RECONSTRUCTION FOR THE COEFFICIENTS OF A QUASILINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

CĂTĂLIN I. CÂRSTEA*, GEN NAKAMURA† AND MANMOHAN VASHISTH◇

Abstract. In this paper we consider an inverse coefficients problem for a quasilinear elliptic equation of divergence form $\nabla \cdot \vec{C}(x, \nabla u(x)) = 0$, in a bounded smooth domain Ω. We assume that $\overrightarrow{C}(x, \vec{p}) = \gamma(x)\vec{p} + \vec{b}(x)|\vec{p}|^2 + \mathcal{O}(|\vec{p}|^3)$, by expanding $\overrightarrow{C}(x, \vec{p})$ around $\vec{p} = 0$. We give a reconstruction method for γ and $\vec{b}$ from the Dirichlet to Neumann map defined on $\partial\Omega$.

Keywords: non-linear equation, inverse problems, reconstruction, Dirichlet to Neumann map

Mathematics Subject Classifications (2010): 35J66, 65M32

1. Introduction and statement of the main result

First of all, we set up a boundary value problem for a quasilinear elliptic equation of divergence form. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded open set with smooth boundary $\partial\Omega$. We consider the following quasilinear elliptic boundary value problem (BVP)

$$\begin{cases} \nabla \cdot \overrightarrow{C}(x, \nabla u(x)) = 0, \ x \in \Omega, \\ u(x) = \epsilon f(x), \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\overrightarrow{C}(x, \nabla u(x))$ is given by

$$\overrightarrow{C}(x, \nabla u(x)) := \gamma(x)\nabla u(x) + |\nabla u(x)|^2 \vec{b}(x) + \overrightarrow{R}(x, \nabla u(x)) \tag{1.2}$$

with $\gamma, \vec{b} \in C^\infty(\overline{\Omega})$ and, for vector $q := (q_1, q_2, \cdots, q_n) \in \mathbb{R}^n$, $\overrightarrow{R}(x, q) \in C^\infty(\overline{\Omega} \times H)$ with $H := \{q \in \mathbb{R}^n : |q| \leq h\}$ for a constant $h > 0$. Throughout this paper we assume $\gamma(x) \geq C_1$ for some constant $C_1 > 0$ and there exists a constant $C_2 > 0$ such that

$$|\partial_q^\alpha \partial_x^\beta \overrightarrow{R}(x, q)| \leq C_2 |q|^{3-|\alpha|} \tag{1.3}$$

holds for all $(x, q) \in \overline{\Omega} \times H$ and multi-indices α, β with $|\alpha| \leq 3$.

Under the above setup, we have the following well-posedness result for the above (BVP) which is proved in [5].

Theorem 1.1. ([5]) *Let $n < p < \infty$. There exists ϵ and $\delta < h/2$ such that for any $f \in W^{2-1/p, p}(\partial\Omega)$ satisfying $\|f\|_{W^{2-1/p, p}(\partial\Omega)} < \epsilon$, (BVP) admits a unique solution u such that $\|u\|_{W^{2,p}(\Omega)} < \delta$. Moreover, there exists $C_3 > 0$ independent of f such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_3 \|f\|_{W^{2-1/p, p}(\partial\Omega)}. \tag{1.4}$$

Here $W^{2,p}(\Omega)$ and $W^{2-1/p, p}(\partial\Omega)$ are the usual L^p-Sobolev spaces of order 2 and $2 - 1/p$ in Ω and on $\partial\Omega$, respectively.

Based on the well-posedness of (BVP), we define the Dirichlet to Neumann (DN in short) map $\Lambda_{\overrightarrow{C}}(\epsilon f)$ by

$$\Lambda_{\overrightarrow{C}}(\epsilon f) := \nu(x) \cdot \overrightarrow{C}(x, \nabla u)|_{\partial\Omega}, \ f \in W^{2-1/p, p}(\partial\Omega), \tag{1.5}$$

where u is solution to the (BVP) and ν is the unit normal vector of $\partial\Omega$ directed into the exterior of Ω.

