Some Inverse Problems in Hyperbolic Partial Differential Equations

A Thesis

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by

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Declaration of Authorship

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work was done under the supervision of Dr. Venkateswaran P. Krishnan at TIFR Centre for Applicable Mathematics, Bangalore.

MANMOHAN VASHISTH

In my capacity as supervisor of the candidates thesis, I certify that the above statements are true to the best of my knowledge.

VENKATESWARAN P. KRISHNAN

Date:

Dedicated to

my beloved parents and teachers for their Belief,

Support, Encouragement and Love.

Abstract

In this thesis, we study three inverse problems related to linear and non-linear hyperbolic partial differential equations (PDEs). Our particular interest is in unique determination of the coefficients appearing in these PDEs from the knowledge of functionals that depend on the coefficients and solutions of the PDEs.

In the first problem, we address the inverse problem of determining the density coefficient of a medium by probing it with an external point source and by measuring the responses at a single point for a certain period of time.

In the second problem, we consider the inverse problem of determining time-dependent vector and scalar potentials appearing in the wave equation in space dimension $n \geq 3$ from information about the solution on a suitable subset of the boundary cylinder.

In the third problem, we consider an inverse boundary value problem for a non-linear wave equation of divergence form with space dimension $n \geq 3$. We study the unique determination of quadratic non-linearity appearing in the wave equation from measurements of the solution at the boundary of spacial domain over finite time interval.

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Contents

Chapter 1

Introduction

We consider inverse problems related to linear and non-linear hyperbolic PDEs. These problems arise in the study of seismic imaging, thermo acoustic and photo acoustic tomography (TAT), ultrasound imaging, vibrating string, piezoelectricity, love waves etc. (see [22, 29, 37, 46, 62]). We briefly describe the three problems related to these PDEs as follows:

- *An inverse problem for the wave equation with source and receiver at distinct points [74]*: In this work, we study the inverse problem of determining the coefficient q appearing in the wave equation

$$\begin{aligned} (\Box - q(x))u(x,t) &= \delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad\qquad\quad x \in \mathbb{R}^3 \end{aligned} \tag{1.0.1}$$

  where $\Box := \partial_t^2 - \Delta_x$ denotes the wave operator and q is a real-valued C^1 function on $\mathbb{R}^3$. We call the point of disturbance given by $\delta(x,t)$ on the right hand side of (1.0.1) as the source point and the point where we measure the response we call as the receiver point. Inverse problem of our interest is whether we can determine the potential $q(x)$ uniquely from the measured response of the solution $u(a,t)$ for $0 \leq t \leq T$, at a single point. Rakesh in [52] studied the above problem when $a = 0$ (i.e. when source and receiver are at the same point) for a class of potentials which consists of either the potentials which are comparable or the potentials which are radially symmetric with respect to a point different from the source location. In our work, we consider the case when source and receiver are at *distinct* points.

- *An inverse problem for the relativistic Schrödinger equation with partial boundary data [36]*: In this joint work with Venkateswaran P. Krishnan, we establish the unique determination of the time-dependent vector and scalar potentials $\mathcal{A} := (A_0, A_1, \cdots, A_n)$ and q respectively, appearing in the following relativistic Schrödinger

$$L_{\mathcal{A},q}u(t,x) := \Big((\partial_t + A_0(t,x))^2 - \sum_{j=1}^{n}(\partial_j + A_j(t,x))^2 + q(t,x)\Big)u(t,x)$$

from the measurements of the solution on a part of the boundary. Let $\Omega \subset \mathbb{R}^n$ for $n \geq 3$ be a bounded open set with C^2 boundary. Let $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$. For a fixed $\omega \in \mathbb{S}^{n-1}$, define

$$\partial\Omega_{\pm,\omega} := \{x \in \partial\Omega : \ \pm\omega \cdot \nu(x) \geq 0\}$$

and denote by $\Sigma_{\pm,\omega} := (0,T) \times \partial\Omega_{\pm,\omega}$. Consider the following initial boundary value problem

$$\begin{cases} L_{\mathcal{A},q}u(t,x) = 0; \ (t,x) \in Q \\ u(0,x) = \phi(x), \ \partial_t u(0,x) = \psi(x); \quad x \in \Omega \\ u(t,x) = f(t,x); \ (t,x) \in \Sigma \end{cases} \tag{1.0.2}$$

where $A_j \in C_c^\infty(Q)$, for $j = 0, 1, \cdots, n$ and $q \in L^\infty(Q)$. For $\phi \in H^1(\Omega)$, $\psi \in L^2(\Omega)$ and $f \in H^1(\Sigma)$ such that the compatibility condition $f(0,x) = \phi(x)$ $\forall\ x \in \partial\Omega$ is satisfied, it is shown in [32, 38] that Equation (1.0.2) admits a unique solution u with

$$u \in C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \text{ and } \partial_\nu u \in L^2(\Sigma).$$

Therefore we can define our input-output operator

$$\Lambda_{\mathcal{A},q} : H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma) \ \to \ H^1(\Omega) \times L^2(G)$$

by

$$\Lambda_{\mathcal{A},q}(\phi,\psi,f) = (\partial_\nu u|_G, u|_{t=T}) \tag{1.0.3}$$

where u is the solution to (3.2.3) and $G = (0,T) \times G'$ with G' a neighbourhood of $\partial\Omega_{-,\omega}$ in $\partial\Omega$. We are interested in the unique-determination of the coefficients $\mathcal{A}$ and q appearing in (1.0.2) from the knowledge of the input-output operator $\Lambda_{\mathcal{A},q}$. As follows from the work of [64], one cannot determine the coefficients uniquely given $\Lambda_{\mathcal{A},q}$ because of gauge invariance (see 3.2.2). In our work, we prove unique determination of scalar potential q, and the vector potential term $\mathcal{A}$ modulo a gauge invariant term given the input-output operator $\Lambda_{\mathcal{A},q}$. For more details, we refer to Theorem 3.2.2 in Chapter 3.

- *Inverse boundary value problem for a non-linear hyperbolic partial differential equations [44]*: In this joint work with Gen Nakamura (Emeritus Professor at Hokkaido University, Sapporo, Japan), we consider the inverse problem of determining the quadratic non-linearity appearing in the wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \nabla_x \cdot \vec{C}(x, \nabla_x u(t,x)) = 0, \ (t,x) \in Q_T := (0,T) \times \Omega, \\ u(0,x) = \partial_t u(0,x) = 0, \ x \in \Omega, \\ u = \epsilon f(t,x), \ (t,x) \in \partial Q_T := (0,T) \times \partial\Omega, \end{cases} \tag{1.0.4}$$

where $\vec{C}(x, \nabla_x u)$ is given by

$$\begin{aligned} \vec{C}(x,\nabla_x u) := \gamma(x)\nabla_x u \\ &+ \underbrace{\left(\sum_{k,l=1}^n c_{kl}^1 \partial_k u \partial_l u, \sum_{k,l=1}^n c_{kl}^2 \partial_k u \partial_l u, \cdots, \sum_{k,l=1}^n c_{kl}^n \partial_k u \partial_l u \right)}_{\vec{P}} \\ &+ \vec{R}(x, \nabla_x u), \end{aligned} \tag{1.0.5}$$

here $c \le \gamma(x) \in C^\infty(\overline{\Omega})$ for some constant $c > 0$ and each $c_{kl}^i \in C^\infty(\overline{\Omega})$. Now for $f \in C_c^\infty(\partial Q_T)$, we have for any $T > 0$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the Equation (1.0.4) has a unique solution u (see [44, 46]) satisfying

$$u \in \bigcap_{j=0}^m C^j\left([0,T]; H^{m-j}(\Omega)\right).$$

We define the Dirichlet to Neumann (DN) map $\Lambda_{\vec{C}}^{T}$ by

$$\Lambda_{\vec{C}}^{T}(\epsilon f) = \nu(x) \cdot \vec{C}(x, \nabla_x u^f)|_{\partial Q_T},\ f \in C_c^{\infty}(\partial Q_T),\ 0 < \epsilon < \epsilon_0 \tag{1.0.6}$$

where $u^f(t,x)$ is the solution to (1.0.4) and $\nu(x)$ is the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. The inverse problem we consider is the uniqueness of identifying $\vec{C}$ from the DN map $\Lambda_{\vec{C}}^{T}$. More precisely, we show that if the DN maps $\Lambda_{\vec{C}_i}^{T}$, $i = 1, 2$ given by (1.0.6) for $\vec{C} = \vec{C}_i$, $i = 1, 2$ are same, then $(\gamma_i, \vec{P}_i)$, $i = 1, 2$ are the same, where $(\gamma_i, \vec{P}_i)$, $i = 1, 2$ are $(\gamma, \vec{P})$ associated to $\vec{C}_i$, $i = 1, 2$. For more details, we refer to Theorem 4.1.2 in Chapter 4.

This thesis is organized as follows. In Chapter 2, we give details of our work on inverse problems for the wave equation with source and receiver at distinct points. Chapter 3 provides the details of our work related to unique-determination of the time-dependent vector and scalar potentials in (1.0.2) from the measurements of the solutions on a part of the boundary. In Chapter 4, we give details of our work on determining the non-linearity appearing in the non-linear wave equation (1.0.4).

Chapter 2

An Inverse Problem for the Wave Equation with Source and Receiver at Distinct Points

2.1 Introduction and statement of the main result

In this chapter, we address the inverse problem of determining the density coefficient of a medium by probing it with an external point source and by measuring the responses at a single point for a certain period of time.

More precisely, consider the following initial value problem (IVP):

$$\begin{aligned} (\Box - q(x))u(x,t) &= \delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3, \end{aligned} \tag{2.1.1}$$

where $\Box := \partial_t^2 - \Delta_x = \partial_t^2 - \left(\partial_{x_1}^2 + \partial_{x_3}^2 + \partial_{x_3}^2\right)$ denotes the wave operator.

In Equation (2.1.1), we assume that the coefficient q is real-valued and is a $C^1(\mathbb{R}^3)$ function. The inverse problem we address is the uniqueness of determination of the coefficient q from the knowledge of $u(e,t)$ where $e = (1,0,0)$ for $t \in [0,T]$ with $T > 1$. Motivation for studying such problems arises in geophysics see [73] and references therein. Geophysicists determine properties of the earth structure by sending waves from the surface of the earth and measuring the corresponding scattered responses. Note that in the problem we consider here, the point source is located at the origin, whereas the responses are measured at a different point. Since the given data depends on one variable whereas the coefficient to be determined depends on three variables, some additional restrictions on the coefficient q are required to make the inverse problems tractable. There are several results related to inverse problems with under-determined data, we refer to [52, 53, 59, 61].

There has been extensive work in the literature in the context of formally determined inverse problems involving the wave equation. For a partial list of works in this direction, we refer to [7, 8, 10, 35, 41, 49–51, 54, 55, 60, 63, 67] .

We now state the main results of this chapter.

Theorem 2.1.1. *Suppose* $q_i \in C^1(\mathbb{R}^3)$, $i = 1, 2$ *with* $q_1(x) \geq q_2(x)\ \forall\ x \in \mathbb{R}^3$. *Let* $u_i(x,t)$ *be the solution to the IVP*

$$\begin{aligned} (\Box - q_i(x))u_i(x,t) &= \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3. \end{aligned}$$

If $u_1(e,t) = u_2(e,t)$, *for all* $t \in [0,T]$ *with* $T > 1$ *and where* $e = (1,0,0)$, *then* $q_1(x) = q_2(x)$ *for all* x *with* $|e-x| + |x| \leq T$.

