Department of Mathematics, IIT Jammu, India
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Marks: 30
Time: 10:00 AM to 11:30 AM

Calculus (SMD001UM1) Mid-sem exam

- Attempt any 6 questions.
- Answers to the **true-false** questions will be considered **only if** correct justification is provided.

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1. True or false: Let $r \in \mathbb{Q} \cap (0,1)$. Then for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $r^n < \epsilon$ for all $n \geq N$. **(5 marks)**

2. True or false: The sequence
$$a_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \cdots \frac{1}{\sqrt{n}} - \sqrt{n}$$
converges. **(5 marks)**

3. Show that the series $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$ and $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ are divergent. **(5 marks)**

4. Let $\{a_n\}$ be a sequence of positive real numbers such that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ is convergent. Does the converse hold? **(5 marks)**

5. Let $a_1 > 0$ is fixed and define the sequence $\{a_n\}$ by $a_{n+1} := \dfrac{1}{2}\left(1 + \dfrac{1}{a_n}\right)$ for $n \in \mathbb{N}$. Then show that $\{a_n\}$ converges to 1. **(5 marks)**

6. Evaluate the following limits.
   (a) $\displaystyle\lim_{x \to 0} x^{\frac{1}{3}} \log|x|$. **(2.5 marks)**
   (b) $\displaystyle\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^3}$. **(2.5 marks)**

7. Check the uniform continuity of the following functions.
   (a) $f(x) = x \cos x$, on $\mathbb{R}$. **(2.5 marks)**
   (b) $f(x) = \tan^{-1} x$, on $\mathbb{R}$. **(2.5 marks)**

8. Let $S := \{x \in \mathbb{R} : |x| > 3\}$ and suppose $f : S \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$, for all $x \in S$. Then is it true that f has to be constant on S? Justify your answer. **(5 marks)**

9. Show that for any real number b, the equation $x^3 - 6x^2 + b = 0$, has at most one root in $[-1, 0]$. Also find the conditions on b for which it has exactly one real root in $[-1, 0]$. **(5 marks)**

10. True or false: Let $f : (a, b) \to \mathbb{R}$ is a differentiable function. Then f is strictly decreasing on (a, b) if and only if $f'(x) < 0$, for all $x \in (a, b)$. **(5 marks)**