The Versatile Continuous Order.

Conference Paper · April 1987

DOI: 10.1007/3-540-19020-1_7 · Source: DBLP

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THE VERSATILE CONTINUOUS ORDER

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Abstract

In this paper we survey some of the basic properties of continuously ordered sets, especially those properties that have led to their employment as the underlying structures for constructions in denotational semantics. The earlier sections concentrate on the order-theoretic aspects of continuously ordered sets and then specifically of domains. The last two sections are concerned with two natural topologies for sets with continuous orders, the Scott and Lawson topologies.

The purpose of this article is to survey some of the principal aspects of the theory of domains and continuous orders. A significant portion of the theory has arisen in an attempt to find suitable mathematical structures to serve as a framework for modeling concepts and constructions from theoretical computer science. In the course of our presentation, we try to include (in a rather sketchy way) some of this background.

There are certain distinctive and novel mathematical ideas that have grown out of the theory of continuous orders that appear to be worthwhile mathematical contributions independent of whatever future role the theory may play in the field of theoretical computer science. A basic aim of this paper is to point out explicitly some of these unique features of the theory. The knowledgable reader will not find much new herein (although here and there we include some improvements on earlier results); rather we seek to present some of the basic ideas for the non-expert in an illustrative and comprehensible manner. A significant departure from the treatment given in [**COMP**] is our emphasis on continuous partially ordered sets instead of continuous lattices. This is consistent with the way that the theory has developed since the publication of that book.

I. Complete Partial Orders

Plato envisioned an "ideal" world where everything existed in perfection; in contrast, the objects of our universe were viewed as approximations to the ideal. Similarly in mathematics we postulate ideal abstractions (e.g. the points, lines, etc., of geometry) of physical objects or phenomena. An idea of ordering comes into play when we consider whether one approximation to an ideal is better or worse than (or incomparable to) another. It is often the case that the approximations determine the ideal in the sense that the ideal is the limiting case of increasingly finer approximations (we could consider, for example, algorithms that compute the number π to increasingly better accuracy). It would, of course, be pretentious to claim that a certain mathematical model captured these philosophical and metamathematical ideas better than any other or that one specific approach was the most useful for treating them. These ideas are helpful, however, for gaining intuitive insight into many of the mathematical structures and constructions that appear in the following.

We first introduce the mathematical concepts that model the idea of increasingly better approximations or stages of a computation.

Let $(U, \leq)$ be a partially ordered set. An ω-**chain** is an increasing sequence, a **chain** is a totally ordered subset, and a **directed** set is one for which any two elements have an upper bound.

An ω-chain: $x_1 \leq x_2 \leq x_3 \leq \ldots$

A chain C: $x, y \in C \Rightarrow x \leq y$ or $y \leq x$

A directed set D: $x, y \in D \Rightarrow \exists z \in D$ such that $x, y \leq z$.

Completeness in the current context means the existence of "ideal" objects or the limits of computations. From this viewpoint, a distinctive feature of the following systems is that they incorporate *both* the ideal objects and the approximations. It is a standard (though non-trivial) result that completeness with respect to chains is equivalent to completeness with respect to directed sets (see e.g., [**Ma76**]). Of course, ω-completeness is a slightly weaker concept, but is more suitable for recursive considerations.

A **chain-complete partially ordered set** (CPO) is a partially ordered set U such that every directed subset (equivalently, every chain) of U has a least upper bound in U. Frequently one postulates a least element $\perp$ also.

We can view the suprema of directed sets as "ideal" elements in our universe U, and the elements of the directed sets as increasingly better approximations or as stages in a "computation".

A **continuous** morphism (or CPO-morphism) is an order preserving function which also preserves suprema of directed sets (equivalently, suprema of chains). These may be viewed as the "computation-preserving" functions.

Ideal Completions

Let $(P, \leq)$ be a partially ordered set. A subset I of P is an **ideal** if

(i) $I = \downarrow I$, where $\downarrow I = \{y \in I \colon \exists x \in I$ such that $y \leq x\}$,

(ii) I is directed.

Let $Id(P)$ denote the set of ideals of P ordered by inclusion. Then $Id(P)$ is a CPO and the mapping $i: P \to Id(P)$ defined by $x \mapsto \downarrow x$ is an order embedding. The pair $(Id(P), i)$ is called the **ideal completion** of P.

The ideal completion can be characterized alternately as arising from the adjoint functor to the forgetful functor from the category of CPO's and continuous morphisms to the category of partially ordered sets and order preserving functions. Or it may be characterized in terms of the universal property that any order preserving mapping α from P to a CPO Q extends uniquely to a CPO mapping $\hat{\alpha}$ from $Id(P)$ to Q, which sends an ideal I to $\sup \alpha(I)$ (see the paper of Markowsky and Rosen [**MR76**] for this and other results on the ideal completion).

$$\begin{tikzcd} Id(P) \arrow[dr, "\exists ! \hat{\alpha}"] & \\ P \arrow[u, "i"] \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

Example. *The Cantor Tree.*

Consider the set P of all finite strings of $\{0, 1\}$ (including the empty string). We take the prefix order and form the ideal completion $Id(P)$. $Id(P)$ can be identified with all finite and infinite strings of $\{0, 1\}$. The longer finite strings give increasingly better approximations to the infinite strings. Alternately $Id(P)$ can be identified with the finite and infinite points in the free dyadic tree (with P being embedded as the finite points).

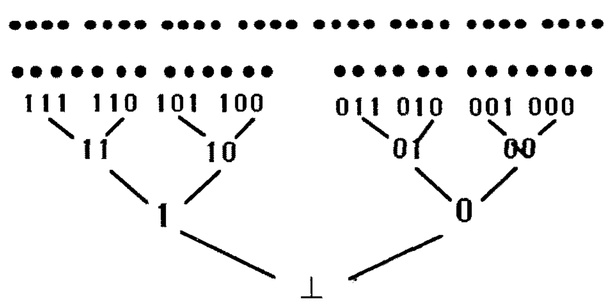

Figure 1. The Cantor Tree

Such constructions as those arising in the previous example also have other interesting interpretations. Suppose that $(X_n, f_n^m : X_n \to X_m)$ is an inverse system of finite sets indexed by the integers. For $m < n$, we define $x < y$ for $x \in X_m$

and $y \in X_n$ if $f_n^m(y) = x$. (In the previous example the sets X_m are the strings of length m and f_n^m sends a string of length n to the prefix of length m.) The ideal points then arise at the top and may be viewed as the (inverse) limit of the system.

Special classes of partially ordered sets often have important alternate characterizations in terms of their ideal completions. If P itself is a CPO and we consider the identity mapping on P, then this extends to a continuous morphism SUP: $Id(P) \to P$ which sends an ideal to its supremum. Observe that the composition $\mathrm{SUP} \circ i = 1_P$ and $i \circ \mathrm{SUP} \geq 1_{Id(P)}$. The existence of an order preserving function from $Id(P)$ to P with these composition properties characterizes CPO's (we leave the straightforward verification of this assertion as an exercise for the reader). Note, however, in this case that the inclusion mapping $i: P \to Id(P)$ is not continuous.

An important class of CPO's are what have been called domains. There is some variation in the way they are defined. We define a **domain** to be a partially ordered set which is order-isomorphic to the ideal completion of a partially ordered set. Frequently one restricts oneself to **countably based domains**, which are order-isomorphic to the ideal completion of a countable partially ordered set. The countability restriction lends itself to recursive considerations.

Fixed Points

The Banach Contraction Theorem states that a contraction in a complete metric space has a unique fixed point and is used (among other things) to guarantee the existence of solutions of differential equations. A somewhat analogous role is played by the Tarski Fixed Point Theorem for CPO's and shows why chain-completeness is an appropriate form of completeness in the current context.

Proposition. *Let P be a CPO with $\perp$, and let $f: P \to P$ be a continuous morphism. Then f has a least fixed point.*

Proof. $\perp \leq f(\perp) \leq f^2(\perp) \ldots \leq f^n(\perp) \ldots$ The sup of this sequence is the least fixed point. ∎

This proposition is the basis for inductive or recursive constructions and solutions of equations in CPO's. The construct to be defined or solution to some equation is given by the least fixed point of an appropriate continuous function.

Example. Suppose we wish to solve the equation $W = 01W$ on the set of finite and infinite words with alphabet $\{0, 1\}$ (which may be identified with the Cantor tree, the ideal completion of the free dyadic tree). To find a solution, we consider the continuous transformation $T(W) = 01W$. The solution is the least fixed point of the transformation T, and consists of the infinite word consisting of repeated 01's.

More generally one can order a (universal) set of countably based domains by inclusion to form a domain of domains and use the Tarski theorem for finding solutions to domain equations (see [**WL84**] for a nice presentation of this approach).

