cos^4 3\alpha + \ldots$ *to n terms.* ◊

§§ Solution.

Let $S = \cos^4\alpha + \cos^4 2\alpha + \cos^4 3\alpha + \ldots$ *to n terms*

$$\begin{aligned} \because 4\cos^4\alpha &= \left(2\cos^2\alpha\right)^2 = (1+\cos 2\alpha)^2 \\ &= 1 + 2\cos 2\alpha + \cos^2 2\alpha \\ \therefore 8\cos^4\alpha &= 2 + 4\cos 2\alpha + (1+\cos 4\alpha) \\ &= 3 + 4\cos 2\alpha + \cos 4\alpha \\ \therefore 8S &= 3n + 4(\cos 2\alpha + \cos 4\alpha + \ldots \textit{ to } n \textit{ terms}) \\ &\quad + (\cos 4\alpha + \cos 8\alpha + \ldots \textit{ to } n \textit{ terms}) \\ &= 3n + 4\cos[2\alpha + (n-1)\alpha]\sin n\alpha\operatorname{cosec}\alpha \\ &\quad + \cos[4\alpha + (n-1)2\alpha]\sin 2n\alpha\operatorname{cosec}2\alpha \end{aligned}$$

$$\therefore S = \frac{1}{8}\{3n + 4\cos(n+1)\alpha \sin n\alpha \operatorname{cosec} \alpha \\ + \cos(2n+2)\alpha \sin 2n\alpha \operatorname{cosec} 2\alpha\}. \qquad \blacksquare$$

§ Problem 19.1.19.

$$\cos\theta\cos 2\theta\cos 3\theta + \cos 2\theta\cos 3\theta\cos 4\theta \\ + \ldots \text{ to } n \text{ terms}. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
S &= \cos\theta\cos 2\theta\cos 3\theta + \cos 2\theta\cos 3\theta\cos 4\theta + \ldots \text{ to } n \text{ terms} \\
\therefore 2S &= \cos 2\theta\,(\cos 2\theta + \cos 4\theta) + \cos 3\theta\,(\cos 2\theta + \cos 6\theta) + \ldots \text{ to } n \text{ terms} \\
\therefore 4S &= 2\cos 2\theta\,(\cos 2\theta + \cos 3\theta + \ldots \text{ to } n \text{ terms}) \\
&\quad + \cos 2\theta + \cos 6\theta + \cos 3\theta + \cos 9\theta + \ldots \text{ to } 2n \text{ terms} \\
&= (2\cos 2\theta + 1)\,(\cos 2\theta + \cos 3\theta + \ldots \text{ to } n \text{ terms}) \\
&\quad + (\cos 6\theta + \cos 9\theta + \ldots \text{ to } n \text{ terms}) \\
&= (2\cos 2\theta + 1)\cos\left[2\theta + \frac{n-1}{2}\theta\right]\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2} \\
&\quad + \cos\left[6\theta + \frac{n-1}{2}\cdot 3\theta\right]\sin\frac{3n\theta}{2}\operatorname{cosec}\frac{3\theta}{2} \\
&= \sin\frac{n\theta}{2}\left(2\cos 2\theta\cos\frac{n+3}{2}\theta + \cos\frac{n+3}{2}\theta\right)\operatorname{cosec}\frac{\theta}{2} \\
&\quad + \sin\frac{3n\theta}{2}\cos\frac{3n+9}{2}\theta\operatorname{cosec}\frac{3\theta}{2} \\
\therefore S &= \frac{1}{4}\sin\frac{n\theta}{2}\left(\cos\frac{n-1}{2}\theta + \cos\frac{n+7}{2}\theta + \cos\frac{n+3}{2}\theta\right)\operatorname{cosec}\frac{\theta}{2} \\
&\quad + \frac{1}{4}\sin\frac{3n\theta}{2}\cos\frac{3n+9}{2}\theta\operatorname{cosec}\frac{3\theta}{2}. \qquad \blacksquare
\end{aligned}$$

§ Problem 19.1.20.

$$\sin\alpha\sin(\alpha+\beta) - \sin(\alpha+\beta)\sin(\alpha+2\beta) \\ + \ldots \text{ to } 2n \text{ terms}. \qquad \lozenge$$

§§ Solution.

$$\begin{aligned}
S &= \sin\alpha\sin(\alpha+\beta) - \sin(\alpha+\beta)\sin(\alpha+2\beta) + \ldots \text{ to } 2n \text{ terms} \\
\therefore 2S &= \cos\beta - \cos(2\alpha+\beta) - \cos\beta + \cos(2\alpha+3\beta) + \ldots \text{ to } 2n \text{ terms} \\
&= -\{\cos(2\alpha+\beta) - \cos(2\alpha+3\beta) + \ldots \text{ to } 2n \text{ terms}\} \\
&= -\{\cos(2\alpha+\beta) + \cos(2\alpha+3\beta+\pi) + \ldots \text{ to } 2n \text{ terms}\} \\
&= -\cos\left[(2\alpha+\beta) + \frac{2n-1}{2}(2\beta+\pi)\right]\sin n(2\beta+\pi)\operatorname{cosec}\left(\beta+\frac{\pi}{2}\right) \\
&= -\cos\left(2\alpha + 2n\beta + n\pi - \frac{\pi}{2}\right)\sin(2n\beta + n\pi)\sec\beta \\
&= -\sin(2\alpha + 2n\beta + n\pi)\sin(2n\beta + n\pi)\sec\beta \\
&= -\sin(2\alpha + 2n\beta)\cos n\pi\sin 2n\beta\cos n\pi\sec\beta \\
\therefore S &= -\frac{1}{2}\sin(2\alpha+2n\beta)\sin 2n\beta\sec\beta, \ \because \cos^2 n\pi = +1. \qquad \blacksquare
\end{aligned}$$

§ Problem 19.1.21. *From the sum of the series*

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots \text{ to } n \text{ terms},$$

deduce (by making α very small) the sum of the series

$$1 + 2 + 3 + \ldots + n. \qquad \lozenge$$

§§ Solution.

