The weakness of finding descending sequences in ill-founded linear orders

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We prove that the Weihrauch degree of the problem of finding a bad sequence in a non-well quasi order (BS) is strictly above that of finding a descending sequence in an ill-founded linear order (DS). This corrects our mistaken claim in [8], which stated that they are Weihrauch equivalent. We prove that König's lemma KL and the problem $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega}$ of enumerating a given non-empty countable closed subset of $2^{\mathbb{N}}$ are not Weihrauch reducible to DS either, resolving two main open questions raised in [8].

1 Introduction and background

A quasi-order $(Q, \preceq)$ is called *well quasi-order* (abbreviated wqo) if, for every infinite sequence $(q_n)_{n\in\mathbb{N}}$ of elements of Q, there are i, j with $i < j$ such that $q_i \preceq_Q q_j$. This can be restated by saying that a quasi-order is a wqo if it contains no infinite bad sequences, where a sequence $(q_n)_{n<\alpha}$ is called bad if $q_i \npreceq_Q q_j$ for every $i < j < \alpha$. Equivalently, wqo's can be defined as quasi-orders that contain no infinite descending sequence and no infinite antichain. There is an extensive literature on the theory of wqo's. For an overview, we refer the reader to [10].

We study the difficulty of solving the following computational problems:

- given a countable ill-founded linear order, find an infinite Descending Sequence in it (DS), and

- given a countable non-well quasi-order, find a Bad Sequence in it (BS).

A suitable framework for this is the Weihrauch lattice (see [2] for a self-contained introduction). Several results on DS were proved in our previous paper [8]; however, [8, Prop. 4.5] falsely claims that DS and BS are Weihrauch equivalent. In Theorem 3 we refute our claim by proving that the first-order part of BS is not Weihrauch reducible to DS. On the other hand, the deterministic part and the k-finitary parts of BS are Weihrauch equivalent to the corresponding parts of DS (Theorem 16, Corollary 18).

We will also resolve (negatively) two main open questions raised in [8, Questions 6.1 and 6.2], namely whether KL and $\mathsf{wList}_{2^{\mathbb{N}},\leq\omega}$ are Weihrauch reducible to DS (Corollaries 10, 11). The core of our proof (Theorem 8) is

$$\lim \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{f \mid \widehat{\mathsf{ACC}_{\mathbb{N}}} \times f \leq_{\mathrm{W}} \mathsf{DS}\}.$$

That is, even though $\widehat{\mathsf{ACC}_{\mathbb{N}}}$ is fairly weak (in particular it is below $\lim$, KL and DS), DS cannot even compute $\widehat{\mathsf{ACC}_{\mathbb{N}}} \times f$ if $f \nleq_{\mathrm{W}} \lim$. The existence of the maximum above provides an example of a "parallel quotient" [5, Remark 1].

In the rest of this section we briefly introduce relevant notions in Weihrauch complexity, followed with a note that BS is equivalent to its restriction to partial orders (Proposition 2).

A *represented space* $\mathbf{X} = (X, \delta_{\mathbf{X}})$ consists of a set X and a (possibly partial) surjection $\delta_{\mathbf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. Many mathematical objects of interest can be represented in standard ways which we do not spell out here (see e.g. [2, Def. 2.5]), such as: $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}$, $\mathbb{N}^{<\mathbb{N}}$, initial segments of $\mathbb{N}$, the set of binary relations on $\mathbb{N}$, the set of Γ-definable subsets of $\mathbb{N}$ where Γ is a pointclass in the projective hierarchy, countable Cartesian products and countable disjoint unions of represented spaces.

A *problem* f is a (possibly partial) multivalued function between represented spaces $\mathbf{X}$ and $\mathbf{Y}$, denoted $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$. For each $x \in X$, $f(x)$ denotes the set of possible outputs (i.e., f-*solutions*) corresponding to the input x. The *domain* $\mathrm{dom}(f)$ is the set of all $x \in X$ such that $f(x)$ is non-empty. Such x is called an f-*instance*. If $f(x)$ is a singleton for all $x \in \mathrm{dom}(f)$, we say f is *single-valued* and write $f :\subseteq \mathbf{X} \to \mathbf{Y}$. In this case, if y is the f-solution to x, we write $f(x) = y$ instead of (the formally correct) $f(x) = \{y\}$. We say a problem is *computable* (resp. *continuous*) if there is some computable (resp. continuous) $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if p is a name for some $x \in \mathrm{dom}(f)$, then $F(p)$ is a name for an f-solution to x.

A problem f is *Weihrauch reducible* to a problem g, written $f \leq_{\mathrm{W}} g$, if there are computable maps $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if p is a name for some $x \in \mathrm{dom}(f)$, then

1. $\Phi(p)$ is a name for some $y \in \mathrm{dom}(g)$, and

2. if q is a name for some g-solution of y, then $\Psi(p, q)$ is a name for some f-solution of x.

If Φ and Ψ satisfy the above, we say that $f \leq_{\mathrm{W}} g$ *via* Φ, Ψ.

