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Calculus (SMD001U1M)

Tutorial # 1

1. Let A and B be two nonempty sets. Let $f : A \mapsto B$ be a function and E and F are subsets of A. Then prove that
   (a) $f(E \cup F) = f(E) \cup f(F)$
   (b) $f(E \cap F) \subseteq f(E) \cap f(F)$.

2. Let A and B be two nonempty sets and $f : A \mapsto B$ be a function. For $H \subseteq B$, we define the inverse image of H under f is the subset $f^{-1}(H)$ of A given by $f^{-1}(H) := \{x \in A : \ f(x) \in H\}$. Now let G and H are subsets of B then prove the following
   (a) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$
   (b) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$.

3. Let $x > 0$ and $n \in \mathbb{N}$, then show that $x^{\frac{1}{n}} \leq \dfrac{x+n-1}{n}$.

4. Let $a_i \in \mathbb{R}$ for $1 \leq i \leq n$ and $b_j \in \mathbb{R}$ for $1 \leq j \leq n$ then show that
$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq \left(\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\right)\left(\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}\right).$$

5. What would you choose for m in $|\sqrt{3} - 1.732| < 10^{-m}$, and why?

6. Prove that $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$, for all $n \in \mathbb{N}$ with $n > 1$.

7. Let $x_1 := 1$, $x_2 := 2$ and $x_{n+2} := \frac{1}{2}(x_{n+1} + x_n)$ for all $n \in \mathbb{N}$. Then show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

8. **Bernoulli's Inequality:** If $x > -1$, then $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

9. Recall from the class that for $a \in \mathbb{R}$ and $\epsilon > 0$ we defined the ϵ-neighborhood of a is the set $V_\epsilon(a) := \{x \in \mathbb{R} : \ |x - a| < \epsilon\}$. Now solve the following problems:
   (a) Let $\epsilon > 0$ and $\delta > 0$ and $a \in \mathbb{R}$. Show that $V_\epsilon(a) \cap V_\delta(a)$ and $V_\epsilon(a) \cup V_\delta(a)$ are $\gamma-$neighborhoods of a for some appropriate value of γ. Also find γ in each case.
   (b) Let $\epsilon_n = \dfrac{1}{n}$ for $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Let $V_{\epsilon_n}(a)$ be neighborhoods of a for $n \in \mathbb{N}$. Then is it true that $\displaystyle\bigcap_{n=1}^{\infty} V_{\epsilon_n}(a)$ is a neighborhood of a? What about $\displaystyle\bigcup_{n=1}^{k} V_{\epsilon_n}(a)$?
   (c) Let $a, b \in \mathbb{R}$ with $a \neq b$. Then can you find ϵ-neighborhoods $V_\epsilon(a)$ of a and $V_\epsilon(b)$ of b such that $V_\epsilon(a) \cap V_\epsilon(b) = \phi$?

10. Prove that a non-empty subset A of $\mathbb{R}$ is bounded if and only if there is a positive real number M such that $-M \leq a \leq M$ for all $a \in A$ (i.e. $|a| \leq M$).

11. If A is a non-empty subset of $\mathbb{R}$ which is bounded above then show that $-A := \{-x : x \in A\}$ is bounded below and $\sup(A) = -\inf(-A)$.

12. Let A and B be two nonempty bounded above subsets of $\mathbb{R}$ then prove that $A + B := \{a + b : \ a \in A \text{ and } b \in B\}$ is also bounded above subset of $\mathbb{R}$ and $\sup(A + B) = \sup A + \sup B$. How about $A - B := \{a - b : \ a \in A \text{ and } b \in B\}$? Is it bounded above?

13. Let $\alpha \in \mathbb{R}$ be non-negative and $n \in \mathbb{N}$. Then there exists a unique non-negative $x \in \mathbb{R}$ such that $x^n = \alpha$.

14. (Nested Interval Theorem) Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Then $\bigcap\limits_{n \in \mathbb{N}} J_n \neq \phi$.

15. Find the supremum (l.u.b.) and infimum (g.l.b) if they exist, of each of the following sets.

    (a) $\left\{1 - \dfrac{1}{n^2} : \ n \in \mathbb{N}\right\}$ $\quad (b)$ $\{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$ $\quad (c)$ $\{x + x^{-1} : \ x > 0\}$

    (d) $\left\{\dfrac{m + n}{mn} : \ m, n \in \mathbb{N}\right\}$ $\quad (e)$ $\{r \in Q : \ r < a, \ \}$ for $a \in \mathbb{R}$ fixed

    (f) $\bigcup\limits_{n \in \mathbb{N}} [2n, 2n + 1]$ $\quad (g)$ $\{n^{-1} : \ n \in \mathbb{N} \text{ and } n \text{ is prime}\}$ $\quad (h)$ $\left\{\sin\left(\dfrac{n\pi}{3}\right) : \ n \in \mathbb{N}\right\}$

    (i) $\bigcap\limits_{n \in \mathbb{N}} \left(1 - \dfrac{1}{n}, 1 + \dfrac{1}{n}\right)$ $\quad (j)$ $\left\{1 - \dfrac{1}{3^n} : \ n \in \mathbb{N}\right\}$ $\quad (k)$ $\left\{1 - \dfrac{(-1)^n}{n} : \ n \in \mathbb{N}\right\}$

    (l) $\{x \in \mathbb{R} : \ |x + 1| + |x - 2| = 7\}$.