On the density of the Weihrauch degrees

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Given an instance of a problem, produce a solution.

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Can be formalized using partial multi-valued function $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$

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Even $\forall\exists$ theorems can be seen as computational problems!

$$(\forall X)(\varphi(X) \to (\exists Y)\psi(X, Y))$$

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Diagram of a "generic" $g \le f$:

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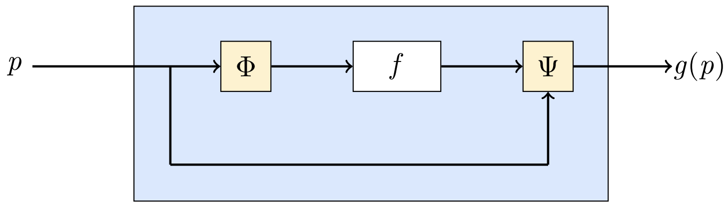

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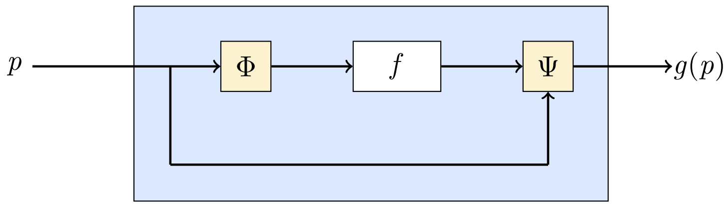

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According to the properties of Φ and Ψ we get different reductions.

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The forward and backward functional are partial computable functions on $\mathbb{N}^{\mathbb{N}}$.

$g \le_{\mathrm{W}} f :\Longleftrightarrow$ there are computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

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We can extend the definition of Weihrauch reducibility to problem on represented spaces, but problems on $\mathbb{N}^\mathbb{N}$ are enough to study the Weihrauch degrees.

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Proposition (Pauly; Brattka, Gherardi)

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The existence of a "natural" top element is equivalent to a (relatively weak) form of choice.

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No non-trivial countable suprema exists, i.e.

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Theorem (Lempp, Marcone, V.)

For every $\kappa \le \mathfrak{c}$ with $\mathrm{cof}(\kappa) > \omega$, there is a chain of order type κ in $\mathcal{W}$ that admits a supremum.

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$$\begin{tikzcd}
\mathrm{id} \\
\emptyset \arrow[u]
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$$\begin{tikzcd}
{} \\
\mathsf{lim}'' \arrow[u, dashed, no head] \\
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$$
\begin{tikzcd}[column sep=small]
 & \vdots & \\
 & \mathsf{C}_{\mathbb{N}^\mathbb{N}} \arrow[u, dash, dashed] & \\
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 & \mathsf{lim}'' \arrow[u, dotted] & \\
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 & \mathsf{C}_\mathbb{R} \arrow[u] & \\
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C_X: find a point in a (non-empty) Π^0_1 subset of X

$\mathsf{C}_{2^\mathbb{N}}$: find a path in an infinite $T \subseteq 2^{<\mathbb{N}}$

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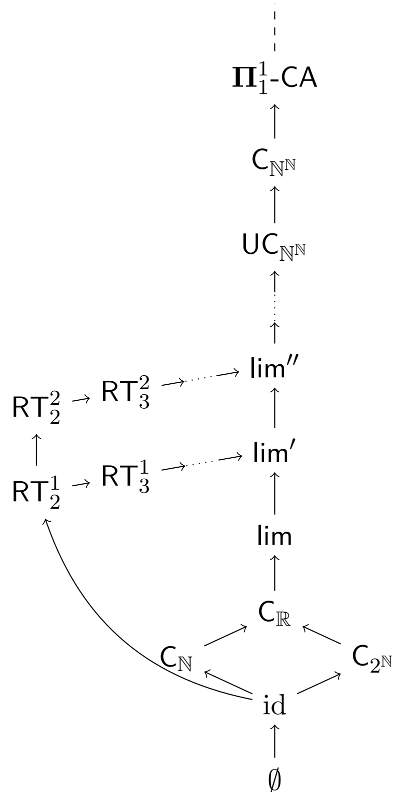

