UNIQUE DETERMINATION OF THE DAMPING COEFFICIENT IN THE WAVE EQUATION USING POINT SOURCE AND RECEIVER DATA

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Abstract. In this article, we consider the inverse problems of determining the damping coefficient appearing in the wave equation. We prove the unique determination of the coefficient from the data coming from a single coincident source-receiver pair. Since our problem is under-determined, so some extra assumption on the coefficient is required to prove the uniqueness.

Keywords : Inverse problems, wave equation, point source-receiver, damping coefficient

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1. Introduction

We consider the following initial value problem (IVP),

$$\begin{aligned} (\Box - q(x)\partial_t)u(x,t) &= \delta(x,t) \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0 \qquad\qquad\quad x \in \mathbb{R}^3 \end{aligned} \tag{1}$$

where $\Box := \partial_t^2 - \Delta_x$ denotes the wave operator and the coefficient $q \in C^\infty(\mathbb{R}^3)$ is known as damping coefficient. In this paper, we study the problem of determination of coefficient q appearing in (1) from the knowledge of solution measured at a single point for a certain period of time. We are interested in the uniqueness of determination of coefficients q from the knowledge of $u(0,t)$ for $t \in [0,T]$ with $T > 0$ in Equation (1). The problem studied here is motivated by geophysics, where geophysicists wish to determine the properties of earth structure by sending the waves from the surface of the earth and measuring the corresponding scattered responses (see [2, 24] and references therein). Since the coefficient to be determined here depends on three variables while the given data depends on one variable as far as the parameter count is concerned, the problem studied here is under-determined. Thus some extra assumptions on coefficient q are required in order to make the inverse problem solvable. We prove the uniqueness result for the radial coefficient.

There are several results related to the inverse problems for the wave equation with point source. We list them here. Romanov in [18] considered the problem for determining the damping and potential coefficient in the wave equation with point source and proved unique determination of these coefficients by measuring the solution on a set containing infinite points. In [12] the problem of determining the radial potential from the knowledge of solution measured on a unit sphere for some time interval is studied. Rakesh and Sacks in [16] established the uniqueness for angular controlled coefficient potential in the wave equation from the knowledge of solution and its radial derivative measured on a unit sphere. In the above mentioned works the measurement set is an infinite set. Next we mention the work where uniqueness is established from the measurement of solution at a single point. Determination of the potential from the data coming from a single coincident source-receiver pair is considered in [15] and the uniqueness result is established for the potentials which are either radial with respect a point different from source location or the potentials which are comparable. Recently author in [25] extended the result of [15] to a separated point source and receiver data. To the best of our understanding, very few results exist in the literature involving the recovery of the damping coefficient from point source and receiver data. Our result, Theorem 1.1, is work in this direction. In the 1-dimensional inverse problems context, several results exist involving the uniqueness of recovery of the coefficient which depends on the space variable corresponding to the first order derivative; see [9, 10, 11, 13, 19, 22]. We refer to [1, 3, 8, 14, 17] and references therein for more works related to the point source inverse problems for the wave equation.

We now state the main results of this article.

Theorem 1.1. *Suppose $q_i(x) \in C^1(\mathbb{R}^3)$, $i = 1, 2$ with $q_i(x) = A_i(|x|)$ for some C^1 function A_i on $[0, \infty)$. Let u_i be the solution of the IVP*

$$\begin{aligned} (\Box - q_i(x)\partial_t)u_i(x,t) &= \delta(x,t) \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u_i(x,t)|_{t<0} &= 0 \qquad\qquad\quad x \in \mathbb{R}^3. \end{aligned} \tag{2}$$

If $u_1(0,t) = u_2(0,t)$ for all $t \in [0,T]$ for some $T > 0$, then $q_1(x) = q_2(x)$ for all x with $|x| \leq T/2$, provided $q_1(0) = q_2(0)$.

The proof of the above theorem is based on an integral identity derived using the solution to an adjoint problem as used in [21] and [23]. This idea was used in [4, 17, 25] as well.