II. A New Kind of Order

A significant contribution of the theory of continuous orders has been the explicit definition and use of a new order relation, one that sharpens the traditional notion of order.

Let P be a CPO, x,y in P. We say x is **essentially below** y (traditionally, "way-below"), written $x \ll y$, if given a directed set $D \subseteq P$ such that $y \leq \sup D$, then $x \leq d$ for some $d \in D$.

If we think of D as a computation of y, then D yields x at some "finite" stage. In the Cantor tree, $x \ll y$ if and only if x is a finite prefix of y. (Hence the alternate terminology x is "finite" in y.) In this context the idea of essentially below is that any method of "approximating" or "computing" y must give x at some stage along the way.

A partially ordered set P is a **continuous CPO** if it is a CPO (complete with respect to directed sups) and satifies

$$y \in P \Rightarrow y = \sup\{x : x \ll y\} = \sup \Downarrow y,$$

and the set on the right is directed.

Every element of P can be "approximated" by or "computed" from the elements essentially below it.

We remark that if $\Downarrow y$ contains a directed set with supremum y, then from the definition of $\ll$ it follows that this set must be cofinal in $\Downarrow y$, and hence that $\Downarrow y$ is itself directed.

The following example of D. Scott [**Sc76**] nicely illustrates these ideas.

Example. *A Data Type Structure.*

Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$, and let D^* consist of all closed intervals $[\underline{x}, \overline{x}]$ where $\underline{x} \leq \overline{x}$ for $\underline{x}, \overline{x} \in \mathbb{R}^*$. We partial order D^* by reverse inclusion. Then D^* is a complete semilattice containing the least element $\perp = \mathbb{R}^*$, the "perfect" reals–the one-point intervals–$x = [x, x]$, and the "approximate" reals $[\underline{x}, \overline{x}]$ with $\underline{x} < \overline{x}$.

The interval $[\underline{y}, \overline{y}]$ is a "better approximation" to $r \in R^*$ or "contains more information" than $[\underline{x}, \overline{x}]$ if $\underline{y} \leq r \leq \overline{y}$ and $[\underline{x}, \overline{x}] \leq [\underline{y}, \overline{y}]$, i.e., $[\underline{y}, \overline{y}] \subseteq [\underline{x}, \overline{x}]$.

If $[\underline{x}, \overline{x}] \in D^*$ and $\underline{x} < r < \overline{x}$, then any algorithm for computing r will eventually give an upper bound $\leq \overline{x}$ and a lower bound $\geq \underline{x}$. Hence the interval of error will be greater than $[\underline{x}, \overline{x}]$ in the order on D^*. Thus $[\underline{x}, \overline{x}] \ll [r, r] = r$.

Other Completeness Properties

The requirement that the set $\Downarrow y = \{x : x \ll y\}$ be directed is sometimes difficult to check and is often not preserved by standard constructions on CPO's. Thus it is frequently convenient or necessary to work with partially ordered sets with stronger completeness properties so that the directedness happens automatically.

It is immediate from the definition that if $a \ll y$, $b \ll y$, and $c = \sup\{a, b\}$, then $c \ll y$. Thus if any two elements below y have a supremum, it follows that the set of elements essentially below y *must* be directed. We consider some important classes where this holds.

A CPO is said to be **bounded complete** if any two elements that are bounded above have a least upper bound. Note that since a CPO is already chain-complete, it follows that any non-empty subset which has an upper bound must have a least upper bound. It follows that any interval $[a, b] = \{x : a \leq x \leq b\}$ is a complete lattice in the restricted order, and indeed this property is an alternate characterization of bounded complete CPO's. Thus the class of **bounded complete continuous CPO's** is a convenient class of continuous CPO's.

A semilattice is said to be **complete** if it is chain complete (i.e., a CPO) and every non-empty subset has an infimum. Another important subclass is the class of **continuous complete semilattices**, continuous CPO's which are also complete semilattices. It is an elementary exercise to show that complete semilattices are bounded complete and conversely that a bounded complete CPO with bottom $\perp$ is a complete semilattice. (Similarly a CPO P is bounded complete iff $P_\perp$, the set P with a bottom element adjoined, is a complete semilattice.) Hence the complete semilattices are precisely the bounded complete CPO's with $\perp$. These are sometimes alternately referred to as being **consistently complete**. It is sometimes required as part of the definition of a domain that it be consistently complete.

Another important class is the class of **continuous lattices**, complete lattices with a continuous order. These are the bounded complete continuous CPO's that have both a top and a bottom. A CPO is a bounded complete continuous CPO iff when a top and bottom are adjoined to it, a continuous lattice results.

Rounded Ideals

We consider some of the basic properties of the relation $\ll$ in a continuous CPO.

(1) $a \ll b \Rightarrow a \leq b$

(2) $a \ll d,\ b \ll d \Rightarrow \exists c$ such that $a, b \leq c$ and $c \ll d$

(3) $a \leq b \ll c \leq d \Rightarrow a \ll d$

(4) $a \ll c \Rightarrow \exists b$ such that $a \ll b \ll c$

(5) $\perp \ll a$

The fourth property plays a crucial role in the theory and is referred to as the "interpolation" property.

An arbitrary relation on a poset P that satisfies the preceding axioms is called an **auxiliary** relation. If P is a poset equipped with an auxiliary relation, then we can consider the set of "rounded" ideals, i.e., those ideals I satisfying $x \in I \Rightarrow \exists y \in I$ such that $x \ll y$. These ideals ordered by inclusion form a completion, called the **rounded ideal completion**. This completion is a continuous CPO and P maps into the completion by sending x to $\{y : y \ll x\}$. This completion generalizes the ideal completion. (Indeed the ideal completion arises if $\ll$ is chosen as $\leq$.)

Example. The data structure example is the rounded ideal completion of the approximate rational data structure consisting of those intervals $[p, q]$ with p, q rational and $p < q$ with the inherited order and essentially below relation.

If one wishes to consider the notion of a recursively defined continuous CPO, then one can consider those continuous CPO's which arise as the rounded ideal completion of a countable set with a recursively defined partial order and auxiliary relation. More directly, we say that a continuous CPO P is **countably based** if there exists a countable subset B of P such that $p \ll q$ in P implies there exists $b \in B$ with $p \ll b \ll q$.

Locally Compact Spaces

We consider another illustrative example of naturally occuring continuous orders. The results of the next two sections are mainly drawn from [**HL78**] or [**COMP**, Chapter V].

Let X be a topological space, let $O(X)$ denote the lattice of open sets, and let $U, V \in O(X)$. Then $U \ll V$ iff for every open cover of V, there is a finite subcollection that covers U. In this context it seems appropriate to say that U is **compact** in V.

We say that X is **core compact** if given $x \in V \in O(X)$, there exists U open, $x \in U \subseteq V$, such that U is compact in V.

Theorem. *X is core compact $\Leftrightarrow$ $O(X)$ is a continuous lattice.*

For Hausdorff spaces, these are precisely the locally compact spaces. They appear to be the appropriate generalization of local compactness to the non-Hausdorff setting (in the sense that many basic mapping properties of locally compact spaces are retained in this setting). For example, X is core compact iff $1_X \times f: X \times Y \to X \times Z$ is a quotient mapping whenever $f: Y \to Z$ is a quotient mapping [**DK70**]. Also appropriate modifications of the compact-open topology for function spaces exist so that one gets an equivalence between $[X \times Y, Z]$ and $[X, [Y, Z]]$ if Y is core compact (see [**COMP**, Chapter II] and for later developments, [**SW85**] or [**LP85**]). Of course this equivalence is closely related to cartesian closedness, a topic to which we return at a later point.

Spectral Theory

Spectral theory seeks to represent a lattice as the lattice of open sets of a topological space. However, the constructions are more intuitive if one works with the lattice of closed sets. We take this approach initially, and set everything on its head at a later stage.

Suppose that X is a T_1-space, and let L be the lattice of closed sets. We let $\hat{X}$ denote the set of atoms in L (which correspond to the singleton subsets of X) and topologize $\hat{X}$ by defining a closed set to be all the atoms below a fixed member of the lattice L, i.e., $\{\{x\}: \{x\} \subseteq A\}$ where A is a closed subset of X. Then the mapping from X to $\hat{X}$ which sends an element to the corresponding singleton set is a homeomorphism. Thus X may be recovered (up to homeomorphism) from the lattice of closed sets.

The situation becomes more complex (and more interesting) for a T_0-space X. In this case we let an element of X correspond to the closure of the corresponding singleton set in the lattice L of closed sets. The fact that X is T_0 is precisely the condition needed for this correspondence to be one-to-one. But how does one distinguish in a lattice-theoretic way the closed sets that arise in this fashion? One easily verifies that sets that are closures of points are **irreducible**, i.e., not the union of two strictly smaller closed sets. We are thus led to define the cospectrum, Cospec(L), to be the set of coprime elements (p is **coprime** if $p \leq \sup\{x, y\}$ implies $p \leq x$ or $p \leq y$) equipped with the **hull-kernel** topology with closed sets of the form $hk(a) = \{p \in L\colon p \text{ is coprime}, p \leq a\}$.