Weihrauch reducibility forms a preorder on problems. We say f and g are *Weihrauch equivalent*, written $f \equiv_{\mathrm{W}} g$, if $f \leq_{\mathrm{W}} g$ and $g \leq_{\mathrm{W}} f$. The $\equiv_{\mathrm{W}}$-equivalence classes (*Weihrauch degrees*) are partially ordered by $\leq_{\mathrm{W}}$. Among the numerous algebraic operations in the Weihrauch degrees, we consider:

- for problems $f_i :\subseteq \mathbf{X}_i \rightrightarrows \mathbf{Y}_i$, the *parallel product*

$$f_0 \times f_1 :\subseteq \mathbf{X}_0 \times \mathbf{X}_1 \rightrightarrows \mathbf{Y}_0 \times \mathbf{Y}_1 \text{ defined by } (x_0, x_1) \mapsto f_0(x_0) \times f_1(x_1),$$

  i.e., given an f_0-instance and an f_1-instance, solve both,

- for a problem $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, the *(infinite) parallelization*

$$\widehat{f} :\subseteq \mathbf{X}^{\mathbb{N}} \rightrightarrows \mathbf{Y}^{\mathbb{N}} \text{ defined by } (x_i)_i \mapsto \prod_i f(x_i),$$

  i.e., given a countable sequence of f-instances, solve all of them.

These operations are defined on problems, but they all lift to the Weihrauch degrees. Parallelization even forms a closure operator, i.e., $f \leq_{\mathrm{W}} \widehat{f}$, $f \leq_{\mathrm{W}} g$ implies $\widehat{f} \leq_{\mathrm{W}} \widehat{g}$, and $\widehat{\widehat{f}} \equiv_{\mathrm{W}} \widehat{f}$.

The Weihrauch degrees also support a number of interior operators, which have been used to separate degrees of interest (see e.g. [13, §3.1]). For any problem f and any represented space $\mathbf{X}$,

$$\mathrm{Det}_{\mathbf{X}}(f) := \max_{\leq_{\mathrm{W}}} \{g \leq_{\mathrm{W}} f \mid g \text{ has codomain } \mathbf{X} \text{ and is single-valued}\}$$

exists [8, Thm. 3.2]. We call $\mathrm{Det}_{\mathbb{N}^{\mathbb{N}}}(f)$ the *deterministic part of* f and denote it by $\mathrm{Det}(f)$ for short. Observe that [8, Prop. 3.6] can be generalized slightly:

Proposition 1. $\mathrm{Det}(f) \leq_{\mathrm{W}} \widehat{\mathrm{Det}_{\mathbf{2}}(f)}$.

Proof. Suppose g is single-valued, has codomain $\mathbb{N}^{\mathbb{N}}$, and $g \leq_{\mathrm{W}} f$. Define a single-valued problem h as follows: Given $n, m \in \mathbb{N}$ and a g-instance x, produce 1 if $g(x)(n) \geq m$, otherwise produce 0. It is easy to see that $g \leq_{\mathrm{W}} \widehat{h}$ and $h \leq_{\mathrm{W}} f$. The latter implies $h \leq_{\mathrm{W}} \mathrm{Det}_{\mathbf{2}}(f)$ and so $g \leq_{\mathrm{W}} \widehat{h} \leq_{\mathrm{W}} \widehat{\mathrm{Det}_{\mathbf{2}}(f)}$. □

For any problem f and $\mathbf{X} = \mathbb{N}$ or $\mathbf{k}$, it is also known that

$$\max_{\leq_{\mathrm{W}}} \{ g \leq_{\mathrm{W}} f \mid g \text{ has codomain } \mathbf{X} \}$$

exists. For $\mathbf{X} = \mathbb{N}$ we call it the *first-order part of f* [6, Thm. 2.2], denoted by ^1f, while for $\mathbf{X} = \mathbf{k}$ we call it the $\mathbf{k}$*-finitary part of f* [4, Prop. 2.9], denoted by $\mathrm{Fin}_{\mathbf{k}}(f)$. We have $\mathrm{Det}_{\mathbb{N}}(f) \leq_{\mathrm{W}} {}^1f$ and $\mathrm{Det}_{\mathbf{k}}(f) \leq_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}(f) \leq_{\mathrm{W}} {}^1f$.

To study the problems DS and BS from the point of view of Weihrauch reducibility, we need to introduce the represented spaces of linear orders and quasi-orders. We only work with countable linear orders/quasi-orders with domain contained in $\mathbb{N}$. We represent a linear order $(L, \leq_L)$ with the characteristic function of the set $\{ \langle n, m \rangle : n \leq_L m \}$. Likewise, we represent a quasi-order $(Q, \preceq_Q)$ with the characteristic function of the set $\{ \langle n, m \rangle : n \preceq_Q m \}$.

We conclude this section observing a fact about BS which was implicit in [8].

Proposition 2. BS is Weihrauch equivalent to its restriction to partial orders.

Proof. Given a non-well quasi-order $(Q, \preceq_Q)$ where $Q \subseteq \mathbb{N}$, compute the set $S = \{ a \in Q \mid (\forall b <_{\mathbb{N}} a)(a \not\preceq_Q b \text{ or } b \not\preceq_Q a) \}$. The restriction $(S, \preceq_Q)$ is a non-well partial order because it is isomorphic to the partial order of $\preceq_Q$-equivalence classes. □

Henceforth we will use Proposition 2 without mention.

2 Separating BS and DS

We shall separate BS and DS by separating their first-order parts.

Theorem 3. $^1\mathsf{BS} \not\leq_{\mathrm{W}} {}^1\mathsf{DS}$ and so $\mathsf{DS} <_{\mathrm{W}} \mathsf{BS}$.