The article is organized as follows. In Section 2, we state the existence and uniqueness results for the solution of Equation (1), the proof of which is given in [5, 8, 20]. Section 3 contains the proof of Theorem 1.1.

2. Preliminaries

Proposition 2.1. [5, pp.139,140] *Suppose $q \in C^\infty(\mathbb{R}^3)$ and $u(x,t)$ satisfies the following initial value problem*

$$\begin{aligned} Pu(x,t) := (\Box - q(x)\partial_t)u(x,t) = \delta(x,t),\ (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} \ = \ 0, \qquad\qquad x \in \mathbb{R}^3 \end{aligned} \tag{3}$$

then $u(x,t)$ is given by

$$u(x,t) = \frac{R(x,t)\delta(t-|x|)}{4\pi|x|} + v(x,t) \tag{4}$$

where $v(x,t) = 0$ for $t < |x|$ and in the region $t > |x|$, $v(x,t)$ is a C^∞ solution of the characteristic boundary value problem (Goursat Problem)

$$\begin{aligned} &Pv(x,t) = 0,\ \ for\ t > |x| \\ &v(x,|x|) = -\frac{R(x,|x|)}{8\pi}\int_0^1 \frac{PR(sx, s|x|)}{R(sx, s|x|)}ds, \quad \forall x \in \mathbb{R}^3 \end{aligned} \tag{5}$$

and $R(x,t)$ is given by [5, pp. 134]

$$R(x,t) = \exp\left(-\frac{1}{2}\int_0^1 q(sx)t ds\right). \tag{6}$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We will first prove an integral identity which will be used to prove our main result.

Lemma 3.1. *Let $u_i(x,t)$ for $i = 1, 2$ be the solution to Equation* (2)*. Then the following integral identity holds for all $\sigma \geq 0$*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}} q(x)\partial_t u_2(x,t)u_1(x, 2\sigma - t)dtdx = u(0, 2\sigma) \tag{7}$$

where $q(x) := q_1(x) - q_2(x)$ *and* $u(x,t) = (u_1 - u_2)(x,t)$.

Proof. Here we have u satisfies the following IVP

$$\begin{aligned} \Box u(x,t) - q_1(x)\partial_t u(x,t) &= q(x)\partial_t u_2(x,t) \qquad && (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0 && x \in \mathbb{R}^3. \end{aligned} \tag{8}$$

Multiplying Equation (8) by $u_1(x, 2\sigma - t)$ and integrating over $\mathbb{R}^3 \times \mathbb{R}$, we have

$$\begin{aligned} \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t u_2(x,t)u_1(x,2\sigma - t)dtdx &= \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} \left(\Box u(x,t) - q_1(x)\partial_t u(x,t)\right) u_1(x, 2\sigma - t)dtdx \\ &= \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} u(x,t)\left(\Box u(x,t) - q_1(x)\partial_t u_1(x, 2\sigma - t)\right) \mathrm{d}x\mathrm{d}t \end{aligned}$$

where in the last step above we have used integration by parts and the properties of v in Proposition 2.1. Thus finally using the fact that u_1 is solution to (2), we get

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t u_2(x,t)u_1(x,2\sigma - t)dtdx = u(0, 2\sigma); \text{ for all } \sigma \geq 0.$$

This completes the proof of the lemma. $\square$

Using Lemma 3.1 and the fact that $u(0,t) = 0$ for all $t \in [0,T]$, we see that

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t u_2(x,t)u_1(x,2\sigma - t)dtdx = 0; \text{ for all } \sigma \in [0, T/2].$$

Now using Equation (4), we get

$$\begin{aligned} &\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t\left(\frac{R_2(x,t)\delta(t-|x|)}{4\pi|x|} + v_2(x,t)\right)\left(\frac{R_1(x,2\sigma - t)\delta(2\sigma - t - |x|)}{4\pi|x|}\right)dtdx \\ &+ \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)v_1(x,2\sigma - t)\partial_t\left(\frac{R_2(x,t)\delta(t-|x|)}{4\pi|x|} + v_2(x,t)\right)dtdx = 0. \end{aligned}$$

