1.4 Linear Transformation, Matrices, and Change of Basis

27 January 2024 14:26

Definition (Linear transformation):

Let U and V be vector spaces over same field $\mathbb{F}$. A linear transformation is a function $T: U \to V$ with the following properties:

(Additivity) (i) $T(u_1 + u_2) = T(u_1) + T(u_2)$, $\forall\, u_1, u_2 \in U$

(Scalar homogeneity) (ii) $T(\alpha u) = \alpha \cdot T(u)$, $\forall\, \alpha \in \mathbb{F}$ and $u \in U$

Remark: Suppose $T: U \to V$ is a linear transformation where U is a finite dimentional vector space with basis $\{u_1, \dots, u_n\}$. Let u be any element in U. Then $\exists$ unique scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $u = \sum_{j=1}^{n} \alpha_j \cdot u_j$. Now,

$$T(u) = T\left(\sum_{j=1}^{n} \alpha_j \cdot u_j\right) = \alpha_1 T(u_1) + \cdots + \alpha_n T(u_n).$$

In particular, suppose $L: U \to U$ is a linear transformation, and $L(u_j) = T(u_j)$ for all $j = 1, 2, \dots, n-1, n$. Then $T = L$.

> **Recall:** Let V be a vector space over a field $\mathbb{F}$ and U be a nonempty subset of V.
>
> (i) Suppose U is finite, say $U = \{u_1, \dots, u_n\}$ for some $n \in \mathbb{N}$, with $u_j \neq u_k$ for $j \neq k$. Then U is a basis for V $\iff$ for each $v \in V$ $\exists$ unique $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $v = \sum_{j=1}^{n} \alpha_i u_i$.
>
> (ii) If U is infinite, then U is a basis for V $\iff$ for each non-zero $v \in V$, $\exists$ unique $m \in \mathbb{N}$, $u_1, u_2, \dots, u_m \in U$, and nonzero $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{F}$ $\ni$ $v = \sum_{j=1}^{m} \alpha_j u_j$

Lemma (Linear transformation with a matrix):

Let U and V be finite dimentional vector spaces over a field $\mathbb{F}$. Suppose $R = \{u_1, \dots, u_n\}$ is a basis for U and $S = \{v_1, \dots, v_m\}$ is a basis for V. Let A be an $m \times n$ matrix over $\mathbb{F}$. We define a mapping $T_A : U \to V$ as follows: for $u \in U$. Let $T_A(u)$ be an element of V whose vectors of components $[T_A(u)]_S$ with respect to S is $A[u]_R$, that is $[T_A(u)]_S = A[u]_R$

Recall: If V is a vector space over $\mathbb{F}$ with basis $S = \{v_1, \ldots, v_n\}$, then $\exists$ unique $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ for a given $v \in V$ such that $\alpha_1 v_1 + \cdots + \alpha_n v_n = v$. We write this in matrix as $[v]_S = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{n \times 1}$

Remark: The above lemma says that we can construct a linear transformation, between two vector spaces over same field, for given matrix.

For converse see next lemma.

Lemma (Matrix representing a linear transformation):

Let U and V be finite dimentional vector spaces over a field $\mathbb{F}$. Suppose $R = \{u_1, u_2, \ldots, u_n\}$ is a basis for U. $S = \{v_1, \ldots, v_m\}$ is a basis for V and $T : U \to V$ is a linear transformation. Since $T(u_j) \in V$ for each j, there are unique scalars α_{ij}, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that

$$T(u_1) = \alpha_{11} v_1 + \alpha_{21} v_2 + \cdots + \alpha_{m1} v_m,$$
$$T(u_2) = \alpha_{12} v_1 + \alpha_{22} v_2 + \cdots + \alpha_{m2} v_m,$$
$$\vdots$$
$$T(u_n) = \alpha_{1n} v_1 + \alpha_{2n} v_2 + \cdots + \alpha_{mn} v_m.$$

Let A be the $m \times n$ matrix whose k^{th} coloumn consists of the scalar $\alpha_{1k}, \alpha_{2k}, \ldots \alpha_{m-1,k}, \alpha_{mk}$ in the expansion of $T(u_k)$:

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,1} & \alpha_{m-1\,2} & \cdots & \alpha_{m-1\,n} \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}$$

Then $[T(u)]_S = A[u]_R \quad \forall u \in U$.

Moreover, A is the unique matrix satisfying the above relation $[T(u)]_S = A \cdot [u]_R$

Example: Define a basis R for $\mathbb{R}^2$ by $R = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$, and a basis S for $\mathbb{R}^3$ by $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$.

Define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + y \\ x - 3y \end{bmatrix}$. Find the matrix A that represents T with respect to R and S.

Solution: By definition,

$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$, and $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_{21} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_{31} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Leftrightarrow \begin{aligned} 1a_{11} + 0a_{21} + 1a_{31} &= 3 \\ 0a_{11} + 1a_{21} + 0a_{31} &= 3 \\ 2a_{11} + 1a_{21} + 3a_{31} &= -1 \end{aligned}$$

$\Rightarrow a_{11} = 13$, $a_{21} = 3$ and $a_{31} = -10$

and $\begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = a_{12}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_{22}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_{32}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$\Rightarrow a_{12} = 2$, $a_{22} = 2$ and $a_{32} = -4$

Hence

$$A = \begin{bmatrix} 13 & 2 \\ 3 & 2 \\ -10 & -4 \end{bmatrix}$$

Now let us define linear transformation T_A from A.

$$A[u]_R = \begin{bmatrix} 13 & 2 \\ 3 & 2 \\ -10 & -4 \end{bmatrix}_{3\times 2}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}_{2\times 1} \quad \text{where } u = \alpha_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 13\alpha_1 + 2\alpha_2 \\ 3\alpha_1 + 2\alpha_2 \\ -10\alpha_1 - 4\alpha_2 \end{bmatrix}_{3\times 1}$$

$$\therefore T(u) = (13\alpha_1 + 2\alpha_2)\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (3\alpha_1 + 2\alpha_2)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-10\alpha_1 - 4\alpha_2)\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Remark: These lemmas shows there is a complete correspond-ence between matricies and linear transformation between finite dimentional vector spaces.