Recall from [8, Thm. 4.10] that $^1\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Pi}_1^1-\mathsf{Bound}$, which is the problem of producing an upper bound for a finite subset of $\mathbb{N}$ (given via a Π_1^1 code). Observe that $\mathbf{\Pi}_1^1-\mathsf{Bound}$ is *upwards closed*, i.e., if $n \in g(x)$ then $m \in g(x)$ for all $m > n$.

Lemma 4. Let f be a problem with codomain $\mathbb{N}$. The following are equivalent:

1. there exists an upwards closed problem g with codomain $\mathbb{N}$ such that $f \leq_{\mathrm{W}} g$;

2. there is a computable procedure which takes as input any $x \in \mathrm{dom}(f)$ and produces a sequence $p_x \in \mathbb{N}^{\mathbb{N}}$ of guesses for f-solutions to x which is correct cofinitely often.

Proof. For 1. ⇒ 2., let g be upwards closed and assume $f \leq_{\mathrm{W}} g$ via Φ and Ψ. Given $x \in \mathrm{dom}(f)$, run the computations $(\Psi(x, m))_{m \in \mathbb{N}}$ in parallel. Once some $\Psi(x, m)$ halts, we output its result and cancel $\Psi(x, n)$ for all $n < m$. This produces a sequence of numbers. The fact that g is upwards closed guarantees that cofinitely many elements of this sequence are elements of $f(x)$.

For the converse direction, for every $x \in \mathrm{dom}(f)$, let $p_x \in \mathbb{N}^{\mathbb{N}}$ be as in the hypothesis. Define $M_x := \max\{ m \mid p_x(m) \notin f(x) \}$ and let $g(x) := \{ n \mid n > M_x \}$. Clearly g is upwards closed. The fact that $f \leq_{\mathrm{W}} g$ follows from the fact that $x \mapsto p_x$ is computable. □

Given a non-well quasi-order $(Q, \preceq_Q)$, we say that a finite sequence σ is extendible to an infinite $\preceq_Q$-bad sequence (or, more compactly, σ is $\preceq_Q$-extendible) if there is a $\preceq_Q$-bad sequence $(q_n)_{n \in \mathbb{N}}$ such that $(\forall i < |\sigma|)(\sigma(i) = q_i)$. We omit the order whenever there is no ambiguity.

Observe that ^{1}BS can compute the problem "given a non-well partial order $(P, \leq_P)$, produce an element of P that is extendible to an infinite bad sequence". In light of Lemma 4, to prove Theorem 3 it suffices to show that one cannot computably "guess" solutions for BS. In other words, given a computable procedure which tries to guess extendible elements in a non-wqo, we want to construct a non-wqo P on which the procedure outputs a non-extendible element infinitely often. This would imply that ^{1}BS $\not\leq_W \Pi_1^1-$Bound. The non-wqos P we construct will be "tree-like" in the following sense:

Definition 5. A *tree decomposition* of a partial order $(P, \leq_P)$ consists of a tree $T \subseteq 2^{<\mathbb{N}}$ and a function $\iota \colon T \to P$ such that:

1. If $w_1, w_2 \in T$ and w_1 is a proper prefix of w_2 (written $w_1 \sqsubset w_2$), then $\iota(w_1) <_P \iota(w_2)$.

2. P is partitioned into finite P-intervals, where each interval has the form

$$(wb] = \{v \in P \mid \iota(w) <_P v \leq_P \iota(wb)\}$$

for some vertex $wb \in T$ (with final entry b), or $(\varepsilon] = \{\iota(\varepsilon)\}$ (where ε denotes the root of $2^{<\mathbb{N}}$). For $v \in P$ let $\lceil v \rceil \in T$ be uniquely defined by $v \in (\lceil v \rceil]$.

3. If $w_1, w_2 \in T$ are incompatible, so are $\iota(w_1)$ and $\iota(w_2)$ (i.e. they have no common upper bound in P).

The following lemma is straightforward.

Lemma 6. If $\iota \colon T \to P$ is a tree decomposition, then P has no infinite descending sequences. Moreover, T is wqo (i.e. it has finite width) if and only if P is wqo. In other words, T has an infinite antichain iff so does P.

Proof. The fact that every partial order that admits a tree decomposition does not have an infinite descending sequence follows from the fact that if $(v_n)_{n \in \mathbb{N}}$ is an infinite descending sequence in P, then since every interval $(\lceil v_n \rceil]$ is finite, up to removing duplicates, the sequence $(\lceil v_n \rceil)_{n \in \mathbb{N}}$ would be an infinite descending sequence in T.

If $(w_n)_{n \in \mathbb{N}}$ is an infinite antichain in T then, by definition of tree decomposition, $(\iota(w_n))_{n \in \mathbb{N}}$ is an infinite antichain in P. Conversely, if $(v_n)_{n \in \mathbb{N}}$ is an infinite antichain in P, then for every n, for all but finitely m, $\lceil v_n \rceil$ is $\sqsubseteq$-incomparable with $\lceil v_m \rceil$. In particular, we can obtain an infinite antichain in T by choosing a subsequence $(v_{n_i})_{i \in \mathbb{N}}$ such that, for every $i \neq j$, $\lceil v_{n_i} \rceil$ and $\lceil v_{n_j} \rceil$ are $\sqsubseteq$-incomparable. $\qquad\square$

Lemma 7. There is no computable procedure that, given in input a partial order which admits a tree decomposition, outputs an infinite sequence of elements of that partial order such that if the input is not wqo, then cofinitely many elements in the output are extendible to a bad sequence.