This gives

$$
\begin{aligned}
&\underbrace{\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} \frac{q(x)\partial_t R_2(x,t)R_1(x,2\sigma-t)\delta(t-|x|)\delta(2\sigma-t-|x|)}{16\pi^2|x|^2}dtdx}_{I_1}\\
&+\underbrace{\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} \frac{q(x)R_2(x,t)R_1(x,2\sigma-t)\partial_t\delta(t-|x|)\delta(2\sigma-t-|x|)}{16\pi^2|x|^2}dtdx}_{I_2}\\
&+\underbrace{\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t\Big(\frac{R_2(x,t)\delta(t-|x|)}{4\pi|x|}\Big)v_1(x,2\sigma-t)dtdx}_{I_3}\\
&+\underbrace{\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} \frac{q(x)\partial_t v_2(x,t)R_1(x,2\sigma-t)\delta(2\sigma-t-|x|)}{4\pi|x|}dtdx}_{I_4}\\
&+\underbrace{\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t v_2(x,t)v_1(x,2\sigma-t)dtdx}_{I_5}=0;\ \text{for all } \sigma\in[0,T/2].
\end{aligned}
\tag{9}
$$

In a compact form, this can be written as

$$
I_1+I_2+I_3+I_4+I_5=0. \tag{10}
$$

Next we simplify each I_j with $j=1,2,.....,5$. We will use the fact that $v_i(x,t)=0$ for $t<|x|$.

We have

$$
\begin{aligned}
I_1&=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} \frac{q(x)\partial_t R_2(x,t)R_1(x,2\sigma-t)\delta(t-|x|)\delta(2\sigma-t-|x|)}{16\pi^2|x|^2}dtdx\\
&=\int\limits_{|x|=\sigma}\frac{q(x)\partial_t R_2(x,|x|)R_1(x,|x|)}{16\pi^2|x|^2}dS_x\\
&=-\int\limits_{|x|=\sigma}\frac{q(x)R_1(x,|x|)R_2(x,|x|)}{32\pi^2|x|^2}\Big(\int\limits_0^1 q_2(sx)ds\Big)dS_x.
\end{aligned}
$$

Next we simplify the integral I_2. We use the following formula [7, Page 231, Eq.(10)]

$$
\int \delta'(r-|x|)\varphi dx=\frac{-1}{|x|^2}\int\limits_{|x|=r}\frac{\partial}{\partial r}\left(\varphi r^2\right)dS_x. \tag{11}
$$

Note that from this formula, by a change of variable, we have

$$
\int \delta'(2r-2|x|)\varphi dx=\frac{-1}{2|x|^2}\int\limits_{|x|=r}\frac{\partial}{\partial r}\left(\varphi r^2\right)dS_x. \tag{12}
$$

Now

$$
\begin{aligned}
I_2 &= \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} \frac{q(x)R_2(x,t)R_1(x,2\sigma-t)\partial_t\delta(t-|x|)\delta(2\sigma-t-|x|)}{16\pi^2|x|^2}dtdx\\
&= \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} \frac{q(x)R_2(x,t)R_1(x,2\sigma-t)\delta^{'}(t-|x|)\delta(2\sigma-t-|x|)}{16\pi^2|x|^2}dtdx\\
&= \int\limits_{\mathbb{R}^3} \frac{q(x)R_2(x,2\sigma-|x|)R_1(x,|x|)\delta'(2\sigma-2|x|)}{16\pi^2|x|^2}dx\\
&= -\frac{1}{32\pi^2\sigma^2}\int\limits_{|x|=\sigma}\frac{\partial}{\partial r}\{q(x)R_1(x,|x|)R_2(x,2\sigma-|x|)\}dS_x.
\end{aligned}
$$

In the last step above, we used Equation (12).