A space is **sober** if every irreducible closed set is the closure of a unique point. In precisely this case the embedding of X into the cospectrum of the closed sets is a homeomorphism. For any topological space X, there is a largest T_0-space $\hat{X}$ having the same lattice of closed (or open) sets, called the **sobrification** of X. The sobrification of X can be obtained by taking $\hat{X}$ to be the cospectrum of the closed sets; X maps to the sobrification by sending a point to its closure. It can be shown that a space is core compact iff its sobrification is locally compact. (A space is **compact** if every open cover has a finite subcover, and **locally compact** if every (not necessarily open) neighborhood of a point contains a compact neighborhood of that point.)

We now dualize the preceding notions to the lattice of open sets. An element $p \in L$, $p \neq 1$ is **prime** (resp. **irreducible**) if $x \wedge y \leq p \Rightarrow x \leq p$ or $y \leq p$ (resp. $x \wedge y = p \Rightarrow x = p$ or $y = p$). It can be shown that the irreducible elements of a continuous lattice order generate (i.e., every element is an infimum of such elements) and that the prime elements of a distributive continuous lattice order generate.

The collection of sets of the form PRIME $L \cap \uparrow x$ for $x \in L$ forms the closed sets for a topology on PRIME L, called the **hull-kernel** topology. PRIME L equipped with the hull-kernel topology is called the **spectrum** of L, and denoted Spec L. The following theorem results by showing that the spectrum is sober (which is always the case) and locally compact when L is continuous.

Theorem. *Given any continuous distributive lattice L, there exists a locally compact sober space X (namely the spectrum) such that L is order-isomorphic to $O(X)$.*

As a consequence of the preceding considerations there results a duality between distributive continuous lattices and locally compact sober spaces.

III. Domains

In this section we turn our attention to domains, which have been the most important structures in the theory of continuous CPO's from the viewpoint of computer science.

In earlier sections we have seen how to obtain a distributive continuous lattice as the lattice of open sets of a locally compact sober space and conversely how to obtain a locally compact sober space as the spectrum of such a lattice. Such inverse constructions (and the dualities to which they often lead) are a pervasive feature of the theory of continuous orders. We turn now to an inverse construction for the ideal completion.

An element $k \in P$ is **compact** if $k \ll k$, i.e., if $\sup D \geq k$ for D directed, then $k \leq d$ for some $d \in D$. A CPO P is **algebraic** if every element is a directed sup of compact elements.

Note that algebraic CPO's are a special subclass of the class of continuous CPO's. In an algebraic CPO the relation $\ll$ is characterized by $x \ll y$ iff there exists a compact element k such that $x \le k \le y$.

For any partially ordered set P, the ideal completion $Id(P)$ is an algebraic poset, and P embeds in $Id(P)$ as the compact elements. Conversely, if Q is an algebraic poset, one may consider the subset of compact elements $K(Q)$ with the inherited order. The operators Id and K are inverse operators between the class of all posets and the class of algebraic CPO's (in the sense that if one is applied and then the other, one obtains a poset order isomorphic to the original.) It follows that the class of algebraic CPO's is precisely the class of domains. Indeed it is customary to define a domain to be an algebraic CPO instead of the way we have done it.

When one is working in the context of algebraic CPO's, properties of continuous CPO's can generally be given alternate characterizations in terms of the partially ordered set of compact elements. For example, an algebraic CPO is countably based iff the set of compact elements is countable.

This "object" correspondence can be extended to a functor for various categories. If continuous morphisms $f: P \to Q$ are considered in the class of domains, then for the corresponding morphisms in the poset category one considers the restrictions of their subgraphs (i.e., the relation $\{(x,y) \in K(P) \times K(Q): y \le f(x)\}$).

If a domain (i.e., algebraic CPO) is a complete semilattice, respectively, complete lattice, then it is called an **algebraic semilattice**, respectively, **algebraic lattice**. Algebraic semilattices are the consistently complete domains. If P is an algebraic semilattice, then (since the sup of two compact elements is again compact) $K(P)$ is also consistently complete. Hence $K(P)$ can be characterized as a partially ordered set with $\perp$ such that $\downarrow x$ is a sup semilattice for each x. Such sets have the property that their ideal completions are algebraic semilattices. Further restricting these inverse constructions, one obtains that the set of compact elements of an algebraic lattice forms a sup semilattice with bottom, and the ideal completion of a sup semilattice with bottom is an algebraic lattice.

Information Systems and Convexity

One is interested in trying to present some version of domain theory in as intuitive and usable form as possible. One recent approach has been in terms of informations systems.

An **information system** is a triple $\mathbf{A} = (A, Con, \Rightarrow)$ for which

1) A is a non-empty set (whose members are called "tokens" and thought of as statements or items of information);
2) Con is a family of *finite* subsets of A (the finite "consistent" subsets) satisfying
   i) $Y \subseteq X \in Con$ implies $Y \in Con$,
   ii) $\{a\} \in Con$ for all $a \in A$;
3) $\Rightarrow$ is a relation (a subset of $Con \times A$) satisfying
   iii) If $X \in Con$ and $a \in A$, then $X \cup \{a\} \in Con$ if $X \Rightarrow a$,

iv) If $X, Y \in Con$ and $c \in A$, and if $X \Rightarrow b$ for all $b \in Y$, and if $Y \Rightarrow c$, then $X \Rightarrow c$.

An arbitrary subset $Y \subseteq A$ is **consistent** if $X \in Con$ for every finite subset $X \subseteq Y$. The set Y is **deductively closed** if $X \subseteq Y$, X finite, and $X \Rightarrow a$ together imply $a \in Y$. The **elements** of $\mathbf{A}$, denoted $|\mathbf{A}|$, are defined by

$$|\mathbf{A}| = \{Y \subseteq A : Y \text{ is consistent and deductively closed}\}.$$

The elements form a consistently complete domain, i.e., an algebraic semilattice. The compact elements are the smallest deductively closed subsets containing a finite set, which is just the set of tokens implied by the finite set. In general, there is no way of recovering the information system from the domain of elements (indeed distinct information systems may lead to isomorphic domains of elements). However, there is associated with each consistently complete domain (=algebraic semilattice) in a canonical way an information system which gives rise to the domain; indeed the construction is an alternate formulation of the inverse constructions between domains and partially ordered sets given in the preceding section. Given an algebraic semilattice D, one defines the tokens to be the compact elements, the sets in Con to be those finite subsets that have an upper bound in D, and the relation $\Rightarrow$ by $F \Rightarrow a$ if $a \leq \sup F$. One verifies that this is an information system whose set of elements is order isomorphic to the original domain.

One may alternately view information systems as abstract convexities. If A is a (not necessarily convex) subset of a convex set X, then one defines Con to be all finite subsets of A whose convex hull is a subset of A; one defines $F \Rightarrow a$ to mean that a is in the convex hull of F. The structure so obtained is an information system; the set of elements consists of all convex sets contained in A. An **abstract convexity** on a set X consists of a collection of subsets that are closed under arbitrary intersections and directed unions. These are all the properties one needs to show that any subset A gives rise to an information system as just defined, and it is not hard to see that the notions of information system and abstract convexity are equivalent notions via this correspondence. Abstract convexities have been investigated in the dissertation of Robert Jamison [**J**]; numerous papers of Marcel van de Vel deal with various aspects of abstract convexities in topological spaces, see e.g. [**VDV85**].

Retracts and Projections

Retracts play an important role in the theory of continuous CPO's. We consider some of their most basic properties.

Let P be a CPO. An **(internal) retraction** is a continuous morphism $r: P \to P$ such that $r \circ r = r$. It was Scott's observation that a continuous retract of a continuous lattice is again a continuous lattice [**Sc72**], and the proof carries over to continuous CPO's.

Proposition. *Let P be a continuous CPO and let $r: P \to P$ be a retraction. Then $r(P)$ is a continuous CPO, and the inclusion $j: r(P) \to P$ is continuous.*

Proof. Let $A = r(P)$. If D is a directed set in A, let $w = \sup D$ in P. Then $r(D) = D$, so by continuity $r(w) = \sup r(D) = \sup D$. Hence $w = r(w)$. Thus A is closed in P with respect to directed sups. This in turn implies that A is a CPO and the inclusion $j: A \to P$ is continuous.

