CS663 Assignment - Question 3

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1 Part A

We are given a matrix $\mathbf{A}$ with dimensions $m \times n$ and we have defined $\mathbf{P} = \mathbf{A}^T\mathbf{A}$ and $\mathbf{Q} = \mathbf{A}\mathbf{A}^T$. We are asked to compute

$$\mathbf{y}^T\mathbf{P}\mathbf{y} = \mathbf{y}^T\mathbf{A}^T\mathbf{A}\mathbf{y}$$

for some vector $\mathbf{y}$. We observe that this can be reduced to

$$\mathbf{y}^T\mathbf{P}\mathbf{y} = \mathbf{v}^T\mathbf{v}$$

where the vector $\mathbf{v}$ is defined as $\mathbf{v} = \mathbf{A}\mathbf{y}$. This represents the expression for the dot product of vector $\mathbf{v}$ with itself, which is same as

$$||\mathbf{v}||^2 \geq 0$$

Hence Proved.

Similarily, we can show that

$$\mathbf{z}^T\mathbf{Q}\mathbf{z} = \mathbf{z}^T\mathbf{A}\mathbf{A}^T\mathbf{z} = \mathbf{u}^T\mathbf{u} = ||\mathbf{u}||^2 \geq 0$$

Here, $\mathbf{u} = \mathbf{A}^T\mathbf{z}$. Hence Proved.

The eigen values λ and ν of both $\mathbf{P}$ and $\mathbf{Q}$ respectively are non-negative, which can be shown as.

$$\mathbf{y}^T\mathbf{P}\mathbf{y} \geq 0$$
$$\mathbf{y}^T\lambda\mathbf{y} \geq 0$$
$$\lambda||\mathbf{y}||^2 \geq 0$$
$$\lambda \geq 0$$

Similarily,

$$\mathbf{z}^T\mathbf{Q}\mathbf{z} \geq 0$$
$$\mathbf{z}^T\nu\mathbf{z} \geq 0$$
$$\nu||\mathbf{z}||^2 \geq 0$$
$$\nu \geq 0$$

2 Part B

We know that λ and μ are the eigen values of the matrices $\mathbf{P}$ and $\mathbf{Q}$ corrosponding to the eigen vector $\mathbf{u}$ and $\mathbf{v}$ respectively. Thus:

$$\mathbf{Pu} = \mathbf{A}^T\mathbf{Au} = \lambda\mathbf{u}.$$

We pre-multiply the equation with $\mathbf{A}$, hence we obtain

$$\mathbf{AA}^T\mathbf{Au} = (\mathbf{AA}^T)\mathbf{Au} = \mathbf{Q}(\mathbf{Au}) = \lambda\mathbf{Au}$$

Hence Proved. Similarily,

$$\mathbf{Qv} = \mathbf{AA}^T\mathbf{v} = \mu\mathbf{v}.$$

We pre-multiply this equation with $\mathbf{A}^T$ and hence we obtain

$$\mathbf{A}^T\mathbf{AA}^T\mathbf{v} = (\mathbf{A}^T\mathbf{A})\mathbf{A}^T\mathbf{v} = \mathbf{P}(\mathbf{A}^T\mathbf{v}) = \mu\mathbf{A}^T\mathbf{v}$$

Hence Proved. As dimension of $\mathbf{P}$ is $n \times n$ and dimension of $\mathbf{Q}$ is $m \times m$ and hence the number of elements in $\mathbf{u}$ and $\mathbf{v}$ are n and m respectively.

3 Part C

As $\mathbf{v}_i$ is the eigen vector of $\mathbf{Q}$ we have a $\lambda \geq 0$(from part A)

$$\mathbf{Qv}_i = \lambda\mathbf{v}_i$$

Now we proceed:

$$\mathbf{Au}_i = \frac{\mathbf{AA}^T\mathbf{v}_i}{||\mathbf{Av}_i||} = \frac{\mathbf{Qv}_i}{||\mathbf{Av}_i||} = \frac{\lambda\mathbf{v}_i}{||\mathbf{Av}_i||}$$

Clearly, we observe that $\gamma_i = \frac{\lambda}{||\mathbf{Av}_i||} \geq 0$Hence Proved!

4 Part D

We have $\mathbf{U} = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|...|\mathbf{v}_m]$ and $\mathbf{V} = [\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_3|...|\mathbf{u}_m]$. We directly consider the product $\mathbf{U\Gamma}$, hence

$$\mathbf{U\Gamma} = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_m]\begin{pmatrix}\gamma_1 & 0 & \cdots & 0\\ 0 & \gamma_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \gamma_m\end{pmatrix}$$

And from the previous part each $\gamma_i\mathbf{v}_i = \mathbf{Au}_i$, we have

$$\mathbf{U\Gamma} = [\mathbf{Au}_1|\mathbf{Au}_2|\mathbf{Au}_3|...|\mathbf{Au}_m]$$

Thus,

$$\mathbf{U\Gamma V}^T = \mathbf{A}[\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_3|...|\mathbf{u}_m][\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]^T = \mathbf{AVV}^T$$

Now we have shown that $\mathbf{u}_i^T\mathbf{u}_j = 0$ for $i \neq j$ and from the previous part we know that if $\mathbf{u}_i^T\mathbf{u}_i = 1$. Hence, we observe that $\mathbf{VV}^T = \mathbf{I}$. Therefore,

$$\mathbf{U\Gamma V}^T = \mathbf{A}$$

Proved!