AN INVERSE PROBLEM FOR THE WAVE EQUATION WITH SOURCE AND RECEIVER AT DISTINCT POINTS

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Abstract. We consider the inverse problem of determining the density coefficient appearing in the wave equation from *separated* point source and point receiver data. Under some assumptions on the coefficients, we prove uniqueness results.

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1. Introduction

We address the inverse problem of determining the density coefficient of a medium by probing it with an external point source and by measuring the responses at a single point for a certain period of time.

More precisely, consider the following initial value problem (IVP), where $\Box = \partial_t^2 - \Delta_x$ denotes the wave operator:

$$\begin{aligned} (\Box - q(x))u(x,t) &= \delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3. \end{aligned} \tag{1}$$

In Equation (1), we assume that the coefficient q is real-valued and is a $C^3(\mathbb{R}^3)$ function. The inverse problem we address is the unique determination of the coefficient q from the knowledge of $u(e,t)$ where $e = (1,0,0)$ for $t \in [0,T]$ with $T > 1$. Motivation for studying such problems arises in geophysics see [27] and references therein. Geophysicist determine properties of the earth structure by sending waves from the surface of the earth and measuring the corresponding scattered responses. Note that in the problem we consider here, the point source is located at the origin, whereas the responses are measured at a different point. Since the given data depends on one variable whereas the coefficient to be determined depends on three variables, some additional restrictions on the coefficient q are required to make the inverse problems tractable.

There are several results related to point source inverse problems involving the wave equation. We briefly mention here the details of works which are closely related to the problem studied in this article. Romanov in [20] considered the problem of determining the damping and density coefficients which are constant outside a bounded, simply connected domain $D \subset \mathbb{R}^3$. By using the expression for fundamental solution, he reduced the problem to an integral geometry problem (whose solution was known by [8]), which gives the determination of these coefficients in D when source and receiver are moving in a plane (say M) chosen in such a way that (i) $M \cap \overline{D} = \emptyset$ and (ii) the line segment joined by source and receiver lies completely outside $\overline{D}$. Rakesh in [17] studied the problem of determining q from the knowledge of $u(0,t)$ for $t \in [0,T]$ and he proved the uniqueness for coefficients which are either comparable or radially symmetric with respect to a point different from the source location. The above mentioned works are related to point source hyperbolic inverse problems with under-determined data. We also mention some related works for the point source hyperbolic inverse problems with formally determined or with over determined data. In [14] Rakesh proved the unique determination of the radially symmetric coefficient $q(x)$ appearing in (1) when $u(a,t)$ is known for all $a \in \mathbb{S}^2$ and $t \in [0,T]$. Rakesh and Sacks in [18] established the uniqueness for angular controlled coefficient appearing in (1) from the knowledge of $u(a,t)$ and $u_r(a,t)$ for all $a \in \mathbb{S}^2$ and

$t \in [0,T]$ where u_r denotes the derivative with respect to $r = |x|$. The problem considered in [18] can be seen as an extension of the work [14] to a set of more general coefficients which is strictly bigger than the set of radial functions but [18] requires more information than [14]. In [19] the problem of determining the density coefficient q with angular controlled is studied. They proved the uniqueness of these coefficients from the knowledge of $u^a(a,t)$ for all $a \in \mathbb{S}^2$ and $t \in [0,T]$, where $u^a(x,t)$ denote the solution to (1) when source is located at $a \in \mathbb{S}^2$. For more works related to the problem studied in this article, we refer to [3, 23, 11, 26, 20, 12, 16, 7, 10, 22] and references therein.

We now state the main results of this article.

Theorem 1.1. *Suppose $q_i \in C^3(\mathbb{R}^3)$, $i = 1,2$ with $q_1(x) \geq q_2(x)$ for all $x \in \mathbb{R}^3$. Let $u_i(x,t)$ be the solution to the IVP*

$$\begin{aligned} (\Box - q_i(x))u_i(x,t) &= \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad\quad x \in \mathbb{R}^3. \end{aligned}$$

If $u_1(e,t) = u_2(e,t)$, for all $t \in [0,T]$ where $T > 1$ and $e = (1,0,0)$, then $q_1(x) = q_2(x)$ for all x with $|e-x| + |x| \leq T$.