Now we state our inverse problem.

Inverse problem: Identify γ and $\vec{b}$ from the knowledge of DN map $\Lambda_{\vec{C}}$.

Remark 1.2. The above (BVP) is the scalar version of displacement boundary value problem for elasticity equation and $\vec{b}$'s correspond to higher order tensors of rank 6. In material science these higher order tensors are becoming important due to the demand to investigate physical phenomena in a smaller scale (see for example [6] using [1] as a guide book for nonlinear elasticity). As a consequence we need to recover these higher order tensors by solving some inverse problems. Hence we can consider our inverse problem as a toy model to reconstruct tensors up to rank 6.

Concerning this inverse problem, its uniqueness is already known in ([5]). Then a next very natural question is about giving a reconstruction for identifying these γ and $\vec{b}$.

Our main result in this paper is the following.

Theorem 1.3. *Knowing the DN map $\Lambda_{\vec{C}}$, we can have point-wise reconstruction for the linear part γ and the coefficient $\vec{b}$ of the quadratic part of $\vec{C}$. (The details of the reconstruction method will be given in the proof of this theorem see Sections 2 and 3).*

Let us locate our results among the well known results on inverse problems for nonlinear scalar elliptic equations using the DN map as their measured data to identify non-linearities or extract some information about them. The first important thing to say is that, as far as we know, the known results are about uniqueness. The major nonlinear scalar equations which have been studied up to now are of the following forms

(i) $-\Delta u + a(x,u) = 0$ ([3], [2]),
(ii) $-\Delta u + b(u, \nabla u) = 0$ ([4]),
(iii) $\nabla \cdot (c(x,u)\nabla u) = 0$ ([9],[10]),
(v) $\nabla \cdot (\vec{C}(x, \nabla u)) = 0$ ([5])

in Ω, with some appropriate conditions on the non-linearities $a(x,u)$, $b(x,\nabla u)$, $c(x,u)$, $\vec{C}(x,\nabla u)$, and we have indicated the contributing papers in the brackets. It should be remarked here that the uniqueness for (ii) was even given with localized DN map. The proof in [5] had one insufficient part which can be corrected by the argument given in this paper. Our main result can be considered as a further development of [5], giving the reconstruction of the linear part and quadratic nonlinear part of $\vec{C}(x, \nabla u)$.

The rest of this paper is organized as follows. In Section 2 we will discuss the ϵ-expansion using which the DN map can be linearized. The linearization of DN map is the DN map for the conductivity equation with conductivity γ. Then by the famous result [7] we reconstruct γ and hence the remaining task is to reconstruct $\vec{b}$. This is done in Section 3.

2. ϵ-expansion of the solution to (BVP)

To prove the theorem, we will use the following ϵ-expansion of solution u to the (BVP)

$$u^f(x) = \epsilon u_1^f(x) + \epsilon^2 u_2^f(x) + \mathcal{O}(\epsilon^3), \tag{2.1}$$

where u_1^f and u_2^f are given as follows. By substituting $\nabla u^f(x) = \epsilon \nabla u_1^f(x) + \epsilon^2 \nabla u_2^f(x) + \mathcal{O}(\epsilon^3)$ in (1.2), we get

$$\begin{aligned}\overrightarrow{C}(x, \nabla_x u^f) &= \gamma(x)\nabla_x u^f(x) + |\nabla u^f(x)|^2 \vec{b}(x) + \overrightarrow{R}(x, \nabla u^f(x)) \\ &= \epsilon\gamma(x)\nabla u_1^f(x) + \epsilon^2 \left(\gamma(x)\nabla u_2^f(x) + |\nabla u_1^f(x)|^2 \vec{b}(x)\right) + \mathcal{O}(\epsilon^3).\end{aligned}$$