Let $y \in A$. If $x \ll y$ in P, we claim $r(x) \ll y$ in A. Let D be a directed set in A with $\sup D \geq y$. Then $x \leq d$ for some $d \in D$. Hence $r(x) \leq r(d) = d$, and thus $r(x) \ll y$ in A. Clearly the set of all $r(x)$ for $x \ll y$ is directed and has supremum $r(y) = y$. ∎

A CPO A is a **retract** of a CPO P if there exist continuous morphisms $r: P \to A$ and $j: A \to P$ such that $r \circ j = 1_A$. In this case the function r is called an **(external) retraction**. Note that $j \circ r$ is an internal retraction on P and that $j: A \to j(A)$ is an order isomorphism. Thus the previous proposition yields

Corollary. *A retract of a continuous CPO is a continuous CPO.*

A special type of (external) retraction is the **projection**, where in addition to the preceding conditions we require that $j \circ r \leq 1_P$. In this case we write $P \overset{r}{\rightleftharpoons} Q$. If r is a projection, then j is unique, is automatically continuous, and is given by $j(y) = \inf\{x: r(x) \geq y\}$.

Continuous CPO's have an alternate characterization in terms of their ideal completions, namely a CPO P is continuously ordered if and only if the mapping $\mathrm{SUP}: Id(P) \to P$ is a projection. The continuous embedding $j: P \to Id(P)$ is given by $j(x) = \Downarrow x$, which is the smallest ideal with supremum greater than or equal to x.

It follows that every continuously ordered set is the retract of a domain and that the class of continuously ordered sets is the smallest class of CPO's that contains the domains and is closed with respect to taking retracts.

IV. The Scott Topology

A distinctive feature of the theory of continuous orders is that many of the considerations are closely interlinked with topological and categorical ideas. The result is that topological considerations and techniques are basic to significant portions of the theory.

The Scott topology is the topology arising from the convergence structure given by $D \to x$ if D is a directed set with $x \leq \sup D$. Thus a set A is **Scott closed** if $A = \downarrow A$ and if $D \subseteq A$ is directed, then $\sup D \in A$. Similarly U is **Scott open** if $U = \uparrow U$ and $\sup D \in U$ for a directed set D implies $d \in U$ for some $d \in D$.

By means of the Scott topology one can pass back and forth between an order-theoretic viewpoint and a topological viewpoint in the study of CPO's. Generally order-theoretic properties have corresponding topological properties and vice-versa. For example, continuous morphisms between CPO's are precisely those functions which are continuous with respect to the Scott topologies.

Example. The Scott-open sets in $\mathbb{R}^*$ consist of open right rays. For a topological space X, the set of Scott-continuous functions $[X, \mathbb{R}^*]$ consists of the lower semicontinuous functions.

Suppose that a CPO P is equipped with the Scott topology, so that it is now a topological space. Then the original order may be recovered from the topological space as the **order of specialization**, which is defined by $x \leq y$ iff $x \in \overline{\{y\}}$. Note that any topological space has an order of specialization, and that this order is a partial order precisely when the space is T_0.

There are useful alternate descriptions of the Scott topology for special classes of CPO's. For a continuous CPO P, let $\Uparrow z = \{x: z \ll x\}$. It follows from the interpolation property that $\Uparrow z$ is a Scott open set. That these form a basis for the Scott topology follows from the fact that each $x \in P$ is the directed supremum of $\Downarrow x$. It follows that a continuous CPO is countably based iff the Scott topology has a countable base. Alternately the Scott open filters also form a basis for the Scott topology in a continuous CPO.

For domains, a basis for the Scott open sets is given by all sets of the form $\uparrow z$, where z is a compact element. The argument is analogous to the continuously ordered case.

Topological Properties

Given a partially ordered set P, there are a host of topologies on P for which the order of specification agrees with the given order. The finest of these is the **Alexandroff discrete** topology, in which every upper set is an open set, and the coarsest of these is the **weak** topology, in which $\{\downarrow x: x \in P\}$ forms a subbasis for the closed sets. The Scott topology is the finest topology giving back the original order with the additional property that directed sets converge to their suprema. It is this wealth of topologies that makes the study of CPO's from a topological viewpoint (as opposed to an order-theoretic viewpoint) both richer and more complex.

What spaces arise by equipping continuous CPO's with the Scott topology? A result of Scott's [**Sc72**] asserts that continuous lattices equipped with the Scott topology are precisely the injective T_0-spaces (a continuous function from a subspace A of a T_0-space X into L extends to a continuous function on all of X). In general, a continuous CPO equipped with the Scott topology gives rise to a locally compact, sober (T_0-)space. (A base of compact neighborhoods of x in this case is given by $\uparrow z$ for all $z \ll x$.) Indeed, the lattice of Scott-open sets in this case is a completely distributive lattice (a lattice is completely distributive if arbitrary joins distribute over arbitrary meets and vice-versa; these are a special class of distributive continuous lattices). Conversely the spectrum of a completely distributive lattice turns out to be a continuous CPO (with respect to the order of specification) equipped with the Scott topology. Hence another characterization of continuous CPO's equipped with their Scott topologies is that they are the spectra of completely distributive lattices (see [**La79**] or [**Ho81a**]).

There is an inclusion functor from category of sober spaces into the category of T_0-spaces and there is a functor from the category of T_0-spaces into the category of partially ordered sets which sends a space to the order of specialization. Both of these functors have adjoints. The adjoint functor for the order of specialization functor equips a partially ordered set with the Alexandroff discrete topology. The functor sending a space to its sobrification is the adjoint of the inclusion of sober spaces into T_0-spaces. The composition of these two functors sends a partially ordered set to the sobrification of the Alexandroff discrete topology, which turns out to be the ideal completion equipped with the Scott topology. Thus the sobrification of the Alexandroff discrete topology gives a *topological analog of the ideal completion*. The topological retracts of these sobrified Alexandroff discrete spaces are the retracts in our earlier sense and, as we have seen previously, are the continous CPO's. (These results appear in [**Ho81b**].)

Function Spaces

A crucial and characteristic property of countably based continuous CPO's is that they are closed under a wide variety of set-theoretic operations. This allows one to carry along a recursive theory. Such constructions break down in the category of sets because one obtains sets of larger cardinality. Also one can employ these stability features of continuous CPO's to obtain examples which reproduce isomorphic copies of themselves under a variety of set-theoretic operations. (This is essentially the idea of solving domain equations.) It is these features that provide strong motivation for moving from the category of sets to some suitable category of domains or continuous CPO's.

One of the most basic constructs is that of a function space. If X and Y are CPO's, then $[X \to Y]$ denotes the set of continuous morphisms (the order preserving functions which preserve suprema of directed sets) from X to Y. For a directed family of continuous morphisms, the pointwise supremum is again continuous. So the set $[X \to Y]$ with the pointwise order is again a CPO.

For topological spaces X and Y let $[X \to Y]$ denote the set of continuous functions from X to Y. If X or Y is a CPO, then we identify it with the topological space arising from the Scott topology. If Y is a CPO, then $[X \to Y]$ is also a CPO with respect to the pointwise order on functions. One verifies that the supremum of a directed family of continuous functions is again continuous, so directed suprema are computed pointwise in $[X \to Y]$. If X and Y are both CPO's equipped with the Scott topology, then the function space $[X \to Y]$ is just the set of continuous morphisms of the previous paragraph.

Suppose additionally that X is a continuous CPO. Let $f: X \to Y$ be a (not necessarily continuous) order preserving function. Then there exists a largest continuous morphism $\underline{f}: X \to Y$ which satisfies $\underline{f} \leq f$; $\underline{f}$ is given by $\underline{f}(x) = \sup\{f(z): z \ll x\}$. Thus if Y^X denotes the set of all order-preserving functions from X to Y, the mapping $f \to \underline{f}$ from Y^X to $[X \to Y]$ is a projection. If X is an algebraic CPO, then $\underline{f}$ is the unique continuous extension of the restriction of f to the set of compact elements $K(X)$.

Under what conditions will $[X \to Y]$ be a continuous CPO? Let us first consider the case that $Y = 2$, where $2 = \{0, 1\}$ denotes the *two-element chain* with $0 < 1$ equipped with the Scott topology (sometimes called the Sierpinski space). Then $f: X \to 2$ is continuous iff f is the characteristic function of an open set of X. Hence there is a natural order isomorphism between $O(X)$, the lattice of open sets, and $[X \to 2]$. Since $O(X)$ is a continuous lattice iff X is core compact, we conclude that the same is true for $[X \to 2]$.

More generally, let us suppose that X is core compact and that Y is a continuous CPO with $\perp$. Let $f \in [X \to Y]$, $a \in X$, and $f(a) = b$. Let $z \ll b$. Pick U open in X containing a such that $f(U) \subseteq \Uparrow z$ (which we can do since f is Scott continuous). Pick V open with $a \in V$ such that $V \ll U$. Define $g \in [X \to Y]$ by $g(x) = z$ if $x \in V$ and $g(x) = \perp$ otherwise. It is straightforward to verify that $g \ll f$ in $[X \to Y]$ (see [**COMP**, Exercise II.4.20]) and that f is the supremum of such functions. However, one needs additional hypotheses on X and/or Y to be able to get a directed set of such functions. If S is a continuous complete semilattice, then one can take finite suprema of such functions g and obtain the principal implication of

Theorem. *Let S be a non-trivial CPO equipped with the Scott topology. Then $[X \to S]$ is a continuous complete semilattice iff X is core compact and S is a continuous complete semilattice.*

The proof of the reverse implication follows from the fact that $O(X) \cong [X \to 2]$ and S are both retracts of $[X \to S]$ (see [**COMP**, Section II.4]).