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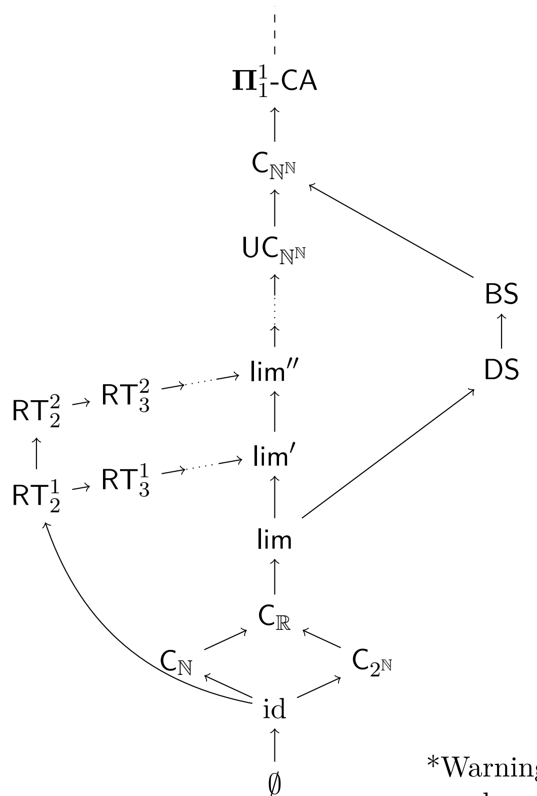

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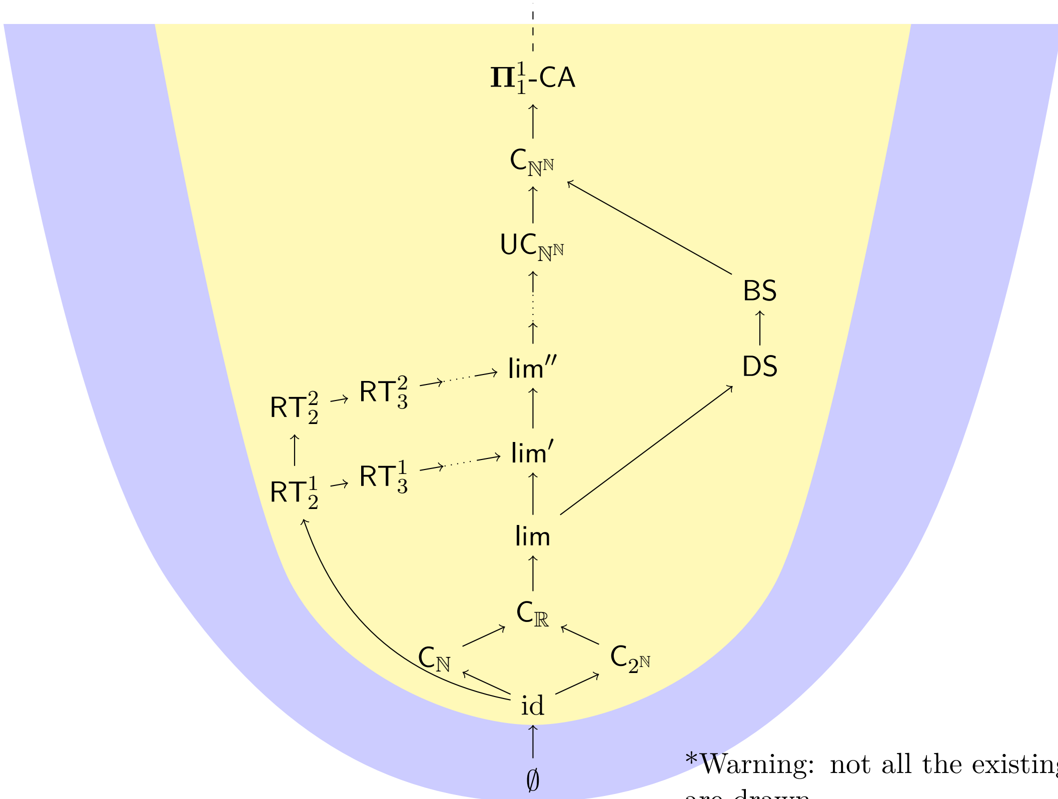

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Theorem (Lempp, Marcone, V.)

There are no cofinal chains in $\mathcal{W}$.

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We are heavily exploiting the complexity of the domain!

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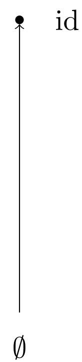
It follows that: $B \le_{\mathrm{M}} A$ iff $d_A \le_{\mathrm{W}} d_B$

This embedding reverses the Medvedev order!

Weihrauch degrees



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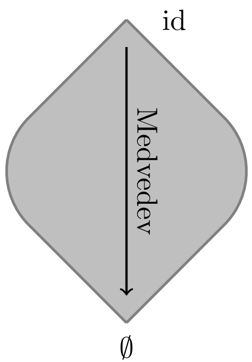