Now comparing the various powers of ϵ on both sides, u_1^f is the solution to

$$\begin{cases} L_\gamma u^f(x) := \nabla \cdot \left(\gamma(x)\nabla u_1^f(x)\right) = 0, \ x \in \Omega, \\ u_1^f(x) = f(x), \ x \in \partial\Omega, \end{cases} \tag{2.2}$$

and u_2^f solves

$$\begin{cases}\nabla\cdot\Big(\gamma(x)\nabla u_2^f(x)\Big)+\nabla\cdot\Big(\vec{b}(x)|u_1^f(x)|^2\Big)=0,\ x\in\Omega,\\ u_2^f(x)=0,\ x\in\partial\Omega.\end{cases}\tag{2.3}$$

As for the justification of the above expansion, we refer to [5].

Next, the ϵ-expansion of the DN map is

$$\begin{aligned}\Lambda_{\vec{C}}(\epsilon f)\Big|_{\partial\Omega}&=\epsilon\Big(\gamma(x)\partial_\nu u_1^f(x)\Big)\Big|_{\partial\Omega}+\epsilon^2\Big(\gamma(x)\partial_\nu u_2^f(x)+\nu(x)\cdot\vec{b}(x)|\nabla_x u_1^f(x))|^2\Big)\Big|_{\partial\Omega}+\mathcal{O}(\epsilon^3)\\ &=:\epsilon g_1(x)+\epsilon^2 g_2(x)+\mathcal{O}(\epsilon^3).\end{aligned}\tag{2.4}$$

Hence we can know

$$\Lambda_\gamma(f):=\Big(\gamma(x)\partial_\nu u_1^f(x)\Big)\Big|_{\partial\Omega}=g_1(x)$$

and

$$\Big(\gamma(x)\partial_\nu u_2^f(x)+\nu(x)\cdot\vec{b}(x)|\nabla_x u_1^f(x))|^2\Big)\Big|_{\partial\Omega}=g_2(x).$$

Note that Λ_γ is the DN map for (2.2). Also, since $W^{2-1/p,\,p}(\partial\Omega)$ is dense in the L^2-Sobolev space $H^{1/2}(\partial\Omega)$ of order $1/2$ on $\partial\Omega$ and the boundary value problem 2.2 with Dirichlet data $f\in H^{1/2}(\partial\Omega)$ is well-posed in L^2-Sobolev space $H^1(\Omega)$ of order 1 in Ω, $\Lambda_\gamma(f)$ can be defined for $f\in H^{1/2}(\partial\Omega)$. It is well-known from the work of [7] that γ can be reconstructed from the knowledge of Λ_γ. Once knowing $\gamma(x)$, we also know $u_1^f(x)$ in Ω for every given $f\in H^{1/2}(\partial\Omega)$.

For readers' convenience, we will briefly give a summary of the reconstruction given in [7]. It consists of the following five steps:

Step 1. By the determination at the boundary, reconstruct γ and $\nabla\gamma$ at $\partial\Omega$ (see for example [8]).

Step 2. Compute the DN map $\tilde{\Lambda}_q: H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)$ defined by $\tilde{\Lambda}_q f=\partial_\nu v\big|_{\partial\Omega}$, where $v\in H^1(\Omega)$ is the solution to boundary value problem: $(\Delta-q)v=0$ in Ω, $v\big|_{\partial\Omega}=g\in H^{1/2}(\partial\Omega)$ with $q=(\Delta\sqrt{\gamma})/\sqrt{\gamma}$, and $H^{-1/2}(\partial\Omega)$ is the dual space of $H^{1/2}(\partial\Omega)$.

