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Calculus (Tutorial # 4)

Limits and continuity

1. Using the $\epsilon - \delta$ definition, check the continuity of the following functions defined on their respective domains.
   (a) $f(x) = \frac{1}{x}$, for $x \in (0, \infty)$.
   (b) $f(x) = \sqrt{x}$, for $x \geq 0$.
   (c) $f(x) = x^n$ for $x \in \mathbb{R}$ and fixed $n \in \mathbb{N}$.
   (d) $f(x) = x^{1/n}$ for $x \geq 0$ and fixed $n \in \mathbb{N}$.
   (e) $f(x) = x \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$.
   (f) $f(x) = \sin x$, $x \in \mathbb{R}$.

2. Let $f : (0, \infty) \to \mathbb{R}$ be $f(x) = x^2$. For fixed $\varepsilon > 0$ and arbitrary $a \in (0, \infty)$, show that
$$|x - a| \geq \frac{\varepsilon}{2a} \quad \not\Longrightarrow \quad |f(x) - f(a)| < \varepsilon.$$
   Therefore δ should be taken to be $< \frac{\varepsilon}{2a}$ so that
$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon. \tag{1}$$
   Conclude that there is no δ which is independent of $a \in \mathbb{R}$ for which (1) holds.

3. Let $f : (a, b) \to \mathbb{R}$ be a continuous function at $c \in (a, b)$ such that $f(c) \neq 0$. Then using $\epsilon - \delta$ definition show that there exists a $\delta > 0$ such that $|f(x)| > \dfrac{|f(c)|}{2}$, for all $x \in (c - \delta, c + \delta)$.

4. Check the continuity of the
   (a) **Thomae's function** $f : (0, 1) \to \mathbb{R}$ given by
$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \\ \dfrac{1}{q} & \text{for } \ x := \dfrac{p}{q} \text{ with } p, q \in \mathbb{N},\ p, q \text{ have no common factor.} \end{cases}$$
   (b) **Dirichlet's function** $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \begin{cases} 1 & \text{if } \ x \in \mathbb{Q} \\ -1 & \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by
$$f(x) = \begin{cases} \alpha & \text{if } \ x < 0 \\ ax^2 - bx + c & \text{if } \ x \geq 0. \end{cases}$$
   Find the value(s) of α which ensure the continuity of f at 0.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then prove that if f is continuous at one point then it is continuous on $\mathbb{R}$ and $\exists\, \lambda \in \mathbb{R}$ such that $f(x) = \lambda x$. Does $\exists$ a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ but f is not continuous?

7. Answer the following questions with justification.

   (a) Can you construct a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous only at five points?

   (b) Can you construct a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point?

   (c) Can you construct a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous only at the set of natural numbers?

   (d) Construct a continuous bijection $f : [a, b] \to [c, d]$ such that f^{-1} is also continuous.

8. Answer the following questions with justification.

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions such that $\lim\limits_{x \to a} f(x) = l_1$ exists and $\lim\limits_{y \to l_1} g(y) = l_2$ exists. Then is it true that $\lim\limits_{x \to a} (gof)(x)$ exists? What would be your conclusion if g is continuous at l_1?

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions such that f is continuous at a and $\lim\limits_{x \to f(a)} g(x) = l$ exists. Then is it true that gof is continuous at a? What would be your conclusion if g is continuous at $f(a)$?

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions such that both f and g are uniformly continuous on $\mathbb{R}$. Then is it true that gof is uniformly continuous?

9. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing function. Assume that $a \in I$ is not an endpoint of I. Then prove the following:

   (a) $\lim\limits_{x \to a-} f(x)$ exists and $\lim\limits_{x \to a-} f(x) = \sup\{f(x) : x \in I \text{ and } x < a\}$.

   (b) $\lim\limits_{x \to a+} f(x)$ exists and $\lim\limits_{x \to a+} f(x) = \inf\{f(x) : \ x \in I \text{ and } x > a\}$.

   (c) Define the jump $j_f(a)$ of f at a by

   $$j_f(a) := \lim_{x \to a+} f(x) - \lim_{x \to a-} f(x)$$

   Then prove that f is continuous at a if and only if $j_f(a) = 0$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x}{1+x} & \text{if } x \geq 0 \\ \dfrac{x}{1-x} & \text{if } x < 0. \end{cases}$$

Prove the following:

   (a) f is continuous and bounded.

   (b) $\inf\{f(x) \mid x \in \mathbb{R}\} = -1$ and $\sup\{f(x) \mid x \in \mathbb{R}\} = 1$.

   (c) There are no a and b in $\mathbb{R}$ such that $f(a) = 1$ and $f(b) = -1$.

11. Give examples of functions of each of the following type:

    (a) $f : (0, 1) \to \mathbb{R}$ is continuous, but not bounded above as well as below.

    (b) $f : (0, 1) \to \mathbb{R}$ is continuous, bounded above, not bounded below and does not attain maximum value.

    (c) $f : \mathbb{R} \to \mathbb{R}$ which is bounded but does not attain maximum as well as minimum value.

12. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Which of the following intervals can appear as image $f([a, b])$? Give an example for each of them.

$$(\alpha, \beta), \quad (\alpha, \beta], \quad [\alpha, \beta], \quad (\alpha, \infty), \quad (-\infty, \infty).$$

13. Let $f : (a, b) \to \mathbb{R}$ be a continuous function. Which of the following intervals can appear as image $f((a, b))$? Give an example for each of them.

$$(\alpha, \beta), \quad (\alpha, \beta], \quad [\alpha, \beta], \quad (\alpha, \infty), \quad (-\infty, \infty).$$

14. Answer the following questions with justification.

    (a) Does $\exists$ a continuous function $f : [0, 1] \to (0, \infty)$ which is onto?

    (b) Does $\exists$ a continuous function $f : [a, b] \to (0, 1)$ which is onto?

    (c) Construct a continuous function from $(0, 1)$ onto $[0, 1]$. Can such a function be one-one?

    (d) Does $\exists$ a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}\{x\}$ has exactly two elements?

15. Let $f(x) = xe^x$. Then show that $f : (0, 1) \to (0, e)$ is bijective.

16. Use Intermediate Value Theorem to prove that for any $M \in \mathbb{R}$, $\exists\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(x) = M$.

17. Does $\exists$ a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}$? Justify.

18. Is there a continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$f(a)f(b) < 0 \qquad \text{for all } a, b \in [0, 1]?$$

    Justify.

19. Let $f : [a, b] \to [a, b]$ be a continuous function. Then show that $\exists\, x \in [a, b]$ such that $f(x) = x$.

20. Prove that any polynomial function $p : \mathbb{R} \to \mathbb{R}$ of odd degree is onto.

21. Prove that any continuous one to one function $f : \mathbb{R} \to \mathbb{R}$ is either strictly increasing or strictly decreasing.

22. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ is a monotone function such that $f(I)$ is also an interval. Then show that f is continuous on I.

23. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $f(x+1) = f(x)$ for all x, then show that f is uniformly continuous.

24. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then f maps Cauchy sequences in I to Cauchy sequences in $\mathbb{R}$. Does the converse hold?

25. A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if and only if whenever $\{a_n\}$ and $\{b_n\}$ are sequences such that $|a_n - b_n| \to 0$ as $n \to \infty$, then we have $|f(a_n) - f(b_n)| \to 0$ as $n \to \infty$.

26. Let $f : [0, \infty) \to \mathbb{R}$ be continuous function. Assume that f is uniformly continuous on $[a, \infty)$ for some $a > 0$. Then show that f is uniformly continuous on $[0, \infty)$.

27. Answer the following questions with justification.

    (a) True or false: A uniformly continuous function on a bounded subset of $\mathbb{R}$ is bounded.

    (b) Ture or false: Let f and g be two uniformly continuous functions on an interval $I \subseteq \mathbb{R}$. Then the product $f.g$ is also uniformly continuous on I.

    (c) True or false: Let f is uniformly continuous function on an interval $I \subseteq \mathbb{R}$ be such that $f(x) \neq 0$ for each $x \in I$. Then $\dfrac{1}{f}$ is also uniformly continuous function on I.

28. Determine which of the following functions are uniformly continuous on the given domain?

    (a) $f(x) = \sin x$, for $x \in \mathbb{R}$.

    (b) $f(x) = \tan x$, for $x \in (-\pi/2, \pi/2)$.

    (c) $f(x) = \sec x$, for $x \in (0, \pi/2)$.

    (d) $f(x) = \sqrt{x}$, for $x \geq 0$.

    (e) $f(x) = \dfrac{1}{x}$, for $x \geq \frac{1}{20}$.

    (f) $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \in (0, \infty)$ (or $(0, 1)$).

    (g) $f(x) = \sin x^2$, for $x \in \mathbb{R}$

    (h) $f(x) = \sin(\sin x^2)$, for $x \in \mathbb{R}$.

    (i) $f(x) = x^{\frac{1}{3}} \log(1 + |x|)$, for $x \in \mathbb{R}$.

    (j) $f(x) = \dfrac{1}{x+1} \cos x^2$, for $x \in [0, \infty)$.

    (k) $f(x) = x \sin\left(\frac{1}{x}\right)$, for $x \neq 0$ and $f(0) = 0$.

29. We say that a function $f : I \to \mathbb{R}$ is *Lipshitz* on I, if there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in I$. Now prove that any Lipschitz function f on I is uniformly continuous on I. Does the converse hold?

30. A function $f : (a, b) \to \mathbb{R}$ is said to be *Convex* if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \text{ for all } x, y \in (a, b) \text{ and for any } 0 < \alpha < 1.$$

Then

(a) Prove that every convex function on I is continuous on I. Does the converse hold?

(b) Every increasing convex function of a convex function is convex.

(c) Suppose $f : (a, b) \to \mathbb{R}$ be a continuous function on (a, b) is such that

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x) + f(y)}{2}, \ \forall\ x, y \in (a, b).$$

Then f is convex on (a, b). Does the converse hold?