Next we have

$$
I_3 = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t\Big(\frac{R_2(x,t)\delta(t-|x|)}{4\pi|x|}\Big)v_1(x,2\sigma-t)dxdt.
$$

We can view the derivative above as a limit of the difference quotients in the distribution topolgy [6, pp.48]. Combining this with the fact that v_1 is C^2 in $\{(x,t) : |x| \leq t\}$, we get,

$$
\begin{aligned}
I_3 &= -\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\frac{R_2(x,t)\delta(t-|x|)}{4\pi|x|}\partial_t\Big(v_1(x,2\sigma-t)\Big)dxdt\\
&= \int\limits_{\mathbb{R}^3}\frac{q(x)R_2(x,|x|)\partial_t v_1(x,2\sigma-|x|)}{4\pi|x|}dx.
\end{aligned}
$$

Again using the fact that $v_1(x,t) = 0$ for $t < |x|$, we get,

$$
I_3 = \int\limits_{|x|\leq\sigma}\frac{q(x)R_2(x,|x|)\partial_t v_1(x,2\sigma-|x|)}{4\pi|x|}dx.
$$

Next we simplify I_4. Similiar to I_3, we have

$$
\begin{aligned}
I_4 &= \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}}\frac{q(x)\partial_t v_2(x,t)R_1(x,2\sigma-t)\delta(2\sigma-t-|x|)}{4\pi|x|}dtdx\\
&= \int\limits_{|x|\leq\sigma}\frac{q(x)R_1(x,|x|)\partial_t v_2(x,2\sigma-|x|)}{4\pi|x|}dx.
\end{aligned}
$$

Finally, we have

$$I_5 = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\partial_t v_2(x,t)v_1(x,2\sigma-t)dtdx$$
$$= \int\limits_{|x|\leq\sigma}\int\limits_{|x|}^{2\sigma-|x|} q(x)\partial_t v_2(x,t)v_1(x,2\sigma-t)dtdx.$$

Now, we use the fact that q_i is a radial function, that is, $q_i(x) = A_i(|x|)$. Then note that

$$R_i(x,|x|) = \exp\left(-\frac{|x|}{2}\int\limits_0^1 q_i(sx)ds\right) = \exp\left(-\frac{|x|}{2}\int\limits_0^1 A_i(s|x|)ds\right)$$

is also radial. For simplicity, we denote $R(x,|x|)$ by $R(|x|)$.

With this, we have

$$I_1 = -\frac{A(\sigma)R_1(\sigma)R_2(\sigma)}{8\pi}\int\limits_0^1 A_2(s\sigma)ds.$$

Next we consider I_2. First let us consider the derivative:

$$D_r := \frac{\partial}{\partial r}\left(A(r)R_1(x,r)R_2(x,2\sigma-r)\right).$$

After a routine calculation, we get,

$$\begin{aligned}D_r &= A'(r)R_1(x,r)R_2(x,r) - \frac{1}{2}A(r)^2R_1(x,r)R_2(x,2\sigma-r)\\ &- \sigma A(r)R_1(x,r)R_2(x,2\sigma-r)\int\limits_0^1 A_2'(rs)sds\\ &= A'(r)R_1(x,r)R_2(x,r) - \frac{1}{2}A(r)^2R_1(x,r)R_2(x,2\sigma-r)\\ &- A(r)R_1(x,r)R_2(x,2\sigma-r)\left[\frac{\sigma}{r}\left(A_2(r) - \int\limits_0^1 A_2(rs)ds\right)\right].\end{aligned}$$

On $|x| = \sigma$, we have

$$\begin{aligned}D_r|_{|x|=\sigma} &= R_1(\sigma)R_2(\sigma)\left[A'(\sigma) - \frac{1}{2}A(\sigma)^2 - A(\sigma)A_2(\sigma) + A(\sigma)\int\limits_0^1 A_2(s\sigma)ds\right]\\ &= R_1(\sigma)R_2(\sigma)\left[A'(\sigma) - \frac{1}{2}A(\sigma)(A_1+A_2)(\sigma) + A(\sigma)\int\limits_0^1 A_2(s\sigma)ds\right].\end{aligned}$$

Hence

$$I_2 = -\frac{1}{8\pi}\left(R_1(\sigma)R_2(\sigma)\left[A'(\sigma) - \frac{1}{2}A(\sigma)(A_1+A_2)(\sigma) + A(\sigma)\int\limits_0^1 A_2(s\sigma)ds\right]\right).$$