Step 3. For any fixed $\xi\in\mathbb{R}^n$, let $\zeta\in\mathbb{C}^n$ be such that $\zeta\cdot\zeta=0$, $(\xi+\zeta)\cdot(\xi+\zeta)=0$ and define $t(\xi,\zeta)$ by

$$t(\xi,\zeta):=\langle(\tilde{\Lambda}_q-\tilde{\Lambda}_0)e^{-ix\cdot(\zeta+\xi)}\big|_{\partial\Omega},(2^{-1}I+S_\zeta\tilde{\Lambda}_q-B_\zeta)^{-1}e^{ix\cdot\zeta}\big|_{\partial\Omega}\rangle,$$

where S_ζ, B_ζ are the traces of single layer and double layer potentials of $G_\zeta:=e^{ix\cdot\zeta}(\Delta+2i\zeta\cdot\nabla)^{-1}$ to $\partial\Omega$, respectively. Here we have denoted $\tilde{\Lambda}_q$ when $q=0$ by $\tilde{\Lambda}_0$.

Step 4. Compute the Fourier transform of q extended by 0 outside Ω by the inversion formula:

$$\lim_{|\zeta|\to\infty}t(x,\zeta)=\int_\Omega e^{-x\cdot\xi}q(x)\,dx.$$

Step 5. Solve $(\Delta-q)z=0$ in Ω, $z\big|_{\partial\Omega}=\gamma^{1/2}\big|_{\partial\Omega}$ to get $\gamma=z^2$.

3. Reconstruction of $\vec{b}(x)$

Based on what we have obtained in the previous section, in this section we will give a reconstruction for identifying $\vec{b}(x)$. Let us start this by deriving an integral identity. Take any solution w of $L_\gamma w=0$ in Ω, with enough regularity, and let $\beta_w(x):=\gamma^{-\frac12}(x)\chi_\Omega\vec{b}(x)\cdot\nabla w(x)$, where χ_Ω is the characteristic function of Ω. By multiplying (2.3) by w and integrating over Ω, we have

$$\int\beta_w(x)\gamma^{\frac12}(x)|\nabla u_1^f(x)|^2\mathrm{d}x=\int_{\partial\Omega}\Big(\gamma(x)\partial_\nu u_2^f(x)+\nu(x)\cdot\vec{b}(x)|\nabla_x u_1^f(x))|^2\Big)w(x)\mathrm{d}S_x\,.\tag{3.1}$$

Here and hereafter $\int\,dx$ denotes the integration over $\mathbb{R}^n$ and dS_x denotes the standard measure on $\partial\Omega$.

We will polarize (3.1) as follows. Consider $u_2(x) = u_2^{f+g}(x) - u_2^{f-g}(x)$. Then from equations (2.3) and (3.1), we get

$$4\int \beta_w(x)\gamma^{\frac{1}{2}}(x)\nabla u_1^f(x)\cdot\nabla u_1^g(x)dx = \int_{\partial\Omega}\Big(\gamma(x)\partial_\nu u_2(x) + 4\nu(x)\cdot\vec{b}(x)\nabla_x u_1^f(x)\cdot\nabla u_1^g(x)\Big)w(x)\mathrm{d}S_x. \tag{3.2}$$

The right hand side of equation (3.2) is known for all f and g.

We can choose u_1^f and u_1^g to be complex geometric optics solutions

$$u_1^f(x) = v_1(x) = e^{\zeta_1\cdot x}\gamma^{-\frac{1}{2}}(x)\left(1 + r_1(x,\zeta_1)\right),\ \text{and}\ u_1^g(x) = v_2(x) = e^{\zeta_2\cdot x}\gamma^{-\frac{1}{2}}(x)\left(1 + r_2(x,\zeta_2)\right) \tag{3.3}$$

where r_i, $i = 1,2$ satisfy the equations

$$\triangle r_i + \zeta_i\cdot\nabla r_i - qr_i = q \text{ in } \mathbb{R}^n, \quad q = \frac{\triangle\gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}, \tag{3.4}$$

and the estimate

$$\|r_i\|_{H^\sigma(\Omega)} \le \frac{C}{|\zeta_i|},\ \text{for any}\ \sigma > \frac{n}{2}. \tag{3.5}$$