We point out a subtle yet important aspect regarding Lemma 7: The procedure only has access to the partial order, not to a tree decomposition of it.

Proof. Fix a computable "guessing" procedure g that receives as input a partial order (admitting a tree decomposition) and outputs an infinite sequence of elements in that partial order. We shall build a partial order P together with a tree decomposition $\iota \colon T \to P$ in stages such that, infinitely often, g outputs an element of P that does not extend to an infinite bad sequence.

Start with $T_0 = \{\varepsilon\}$ and P_0 having a single element v_ε, with $\iota_0(\varepsilon) = v_\varepsilon$. In stage s, we have built a finite tree decomposition $\iota_s \colon T_s \to P_s$ and wish to extend it to some $\iota_{s+1} \colon T_{s+1} \to P_{s+1}$. The tree T_{s+1} will always be obtained by giving each leaf in T_s a single successor, and then adding two successors to exactly one of the new leaves. To decide which leaf gets two successors, say a finite extension Q of P_s is *suitable* for $\iota_s \colon T_s \to P_s$ if for every $v \in Q \setminus P_s$, there is exactly one leaf $w \in T_s$ such that $\iota_s(w) <_Q v$. Pick the left-most leaf σ of T_s with the following property:

> There is some suitable extension Q of P_s such that, when given Q, the guessing procedure g would guess an element of Q which is comparable with $\iota_s(\sigma)$.

To see that such σ must exist, consider extending P_s by adding an "infinite comb" (i.e. a copy of $\{0^n 1^i \mid n \in \mathbb{N}, i \in \{0,1\}\}$) above the ι_s-image of a single leaf in T_s. The resulting partial order Q is non-wqo, admits a tree decomposition (obtained by extending T_s and ι_s in the obvious way), and its finite approximations (extending P_s) are suitable for ι_s. Hence, by hypothesis, g eventually guesses some element, which must be comparable with $\iota_s(\sigma)$, for some leaf $\sigma \in T_s$ (because all elements of Q are).

Having identified σ, we fix any corresponding suitable extension Q of P_s. In order to extend ι_s, we further extend Q to Q' by adding a new maximal element v_w to Q for each leaf $w \in T_s$ as follows: v_w lies above all $v \in Q \setminus P_s$ such that $\iota_s(w) <_P v$, and is incomparable with all other elements (including the other new maximal elements $v_{w'}$). To extend T_s, we add a new leaf $\tau ^\frown 0$ to T_s for each leaf τ, obtaining a tree T'. We extend ι_s to yield a tree decomposition $\iota' \colon T' \to Q'$ in the obvious way.

Finally, we add two successors to $\sigma ^\frown 0$ in T', i.e., define $T_{s+1} = T' \cup \{\sigma ^\frown 00, \sigma ^\frown 01\}$. We also add two successors v_1, v_2 to $\iota'(\sigma ^\frown 0)$ in Q' to obtain P_{s+1}, and extend ι' to ι_{s+1} by setting $\iota_{s+1}(\sigma ^\frown 0i) = v_i$. This concludes stage s.

It is clear from the construction that $\iota \colon T \to P$ is a tree decomposition. Let us discuss the shape of the tree T. In stage s, we introduced a bifurcation above a leaf σ_s of T_s. These are the only bifurcations in T. Observe that, whenever $s' < s$, σ_s is either above or to the right of $\sigma_{s'}$, because every suitable extension of P_s is also a suitable extension of $P_{s'}$ and, at stage s', the chosen leaf was the left-most. Therefore T has a unique non-isolated infinite path $p = \lim_s \sigma_s$, and a vertex w in T is extendible to an infinite antichain in T if and only if it does not belong to p.

We may now apply Lemma 6 to analyze P. First, since T is not wqo, neither is P. Second, we claim that if $v <_P \iota(\sigma)$ for some vertex σ on p, then v is not extendible to an infinite bad sequence. To prove this, suppose v is extendible. Then so is $\iota(\sigma)$. The proof of Lemma 6 implies that $\lceil \iota(\sigma) \rceil = \sigma$ is extendible to an infinite antichain in T. So σ cannot lie on p, proving our claim.

To complete the proof, observe that our construction of ι ensures that for each s, $g(P)$ eventually outputs a guess which is below $\iota(\sigma_s ^\frown 0)$. Whenever $\sigma_s ^\frown 0$ lies along p (which holds for infinitely many s), this guess is wrong by the above claim. $\qquad\square$

We may now complete the proof of Theorem 3.

Theorem 3. Suppose towards a contradiction that $^1\mathsf{BS} \leq_W \mathbf{\Pi}_1^1-\mathsf{Bound}$. Since the problem of finding an element in a non-wqo which extends to an infinite bad sequence is first-order, it is Weihrauch reducible to $\mathbf{\Pi}_1^1-\mathsf{Bound}$ as well. Now $\mathbf{\Pi}_1^1-\mathsf{Bound}$ is upwards closed, so there is a computable guessing procedure for this problem (Lemma 4). However such a procedure cannot exist, even for partial orders which admit a tree decomposition (Lemma 7).   □

3   Separating KL and DS

Recall the following problems [2, §6, Def. 7.4(7), Thm. 8.10]:

   $\mathsf{ACC}_\mathbb{N}$: Given an enumeration of a set $A \subseteq \mathbb{N}$ of size at most 1, find a number not in A.

   lim: Given a convergent sequence in $\mathbb{N}^\mathbb{N}$, find its limit.

