1. Figuring Out the Tips

a) In the case of 6 guests, the answer is $\boxed{\text{no.}}$

For example, 1) If all guests give 3 as their tip, then each plate will have a tip of 3

2) If G_1, G_3, G_5 give tips of 4 and G_2, G_4, G_6 give tips of 2, then each plate will have a tip of 3. So, that's why the answer is no.

b) $\boxed{\text{Yes}}$, in the case of 5 guests you can figure out each guest tips.

$$\begin{cases} \frac{1}{2}G_1 + \frac{1}{2}G_5 = P_1 \\ \frac{1}{2}G_1 + \frac{1}{2}G_2 = P_2 \\ \frac{1}{2}G_2 + \frac{1}{2}G_3 = P_3 \\ \frac{1}{2}G_3 + \frac{1}{2}G_4 = P_4 \\ \frac{1}{2}G_4 + \frac{1}{2}G_5 = P_5 \end{cases} \quad \left\{ \begin{array}{ccccc|c} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & P_1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & P_2 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & P_3 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & P_4 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & P_5 \end{array} \right]$$

By looking at the columns of this matrix, we can see that they're linearly independent. So, this matrix is invertible and only a unique solution exists.

c) When n is an even number, there's multiple ways to set guest tips, such that they cancel out or add up to a common plate tip. But, that is not the case when n is an odd number In other words, we can figure out everyone's tip when there are odd number of guests and can't when there are even.

2. Show It!

a) Let $\vec{x}_1$ and $\vec{x}_2$ be unique solutions to $A\vec{x} = \vec{b}$

That means,

$A\vec{x}_1 = \vec{b}$ $\qquad$ $A\vec{x}_2 = \vec{b}$

$$A\vec{x}_1 = A\vec{x}_2$$
$$A(\vec{x}_1 - \vec{x}_2) = \vec{0}$$
$$[\vec{c}_1 \ \vec{c}_2 \ \dots \ \vec{c}_n](\vec{x}_1 - \vec{x}_2) = \vec{0}$$

Since $\vec{x}_1 \neq \vec{x}_2$, $\vec{x}_1 - \vec{x}_2 \neq 0$

Let $\vec{u} = \vec{x}_1 - \vec{x}_2$

$$[\vec{c}_1 \ \vec{c}_2 \ \dots \ \vec{c}_n]\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \vec{0}$$

$$u_1\vec{c}_1 + u_2\vec{c}_2 + \dots + u_n\vec{c}_n = \vec{0}$$

Let $u_1 \neq 0$, then

$$\vec{c}_1 = -\frac{u_2}{u_1}\vec{c}_2 + \dots + \frac{u_n}{u_1}\vec{c}_n$$

Thus, A is <u>linearly dependent.</u>

c) Given that set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a linearly dependent set in $\mathbb{R}^n$, we know that

For some $a_1, a_2, \dots, a_k \in \mathbb{R}$

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k = \vec{0}$$

Let A be any $n \times n$ matrix

$$A(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k) = A(\vec{0})$$
$$A(a_1\vec{v}_1) + A(a_2\vec{v}_2) + \dots + A(a_k\vec{v}_k) = \vec{0}$$
$$a_1 A\vec{v}_1 + a_2 A\vec{v}_2 + \dots + a_k A\vec{v}_k = \vec{0}$$

The equation still holds true, meaning that some vector is $\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_k\}$ can be represented by a linear combination of some other vector in the set.

Thus, the set $\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_k\}$ is <u>linearly dependent.</u>

3. Quadrocopter Transformations

a)

$$R_x(30^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(30^\circ) & -\sin(30^\circ) \\ 0 & \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$R_z(60^\circ) = \begin{bmatrix} \cos(60^\circ) & -\sin(60^\circ) & 0 \\ \sin(60^\circ) & \cos(60^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+0+0 & 0-\frac{3}{4}+0 & 0+\frac{\sqrt{3}}{4}+0 \\ \frac{\sqrt{3}}{2}+0+0 & 0+\frac{\sqrt{3}}{4}+0 & 0-\frac{1}{4}+0 \\ 0 & \frac{1}{2} & 0+0+\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

resulting matrix

b) We'll multiply the 2 matrices now in reverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+0+0 & -\frac{\sqrt{3}}{2}+0+0 & 0 \\ 0+\frac{3}{4}+0 & 0+\frac{\sqrt{3}}{4}+0 & -\frac{1}{2} \\ 0+\frac{\sqrt{3}}{4}+0 & 0+\frac{1}{4}+0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

computed matrix

c) Intended: $\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{4} \\ \frac{3\sqrt{3}-1}{4} \\ \frac{\sqrt{3}+1}{2} \end{bmatrix}$

Actual: $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{4} \\ \frac{1+3\sqrt{3}}{4} \end{bmatrix}$

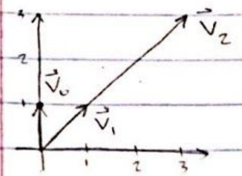
The expected and actual positions are <u>different</u>.

d) <u>$\|\vec{3}\| = 1$</u>. The distance is still the <u>same</u>.