The expressions for u_1^f and u_1^g in (3.3) and the estimate in (3.5) follow from the work of [12]. Now let $\xi\in\mathbb{R}^n$ be any vector and choose $\eta, k\in\mathbb{S}^{n-1}$ such that

$$k\cdot\xi = k\cdot\eta = \xi\cdot\eta = 0.$$

Using these, define $\zeta_1,\zeta_2\in\mathbb{C}^n$ by

$$\zeta_1 := rk - i\left(\frac{\xi}{2} + s\eta\right),\quad \zeta_2 := -rk - i\left(\frac{\xi}{2} - s\eta\right), \tag{3.6}$$

where r and s are chosen such that

$$r^2 = \frac{|\xi|^2}{4} + s^2.$$

With this, we have

$$\zeta_i\cdot\zeta_i = 0,\ \zeta_1 + \zeta_2 = -i\xi.$$

Note that

$$\nabla v_i = e^{\zeta_i\cdot x}\left[\zeta_i\gamma^{-\frac{1}{2}}(1+r_i) + \nabla(\gamma^{-\frac{1}{2}})(1+r_i) + \gamma^{-\frac{1}{2}}\nabla r_i\right],$$

so

$$\begin{aligned}\nabla v_1\cdot\nabla v_2 &= e^{-i\xi\cdot x}\Big[\gamma^{-1}\zeta_1\cdot\zeta_2 + \gamma^{-\frac{1}{2}}(\zeta_1+\zeta_2)\cdot\nabla(\gamma^{-\frac{1}{2}})(1+r_1)(1+r_2)\\ &\qquad + |\nabla(\gamma^{-\frac{1}{2}})|^2 + \gamma^{-1}(\zeta_1\cdot\nabla r_2 + \zeta_2\cdot\nabla r_1)\Big] + \mathcal{O}(s^{-1})\\ &= e^{-i\xi\cdot x}\left[\left(-\frac{1}{2}|\xi|^2\right)\gamma^{-1} - i\gamma^{-\frac{1}{2}}\xi\cdot\nabla(\gamma^{-\frac{1}{2}}) + |\nabla(\gamma^{-\frac{1}{2}})|^2 + \gamma^{-1}(\zeta_1\cdot\nabla r_2 + \zeta_2\cdot\nabla r_1)\right] + \mathcal{O}(s^{-1}).\end{aligned}$$

Consider the the term

$$\zeta_1\cdot\nabla r_2 = (-i\xi - \zeta_2)\cdot\nabla r_2 = -\xi\cdot\nabla r_2 - q + \triangle r_2 - qr_2 = -q + \mathcal{O}(s^{-1}).$$

Then

$$\nabla v_1\cdot\nabla v_2 = e^{-i\xi\cdot x}\left[\left(-\frac{1}{2}|\xi|^2\right)\gamma^{-1} + i\gamma^{-\frac{3}{2}}\xi\cdot\nabla(\gamma^{\frac{1}{2}}) + \gamma^{-2}|\nabla(\gamma^{\frac{1}{2}})|^2 - 2q\gamma^{-1}\right] + \mathcal{O}(s^{-1}).$$

Taking the limit $s \to \infty$ in (3.2), we get

$$\int e^{-i\xi\cdot x}\beta_w\left[\left(-\frac{1}{2}|\xi|^2\right)\gamma^{-\frac{1}{2}}+i\gamma^{-1}\xi\cdot\nabla(\gamma^{\frac{1}{2}})+\gamma^{-\frac{3}{2}}|\nabla(\gamma^{\frac{1}{2}})|^2-2q\gamma^{-\frac{1}{2}}\right]dx = \text{ known.}$$