Theorem 1.2. *Suppose $q_i \in C^3(\mathbb{R}^3)$, $i = 1,2$ with $q_i(x) = a_i(|x| + |x-e|)$ with $e = (1,0,0)$, for some C^3 functions a_i on $(1-\epsilon, \infty)$ for some $0 < \epsilon < 1$. Let u_i be the solution to the IVP*

$$\begin{aligned} (\Box - q_i(x))u_i(x,t) &= \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad\quad x \in \mathbb{R}^3. \end{aligned}$$

If $u_1(e,t) = u_2(e,t)$, for all $t \in [0,T]$ where $T > 1$ and $e = (1,0,0)$, then $q_1(x) = q_2(x)$ for all x with $|e-x| + |x| \leq T$.

Remark 1.3. From Proposition 2.1, the solution $u(x,t)$ of (1.1) is supported in $t \geq |x|$ hence $u(e,t) = 0$ for $t < 1$. So $u(e,t)$ has no information about q if $t < 1$, hence we require $T > 1$ in Theorems 1.1 and 1.2. Further note that ellipsoids $|x| + |x-e| \leq t$ are empty if $t < 1$.

To the best of our knowledge, our results, Theorems 1.1 and 1.2, which treat *separated* source and receiver, have not been studied earlier. Our result generalize the work [17], who considered the aforementioned inverse problem but with *coincident* source and receiver; see also [26].

The proofs of the above theorems are based on an integral identity derived using the solution to an adjoint problem as used in [24] and [26]. Recently this idea was used in [19] as well.

The article is organized as follows. In Section 2, we state the existence and uniqueness results for the solution of Equation (1), the proof of which is given in [4, 9, 22]. Sections 3 contains the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

Proposition 2.1. [4, pp.139,140] *Suppose q is a C^3 function on $\mathbb{R}^3$ and $u(x,t)$ satisfies the following IVP*

$$\begin{aligned} (\Box - q(x))u(x,t) &= \delta(x,t), \; (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad\quad x \in \mathbb{R}^3 \end{aligned} \tag{2}$$

then $u(x,t)$ is given by

$$u(x,t) = \frac{\delta(t-|x|)}{4\pi|x|} + v(x,t) \tag{3}$$

where $v(x,t)=0$ for $t<|x|$ and in the region $t>|x|$, $v(x,t)$ is a C^2 solution of the characteristic boundary value problem (Goursat Problem)

$$\begin{aligned}(\Box - q(x))v(x,t) &= 0, \qquad t>|x| \\ v(x,|x|) &= \frac{1}{8\pi}\int_0^1 q(sx)ds.\end{aligned} \tag{4}$$

We will use the following version of this proposition. Consider the following IVP

$$\begin{aligned}(\Box - q(x))U(x,t) = \delta(x-e,t),&\ \ (x,t)\in\mathbb{R}^3\times\mathbb{R}\\ U(x,t)|_{t<0} \ = \ 0, \qquad\qquad &\qquad x\in\mathbb{R}^3.\end{aligned} \tag{5}$$

Now we have

$$U(x,t)=\frac{\delta(t-|x-e|)}{4\pi|x-e|}+V(x,t) \tag{6}$$

where $V(x,t)=0$ for $t<|x-e|$ and for $t>|x-e|$, $V(x,t)$ is a C^2 solution to the following Goursat Problem

$$\begin{aligned}(\Box - q(x))V(x,t) &= 0, \qquad t>|x-e| \\ V(x,|x-e|) &= \frac{1}{8\pi}\int_0^1 q(sx+(1-s)e)ds.\end{aligned} \tag{7}$$

We can see this by translating source by $-e$ in Equation (5) and using the above proposition.

3. Proof of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. We will first show the following three lemmas which will be used in the proof of the main results.

Lemma 3.1. *Suppose $q_i's$ $i=1,2$ be C^3 real-valued functions on $\mathbb{R}^3$. Let u_i be the solution to Equation (1) with $q=q_i$ and denote $u(x,t):=u_1(x,t)-u_2(x,t)$ and $q(x):=q_1(x)-q_2(x)$. Then we have the following integral identity*

$$u(e,\tau)=\int_{\mathbb{R}}\int_{\mathbb{R}^3} q(x)u_2(x,t)U(x,\tau-t)dxdt, \text{ for all } \tau\in\mathbb{R} \tag{8}$$

where $U(x,t)$ is the solution to the following IVP

$$\begin{aligned}(\Box - q_1(x))U(x,t) = \delta(x-e,t),&\quad (x,t)\in\mathbb{R}^3\times\mathbb{R}\\ U(x,t)|_{t<0} = 0, \qquad\qquad &\qquad x\in\mathbb{R}^3.\end{aligned} \tag{9}$$

