PROJECT REPORT

Simulation of wave propagation using shallow water equations

Manjusha Nair

Geetha Natarajan

Aim:

Simulation of wave propagation using shallow water equations.

Introduction:

Shallow water equations are set of hyperbolic partial difference equations describing the flow below a pressure surface in a fluid. Shallow water equations are used to model waves, that is propagation of disturbance in water, where the wavelength is significantly larger than the depth of the medium.

Shallow water equations are derived from Navier-stroke's equation and this is in the case where horizontal length scale is much greater than the vertical scale. Hence the velocity can be considered negligible where the surface is flat or even and the surface does not change its depth.

Conservative Equations:

Shallow water equations exhibit rich variety of features, because of which they have infinitely many laws of conservation. For example, Non-conservative form and conservative form.

In our project we will be focusing on conservative form of equation. In case of conservative form, shallow water equations are derived from equations of conservation of mass and conservation of linear momentum which is Navier-stroke's equation.

Below equations represent the conservative form.

$$\frac{\partial(\rho\eta)}{\partial t} + \frac{\partial(\rho\eta u)}{\partial x} + \frac{\partial(\rho\eta v)}{\partial y} = 0, \tag{1}$$

$$\frac{\partial(\rho\eta u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho\eta u^2 + \frac{1}{2}\rho g\eta^2\right) + \frac{\partial(\rho\eta uv)}{\partial y} = 0, \tag{2}$$

$$\frac{\partial(\rho\eta v)}{\partial t} + \frac{\partial(\rho\eta uv)}{\partial x} + \frac{\partial}{\partial y}\left(\rho\eta v^2 + \frac{1}{2}\rho g\eta^2\right) = 0. \tag{3}$$

Where:

ρ : fluid density

g : acceleration due to gravity

η : total fluid column height

(u,v) : fluid's horizontal flow velocity averaged across the vertical column

As given, the three equations represent conservative form, in which first formula (1) is derived from conservation of mass, second (2) and third (3) formulas are derived from conservation of linear momentum.

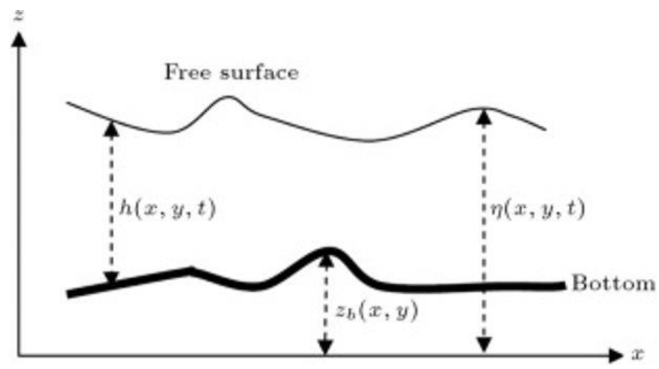

Fig1 : Parameters of shallow water equations in x-direction

Figure1 represent the parameters of shallow water equations in x-direction, where, independent variables are time t and two space co-ordinates x and y. Dependent variables are fluid height or also called as depth h and two dimensional fluid velocity field u and v. With the proper choice of units, the conserved quantities are mass, which is proportional to h and momentum which is proportional to uh and vh. The force acting on the fluid is the gravity represented by gravitational constant g , rho is the fluid density, Eta is total fluid column height, which is nothing but the instantaneous fluid depth as a function of x,y and t.

To simplify the equations three vectors (4) are introduced and by replacing these vectors in the equations (1), (2) and (3) we arrive at the notation (5) which states that the shallow water equations are an instance of a hyperbolic conservation law.

$$
\begin{aligned}
U &= \begin{pmatrix} h \\ uh \\ vh \end{pmatrix} \\
F(U) &= \begin{pmatrix} uh \\ u^2h + \frac{1}{2}gh^2 \\ uvh \end{pmatrix} \\
G(U) &= \begin{pmatrix} vh \\ uvh \\ v^2h + \frac{1}{2}gh^2 \end{pmatrix}
\end{aligned}
\tag{4}
$$

$$
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = 0 \tag{5}
$$

Lax-Wendroff Methods:

There are different ways of solving the shallow-water equations and in our case we will be using Lax-Wendroff method. By solving the equation (4) and (5) numerically by using Lax-Wendroff method we arrive the equations (6), (7) and (8).