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha$$
$$= \frac{\sin\left[\alpha + \frac{n-1}{2}\alpha\right]\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}} = \frac{\sin\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

Dividing by α, we have

$$\frac{\sin\alpha}{\alpha} + 2\cdot\frac{\sin 2\alpha}{2\alpha} + \ldots + n\cdot\frac{\sin n\alpha}{n\alpha}$$
$$= \frac{\frac{n+1}{2}\cdot\frac{\sin\frac{n+1}{2}\alpha}{\frac{n+1}{2}\alpha}\cdot\frac{n}{2}\cdot\frac{\sin\frac{n\alpha}{2}}{\frac{n\alpha}{2}}}{\frac{1}{2}\cdot\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}}$$

Hence, making α very small, we have

$$1 + 2 + \ldots + n = \frac{\frac{n+1}{2}\cdot\frac{n}{2}}{\frac{1}{2}} = \frac{n(n+1)}{2}.$$

■

§ Problem 19.1.22. *From the result of the example of* Art. 241*, deduce the sum of*

$$1 + 3 + 5 \ldots \textit{ to } n \textit{ terms}.$$

◊

§§ Solution.

$$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha = \frac{\sin^2 n\alpha}{\sin\alpha}.$$

Dividing by α, we have

$$\frac{\sin\alpha}{\alpha} + 3\cdot\frac{\sin 3\alpha}{3\alpha} + \ldots + (2n-1)\frac{\sin(2n-1)\alpha}{(2n-1)\alpha} = \frac{n^2\left(\frac{\sin n\alpha}{n\alpha}\right)^2}{\frac{\sin\alpha}{\alpha}}.$$

Hence, making α very small, we have

$$1 + 3 + 5 \ldots \textit{ to } n \textit{ terms } = n^2.$$

■

§ Problem 19.1.23. *If* $\alpha = \frac{2\pi}{17}$*,*

prove that $\quad 2(\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha)$

and $\quad 2(\cos 3\alpha + \cos 5\alpha + \cos 6\alpha + \cos 7\alpha)$

are the roots of the equation

$$x^2 + x - 4 = 0.$$

◊

§§ Solution. $\because \alpha = \frac{2\pi}{17}$, $\therefore 17\alpha = 2\pi$.

Let $\quad p = 2(\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha)$,

and $\quad q = 2(\cos 3\alpha + \cos 5\alpha + \cos 6\alpha + \cos 7\alpha)$.

Then, by the relation $17\alpha = 2\pi$, we have

$$p = 2(\cos\alpha + \cos 15\alpha + \cos 13\alpha + \cos 9\alpha)$$

and $\quad q = 2(\cos 3\alpha + \cos 5\alpha + \cos 11\alpha + \cos 7\alpha)$.

$$\therefore p + q = 2(\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots \textit{ to } 8 \textit{ terms})$$

$$= \frac{2\cos(\alpha + 7\alpha)\sin 8\alpha}{\sin\alpha}$$
$$= \frac{2\cos 8\alpha\sin 8\alpha}{\sin\alpha} = \frac{\sin 16\alpha}{\sin\alpha} = -1.$$

$$\therefore pq = 2\begin{bmatrix} \cos 4\alpha + \cos 2\alpha + \cos 18\alpha + \cos 12\alpha + \cos 16\alpha + \cos 10\alpha \\ + \cos 12\alpha + \cos 6\alpha + \cos 6\alpha + \cos 4\alpha + \cos 20\alpha + \cos 10\alpha \\ + \cos 18\alpha + \cos 8\alpha + \cos 14\alpha + \cos 4\alpha + \cos 12\alpha + \cos 10\alpha \\ + \cos 26\alpha + \cos 4\alpha + \cos 24\alpha + \cos 2\alpha + \cos 20\alpha + \cos 2\alpha \\ + \cos 8\alpha + \cos 6\alpha + \cos 22\alpha + \cos 8\alpha + \cos 20\alpha + \cos 6\alpha \\ + \cos 16\alpha + \cos 2\alpha \end{bmatrix}$$

$$\therefore pq = 8\left(\cos 2\alpha + \cos 4\alpha + \cos 6\alpha + \ldots \textit{ to } 8 \textit{ terms}\right),$$
$$\because 17\alpha = 2\pi \textit{ and } 34\alpha = 4\pi$$
$$= \frac{8\cos(2\alpha + 7\alpha)\sin 8\alpha}{\sin\alpha}$$
$$= \frac{8\cos 8\alpha\sin 8\alpha}{\sin\alpha} = \frac{4\sin 16\alpha}{\sin\alpha} = -4.$$

Hence the equation whose roots are p and q is $x^2 + x - 4 = 0$. ■

§ Problem 19.1.24. *$ABCD\ldots$ is a regular polygon of n sides which is inscribed in a circle, whose center is O and whose radius is r and P is any point on the arc AB such that $\angle POA$ is θ. Prove that*

$$PA \cdot PB + PA \cdot PC + PA \cdot PD + \ldots + PB \cdot PC + \ldots$$
$$= r^2\left[2\cos^2\left(\frac{\theta}{2} - \frac{\pi}{2n}\right)\operatorname{cosec}^2\frac{\pi}{2n} - n\right].$$ ◊

§§ Solution. Each of the angles $\angle AOB,\ \angle BOC\ldots = \dfrac{2\pi}{n} = \alpha$, say.