If additionally X is compact, then one only needs that S is bounded complete. (Apply the preceding theorem to $[X \to S_\perp]$ and note that if $f(X) \subseteq S$, then by compactness the sup of finitely many of the g's constructed earlier must also not take on the value $\perp$, and hence give an element essentially below f in $[X \to S]$.)

Problem. Suppose P is a continuous CPO and that $[X \to P]$ is a continuous CPO for all core compact spaces X. Is P a continuous complete semilattice?

If P is a CPO without a bottom element, then it is frequently more appropriate to consider the function space $[X \rightharpoonup P]$ of all continuous partial functions with domain some open subset of X. This function space corresponds to $[X \to P_\perp]$ (by extending a partial function to be $\perp$ where not defined), and by this device it can be treated as a full function space. Hence the preceding theorem yields that if X is core compact and P is a bounded complete continuous CPO, then $[X \rightharpoonup P]$ is a continuous complete semilattice.

It is frequently desirable to model the notion of self-application (we may think of programs that act on other programs, including themselves, or programming languages that incorporate the λ-calculus, where objects are also functions and vice-versa). This involves building spaces X homeomorphic to $[X \to X]$. These can be constructed in suitable subcategories of continuously ordered sets by using projective limit constructions, where the bonding maps are projections. This was the original approach of Scott in [**Sc72**]. In these constructions one needs to know that the function space $[X \to X]$ is back in the category that one is considering, that one has natural projections from $[X \to X]$ to X, and that taking function space is preserved by inverse limits. Similar remarks apply for trying to solve other types of domain equations by the technique of projective limits (see [**COMP**, Chapter IV]). The preceding theorem shows that continuous complete semilattices form a good category in this regard (since $[X \to X]$ is another such).

Cartesian Closedness

Let X, Y, Z be sets and let $\alpha: X \times Y \to Z$, and define $\hat{\alpha}: X \to [Y \to Z]$ by $\hat{\alpha}(x)(y) = \alpha(x, y)$. This induces the exponential (or currying) function $E_{XYZ} = E: [X \times Y \to Z] \to [X \to [Y \to Z]]$ sending $\alpha: X \times Y \to Z$ to the associated function $\hat{\alpha}: X \to [Y \to Z]$, and E_{XYZ} is a bijection (a type of exponential law). In general, we call a category **cartesian closed** if products and function spaces are again in the category and the exponential function is always a bijection. This is a convenient property for constructions such as in the preceding section and for other purposes.

Note that E restricted to the category of CPO's and continuous morphisms is still a bijection, for if X, Y, Z are all CPO's, then one verifies directly that α preserves directed sups if and only if $\hat{\alpha}$ does (where $[Y \to Z]$ is given the pointwise order). Hence the category of CPO's and continuous morphisms is also cartesian closed.

Again things rapidly become more complicated when one moves to a topological viewpoint. First of all, one has to have a means of topologizing the function spaces $[Y \to Z]$. In this regard we recall certain basic notions from topology (see e.g. [**Du**, Chapter XII]).

A topology τ on $[Y \to Z]$ is **splitting** if for every space X, the continuity of $\alpha: X \times Y \to Z$ implies that of the associated function $\hat{\alpha}: X \to [Y \to Z]_\tau$ (where $\hat{\alpha}(x)(y) = \alpha(x, y)$). A topology τ on $[Y \to Z]$ is called **admissible** (or **conjoining**) if for every space X, the continuity of $\hat{\alpha}: X \to [Y \to Z]_\tau$ implies that of $\alpha: X \times Y \to Z$. Thus for fixed Y, Z we have that E_{XYZ} is a bijection for all X if and only if the topology τ on $[Y \to Z]$ is both splitting and admissible.

We list some basic facts about splitting and admissible topologies. A topology τ is admissible iff the evaluation mapping $\epsilon: [X \to Y]_\tau \times X \to Y$ defined by $\epsilon(f, x) = f(x)$ is continuous. A topology larger than an admissible topology is again admissible, and a topology smaller than a splitting topology is again splitting. Any admissible topology is larger than any splitting topology, and there is always a unique largest splitting topology. Thus a function space can have at most one topology that is both admissible and splitting, and such a topology is the largest splitting topology and the smallest admissible topology.

A standard function space topology is the compact-open topology. We need a slight modification of this that is suitable for core compact spaces. Let X and Y be spaces, let H be a Scott open set in the lattice $O(X)$ of open sets on X, and let V be an open subset of Y. We define the **Isbell topology** on $[X \to Y]$ by taking as a subbase for the open sets all sets of the form

$$N(H, V) = \{f \in [X \to Y]: f^{-1}(V) \in H\}.$$

If X is locally compact, then the Isbell topology is just the compact-open topology. The next theorem asserts that the core compact spaces are the exponentiable spaces (see [**Is75**], [**SW85**], or [**LP85**]).

Theorem. *Let Y be a core compact space. Then for any space Z the space $[Y \to Z]$ admits an (unique) admissible, splitting topology, the Isbell topology, and with respect to this topology the exponential function E_{XYZ} is a bijection for all X.*

What happens if Y is not core compact? Then results of Day and Kelly [**DK70**] show that the Scott topology on $[Y \to 2]$ is not admissible, but it is the inf of admissible topologies. Thus there is no smallest admissible topology on $[Y \to 2]$, hence no topology that is both admissible and splitting. In this case there is no topology on $[Y \to Z]$ such that E_{XYZ} is a bijection for all X. Thus any category of topological spaces which contains 2, is closed with respect to taking function spaces with respect to some appropriate topology, and is cartesian closed must be some subcategory of core compact spaces. These considerations reduce the search for a largest cartesian closed category in Top to the following problem:

Problem. Is there a largest collection of core compact spaces containing 2 which is closed with respect to taking finite products and function spaces equipped with the Isbell topology (since this is the one that yields that the exponential function is a bijection)?

Suppose now that Z is a CPO equipped with the Scott topology. Then $[Y \to Z]$ is again a CPO, and one can investigate how the Scott and Isbell topology compare on $[Y, Z]$. A direct argument from the definition of the Isbell topology yields that a directed set of functions converges to its pointwise supremum in the Isbell topology, and hence the Isbell topology is coarser than the Scott topology. Since we have seen that the Isbell topology is an admissible topology if Y is core compact, it follows that the Scott topology is also admissible. Gierz and Keimel [**KG82**] have shown that if Y is locally compact and Z is a continuous lattice, then the compact-open and Scott topology agree on $[Y \to Z]$. Analogously Schwarz and Weck [**SW85**] have shown that if Y is core compact and Z is a continuous lattice, then the Isbell topology agrees with the Scott topology on $[Y \to Z]$. In the later section on supersober and compact ordered spaces we generalize these results.

If Y is core compact and second countable (i.e., the topology has a countable base) and if Z is also second countable, then $[Y \to Z]$ equipped with the Isbell topology is second countable (see [**LP 85**, Proposition 2.17]). Hence if Y is core compact and second countable (e.g., Y is a countably based continuous CPO), Z is a countably based continuous CPO, and $[Y \to Z]$ is a continuous CPO on which the Scott and Isbell topologies agree, then $[Y \to Z]$ is a countably based continuous CPO (since being countably based is equivalent to the second countability of the Scott topology).

Strongly algebraic and finitely continuous CPO's

The category of finite partially ordered sets and order preserving functions is cartesian closed. The full subcategories with objects lattices or (meet) semilattices are also cartesian closed. One can extend these categories by taking projective limits where the bonding mappings are projections. For the finite lattices (resp. semilattices), one gets the algebraic lattices (resp. the algebraic semilattices). For all finite partially ordered sets one obtains objects which are called **strongly algebraic** CPO's. They form a larger cartesian closed category than the algebraic semilattices and were introduced by Plotkin [**Pl76**] to have a cartesian closed category available where one could carry out certain power domain constructions and remain in the category. The morphisms in these categories (as earlier) are the Scott continuous morphisms, and the function spaces are the CPO's arising from the pointwise order of functions. In the section on supersober and compact ordered spaces we will relate these function spaces to the topological considerations of the previous section.

One can consider all retracts of strongly algebraic CPO's and obtain an even larger cartesian closed category. These objects have been called **finitely continuous** CPO's by Kamimura and Tang and studied in several of their papers (see in particular [**KT86**]). A CPO P is a finitely continuous CPO iff there exists a directed family D of continuous functions from P into P with supremum the identity function on P such that the $f(P)$ is finite for each $f \in D$. The strongly algebraic CPO's are characterized by requiring in addition that each member of D be a projection. We take these characterizations for our working definition of these concepts. Frequently one's attention is restricted to the countably based case. Here the the directed family of functions, respectively, projections with finite range may be replaced by an increasing *sequence* of functions.

Let S be a continuous complete semilattice and let F be a finite subset of S containing $\perp$. Enlarge F to G by adjoining the supremum of each subset of F that is bounded above. Then G is still finite and has the property that $\Downarrow x \cap G$ has a largest element for each $x \in S$. The mapping that sends x to the largest element of $\Downarrow x \cap G$ is continuous and below the identity mapping. Furthermore, the family of all such mappings for all finite sets is a directed family whose supremum is the identity. Hence S is a finitely continuous CPO.

We list some basic properties of finitely continuous CPO's.

Proposition. *A retract of a finitely continuous CPO is again a finitely continuous CPO.*

Proof. Let $r : P \to Q$ be a retract with inclusion $i : Q \to P$. Let D be the directed family of functions in $[P \to P]$ with finite range and with supremum 1_P. Then $\{r \circ f \circ i : f \in D\}$ gives the desired directed family on Q. ∎

Proposition. *Let P and Q be finitely continuous CPO's. Then $[P \to Q]$ is a finitely continuous CPO.*

Proof. Let D be the directed family for P and D' for Q. Then $\{h \mapsto f' \circ h \circ f : f \in D, f' \in D'\}$ gives the desired directed family on $[P \to Q]$. ∎

It follows directly from the last proposition that the finitely continuous CPO's form a cartesian closed subcategory of the CPO category.

Plotkin [**Pl76**] gave an alternate characterization of strongly algebraic CPO's in terms of the partially ordered set of compact elements, which we do not pursue here. Smyth [**Sm83a**] used these to derive the following significant result:

Theorem. *Let P be a countably based algebraic CPO with $\perp$. If $[P \to P]$ is also an algebraic CPO, then P is a strongly algebraic CPO.*

This theorem shows that the largest cartesian closed full subcategory of countably based algebraic CPO's consists of the strongly algebraic CPO's.

Problem. Do the finitely continuous CPO's form the largest cartesian closed full subcategory contained in the category of continuous CPO's?

Problem. Give an internal description of a finitely continuous CPO that one can apply directly to determine whether a given continuous CPO is finitely continuous.

Problem. Find a topological description of the spaces obtained by endowing a finitely continuous CPO with the Scott topology.

V. Dual and Patch Topologies

Given a T_0-topology, each open set is an upper set and each closed set is a lower set with respect to the order of specification $x \leq y \Leftrightarrow x \in \overline{\{y\}}$. There are methods for creating "dual" topologies from the given topology in which open sets in the new dual topology are now lower sets (with respect to the original order of specification). "Patch" topologies then arise as the join of a topology and its dual.

Suppose $d: X \times X \to \mathbb{R}^+$ satisfies the triangular inequality. We use d to generate a topology on X by declaring a set U open if for each $x \in U$, there exists a *positive* number r such that $N(x;r) \subseteq U$, where $N(x;r) = \{y: d(x,y) \leq r\}$. (This is slightly at variance with the usual approach, but allows us momentarily a useful generalization.) Then $d^*(x,y) = d(y,x)$ gives rise to a **dual** topology.

Example. Define $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by $d(x,y) = \max\{0, x-y\}$. Then d generates the Scott topology on $\mathbb{R}$, d^* gives the reverse of the Scott topology (the Scott topology on the order dual), and the join of the two topologies is the usual topology.

The situation can be considerably generalized by considering functions satisfying the triangular inequality into much more general semigroups than the positive reals $\mathbb{R}^+$ (see e.g. [**Ko87**]). In this case we need to specify an ordered semigroup S and a subset of positive elements S^+ for the codomain of the "distance" function. Suppose that P is a continuous CPO. We set S equal to the power set of P with addition being the operation of union. We let S^+, the set of positive elements, be the cofinite subsets. We define the metric d by $d(x,y) = \Downarrow x \setminus \downarrow y$, and then define the open sets precisely as in the earlier paragraph for real metrics. This metric is called the **canonical generalized metric** for a continuous CPO.

Proposition. *The topology generated by d is the Scott topology.*

Proof. Consider the set $N(x;A)$, where A is a cofinite subset. Let F be the complement of A. Then one verifies directly that

$$N(x;A) = \{y: F \cap \Downarrow x \subseteq \downarrow y\} = \bigcap \{\uparrow z: z \in F \cap \Downarrow x\}.$$

Such sets contain x in their interior in the Scott topology if P is a continuously ordered set (since $x \in \Uparrow z \subseteq \uparrow z$); hence the metric open sets are Scott open. Conversely if $x \in U$ where U is Scott open, pick $z \in U$ such that $z \ll x$ (this is possible since $\Downarrow x$ is directed). Let $A = P \setminus \{z\}$. Then $N(x;A) = \uparrow z \subseteq U$. Hence U is metric open. ∎

An approach that has received more attention has been the following (see [**Sm83b**]). Let X be a T_0-topological space. A set is said to be **saturated** if it is the intersection of open sets (this is equivalent to being an upper set in the order of specification). One defines the **dual** topology by taking as a subbasis for the closed sets all saturated compact sets. The join of these two topologies is called the **patch** topology.

For a partially ordered set P, the **weak** topology is defined by taking as a subbase for the closed sets all principal lower sets $\downarrow x$ for $x \in P$. The **weak**^{d} topology is defined to be the weak topology on the dual of P, the set P with the order reversed. All sets of the form $\uparrow x$ form a subbasis for the closed sets for the weak^d topology.

Proposition. *Let P be a continuous CPO. Then the dual topology for the canonical generalized metric and the dual topology for the Scott topology both agree and both yield the weak^d topology.*

Proof. Let $d(x,y) = \Downarrow x \setminus \downarrow y$. Let A be cofinite in P and let F be its complement. Then

$$\begin{aligned} x \in & P \setminus \bigcup\{\uparrow z : z \in F \setminus \downarrow x\} \subseteq P \setminus \bigcup\{\Uparrow z : z \in F \setminus \downarrow x\} \\ &= \{y : \Downarrow y \cap F \subseteq \downarrow x\} = \{y : \Downarrow y \setminus \downarrow x \subseteq A\} = N_{d^*}(x, A). \end{aligned}$$

Since the first set is open in the weak^d topology, it follows that dual open sets are open in the weak^d topology.

Conversely we show that a subbasic open set $U = P \setminus \uparrow w$ is open in the dual topology. Let $x \in U$. Then there exists $z \ll w$ such that $z \not\leq x$. Let $F = \{z\}$ and let $A = P \setminus F$. Then as before $N_{d^*}(x, A) = \{y : \Downarrow y \setminus \downarrow x \subseteq A\} = P \setminus \Uparrow z \subseteq P \setminus \uparrow w$. Thus U is open in the dual topology.

We turn now to the second case. Since $\uparrow x$ is trivially compact in the Scott topology, it follows that every weak^d open set is open in the dual topology. Conversely let A be an upper set which is compact in the Scott topology and pick $y \notin A$. For each $x \in A$, pick $z_x \ll x$ such that $z_x \not\leq y$. Since $\{\Uparrow z_x\}$ is an open cover of A, there exist finitely many such that $A \subseteq \bigcup\{\uparrow z_i : 1 \leq i \leq n\}$. Note that the righthand set is closed in the weak^d topology and misses y. Since y was arbitrary, it follows that A is the intersection of sets closed in the weak^d topology, and hence is itself closed in the weak^d topology. ∎

The Lawson Topology

The **Lawson** topology on a CPO is obtained by taking the join of the Scott topology and the weak^d topology. It follows from the last proposition of the preceding section that if P is a continuous CPO, then the Lawson topology is the patch topology defined from the canonical generalized metric and it is also the patch topology arising from the Scott topology.

The Lawson topology on a continuous CPO P is Hausdorff, for if $x \not\leq y$, then there exists $z \ll x$ such that $z \not\leq y$, and $\Uparrow z$ and $P \setminus \uparrow z$ are disjoint neighborhoods of x and y resp. Indeed the set $\Uparrow z \times P \setminus \uparrow z$ misses the graph of the order relation

$\leq$, so that the order relation is closed in $P \times P$. Such spaces (in which the order is closed) are called **partially ordered spaces**.

If P is an algebraic CPO, then the Lawson topology is generated by taking all sets $\uparrow x$ for compact elements x to be *both* open and closed. It follows that P with the Lawson topology is a 0-dimensional space. Hence it is the continuous (as opposed to the algebraic) CPO's that can give rise to continuum-like properties with respect to the Lawson topology.