It follows that

$$\frac{1}{2}\triangle(\gamma^{-\frac{1}{2}}\beta_w)+\nabla\cdot\left(\gamma^{-1}\nabla(\gamma^{\frac{1}{2}})\beta_w\right)+\left(\gamma^{-\frac{3}{2}}|\nabla(\gamma^{\frac{1}{2}})|^2-2q\gamma^{-\frac{1}{2}}\right)\beta_w = \text{ known,}$$

in $\mathbb{R}^n$, in the sense of distributions, and where we have extended γ so that it is smooth in $\mathbb{R}^n$ and the support of $\gamma-1$ is compact. Since

$$\triangle(\gamma^{-\frac{1}{2}}\beta_w)=\gamma^{-\frac{1}{2}}\triangle\beta_w-2\gamma^{-1}\nabla(\gamma^{\frac{1}{2}})\cdot\nabla\beta_w+\left(2\gamma^{-\frac{3}{2}}|\nabla(\gamma^{\frac{1}{2}})|^2-\gamma^{-1}\triangle(\gamma^{\frac{1}{2}})\right)\beta_w$$

and

$$\nabla\cdot\left(\gamma^{-1}\nabla(\gamma^{\frac{1}{2}})\beta_w\right)=\gamma^{-1}\nabla(\gamma^{\frac{1}{2}})\cdot\nabla\beta_w+\left(\gamma^{-1}\triangle(\gamma^{\frac{1}{2}})-2\gamma^{-\frac{3}{2}}|\nabla\gamma^{\frac{1}{2}})|^2\right)\beta_w,$$

we can conclude that β_w satisfies

$$\triangle\beta_w-3q\beta_w = \text{ known in } \mathbb{R}^n \tag{3.7}$$

in the sense of distributions.

Next we will show that β_w can be known. Since we do know that β_w does exist and satisfies (3.7), we only need to show such β_w is unique. For this it is enough to show that if $f \in L^2(\mathbb{R}^n)$, with compact support, satisfies

$$\triangle f-3qf=0 \text{ in } \mathbb{R}^n,$$

then $f=0$. To start proving this, note that by the interior regularity of solutions of elliptic equations, $f\in C^\infty(\mathbb{R}^n)$. Further, by recalling f is compactly supported, we have $f\in C_0^\infty(\mathbb{R}^n)$.

Now by the limiting absorption principle, for any fixed $\delta>1/2$ and any given $\psi\in L^2_\delta(\mathbb{R}^n)$ there exists a unique $\phi\in L^2_{-\delta}(\mathbb{R}^n)$ such that

$$\triangle\varphi-3q\varphi=\psi \text{ in } \mathbb{R}^n,$$

where

$$L^2_{\pm\delta}(\mathbb{R}^n):=\{\eta\in L^2_{\text{loc}}(\mathbb{R}^n):\|\eta\|_{\pm\delta}:=\big(\int_{\mathbb{R}^n}(1+|x|^2)^{\pm\delta}|\eta(x)|^2\,dx\big)^{1/2}<\infty\}$$

(see Theorem 3.6 in page 413 of [13] for the details). This implies

$$\langle\psi,f\rangle=\langle\Delta\phi-3q\phi,f\rangle=\langle\phi,\Delta f-3qf\rangle=0.$$

Then, since $L^2_\delta(\mathbb{R}^n)$ is dense in $L^2_{\text{loc}}(\mathbb{R}^n)$, we immediately have $f=0$. Summing up we have obtained the following

$$\gamma^{\frac{1}{2}}\beta_w=\vec{b}\cdot\nabla w=\text{known in }\Omega\text{ for all }w\text{ solving }L_\gamma w=0\text{ in }\Omega\text{ with enough regularity.} \tag{3.8}$$

Now let $\{w_j\}_{1\le j\le n}$ be solutions of $L_\gamma w_j=0$ in Ω, with enough regularity, such that $\{\nabla w_j(x)\}_{1\le j\le n}$ are linearly independent for a.e. every $x\in\Omega$ (see Lemma 3.1 of [5] for such $\{w_j\}_{1\le j\le n}$). Therefore, we have that $\vec{b}(x)\cdot\nabla w_j(x)$ is known for all $1\le j\le n$ and $x\in\Omega$. We will denote this known value by $F_{w_j}(x)$. Thus we have the following system of equations

