Vectors

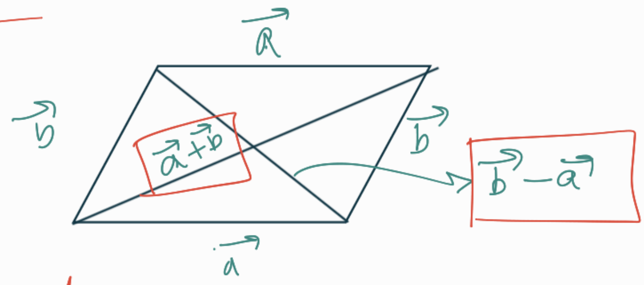
* Collinear or Parallel vectors

$\vec{a} \parallel \vec{b} \Rightarrow \vec{a} = \lambda \vec{b}, \quad \lambda \in R.$

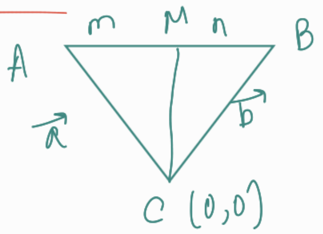
for e.g. → $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \quad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

if, $\vec{a} \parallel \vec{b} \Rightarrow \boxed{\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}}$

* Parallelogram



* Section formula



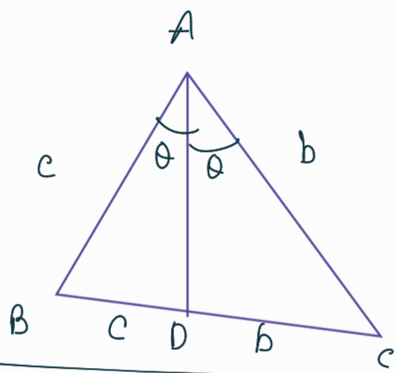
$\vec{CM} = \dfrac{\vec{a}+\vec{b}}{2}$

$\vec{CM} = \dfrac{\vec{AC} + \vec{BC}}{2}$

* $AM : MB = m : n$ {Internally}

* $\vec{MC} = \dfrac{m(\vec{CB}) + n(\vec{AC})}{m+n}$

*

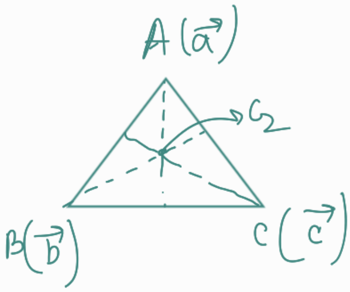


$\boxed{BD : DC = c : b}$

* Centroid

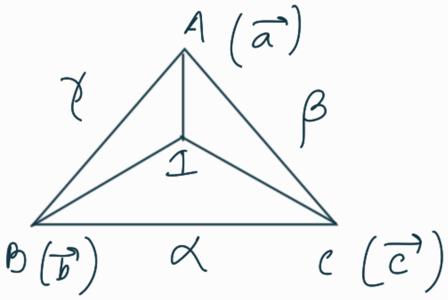
* $\vec{G} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$.

* $\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$.



* Incentre

$$\vec{I} = \left(\frac{\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}}{\alpha+\beta+c}\right)$$



** $BC(\overrightarrow{IA}) + AC(\overrightarrow{IB}) + AB(\overrightarrow{IC}) = 0.$

* Circumcentre :-

$HG : GP = 2:1.$

* Coplanar Vectors

$\vec{a}, \vec{b}, \vec{c}$ are coplanar

$$\Rightarrow \begin{vmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{vmatrix} = 0.$$

* $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar $\Rightarrow$ A, B, C, D are also coplanar.

* Product of Vectors

(i) $\vec{a}\cdot\vec{b} = \vec{a}\,\vec{b}\cos\theta$ (ii) $\theta = 90° \Rightarrow (\vec{a}\cdot\vec{b}) = 0$

(iii) $0 \leq \theta < 90° \Rightarrow \vec{a}\cdot\vec{b} \geq 0$ (iv) $\vec{a}\cdot\vec{b} = 0$

(v) $\vec{a}^2 = \vec{a}\cdot\vec{a}$

⋮ (Basics)

imp

$$\cos^2\theta = \left(\frac{\vec{a}\cdot\vec{b}}{ab}\right)^2 = \left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

$$\Rightarrow (a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

Equality when $\Rightarrow$

$$\boxed{\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}}$$

** $\boxed{\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} \geq -\frac{3}{2}}$, only when $\vec{a}+\vec{b}+\vec{c} = 0$

* Geometrical significance of Dot Product

(i) projection of B on A $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = OP$

(ii) projection vector of $\vec{b}$ on $\vec{a} = (\overrightarrow{OP})\cdot\hat{a}$.

* Cross Products

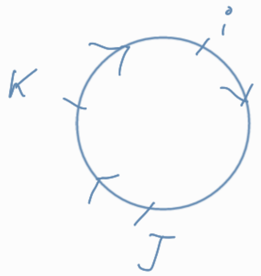
$\vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$, $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{a}|$

$\vec{a}\times\vec{a} = 0$, $\vec{a}\times\vec{b}$ is $\perp^r$ to both $\vec{a}$ and $\vec{b}$.