Theorem 2.1.2. *Suppose* $q_i \in C^1(\mathbb{R}^3)$, $i = 1, 2$ *with* $q_i(x) = a_i(|x| + |x-e|)$ *with* $e = (1,0,0)$, *for some* C^1 *functions* a_i *on* $(1-\epsilon, \infty)$ *for some* $0 < \epsilon < 1$. *Let* u_i *be the solution to the IVP*

$$\begin{aligned} (\Box - q_i(x))u_i(x,t) &= \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3. \end{aligned}$$

If $u_1(e,t) = u_2(e,t)$, *for all* $t \in [0,T]$ *with* $T > 1$ *and where* $e = (1,0,0)$, *then* $q_1(x) = q_2(x)$ *for all* x *with* $|e-x| + |x| \leq T$.

To the best of our knowledge, our results, Theorem 2.1.1 and 2.1.2, which treat *separated* source and receiver, have not been studied earlier. Our result generalize the work [52] who considered the aforementioned inverse problem but with *coincident* source and receiver; see also [67].

The proofs of the above theorems are based on an integral identity derived using the solution to an adjoint problem as used in [65] and [67]. Recently this idea was used in [56] as well.

The chapter is organized as follows. In §2.2 we state the existence and uniqueness results for the solution of Equation (2.1.1), the proof of which is given in [20, 39, 60]. §2.3 contains the proofs of Theorems 2.1.1 and 2.1.2.

2.2 Preliminaries

Proposition 2.2.1. [20, pp.139,140] Suppose $q(x)$ is a C^1 function on $\mathbb{R}^3$ and $u(x,t)$ satisfies the following IVP

$$\begin{aligned} (\Box - q(x))u(x,t) &= \delta(x,t), \ \ (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3 \end{aligned} \tag{2.2.1}$$

then $u(x,t)$ is given by

$$u(x,t) = \frac{\delta(t-|x|)}{4\pi|x|} + v(x,t) \tag{2.2.2}$$

where $v(x,t) = 0$ for $t < |x|$ and in the region $t > |x|$, $v(x,t)$ is a C^2 solution of the characteristic boundary value problem (Goursat Problem)

$$\begin{aligned} (\Box - q(x))v(x,t) &= 0, \qquad t > |x| \\ v(x,|x|) &= \frac{1}{8\pi}\int_0^1 q(sx)ds. \end{aligned} \tag{2.2.3}$$

□

We will use the following version of this proposition. Consider the following IVP

$$\begin{aligned} (\Box - q(x))U(x,t) &= \delta(x-e,t), \ \ (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ U(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3. \end{aligned} \tag{2.2.4}$$

Now we have

$$U(x,t) = \frac{\delta(t-|x-e|)}{4\pi|x-e|} + V(x,t) \tag{2.2.5}$$

where $V(x,t) = 0$ for $t < |x-e|$ and for $t > |x-e|$, $V(x,t)$ is a C^2 solution to the following Goursat Problem

$$\begin{aligned} (\Box - q(x))V(x,t) &= 0, \qquad t > |x-e| \\ V(x,|x-e|) &= \frac{1}{8\pi}\int_0^1 q(sx+(1-s)e)ds. \end{aligned} \tag{2.2.6}$$

We can see this by translating source by $-e$ in Equation (2.2.4) and using the above proposition.

2.3 Proof of Theorems 2.1.1 and 2.1.2

In this section, we prove Theorems 2.1.1 and 2.1.2. We will first show the following three lemmas which will be used in the proof of the main results.

Lemma 2.3.1. *Suppose $q_i's$ $i = 1,2$ be C^1 real-valued functions on $\mathbb{R}^3$. Let u_i be the solution to Equation (2.1.1) with $q = q_i$ and denote $u(x,t) := u_1(x,t) - u_2(x,t)$ and $q(x) := q_1(x) - q_2(x)$. Then we have the following integral identity*

$$u(e,\tau) = \int_{\mathbb{R}}\int_{\mathbb{R}^3} q(x)u_2(x,t)U(x,\tau-t)dxdt \text{ for all } \tau \in \mathbb{R} \tag{2.3.1}$$

where $U(x,t)$ is the solution to the following IVP

$$\begin{aligned} (\Box - q_1(x))U(x,t) &= \delta(x-e,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ U(x,t)|_{t<0} &= 0, \qquad\qquad\quad x \in \mathbb{R}^3. \end{aligned} \tag{2.3.2}$$

Proof. Since each u_i for $i = 1,2$ satisfies the following IVP,

$$\begin{aligned} (\Box - q_i(x))u_i(x,t) &= \delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3, \end{aligned}$$

we have that u satisfies the following IVP

$$\begin{aligned} (\Box - q_1(x))u(x,t) &= q(x)u_2(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad x \in \mathbb{R}^3. \end{aligned} \tag{2.3.3}$$

Now since

$$u(e,\tau) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} u(x,t)\delta(x-e,\tau-t)dtdx,$$

using (2.3.2), we have

$$u(e,\tau) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} u(x,t)(\Box - q_1(x))U(x,\tau-t)dtdx.$$

Now by using integration by parts and Equations (2.3.2) and (2.3.3), also taking into account that $u(x,t)=0$ for $t<|x|$ and that $U(x,t)=0$ for $|x-e|>t$, we get

$$\begin{aligned}u(e,\tau) &= \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} U(x,\tau-t)(\Box - q_1(x))u(x,t)dtdx\\ &= \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)u_2(x,t)U(x,\tau-t)dtdx.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.3.2. *Suppose q_i's are as in Lemma 2.3.1 and u_i is the solution to Equation (2.1.1) with $q=q_i$ and if $u(e,t) := (u_1-u_2)(e,t) = 0$ for all $t\in[0,T]$, then there exists a constant $K>0$ depending on the bounds on v_2, V and T such that the following inequality holds*

$$\left|\int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x - |x|e|}dS_x\right| \leq K \int\limits_{|x-e|+|x|\leq 2\tau} \frac{|q(x)|}{|x||x-e|}dx, \;\; \forall \tau\in(1/2,T/2]. \tag{2.3.4}$$

Here dS_x is the surface measure on the ellipsoid $|x-e|+|x|=2\tau$ and v_2, V are solutions to the Goursat problem (see Equations (2.2.3) and (2.2.6)) corresponding to $q=q_i$.

Proof. From Lemma 2.3.1, we have

$$u(e,2\tau) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^3} q(x)u_2(x,t)U(x,2\tau-t)dxdt, \;\text{ for all } \tau\in\mathbb{R}.$$

Now since $u(e, 2\tau) = 0$ for all $\tau \in [0, T/2]$, and using Equations (2.2.2) and (2.2.5), we get

$$\begin{aligned}
0 =& \int_{\mathbb{R}^3}\int_{\mathbb{R}} q(x)\frac{\delta(t-|x|)\delta(2\tau - t - |x-e|)}{16\pi^2|x||x-e|}dtdx \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}} q(x)\frac{\delta(t-|x|)V(x, 2\tau - t)}{4\pi|x|}dtdx \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}} q(x)\frac{\delta(2\tau - t - |x-e|)}{4\pi|x-e|}v_2(x,t)dtdx \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}} q(x)V(x, 2\tau - t)v_2(x,t)dtdx.
\end{aligned}$$

Now using the fact that $v_2(x,t) = 0$ for $t < |x|$, $V(x,t) = 0$ for $t < |x-e|$ and

$$\int_{\mathbb{R}^n} \phi(x)\delta(P)dx = \int_{P(x)=0} \frac{\phi(x)}{|\nabla_x P(x)|}dS_x$$

where dS_x is the surface measure on the surface $P = 0$, we have that

$$\begin{aligned}
0 =& \frac{1}{16\pi^2}\int_{|x-e|+|x|=2\tau} \frac{q(x)}{|x||x-e||\nabla_x(2\tau - |x| - |x-e|)|}dS_x \\
&+ \frac{1}{4\pi}\int_{|x|+|x-e|\le 2\tau} \frac{q(x)V(x, 2\tau - |x|)}{|x|}dx \\
&+ \frac{1}{4\pi}\int_{|x|+|x-e|\le 2\tau} \frac{q(x)v_2(x, 2\tau - |x-e|)}{|x-e|}dx \\
&+ \int_{|x|+|x-e|\le 2\tau}\int_{|x|}^{2\tau-|x-e|} q(x)V(x, 2\tau - t)v_2(x,t)dtdx.
\end{aligned}$$

For simplicity, denote

$$\begin{aligned}
F(\tau, x) :=& \frac{1}{4\pi}\Big(|x-e|V(x, 2\tau - |x|) + |x|v_2(x, 2\tau - |x-e|) \\
&+ 4\pi|x||x-e|\int_{|x|}^{2\tau-|x-e|} V(x, 2\tau - t)v_2(x,t)dt\Big)
\end{aligned}$$

and using

$$|\nabla_x(2\tau - |x| - |x-e|)| = \Big|\frac{x}{|x|} + \frac{x-e}{|x-e|}\Big| = \Big|\frac{|x-e|x + (x-e)|x|}{|x||x-e|}\Big|.$$

We have

$$\frac{1}{16\pi^2}\int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x - |x|e|}dS_x = -\int\limits_{|x|+|x-e|\le 2\tau} \frac{q(x)}{|x||x-e|}F(\tau,x)dx.$$

Note that $\tau \in [0, T/2]$ with $T < \infty$. Now using the boundedness of v_2 and V on compact subsets, we have $|F(\tau,x)| \le K$ on $|x| + |x-e| \le T$.

Therefore, finally we have

$$\Bigg|\int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x - |x|e|}dS_x\Bigg| \le K\int\limits_{|x-e|+|x|\le 2\tau} \frac{|q(x)|}{|x||x-e|}dx,\ \forall\tau \in (1/2, T/2].$$

The lemma is proved. $\square$

Lemma 2.3.3. *Consider the solid ellipsoid $|e-x|+|x| \le r$, where $e = (1,0,0)$ and $x = (x_1, x_2, x_3)$, then we have its parametrization in prolate-spheriodal co-ordinates (ρ, θ, ϕ) given by*

$$\begin{aligned} x_1 &= \frac{1}{2} + \frac{1}{2}\cosh\rho\cos\phi \\ x_2 &= \frac{1}{2}\sinh\rho\sin\theta\sin\phi \\ x_3 &= \frac{1}{2}\sinh\rho\cos\theta\sin\phi \end{aligned} \tag{2.3.5}$$

with $\cosh\rho \le r$, $\theta \in (0, 2\pi)$, $\phi \in (0,\pi)$ *and the surface measure dS_x on $|e-x| + |x| = r$ and volume element dx on $|e-x| + |x| \le r$, are given by*

$$\begin{aligned} dS_x &= \frac{1}{4}\sinh\rho\sin\phi\sqrt{\cosh^2\rho - \cos^2\phi}\,d\theta d\phi, \\ &\quad \textit{with}\ \cosh\rho = r,\ \theta \in [0,2\pi]\ \textit{and}\ \phi \in [0,\pi] \\ dx &= \frac{1}{8}\sinh\rho\sin\phi(\cosh^2\rho - \cos^2\phi)d\rho d\theta d\phi, \\ &\quad \textit{with}\ \cosh\rho \le r,\ \theta \in [0,2\pi]\ \textit{and}\ \phi \in [0,\pi]. \end{aligned} \tag{2.3.6}$$

Proof. The above result is well known, but for completeness, we will give the proof. The solid ellipsoid $|e-x|+|x| \le r$ in explicit form can be written as

$$\frac{(x_1-1/2)^2}{r^2/4}+\frac{x_2^2}{(r^2-1)/4}+\frac{x_3^2}{(r^2-1)/4} \le 1.$$

From this, we see that

$$\begin{aligned} x_1 &= \frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi \\ x_2 &= \frac{1}{2}\sinh\rho\sin\theta\sin\phi \\ x_3 &= \frac{1}{2}\sinh\rho\cos\theta\sin\phi \end{aligned}$$

with $\cosh\rho \le r$, $\theta \in [0,2\pi]$ and $\phi \in [0,\pi]$. This proves the first part of the lemma.