Hence we have

$$PA = 2r\sin\frac{\theta}{2},\ PB = 2r\sin\frac{\alpha - \theta}{2},\ PC = 2r\sin\frac{2\alpha - \theta}{2},\ \ldots$$

$$\therefore PA + PB + PC + \ldots$$
$$= 2r\left[\sin\frac{\theta}{2} + \sin\left(\frac{\pi}{n} - \frac{\theta}{2}\right) + \sin\left(\frac{2\pi}{n} - \frac{\theta}{2}\right) + \ldots \textit{ to } n \textit{ terms}\right]$$
$$\because \sin\frac{\theta}{2} = \sin\left(\pi - \frac{\theta}{2}\right) = \textit{the last term},$$
$$= 2r\left[\sin\left(\frac{\pi}{n} - \frac{\theta}{2}\right) + \sin\left(\frac{2\pi}{n} - \frac{\theta}{2}\right) + \ldots \textit{ to } n \textit{ terms}\right]$$
$$= 2r\left[\sin\left(\frac{\pi}{n} - \frac{\theta}{2}\right) + \frac{n-1}{2}\cdot\frac{\pi}{n}\right]\sin\frac{n}{2}\cdot\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{2n}$$
$$= 2r\sin\left(\frac{\pi}{n} - \frac{\theta}{2} + \frac{\pi}{2} - \frac{\pi}{2n}\right)\sin\frac{\pi}{2}\operatorname{cosec}\frac{\pi}{2n}$$
$$= 2r\cos\left(\frac{\pi}{2n} - \frac{\theta}{2}\right)\operatorname{cosec}\frac{\pi}{2n}$$
$$\therefore (PA + PB + PC + \ldots)^2 = 4r^2\cos^2\left(\frac{\theta}{2} - \frac{\pi}{2n}\right)\operatorname{cosec}^2\frac{\pi}{2n} \tag{19.1}$$
$$\therefore PB^2 + PC^2 + \ldots$$
$$= 4r^2\left[\sin^2\left(\frac{\pi}{n} - \frac{\theta}{2}\right) + \sin^2\left(\frac{2\pi}{n} - \frac{\theta}{2}\right) + \ldots \textit{ to } n \textit{ terms}\right]$$
$$= 2r^2\left[1 - \cos\left(\frac{2\pi}{n} - \theta\right) + 1 - \cos\left(\frac{4\pi}{n} - \theta\right) + \ldots\right]$$
$$= 2r^2\left[n - \cos\left(\frac{2\pi}{n} - \theta + \frac{n-1}{2}\cdot\frac{2\pi}{n}\right)\sin\pi\operatorname{cosec}\frac{\pi}{n}\right]$$

$$\therefore PB^2 + PC^2 + \ldots = 2r^2 n, \because \sin\pi = 0 \tag{19.2}$$

Hence, subtracting Eq. (19.2) from Eq. (19.1) and canceling 2 on each side, we have

$$\begin{aligned} &PA \cdot PB + PA \cdot PC + PA \cdot PD + \ldots + PB \cdot PC + \ldots \\ &= r^2 \left[2\cos^2 \left(\frac{\theta}{2} - \frac{\pi}{2n} \right) \operatorname{cosec}^2 \frac{\pi}{2n} - n \right]. \end{aligned}$$ ∎

§ Problem 19.1.25. *Two regular polygons, each of n sides, are circumscribed to and inscribed in a given circle. If an angular point of one of them be joined to each of the angular points of the other, then the sum of the squares of the straight lines so drawn is to the sum of the areas of the polygons as*

$$2 : \sin\frac{2\pi}{n}.$$ ◊

§§ Solution. Let O be the center of the circle and let the angular point A of the in-polygon be joined to each of the angular points $(A',\ B',\ \ldots)$ of the circum-polygon.

Let $OA = r$ and $OA' = r' \left(= r\sec\frac{\pi}{n} \right)$.

Let the $\angle AOK = \alpha$, where K is the point of contact of the side $A'B'$ of the circum-polygon with the circle. We have

$$\begin{aligned} AA'^2 &= r^2 + r'^2 - 2rr' \cos\left(\frac{\pi}{n} - \alpha \right) \\ \text{and } AB'^2 &= r^2 + r'^2 - 2rr' \cos\left(\frac{3\pi}{n} - \alpha \right) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

For the sum of the series of cosines vanishes, since the difference of the angles $= \dfrac{2\pi}{n}$,

$$\begin{aligned} \therefore AA'^2 + AB'^2 + \ldots &= n\left(r^2 + r'^2\right) \\ &= nr^2 \left(1 + \sec^2\frac{\pi}{n}\right) = nr^2 \frac{\cos^2\frac{\pi}{n} + 1}{\cos^2\frac{\pi}{n}}. \end{aligned}$$

Also, the sum of the areas of the polygons

$$\begin{aligned} &= \frac{n}{2} r^2 \sin\frac{2\pi}{n} + nr^2 \tan\frac{\pi}{n} \\ &= nr^2 \left(\sin\frac{\pi}{n}\cos\frac{\pi}{n} + \frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}} \right) \\ &= nr^2 \sin\frac{\pi}{n} \cdot \frac{\cos^2\frac{\pi}{n} + 1}{\cos\frac{\pi}{n}}. \end{aligned}$$

$$\text{Hence the required ratio } = \frac{1}{\sin\frac{\pi}{n}\cos\frac{\pi}{n}} = \frac{2}{\sin\frac{2\pi}{n}}.$$ ∎

§ Problem 19.1.26. $A_1,\ A_2,\ \ldots A_{2n+1}$ *are the angular points of a regular polygon inscribed in a circle and O is any point on the circumference between A_1 and A_{2n+1}; prove that*

$$OA_1 + OA_3 + \ldots + OA_{2n+1} = OA_2 + OA_4 + \ldots + OA_{2n}.$$ ◊

§§ Solution. Let C be the center and r be the radius of the circle; the $\angle OCA_1 = \alpha$, and the $\angle A_1CA_2 = \theta = \dfrac{2\pi}{2n+1}$. We have

$$\begin{aligned}
&OA_1 + OA_3 + \ldots + OA_{2n+1}\\
&\quad = 2r\sin\frac{\alpha}{2} + 2r\sin\left(\frac{\alpha}{2}+\theta\right) + \ldots \textit{ to } (n+1) \textit{ terms}\\
&\quad = 2r\sin\left(\frac{\alpha}{2}+\frac{n\theta}{2}\right)\sin\frac{(n+1)\theta}{2}\operatorname{cosec}\frac{\theta}{2}\\
&OA_2 + OA_4 + \ldots + OA_{2n}\\
&\quad = 2r\sin\frac{\alpha+\theta}{2} + 2r\sin\frac{\alpha+3\theta}{2} + \ldots \textit{ to } n \textit{ terms}\\
&\quad = 2r\sin\left[\frac{\alpha+\theta}{2}+\frac{(n-1)\theta}{2}\right]\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}\\
&\quad = 2r\sin\left(\frac{\alpha}{2}+\frac{n\theta}{2}\right)\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}.
\end{aligned}$$