$(\vec{a}\times\vec{b})\cdot\vec{a} = 0$, $(\vec{a}\times\vec{b})\cdot\vec{b} = 0$

* $\vec{a}\times\vec{b} = 0 \Rightarrow \vec{a}\parallel\vec{b} \Rightarrow \vec{a} = \lambda\vec{b}$
* $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$

(*) $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$.

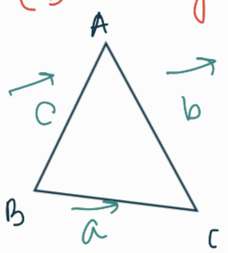


* Lagrange's Identity :-

$(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = a^2b^2$.

* Area

(i) Triangle →



Area of $\triangle ABC = \frac{1}{2}|\vec{a}\times\vec{b}|$

(ii) Parallelogram :- Area of parallelogram — $(\vec{a}\times\vec{b})$

where $\vec{a}$ and $\vec{b}$ are adjacent sides.

(iii) for any quadrilateral :-

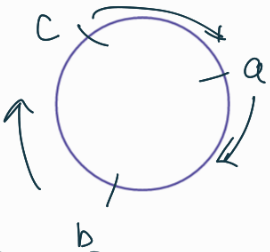
$\frac{1}{2}|\vec{d_1}\times\vec{d_2}|$ (where d_1 and d_2 are diagonals).

Scalar Triple Products

* $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$

* $$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

* $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]$



* $[\vec{a}\ \vec{c}\ \vec{c}] = [\vec{b}\ \vec{b}\ \vec{c}] = 0.$

or, $\vec{a} \parallel \vec{b}$, $[\vec{a}\ \vec{b}\ \vec{c}] = 0.$

* $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

* $$\vec{p} = p_1\vec{a} + p_2\vec{b} + p_3\vec{c}$$
$$\vec{q} = q_1\vec{a} + q_2\vec{b} + q_3\vec{c}$$
$$\vec{r} = r_1\vec{a} + r_2\vec{b} + r_3\vec{c}$$

$$[\vec{p}\ \vec{q}\ \vec{r}] = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$

NOTE :- if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar $\Rightarrow \vec{p}, \vec{q}, \vec{r}$ are also coplanar

if $\vec{p}, \vec{q}, \vec{r}$ are coplanar then, $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = 0.$

* Result

(i) $$[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{p}\ \vec{q}\ \vec{r}] = \begin{vmatrix} \vec{a}\cdot\vec{p} & \vec{a}\cdot\vec{q} & \vec{a}\cdot\vec{r} \\ \vec{b}\cdot\vec{p} & \vec{b}\cdot\vec{q} & \vec{b}\cdot\vec{r} \\ \vec{c}\cdot\vec{p} & \vec{c}\cdot\vec{q} & \vec{c}\cdot\vec{r} \end{vmatrix}$$

(*) Geometrical Significance of STP / Box Product

Volume of '||piped' $\Rightarrow V = \left| [\vec{a} \;\; \vec{b} \;\; \vec{c}] \right|$ = Area of base × height

volume of tetrahedron $= \frac{1}{3} \times$ area of base × height

$= \frac{1}{6} [\vec{a} \;\; \vec{b} \;\; \vec{c}]$

(*) Vector Triple Product

(*) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$

(*) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$

(*) $\vec{a} \times (\vec{b} \times \vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$ and it is $\perp^r$ to $\vec{a}$.

(*) $(\vec{a} \times \vec{b}) \times \vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$ and $\perp^r$ with $\vec{c}$.

(*) $[\vec{a} \times \vec{b} \;\; \vec{b} \times \vec{c} \;\; \vec{c} \times \vec{a}] = [\vec{a} \;\; \vec{b} \;\; \vec{c}]^2$

(*) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$

(*) $\underbrace{(\vec{a} \times \vec{b})}_{\vec{r}} \times (\vec{c} \times \vec{d})$

$\vec{r} \times (\vec{c} \times \vec{d})$

$= (\vec{r} \cdot \vec{d}) \vec{c} - (\vec{r} \cdot \vec{c}) \vec{d} = [\vec{a} \; \vec{b} \; \vec{d}] \vec{c} - [\vec{a} \; \vec{b} \; \vec{c}] \vec{d}$

(*) $[\vec{a} \times (\vec{b} \times \vec{c}) \quad \vec{b} \times (\vec{c} \times \vec{a}) \quad \vec{c} \times (\vec{a} \times \vec{b})] = 0$

(*) Linearly dependent Vectors

(i) $\vec{a} = \lambda \vec{b}$ or $\vec{b} = \lambda \vec{a}$

$\Rightarrow \vec{a}$ and $\vec{b}$ are linearly dependent

$\Rightarrow a \parallel \vec{b}$

(ii) $\vec{c} = x\vec{a} + y\vec{b}$ or $\vec{b} = x\vec{a} + y\vec{c}$

or, $\vec{a} = x\vec{b} + y\vec{c}$

$\vec{a}, \vec{b}, \vec{c}$ are linearly dependent $\Leftrightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar

(iii) More than 3 vectors are always linearly dependent

(iv) $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar