Discrete commutative hypergroups
&
Polynomial Hypergroups

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Chapter 1

Discrete Hypergroups

1.1 Basic definitions

Let K be a set.

Recall 1.1.1. *Definition of Linear spaces, Normed spaces, complete space and Banach space (Complete Normed linear space)*

Definition 1.1.2 (Dirac function on K). *For $x \in K$, the Dirac function on K is denoted by ε_x , and given by*

$$\varepsilon_x(x) = 1 \quad and \quad \varepsilon_x(y) = 0 \quad if \quad y \in K, y \neq x.$$

Definition 1.1.3. *Let ℓ^1 be the space of all functions $f : K \to \mathbb{C}$ of the form $f = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$, where $a_n \in \mathbb{C}$ such that $\sum_{n=1}^{\infty} |a_n| < \infty$, and x_n are distinct points of K.*

Example 1.1.4 (Belongs to ℓ^1). *Let K be the set of natural numbers $\mathbb{N}$, and consider the function $f : K \to \mathbb{C}$ defined by:*

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} \varepsilon_n$$

Here, $a_n = \frac{1}{2^n}$, $x_n = n$, and ε_n is the function that takes the value 1 at n and 0 elsewhere. The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges absolutely, so f belongs to ℓ^1.

Example 1.1.5 (Does not belong to ℓ^1). *Let K again be the set of natural numbers $\mathbb{N}$, and consider the function $g : K \to \mathbb{C}$ defined by:*

$$g = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \varepsilon_{2n}$$

Here, $a_n = \frac{(-1)^n}{n}$, $x_n = 2n$, and ε_{2n} is the function that takes the value 1 at $2n$ and 0 elsewhere. The series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right|$ diverges, so g does not belong to ℓ^1.

Question 1.1.6. *Can you write what is $f(n)$ for f in the example 1.1.4?*

Note 1.1.7. *The norm of f is $\|f\| = \sum_{n=1}^{\infty} |a_n|$.*

Exercise 1.1.8. *Prove that space ℓ^1 [see:1.1.3] is a Banach space.*
Hint: *A normed vector space V is a Banach space if and only if every absolutely summable series is summable.*

Definition 1.1.9 (Finite convex combination of Dirac functions). *If $f \in \ell^1$ is a finite convex combination of Dirac functions, we write $f \in \ell^1_{co}$,*
i.e.,

$$f = \sum_{n=1}^{N} \alpha_n \varepsilon_{x_n},$$

where $\alpha_n \geq 0$ and $\sum_{n=1}^{N} \alpha_n = 1$.

Note 1.1.10. *We set $suppf = \{x \in K : f(x) \neq 0\}$ (support of f).*

Question 1.1.11.

- *Why do we set $suppf$ as mentioned in 1.1.10? Can we see it anywhere else?*
- *Can you write $suppf$ for examples 1.1.4 and* **??***?*

Now let us assume that a map $\omega^* : K \times K \to \ell^1_{\mathrm{co}}$ is given.

Example 1.1.12. $\omega^* : K \times K \to \ell^1_{co}$ *given by*

$$\omega^*(i, j) = \frac{1}{2}\varepsilon_i + \frac{1}{2}\varepsilon_j$$

Question 1.1.13. *(i) Can we find out $\omega^*(2, 3)$ in the example 1.1.12?*

(ii) Dose $\omega^*(f, g) = \varepsilon_{n+m}$ belongs to ℓ^1?

Definition 1.1.14. *We define* $\omega : \ell^1 \times \ell^1 \to \ell^1$, $\omega(f,g) = \sum_{n,m=1}^{\infty} a_n b_m \omega^*(x_n, y_m)$, *where* $f = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$ *and* $g = \sum_{m=1}^{\infty} b_m \varepsilon_{y_m}$.

Remark 1.1.15. *We can apply any rearrangement of the series, and we obtain* $\|\omega(f,g)\| \leq \sum_{n,m=1}^{\infty} |a_n| \cdot |b_m| \|\omega^*(x_n, y_m)\| = \|f\| \cdot \|g\|$.