   KL: Given an infinite finitely branching subtree of $\mathbb{N}^{<\mathbb{N}}$, find an infinite path through it.

It is known that $\widehat{\mathsf{ACC}_\mathbb{N}} <_W \mathsf{lim} <_W \mathsf{KL}$ (see [2]). For our separation of $\mathsf{KL} \not\leq_W \mathsf{DS}$ in Corollary 10, all we need to know about KL are the facts stated before the corollary. The core of our proof is the following.

Theorem 8. Let f be a problem. The following are equivalent:

1. $\widehat{\mathsf{ACC}_\mathbb{N}} \times f \leq_W \mathsf{DS}$

2. $f \leq_W \mathsf{lim}$.

Proof. The implication from 2. to 1. follows from $\mathsf{lim} \leq_W \mathsf{DS}$ (as shown in [8, Thm. 4.16]), as lim is closed under parallel product. For the other direction, we consider a name x for an input to f together with witnesses Φ, Ψ for the reduction. We show that, from them, we can uniformly compute an input q to $\widehat{\mathsf{ACC}_\mathbb{N}}$ together with an enumeration of a set W such that W is the well-founded part of the (ill-founded) linear order L built by Φ on (q, x). We can then use lim to obtain the characteristic function of W. Having access to this lets us find an infinite descending sequence in L greedily by avoiding ever choosing an element of W. From such a descending sequence Ψ then computes a solution to f for x.

It remains to construct $q = \langle q_0, q_1, \ldots \rangle$ and W to achieve the above. At the beginning, W is empty, and we extend each q_i in a way that removes no solution from its $\mathsf{ACC}_\mathbb{N}$-instance. As we do so, for each $i \notin W$ (in parallel), we monitor whether the following condition has occurred:

   L (as computed by the finite prefix of (q, x) built/observed thus far) contains i and some (finite) descending sequence ℓ such that

   1. ℓ is L-above i (i.e. $i <_L \min_L \ell$);
   2. the functional Ψ, upon reading the current prefix of (q, x) and ℓ, produces some output m for the i-th $\mathsf{ACC}_\mathbb{N}$-instance.

Once the above occurs for i (if ever), we remove m as a valid solution to q_i. This means that ℓ cannot be extendible to an infinite descending sequence in L, so i must be in the well-founded part of L. Hence we shall enumerate i into W. This completes our action for i, after which we return to monitoring the above condition for numbers not in W. This completes the construction.

It is clear that each q_i is an $\mathsf{ACC}_\mathbb{N}$-instance (with solution set $\mathbb{N}$ if the condition is never triggered, otherwise with solution set $\mathbb{N} \setminus \{m\}$). Hence $L = \Phi(q, x)$ is an ill-founded linear order. As argued above, W is contained in the well-founded part of L. Conversely, suppose

i lies in the well-founded part of L. Fix an infinite descending sequence r which lies above i. Then Ψ has to produce all $\mathsf{ACC}_{\mathbb{N}}$ answers upon receiving (q, x) and r, including an answer to q_i. This answer is determined by finite prefixes only, and after having constructed a sufficiently long prefix of q, some finite prefix ℓ of r will trigger the condition for i (unless something else triggered it previously), which ensures that i gets placed into W. This shows that W is exactly the well-founded part of L, thereby concluding the proof. $\qquad\square$

Corollary 9. If f is a parallelizable problem (i.e., $f \equiv_{\mathrm{W}} \widehat{f}$) with $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} f \leq_{\mathrm{W}} \mathsf{DS}$, then $f \leq_{\mathrm{W}} \mathsf{lim}$.

Proof. Since $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} f \leq_{\mathrm{W}} \mathsf{DS}$ and f is parallelizable, we have $\widehat{\mathsf{ACC}_{\mathbb{N}}} \times f \leq_{\mathrm{W}} f \leq_{\mathrm{W}} \mathsf{DS}$. By the previous theorem, $f \leq_{\mathrm{W}} \mathsf{lim}$. $\qquad\square$

Since KL is parallelizable, $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{KL}$, yet $\mathsf{KL} \not\leq_{\mathrm{W}} \mathsf{lim}$, we obtain a negative answer to [8, Question 6.1]:

Corollary 10. $\mathsf{KL} \not\leq_{\mathrm{W}} \mathsf{DS}$.

Similarly, consider the problem $\mathsf{wList}_{2^{\mathbb{N}}, \leq \omega}$ of enumerating all elements (possibly with repetition) of a given non-empty countable closed subset of $2^{\mathbb{N}}$. Since $\mathsf{wList}_{2^{\mathbb{N}}, \leq \omega}$ is parallelizable, $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{lim} \leq_{\mathrm{W}} \mathsf{wList}_{2^{\mathbb{N}}, \leq \omega}$, yet $\mathsf{wList}_{2^{\mathbb{N}}, \leq \omega} \not\leq_{\mathrm{W}} \mathsf{lim}$ [9, Prop. 6.12, 6.13, Cor. 6.16], we obtain a negative answer to [8, Question 6.2]:

Corollary 11. $\mathsf{wList}_{2^{\mathbb{N}}, \leq \omega} \not\leq_{\mathrm{W}} \mathsf{DS}$.