$$\begin{bmatrix}\frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} & \cdots\cdots & \frac{\partial w_1}{\partial x_n}\\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} & \cdots\cdots & \frac{\partial w_2}{\partial x_n}\\ \frac{\partial w_3}{\partial x_1} & \frac{\partial w_3}{\partial x_2} & \cdots\cdots & \frac{\partial w_3}{\partial x_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial w_n}{\partial x_1} & \frac{\partial w_n}{\partial x_2} & \cdots\cdots & \frac{\partial w_n}{\partial x_n}\end{bmatrix}\begin{bmatrix}b_1(x)\\ b_2(x)\\ b_3(x)\\ \vdots\\ b_n(x)\end{bmatrix}=\begin{bmatrix}F_{w_1}(x)\\ F_{w_2}(x)\\ F_{w_3}(x)\\ \vdots\\ F_{w_n}(x)\end{bmatrix},\quad x\in\Omega.$$

Since the matrix

$$A(x):=\left(\left(\frac{\partial w_i}{\partial x_j}\right)\right)_{1\le i,j\le n}$$

is invertible for each $x \in \Omega$, therefore we obtain that

$$\vec{b}(x) = A^{-1}(x)\vec{F}(x),\ x \in \Omega,$$

where

$$\vec{F}(x) := \begin{bmatrix} F_{w_1}(x) \\ F_{w_2}(x) \\ F_{w_3}(x) \\ \vdots \\ F_{w_n}(x) \end{bmatrix}.$$

This gives the reconstruction for $\vec{b}$ in Ω.

Acknowledgement

The work of first author was supported by the Sichuan University. Second author was partially supported by Grant-in-Aid for Scientific Research (15K21766, 15H05740) of the Japan Society for the Promotion of Science doing the research of this paper. The work of third author was supported by NSAF grant (No. U1530401).

References

[1] P. Ciarlet, Lectures on Three-Dimensioanal Elasticity, Published for TIFR, Springer-Verlag, Berline-Heidelberg New York, 1983.
[2] V. Isakov and J. Sylvester, Global uniqueness for a semi linear elliptic inverse problem, Comm. Pure Appl. Math., 47 (1994), 1403-1410.
[3] V. Isakov and A. Nachman, Global uniqueness for a two-dimensional semilinear elliptic inverse problem, Trans. Amer. Math. Soc. 347 (1995), no. 9, 3375-3390.
[4] V. Isakov, Uniqueness of recovery of some quasilinear partial differential equations, Comm. Part. Diff. Equat., 26 (2001), 1947-1973.
[5] H. Kang and G. Nakamura, Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map, Inverse Problems 18 (2002), no. 4, 1079-1088.
[6] V.A. Lubarda, New estimates of the third order elastitic constants for isotropic aggregates of cubic crystal, J. Mech. Phys. Solids, 45 (1997), no. 4, 471-490.
[7] A. Nachman, The inverse reconstructions from boundary measurements, Ann. of Math., 128 (1988), pp 531-587.
[8] G. Nakamura and K. Tanuma, Local determination of conductivity at the boundary from the Dirichlet-to-Neumann map, Inverse Problems 17(2001), 405-419.
[9] Z. Sun, On a quasilinear inverse boundary value problem, Math. Z. 221 (1996), no. 2, 293-305.
[10] Z. Sun and G. Uhlmann, Inverse problems in quasilinear anisotropic media, Amer. J. Math. 119 (1997), no. 4, 771-797.
[11] Z. Sun, An inverse boundary-value problem for semilinear elliptic equations, Electron. J. Differential Equations 2010, No. 37, 5 pp.
[12] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. (2) 125 (1987), no. 1, 153-169.
[13] D. Yafaev, Mathematical scattering theory, Analytic theory, Mathematical Survey Monographs, 158, AMS, Providence, RI 2010.

*School of Mathematics, Sichuan University, Chengdu, Sichuan, 610064, P.R.China.
E-mail:catalin.carstea@gmail.com

†Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
E-mail: nakamuragenn@gmail.com

◇ Beijing Computational Science Research Center, Beijing 100193, China.
E-mail: mvashisth@csrc.ac.cn