Question 1.1.16. *Is the remark 1.1.15 true? Can we say* $\omega : \ell^1 \times \ell^1 \to \ell^1$ *in definition 1.1.14 is well-defined?*

Recall 1.1.17. *What is Bilinear extension?*

Remark 1.1.18. *The map* $\omega : \ell^1 \times \ell^1 \to \ell^1$ *is the bilinear extension of the given map* $\omega^* : K \times K \to \ell^1_{co}$.

Let us assume that the map $\sim^* : K \to K$, $x \mapsto \tilde{x}$, is given.

Definition 1.1.19. *We define* $\sim : \ell^1 \to \ell^1$ *by setting*

$$f = \sum_{n=1}^{\infty} a_n \varepsilon_{\tilde{x}_n}$$

for $f = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$.

Remark 1.1.20. *The map* $\sim$ *is the extension of* $\sim^*$ *from* K *to* ℓ^1.

Now, the stage is ready to give the definition of a discrete hypergroup.

Note 1.1.21. *The definition of a discrete hypergroup reads much simpler than that of a general hypergroup.*

1.2 Hypergroup Definition

Definition 1.2.1 (Discrete Hypergroup). *We call a triplet* $(K, \omega, \sim)$ *a discrete hypergroups if the following conditions holds.*

(H1) $\omega : K \times K \to \ell^1_{co}$ *is a mapping fulfilling the associativity law*

$$\omega(\varepsilon_x, \omega^*(y,z)) = \omega(\omega^*(x,y), \varepsilon_z)$$

for all $x, y, z \in K$.

(H2) $\sim: K \to K$ *is a bijective mapping such that* $\tilde{\tilde{x}} = x$ *and*

$$\omega(x, y)\tilde{} = \omega(\tilde{y}, \tilde{x})$$

for all $x, y \in K$.

(H3) There exists a (necessarily unique) element $e \in K$ *such that*

$$\omega(e, x) = \varepsilon_x = \omega(x, e)$$

for all $x \in K$.

(H4) We have $e \in \operatorname{supp} \omega(x, \tilde{y})$ *if, and only if* $x = y$.

Remark 1.2.2. *If* $\omega(x, y) = \omega(y, x)$ *for all* $x, y \in K$, *the hypergroup* $(K, \omega, \sim)$ *is called* ***commutative***.

Note 1.2.3.

- *The mapping* ω *(and its extension to* ℓ^1*) is called* ***convolution***.
- *The mapping* $\sim$ *(and its extension to* ℓ^1*) is called* ***involution***.
- *The element* e *is called the* ***unit element*** .
- *In the literature, the convolution* $\omega(x, y)$ *is often written as* $\varepsilon_x * \varepsilon_y$ *or* $\delta_x * \delta_y$.
- *If any confusion can be excluded, we shall use the notation* K *instead of* $(K, \omega, \sim)$.

Recall 1.2.4.

- *A topology on a set* X *may be defined as a collection* τ *of subsets of* X*, called open sets, and satisfying the following axioms:*

  1. *The empty set and* X *itself belong to* τ.
  2. *Any arbitrary (finite or infinite) union of members of* τ *belongs to* τ.
  3. *The intersection of any finite number of members of* τ *belongs to* τ.

  (This definition of a topology is the most commonly used; the set τ *of the open sets is commonly called a topology on* X.*)*

- *Topological groups are the combination of groups and topological spaces.*

- *A topological group G is called a discrete group if there is no limit point in it (i.e., for each element in G, there is a neighborhood that only contains that element). Equivalently, the group G is discrete if and only if its identity is isolated.*

Exercise 1.2.5. *Prove that every discrete group G is also a discrete hypergroup by setting $\omega^*(x,y) = \varepsilon_{xy}$ and $\tilde{x} = x^{-1}$.*

For our further investigations, we need an extension of the convolution ω and the involution $\sim$ to the powerset of K.

Definition 1.2.6. *We define, for $A, B \subseteq K$*

$$\omega(A,B) := \bigcup_{x \in A, y \in B} supp\ \omega(x,y) \tag{1.1}$$

and

$$\tilde{A} = \{\tilde{x} : x \epsilon A\} \tag{1.2}$$

Note 1.2.7. *In definition 1.2.6 the domain of ω is $2^K \times 2^K$.*

Lemma 1.2.8. *Let A, B, and C be subsets of K. Then we have:*

(i) $\omega(\omega(A,B),C) = \omega(A,\omega(B,C))$.

(ii) $\omega(A,B) \cap C = \varnothing$ *if, and only if* $\omega(\tilde{A},C) \cap B = \varnothing$.

Proof. (i) Let $a \in A$, $b \in B$, and $c \in C$ such that $\omega(a,b) = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$. Using the definition, we obtain

$$\omega(\omega(a,b),\varepsilon_c) = \sum_{n=1}^{N} a_n \omega(x_n, c)$$

.

$$\begin{aligned} \text{supp } \omega(\omega(a,b),\varepsilon_c) &= \bigcup_{n=1}^{N} \text{supp } \omega(x_n,c) \\ &= \omega(\text{supp } \omega(a,b),\{c\}). \end{aligned}$$

Now applying the definition 1.2.6 we obtain

$$\begin{aligned}\omega(\omega(\{a\},\{b\}),\{c\}) &= \omega(\text{supp }\omega(a,b),\{c\})\\ &= \text{supp }\omega(\omega(a,b),\varepsilon_c).\end{aligned}$$

In an analogous way we get

$$\omega(\{a\},\omega(\{b\},\{c\})) = \text{supp }\omega(\varepsilon_a,\omega(b,c)).$$

The associativity law (H1) implies

$$\omega(\omega(\{a\},\{b\}),\{c\}) = \omega(\{a\},\omega(\{b\},\{c\})).$$

Now (i) follows easily.

(ii) In view of Axiom (H4), we have $e \in \omega(\tilde{A},B)$ if, and only if there is an $x \in A \cap B$. Axiom (H3) yields

$$\omega(A,B)^{\sim} = \omega(\tilde{B},\tilde{A}).$$

Therefore we have

$$\omega(A,B)\cap C \neq \varnothing \Leftrightarrow e \in \omega(\tilde{C},\omega(A,B)).$$

and since

$$\omega(\tilde{C},\omega(A,B)) = \omega(\omega(\tilde{C},A),B) = \omega(\omega(\tilde{A},C)^{\sim},B)$$

we obtain the equivalence

$$\omega(A,B)\cap C \neq \varnothing \Leftrightarrow \omega(\tilde{A},C)\cap B \neq \varnothing.$$

This completes the proof.

□

Question 1.2.9. *Do you have any previous exposure to Haar measure? What is it?*

1.3 Left and Right translations and Haar measure

Let $f : K \to \mathbb{C}$ be a function, and let $x \in K$.

Definition 1.3.1 (Left translation). *The left-translation, $L_x f : K \to \mathbb{C}$, is defined by*

$$L_x f(y) = \sum_{n=1}^{N} a_n f(u_n)$$

where $\omega^*(x, y) = \sum_{n=1}^{N} a_n \varepsilon_{u_n}$.

Recall 1.3.2. *A measure μ on (X, Σ) is discrete (with respect to ν) if and only if μ has the form*

$$\mu = \sum_{i=1}^{\infty} a_i \delta_{s_i}$$

with $a_i \in \mathbb{R}_{>0}$ and Dirac measures δ_{s_i} on the set $S = \{s_i\}_{i \in \mathbb{N}}$ defined as

$$\delta_{s_i}(X) = \begin{cases} 1 & \text{if } s_i \in X, \\ 0 & \text{if } s_i \notin X. \end{cases}$$

for all $X \in \Sigma$ and $i \in \mathbb{N}$. See

Remark 1.3.3.