Now
$$\frac{(n+1)\theta}{2} + \frac{n\theta}{2} = \frac{(2n+1)\theta}{2} = \pi.$$
$$\therefore \sin\frac{(n+1)\theta}{2} = \sin\frac{n\theta}{2}.$$

Hence the series are equal. ■

§ Problem 19.1.27. *If perpendiculars be drawn on the sides of a regular polygon of n sides from any point on the inscribed circle whose radius is a, prove that*

$$\frac{2}{n}\Sigma\left(\frac{p}{a}\right)^2 = 3, \textit{ and } \frac{2}{n}\Sigma\left(\frac{p}{a}\right)^3 = 5.$$ ◊

§§ Solution. Let KA, KB, $KC\ldots$ be the perpendiculars from K, the center of the circle, on the sides of the polygon.

Let O be the point on the circumference of the circle (take O between A and B) and OM, ON $\ldots$ be the perpendiculars on the sides (produced) passing through B, C, $\ldots$; also let the $\angle AKO = \alpha$.

We have the $\angle AKB = \dfrac{2\pi}{n} = \angle BKC = \ldots$

$$\therefore p_1 = OM = KB - OK\cos\angle OKB = a\left[1-\cos\left(\frac{2\pi}{n}-\alpha\right)\right] \tag{19.3}$$

Similarly, $$p_2 = a\left[1-\cos\left(\frac{4\pi}{n}-\alpha\right)\right], \textit{ and so on.}$$

From Eq. (19.3):

$$\begin{aligned}
\left(\frac{p_1}{a}\right)^2 &= 1 - 2\cos\left(\frac{2\pi}{n}-\alpha\right) + \cos^2\left(\frac{2\pi}{n}-\alpha\right)\\
\therefore 2\left(\frac{p_1}{a}\right)^2 &= 3 - 4\cos\left(\frac{2\pi}{n}-\alpha\right) + \cos\left(\frac{4\pi}{n}-2\alpha\right)\\
\therefore 2\Sigma\left(\frac{p_1}{a}\right)^2 &= 3n - 4\left[\cos\left(\frac{2\pi}{n}-\alpha\right) + \cos\left(\frac{4\pi}{n}-\alpha\right) + \ldots \textit{ to } n \textit{ terms}\right]\\
&\quad + \left[\cos\left(\frac{4\pi}{n}-2\alpha\right) + \cos\left(\frac{8\pi}{n}-2\alpha\right) + \ldots \textit{ to } n \textit{ terms}\right]\\
&= 3n, \because \textit{ each series vanishes } (\textit{Art. } 243)\,.
\end{aligned}$$

$$\therefore \frac{2}{n}\Sigma\left(\frac{p}{a}\right)^2 = 3.$$

Again,

$$\left(\frac{p_1}{a}\right)^3 = 1 - 3\cos\left(\frac{2\pi}{n}-\alpha\right) + 3\cos^2\left(\frac{2\pi}{n}-\alpha\right) - \cos^3\left(\frac{2\pi}{n}-\alpha\right)$$

$$= 1 - 3\cos\left(\frac{2\pi}{n} - \alpha\right) + \frac{3}{2}\left[1 + \cos\left(\frac{4\pi}{n} - \alpha\right)\right]$$
$$- \frac{1}{4}\left[\cos 3\left(\frac{2\pi}{n} - \alpha\right) + 3\cos\left(\frac{2\pi}{n} - \alpha\right)\right]$$
$$\therefore \Sigma\left(\frac{p}{a}\right)^3 = n + \frac{3}{2}n. \because \textit{ the series vanishes as before,}$$
$$\therefore \frac{2}{n}\Sigma\left(\frac{p}{a}\right)^3 = 5.$$

■

Chapter 20

Elimination

20.1 Elimination of Unknown Quantity

Eliminate θ from the equations :

§ Problem 20.1.1. $a\cos\theta + b\sin\theta = c,$ *and* $b\cos\theta - a\sin\theta = d.$ ◊

§§ Solution. Solving for $\cos\theta$ and $\sin\theta$, we have

$$\frac{\cos\theta}{ac+bd} = \frac{\sin\theta}{bc-ad} = \frac{1}{a^2+b^2}$$

$$\therefore 1 = \cos^2\theta + \sin^2\theta = \frac{(ac+bd)^2 + (bc-ad)^2}{(a^2+b^2)^2}$$

$$= \frac{a^2c^2 + b^2d^2 + 2abcd + b^2c^2 + a^2d^2 - 2abcd}{(a^2+b^2)^2}$$

$$= \frac{\left(a^2+b^2\right)c^2 + \left(a^2+b^2\right)d^2}{(a^2+b^2)^2} = \frac{c^2+d^2}{a^2+b^2}$$

$$\therefore a^2+b^2 = c^2+d^2.$$

Otherwise thus :