$$
Hx_{i+1/2}^{n+1/2} = \frac{H_{i+1}^{n} + H_{i}^{n}}{2} + (2 * \Delta t)\frac{U_{i+1}^{n} - U_{i}^{n}}{\Delta x} \tag{6}
$$

$$
Hy_{j+1/2}^{n+1/2} = \frac{H_{j+1}^{n} + H_{j}^{n}}{2} + (2 * \Delta t)\frac{V_{j+1}^{n} - V_{j}^{n}}{\Delta y} \tag{7}
$$

$$
H_{i,j}^{n+1} = H_{i,j}^{n} + \Delta t\frac{Ux_{i+1/2}^{n+1/2} - Ux_{i-1/2}^{n+1/2}}{\Delta x} + \Delta t\frac{Vy_{i+1/2}^{n+1/2} - Vy_{i-1/2}^{n+1/2}}{\Delta y} \tag{8}
$$

Lax-Wendroff method involves in first calculating a half step represented by equations (6), (7) and then using the result from the half step to calculate the full step which is represented by the equation (8).

(Since H is two-dimensional, the step must be calculated twice for each direction).

Assumptions:

In this project, we will be working on simulation of propagation of waves when the disturbance is introduced in a confined container. For example, figure 2 demonstrate the wave propagation when a droplet is introduced.

Fig2: Waves propagate by introducing droplet.

To achieve this wave propagation, we assume:

a. That we have a confined container with specified boundaries or walls which can be reflective.
b. We assume the topography to be flat.
c. Velocity, frictional force, viscous force to be negligible and
d. We have a constant depth.

(Note that Conservative form is preferred over non-conservative form due to our assumptions in this scenario).

Implementation:

1. We start with analysing the initial variables or parameters (required for equations 1,2 and 3). Below code represent the initial declaration of values and assumptions.

```
%-------------------------------------------------
%Simmulation of Wave Propagation
%using Shallow water Equations


%-------------------------------------------------
    %Initial declarations

% Grid size of confined container
% define the grid size
n = 100;
dx = 0.1;
dy = 0.1;
%gravitational constant
g = 9.8;
%Timestep
time = 0.009;
% Assuming flat topology for our model for initial condition
height = ones(n+2, n+2);
% Initialize the momentum in x and y directions
UXmomentum=zeros(n+2, n+2);
VYmomentum=zeros(n+2, n+2);
```

```
%Creating empty matrix for plotting, calculating momentums and storing
%values
Heightx = zeros(n+1,n+1);
Heighty = zeros(n+1,n+1);
Ux = zeros(n+1,n+1);
Uy = zeros(n+1,n+1);
Vx = zeros(n+1,n+1);
Vy = zeros(n+1,n+1);
[x,y] = meshgrid( linspace(-3,3,10) );
```

As you can see we have initial declarations like gravitational constant, mesh grid, timestep, two dimensional vectors for mass and momentum. Since we have assumed flat topography, initially there is no displacement hence we have declared height which acts as a droplet in our case.

2. Our next step is to proceed with the initial displacement which is nothing but a single droplet in our case. To introduce this displacement as a height factor, we experimented with random values, mathematical operations (for example adding random values, using sin , exponential functions) and finally we were able to achieve good results with log function. Below code represent the creation of initial displacement or droplet.

```
            %Plot the initial displacement or droplet

%------------------------CASE 1-------------------------------
%----------------------Single Displacement--------------------

 % create initial displacement or disturbance of water(droplet)
 height(11:20,21:30) = height(11:20,21:30) + (log(x.^2 + y.^2))./(log(x.^2 + y.^2));
 % Plot initial displacement
 graph = surf(height);
 axis([1 n 1 n 0.5 3]);
 shading interp;
 hold all;
 % plot the momentums and height of water
     title("Simulation of wave propogation using shallow water equation - Single displacement");
     set(graph, 'zdata', height);
     xlabel("Momentum of waves in x-direction");
     ylabel("Momentum of waves in y-direction");
     zlabel("Height of the waves");
```

Below Fig3 represent the single displacement or droplet at timestep = 0.

