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9/29/2021 Homework - 1

Q1:

a) $E(x) = \int_{-\infty}^{\infty} x\, p(x)\, dx$

Now given is, $p(x) = \begin{cases} \frac{x}{10}, & x = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$

$\therefore E(x) = \sum_{x=1}^{4} x\, p(x)$

$\therefore E(x) = 1 \cdot \frac{1}{10} + 2 \cdot \frac{2}{10} + 3 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10}$

$\therefore E(x) = \frac{1}{10} + \frac{4}{10} + \frac{9}{10} + \frac{16}{10} = \frac{30}{10} = 3$

Hence the expected value of X is 3.

$Var(x) = E[x^2] - E[x]^2$

$\therefore Var(x) = \sum_{x=1}^{4} x^2 \cdot p(x) - (3)^2$

$\cancel{Var(x) = 1^2 \cdot \frac{1}{10} + 2 \cdot \frac{2^2}{10}} + 3$

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$$\therefore Var(X) = \left(1^2 \cdot \frac{1}{10} + 2^2 \cdot \frac{2}{10} + 3^2 \cdot \frac{3}{10} + 4^2 \cdot \frac{4}{10}\right) - 9$$

$$Var(X) = 10 - 9 = 1$$

$$\boxed{\text{Hence, } E[X] = 3 \text{ and } Var(X) = 1}$$

b)

$$P(x|y) = \frac{P(x) \cdot P(y|x)}{\sum_{x=1}^{4} P(x) \cdot P(y|x)}$$

$$\therefore P(x|y) = \frac{\frac{x}{10}\left(\frac{x+1}{2x}\right)^{y}\left(1-\frac{x+1}{2x}\right)^{1-y}}{\sum_{x=1}^{4} \frac{x}{10}\left(\frac{x+1}{2x}\right)^{y}\left(1-\frac{x+1}{2x}\right)^{1-y}}$$

Now, we create a PMF table to calculate the marginal probabilities.

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$x \backslash y$	0	1	$P(X=x)$
1	0	1/10	1/10
2	1/20	3/20	1/5
3	1/10	1/5	3/10
4	3/20	1/4	2/5
$P(Y=y)$	$\frac{3}{10}$	$\frac{7}{10}$	

$\therefore P(x \mid y=0) =$ 0 for $x = 1$

$\frac{1}{6}$ for $x = 2$

$1/3$ for $x = 3$

$1/2$ for $x = 4$

$P(x \mid y=1) =$ $1/7$ for $x = 1$

$\frac{3}{14}$ for $x = 2$

$2/7$ for $x = 3$

$\frac{5}{14}$ for $x = 4$

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(c) Find, $E[X|Y=1]$,

$$\therefore E[X|Y=1] = \sum_{\substack{x \in \{1,2,3,4\} \\ y \in \{1\}}} x \cdot P(x|y=1)$$

$$= \frac{1}{7} + \frac{2 \times 3}{14} + \frac{3 \times 2}{7} + \frac{4 \times 5}{14}$$

$$= \frac{1}{7} + \frac{3}{7} + \frac{6}{7} + \frac{10}{7}$$

$$= \frac{20}{7}$$

$\therefore$ Conditional Expectation $= \frac{20}{7}$

Q2 :

We define covariance for two random variables as ;

$$Cov[X, Y] = E[(X - E[X])(Y - E[Y])]$$

$$= E[XY - XE[Y] - YE[X] + E[X]E[Y]]$$

$$= E[XY] - E[X]E[Y] - \cancel{E[Y]E[X]} + \cancel{E[X]E[Y]}$$

$$= E[XY] - E[X]E[Y]$$

now since X and Y are independent,

$$E[XY] = E[X][Y]$$

(the expected value of Y in no way affect the expected value of X since they are independent)

$$= E[X]E[Y] - E[X]E[Y]$$

$$\therefore Cov[X, Y] = 0$$

Hence Proved.

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Q3 Given, a R.V $X = \{a, b, c\}$ with PMF $p(x)$.