Note that KL is a parallelization of a first-order problem (such as RT_2^1). Using a recent result of Pauly and Soldà [12], we can characterize (up to *continuous* Weihrauch reducibility $\leq_{\mathrm{W}}^*$) the parallelizations of first-order problems which are reducible to DS.

Corollary 12. If $\widehat{f} \leq_{\mathrm{W}}^* \mathsf{DS}$, then $^1 f \leq_{\mathrm{W}}^* \mathsf{C}_{\mathbb{N}}$. Therefore, for any first-order f,

$$\widehat{f} \leq_{\mathrm{W}}^* \mathsf{DS} \qquad \text{if and only if} \qquad f \leq_{\mathrm{W}}^* \mathsf{C}_{\mathbb{N}}.$$

Proof. If $^1 f$ is continuous, the conclusion of the first statement is satisfied. Otherwise, $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{W}}^* f$ by [12, Thm. 1]. The relativization of Theorem 8 then implies $\widehat{f} \leq_{\mathrm{W}}^* \mathsf{lim}$. We conclude $^1 f \leq_{\mathrm{W}}^* {}^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. The second statement then follows from $\widehat{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{lim} \leq_{\mathrm{W}} \mathsf{DS}$. $\qquad\square$

4 The finitary part and deterministic part of BS

In this section, we show that BS and DS cannot be separated by looking at their respective finitary or deterministic parts. Recall from [8, Thms. 4.16, 4.31] that $\mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$ and $\mathrm{Fin}_{\mathbf{k}}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{RT}_k^1$. Since both the deterministic and the finitary parts are monotone, this implies that $\mathsf{lim} \leq_{\mathrm{W}} \mathrm{Det}(\mathsf{BS})$ and $\mathsf{RT}_k^1 \leq_{\mathrm{W}} \mathrm{Fin}_{\mathbf{k}}(\mathsf{BS})$, so we only need to show that the converse reductions hold.

To this end, we first introduce a technical lemma. For a fixed partial order $(P, \leq_P)$, we can define the following quasi-order on the (finite or infinite) $\leq_P$-bad sequences:

$$\alpha \trianglelefteq^P \beta :\Longleftrightarrow \alpha = \beta \text{ or } (\exists i < |\alpha|)(\forall j < |\beta|)(\alpha(i) \leq_P \beta(j)).$$

We just write $\trianglelefteq$ when the partial order is clear from the context.

Lemma 13. Let $(P, \leq_P)$ be a non-well partial order and let α, β be finite $\leq_P$-bad sequences. If $\alpha \trianglelefteq \beta$ and α is extendible to an infinite $\leq_P$-bad sequence, then so is β. If α is not extendible then there is an infinite $\leq_P$-bad sequence $B \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \trianglelefteq B$. (Hence $\alpha \trianglelefteq \beta$ for every initial segment β of B.)

Proof. To prove the first part of the theorem, fix $\alpha \trianglelefteq \beta$ and let $A \in \mathbb{N}^{\mathbb{N}}$ be an infinite $\leq_P$-bad sequence extending α. Let also $i < |\alpha|$ be a witness for $\alpha \trianglelefteq \beta$. For every $j > i$ and every $k < |\beta|$, $\beta(k) \not\leq_P A(j)$ (otherwise $A(i) = \alpha(i) \leq_P \beta(k) \leq_P A(j)$ would contradict the fact that A is a $\leq_P$-bad sequence), which implies that β is extendible.

Assume now that α is non-extendible and let $F \in \mathbb{N}^{\mathbb{N}}$ be a $\leq_P$-bad sequence. We show that there is $i < |\alpha|$ and infinitely many k such that $\alpha(i) <_P F(k)$. This is enough to conclude the proof, as we could take B as any subsequence of F with $\alpha(i) <_P B(k)$ for every k (i.e. $\alpha \trianglelefteq B$).

Assume that, for every $i < |\alpha|$ there is k_i such that for every $k \geq k_i$, $\alpha(i) \not\leq_P F(k)$ (since P is a partial order, there can be at most one k such that $\alpha(i) = F(k)$). Since α is finite, we can take $k := \max_{i < |\alpha|} k_i$ and consider the sequence $\alpha^\frown(F(k+1), F(k+2), \ldots)$. We have now reached a contradiction as this is an infinite $\leq_P$-bad sequence extending α. $\square$

Let $(P, \leq_P)$ be a partial order. We call a $A \subseteq P$ *dense* if for every $w \in P$ there is some $u \geq_P w$ with $u \in A$. We call it *upwards-closed*, if $w \in A$ and $w \leq_P u$ implies $u \in A$. By $\boldsymbol{\Sigma}_1^1-\mathsf{DUCC}$ we denote the following problem: Given a partial order P and a dense upwards-closed subset $A \subseteq P$ (given via a Σ_1^1 code), find some element of A. We can think of a Σ_1^1 code for $(P, \leq_P)$ as a sequence $(T_n)_{n \in \mathbb{N}}$ of subtrees of $\mathbb{N}^{<\mathbb{N}}$ such that, for every $n, m \in \mathbb{N}$, $n \leq_P m$ iff $T_{\langle n, m \rangle}$ is ill-founded (and $n \in P$ iff $n \leq_P n$). We refer to [8] for a more detailed discussion on various presentations of orders.