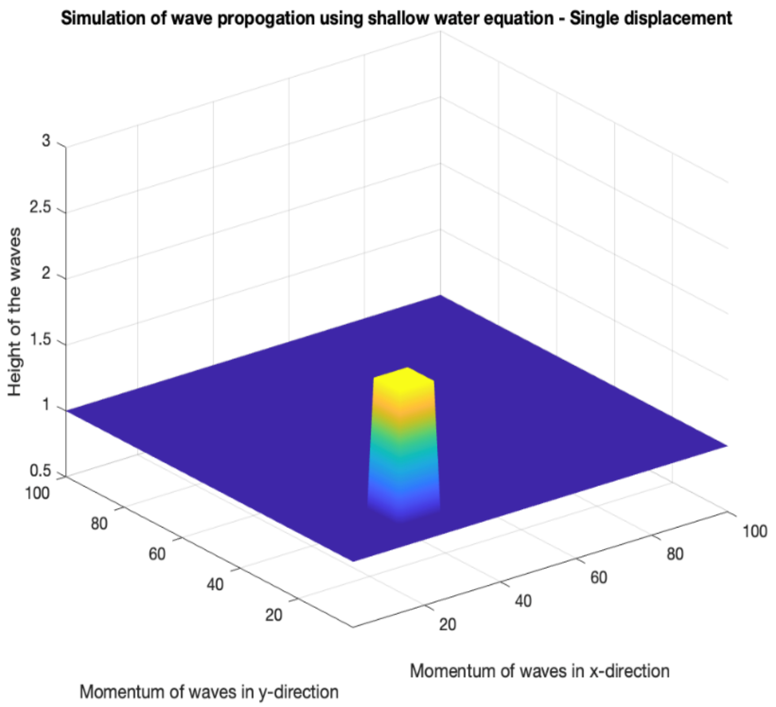

Fig3: Single displacement (or droplet) at the surface of water at time step = 0

3. At this stage we will be using Lax-Wendroff method to simulate the wave propagation. Here we will be first calculating the half step and then we will be using the results from the half step to calculate the full step and then update height and momentum vectors.
   Below code represents the loop where we will be using Shallow water equations to simulate waves.

```
%Using the Conservative equation for wave propogation
for t=1:1000

    %Boundary condition 1 - Lax-Wendroff equation half step calculation
    p = 1:n+1;
    q = 1:n+1;

    % height in x and y directions
    Heightx(p,q) = (height(p+1,q+1)+height(p,q+1))/2 - ...
               time/(2*dx)*(UXmomentum(p+1,q+1)-...
               UXmomentum(p,q+1));
    Heighty(p,q) = (height(p+1,q+1)+height(p+1,q))/2 - ...
              time/(2*dy)*(VYmomentum(p+1,q+1)-...
              VYmomentum(p+1,q));

    % waves in x and y direction/momentum
    Ux(p,q) = (UXmomentum(p+1,q+1)+UXmomentum(p,q+1))/2 - ...
              time/(2*dx)*(UXmomentum(p+1,q+1).^2./height(p+1,q+1) - ...
              UXmomentum(p,q+1).^2./height(p,q+1) + ...
              g/2*height(p+1,q+1).^2 - g/2*height(p,q+1).^2);
```

```
Uy(p,q) = (UXmomentum(p+1,q+1)+UXmomentum(p+1,q))/2 - ...
          time/(2*dy)*( (VYmomentum(p+1,q+1).*UXmomentum(p+1,q+1)./height(p+1,q+1)) - ...
          (VYmomentum(p+1,q).*UXmomentum(p+1,q)./height(p+1,q)) );
Vx(p,q) = (VYmomentum(p+1,q+1)+VYmomentum(p,q+1))/2 - ...
           time/(2*dx)*((UXmomentum(p+1,q+1).*VYmomentum(p+1,q+1)./height(p+1,q+1)) - ...
           (UXmomentum(p,q+1).*VYmomentum(p,q+1)./height(p,q+1)));

Vy(p,q) = (VYmomentum(p+1,q+1)+VYmomentum(p+1,q))/2 - ...
           time/(2*dy)*((VYmomentum(p+1,q+1).^2./height(p+1,q+1) + ...
           g/2*height(p+1,q+1).^2) - (VYmomentum(p+1,q).^2./height(p+1,q) + ...
           g/2*height(p+1,q).^2));
```

```
%Boundary condition2
p = 2:n+1;
q = 2:n+1;

% height in x and y direction
height(p,q) = height(p,q) - (time/dx)*(Ux(p,q-1)-Ux(p-1,q-1)) - (time/dy)*(Vy(p-1,q)-Vy(p-1,q-1));

% x-axis momentum
UXmomentum(p,q) = UXmomentum(p,q) - (time/dx)*((Ux(p,q-1).^2./Heightx(p,q-1) + g/2*Heightx(p,q-1).^2) - ...
        (Ux(p-1,q-1).^2./Heightx(p-1,q-1) + g/2*Heightx(p-1,q-1).^2)) ...
         - (time/dy)*((Vy(p-1,q).*Uy(p-1,q)./Heighty(p-1,q)) - ...
        (Vy(p-1,q-1).*Uy(p-1,q-1)./Heighty(p-1,q-1)));
```

```
    % y-axis momentum
    VYmomentum(p,q) = VYmomentum(p,q) - (time/dx)*((Ux(p,q-1).*Vx(p,q-1)./Heightx(p,q-1)) - ...
            (Ux(p-1,q-1).*Vx(p-1,q-1)./Heightx(p-1,q-1))) ...
            - (time/dy)*((Vy(p-1,q).^2./Heighty(p-1,q) + g/2*Heighty(p-1,q).^2) - ...
            (Vy(p-1,q-1).^2./Heighty(p-1,q-1) + g/2*Heighty(p-1,q-1).^2));

    % plot the momentums and height of water
    title("Simulation of wave propogation using shallow water equation - Single displacement");
    set(graph, 'zdata', height);
    xlabel("Momentum of waves in x-direction");
    ylabel("Momentum of waves in y-direction");
    zlabel("Height of the waves");
    grid off;
    drawnow

end
```

