Binomial Theorem

(i) General expansion ⇒

$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 \dots + {}^nC_n x^0 y^n$

Putting $x=y=1 \rightarrow$

$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 \dots + {}^nC_n$ — (i)

Putting $x=1$ and $y=-1 \rightarrow$

$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 \dots + {}^nC_n(-1)^n$ — (ii)

(i)+(ii) $\rightarrow$

$2^n = 2({}^nC_0 + {}^nC_2 + \dots)$

$\boxed{{}^nC_0 + {}^nC_2 + \dots = 2^{n-1}}$

Similarly, (i − ii) $\rightarrow$

$\boxed{{}^nC_1 + {}^nC_3 \dots = 2^{n-1}}$

(ii) General Term binomial expansion :-

$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y^1 \dots + {}^nC_n x^0 y^n$

General Terms $\rightarrow$

$T_1 = {}^nC_0 x^n y^0$, $T_2 = {}^nC_1 x^{n-1} y^1$

Similarly, General form ⇒ $T_{r+1} = {}^nC_r x^{n-r} y^r$.

(iii) Application of Binomial Theorem

(a) $(1+x)^n - 1 - nx$ is divisible by x^r.

(b) $x^n - y^n$ is divisible by $x-y$.

(c) $x^{2n+1} + y^{2n+1}$ is divisible by by $x+y$.

NOTE

${}^nC_0 \cdot {}^nC_1 \cdot \dots {}^nC_n$

⇒ ${}^{100}C_{10} = {}^{100}C_{90}$ ∵ $\boxed{{}^nC_r = {}^nC_{n-r}}$

If n is even then, ${}^nC_r$ is max when $r = \frac{n}{2}$.

${}^6C_0 < {}^6C_1 < {}^6C_2 < \boxed{{}^6C_3} < {}^6C_4 < {}^6C_5 < {}^6C_6$

max/peak value

If n is odd, then

for e.g –

${}^7C_0 < {}^7C_1 < {}^7C_2 < \boxed{{}^7C_3 = {}^7C_4} < {}^7C_5 < {}^7C_6 < {}^7C_7$

${}^nC_r$ is max. when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

** Result $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$ | $\dfrac{^nC_{r+1}}{^nC_r} = \dfrac{n-r}{r+1}$

** Binomial coefficient Problems

(i) $^nC_0 + {^nC_1} + {^nC_2} + {^nC_3} \cdots + {^nC_n} = 2^n$

(ii) $^nC_0 - {^nC_1} + {^nC_2} - {^nC_3} \cdots + = 0$

(iii) $^nC_0 + {^nC_2} + {^nC_4} + \cdots =$

** Result

(i) $^nC_r + {^nC_{r+1}} = {^{n+1}C_{r+1}}$

(ii) $^{n+1}C_{r+1} - {^n}_{r+1} = {^nC_r}$

Same

(iii) $\sum_{r=0}^{n} {^nC_r}\, a^{n-r} x^r = (a+x)^n$.

(iv) $\sum_{r=0}^{n} {^nC_r}\, x^r = \sum_{r=0}^{n} {^nC_r} \cdot 1^{n-r} x^r = (1+x)^n$

(v) $\sum_{r=0}^{n} r \times {^nC_r} = n \times 2^{n-1}$

(vi) $\sum_{r=0}^{n} r(r-1)\, {^nC_r} = n(n-1)\, 2^{n-2}$

(vii) $^nC_r = \dfrac{n}{r} \times {^{n-1}C_{r-1}}$

(viii) $\sum_{r=0}^{n} \dfrac{^nC_r}{r+1} = \dfrac{2^{n+1}-1}{n+1}$.

(ix) $^mC_0\, {^nC_r} + {^mC_1}\, {^nC_{r-1}} + {^mC_2}\, {^nC_{r-2}} \cdots + {^mC_r}\, {^nC_0} = {^{m+n}C_r}$

(x) $({^nC_0})^2 + ({^nC_1})^2 + ({^nC_2})^2 \cdots + ({^nC_n})^2 = {^{2n}C_n}$