Squaring both sides of each of the given equations, we have

$$a^2\cos^2\theta + 2ab\cos\theta\sin\theta + b^2\sin^2\theta = c^2,$$

and

$$b^2\cos^2\theta - 2ab\cos\theta\sin\theta + a^2\sin^2\theta = d^2.$$

Hence, by addition, we have

$$a^2\left(\cos^2\theta + \sin^2\theta\right) + b^2\left(\cos^2\theta + \sin^2\theta\right) = c^2 + d^2$$

$$\therefore a^2+b^2 = c^2+d^2.$$

■

§ Problem 20.1.2. $x = a\cos(\theta-\alpha)$ *and* $y = b\cos(\theta-\beta).$ ◊

§§ Solution. From the given equations, we have

$$\frac{x}{a} = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

and
$$\frac{y}{b} = \cos\theta\cos\beta + \sin\theta\sin\beta.$$

$$\therefore\ \frac{x}{a}\sin\beta - \frac{y}{b}\sin\alpha = \cos\theta\sin\left(\beta-\alpha\right)) \tag{20.1}$$

$$\therefore\ \frac{x}{a}\cos\beta - \frac{y}{b}\cos\alpha = \sin\theta\sin\left(\alpha-\beta\right)) \tag{20.2}$$

Squaring and adding Eq. (20.1) and Eq. (20.2), we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right) = \sin^2\left(\alpha-\beta\right)$$

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\left(\alpha-\beta\right) = \sin^2\left(\alpha-\beta\right).$$ ■

§ Problem 20.1.3. *$a\cos 2\theta = b\sin\theta$ and $c\sin 2\theta = d\cos\theta$.* ◊

§§ Solution.

$$a\cos 2\theta = b\sin\theta \tag{20.3}$$

$$c\sin 2\theta = d\cos\theta \tag{20.4}$$

From Eq. (20.4), we have

$$2c\sin\theta\cos\theta = d\cos\theta$$

$$\therefore\ \sin\theta = \frac{d}{2c} \tag{20.5}$$

But Eq. (20.3) is
$$a\left(1-2\sin^2\theta\right) = b\sin\theta \tag{20.6}$$

Substituting from Eq. (20.5) in Eq. (20.6), we have

$$a\left(1-\frac{d^2}{2c^2}\right) = \frac{bd}{2c}$$

$$\therefore\ a\left(2c^2-d^2\right) = bdc.$$ ■

§ Problem 20.1.4. *$a\sin\alpha - b\cos\alpha = 2b\sin\theta$ and $a\sin 2\alpha - b\cos 2\theta = a$.* ◊

§§ Solution. The second equation may be written

$$2a\sin\alpha\cos\alpha - b + 2b\sin^2\theta = a \tag{20.7}$$

From the first equation, we have

$$\sin\theta = \frac{a\sin\alpha - b\cos\alpha}{2b} \tag{20.8}$$

Hence, by substituting from Eq. (20.8) in Eq. (20.7), we have

$$2a\sin\alpha\cos\alpha - b + \frac{\left(a\sin\alpha - b\cos\alpha\right)^2}{2b} = a$$

$$\therefore\ 4ab\sin\alpha\cos\alpha - 2b^2 + a^2\sin^2\alpha - 2ab\sin\alpha\cos\alpha + b^2\cos^2\alpha = 2ab$$

$$\therefore\ \left(a\sin\alpha + b\cos\alpha\right)^2 = 2ab + 2b^2$$

$$\therefore\ a\sin\alpha + b\cos\alpha = \sqrt{2b(a+b)}.$$ ■

§ Problem 20.1.5. *$x\sin\theta - y\cos\theta = \sqrt{x^2+y^2}$ and $\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \frac{1}{x^2+y^2}$.* ◊

§§ Solution. Squaring the first equation, we have

$$x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta = x^2+y^2$$
$$\therefore x^2\left(1-\sin^2\theta\right) + y^2\left(1-\cos^2\theta\right) + 2xy\sin\theta\cos\theta = 0$$
$$\therefore x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta = 0$$
$$\therefore (x\cos\theta + y\sin\theta)^2 = 0$$
$$\therefore x\cos\theta + y\sin\theta = 0,\ i.e.\ \tan\theta = -\frac{x}{y}.$$
$$\therefore \sin^2\theta = \frac{x^2}{x^2+y^2}\ and\ \cos^2\theta = \frac{y^2}{x^2+y^2}.$$

Substituting these values of $\sin^2\theta$ and $\cos^2\theta$ in the second equation, we have

$$\frac{1}{x^2+y^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = \frac{1}{x^2+y^2}$$
$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

■

§ Problem 20.1.6.

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

and $$x\sin\theta - y\cos\theta = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}.$$ ♢

§§ Solution. Squaring the given equations, we have

$$\frac{x^2}{a^2}\cos^2\theta + \frac{2xy}{ab}\sin\theta\cos\theta + \frac{y^2}{b^2}\sin^2\theta = 1 = \sin^2\theta + \cos^2\theta \tag{20.9}$$
$$\text{and } x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta = a^2\sin^2\theta + b^2\cos^2\theta \tag{20.10}$$

From Eq. (20.9) and Eq. (20.10), we have

$$\frac{x^2-a^2}{a^2}\cos^2\theta + \frac{2xy}{ab}\sin\theta\cos\theta + \frac{y^2-b^2}{b^2}\sin^2\theta = 0 \tag{20.11}$$
$$\text{and } \frac{x^2-a^2}{ab}\sin^2\theta - \frac{2xy}{ab}\sin\theta\cos\theta + \frac{y^2-b^2}{ab}\cos^2\theta = 0 \tag{20.12}$$

From Eq. (20.11) and Eq. (20.12), by addition, we have

$$\frac{x^2-a^2}{a}\left(\frac{\cos^2\theta}{a}+\frac{\sin^2\theta}{b}\right) + \frac{y^2-b^2}{b}\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right) = 0$$
$$\therefore \frac{x^2}{a} - a + \frac{y^2}{b} - b = 0;\ i.e.\ \frac{x^2}{a}+\frac{y^2}{b} = a+b.$$

■

§ Problem 20.1.7. $\sin\theta - \cos\theta = p$ *and* $\operatorname{cosec}\,\theta - \sin\theta = q$. ♢

§§ Solution.

$$\sin\theta - \cos\theta = p \tag{20.13}$$
$$\operatorname{cosec}\,\theta - \sin\theta = q \tag{20.14}$$