Below figures demonstrates the wave propagation at different time step.

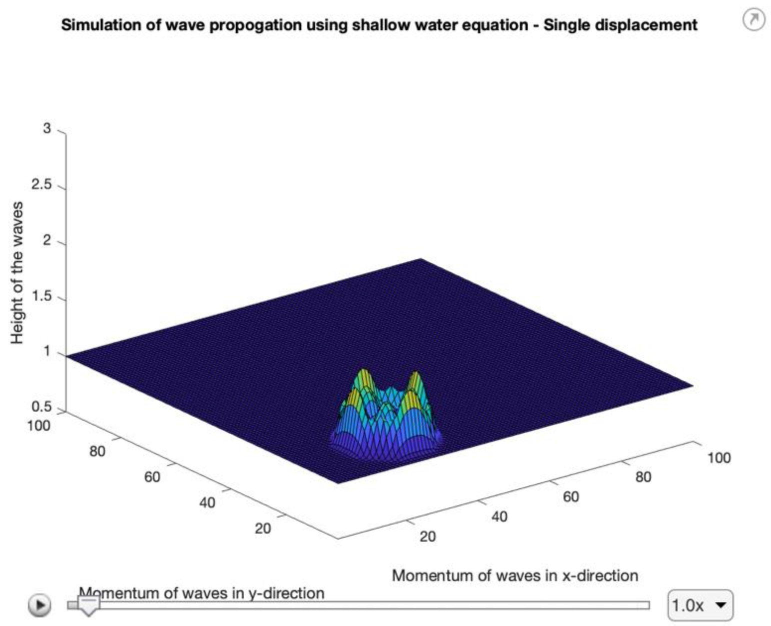

Fig4 : Time instance when disturbance is occurred in turn resulting in splash view