$p(a) = 0.1$, $p(b) = 0.2$, $p(c) = 0.7$

$$f(x) = \begin{cases} 10 & x = a \\ 5 & x = b \\ \frac{10}{7} & x = c \end{cases}$$

a) $$E[F(X)] = \sum_{x \in Val(X)} f(x) \cdot p(x)$$

$$= f(a) \cdot p(a) + f(b) \cdot p(b) + f(c) \cdot p(c)$$

$$= 10 \times 0.1 + 5 \times 0.2 + \frac{10}{7} \times 0.7$$

$$= 1 + 1 + 1$$

$$\therefore \boxed{E[f(x)] = 3}$$

b) $$E\left[\frac{1}{P(X)}\right] = \sum_{x \in Val(X)} \frac{1}{P(X)} \cdot p(x)$$

$$= \frac{1}{p(a)} p(a) + \frac{1}{p(b)} p(b) + \frac{1}{p(c)} p(c)$$

$$\boxed{\therefore E\left[\frac{1}{P(X)}\right] = 3}$$

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Q4.

(a) Assuming that $x_1, x_2, \ldots x_n$ are i.i.d

$$x_1, x_2 \ldots x_n \sim N(\mu, \sigma^2)$$

$$\therefore p(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Calculating the likelihood,

$$ML = \prod_{i=1}^{N} p(x_i) = \prod_{i=1}^{N} \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$\therefore ML = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\mu)^2\right)}$$

taking log on both sides,

$$\log(ML) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{2}\log(2\pi\sigma^2)$$

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Let us differentiate w.r.t μ to find its maximum likelihood:

$$\frac{\partial \log(ML)}{\partial \mu} = 0$$

$$\cancel{\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu) - 0 = 0}$$

$$\cancel{\sum_{i=1}^{n} (x_i - \mu) = 0}$$

$$\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu) - 0 = 0$$

$$\therefore \sum_{i=1}^{n} (x_i - \mu) = 0$$

$$\therefore \sum_{i=1}^{n} x_i - n\mu = 0$$

$$\boxed{\begin{aligned} \therefore \mu &= \frac{\sum_{i=1}^{n} x_i}{n} \\ \therefore \mu &= \bar{x} \end{aligned}}$$

Maximum Likelihood Estimator of μ is the sample mean $\bar{x}$

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Let us differentiate w.r.t σ^2 to find its MLE.

$$\frac{\partial(\log(ML))}{\partial\sigma^2} = 0$$

$$\therefore \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{\cancel{2}}\cdot\frac{1}{2\cancel{\pi}\sigma^2}\cdot 2\cancel{\pi} = 0$$

$$\therefore \frac{1}{\cancel{2}\sigma^{4}}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{n}{\cancel{2}\sigma^2}$$

$$\therefore \sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$$

Now we know that MLE of μ is $\bar{x}$ so we can write ~~μ insto~~ $\bar{x}$ instead of μ.

$$\boxed{\therefore \text{MLE of } \sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}}$$

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(b) The MLE for μ is unbiased, let us see how :

we can say that $\bar{x}$ is unbiased estimator of μ if,

$$E[\bar{x}] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$$

$$\frac{1}{n} E\left[\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E[x_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu \quad (\text{since } E[x_i] = \mu)$$

$$= \frac{1}{n} \times n\mu$$

$$\therefore E[\bar{x}] = \mu$$

Since $E[\bar{x}]$ is the population parameter μ, we can say that the MLE of μ is unbiased.

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The MLE of the variance σ^2 is a biased (d) ~~est~~ estimator, let us see how :

$$E[s^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$$

$$\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}\right) = E\left[\frac{1}{n}\sum_{i=n}^{n}\left((x_i-\mu)-(\bar{x}-\mu)\right)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=n}^{n}\left((x_i-\mu)^2+(\bar{x}-\mu)^2 - 2(x_i-\mu)(\bar{x}-\mu)\right]\right.$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}\left((x_i-\mu)^2\right)+\frac{1}{n}(\bar{x}-\mu)^2\sum_{i=1}^{n}1 - \frac{2}{n}(\bar{x}-\mu)\sum_{i=1}(x_i-\mu)\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{2}{n}(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\mu) + (\bar{x}-\mu)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{2}{n}(\bar{x}-\mu)\cdot n(\bar{x}-\mu) + (\bar{x}-\mu)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 - (\bar{x}-\mu)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right] - E\left[(\bar{x}-\mu)^2\right]$$

$$= \sigma^2 - E\left[(\bar{x}-\mu)^2\right]$$

$\therefore$

$$\therefore E[s^2] < \sigma^2$$

Hence σ^2 is an ~~unbiased~~ biased estimator.