- *Keep in mind that in the definition 1.3.1 $N = N(x, y)$ and $a_n = a_n(x, y)$. That is they depend on x and y.*

- *Since $L_x \varepsilon_u(y) = \omega(x, y)(u)$ for each $u \in K$, it makes sense to write $L_x f(y) = \omega(x, y)(f)$.*

- *The notation $\omega(x, y)(f)$ emphasizes that $\omega(x, y)$ can be viewed as a functional or a discrete measure, while the notation $L_x f(y)$ reminds us of the role of operators that the family $(L_x)_{x \in K}$ will play.*

Definition 1.3.4 (right-translation). *The right-translation, $R_x f : K \to \mathbb{C}$, is defined by*

$$R_x f : K \to \mathbb{C}, R_x f(y) = \sum_{n=1}^{M} b_n f(v_n),$$

where now $\omega(y, x) = \sum_{n=1}^{M} b_n \varepsilon_{v_n}$.

Question 1.3.5. • *What do you say about M and a_n in the definition 1.3.4? See remark 1.3.3.*

- *What is the difference in Left and Right translation?*

Recall 1.3.6. *Lemma 1.2.8 (ii).*

Question 1.3.7. *What happens to supp $f \cap$ supp $\omega(x, y)$ when $L_x f(y) \neq 0$?*

Proposition 1.3.8. *For any function $f : K \to \mathbb{C}$ and $x \in K$, one has*

$$\textit{supp}\ L_x f \subseteq \omega(\{\tilde{x}\}, \textit{supp}\ f)$$

In particular, if supp f is finite, then supp $L_x f$ is finite as well. Furthermore, if f is bounded, then $L_x f$ is bounded, and $\|L_x f\|_1 \leq \|f\|_1$ (where, as usual, $\|f\|_1 = \sup_{x \in K} |f(x)|$).

Proof. Left out for later days. Its easy just solve just previous question and see recall. □

Note 1.3.9. *Two mathematical objects a and b are called "equal **up to** an equivalence relation R" if a and b are related by R, that is, if aRb holds, which means that the equivalence classes of a and b with respect to R are equal.*

Recall 1.3.10. ***Haar's theorem:*** *There is, up to a positive multiplicative constant, a unique countably additive, nontrivial measure μ on the Borel subsets of G satisfying the following properties:*

- *The measure μ is left-translation-invariant: $\mu(gS) = \mu(S)$ for every $g \in G$ and all Borel sets $S \subseteq G$.*

- *The measure μ is finite on every compact set: $\mu(K) < \infty$ for all compact $K \subseteq G$.*

- *The measure μ is outer regular on Borel sets $S \subseteq G$: $\mu(S) = \inf\{\mu(U) : S \subseteq U, U \text{ open}\}$.*

- *The measure μ is inner regular on open sets $U \subseteq G$: $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}$.*

Such a measure on G is called a left Haar measure.

Note 1.3.11. *In the discrete case, the Haar measure is completely determined by its values at the points* $x \in K$.

Definition 1.3.12 (left–invariant)**.** *A positive function* $h : K \to [0, \infty[$ *is called left-invariant if, for each* $f : K \to \mathbb{C}$ *with* $|supp\ f| < \infty$ *and* $y \in K$,

$$\sum_{x \in K} L_y f(x) h(x) = \sum_{x \in K} f(x) h(x).$$

Definition 1.3.13 (Haar function)**.** *A left-invariant positive function* $h : K \to [0, \infty[$, $h \neq 0$, *is called a Haar function.*

Remark 1.3.14. *There is an analogous definition of right–invariance.*

Question 1.3.15.

- *Can you define right–invariance?*
- *We have defined Haar function but does there exists such a function in general?*

Note 1.3.16. *For discrete hypergroups the existence of a Haar function is true.*

Theorem 1.3.17.

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