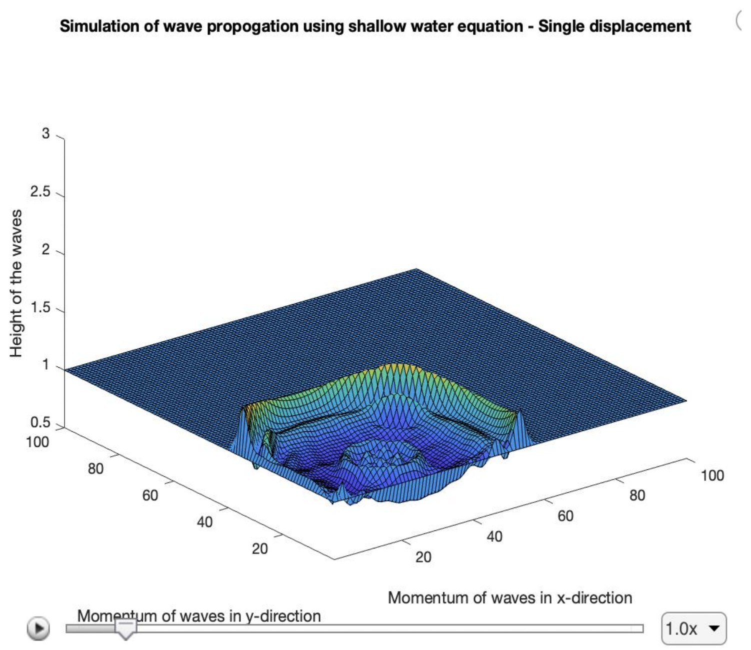

Fig5 : Time instance when the waves start to propagate

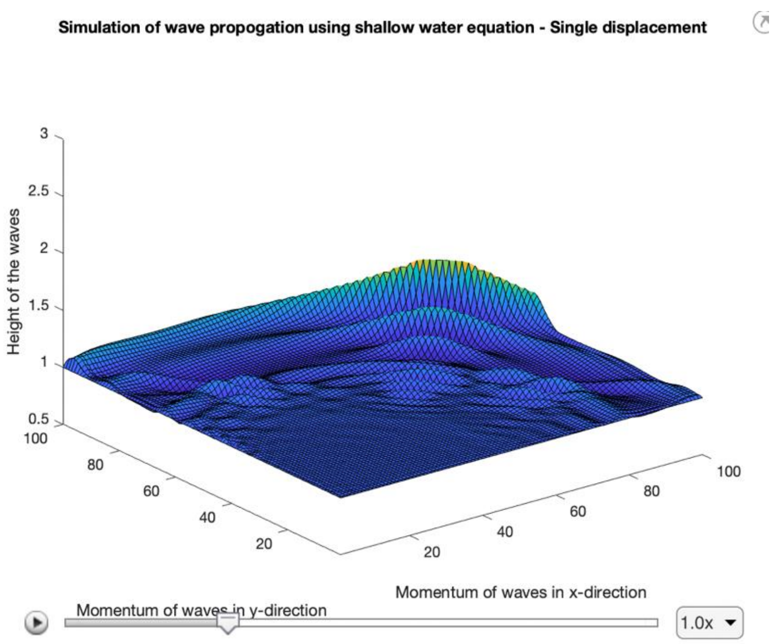

Fig6 : Time instance where waves propagates to walls of the container

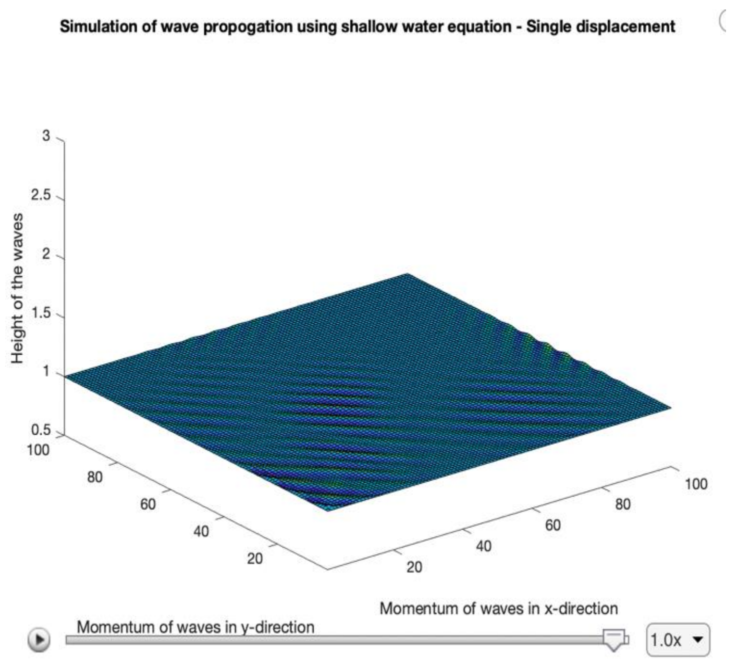

Fig7 : Time instance at the end of the propagation of the wave

4. Further to test the numerical stability and to analyse on how the Shallow Water Equations work, we introduced multiple droplets or multiple displacements as height vector at different positions.

