Homework 1

Deadline: 09-30-2021

Question 1. A biased four-sided die is rolled, and the down face is a random variable X described by the following probability mass function (pmf).

$$p(x) = \begin{cases} \frac{x}{10}, & x = 1, 2, 3, 4. \\ 0, & otherwise \end{cases}$$

Given the random variable X a biased coin is flipped and the random variable Y is 1 or zero according to whether the coin shows heads or tails. The conditional pmf is

$$p(y|x) = \left(\frac{x+1}{2x}\right)^y \left(1 - \frac{x+1}{2x}\right)^{1-y},$$

where $y \in \{0, 1\}$.

a) Find the expectation $E(X)$ and the variance $Var(X)$. [5 pts]
b) Find the conditional pmf $p(x|y)$. [5 pts]
c) Find the conditional expectation $E[X|Y = 1]$; i.e., the expectation with respect to the conditional pmf $p(x|y = 1)$. [5 pts]

Question 2. Show that if two variables x and y are independent, then their covariance is zero. [5 pts]

Question 3. Let X be a random variable on X = {a, b, c} with the probability mass function p(x). Let p(a) = 0.1, p(b) = 0.2, and p(c) = 0.7 and some function f(x) be

$$f(x) = \begin{cases} 10\ ;\ x = a \\ 5\ ; x = b \\ \frac{10}{7}\ ; x = c \end{cases}$$

a) What is E[f(X)]? [5 pts]

b) What is E[1/p(X)]? [5 pts]

Question 4. Let $X_1, X_2, X_3, \dots, X_n$ be a random sample from a normal distribution with unknown mean μ and variance σ^2.

a) Find maximum likelihood estimators of mean μ and variance σ^2. [10 pts]
b) Are the MLEs unbiased for their respective parameters? [10 pts]

Question 5. Show that if two random variables X and Y are independent, then E[XY] = E[X]E[Y] – [5 pts]

Question 6. Conditional independence

a) Let H ∈ {1, . . . , K} be a discrete random variable, and let e1 and e2 be the observed values of two other random variables E1 and E2. Suppose we wish to calculate the vector P(H|e1, e2) = (P(H = 1|e1, e2), . . . , P (H = K|e1, e2)) [5 pts]

   Which of the following sets of numbers are sufficient for the calculation?
   i. P(e1, e2), P(H), P(e1|H), P(e2|H)
   ii. P(e1, e2), P(H), P(e1, e2|H)
   iii. P(e1|H), P(e2|H), P(H)

b) Now suppose we now assume E1 ⊥ E2|H (i.e., E1 and E2 are conditionally independent given H).
   Which of the above 3 sets are sufficient now? [5 pts]
   Show your calculations as well as giving the final result. Hint: use Bayes rule.

Question 7. Show that the variance of a sum is var[X + Y] = var[X] + var[Y] + 2cov[X, Y], where cov[X, Y] is the covariance between X and Y. [10 pts]

Question 8. Let $D = \{x_1, x_2, x_3, \dots, x_n$ be random sample with joint probability

$$p(x|\theta_0) = \begin{cases} e^{-\sum_{i=1}^{n}(x_i-\theta_0)}, & x \geq \theta_0. \\ 0, & otherwise \end{cases}$$

Determine θ_{MLE} the maximum likelihood estimate of θ_0. [15 pts]

Question 9. Suppose A is a real $m \times n$ matrix.

a) Prove that the symmetric matrix $A^T A$ has the property $x^T(A^T A)x \geq 0$ for every vector x in R^n. [5 pts]
b) According to part a), the matrix $A^T A$ is positive semidefinite at least and possibly positive definite. Under what condition A on is $A^T A$ positive definite? [5 pts]