Proof. Since each u_i for $i=1,2$ satisfies the following IVP,

$$\begin{aligned}(\Box - q_i(x))u_i(x,t) = \delta(x,t),&\ \ (x,t)\in\mathbb{R}^3\times\mathbb{R}\\ u_i(x,t)|_{t<0} \ = \ 0, \qquad &\qquad x\in\mathbb{R}^3,\end{aligned}$$

we have that u satisfies the following IVP

$$\begin{aligned}(\Box - q_1(x))u(x,t) = q(x)u_2(x,t),&\ \ (x,t)\in\mathbb{R}^3\times\mathbb{R}\\ u(x,t)|_{t<0} \ = \ 0, \qquad &\qquad x\in\mathbb{R}^3.\end{aligned} \tag{10}$$

Now since

$$u(e,\tau)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} u(x,t)\delta(x-e,\tau-t)dtdx,$$

using (9), we have

$$u(e,\tau)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} u(x,t)(\Box-q_1(x))U(x,\tau-t)dtdx.$$

Now by using integration by parts and Equations (9) and (10), also taking into account that $u(x,t)=0$ for $t<|x|$ and that $U(x,t)=0$ for $|x-e|>t$, we get

$$\begin{aligned}u(e,\tau)&=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} U(x,\tau-t)(\Box-q_1(x))u(x,t)dtdx\\&=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)u_2(x,t)U(x,\tau-t)dtdx.\end{aligned}$$

This completes the proof of the lemma. ☐

Lemma 3.2. *Suppose $q_i's$ are as in Lemma 3.1 and u_i is the solution to Equation (1) with $q=q_i$ and if $u(e,t):=(u_1-u_2)(e,t)=0$ for all $t\in[0,T]$, then there exists a constant $K>0$ depending on the bounds on v_2, V and T such that the following inequality holds*

$$\left|\int\limits_{|x-e|+|x|=2\tau}\frac{q(x)}{|2\tau x-|x|e|}dS_x\right|\le K\int\limits_{|x-e|+|x|\le 2\tau}\frac{|q(x)|}{|x||x-e|}dx,\ \forall\tau\in(1/2,T/2].\tag{11}$$

Here dS_x is the surface measure on the ellipsoid $|x-e|+|x|=2\tau$ and v_2, V are solutions to the Goursat problem (see Equations (4) and (7)) corresponding to $q=q_i$.

Proof. From Lemma 3.1, we have

$$u(e,2\tau)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^3} q(x)u_2(x,t)U(x,2\tau-t)dxdt,\ \text{for all }\tau\in\mathbb{R}.$$

Now since $u(e,2\tau)=0$ for all $\tau\in[0,T/2]$, and using Equations (3) and (6), we get

$$\begin{aligned}0=&\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\frac{\delta(t-|x|)\delta(2\tau-t-|x-e|)}{16\pi^2|x||x-e|}dtdx\\&+\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\frac{\delta(t-|x|)V(x,2\tau-t)}{4\pi|x|}dtdx\\&+\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\frac{\delta(2\tau-t-|x-e|)}{4\pi|x-e|}v_2(x,t)dtdx\\&+\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)V(x,2\tau-t)v_2(x,t)dtdx.\end{aligned}$$

Now using the fact that $v_2(x,t)=0$ for $t<|x|$, $V(x,t)=0$ for $t<|x-e|$ and

$$\int_{\mathbb{R}^n} \phi(x)\delta(P)dx = \int_{P(x)=0} \frac{\phi(x)}{|\nabla_x P(x)|}dS_x$$

where dS_x is the surface measure on the surface $P=0$, we have that

$$\begin{aligned}
0 =& \frac{1}{16\pi^2} \int_{|x-e|+|x|=2\tau} \frac{q(x)}{|x||x-e||\nabla_x(2\tau-|x|-|x-e|)|}dS_x \\
&+ \frac{1}{4\pi} \int_{|x|+|x-e|\le 2\tau} \frac{q(x)V(x,2\tau-|x|)}{|x|}dx \\
&+ \frac{1}{4\pi} \int_{|x|+|x-e|\le 2\tau} \frac{q(x)v_2(x,2\tau-|x-e|)}{|x-e|}dx \\
&+ \int_{|x|+|x-e|\le 2\tau} \int_{|x|}^{2\tau-|x-e|} q(x)V(x,2\tau-t)v_2(x,t)dtdx.
\end{aligned}$$