   Below code demonstrate the use of height variable to add three displacements (three droplets) in the container.

```
%-------------------CASE 2---------------------------
%-----------------Multiple Displacement-------------

% create initial displacement or disturbance of water(droplet)
height(2:5,5:10) = height(2:5,5:10) + 1;
height(25:35,35:45) = height(25:35,35:45) + 1;
height(60:70,70:80) = height(60:70,70:80) + 1;
% Plot initial displacement
graph = surf(height);
axis([1 n 1 n 0.5 3]);
colormap winter;
shading interp;
hold all;
%plot the momentums and height of water
     title("Simulation of wave propogation using shallow water equation - multiple displacement");
     set(graph, 'zdata', height);
     xlabel("Momentum of waves in x-direction");
     ylabel("Momentum of waves in y-direction");
     zlabel("Height of the waves");
```

Below Fig8, represents three droplets or displacement at time step = 0

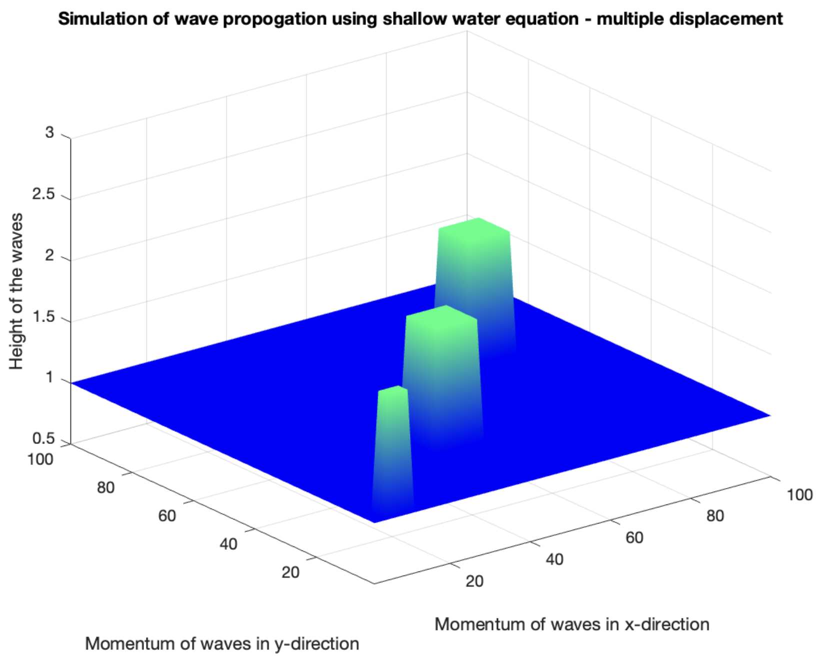

Fig8 : Multiple displacement (or droplet) at the surface of water at time step = 0

Below figures demonstrates the wave propagation at different time step.

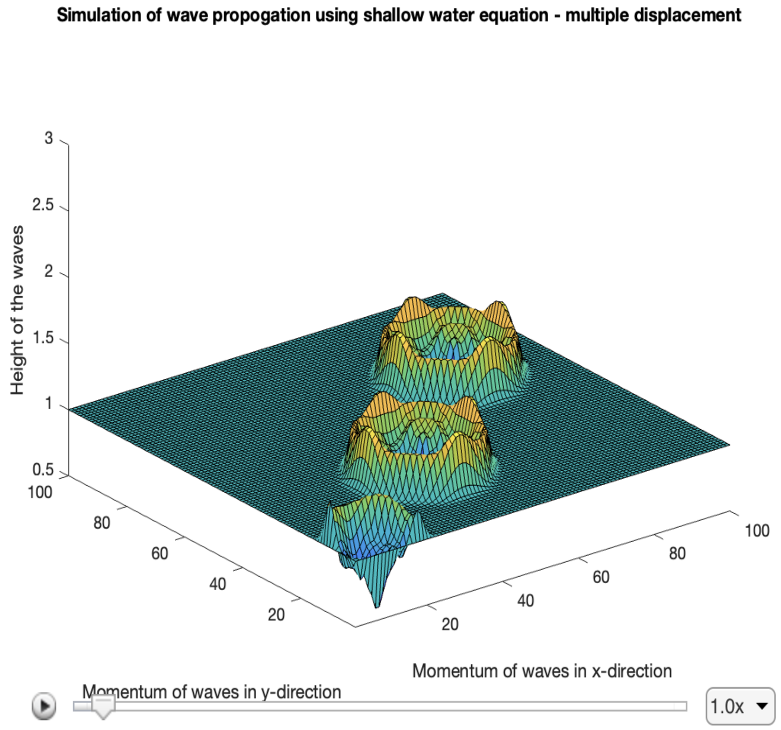

Fig9: Time instance when disturbance is occurred in turn resulting in splash view