Proposition 14. $^1\mathsf{BS} \leq_{\mathrm{W}} \boldsymbol{\Sigma}_1^1-\mathsf{DUCC}$.

Proof. Let f be a problem with codomain $\mathbb{N}$ and assume $f \leq_{\mathrm{W}} \mathsf{BS}$ via Φ, Ψ. Fix $x \in \mathrm{dom}(f)$. Let $(P, \leq_P)$ denote the non-well partial order defined by $\Phi(x)$. We say that a finite $\leq_P$-bad sequence β is *sufficiently long* if $\Psi(x, \beta)$ returns a natural number in at most $|\beta|$ steps.

To show that $f \leq_{\mathrm{W}} \boldsymbol{\Sigma}_1^1-\mathsf{DUCC}$, it is enough to notice that Lemma 13 implies that the set of sufficiently long finite extendible bad sequences is $\Sigma_1^{1,x}$, non-empty, dense, and upwards-closed with respect to $\trianglelefteq^P$. $\square$

Lemma 15. $\boldsymbol{\Sigma}_1^1-\mathsf{DUCC} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}_1^1-\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$, where the latter denotes the restriction of the former to $2^{<\mathbb{N}}$ with the prefix ordering $\sqsubseteq$.

Proof. Clearly, we only need to show that $\boldsymbol{\Sigma}_1^1-\mathsf{DUCC} \leq_{\mathrm{W}} \boldsymbol{\Sigma}_1^1-\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$. Let the input be $((P, \leq_P), A)$ where A is a $\boldsymbol{\Sigma}_1^1$, non-empty, dense, and upwards-closed subset of the partial order $(P, \leq_P)$. We uniformly define a computable labelling $\lambda: 2^{<\mathbb{N}} \to P$ such that $\lambda^{-1}(A)$ is non-empty, dense, and upwards-closed. This suffices to prove the claimed reduction, as the preimage of A via λ is (uniformly) $\boldsymbol{\Sigma}_1^1$ and, given $\sigma \in \lambda^{-1}(A)$, we can simply compute $\lambda(\sigma) \in A$.

Let $(x_n)_{n \in \mathbb{N}}$ be an enumeration of P. We define an auxiliary computable function $\overline{\lambda}: P \times 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ as follows: for every i, $\overline{\lambda}(x, 0^i) := \overline{\lambda}(x, 0^i 1) := x$. To define $\overline{\lambda}(x, 0^i 1 b^\frown \sigma)$ we distinguish two cases:

$$\overline{\lambda}(x, 0^i 1 b^\frown \sigma) := \begin{cases} \overline{\lambda}(x, \sigma) & \text{if } b = 0 \text{ or } x \not\leq_P x_i \\ \overline{\lambda}(x_i, \sigma) & \text{if } b = 1 \text{ and } x \leq_P x_i. \end{cases}$$

We then define $\lambda(\sigma) := \overline{\lambda}(x_0, \sigma)$. It is clear that λ is computable and total. Let us show that $\lambda^{-1}(A)$ is a valid input for $\mathbf{\Sigma}_1^1-\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$. Observe first that, for every $\sigma \sqsubseteq \tau$, $\lambda(\sigma) \leq_P \lambda(\tau)$, which implies that $\lambda^{-1}(A)$ is upwards-closed.

To prove the density, fix $\sigma \in 2^{<\mathbb{N}}$ and assume $\lambda(\sigma) \notin A$. Since A is dense, there is i such that $\lambda(\sigma) \leq_P x_i \in A$. Let $\tau \sqsupseteq \sigma$ be such that, for every ρ, $\lambda(\tau^\frown\rho) = \lambda(\rho)$. Notice that such τ always exists: indeed, if $\overline{\sigma}$ is the longest tail of σ of the form 0^j or $0^j 1$ for some j, then $\tau := \sigma^\frown d0$, where $d = 1$ if $\overline{\sigma} = 0^j$ and $d = \varepsilon$ otherwise, satisfies the above requirement. In particular, $\lambda(\tau^\frown 0^i 11) = x_i$. This proves that $\lambda^{-1}(A)$ is dense and therefore concludes the proof. $\qquad\square$

Theorem 16. $\mathrm{Fin}_\mathbf{k}(\mathsf{BS}) \equiv_W \mathrm{Fin}_\mathbf{k}(\mathbf{\Sigma}_1^1-\mathsf{DUCC}) \equiv_W \mathrm{Fin}_\mathbf{k}(\mathsf{DS}) \equiv_W \mathsf{RT}_k^1$.

Proof. We have $\mathsf{RT}_k^1 \equiv_W \mathrm{Fin}_\mathbf{k}(\mathsf{DS}) \leq_W \mathrm{Fin}_\mathbf{k}(\mathsf{BS}) \leq_W \mathrm{Fin}_\mathbf{k}(\mathbf{\Sigma}_1^1-\mathsf{DUCC})$ by [8, Thm. 4.31] and Proposition 14. It remains to show that $\mathrm{Fin}_\mathbf{k}(\mathbf{\Sigma}_1^1-\mathsf{DUCC}) \leq_W \mathsf{RT}_k^1$. By Lemma 15, it is enough to show that $\mathrm{Fin}_\mathbf{k}(\mathbf{\Sigma}_1^1-\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)) \leq_W \mathsf{RT}_k^1$.