For simplicity, denote

$$\begin{aligned}
F(\tau,x) := \frac{1}{4\pi}\Big(&|x-e|V(x,2\tau-|x|) + |x|v_2(x,2\tau-|x-e|) \\
&+ 4\pi|x||x-e| \int_{|x|}^{2\tau-|x-e|} V(x,2\tau-t)v_2(x,t)dt\Big)
\end{aligned}$$

and using

$$|\nabla_x(2\tau-|x|-|x-e|)| = \left|\frac{x}{|x|}+\frac{x-e}{|x-e|}\right| = \left|\frac{|x-e|x+(x-e)|x|}{|x||x-e|}\right|.$$

We have

$$\frac{1}{16\pi^2} \int_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x-|x|e|}dS_x = -\int_{|x|+|x-e|\le 2\tau} \frac{q(x)}{|x||x-e|}F(\tau,x)dx.$$

Note that $\tau\in[0,T/2]$ with $T<\infty$. Now using the boundedness of v_2 and V on compact subsets, we have $|F(\tau,x)|\le K$ on $|x|+|x-e|\le T$.

Therefore, finally we have

$$\left|\int_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x-|x|e|}dS_x\right| \le K \int_{|x-e|+|x|\le 2\tau} \frac{|q(x)|}{|x||x-e|}dx,\ \forall\tau\in(1/2,T/2].$$

The lemma is proved. ∎

Lemma 3.3. *Consider the solid ellipsoid $|e-x|+|x|\le r$, where $e=(1,0,0)$ and $x=(x_1,x_2,x_3)$, then we have its parametrization in prolate-spheroidal co-ordinates (ρ,θ,ϕ) given by*

$$\begin{aligned} x_1 &= \frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi \\ x_2 &= \frac{1}{2}\sinh\rho\sin\theta\sin\phi \\ x_3 &= \frac{1}{2}\sinh\rho\cos\theta\sin\phi \end{aligned} \tag{12}$$

with $\cosh\rho\le r$, $\theta\in(0,2\pi)$, $\phi\in(0,\pi)$ *and the surface measure dS_x on $|e-x|+|x|=r$ and volume element dx on $|e-x|+|x|\le r$, are given by*

$$\begin{aligned} dS_x &= \frac{1}{4}\sinh\rho\sin\phi\sqrt{\cosh^2\rho-\cos^2\phi}\,d\theta d\phi, \\ &\quad \textit{with } \cosh\rho=r,\ \theta\in[0,2\pi] \textit{ and } \phi\in[0,\pi] \\ dx &= \frac{1}{8}\sinh\rho\sin\phi(\cosh^2\rho-\cos^2\phi)d\rho d\theta d\phi, \\ &\quad \textit{with } \cosh\rho\le r,\ \theta\in[0,2\pi] \textit{ and } \phi\in[0,\pi]. \end{aligned} \tag{13}$$

Proof. The above result is well known, but for completeness, we will give the proof. The solid ellipsoid $|e-x|+|x|\le r$ in explicit form can be written as

$$\frac{(x_1-1/2)^2}{r^2/4}+\frac{x_2^2}{(r^2-1)/4}+\frac{x_3^2}{(r^2-1)/4}\le 1.$$

From this, we see that

$$\begin{aligned} x_1 &= \frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi \\ x_2 &= \frac{1}{2}\sinh\rho\sin\theta\sin\phi \\ x_3 &= \frac{1}{2}\sinh\rho\cos\theta\sin\phi \end{aligned}$$

with $\cosh\rho\le r$, $\theta\in[0,2\pi]$ and $\phi\in[0,\pi]$. This proves the first part of the lemma.

Now the parametrization of ellipsiod $|e-x|+|x|=r$, is given by

$$F(\theta,\phi)=\left(\frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi,\frac{1}{2}\sinh\rho\sin\theta\sin\phi,\frac{1}{2}\sinh\rho\cos\theta\sin\phi\right)$$

with $\theta\in(0,2\pi)$, $\phi\in(0,\pi)$ and $\cosh\rho=r$.