Now the parametrization of ellipsiod $|e-x|+|x| = r$, is given by

$$F(\theta,\phi)=\Big(\frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi, \frac{1}{2}\sinh\rho\sin\theta\sin\phi, \frac{1}{2}\sinh\rho\cos\theta\sin\phi\Big)$$

with $\theta \in (0,2\pi)$, $\phi \in (0,\pi)$, and $\cosh\rho = r$.

Next, we have

$$\begin{aligned} \frac{\partial F}{\partial\theta} &= \Big(0, \frac{1}{2}\sinh\rho\cos\theta\sin\phi, -\frac{1}{2}\sinh\rho\sin\theta\sin\phi\Big) \\ \frac{\partial F}{\partial\phi} &= \Big(-\frac{1}{2}\cosh\rho\sin\phi, \frac{1}{2}\sinh\rho\sin\theta\cos\phi, \frac{1}{2}\sinh\rho\cos\theta\cos\phi\Big). \end{aligned}$$

We have $dS_x = |\frac{\partial F}{\partial\theta}\times\frac{\partial F}{\partial\phi}|d\theta d\phi$, simple computation will gives us

$$\begin{aligned} dS_x &= \frac{1}{4}\sinh\rho\sin\phi\sqrt{\cosh^2\rho-\cos^2\phi}d\theta d\phi, \\ &\text{with } \cosh\rho = r,\ \theta \in (0,2\pi)\ \textit{and}\ \phi \in (0,\pi). \end{aligned}$$

Last part of the lemma follows from change of variable formula, which is given by

$$dx = \Big|\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\theta,\phi)}\Big|d\theta d\phi d\rho;\ \text{with } \cosh\rho \le r,\ \theta \in [0,2\pi] \text{ and } \phi \in [0,\pi]$$

where $\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\theta,\phi)}$ is given by

$$\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\theta,\phi)} = \det\begin{vmatrix} \frac{\partial x_1}{\partial\rho} & \frac{\partial x_1}{\partial\theta} & \frac{\partial x_1}{\partial\phi} \\ \frac{\partial x_2}{\partial\rho} & \frac{\partial x_2}{\partial\theta} & \frac{\partial x_3}{\partial\phi} \\ \frac{\partial x_3}{\partial\rho} & \frac{\partial x_3}{\partial\theta} & \frac{\partial x_3}{\partial\phi} \end{vmatrix}.$$

This gives

$$dx = \frac{1}{8}\sinh\rho\sin\phi(\cosh^2\rho - \cos^2\phi)d\rho d\theta d\phi,$$
$$\text{with } \cosh\rho \le r,\ \theta\in[0,2\pi] \text{ and } \phi\in[0,\pi].$$

□

2.3.1 Proof of Theorem 2.1.1

We first consider the surface integral in Equation (2.3.4) and denote it by $Q(2\tau)$:

$$Q(2\tau) := \int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x - |x|e|} dS_x. \tag{2.3.7}$$

We have

$$\begin{aligned} |2\tau x - |x|e| = |(2\tau x_1 - |x|, 2\tau x_2, 2\tau x_3)| &= \sqrt{(2\tau x_1 - |x|)^2 + 4\tau^2 x_2^2 + 4\tau^2 x_3^2} \\ &= \sqrt{4\tau^2|x|^2 + |x|^2 - 4\tau x_1|x|}. \end{aligned}$$

From Equation (2.3.5) and using the fact that $\cosh\rho = 2\tau$, we have

$$\begin{aligned} |2\tau x - |x|e| &= \frac{1}{2}\sqrt{(2\tau+\cos\phi)\{(4\tau^2+1)(2\tau+\cos\phi) - 4\tau(1+2\tau\cos\phi)\}} \\ &= \frac{1}{2}\sqrt{(2\tau+\cos\phi)(8\tau^3+4\tau^2\cos\phi+2\tau+\cos\phi-4\tau-8\tau^2\cos\phi)} \\ &= \frac{1}{2}\sqrt{(2\tau+\cos\phi)(8\tau^3-4\tau^2\cos\phi-2\tau+\cos\phi)} \\ &= \frac{1}{2}\sqrt{(4\tau^2-\cos^2\phi)(4\tau^2-1)}. \end{aligned}$$

Using the above expression for $|2\tau x - |x|e|$ and Equation (2.3.6), we get

$$Q(2\tau) = \frac{1}{8}\int_0^{\pi}\int_0^{2\pi} \frac{q(\rho,\theta,\phi)\sinh\rho\sin\phi\sqrt{\cosh^2\rho-\cos^2\phi}}{\sqrt{(4\tau^2-\cos^2\phi)(4\tau^2-1)}}d\theta d\phi,$$

where we have denoted

$$q(\rho,\theta,\phi) = q\left(\frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi, \frac{1}{2}\sinh\rho\sin\theta\sin\phi, \frac{1}{2}\sinh\rho\cos\theta\sin\phi\right).$$

On using $\cosh\rho = 2\tau$, $\sinh\rho = \sqrt{4\tau^2-1}$ and $\rho = \ln(2\tau+\sqrt{4\tau^2-1})$, we get

$$Q(2\tau) = \int_0^{\pi}\int_0^{2\pi} q(\ln(2\tau+\sqrt{4\tau^2-1}),\theta,\phi)\sin\phi d\theta d\phi. \tag{2.3.8}$$

Now consider the integral

$$\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx.$$

Again using Equations (2.3.5) and (2.3.6) in the above integral, we have

$$\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx = \frac{1}{2}\int_{\cosh\rho\leq 2\tau}\int_0^{\pi}\int_0^{2\pi} q(\rho,\theta,\phi)\sinh\rho\sin\phi d\theta d\phi d\rho.$$

After substituting $\cosh\rho = r$ and $\rho = \ln(r+\sqrt{r^2-1})$, we get

$$\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx = \frac{1}{2}\int_1^{2\tau}\int_0^{\pi}\int_0^{2\pi} q(\ln(r+\sqrt{r^2-1}),\theta,\phi)\sin\phi d\theta d\phi dr.$$

Now using (2.3.8), we get

$$\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx \leq C\int_1^{2\tau} Q(r)dr. \tag{2.3.9}$$

Now applying this inequality in Equation (2.3.4) and noting that $q(x) = q_1(x) - q_2(x) \geq 0$, we get

$$Q(2\tau) \leq CK \int_1^{2\tau} Q(r)dr. \tag{2.3.10}$$

Now Equation (2.3.10) holds for all $\tau \in [1/2, T/2]$ and since $q(x) \geq 0$ for all $x \in \mathbb{R}^3$, by Gronwall's inequality, we have

$$Q(2\tau) = 0, \quad \tau \in [1/2, T/2].$$

Now from Equation (2.3.7), again using $q(x) \geq 0$, we have $q(x) = 0$, for all $x \in \mathbb{R}^3$ such that $|x| + |x - e| \leq T$. The proof is complete.

2.3.2 Proof of Theorem 2.1.2

Again first, we consider the surface integral in (2.3.4) and denote it by $Q(2\tau)$:

$$Q(2\tau) := \int\limits_{|x|+|x-e|=2\tau} \frac{q(x)}{|2\tau x - |x|e|} dS_x.$$

and $q(x) := a(|x| + |x - e|)$. Now from Equations (2.3.5), (2.3.6) and (2.3.8) and hypothesis $q_i(x) = a_i(|x| + |x - e|)$ of the theorem, we get

$$Q(2\tau) = \frac{1}{8} \int_0^{\pi} \int_0^{2\pi} a(2\tau) \sin\phi d\theta d\phi = \frac{\pi}{2} a(2\tau). \tag{2.3.11}$$

Now consider the integral

$$\int\limits_{|x|+|x-e|\leq 2\tau} \frac{q(x)}{|x||x - e|} dx.$$

Again using (2.3.5) and (2.3.6) in the above integral, we have

$$\int\limits_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x - e|} dx = \frac{1}{2} \int\limits_{\cosh\rho\leq 2\tau} \int_0^{\pi} \int_0^{2\pi} q(\rho, \theta, \phi) \sinh\rho \sin\phi d\theta d\phi d\rho.$$

After substituting $\cosh\rho = r$ and $\rho = \ln(r + \sqrt{r^2 - 1})$, we get

$$\left|\int\limits_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx\right| = \left|\frac{1}{2}\int\limits_1^{2\tau}\int\limits_0^{\pi}\int\limits_0^{2\pi} a(r)\sin\phi d\theta d\phi dr\right|$$
$$\leq C\int\limits_1^{2\tau}|a(r)|dr.$$

Now using this inequality and Equation (2.3.11) in (2.3.4), we see

$$|a(2\tau)| \leq C\int\limits_1^{2\tau}|a(r)|dr. \tag{2.3.12}$$

Now Equation (2.3.12) holds for all $\tau \in [1/2, T/2]$, so using Gronwall's inequality, we have

$$a(2\tau) = 0, \qquad \tau \in [1/2, T/2].$$

Thus, we have $q(x) = 0$, for all $x \in \mathbb{R}^3$ such that $|x| + |x - e| \leq T$.

Chapter 3

An Inverse Problem for the Relativistic Schrödinger Equation with Partial Boundary Data

3.1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain with C^2 boundary. For $T > \operatorname{diam}(\Omega)$, let $Q := (0,T) \times \Omega$ and denote its lateral boundary by $\Sigma := (0,T) \times \partial\Omega$. Consider the linear hyperbolic partial differential operator of second order with time-dependent coefficients:

$$\mathcal{L}_{\mathcal{A},q} u := \Big\{ (\partial_t + A_0(t,x))^2 - \sum_{j=1}^{n} (\partial_j + A_j(t,x))^2 + q(t,x) \Big\} u = 0, \ \ (t,x) \in Q. \tag{3.1.1}$$

We denote $\mathcal{A} = (A_0, \cdots, A_n)$ and $A = (A_1, \cdots, A_n)$. Then $\mathcal{A} = (A_0, A)$. We assume that $\mathcal{A}$ is $\mathbb{R}^{1+n}$ valued with coefficients in $C_c^\infty(Q)$ and $q \in L^\infty(Q)$. The operator (3.1.1) is known as the relativistic Schrödinger equation and appears in quantum mechanics and general relativity [40, Chap. XII]. In this paper, we study an inverse problem related to this operator. More precisely, we are interested in determining the coefficients in (3.1.1) from certain measurements made on suitable subsets of the topological boundary of Q.

Starting with the work of Bukhgeìm and Klibanov [8], there has been extensive work in the literature related to inverse boundary value problems for second order linear hyperbolic PDE. For the case when $\mathcal{A}$ is 0 and q is time-independent, the unique determination of q from full lateral boundary Dirichlet to Neumann data was addressed by Rakesh and Symes in [55]. Isakov in [27] considered the same problem with an additional time-independent time derivative perturbation, that is, with $\mathcal{A} = (A_0(x), 0)$ and $q(x)$ and proved uniqueness results. The results in [55] and [27] were proved using geometric optics solutions inspired by the work of Sylvester and Uhlmann [72]. For the case of time-independent coefficients, another powerful tool to prove uniqueness results is the boundary control (BC) method pioneered by Belishev, see [2–4]. Later it was developed by Belishev, Kurylev, Katchalov, Lassas, Eskin and others; see [32] and references therein. Eskin in [16, 18] developed a new approach based on the BC method for determining the time-independent vector and scalar potentials assuming $A_0 = 0$ in (3.1.1). Hyperbolic inverse problems for time-independent coefficients have been extensively studied by Yamamoto and his collaborators as well; see [1, 6, 12, 24, 25]

Inverse problems involving time-dependent first and zeroth order perturbations focusing on the cases $\mathcal{A} = 0$ or when $\mathcal{A}$ is of the form $(A_0, 0)$ have been well studied in prior works. We refer to [57, 58, 66] for some works in this direction.

Eskin in [17] considered full first and zeroth order time-dependent perturbations of the wave equation in a Riemannian manifold set-up and proved uniqueness results (for the first order term, uniqueness modulo a natural gauge invariance) from boundary Dirichlet-to-Neumann data, under the assumption that the coefficients are analytic in time. Salazar removed the analyticity assumption of Eskin in [64], and proved that the unique determination of vector and scalar potential modulo a natural gauge invariance is possible from Dirichlet-to-Neumann data on the boundary. In a recent work of Stefanov and Yang in [47], proved stability estimates for the recovery of light ray transforms of time-dependent first- and zeroth-order perturbations for the wave equation in a Riemannian manifold setting from certain local Dirichlet to Neumann map. Their results, in particular, would give uniqueness results in suitable subsets of the domain recovering the vector field term up to a natural gauge invariance and the zeroth-order potential term from this data.

For the case of time-dependent perturbations, if one is interested in global uniqueness results in a finite time domain, extra information in addition to Dirichlet-to-Neumann data is required to prove uniqueness results. Isakov in [26] proved unique-determination of time-dependent potentials (assuming $\mathcal{A} = 0$ in (3.1.1)) from the data set given by the Dirichlet-to-Neumann data as well as the solution and the time derivative of the solution on the domain at the final time. Recently Kian in [33] proved unique determination of time-dependent damping coefficient $A_0(t,x)$ (with $\mathcal{A}$ of the form, $\mathcal{A} = (A_0, 0)$) and the potential $q(t,x)$ from partial Dirichlet to Neumann data together with information of the solution at the final time.

In this article, we prove unique determination of time-dependent vector and scalar potentials $\mathcal{A}(t,x)$ and $q(t,x)$ appearing in (3.1.1) (modulo a gauge invariance for the vector potential) from partial boundary data. Our work extends the results of Salazar [64] in the sense that in [64], the uniqueness results are shown with full Dirichlet to Neumann boundary data and assuming such boundary measurements are available for all time. Additionally, it extends the recent work of Kian [33], since we consider the full time-dependent vector field perturbation, whereas Kian assumes only a time derivative perturbation.

The chapter is organized as follows. In §3.2, we state the main result of the article. In §3.3, we prove the Carleman estimates required to prove the existence of geometric optics (GO) solutions, and in §3.4, we construct the required GO solutions. In §3.5, we derive the integral identity using which, we prove the main theorem in §3.6.

3.2 Statement of the main result

In this section, we state the main result of this article. We begin by stating precisely what we mean by gauge invariance.

3.2.1 Gauge Invariance

Definition 3.2.1. The vector potentials $\mathcal{A}^{(1)}, \mathcal{A}^{(2)} \in C_c^\infty(Q)$ are said to be gauge equivalent if there exists a $g(t,x) \in C^\infty(\overline{Q})$ such that $g(t,x) = e^{\Phi(t,x)}$ with $\Phi \in$

$C_c^\infty(Q)$ and

$$(\mathcal{A}^{(2)} - \mathcal{A}^{(1)})(t,x) = -\frac{\nabla_{t,x} g(t,x)}{g(t,x)} = -\nabla_{t,x}\Phi(t,x).$$

Now we state the following proposition proof of which is given in [64].

Proposition 3.2.2. *[64] Suppose $u(t,x)$ is a solution to the following IBVP*

$$\begin{aligned} \Big[(\partial_t + A_0^{(1)}(t,x))^2 - \sum_{j=1}^{n}(\partial_j + A_j^{(1)}(t,x))^2 + q_1(t,x)\Big]u(t,x) &= 0 \quad \text{in } Q \\ u(0,x) = \partial_t u(0,x) &= 0 \quad \text{in } \Omega \\ u|_\Sigma &= f \end{aligned} \tag{3.2.1}$$

and $g(t,x)$ is as defined above, then $v(t,x) = g(t,x)u(t,x)$ satisfies the following IBVP

$$\begin{aligned} \Big[(\partial_t + A_0^{(2)}(t,x))^2 - \sum_{j=1}^{n}(\partial_j + A_j^{(2)}(t,x))^2 + q_1(t,x)\Big]v(t,x) &= 0 \quad \text{in } Q \\ v(0,x) = \partial_t v(0,x) &= 0 \quad \text{in } \Omega \\ v|_\Sigma &= f \end{aligned} \tag{3.2.2}$$

with $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ gauge equivalent. In addition if Λ_1 and Λ_2 are Dirichlet to Neumann operator associated with (3.2.1) and (3.2.2) respectively, then

$$\Lambda_1 = \Lambda_2.$$

3.2.2 Statement of the main result

We introduce some notation. Following [9], fix an $\omega_0 \in \mathbb{S}^{n-1}$, and define the ω_0-shadowed and ω_0-illuminated faces by

$$\partial\Omega_{+,\omega_0} := \{x \in \partial\Omega : \ \nu(x)\cdot\omega_0 \geq 0\}, \quad \partial\Omega_{-,\omega_0} := \{x \in \partial\Omega : \ \nu(x)\cdot\omega_0 \leq 0\}$$

of $\partial\Omega$, where $\nu(x)$ is outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Corresponding to $\partial\Omega_{\pm,\omega_0}$, we denote the lateral boundary parts by $\Sigma_{\pm,\omega_0} := (0,T)\times\partial\Omega_{\pm,\omega_0}$. We denote by $F = (0,T)\times F'$ and $G = (0,T)\times G'$ where F' and G' are small enough open neighbourhoods of $\partial\Omega_{+,\omega_0}$ and $\partial\Omega_{-,\omega_0}$ respectively in $\partial\Omega$.

Consider the IBVP

$$\begin{cases} \mathcal{L}_{\mathcal{A},q}u(t,x) = 0; \ (t,x) \in Q \\ u(0,x) = \phi(x), \ \partial_t u(0,x) = \psi(x); \ x \in \Omega \\ u(t,x) = f(t,x), \ (t,x) \in \Sigma. \end{cases} \tag{3.2.3}$$

For $\phi \in H^1(\Omega), \psi \in L^2(\Omega)$ and $f \in H^1(\Sigma)$, (3.2.3) has a unique solution $u \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^1(\Omega))$ and furthermore $\partial_\nu u \in L^2(\Sigma)$; see [32, 38]. Thus we have $u \in H^1(Q)$. Therefore we can define our input-output operator $\Lambda_{\mathcal{A},q}$ by

$$\Lambda_{\mathcal{A},q}(\phi,\psi,f) = (\partial_\nu u|_G, u|_{t=T}) \tag{3.2.4}$$

where u is the solution to (3.2.3). The operator

$$\Lambda_{\mathcal{A},q} : H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma) \ \rightarrow \ H^1(\Omega) \times L^2(G)$$

is a continuous linear map which follows from the well-posedness of the IBVP given by Equation (3.2.3) (see [32, 38]). A natural question is whether this input-output operator uniquely determines the time-dependent perturbations $\mathcal{A}$ and q. We now state our main result.

Theorem 3.2.3. *Let $\left(\mathcal{A}^{(1)}, q_1\right)$ and $\left(\mathcal{A}^{(2)}, q_2\right)$ be two sets of vector and scalar potentials such that each $A_j^{(i)} \in C_c^\infty(Q)$ and $q_i \in L^\infty(Q)$ for $i = 1,2$ and $0 \leq j \leq n$. Let u_i be solutions to (3.2.3) when $(\mathcal{A}, q) = (\mathcal{A}^{(i)}, q_i)$ and $\Lambda_{\mathcal{A}^{(i)},q_i}$ for $i = 1,2$ be the input-output operators defined by (3.2.4) corresponding to u_i. If*

$$\Lambda_{\mathcal{A}^{(1)},q_1}(\phi,\psi,f) = \Lambda_{\mathcal{A}^{(2)},q_2}(\phi,\psi,f), \ \textit{for all } (\phi,\psi,f) \in H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma),$$

then there exists a function $\Phi \in C_c^\infty(Q)$ such that

$$(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})(t,x) = \nabla_{t,x}\Phi(t,x); \ \textit{and } q_1(t,x) = q_2(t,x), \ (t,x) \in Q.$$

Remark 3.2.4. We have stated the above result for vector potentials in $C_c^\infty(Q)$ for simplicity. We can, in fact, prove for the case in which the vector potentials $\mathcal{A} \in W^{1,\infty}(Q)$ provided they are identical on the boundary ∂Q, see [33].

3.3 Carleman Estimate

We denote by $H^1_{\mathrm{scl}}(Q)$, the semiclassical Sobolev space of order 1 on Q with the following norm

$$||u||_{H^1_{\mathrm{scl}}(Q)} = ||u||_{L^2(Q)} + ||h\nabla_{t,x}u||_{L^2(Q)},$$

and for $Q = \mathbb{R}^{1+n}$ we denote by $H^s_{\mathrm{scl}}(\mathbb{R}^{1+n})$, the Sobolev space of order s with the norm given by

$$||u||^2_{H^s_{\mathrm{scl}}(\mathbb{R}^{1+n})} = ||\langle hD\rangle^s u||^2_{L^2(\mathbb{R}^{1+n})} = \int\limits_{\mathbb{R}^{1+n}} (1 + h^2\tau^2 + h^2|\xi|^2)^s |\widehat{u}(\tau,\xi)|^2 \,\mathrm{d}\tau\mathrm{d}\xi.$$

In this section, we derive a Carleman estimate involving boundary terms for (3.1.1) conjugated with a linear weight. We use this estimate to control boundary terms over subsets of the boundary where measurements are not available. Our proof follows from modifications of the Carleman estimate given in [33]. Since we work in a semiclassical setting, we prefer to give the proof for the sake of completeness.

Theorem 3.3.1. *Let $\varphi(t,x) := t + x\cdot\omega$, where $\omega \in \mathbb{S}^{n-1}$ is fixed. Assume that $A_j \in C^\infty_c(Q)$ for $0 \le j \le n$ and $q \in L^\infty(Q)$. Then the Carleman estimate*

$$\begin{aligned}
&h\left(e^{-\varphi/h}\partial_\nu\varphi\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{+,\omega})} + h\left(e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot), e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot)\right)_{L^2(\Omega)}\\
&+\|e^{-\varphi/h}u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\partial_t u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\nabla_x u\|^2_{L^2(Q)}\\
&\le C\Big(\|he^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}u\|^2_{L^2(Q)} + \left(e^{-\varphi(T,\cdot)/h}u(T,\cdot), e^{-\varphi(T,\cdot)/h}u(T,\cdot)\right)_{L^2(\Omega)} \qquad (3.3.1)\\
&+h\left(e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot), e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot)\right)_{L^2(\Omega)} + h\left(e^{-\varphi/h}\left(-\partial_\nu\varphi\right)\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{-,\omega})}\Big)
\end{aligned}$$

holds for all $u \in C^2(Q)$ with

$$u|_\Sigma = 0,\ u|_{t=0} = \partial_t u|_{t=0} = 0,$$

and h small enough.

Proof. To prove the estimate (3.3.1), we will use a convexification argument used in [33]. Consider the following perturbed weight function

$$\widetilde{\varphi}(t,x) = \varphi(t,x) - \frac{ht^2}{2\varepsilon}. \qquad (3.3.2)$$

We first consider the conjugated operator

$$\Box_{\varphi,\varepsilon} := h^2 e^{-\widetilde{\varphi}/h} \Box e^{\widetilde{\varphi}/h}. \tag{3.3.3}$$

For $v \in C^2(Q)$ satisfying $v|_\Sigma = v|_{t=0} = \partial_t v|_{t=0} = 0$, consider the L^2 norm of $\Box_{\varphi,\varepsilon}$:

$$\int_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t.$$

Expanding (3.3.3), we get,

$$\Box_{\varphi,\varepsilon} v(t,x) = \Big(h^2\Box + h\Box\widetilde{\varphi} + \left(|\partial_t\widetilde{\varphi}|^2 - |\nabla_x\widetilde{\varphi}|^2\right) + 2h\left(\partial_t\widetilde{\varphi}\partial_t - \nabla_x\widetilde{\varphi}\cdot\nabla_x\right)\Big)v(t,x).$$

We write this as

$$\Box_{\varphi,\varepsilon} v(t,x) = P_1 v(t,x) + P_2 v(t,x),$$

where

$$\begin{aligned} P_1 v(t,x) &= \Big(h^2\Box + h\Box\widetilde{\varphi} + \left(|\partial_t\widetilde{\varphi}|^2 - |\nabla_x\widetilde{\varphi}|^2\right)\Big)v(t,x) \\ &= \left(h^2\Box + \frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)v(t,x), \end{aligned}$$

and

$$\begin{aligned} P_2 v(t,x) &= 2h\left(\partial_t\widetilde{\varphi}\partial_t - \nabla_x\widetilde{\varphi}\cdot\nabla_x\right)v(t,x) \\ &= 2h\left(\left(1 - \frac{ht}{\varepsilon}\right)\partial_t - \omega\cdot\nabla_x\right)v(t,x). \end{aligned}$$

Now

$$\begin{aligned} \int_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t &\geq 2\int_Q \mathrm{Re}\left(P_1 v(t,x)\overline{P_2 v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &= 4h^3 \int_Q \mathrm{Re}\left(\Box v(t,x)\left(1 - \frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &\quad - 4h^3 \int_Q \mathrm{Re}\left(\Box v(t,x)\omega\cdot\overline{\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &\quad + 4h \int_Q \mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)v(t,x)\left(1 - \frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&- 4h \int_Q \mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right) v(t,x)\omega \cdot \overline{\nabla_x v(t,x)}\right) \mathrm{d}x\mathrm{d}t \\
:= {} & I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We first simplify I_1. We have

$$
\begin{aligned}
I_1 &= 4h^3 \int_Q \mathrm{Re}\left(\Box v(t,x)\left(1 - \frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= 2h^3 \int_Q \frac{\partial}{\partial t}|\partial_t v(t,x)|^2\left(1 - \frac{ht}{\varepsilon}\right)\mathrm{d}x\mathrm{d}t - 4h^3 \int_Q \mathrm{Re}\left(\Delta v(t,x)\left(1 - \frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= 2h^3\left(1 - \frac{hT}{\varepsilon}\right)\int_\Omega \left(|\partial_t v(T,x)|^2 + |\nabla_x v(T,x)|^2\right)\mathrm{d}x + \frac{2h^4}{\varepsilon}\int_Q \left(|\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

In the above steps, we used integration by parts combined with the hypotheses that $v|_\Sigma = v|_{t=0} = \partial_t v|_{t=0} = 0$. Note that $v|_\Sigma = 0$ would imply that $\partial_t v = 0$ on Σ.

Now we consider I_2. We have

$$
I_2 = -4h^3 \int_Q \mathrm{Re}\left(\Box v(t,x)\omega \cdot \overline{\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t.
$$

We have

$$
\begin{aligned}
I_2 &= -4h^3\mathrm{Re}\int_Q \partial_t^2 v(t,x)\overline{\omega \cdot \nabla_x v(t,x)}\mathrm{d}x\mathrm{d}t + 4h^3\mathrm{Re}\int_Q \Delta v(t,x)\overline{\omega \cdot \nabla_x v(t,x)}\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int_Q \partial_t\left(\partial_t v(t,x)\overline{\omega \cdot \nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t + 4h^3\mathrm{Re}\int_Q \partial_t v(t,x)\overline{\omega \cdot \nabla_x \partial_t v(t,x)}\mathrm{d}x\mathrm{d}t \\
&\quad + 4h^3\mathrm{Re}\int_Q \nabla_x \cdot \left(\nabla_x v(t,x)\overline{\omega \cdot \nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t - 4h^3\mathrm{Re}\int_Q \nabla_x v(t,x) \cdot \nabla_x\left(\overline{\omega \cdot \nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int_\Omega \partial_t v(T,x)\overline{\omega \cdot \nabla_x v(T,x)}\mathrm{d}x + 2h^3\int_Q \nabla_x \cdot \left(\omega|\partial_t v(t,x)|^2\right)\mathrm{d}x\mathrm{d}t \\
&\quad + 2h^3\mathrm{Re}\int_\Sigma \partial_\nu v(t,x)\overline{\omega \cdot \nabla_x v(t,x)}\mathrm{d}S_x\mathrm{d}t - 2h^3\int_Q \nabla_x \cdot \left(\omega|\nabla_x v|^2\right)\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int_\Omega \partial_t v(T,x)\overline{\omega \cdot \nabla_x v(T,x)}\mathrm{d}x + 2h^3\int_\Sigma \omega \cdot \nu|\partial_t v(t,x)|^2\mathrm{d}S_x\mathrm{d}t + 2h^3\int_\Sigma \omega \cdot \nu|\partial_\nu v|^2\mathrm{d}S_x\mathrm{d}t.
\end{aligned}
$$

In deriving the above equation, we used the fact that

$$2h^3\mathrm{Re}\int\limits_{\Sigma} \partial_\nu v(t,x)\overline{\omega\cdot\nabla_x v(t,x)}\mathrm{d}S_x\mathrm{d}t = 2h^3\int\limits_{\Sigma} \omega\cdot\nu|\partial_\nu v|^2\mathrm{d}S_x\mathrm{d}t,$$

since $v=0$ on Σ. Also note that $|\nabla_x v| = |\partial_\nu v|$.

Next we consider I_3. We have

$$\begin{aligned}
I_3 &= 4h\int\limits_{Q} \mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2}-\frac{2ht}{\varepsilon}-\frac{h^2}{\varepsilon}\right)v(t,x)\left(1-\frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t\\
&= 2h\int\limits_{Q}\left(\frac{h^2t^2}{\varepsilon^2}-\frac{2ht}{\varepsilon}-\frac{h^2}{\varepsilon}\right)\left(1-\frac{ht}{\varepsilon}\right)\partial_t|v(t,x)|^2\mathrm{d}x\mathrm{d}t\\
&= 2\int\limits_{\Omega}\left(\frac{h^3T^2}{\varepsilon^2}-\frac{2h^2T}{\varepsilon}-\frac{h^3}{\varepsilon}\right)\left(1-\frac{hT}{\varepsilon}\right)|v(T,x)|^2\mathrm{d}x\\
&\quad -2\int\limits_{Q}\left[\left(\frac{2h^3t}{\varepsilon^2}-\frac{2h^2}{\varepsilon}\right)\left(1-\frac{ht}{\varepsilon}\right)-\frac{h^2}{\varepsilon}\left(\frac{h^2t^2}{\varepsilon^2}-\frac{2ht}{\varepsilon}-\frac{h^2}{\varepsilon}\right)\right]|v(t,x)|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Finally, we consider I_4. This is

$$\begin{aligned}
I_4 &:= -4h\int\limits_{Q}\mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2}-\frac{2ht}{\varepsilon}-\frac{h^2}{\varepsilon}\right)v(t,x)\omega\cdot\overline{\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t\\
&= -4h\int\limits_{Q}\left(\frac{h^2t^2}{\varepsilon^2}-\frac{2ht}{\varepsilon}-\frac{h^2}{\varepsilon}\right)\nabla\cdot\left(\omega|v|^2\right)\mathrm{d}x\mathrm{d}t\\
&= 0 \quad \text{ since } v=0 \text{ on } \Sigma.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int\limits_{Q}|\Box_{\varphi,\varepsilon}v(t,x)|^2\mathrm{d}x\mathrm{d}t &\geq 2h^3\left(1-\frac{hT}{\varepsilon}\right)\int\limits_{\Omega}\left(|\partial_t v(T,x)|^2+|\nabla_x v(T,x)|^2\right)\mathrm{d}x\\
&\quad+\frac{2h^4}{\varepsilon}\int\limits_{Q}\left(|\partial_t v(t,x)|^2+|\nabla_x v(t,x)|^2\right)\mathrm{d}x\mathrm{d}t+2h^3\int\limits_{\Sigma}\omega\cdot\nu|\partial_t v(t,x)|^2\mathrm{d}S_x\mathrm{d}t\\
&\quad-4h^3\mathrm{Re}\int\limits_{\Omega}\partial_t v(T,x)\overline{\omega\cdot\nabla_x v(T,x)}\mathrm{d}x+2h^3\int\limits_{\Sigma}\omega\cdot\nu|\partial_\nu v|^2\mathrm{d}S_x\mathrm{d}t\\
&\quad+2\int\limits_{\Omega}\left(\frac{h^3T^2}{\varepsilon^2}-\frac{2h^2T}{\varepsilon}-\frac{h^3}{\varepsilon}\right)\left(1-\frac{hT}{\varepsilon}\right)|v(T,x)|^2\mathrm{d}x
\end{aligned}$$

$$- 2\int\limits_Q \left[\left(\frac{2h^3t}{\varepsilon^2} - \frac{2h^2}{\varepsilon}\right)\left(1 - \frac{ht}{\varepsilon}\right) - \frac{h^2}{\varepsilon}\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)\right] |v(t,x)|^2 \mathrm{d}x\mathrm{d}t.$$

Choosing ε and h small enough, we have

$$\begin{aligned}\int\limits_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t \geq \frac{2h^4}{\varepsilon}&\left(\int\limits_Q |\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t\right) + ch^3 \int\limits_\Omega |\partial_t v(T,x)|^2 \mathrm{d}x \\ &+ 2h^3 \int\limits_\Sigma \omega\cdot\nu(x)|\partial_\nu v(t,x)|^2 \mathrm{d}S_x\mathrm{d}t - ch^3 \int\limits_\Omega |\nabla_x v(T,x)|^2 \mathrm{d}x \\ &- ch^2 \int\limits_\Omega |v(T,x)|^2 \mathrm{d}x + \frac{ch^2}{\varepsilon}\int\limits_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t. \qquad (3.3.4)\end{aligned}$$

Now we consider the conjugated operator $\mathcal{L}_{\varphi,\varepsilon} := h^2 e^{-\frac{\widetilde{\varphi}}{h}}\mathcal{L}_{\mathcal{A},q} e^{\frac{\widetilde{\varphi}}{h}}$. We have

$$\mathcal{L}_{\varphi,\varepsilon} v(t,x) = h^2 \left(e^{-\widetilde{\varphi}/h}\left(\Box + 2A_0\partial_t - 2A\cdot\nabla_x + \widetilde{q}\right) e^{\widetilde{\varphi}/h} v(t,x)\right),$$

where

$$\widetilde{q} = q + |A_0|^2 - |A|^2 + \partial_t A_0 - \nabla_x \cdot A.$$

We write

$$\mathcal{L}_{\varphi,\varepsilon} v(t,x) = \Box_{\varphi,\varepsilon} v(t,x) + \widetilde{P} v(t,x),$$

where

$$\widetilde{P} v(t,x) = h^2 \left(e^{-\widetilde{\varphi}/h}\left(2A_0\partial_t - 2A\cdot\nabla_x + \widetilde{q}\right) e^{\widetilde{\varphi}/h} v(t,x)\right). \qquad (3.3.5)$$

By triangle inequality,

$$\int\limits_Q |\mathcal{L}_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t \geq \frac{1}{2}\int\limits_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t - \int\limits_Q |\widetilde{P} v(t,x)|^2 \mathrm{d}x\mathrm{d}t. \qquad (3.3.6)$$

We have

$$\begin{aligned}&\int\limits_Q \left|\widetilde{P} v(t,x)\right|^2 \mathrm{d}x\mathrm{d}t = \int\limits_Q \left|\left[h^2\left(2A_0\partial_t - 2A\cdot\nabla_x + \widetilde{q}\right) v + h\left\{2A_0\left(1 - \frac{ht}{\varepsilon}\right) - 2\omega\cdot A\right\} v\right]\right|^2 \mathrm{d}x\mathrm{d}t \\ &\leq Ch^4\left(\|A_0\|^2_{L^\infty(Q)}\int\limits_Q |\partial_t v(t,x)|^2 \mathrm{d}x\mathrm{d}t + \|A\|^2_{L^\infty(Q)}\int\limits_Q |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t\right)\end{aligned}$$

$$
+ Ch^2 \Bigg(\left(h^2 \|\widetilde{q}\|^2_{L^\infty(Q)} + \|A\|^2_{L^\infty(Q)}\right) \int\limits_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t
$$

$$
+ \|A_0\|^2_{L^\infty(Q)} \int\limits_Q \left(1 - \frac{ht}{\varepsilon}\right)^2 |v(t,x)|^2 \mathrm{d}x\mathrm{d}t \Bigg).
$$

Choosing h small enough, we have,

$$
\begin{aligned}
\int\limits_Q \left|\widetilde{P}v(t,x)\right|^2 \mathrm{d}x\mathrm{d}t \leq Ch^4 &\left(\|A_0\|^2_{L^\infty(Q)} \int\limits_Q |\partial_t v(t,x)|^2 \mathrm{d}x\mathrm{d}t + \|A\|^2_{L^\infty(Q)} \int\limits_Q |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t \right) \\
&+ Ch^2 \left(\|A_0\|^2_{L^\infty(Q)} + \|A\|^2_{L^\infty(Q)}\right) \int\limits_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t \\
&+ Ch^4 \|\widetilde{q}\|^2_{L^\infty(Q)} \int\limits_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.3.7}
$$

Using (3.3.4) and (3.3.7) in (3.3.6) and taking ε small enough, we have that there exists a $C > 0$ depending only on ε, T, Ω, $\mathcal{A}$ and q such that

$$
\begin{aligned}
\int\limits_Q |\mathcal{L}_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t \geq Ch^2 &\Bigg\{ h^2 \left(\int\limits_Q |\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t \right) + \int\limits_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t \\
&- \int\limits_\Omega |v(T,x)|^2 \mathrm{d}x \Bigg\} + Ch^3 \int\limits_\Omega |\partial_t v(T,x)|^2 \mathrm{d}x \\
&+ h^3 \int\limits_\Sigma \omega \cdot \nu(x) |\partial_\nu v(t,x)|^2 \mathrm{d}S_x \mathrm{d}t - Ch^3 \int\limits_\Omega |\nabla_x v(T,x)|^2 \mathrm{d}x,
\end{aligned}
$$

and this inequality holds for h small enough. After dividing by h^2, we get

$$
\begin{aligned}
&C \left(\int\limits_Q \left(|h\partial_t v(t,x)|^2 + |h\nabla_x v(t,x)|^2\right) \mathrm{d}x\mathrm{d}t + \int\limits_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t \right) \\
&+ Ch \int\limits_\Omega |\partial_t v(T,x)|^2 \mathrm{d}x + h \int\limits_\Sigma \omega \cdot \nu(x) |\partial_\nu v(t,x)|^2 \mathrm{d}S_x \mathrm{d}t \\
&\leq \frac{1}{h^2} \int\limits_Q |\mathcal{L}_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t + Ch \int\limits_\Omega |\nabla_x v(T,x)|^2 \mathrm{d}x + C \int\limits_\Omega |v(T,x)|^2 \mathrm{d}x.
\end{aligned}
\tag{3.3.8}
$$

Let us now substitute $v(t,x) = e^{-\frac{\widetilde{\varphi}}{h}}u(t,x)$. We have

$$\begin{aligned}
&he^{-\varphi/h}\partial_t u(t,x) = he^{-t^2/2\varepsilon}\partial_t v + e^{-\varphi/h}\left(1-\frac{ht}{\varepsilon}\right)u,\\
&he^{-\varphi/h}\nabla_x u = he^{-t^2/2\varepsilon}\nabla_x v + e^{-\varphi/h}\omega u,\\
&\partial_\nu v(t,x)|_\Sigma = e^{-\varphi/h}\partial_\nu u|_\Sigma, \quad \text{since } u=0 \text{ on } \Sigma.
\end{aligned}$$

Using the above equalities and the triangle inequality, we then have for h small enough,

$$\begin{aligned}
&h\int_\Omega e^{-2\varphi/h}|\partial_t u(T,x)|^2\mathrm{d}x + h\int_{\Sigma_{+,\omega}} e^{-2\varphi/h}|\partial_\nu u(t,x)|^2|\omega\cdot\nu(x)|\mathrm{d}S_x\mathrm{d}t\\
&+h^2\int_Q e^{-2\varphi/h}\left(|\partial_t u(t,x)|^2+|\nabla_x u(t,x)|^2\right)\mathrm{d}x\mathrm{d}t + \int_Q e^{-2\varphi/h}|u(t,x)|^2\mathrm{d}x\mathrm{d}t\\
&\leq C\Bigg(h^2\int_Q e^{-2\varphi/h}|\mathcal{L}_{\mathcal{A},q}u(t,x)|^2\mathrm{d}x\mathrm{d}t + h\int_\Omega e^{-2\varphi/h}|\nabla_x u(T,x)|^2\mathrm{d}x\\
&+\int_\Omega e^{-2\varphi/h}|u(T,x)|^2\mathrm{d}x + h\int_{\Sigma_{-,\omega}} e^{-2\varphi/h}|\partial_\nu u(t,x)|^2|\omega\cdot\nu(x)|\mathrm{d}S_x\mathrm{d}t\Bigg).
\end{aligned}$$

Finally,

$$\begin{aligned}
&h\left(e^{-\varphi/h}\partial_\nu\varphi\partial_\nu u, e^{-\phi/h}\partial_\nu u\right)_{L^2(\Sigma_{+,\omega})} + h\left(e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot), e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot)\right)_{L^2(\Omega)}\\
&+\|e^{-\varphi/h}u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\partial_t u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\nabla_x u\|^2_{L^2(Q)}\\
&\leq C\Bigg(\|he^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}u\|^2_{L^2(Q)} + \left(e^{-\varphi(T,\cdot)/h}u(T,\cdot), e^{-\varphi(T,\cdot)/h}u(T,\cdot)\right)_{L^2(\Omega)}\\
&+h\left(e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot), e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot)\right)_{L^2(\Omega)} + h\left(e^{-\varphi/h}\left(-\partial_\nu\varphi\right)\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{-,\omega})}\Bigg).
\end{aligned}$$

This completes the proof. □

In particular, it follows from the previous calculations that for $u \in C_c^\infty(Q)$,

$$\|u\|_{H^1_{\mathrm{scl}}(Q)} \leq \frac{C}{h}\|\mathcal{L}_\varphi u\|_{L^2(Q)}. \tag{3.3.9}$$

To show the existence of suitable solutions to (3.1.1), we need to shift the Sobolev index by -1 in (3.3.9). This we do in the next lemma.

Lemma 3.3.2. *Let $\varphi(t,x) = t + x\cdot\omega$ and $\mathcal{L}_\varphi := h^2 e^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}e^{\varphi/h}$. There exists an $h_0 > 0$ such that*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leq \frac{C}{h}\|\mathcal{L}_\varphi v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}, \tag{3.3.10}$$

and

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leq \frac{C}{h}\|\mathcal{L}^*_\varphi v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \tag{3.3.11}$$

for all $v \in C_c^\infty(Q)$, $0 < h \leq h_0$.

Proof. We give the proof of the estimate in (3.3.10) and that of (3.3.11) follows similarly. We follow arguments used in [15]. We again consider the convexified weight

$$\widetilde{\varphi}(t,x) = t + x\cdot\omega - \frac{ht^2}{2\varepsilon},$$

and as before consider the convexified operator:

$$\Box_{\varphi,\varepsilon} := h^2 e^{-\widetilde{\varphi}/h}\Box e^{\widetilde{\varphi}/h}.$$

From the properties of pseudodifferential operators, we have

$$\langle hD\rangle^{-1}\left(\Box_{\varphi,\varepsilon}\right)\langle hD\rangle = \Box_{\varphi,\varepsilon} + hR_1$$

where R_1 is a semi-classical pseudo-differential operator of order 1. Now

$$\|\Box_{\varphi,\varepsilon}\langle hD\rangle v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} = \|\langle hD\rangle^{-1}\Box_{\varphi,\varepsilon}\langle hD\rangle v\|_{L^2(\mathbb{R}^{1+n})}.$$

and by the commutator property above, we get

$$\|\Box_{\varphi,\varepsilon}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} = \|\left(\Box_{\varphi,\varepsilon} + hR_1\right)v\|^2_{L^2(\mathbb{R}^{1+n})} \geq \frac{1}{2}\|\Box_{\varphi,\varepsilon}v\|^2_{L^2(\mathbb{R}^{1+n})} - \|hR_1 v\|^2_{L^2(\mathbb{R}^{1+n})}.$$

Let $Q \subset\subset \widetilde{Q}$, and for $v \in C_c^\infty(\widetilde{Q})$, using the estimate in (3.3.4) for C_c^∞ functions combined with estimates for pseudodifferential operators, we have,

$$\|\Box_{\varphi,\varepsilon}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq \frac{Ch^2}{\varepsilon}\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} - h^2\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})}. \tag{3.3.12}$$

Using the expression for $\widetilde{P}$ (see (3.3.5)), we get, for $v \in C_c^\infty(\widetilde{Q})$ and for h small enough,

$$\|\widetilde{P}v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq Ch^2\|v\|^2_{L^2(\mathbb{R}^{1+n})},$$

and therefore

$$\|\widetilde{P}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq Ch^2\|\langle hD\rangle v\|^2_{L^2(\mathbb{R}^{1+n})} = Ch^2\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

Combining this with the estimate in (3.3.12) together with triangle inequality, we get,

$$\|\mathcal{L}_{\varphi,\varepsilon}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq \frac{Ch^2}{\varepsilon}\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \tag{3.3.13}$$

for all $v \in C_c^\infty(\widetilde{Q})$.

Now to complete the proof, for any $u \in C_c^\infty(Q)$, consider $v = \chi\langle hD\rangle^{-1}u$, where $\chi \in C_c^\infty(\widetilde{Q})$ with $\chi \equiv 1$ on Q. Then from (3.3.13), we have

$$\frac{Ch^2}{\varepsilon}\|\chi\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq \|\mathcal{L}_{\varphi,\varepsilon}\langle hD\rangle\chi\langle hD\rangle^{-1}u\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

The operator $\langle hD\rangle\chi\langle hD\rangle^{-1}$ is a semiclassical pseudodifferential operator of order 0, and therefore we have

$$\mathcal{L}_{\varphi,\varepsilon}\langle hD\rangle\chi\langle hD\rangle^{-1}u = \langle hD\rangle\chi\langle hD\rangle^{-1}\mathcal{L}_{\varphi,\varepsilon} + hR_1,$$

where R_1 is a semiclassical pseudodifferential operator of order 1.

$$\frac{Ch^2}{\varepsilon}\|\chi\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq \|\mathcal{L}_{\varphi,\varepsilon}u\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} + h^2\|u\|^2_{L^2(\mathbb{R}^{1+n})}.$$

Finally, write

$$\langle hD\rangle^{-1}u = \chi\langle hD\rangle^{-1}u + (1-\chi)\langle hD\rangle^{-1}u,$$

where χ is as above. Then

$$\|\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq \frac{1}{2}\|\chi\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} - \|(1-\chi)\,\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

Since $(1-\chi)\langle hD\rangle^{-1}$ is a smoothing semiclassical pseudodifferential operator, taking h small enough, and arguing as in the proof of the Carleman estimate, we get,

$$\|\mathcal{L}_{\varphi}u\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq Ch^2\|u\|^2_{L^2(\mathbb{R}^{1+n})}.$$

Cancelling out the h^2 term, we finally have,

$$\|u\|_{L^2(\mathbb{R}^{1+n})} \le \frac{C}{h}\|\mathcal{L}_\varphi u\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

This completes the proof. □

Proposition 3.3.3. *Let φ, $\mathcal{A}$ and q be as in Theorem 3.3.1. For $h > 0$ small enough and $v \in L^2(Q)$, there exists a solution $u \in H^1(Q)$ of*

$$\mathcal{L}_\varphi u = v,$$

satisfying the estimate

$$\|u\|_{H^1_{\mathrm{scl}}(Q)} \le \frac{C}{h}\|v\|_{L^2(\mathbb{R}^{1+n})},$$

where $C > 0$ is a constant independent of h.

Proof. The proof uses standard functional analysis arguments. Consider the space $S := \left\{\mathcal{L}^*_\varphi u : u \in C^\infty_c(Q)\right\}$ as a subspace of $H^{-1}(\mathbb{R}^{1+n})$ and define a linear form L on S by

$$L(\mathcal{L}^*_\varphi z) = \int_Q z(t,x)v(t,x)\mathrm{d}x\mathrm{d}t, \text{ for } z \in C^\infty_c(Q).$$

This is a well-defined continuous linear functional by the Carleman estimate (3.3.11) . We have

$$\left|L(\mathcal{L}^*_\varphi z)\right| \le \|z\|_{L^2(Q)}\|v\|_{L^2(Q)} \le \frac{C}{h}\|v\|_{L^2(Q)}\|\mathcal{L}^*_\varphi z\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}, z \in C^\infty_c(Q).$$

By Hahn-Banach theorem, we can can extend L to $H^{-1}(\mathbb{R}^{1+n})$ (still denoted as L) and it satisfies $\|L\| \le \frac{C}{h}\,||v||_{L^2(Q)}$. By Riesz representation theorem, there exists a unique $u \in H^1(\mathbb{R}^{1+n})$ such that

$$L(f) = \langle f, u\rangle_{L^2(\mathbb{R}^{1+n})} \text{ for all } f \in H^{-1}(\mathbb{R}^{1+n}) \text{ with } \|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \le \frac{C}{h}\|v\|_{L^2(Q)}.$$

Taking $f = \mathcal{L}^*_\varphi z$, for $z \in C^\infty_c(Q)$, we get

$$L(\mathcal{L}^*_\varphi z) = \langle \mathcal{L}^*_\varphi z, u\rangle_{L^2(\mathbb{R}^{1+n})} = \langle z, \mathcal{L}_\varphi u\rangle_{L^2(\mathbb{R}^{1+n})}.$$

Therefore for all $z \in C_c^\infty(Q)$,

$$\langle z, \mathcal{L}_\varphi u \rangle = \langle z, v \rangle.$$

Hence

$$\mathcal{L}_\varphi u = v \text{ in } Q \text{ with } \|u\|_{H^1_{\mathrm{scl}}(Q)} \leq \frac{C}{h}\|v\|_{L^2(Q)}.$$

This completes the proof of the proposition. □

3.4 Construction of geometric optics solutions

In this section we construct geometric optics solutions for $\mathcal{L}_{\mathcal{A},q}u = 0$ and its adjoint operator $\mathcal{L}^*_{\mathcal{A},q}u = \mathcal{L}_{-\mathcal{A},\overline{q}}u = 0$.

Proposition 3.4.1. *Let $\mathcal{L}_{\mathcal{A},q}$ be as in (3.1.1).*

1. *(Exponentially decaying solutions) There exists an $h_0 > 0$ such that for all $0 < h \leq h_0$, we can find $v \in H^1(Q)$ satisfying $\mathcal{L}_{-\mathcal{A},\overline{q}}v = 0$ of the form*

$$v_d(t,x) = e^{-\frac{\varphi}{h}}\left(B_d(t,x) + hR_d(t,x;h)\right), \tag{3.4.1}$$

where $\varphi(t,x) = t + x\cdot\omega$,

$$B_d(t,x) = \exp\left(-\int_0^\infty (1,-\omega)\cdot\mathcal{A}(t+s, x-s\omega)\mathrm{d}s\right) \tag{3.4.2}$$

with $\zeta \in (1,-\omega)^\perp$ and $R_d \in H^1(Q)$ satisfies

$$\|R_d\|_{H^1_{\mathrm{scl}}(Q)} \leq C. \tag{3.4.3}$$

2. *(Exponentially growing solutions) There exists an $h_0 > 0$ such that for all $0 < h \leq h_0$, we can find $v \in H^1(Q)$ satisfying $\mathcal{L}_{\mathcal{A},q}v = 0$ of the form*

$$v_g(t,x) = e^{\frac{\varphi}{h}}\left(B_g(t,x) + hR_g(t,x;h)\right), \tag{3.4.4}$$

where $\varphi(t,x) = t + x \cdot \omega$,

$$B_g(t,x) = e^{-i\zeta\cdot(t,x)} \exp\left(\int_0^{\infty} (1,-\omega)\cdot \mathcal{A}(t+s, x-s\omega)\mathrm{d}s\right) \tag{3.4.5}$$

with $\zeta \in (1,-\omega)^{\perp}$ *and* $R_g \in H^1(Q)$ *satisfies*

$$\|R_g\|_{H^1_{\mathrm{scl}}(Q)} \leq C. \tag{3.4.6}$$

Proof. We have

$$\mathcal{L}_{A,q}v = \Box v + 2A_0\partial_t v - 2A\cdot\nabla_x v + \big(\partial_t A_0 - \nabla_x\cdot A + |A_0|^2 - |A|^2 + q\big)\, v.$$

Letting v of the form

$$v(t,x) = e^{\frac{\varphi}{h}}\left(B_g + hR_g\right),$$

and setting the term involving h^{-1} to be 0, we get,

$$(1,-\omega)\cdot\left(\nabla_{t,x}B_g + (A_0, A)\, B_g\right) = 0.$$

One solution of this equation is

$$B_g(t,x) = \exp\left(\int_0^{\infty} (1,-\omega)\cdot \mathcal{A}(t+s, x-s\omega)\mathrm{d}s\right).$$

Alternately, another solution is

$$B_g(t,x) = e^{-\mathrm{i}\zeta\cdot(t,x)} \exp\left(\int_0^{\infty} (1,-\omega)\cdot \mathcal{A}(t+s, x-s\omega)\mathrm{d}s\right),$$

provided $\zeta \in (1,-\omega)^{\perp}$.

Now we have

$$\mathcal{L}^*_{\mathcal{A},q} = \mathcal{L}_{-\mathcal{A},\overline{q}}.$$

For this adjoint operator, the equation satisfied by B_d is

$$(1,-\omega)\cdot\left(\nabla_{t,x}B_g - (A_0, A)\, B_g\right) = 0.$$

We let

$$B_d(t,x) = \exp\left(-\int_0^\infty (1,-\omega)\cdot\mathcal{A}(t+s,x-s\omega)\mathrm{d}s\right).$$

Now R_g satisfies

$$\mathcal{L}_\varphi R_g = -h\mathcal{L}_{\mathcal{A},q}B_g.$$

Then using the estimate in Proposition 3.3.3, we get that

$$\|R_g\|_{H^1_{\mathrm{scl}}(Q)} \le C\|\mathcal{L}_{\mathcal{A},q}B_g\|_{L^2(Q)}.$$

Similarly,

$$\|R_d\|_{H^1_{\mathrm{scl}}(Q)} \le C\|\mathcal{L}_{-\mathcal{A},\overline{q}}B_d\|_{L^2(Q)}.$$

The proof is complete. □

3.5 Integral Identity

In this section, we derive an integral identity involving the coefficients $\mathcal{A}$ and q using the geometric optics solutions described in the previous section.