This is because rotation matrices don't change the magnitude of the position vector.

4. Image Stitching

a) $\vec{V}_2 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{V}_0 + \vec{V}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$



$\vec{V}_2$ was scaled and rotated from $\vec{V}_0$

b) $\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} \longrightarrow \begin{cases} q_x = R_{xx} P_x + R_{xy} P_y + T_x \\ q_y = R_{yx} P_x + R_{yy} P_y + T_y \end{cases}$

There are 6 unknowns. You need at least 6 independent equations to solve for all unknowns. You need 3 pairs of common points $\vec{p}$ and $\vec{q}$.

c) $\begin{cases} q_{1x} = R_{xx} P_{1x} + R_{xy} P_{1y} + T_x \\ q_{1y} = R_{yx} P_{1x} + R_{yy} P_{1y} + T_y \\ q_{2x} = R_{xx} P_{2x} + R_{xy} P_{2y} + T_x \\ q_{2y} = R_{yx} P_{2x} + R_{yy} P_{2y} + T_y \\ q_{3x} = R_{xx} P_{3x} + R_{xy} P_{3y} + T_x \\ q_{3y} = R_{yx} P_{3x} + R_{yy} P_{3y} + T_y \end{cases}$

$$\begin{bmatrix} P_{1x} & P_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & P_{1x} & P_{1y} & 0 & 1 \\ P_{2x} & P_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & P_{2x} & P_{2y} & 0 & 1 \\ P_{3x} & P_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & P_{3x} & P_{3y} & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{xx} \\ R_{xy} \\ R_{yx} \\ R_{yy} \\ T_x \\ T_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

vector of unknowns

e) I expected the matrix to linearly dependent and that's what I got as the output.
This is because since $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ form a straight line, you can represent $\vec{p}_2$ and $\vec{p}_3$ using $\vec{p}_1$. Thus, the matrix is linearly dependent

5. Properties of Pump Systems

a) $\begin{cases} x_b[n+1] = 0 \\ x_a[n+1] = x_b[n] + x_a[n] \end{cases}$

b) $\vec{x}[n+1] = A\vec{x}[n]$

$$\begin{bmatrix} x_a[n+1] \\ x_b[n+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_a[n] \\ x_b[n] \end{bmatrix}$$

c) 1) $x_a[0] = 0.5$, $x_b[0] = 0.5$

$$\begin{bmatrix} x_a[1] \\ x_b[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2) $x_a[0] = 0.3$, $x_b[0] = 0.7$

$$\begin{bmatrix} x_a[1] \\ x_b[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

d) No, this is because there's infinite number of solutions to the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

e) No, you can't decide which initial state you started with by just observing $\vec{x}[1]$.

What we can tell about matrix A is that:

1) A is not invertible
2) A has infinite number of solutions

Proof by contradiction:

Let $\vec{x}_1[0], \vec{x}_2[0]$ be 2 distinct initial states that lead to same $\vec{x}[1]$, then

$$A\vec{x}_1[0] = A\vec{x}_2[0]$$
$$A^{-1}(A\vec{x}_1[0]) = A^{-1}(A\vec{x}_2[0])$$
$$\vec{x}_1[0] = \vec{x}_2[0] \leftarrow \text{we got a contradiction}$$

So, A is not invertible and thus, has ∞ # of solutions

f) $\begin{cases} x_1[n+1] = 0 \\ x_2[n+1] = 0.4x_1[n] + 0.5x_2[n] + 0.2x_3[n] \\ x_3[n+1] = 0.6x_2[n] + 0.35x_3[n] \end{cases}$

$\vec{x}[n+1] = A\vec{x}[n]$

$$\vec{x}[n+1] = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0.4 & 0.5 & 0.2 \\ 0 & 0.6 & 0.35 \end{bmatrix}}_{A} \vec{x}[n]$$

Sum of entries of the columns:

$C_1 = 0.4 < 1$

$C_2 = 1.1 > 1$

$C_3 = 0.55 < 1$

This means that the total water in the system is decreasing over time.

6. Homework Process

I worked on this homework alone.
I read all the notes, then did the homework little by little each night.