Next, we have

$$\begin{aligned} \frac{\partial F}{\partial\theta} &= \left(0,\frac{1}{2}\sinh\rho\cos\theta\sin\phi,-\frac{1}{2}\sinh\rho\sin\theta\sin\phi\right) \\ \frac{\partial F}{\partial\phi} &= \left(-\frac{1}{2}\cosh\rho\sin\phi,\frac{1}{2}\sinh\rho\sin\theta\cos\phi,\frac{1}{2}\sinh\rho\cos\theta\cos\phi\right). \end{aligned}$$

We have $dS_x = \left|\frac{\partial F}{\partial \theta} \times \frac{\partial F}{\partial \phi}\right| d\theta d\phi$, simple computation will gives us

$$dS_x = \frac{1}{4}\sinh\rho\sin\phi\sqrt{\cosh^2\rho - \cos^2\phi}\,d\theta d\phi,$$
$$\text{with } \cosh\rho = r,\ \theta \in (0, 2\pi)\ and\ \phi \in (0, \pi).$$

Last part of the lemma follows from change of variable formula, which is given by

$$dx = \left|\frac{\partial(x_1, x_2, x_3)}{\partial(\rho, \theta, \phi)}\right| d\theta d\phi d\rho;\ \text{with } \cosh\rho \leq r,\ \theta \in [0, 2\pi] \text{ and } \phi \in [0, \pi]$$

where $\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\theta,\phi)}$ is given by

$$\frac{\partial(x_1, x_2, x_3)}{\partial(\rho, \theta, \phi)} = \det\begin{vmatrix} \frac{\partial x_1}{\partial \rho} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial \phi} \\ \frac{\partial x_2}{\partial \rho} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial \phi} \\ \frac{\partial x_3}{\partial \rho} & \frac{\partial x_3}{\partial \theta} & \frac{\partial x_3}{\partial \phi} \end{vmatrix}.$$

This gives

$$dx = \frac{1}{8}\sinh\rho\sin\phi(\cosh^2\rho - \cos^2\phi)d\rho d\theta d\phi,$$
$$\text{with } \cosh\rho \leq r,\ \theta \in [0, 2\pi] \text{ and } \phi \in [0, \pi].$$

□

3.1. **Proof of Theorem 1.1.** We first consider the surface integral in Equation (11) and denote it $Q(2\tau)$:

$$Q(2\tau) := \int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x - |x|e|} dS_x. \tag{14}$$

We have

$$|2\tau x - |x|e| = |(2\tau x_1 - |x|, 2\tau x_2, 2\tau x_3)| = \sqrt{(2\tau x_1 - |x|)^2 + 4\tau^2 x_2^2 + 4\tau^2 x_3^2}$$
$$= \sqrt{4\tau^2|x|^2 + |x|^2 - 4\tau x_1|x|}.$$

From Equation (12) and using the fact that $\cosh\rho = 2\tau$, we have

$$\begin{aligned}
|2\tau x - |x|e| &= \frac{1}{2}\sqrt{(2\tau + \cos\phi)\{(4\tau^2 + 1)(2\tau + \cos\phi) - 4\tau(1 + 2\tau\cos\phi)\}} \\
&= \frac{1}{2}\sqrt{(2\tau + \cos\phi)(8\tau^3 + 4\tau^2\cos\phi + 2\tau + \cos\phi - 4\tau - 8\tau^2\cos\phi)} \\
&= \frac{1}{2}\sqrt{(2\tau + \cos\phi)(8\tau^3 - 4\tau^2\cos\phi - 2\tau + \cos\phi)} \\
&= \frac{1}{2}\sqrt{(4\tau^2 - \cos^2\phi)(4\tau^2 - 1)}.
\end{aligned}$$

Using the above expression for $|2\tau x - |x|e|$ and Equation (13), we get

$$Q(2\tau) = \frac{1}{8}\int_0^{\pi}\int_0^{2\pi} \frac{q(\rho, \theta, \phi)\sinh\rho\sin\phi\sqrt{\cosh^2\rho - \cos^2\phi}}{\sqrt{(4\tau^2 - \cos^2\phi)(4\tau^2 - 1)}} d\theta d\phi,$$

where we have denoted

$$q(\rho,\theta,\phi)=q\left(\frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi,\frac{1}{2}\sinh\rho\sin\theta\sin\phi,\frac{1}{2}\sinh\rho\cos\theta\sin\phi\right).$$

After using $\cosh\rho=2\tau$, $\sinh\rho=\sqrt{4\tau^2-1}$ and $\rho=\ln(2\tau+\sqrt{4\tau^2-1})$, we get

$$Q(2\tau)=\int_0^{\pi}\int_0^{2\pi}q(\ln(2\tau+\sqrt{4\tau^2-1}),\theta,\phi)\sin\phi d\theta d\phi. \tag{15}$$