From Eq. (20.14), we have

$$1 - \sin^2\theta = q\sin\theta \tag{20.15}$$

Adding Eq. (20.13) and Eq. (20.14), we have

$$\operatorname{cosec}\,\theta - \cos\theta = p+q$$
$$\therefore 1 - \sin\theta\cos\theta = (p+q)\sin\theta \tag{20.16}$$

Squaring Eq. (20.13), we have

$$1 - 2\sin\theta\cos\theta = p^2 \tag{20.17}$$

Multiply Eq. (20.16) by 2 and subtract Eq. (20.17) from the result, we have

$$1 = 2(p+q)\sin\theta - p^2$$

$$\therefore \sin\theta = \frac{p^2+1}{2(p+q)} \tag{20.18}$$

Subtracting this value of $\sin\theta$ in Eq. (20.15), we have

$$1 - \frac{\left(p^2+1\right)^2}{4(p+q)^2} = \frac{q\left(p^2+1\right)}{2(p+q)}$$

$$\therefore 4(p+q)^2 - \left(p^2+1\right)^2 = 2q(p+q)\left(p^2+1\right)$$

$$\therefore \left(p^2+1\right)^2 + 2q\left(p^2+1\right)(p+q) = 4(p+q)^2. \qquad \blacksquare$$

§ Problem 20.1.8. *$x = a\cos\theta + b\cos 2\theta$ and $y = a\sin\theta + b\sin 2\theta$.* $\Diamond$

§§ Solution. We have

$$x + b = a\cos\theta + b\,(1+\cos 2\theta)$$
$$= a\cos\theta + 2b\cos^2\theta$$
$$\therefore x + b = \cos\theta\,(a + 2b\cos\theta) \tag{20.19}$$

and
$$y = \sin\theta\,(a + 2b\cos\theta) \tag{20.20}$$

Squaring and adding Eq. (20.19) and Eq. (20.20), we have

$$(x+b)^2 + y^2 = (a+2b\cos\theta)^2 \tag{20.21}$$

Squaring and adding the original equations, we have

$$x^2 + y^2 = a^2 + b^2 + 2ab\,(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)$$
$$= a^2 + b^2 + 2ab\cos\theta$$
$$\therefore x^2 + y^2 - b^2 = a^2 + 2ab\cos\theta$$
$$= a\,(a + 2b\cos\theta)$$
$$\therefore \left(x^2+y^2-b^2\right)^2 = a^2\,(a+2b\cos\theta)^2 \tag{20.22}$$

Substituting from Eq. (20.22) in Eq. (20.21), we have

$$a^2\left[(x+b)^2 + y^2\right] = \left(x^2+y^2-b^2\right)^2. \qquad \blacksquare$$

§ Problem 20.1.9.

If $\quad m = \operatorname{cosec}\theta - \sin\theta$ *and* $n = \sec\theta - \cos\theta,$

prove that

$$m^{\frac{2}{3}} + n^{\frac{2}{3}} = (mn)^{-\frac{2}{3}}. \qquad \Diamond$$

§§ Solution. From the first equation, we have

$$m = \frac{1}{\sin\theta} - \sin\theta = \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta}.$$

From the second equation, we have

$$n = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$$\therefore mn = \cos\theta\sin\theta.$$

$$\therefore m^{\frac{2}{3}} + n^{\frac{2}{3}} = \frac{\cos^{\frac{4}{3}}\theta}{\sin^{\frac{2}{3}}\theta} + \frac{\sin^{\frac{4}{3}}\theta}{\cos^{\frac{2}{3}}\theta} = \frac{\cos^2\theta + \sin^2\theta}{(\sin\theta\cos\theta)^{\frac{2}{3}}} = \frac{1}{(mn)^{\frac{2}{3}}} = (mn)^{-\frac{2}{3}}. \qquad \blacksquare$$

§ Problem 20.1.10. *Prove that the result of eliminating θ from the equations*

$$x\cos(\theta+\alpha) + y\sin(\theta+\alpha) = a\sin 2\theta,$$

and
$$y\cos(\theta+\alpha) - x\sin(\theta+\alpha) = 2a\cos 2\theta$$

is
$$(x\cos\alpha+y\sin\alpha)^{\frac{2}{3}}+(x\sin\alpha-y\cos\alpha)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}. \qquad \lozenge$$

§§ Solution. The given equations may be written
$$(x\cos\alpha+y\sin\alpha)\cos\theta-(x\sin\alpha-y\cos\alpha)\sin\theta=2a\sin\theta\cos\theta$$
and $(x\sin\alpha-y\cos\alpha)\cos\theta+(x\cos\alpha+y\sin\alpha)\sin\theta=-2a\cos 2\theta$

i.e.
$$m\cos\theta-n\sin\theta=2a\sin\theta\cos\theta \tag{20.23}$$
and
$$m\sin\theta+n\cos\theta=-2a\cos 2\theta \tag{20.24}$$
where $m=x\cos\alpha+y\sin\alpha$ *and* $n=x\sin\alpha-y\cos\alpha$.

Multiplying Eq. (20.23) by $\cos\theta$, Eq. (20.24) by $\sin\theta$ and adding, we have
$$\begin{aligned} m&=2a\left(\sin\theta\cos^2\theta-\cos 2\theta\sin\theta\right)\\ &=2a\sin\theta\left(\cos^2\theta-2\cos^2\theta+1\right) && (20.25)\\ &=2a\sin\theta\left(1-\cos^2\theta\right)=2a\sin^3\theta && (20.26)\end{aligned}$$
Again, multiplying Eq. (20.23) by $\sin\theta$, Eq. (20.24) by $\cos\theta$ and subtracting, we have
$$\begin{aligned} n&=-2a\left(\cos 2\theta\cos\theta+\sin^2\theta\cos\theta\right)\\ &=-2a\cos\theta\left(1-2\sin^2\theta+\sin^2\theta\right)\\ &=-2a\cos\theta\left(1-\sin^2\theta\right)=-2a\cos^3\theta && (20.27)\end{aligned}$$
From Eq. (20.26) and Eq. (20.27), we have
$$\left(\frac{m}{2a}\right)^{\frac{1}{3}}=\sin\theta \tag{20.28}$$
$$\left(\frac{n}{2a}\right)^{\frac{1}{3}}=-\cos\theta \tag{20.29}$$
Squaring and adding Eq. (20.28) and Eq. (20.29), we have
$$\left(\frac{m}{2a}\right)^{\frac{2}{3}}+\left(\frac{n}{2a}\right)^{\frac{2}{3}}=\sin^2\theta+\cos^2\theta=1$$
$$\therefore (x\cos\alpha+y\sin\alpha)^{\frac{2}{3}}+(x\cos\alpha-y\sin\alpha)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}. \qquad \blacksquare$$