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Q5. We define $E[XY]$ as,

$$E[XY] = \int_y \int_x (x \cdot y) \cdot P(x,y) \cdot dx \cdot dy$$

now since x and y are independent variables,

$$P(x,y) = p(x) \cdot p(y)$$

$$\therefore E[XY] = \int_y \int_x (x \cdot y) \cdot p(x) \cdot p(y) \cdot dx \cdot dy$$

$$= \int_y \int_x (x \cdot p(x)) dx \cdot y \cdot p(y) \cdot dy$$

$$= \int_y E[x] \cdot y \cdot p(y) \cdot dy$$

$$= E[x] \cdot \int_y y \cdot p(y) \cdot dy$$

$$E[XY] = E[X] E[Y]$$

Hence Proved.

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Q6.

(a) Since we don't know if E_1, E_2 and H are independent or not,

$$P(H|e_1, e_2) = \left(P(H=1|e_1, e_2) \ldots P(H=k|e_1, e_2)\right)$$

$$\Rightarrow P(H|E_1, E_2) = \frac{P(E_1, E_2|H) \cdot P(H)}{P(E_1, E_2)}$$

(Bayes Rule)

Therefore we need to know,

$P(E_1, E_2|H)$, $P(H)$, $P(e_1, e_2)$

Hence the answer is option (ii)

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b) Now, assuming E_1 and E_2 are independent, the above equations become

$$P(H|E_1, E_2) = \frac{P(E_1|H)\,P(E_2|H)\,P(H)}{\underbrace{P(E_1)\,P(E_2)}}$$

↳ since E_1 and E_2 are independent.

$\boxed{\therefore \text{Option number (i) is sufficient}}$.

Q7. Show that $var[X+Y] = var[X] + var[Y] + 2cov[X, Y]$

Now definition of $var(K) = E[(K - E(K))^2]$

$$\therefore var[x+y] = E[((x+y) - E[x+y])^2]$$

$$= E[((x+y) - (E[x] + E[y]))^2]$$

↳ Property of E,
$E(A+B) = E[A] + E[B]$
linearity of expectation

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$$= E\left[\left((x - E[x]) + (y - E[y])\right)^2\right]$$

$$= E\left[(x - E[x])^2 + (y - E[y])^2 + 2(x - E[x]) \cdot (y - E[y])\right]$$

$$= E\left[(x - E[x])^2\right] + E\left[(y - E[y])^2\right] + 2E\left[(x - E[x])(y - E[y])\right]$$

$$\boxed{\therefore Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)}$$

Hence Proved

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Q9.

a) P.T $x^T(A^TA)x \geq 0$ for every $x \in R^n$ where A is m×n real matrix.

$$\Rightarrow (x^TA^T)(Ax) \geq 0 \quad \left[\begin{array}{l}\text{since } A \text{ is } m\times n, \\ A^TA \text{ is } n\times n\end{array}\right]$$

$$\Rightarrow (Ax)^T(Ax) \geq 0$$

$$\Rightarrow \|Ax\|_2 \geq 0$$

(Ax is a n-dim vector and $(Ax)^T(Ax)$ is just the inner prod/norm)

~~No~~ Now we know that the norm is always ≥ 0 so this equation holds.

Hence Proved.

b) For A^TA to be Positive Definite,

$$x^T(A^TA)x > 0$$

~~i.e~~ $\Rightarrow (Ax)^T(Ax) > 0$

~~The eigen~~ For $Ax > 0$, the eigen values of A needs to greater than zero.

i.e $Ax = \lambda x$

λ here is the eigen value of A, and x is the eigen vector.

If $\forall i = 1 \ldots n$, $\lambda_i > 0$, A is positive definite.

So for A^TA to be PD, A needs to ~~have all eigen values greater~~ be PD.