Now consider the integral

$$\int_{|x-e|+|x|\le 2\tau}\frac{q(x)}{|x||x-e|}dx.$$

Again using Equations (12) and (13) in the above integral, we have

$$\int_{|x-e|+|x|\le 2\tau}\frac{q(x)}{|x||x-e|}dx=\frac{1}{2}\int_{\cosh\rho\le 2\tau}\int_0^{\pi}\int_0^{2\pi}q(\rho,\theta,\phi)\sinh\rho\sin\phi d\theta d\phi d\rho.$$

After substituting $\cosh\rho=r$ and $\rho=\ln(r+\sqrt{r^2-1})$, we get

$$\int_{|x-e|+|x|\le 2\tau}\frac{q(x)}{|x||x-e|}dx=\frac{1}{2}\int_1^{2\tau}\int_0^{\pi}\int_0^{2\pi}q(\ln(r+\sqrt{r^2-1}),\theta,\phi)\sin\phi d\theta d\phi dr.$$

Now using (15), we get

$$\int_{|x-e|+|x|\le 2\tau}\frac{q(x)}{|x||x-e|}dx\le C\int_1^{2\tau}Q(r)dr. \tag{16}$$

Now applying this inequality in Equation (11) and noting that $q(x)=q_1(x)-q_2(x)\ge 0$, we have

$$Q(2\tau)\le CK\int_1^{2\tau}Q(r)dr. \tag{17}$$

Now Equation (17) holds for all $\tau\in[1/2,T/2]$ and since $q(x)\ge 0$, for all $x\in\mathbb{R}^3$, therefore using the Gronwall's inequality, we get

$$Q(2\tau)=0,\quad \tau\in[1/2,T/2].$$

Now from Equation (14), again using $q(x)\ge 0$, we have $q(x)=0$, for all $x\in\mathbb{R}^3$ such that $|x|+|x-e|\le T$. The proof is complete.

3.2. **Proof of Theorem 1.2.** Again first, we consider the surface integral in (11) and denote it by $Q(2\tau)$:

$$Q(2\tau):=\int_{|x|+|x-e|=2\tau}\frac{q(x)}{|2\tau x-|x|e|}dS_x.$$

and $q(x) := a(|x| + |x-e|)$. Now from Equations (12), (13) and (15) and hypothesis $q_i(x) = a_i(|x| + |x-e|)$ of the theorem, we get

$$Q(2\tau) = \frac{1}{8}\int_0^{\pi}\int_0^{2\pi} a(2\tau)\sin\phi d\theta d\phi = \frac{\pi}{2}a(2\tau). \tag{18}$$

Now consider the integral

$$\int_{|x|+|x-e|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx.$$

Again using (12) and (13) in the above integral, we have

$$\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx = \frac{1}{2}\int_{\cosh\rho\leq 2\tau}\int_0^{\pi}\int_0^{2\pi} q(\rho,\theta,\phi)\sinh\rho\sin\phi d\theta d\phi d\rho.$$

After substituting $\cosh\rho = r$ and $\rho = \ln(r+\sqrt{r^2-1})$, we get

$$\begin{aligned}\left|\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx\right| &= \left|\frac{1}{2}\int_1^{2\tau}\int_0^{\pi}\int_0^{2\pi} a(r)\sin\phi d\theta d\phi dr\right| \\ &\leq C\int_1^{2\tau}|a(r)|dr.\end{aligned}$$

Now using this inequality and Equation (18) in (11), we see

$$|a(2\tau)| \leq C\int_1^{2\tau}|a(r)|dr. \tag{19}$$

Now Equation (19) holds for all $\tau \in [1/2, T/2]$, so using the Gronwall's inequality, we have

$$a(2\tau) = 0, \text{ for } \tau \in [1/2, T/2].$$

Thus, we have $q(x) = 0$, for all $x \in \mathbb{R}^3$ such that $|x| + |x-e| \leq T$. This conclude the proof of Theorem 1.2.

3. Conclusion

In this paper, we studied an inverse problem for the wave equation with *separated* point source and point receiver data. Our approach is based on construction of spherical wave solution using the solution to a Goursat problem, combined with the solution to an adjoint problem, we ended up with an integral identity. Then using the prolate-spheroidal co-ordinates and Grownwall's inequality, we completed the proof of the main results.

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