Eliminate θ and ϕ from the equations:

§ Problem 20.1.11. $\sin\theta+\sin\phi=a,\ \cos\theta+\cos\phi=b$ *and* $\theta-\phi=\alpha$. $\lozenge$

§§ Solution. Squaring and adding the first two equations, we have
$$2+2(\sin\theta\sin\phi+\cos\theta\cos\phi)=a^2+b^2$$
$$\therefore a^2+b^2=2+2\cos(\theta-\phi)=2+2\cos\alpha. \qquad \blacksquare$$

§ Problem 20.1.12. $\tan\theta+\tan\phi=x,\ \cot\theta+\cot\phi=y$ *and* $\theta+\phi=\alpha$. $\lozenge$

§§ Solution. We have
$$\frac{x}{y}=\frac{\tan\theta+\tan\phi}{\cot\theta+\cot\phi}=\frac{(\tan\theta+\tan\phi)\tan\theta\tan\phi}{\tan\theta+\tan\phi}=\tan\theta\tan\phi.$$
Also,
$$\tan\alpha=\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{x}{1-\frac{x}{y}}=\frac{xy}{y-x}$$
$$\therefore xy=(y-x)\tan\alpha. \qquad \blacksquare$$

§ Problem 20.1.13.

$$a\cos^2\theta + b\sin^2\theta = c,\ b\cos^2\phi + a\sin^2\phi = d$$

and $$a\tan\theta = b\tan\phi.$$ ♢

§§ Solution.

$$a\cos^2\theta + b\sin^2\theta = c \tag{20.30}$$
$$b\cos^2\phi + a\sin^2\phi = d \tag{20.31}$$
$$a\tan\theta = b\tan\phi \tag{20.32}$$

From Eq. (20.30), we have

$$a\cos^2\theta + b\sin^2\theta = c\left(\cos^2\theta + \sin^2\theta\right)$$
$$\therefore (a-c)\cos^2\theta = (c-b)\sin^2\theta$$
$$\therefore \tan^2\theta = \frac{a-c}{c-b} \tag{20.33}$$

From Eq. (20.31), we have

$$b\cos^2\phi + a\sin^2\phi = d\left(\cos^2\phi + \sin^2\phi\right)$$
$$\therefore (b-d)\cos^2\phi = (d-a)\sin^2\phi$$
$$\therefore \tan^2\phi = \frac{b-d}{d-a} \tag{20.34}$$

From Eq. (20.32), we have

$$a^2\tan^2\theta = b^2\tan^2\phi \tag{20.35}$$

Substituting from Eq. (20.33) and Eq. (20.34) in Eq. (20.35), we have

$$a^2\left(\frac{a-c}{c-b}\right) = b^2\left(\frac{b-d}{d-a}\right)$$
$$\therefore a^2(a-c)(a-d) = b^2(b-c)(b-d).$$ ■

§ Problem 20.1.14.

$$\cos\theta + \cos\phi = a,\ \cot\theta + \cot\phi = b$$

and $$\operatorname{cosec}\,\theta + \operatorname{cosec}\,\phi = c.$$ ♢

§§ Solution.

$$\cos\theta + \cos\phi = a \tag{20.36}$$
$$\cot\theta + \cot\phi = b \tag{20.37}$$
$$\operatorname{cosec}\,\theta + \operatorname{cosec}\,\phi = c \tag{20.38}$$

From Eq. (20.36), we have

$$2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = a \tag{20.39}$$

From Eq. (20.37), we have

$$\frac{\cos\theta\sin\phi + \cos\phi\sin\theta}{\sin\theta\sin\phi} = b$$
$$\therefore \sin(\theta+\phi) = b\sin\theta\sin\phi \tag{20.40}$$

From Eq. (20.38), we have

$$\sin\theta + \sin\phi = c\sin\theta\sin\phi \tag{20.41}$$

From Eq. (20.40) and Eq. (20.41), by division, we have

$$\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi} = \frac{b}{c}$$
$$\therefore \frac{2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2}}{2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}} = \frac{b}{c} \tag{20.42}$$

From Eq. (20.39) and Eq. (20.42), we have

$$2\cos^2\frac{\theta+\phi}{2}=\frac{ab}{c}\ \textit{and}\ 2\cos^2\frac{\theta-\phi}{2}=\frac{ac}{b}$$

$$\therefore 1+\cos(\theta+\phi)=\frac{ab}{c} \tag{20.43}$$

and

$$1+\cos(\theta-\phi)=\frac{ac}{b} \tag{20.44}$$

From Eq. (20.43) and Eq. (20.44), by subtraction, we have

$$2\sin\theta\sin\phi=a\left(\frac{c}{b}-\frac{b}{c}\right)=a\left(\frac{c^2-b^2}{bc}\right).$$

Hence from Eq. (20.40), we have

$$\sin(\theta+\phi)=\frac{a}{2c}\left(c^2-b^2\right).$$

But

$$\cos(\theta+\phi)=\frac{ab}{c}-1.$$

$$\therefore 1=\frac{a^2\left(c^2-b^2\right)^2}{4c^2}+\left(\frac{ab}{c}-1\right)^2$$

which reduces to the form

$$8bc=a\left[4b^2+\left(b^2-c^2\right)^2\right].$$

■

§ Problem 20.1.15.