Let us denote

$$\tilde{A}(\sigma) = A(\sigma)R_1(\sigma)R_2(\sigma).$$

Then

$$I_2 = -\frac{1}{8\pi}\frac{d}{d\sigma}\tilde{A}(\sigma) - \frac{1}{8\pi}\tilde{A}(\sigma)\int_0^1 A_2(s\sigma)ds.$$

Therefore

$$I_1 + I_2 = -\frac{1}{8\pi}\left(2\tilde{A}(\sigma)\int_0^1 A_2(s\sigma)ds + \frac{d}{d\sigma}\tilde{A}(\sigma)\right).$$

Considering the following integrating factor for $I_1 + I_2$

$$\exp\left(2\int_0^\sigma\int_0^1 A_2(ts)dtds\right),$$

we have

$$I_1 + I_2 = -\frac{1}{8\pi}\exp\left(-2\int_0^\sigma\int_0^1 A_2(ts)dtds\right)\frac{d}{d\sigma}\left[\exp\left(2\int_0^\sigma\int_0^1 A_2(ts)dtds\right)\tilde{A}(\sigma)\right].$$

Now from Equation (10), we have

$$\begin{aligned}
&\frac{1}{8\pi}\frac{d}{d\sigma}\left[\tilde{A}(\sigma)\exp\left(2\int_0^\sigma\int_0^1 A_2(st)dsdt\right)\right]\\
&= \exp\left(2\int_0^\sigma\int_0^1 A_2(st)dsdt\right)\left[\int_{|x|\leq\sigma}\frac{q(x)R_2(x,|x|)\partial_t\{R_1v_1\}(x,2\sigma-|x|)}{4\pi|x|}dx\right.\\
&\quad+\int_{|x|\leq\sigma}\frac{q(x)R_1(x,|x|)\partial_t v_2(x,2\sigma-|x|)}{4\pi|x|}dx\\
&\quad\left.+\int_{|x|\leq\sigma}\int_{|x|}^{2\sigma-|x|} q(x)\partial_t v_2(x,t)v_1(x,2\sigma-t)dtdx\right] \text{ for all } \sigma\in[0,T/2].
\end{aligned} \tag{13}$$

Integrating on both sides with respect to σ under the assumption that $\tilde{A}(0)=0$, we get

$$\begin{aligned}
&\exp\left(\int_0^{\tilde{\sigma}}\int_0^1 2A_2(st)dsdt\right)\tilde{A}(\tilde{\sigma})\\
&=\int_0^{\tilde{\sigma}}\exp\left(\int_0^{\sigma}\int_0^1 2A_2(st)dsdt\right)\Bigg\{\int_{|x|\le\sigma}\frac{q(x)R_2(x,|x|)\partial_t v_1(x,2\sigma-|x|)}{4\pi|x|}dx\\
&+\int_{|x|\le\sigma}\frac{q(x)R_1(x,|x|)\partial_t v_2(x,2\sigma-|x|)}{4\pi|x|}dx\\
&+\int_{|x|\le\sigma}\int_{|x|}^{2\sigma-|x|} q(x)\partial_t v_2(x,t)v_1(x,2\sigma-t)dtdx\Bigg\}d\sigma, \quad \text{for all } \tilde{\sigma}\in[0,T/2].
\end{aligned}$$

Now using the fact that $R_i's$ are continuous, non-zero functions, and $v_i's$ are continuous, we have the following inequality:

$$|\tilde{A}(\tilde{\sigma})|\le C\int_0^{\tilde{\sigma}}|\tilde{A}(r)|dr \text{ for all } \tilde{\sigma}\in[0,T/2].$$

Now by Gronwall's inequality, we have $\tilde{A}(\sigma)=0$ for all $\tilde{\sigma}\in[0,T/2]$, which gives us $q_1(x)=q_2(x)$ for all $x\in\mathbb{R}^3$ such that $|x|\le T/2$. This completes the proof.

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