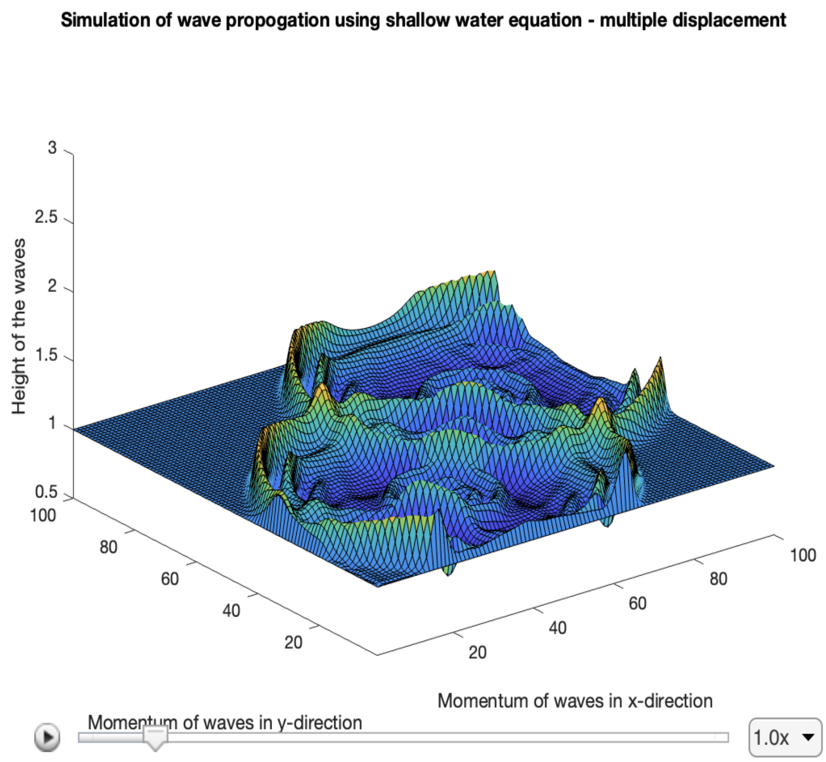

Fig10: Time instance when the waves start to propagate

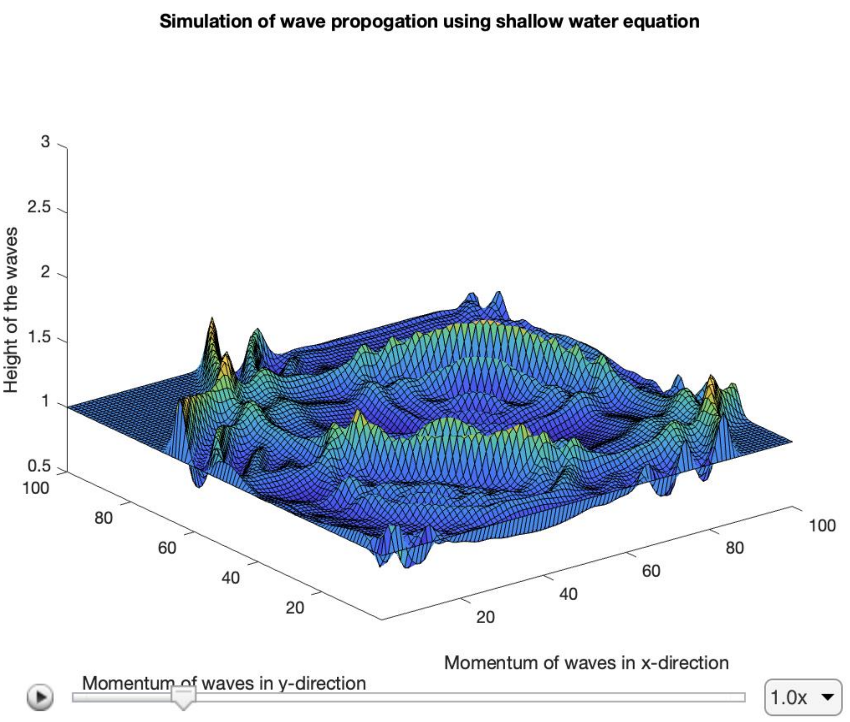

Fig11: Time instance when the waves start to propagate (other view)