$$a\sin\theta=b\sin\phi,\ a\cos\theta+b\cos\phi=c$$

and

$$x=y\tan(\theta+\phi).$$

◊

§§ Solution.

$$a\sin\theta=b\sin\phi \tag{20.45}$$

$$a\cos\theta+b\cos\phi=c \tag{20.46}$$

$$x=y\tan(\theta+\phi) \tag{20.47}$$

From Eq. (20.45), we have

$$a\sin\theta-b\sin\phi=0 \tag{20.48}$$

Squaring and adding Eq. (20.46) and Eq. (20.48), we have

$$a^2+2ab\cos(\theta+\phi)+b^2=c^2$$

$$\therefore \cos(\theta+\phi)=\frac{c^2-a^2-b^2}{2ab} \tag{20.49}$$

Now

$$\tan^2(\theta+\phi)=\sec^2(\theta+\phi)-1.$$

Hence, from Eq. (20.47) and Eq. (20.49), we have

$$\begin{aligned}\frac{x^2}{y^2}&=\left(\frac{2ab}{c^2-a^2-b^2}\right)^2-1=\frac{(2ab)^2-\left(c^2-a^2-b^2\right)^2}{(c^2-a^2-b^2)^2}\\&=\frac{\left(2ab+c^2-a^2-b^2\right)\left(2ab-c^2+a^2+b^2\right)}{(c^2-a^2-b^2)^2}\\&=\frac{\left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right]}{(c^2-a^2-b^2)^2}\\&=\frac{(c+a-b)(c-a+b)(a+b+c)(a+b-c)}{(c^2-a^2-b^2)^2}\end{aligned}$$

$$\therefore x\left(c^2-a^2-b^2\right)=y\sqrt{(a+b+c)(a-b+c)(a+b-c)(-a+b+c)}.$$

Otherwise thus :

Draw a triangle ABC whose sides are a, b and c. Then clearly $\theta = \angle B$ and $\phi = \angle C$.

$$\therefore \frac{x}{y} = \tan(\theta + \phi) = \tan(B + C) = -\tan A.$$

On substituting the values of *Arts.* 164 and 169, we have the same answer as before. ■

§ Problem 20.1.16.

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \ \frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$$

and $$a^2 \sin\frac{\theta}{2}\sin\frac{\phi}{2} + b^2\cos\frac{\theta}{2}\cos\frac{\phi}{2} = c^2. \qquad \Diamond$$

§§ Solution.

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \tag{20.50}$$

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \tag{20.51}$$

$$a^2 \sin\frac{\theta}{2}\sin\frac{\phi}{2} + b^2\cos\frac{\theta}{2}\cos\frac{\phi}{2} = c^2 \tag{20.52}$$

From Eq. (20.50) and Eq. (20.51), by addition, we have

$$\frac{x}{a}\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + \frac{y}{b}\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 1$$

$$\therefore \frac{x}{a}\cos\frac{\theta+\phi}{2} + \frac{y}{b}\sin\frac{\theta+\phi}{2} = \sec\frac{\theta-\phi}{2} \tag{20.53}$$

From Eq. (20.50) and Eq. (20.51), by subtraction, we have

$$\frac{x}{a}\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2} + \frac{y}{b}\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2} = 0$$

$$\therefore \frac{x}{a}\sin\frac{\theta+\phi}{2} = \frac{y}{b}\cos\frac{\theta+\phi}{2} \tag{20.54}$$

From Eq. (20.53) and Eq. (20.54), we have

$$\frac{\left(\frac{x}{a}\right)}{\cos\frac{\theta+\phi}{2}} = \frac{\left(\frac{y}{b}\right)}{\sin\frac{\theta+\phi}{2}} = \frac{\frac{x}{a}\cos\frac{\theta+\phi}{2} + \frac{y}{b}\sin\frac{\theta+\phi}{2}}{\cos^2\frac{\theta+\phi}{2} + \sin^2\frac{\theta+\phi}{2}} = \frac{\sec\frac{\theta-\phi}{2}}{1} \tag{20.55}$$

Again, from Eq. (20.52), we have

$$a^2\left(\cos\frac{\theta-\phi}{2} - \cos\frac{\theta+\phi}{2}\right) + b^2\left(\cos\frac{\theta-\phi}{2} + \cos\frac{\theta+\phi}{2}\right) = 2c^2$$

$$\therefore \left(a^2+b^2\right)\cos\frac{\theta-\phi}{2} + \left(b^2-a^2\right)\cos\frac{\theta+\phi}{2} = 2c^2 \tag{20.56}$$

Also, each fraction in Eq. (20.55)

$$= \sqrt{\frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{\cos^2\frac{\theta+\phi}{2} + \sin^2\frac{\theta+\phi}{2}}} = \frac{\sqrt{b^2x^2 + a^2y^2}}{ab}$$

$$\therefore \sec\frac{\theta-\phi}{2} = \frac{\sqrt{b^2x^2+a^2y^2}}{ab}$$

$$\therefore \cos\frac{\theta-\phi}{2} = \frac{ab}{\sqrt{b^2x^2+a^2y^2}}$$

and $$\cos\frac{\theta+\phi}{2} = \frac{\left(\frac{x}{a}\right)}{\sec\frac{\theta-\phi}{2}} = \frac{bx}{\sqrt{b^2x^2+a^2y^2}}.$$

Hence, substituting these values of $\cos\dfrac{\theta-\phi}{2}$ and $\cos\dfrac{\theta+\phi}{2}$ in Eq. (20.56), we have

$$\frac{ab\left(a^2+b^2\right)+\left(b^2-a^2\right)bx}{\sqrt{b^2x^2+a^2y^2}}=2c^2$$

$$\therefore b^2\left[x\left(b^2-a^2\right)+a\left(a^2+b^2\right)\right]^2=4c^4\left(b^2x^2+a^2y^2\right). \qquad \blacksquare$$

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