If S is a complete semilattice, then one can take all complete subsemilattices which are upper sets or lower sets as a subbase for the closed sets and again obtain the Lawson topology. If S is a continuous complete semilattice, then the Lawson topology is compact and Hausdorff, the operation $(x, y) \mapsto x \wedge y$ is continuous, and S has a basis of neighborhoods at each point which are subsemilattices. Conversely, if a semilattice admits a topology with these properties, then the semilattice is a continuous complete semilattice and the topology is the Lawson topology (see [**COMP**, VI.3]).

Example. Let X be a compact Hausdorff space and let L be the semilattice of closed non-empty subsets ordered by reverse inclusion and with the binary operation of union. Then X is a continuous complete semilattice, the traditional Vietoris topology on L agrees with the Lawson topology, and this is the unique compact Hausdorff topology on L for which the binary operation of union is continuous.

Supersober and Compact Ordered Spaces

A **compact supersober** topological space X is one in which the set of limit points of an ultrafilter is the closure of a unique point. These spaces are in particular sober and also turn out to be locally compact (and hence the lattice of open sets is a continuous lattice). The patch topology on such a space is compact and Hausdorff, and the order of specification is closed in $(X, \text{patch}) \times (X, \text{patch})$. Hence in a natural way a compact ordered space results.

Conversely, if X is a compact ordered space, consider the space $(X, \mathcal{U})$, where $\mathcal{U}$ consists of all open *upper* sets. Then $(X, \mathcal{U})$ is a compact supersober space (with the set of limit points of an ultrafilter being the lower set of the point to which the ultrafilter converged in the original topology). The dual topology consists of all open lower sets, the patch topology is the original topology, and the order of specification is the original order (see [**COMP**, VII.1 Exercises] for the preceding results). Specializing to CPO's and the Scott topology, we obtain

Theorem. *A CPO P is compact supersober with respect to the Scott topology iff the Lawson topology is compact. In this case P is a compact ordered space with respect to the Lawson topology.*

We note that the order dual of a compact partially ordered space is another such. Hence the topology consisting of the open lower sets is also a compact supersober space with dual topology the open upper sets.

The preceding theorem quickly yields

Proposition. *If the Lawson topology is compact for a CPO P, then the same is true for any retract.*

Proof. Let $r: P \to Q$ be a retract with inclusion $i: Q \to P$. We show that Q is compact supersober. Let $\mathcal{U}$ be an ultrafilter in Q. Then the ultrafilter $i(\mathcal{U})$ has a largest limit point $p \in P$. Then $r(p)$ is a limit point of the original ultrafilter in Q, and if q is another limit point, then $i(q) \leq p$, so $q = r(i(q)) \leq r(p)$. ∎

Note that the preceding result is really a topological result, namely that the retract of any compact supersober space is again compact supersober.

It was shown in [**COMP**] that a continuous lattice or continuous complete semilattice is compact in the Lawson topology. This result extents to finitely continuous CPO's.

Proposition. *A finitely continuous CPO is compact in the Lawson topology.*

Proof. Let $\mathcal{F}$ be the directed family of Scott continuous functions with finite range that approximate the identity in the finitely continuous CPO P. Let $\mathcal{U}$ be an ultrafilter in P. Then $f(\mathcal{U})$ is an ultrafilter in the finite set $f(P)$ for each $f \in \mathcal{F}$, and hence contains a singleton set $\{p_f\}$. Since $\mathcal{U}$ is a filter and the family $\mathcal{F}$ is directed, it follows that the family $\{p_f : f \in \mathcal{F}\}$ is directed, and hence has a supremum p. We claim that the ultrafilter converges to p, which will establish the compactness of P.

Let U be a Scott open set containing p. Then $p_f \in U$ for some $f \in \mathcal{F}$. Since $f \leq 1_P$, it follows that p_f is a lower bound for each member of the ultrafilter whose image under f is $\{p_f\}$. Since U is an upper set, U contains each of these sets. Hence the ultrafilter converges to p in the Scott topology. Suppose now that $p \notin \uparrow q$. Pick $f \in \mathcal{F}$ such that $f(q) \not\leq p$. Then $f(q) \not\leq p_f$, which in turn implies $\uparrow q \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $P \setminus \uparrow q \in \mathcal{U}$. Thus the ultrafilter also converges to p in the weak^d topology, and hence in the Lawson topology. ∎

We consider function spaces for the compact supersober continuous CPO's. First we need a lemma.

Lemma. *Let P be a continuous CPO for which the Lawson topology is compact. Suppose that $A \subseteq \Downarrow x$ and $x = \sup A$. If U is Scott open and $x \in U$, then there exists a finite subset F of A such that $a \leq b$ for all $a \in F$ implies $b \in U$.*

Proof. The sets $\uparrow a_1 \cap \uparrow a_2 \cap \cdots \cap \uparrow a_n$ for $a_1, \ldots, a_n \in A$ form a descending family of Lawson closed sets with intersection $\uparrow x \subseteq U$. It follows from compactness that $\uparrow a_1 \cap \uparrow a_2 \cap \cdots \cap \uparrow a_n \subseteq U$ for some finite subset. If $a_i \leq b$ for $1 \leq i \leq n$, then b is in this intersection and hence in U. ∎

Theorem. *Let X be a core compact space, and let P be a continuous CPO with $\perp$. If $[X \to P]$ is a continuous CPO for which the Lawson topology is compact, then the Scott topology on $[X \to P]$ is the Isbell topology (which is the compact-open topology if X is locally compact).*

Proof. We have seen in the section on function spaces that the Scott topology is admissible and that the Isbell topology is coarsest of the admissible topologies. If we show that the Scott topology is also the coarsest of the admissible topologies, then we are done. So suppose that τ is a topology on $[X \to P]$ such that the evaluation mapping ϵ: $[X \to P] \times X \to P$ is continuous.

Let $f \in [X \to Y]$, $a \in X$, and $f(a) = b$. Let $z \ll b$. By joint continuity of the evaluation mapping, there exists a τ-open set W containing f and an open set U in X containing a such that $h(x) \in \Uparrow z$ for all $h \in W$ and $x \in U$. Pick V open with $a \in V$ such that $V \ll U$. Define $g \in [X \to Y]$ by $g(x) = z$ if $x \in V$ and $g(x) = \perp$ otherwise. As we saw in the section on function spaces, $g \ll f$ in $[X \to Y]$ and f is the supremum of such functions. Note also that $W \subseteq \uparrow g$ since $z \leq h(x)$ for $x \in V$, $h \in W$, and $\perp \leq h(x)$ otherwise. By the preceding lemma, for any Scott open set Q containing f, there exists finitely many such g_i such that $\uparrow g_1 \cap \cdots \cap \uparrow g_n \subseteq Q$. Since each $\uparrow g_i$ is a neighborhood of f in the τ-topology (as we have just established), it follows that Q is also. It follows that the τ-topology is finer than the Scott topology. ∎

Adjunctions

Let $f^+: P \to Q$ and $f^-: Q \to P$ be order-preserving functions between the partially ordered sets P and Q. The pair (f^+, f^-) is called an **adjunction** if $y \leq f^+(x) \Leftrightarrow f^-(y) \leq x$. (Such pairs are also sometimes referred to as Galois connections, but many authors prefer to define Galois connections in terms of antitone functions.) Adjunctions can be alternately characterized by the property that $1_Q \leq f^+ \circ f^-$ and $1_P \geq f^- \circ f^+$. Hence f^+ is called the **upper adjoint** and f^- the **lower adjoint**. The mapping f^- is sometimes referred to as a **residuated** mapping.

The upper adjoint f^+ has the property that the inverse of a principal filter $\uparrow q$ in Q is again a principal filter in P (indeed this property characterizes mappings that arise as upper adjoints). Hence if P and Q are CPO's, then f^+ is Scott continuous iff it is Lawson continuous. If Q is a continuous CPO, then f^+ is Scott continuous iff f^- preserves the relation $\ll$ (see [**COMP**, Exercise IV.1.29]). Note that projections are upper adjoints (with the lower adjoint being the inclusion mapping), and hence are continuous in the Lawson topology.

The preceding remarks show that the Scott continuous upper adjoints form a good class of morphisms to consider if one is working with the Lawson topology. If P and Q are both continuous lattices, then these mappings are precisely the Lawson continuous $\wedge$-homomorphisms, which in turn are the mappings that preserve infima of non-empty sets and suprema of directed sets. As we have seen in the previous paragraph, there results a dual category consisting of the same objects with morphisms the lower adjoints which preserve the relation $\ll$. If one restricts to algebraic lattices, then the lower adjoint must preserve the compact elements. Its restriction to the compact elements is a $\vee$-preserving and $\perp$-preserving mapping. In this way one obtains the Hofmann-Mislove-Stralka duality [**HMS**] between the category of algebraic lattices with morphisms the Scott continuous upper adjoints and the category of sup-semilattices with $\perp$ and morphisms preserving $\perp$ and the $\vee$-operation.