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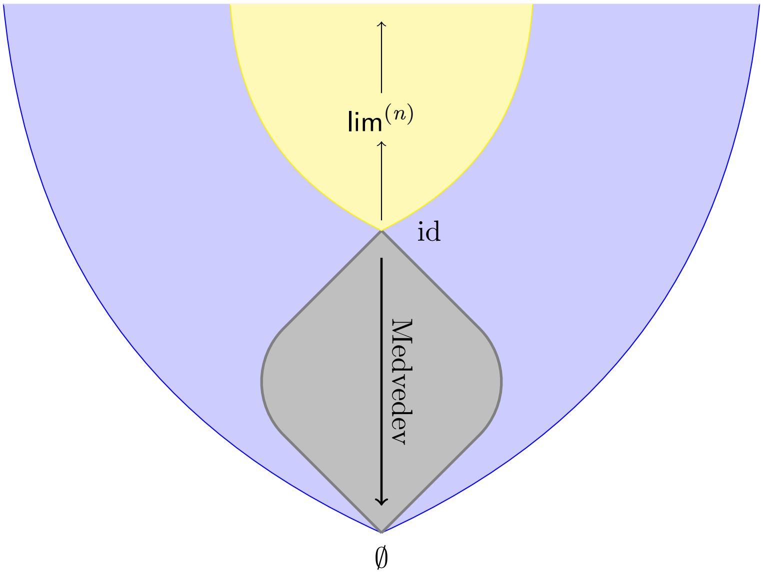

Minimal degrees and minimal covers

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In a poset, $\mathbf{a}$ is a **minimal cover** of $\mathbf{b}$ if $\{\mathbf{c} : \mathbf{b} < \mathbf{c} < \mathbf{a}\} = \emptyset$.

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Empty intervals in the Medvedev degrees have been fully characterized:

For $p \in \mathbb{N}^{\mathbb{N}}$, let $\{p\}^+ := \{(e)^\frown q : p <_T q \text{ and } \Phi_e(q) = p\}$.

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Theorem (Dyment)

For every $A <_{\mathrm{M}} B$, B is a minimal cover of A iff

$$(\exists p \in A)[\, A \equiv_{\mathrm{M}} B \wedge \{p\} \text{ and } B \wedge \{p\}^+ \equiv_{\mathrm{M}} B\,],$$

where $P \wedge Q := (0)^\frown P \cup (1)^\frown Q$ is the meet in the Medvedev degrees.

$$\begin{tikzcd}
 & \{p\}^+ & \\
B \arrow[ur] & & \{p\} \arrow[ul] \\
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Corollary

For every $p \in \mathbb{N}^{\mathbb{N}}$, $\mathrm{id}_{\{p\}}$ is a strong minimal cover of $\mathrm{id}_{\{p\}^+}$.

In particular, id is a SMC of $\mathrm{id}_{\mathrm{NR}}$, where $\mathrm{NR} := \{q : q \not\leq_T 0\} \equiv_{\mathrm{M}} \{0^{\mathbb{N}}\}^+$.

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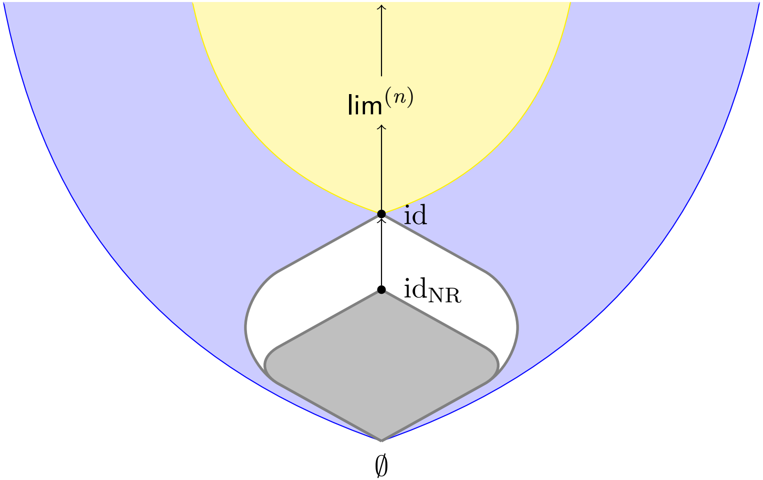

Strong minimal covers in Weihrauch

Theorem (Lempp, Miller, Pauly, Soskova, V.)

The following are equivalent:

1. f is a strong minimal cover of h in the Weihrauch degrees.
2. $h \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}^+}$ and $f \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}}$ for some $p \in \mathbb{N}^{\mathbb{N}}$.

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The first-order theory of the Weihrauch degrees, the first-order theory of the Weihrauch degrees below id, and the third-order theory of true arithmetic are pairwise recursively isomorphic.

Minimal covers in Weihrauch

Theorem (Lempp, Miller, Pauly, Soskova, V.)

Let f and h be partial multi-valued functions on Baire space. The following are equivalent:

1. f is a minimal cover of h in the Weihrauch degrees.
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Corollary (Lempp, Miller, Pauly, Soskova, V.)