Let f be a problem with codomain k and assume $f \leq_W \mathbf{\Sigma}_1^1-\mathsf{DUCC}(2^{<\mathbb{N}}, \cdot)$ via Φ, Ψ. Observe that every $x \in \mathrm{dom}(f)$ induces a coloring $c : 2^{<\mathbb{N}} \to k$ as follows: run $\Psi(x, \sigma)$ in parallel on every $\sigma \in 2^{<\mathbb{N}}$. Whenever we see that $\Psi(x, \sigma)$ returns a number less than k, we define $c(\tau) := \Psi(x, \sigma)$ for every $\tau \sqsubseteq \sigma$ such that $c(\tau)$ is not defined yet. By density of $\Phi(x)$, c is total.

By the Chubb-Hirst-McNicholl tree theorem [3], there is some $\sigma \in 2^{<\mathbb{N}}$ and some color $i < k$ such that i appears densely above σ. We claim that if i appears densely above σ then $i \in f(x)$. To prove this, recall $\Phi(x)$ codes a set which is dense and upwards-closed. By density of both this set and the color i, we may fix some $\tau \sqsupseteq \sigma$ which lies in the set coded by $\Phi(x)$ and has color i. Then fix $\rho \sqsupseteq \tau$ such that $c(\tau)$ was defined to be $\Psi(x, \rho)$. Now ρ lies in the set coded by $\Phi(x)$ as well, so $i = c(\tau) = \Psi(x, \rho)$ lies in $f(x)$.

Since a k-coloring of $2^{<\mathbb{N}}$ can be naturally turned into a k-coloring of $\mathbb{Q}$ (using a canonical computable order isomorphism between them), the problem "given a k-coloring of $2^{<\mathbb{N}}$, find σ and i such that i appears densely above σ" can be solved by RT_k^1, as shown in [11, Cor. 42]. $\qquad\square$

It is immediate from the previous theorem that the finitary parts of BS and DS in the sense of Cipriani and Pauly [4, Definition 2.10] agree as well. Finally, we shall prove that the deterministic parts of BS and DS agree.

Lemma 17. If $\mathrm{Fin}_\mathbf{2}(f) \leq_W \mathsf{RT}_2^1$, then $\mathrm{Det}(f) \leq_W \lim$.

Proof. By algebraic properties of Det and $\mathrm{Fin}_\mathbf{k}$, we have

$$\mathrm{Det}(f) \leq_W \widehat{\mathrm{Det}_\mathbf{2}(f)} \leq_W \widehat{\mathrm{Fin}_\mathbf{2}(f)} \leq_W \widehat{\mathsf{RT}_2^1} \equiv_W \mathsf{KL}.$$

So $\mathrm{Det}(f) \leq_W \mathrm{Det}(\mathsf{KL}) \equiv_W \lim$: Use the fact $\mathsf{KL} \equiv_W \mathsf{WKL} * \lim$ and the choice elimination principle [2, 11.7.25] (see also [8, Thm. 3.9]). $\qquad\square$

Since $\mathrm{Det}(\mathsf{BS}) \geq_W \mathrm{Det}(\mathsf{DS}) \equiv_W \lim$ [8, Thm. 4.16], we conclude that:

Corollary 18. $\mathrm{Det}(\mathsf{BS}) \equiv_W \mathrm{Det}(\mathsf{DS}) \equiv_W \lim$.

Corollary 19. $\mathrm{Det}_\mathbb{N}(\mathsf{BS}) \equiv_W \mathsf{C}_\mathbb{N}$.

Proof. Since $\mathbb{N}$ computably embeds in $\mathbb{N}^\mathbb{N}$, for every problem f we have $\mathrm{Det}_\mathbb{N}(f) \leq_W \mathrm{Det}(f)$. In particular, by Corollary 18, $\mathrm{Det}_\mathbb{N}(\mathsf{BS}) \leq_W \mathrm{Det}(\mathsf{BS}) \equiv_W \lim$. Since $^1\lim \equiv_W \mathsf{C}_\mathbb{N}$ ([1, Prop. 13.10], see also [13, Thm. 7.2]), this implies $\mathrm{Det}_\mathbb{N}(\mathsf{BS}) \leq_W \mathsf{C}_\mathbb{N}$. The converse reduction follows from the fact that $\mathsf{C}_\mathbb{N} \equiv_W \mathrm{Det}_\mathbb{N}(\mathsf{DS})$ [8, Prop. 4.14]. $\qquad\square$

We remark that for establishing $\mathrm{Fin}_{\mathbf{k}}\big(\boldsymbol{\Sigma}_1^1-\mathsf{DUCC}\big) \leq_{\mathrm{W}} \mathsf{RT}_k^1$ in Theorem 16 it was immaterial that the set of correct solutions was provided as a Σ_1^1-set. If we consider any other represented point class $\boldsymbol{\Gamma}$ which is effectively closed under taking preimages under computable functions, and define $\boldsymbol{\Gamma}-\mathsf{DUCC}$ in the obvious way, we can obtain:

Corollary 20. $\mathrm{Fin}_{\mathbf{k}}(\boldsymbol{\Gamma}-\mathsf{DUCC}) \leq_{\mathrm{W}} \mathsf{RT}_k^1$.

This observation could be useful e.g. for exploring the Weihrauch degree of finding bad arrays in non-better-quasi-orders (cf. [7]).

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