Powerdomains

A powerdomain is a CPO together with extra algebraic structure for handling nondeterministic values. Their consideration is motivated by the desire to find semantic models for nondeterministic phenomena. Examples are frequently obtained by taking some appropriate subset of the power set of a given CPO P (hence the terminology "powerdomain"). We think of the subsets as keeping track of the possible outcomes of a nondeterministic computation. Again one is motivated to find categories where powerdomain constructions remain in the category.

We quickly overview some of the standard powerdomain constructions. If P is a CPO with $\perp$, then one can construct the Hoare powerdomain as all non-empty Scott closed subsets. If P is a continuous CPO, then this set is anti-isomorphic to the lattice of open sets, and hence forms a continuous (indeed completely distributive) lattice. The Smyth powerdomain is obtained by taking all the upper sets which are compact in the Scott topology. (We refer to [**Sm73b**] for a nice topological development of these ideas in a general setting.) In the case of a continuous CPO for which the Lawson topology is compact, these are just the closed sets in the weak^d topology, which is again anti-isomorphic to the lattice of weak^d open sets. We have seen previously that in the case that the Lawson topology is compact, this topology is compact supersober, hence locally compact, and hence the lattice of open sets is continuous.

One of the most interesting of the powerdomain constructions is the so-called Plotkin powerdomain. This again lends itself to nice description in the case that D is a continuous CPO for which the Lawson topology is compact (which we assume henceforth). It will also be convenient to assume certain basic facts about compact partially ordered spaces (see [**COMP**, VI.1]). Let $P(D)$ denote the set of all non-empty Lawson closed order-convex subsets. If $A \in P(D)$, then A is compact, and hence $\downarrow A$ and $\uparrow A$ are closed. Since A is order convex, $A = \downarrow A \cap \uparrow A$. Hence $A \in P(D)$ iff it is the intersection of a closed upper and closed lower set. We order $P(D)$ with what is commonly referred to as the Egli-Milner ordering: $A \leq B \Leftrightarrow A \subseteq \downarrow B$ and $B \subseteq \uparrow A$.

Theorem. *$(P(D), \leq)$ is a continuous CPO for which the Lawson topology is compact.*

Proof. Let A_α be a directed family in $P(D)$ (i.e., $\alpha \leq \beta$ implies $A_\beta \subseteq \uparrow A_\alpha$ and $A_\alpha \subseteq \downarrow A_\beta$). Then one verifies directly that the supremum of this family is given by

$$\sup A_\alpha = \overline{\bigcup \downarrow A_\alpha} \cap \bigcap \uparrow A_\alpha = \bigcap (\overline{\bigcup \downarrow A_\alpha} \cap \uparrow A_\alpha).$$

From the first equality it follows that this set is closed and order convex and from the second that it is non-empty (since it is the intersection of a descending family of non-empty compact sets). Thus $P(D)$ is a CPO.

Let A be closed and order-convex. Suppose that F is a finite set such that $A \subseteq \bigcup\{\Uparrow z : z \in F\}$ and $F \subseteq \bigcup\{\Downarrow a : a \in A\}$. We claim the order-convex hull $h(F) = \downarrow F \cap \uparrow F$ satisfies $h(F) \ll A$ in $P(D)$. Suppose that A_α is a directed family in $P(D)$ with $A \leq \sup A_\alpha$. Then

$$\sup A_\alpha = \bigcap (\overline{\bigcup \downarrow A_\alpha} \cap \uparrow A_\alpha) \subseteq \uparrow A \subseteq \bigcup\{\Uparrow z : z \in F\}.$$

Since the latter set is open in the Scott and hence Lawson topology, it follows that $A_\alpha \subseteq \overline{\bigcup \downarrow A_\alpha} \cap \uparrow A_\alpha \subseteq \bigcup\{\Uparrow z : z \in F\}$ for all α sufficiently large.

Conversely, for $z \in F$ there exists $a \in A$ with $z \ll a$. Since $A \leq \sup A_\alpha$, we conclude there exists $b \in \sup A_\alpha$ with $a \leq b$. Then $\Uparrow z$ is a Scott open neighborhood of b, so there exists $c \in \bigcup \downarrow A_\alpha$ such that $z \leq c$. Then $c \in \downarrow A_\gamma$ for some γ. It follows that $z \in \downarrow A_\beta$ for all indices $\beta \geq \gamma$ since $z \leq c \in \downarrow A_\gamma \subseteq \downarrow A_\beta$. Carrying this out for each $z \in F$, we conclude that $F \subset A_\alpha$ for all indices large enough, and hence the same obtains for $h(F)$. We conclude that $h(F) \leq A_\alpha$ for all indices large enough, and hence $h(F) \ll A$.

We next show that the collection of all $h(F)$ as constructed in the preceding is directed. Suppose that we are given $h(F_1)$ and $h(F_2)$. Then $A \subseteq (\bigcup\{\Uparrow y : y \in F_1\}) \cap \bigcup\{\Uparrow z : z \in F_2\}$, and the latter is a Scott open set U. Hence for each $a \in A$ there exists $z_a \ll a$ such that $z_a \in U$. Since A is compact, finitely many of the $\Uparrow z_a$ cover A. Let G be this finite set of z_α. Now given $f \in F_1$, there exists $a \in A$ such that $f \ll a$. Then there exists $g \in F_2$ such that $g \ll a$. Since $\Downarrow a$ is directed, there exists $h \ll a$ with $f, g \leq h$. For each $f \in F_1 \cup F_2$, pick such an h_f. Then the finite set F consisting of G and all the h_f satisfies the earlier conditions and $h(F)$ is above $h(F_1)$ and $h(F_2)$ in the Egli-Milner ordering. The fact that A is the suprema of the $h(F)$ follows fairly directly from the formula in the first paragraph for the calculation of suprema.

We show finally that $P(D)$ is compact. Indeed, more is true. The mapping from the space of non-empty closed subsets $Cl(D)$ with the Vietoris (=Lawson) topology to $P(D)$ which sends A to $h(A)$ is continuous. Since the former is compact, so is the latter. To show continuity, let A_α be a net in $Cl(D)$ converging to some closed set A in the Vietoris topology. Let $h(F) \ll h(A)$ for F finite. Then $\Uparrow h(F)$ is a basic open set containing $h(A)$ in the Scott topology and $A \subseteq h(A) \subseteq \operatorname{int} \uparrow h(F)$. Since the set of all closed sets contained in $\operatorname{int} \uparrow h(F)$ is a neighborhood of A in the Vietoris topology, we conclude that $A_\alpha \subseteq \uparrow h(F)$ for all indices sufficiently large; hence the same holds for $h(A_\alpha)$. For each $z \in F$, there exists $a \in h(A)$ with $z \ll a$. By definition of $h(A)$, there exists $b \in A$ with $a \leq b$. Then $\Uparrow z$ is an open set in D meeting A, and hence $A_\alpha \cap \Uparrow z \neq \emptyset$ for all indices large enough. We repeat this for all $z \in F$, and conclude that $h(F) \subseteq \downarrow h(A_\alpha)$ for all indices sufficiently large. It follows that $h(A_\alpha)$ converges to $h(A)$ in the Scott topology on $P(D)$.

Now let $B \in P(D)$ such that $B \not\leq h(A)$. Then the complement of the upper set of the singleton B in $P(D)$ is a subbasic open set around $h(A)$ in the weak^d topology on $P(D)$. If $h(A) \not\subseteq \uparrow B$, then there exists $a \in A$ with $a \notin \uparrow B$. Since the complement of $\uparrow B$ is an open set meeting A, this complement also meets A_α for all indices large enough. Then $h(A_\alpha) \not\subseteq \uparrow B$. On the other hand, if $B \not\subseteq \downarrow h(A) = \downarrow A$, then there exists $b \in B$ with $\uparrow b \cap A = \emptyset$. Then the complement of $\uparrow b$ is an open set containing A and hence A_α for all α large enough. The complement then contains $h(A_\alpha)$. Thus in either case $B \not\leq h(A_\alpha)$ for large indices. Hence $h(A_\alpha)$ converges to $h(A)$ in both the Scott and weak^d topologies, and hence in the Lawson topology. ∎

We remark that Plotkin introduced the strongly algebraic (countably based) CPO's because the Plotkin powerdomain is another such [**PL76**]. The same is true for finitely continuous CPO's, as has been shown by Kamimura and Tang [**KT87**]. To get the directed family of functions which approximate the identity and have finite range on $P(D)$ from those on D, simply consider $A \mapsto h(f(A))$ for each f in the approximating family on D. The same technique works to obtain projections if D is strongly algebraic, and in the countably based case one obtains a sequence of functions.

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