The Weihrauch degrees above id are dense. In fact, id is the least degree whose upper Weihrauch cone is dense.

Minimal covers and Turing-antichains

Theorem (Lempp, Miller, Pauly, Soskova, V.)

For every family of pairwise Turing incomparable sets $\{p_\alpha\}_{\alpha<\kappa}$ with $\kappa < 2^{\aleph_0}$, there is a multi-valued function h whose minimal covers are exactly those of the form $h \sqcup \mathrm{id}_{\{p_\alpha\}}$.

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Corollary (Lempp, Miller, Pauly, Soskova, V.)

For every cardinal $\kappa \leq 2^{\aleph_0}$, there is a problem h with exactly κ minimal covers.

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Lemma (Lempp, Miller, Pauly, Soskova, V.)

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This can't happen: $h <_{\mathrm{W}} h \sqcup F_\xi <_{\mathrm{W}} f$, against the fact that f is a minimal cover of h.
This will give us the g we are looking for.

The construction

We build ξ in stages. At each stage s, ξ_s is only defined on finitely many points.

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Adding points to $\mathrm{dom}(g)$ “makes it harder to have $g \leq_{\mathrm{W}} f$”

Adding solutions to $g(p)$ “makes it easier to have $g \leq_{\mathrm{W}} f$”

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This concludes the construction.

A FURTHER REFINEMENT

We have showed that

Lemma (Lempp, Miller, Pauly, Soskova, V.)

If $h \leq_{\mathrm{W}} f$ and f is a minimal cover of h then there is g with finite domain such that $f \equiv_{\mathrm{W}} h \sqcup g$.

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If for all $i < k$, $g_i \leq_W h$ then $h \sqcup g \leq_W h$. Hence for some i,

$$h <_W h \sqcup g_i \leq_W f.$$

The claim follows from the fact that f is a minimal cover of h.

Minimal covers in Weihrauch

Lemma (Lempp, Miller, Pauly, Soskova, V.)

If $g \not\leq_{\mathrm{W}} \mathrm{id}$ and g has singleton domain, then for all h such that $g \not\leq_{\mathrm{W}} h$ there is $G <_{\mathrm{W}} g$ such that $G \not\leq_{\mathrm{W}} h$. It follows that $h <_{\mathrm{W}} h \sqcup G <_{\mathrm{W}} h \sqcup g$.

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Combining all the previous lemmas, we can finally characterize the minimal covers in the Weihrauch degrees.

Theorem (Lempp, Miller, Pauly, Soskova, V.)

Let f and h be partial multi-valued functions on Baire space. The following are equivalent:

1. f is a minimal cover of h in the Weihrauch degrees.
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Minimal covers in Weihrauch

Theorem (Lempp, Miller, Pauly, Soskova, V.)

Let f, h be partial multi-valued functions on Baire space. The following are equivalent:

1. f is a strong minimal cover of h in the Weihrauch degrees,
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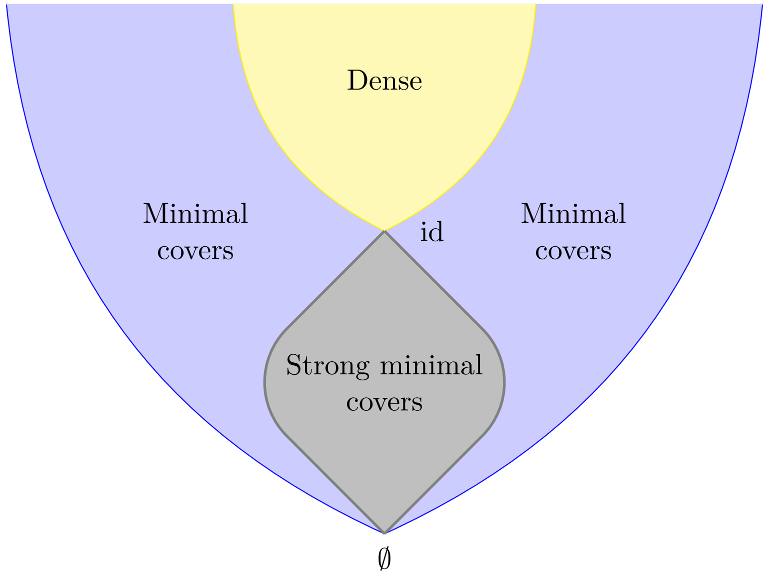
Proof

(2) $\Rightarrow$ (1): easy knowing that the lower cone of id is isomorphic to the dual Medvedev degrees.

(1) $\Rightarrow$ (2): By the characterization, $f \equiv_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$ for some p. Since the top of a SMC is join-irreducible, $f \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}}$. The only possibility is then that $h \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}^+}$.

Overview



Dense
Minimal covers
id
Minimal covers
Strong minimal covers
∅

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