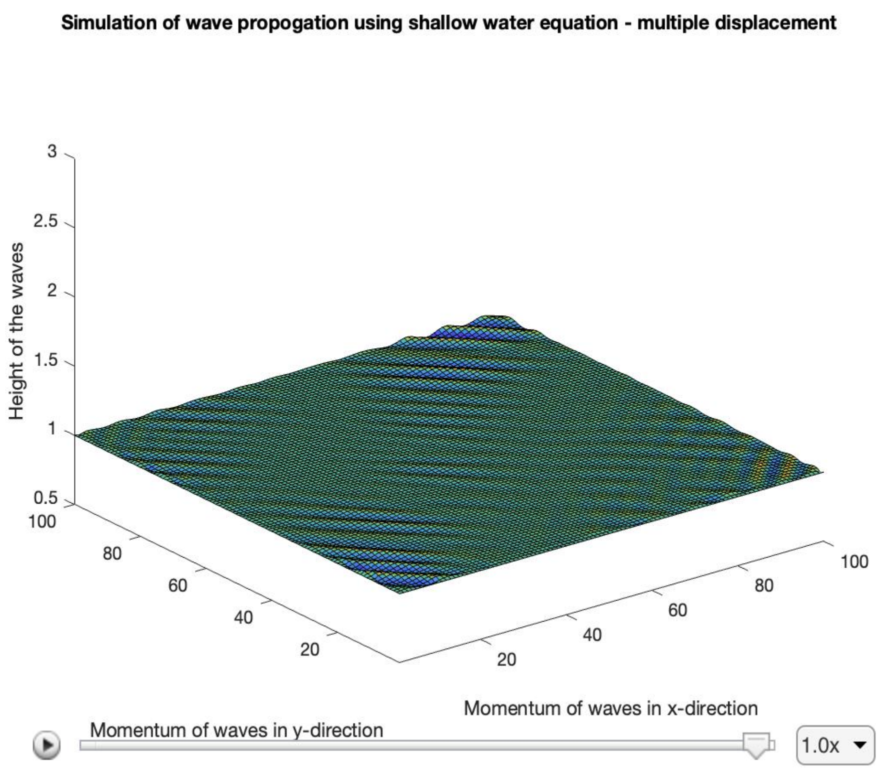

Fig12 : Time instance at the end of the propagation of the wave

Other Approaches:

We want to highlight the different approaches we tried and unfortunately we decided not to include these below approaches since we were not satisfied with the results.

1. Once we had successfully completed wave propagation for single displacement, we wanted to raise the bar and we tried to drop a object or ball in the container. In this case we were able to drop a ball in 2D view, but since we had 3D simulation of waves, introducing 2D ball was complex and we were not satisfied on how the results looked.
2. Second approach we tried to change the shape of the container, we tried to add sphere shape and we had issues in setting the boundary conditions and results were not stable.
3. In third approach we tried to place an object in the container and again we were not happy with the results.

Conclusion:

Finally after trying all the different approaches we finally achieved good result using Lax-Wendroff method of solving Shallow Water Equations numerically. We were able to introduce single and multiple droplets and was able to successfully demonstrate propagation of waves in a confined container with flat topography and constant depth using shallow water equation.

References:

1. https://en.wikipedia.org/wiki/Shallow_water_equations
2. https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations
3. https://en.wikipedia.org/wiki/Momentum#Conservation
4. https://www.mathworks.com/content/dam/mathworks/mathworks-dot-com/moler/exm/book.pdf
5. http://colinrrobinson.com/wpcontent/uploads/2012/03/ShallowWaterEquation
6. https://www.mathworks.com/content/dam/mathworks/mathworks-dot-com/moler/exm/chapters/water.pdf
7. https://www.whoi.edu/fileserver.do?id=136564&pt=10&p=85713