Let u_i be the solutions to the following initial boundary value problems with vector field coefficient $\mathcal{A}^{(i)}$ and scalar potential q_i for $i=1,2$.

$$\begin{cases} L_{\mathcal{A}^{(i)},q_i}u_i(t,x) = 0; \ (t,x)\in Q \\ u_i(0,x) = \phi(x),\ \partial_t u_i(0,x) = \psi(x);\ x\in\Omega \\ u_i(t,x) = f(t,x),\ (t,x)\in\Sigma. \end{cases} \tag{3.5.1}$$

Let us denote

$$\begin{aligned} &u(t,x) := (u_1-u_2)(t,x) \\ &\mathcal{A}(t,x) := \mathcal{A}^{(2)} - \mathcal{A}^{(1)}(t,x) := (A_0(t,x),A_1(t,x),\cdots,A_n(t,x)) \\ &\widetilde{q}_i := \partial_t A_0^{(i)} - \nabla_x\cdot\mathcal{A}^{(i)} + |A_0^{(i)}|^2 - |\mathcal{A}^{(i)}|^2 + q_i \\ &\widetilde{q} := \widetilde{q}_2 - \widetilde{q}_1. \end{aligned} \tag{3.5.2}$$

Then u is the solution to the following initial boundary value problem:

$$\begin{cases} L_{\mathcal{A}^{(1)},q_1}u(t,x) = -2A\cdot\nabla_x u_2 + 2A_0\partial_t u_2 + \widetilde{q}u_2 \\ u(0,x) = \partial_t u(0,x) = 0,\ \ x\in\Omega \\ u|_{\Sigma} = 0. \end{cases} \tag{3.5.3}$$

Let $v(t,x)$ of the form given by (3.4.1) be the solution to following equation

$$L_{-\mathcal{A}^{(1)},\bar{q}_1}v(t,x) = 0 \text{ in } Q. \tag{3.5.4}$$

Also let u_2 of the form given by (3.4.4) be solution to the following equation

$$L_{\mathcal{A}^{(2)},q_2}u_2(t,x) = 0,\ \text{ in } Q. \tag{3.5.5}$$

By the well-posedness result from [32, 38], we have $u\in H^1(Q)$ and $\partial_\nu u \in L^2(\Sigma)$.

Now we multiply (3.5.3) by $\overline{v(t,x)}\in H^1(Q)$ and integrate over Q. We get, after integrating by parts, taking into account the following: $u|_\Sigma = 0$, $u(T,x) = 0$, $\partial_\nu u|_G = 0$, $u|_{t=0} = \partial_t u|_{t=0} = 0$ and $\mathcal{A}^{(1)}$ is compactly supported in Q:

$$\int\limits_Q \mathcal{L}_{\mathcal{A}^{(1)},q_1}u(t,x)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t - \int\limits_Q u(t,x)\overline{\mathcal{L}_{-\mathcal{A}^{(1)},\bar{q}_1}v(t,x)}\mathrm{d}x\mathrm{d}t = \int\limits_\Omega \partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x - \int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t.$$

Now using the fact that $L_{-\mathcal{A}^{(1)},\bar{q}_1}v(t,x) = 0$ in Q and

$$\mathcal{L}_{\mathcal{A}^{(1)},q_1}u(t,x) = -2A\cdot\nabla_x u_2 + 2A_0\partial_t u_2 + \widetilde{q}u_2,$$

we get,

$$\int\limits_Q \left(-2A\cdot\nabla_x u_2 + 2A_0\partial_t u_2 + \widetilde{q}u_2\right)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t = \int\limits_\Omega \partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x - \int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t.$$

Lemma 3.5.1. *Let u_i for $i = 1, 2$ solutions to (3.5.1) with u_2 of the form (3.4.4). Let $u = u_1 - u_2$, and v be of the form (3.4.1). Then*

$$h \int_{\Omega} \partial_t u(T, x)\overline{v(T, x)}\mathrm{d}x \to 0 \text{ as } h \to 0^+. \tag{3.5.6}$$

$$h \int_{\Sigma \setminus G} \partial_\nu u(t, x)\overline{v(t, x)}\mathrm{d}S_x\mathrm{d}t \to 0 \text{ as } h \to 0^+. \tag{3.5.7}$$

Proof. Using (3.4.1), (3.4.3) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&\left| h \int_{\Omega} \partial_t u(T, x)\overline{v(T, x)}\mathrm{d}x \right| \le \int_{\Omega} h \left| \partial_t u(T, x) e^{-\frac{\varphi(T,x)}{h}} \overline{(B_d(T, x) + hR_d(T, x))} \right| \mathrm{d}x \\
&\le C \left(\int_{\Omega} h^2 \left| \partial_t u(T, x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| e^{-i\xi\cdot(T,x)} + h\overline{R_d(T, x)} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\le C \left(\int_{\Omega} h^2 \left| \partial_t u(T, x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(1 + \|hR_d(T, \cdot)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \\
&\le C \left(\int_{\Omega} h^2 \left| \partial_t u(T, x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned}$$

Now using the boundary Carleman estimate (3.3.1), we get,

$$\begin{aligned}
h \int_{\Omega} \left| \partial_t u(T, x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x &\le C \|he^{-\varphi/h} \mathcal{L}_{\mathcal{A}^{(1)},q_1} u\|^2_{L^2(Q)} \\
&= C \|he^{-\varphi/h} \left(2A_0 \partial_t u_2 - 2A \cdot \nabla_x u_2 + \widetilde{q} u_2 \right) \|^2_{L^2(Q)}.
\end{aligned}$$

Now substituting (3.4.4) for u_2, we get,

$$\begin{aligned}
he^{-\varphi/h} \left(2A_0 \partial_t u_2 - 2A \cdot \nabla_x u_2 + \widetilde{q} u_2 \right) &= \left(2A_0 \partial_t \varphi - 2A \cdot \nabla_x \varphi + \widetilde{q} \right) (B_g + hR_g) \\
&+ h \left(2A_0 \partial_t - 2A \cdot \nabla_x + \widetilde{q} \right) (B_g + hR_g)
\end{aligned}$$

Therefore

$$\|he^{-\varphi/h} \left(2A_0 \partial_t u_2 - 2A \cdot \nabla_x u_2 + \widetilde{q} u_2 \right) \|^2_{L^2(Q)} \le C,$$

uniformly in h.

Thus, we have

$$\left(h^2 \int\limits_{\Omega} \left|\partial_t u(T,x) e^{-\frac{\varphi(T,x)}{h}}\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C\sqrt{h}.$$

Therefore

$$h \int\limits_{\Omega} \partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x \to 0 \text{ as } h \to 0^+.$$

For $\varepsilon > 0$, define

$$\partial\Omega_{+,\varepsilon,\omega} = \{x \in \partial\Omega : \nu(x)\cdot\omega > \varepsilon\}.$$

and

$$\Sigma_{+,\varepsilon,\omega} = (0,T) \times \partial\Omega_{+,\varepsilon,\omega}.$$

Next we prove (3.5.7). Since $\Sigma \setminus G \subseteq \Sigma_{+,\varepsilon,\omega}$ for all ω such that $|\omega - \omega_0| \leq \varepsilon$, substituting $v = v_d$ from (3.4.1), in (3.5.7) we have

$$\begin{aligned}\left|\int\limits_{\Sigma\setminus G} \partial_\nu u(t,x)v(t,x)\mathrm{d}S_x\mathrm{d}t\right| &\leq \int\limits_{\Sigma+,\varepsilon,\omega} \left|\partial_\nu u(t,x)e^{-\frac{\varphi}{h}}\left(B_d + hR_d\right)(t,x)\right| \mathrm{d}S_x\mathrm{d}t \\ &\leq C\left(1 + \|hR_d\|^2_{L^2(\Sigma)}\right)^{\frac{1}{2}} \left(\int\limits_{\Sigma+,\varepsilon,\omega} \left|\partial_\nu u(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t\right)\end{aligned}$$

with $C > 0$ is independent of h and this inequality holds for all ω such that $|\omega - \omega_0| \leq \varepsilon$. Next by trace theorem, we have that $\|R_d\|_{L^2(\Sigma)} \leq C\|R_d\|_{H^1_{\mathrm{scl}(Q)}}$.

Using this, we get

$$\left|\int\limits_{\Sigma\setminus G} \partial_\nu u(t,x)v(t,x)\mathrm{d}S_x\mathrm{d}t\right| \leq C\left(\int\limits_{\Sigma+,\varepsilon,\omega} \left|\partial_\nu u(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t\right)^{\frac{1}{2}}.$$

Now

$$\int\limits_{\Sigma_{+,\varepsilon,\omega}} \left|\partial_\nu u(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t = \frac{1}{\varepsilon}\int\limits_{\Sigma_{+,\varepsilon,\omega}} \varepsilon\left|\partial_\nu u(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t$$

$$\leq \frac{1}{\varepsilon} \int\limits_{\Sigma_{+,\varepsilon,\omega}} \partial_\nu \varphi \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t.$$

Using the boundary Carleman estimate (3.3.1), we have

$$\frac{h}{\varepsilon} \int\limits_{\Sigma_{+,\varepsilon,\omega}} \partial_\nu \varphi \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t \leq C \| h e^{-\varphi/h} \mathcal{L}_{\mathcal{A}^{(1)},q_1} u \|^2_{L^2(Q)}.$$

We now proceed as before to conclude that

$$h \int\limits_{\Sigma \setminus G} \partial_\nu u(t,x) \overline{v(t,x)} \mathrm{d}S_x \mathrm{d}t \to 0 \text{ as } h \to 0^+.$$

$\square$

3.6 Proof of Theorem 3.2.3

In this section, we prove the uniqueness results.

3.6.1 Recovery of vector potential $\mathcal{A}$

We consider the integral,

$$\int\limits_{Q} \left(-2A \cdot \nabla_x u_2 + 2A_0 \partial_x u_2 + \widetilde{q} u_2\right)(t,x) \overline{v(t,x)} \mathrm{d}x \mathrm{d}t.$$

Substituting (3.4.4) for u_2 and (3.4.1) for v into the above equation, and letting $h \to 0^+$, we arrive at

$$\int\limits_{Q} \left(-\omega \cdot A + A_0\right) \overline{B_d(t,x)} B_g(t,x) \mathrm{d}x \mathrm{d}t = 0 \text{ for all } \omega \in \mathbb{S}^{n-1} \text{ such that } |\omega - \omega_0| \leq \varepsilon.$$

Denote $\widetilde{\omega} := (1, -\omega)$, $\mathcal{A} = (A_0, A)$, and using the expressions for B_d and B_g, see (3.4.1) and (3.4.4), we get

$$J := \int\limits_{\mathbb{R}^{1+n}} \widetilde{\omega} \cdot \mathcal{A}(t,x) e^{-i\xi \cdot (t,x)} \exp\left(\int\limits_0^\infty \widetilde{\omega} \cdot \mathcal{A}(t+s, x-s\omega) \right) \mathrm{d}x \mathrm{d}t = 0,$$

where $\xi \cdot (1, -\omega) = 0$ for all ω with $|\omega - \omega_0| < \varepsilon$. We decompose $\mathbb{R}^{1+n} = \mathbb{R}(1, -\omega) \oplus (1, -\omega)^{\perp}$. We then get

$$J = \int_{(1,-\omega)^{\perp}} e^{-\mathrm{i}\xi \cdot k} \left(\int_{\mathbb{R}} \widetilde{\omega} \cdot \mathcal{A}(k + \tau(1, -\omega)) \exp \left(\int_{\tau}^{\infty} \widetilde{\omega} \cdot \mathcal{A}(k + s(1, -\omega)) \mathrm{d}s \right) \sqrt{2} \mathrm{d}\tau \right) \mathrm{d}k.$$

Here $\mathrm{d}k$ is the Lebesgue measure on the hyperplane $(1, -\omega)^{\perp}$.

$$\begin{aligned}
&= -\sqrt{2} \int_{(1,-\omega)^{\perp}} e^{-\mathrm{i}\xi \cdot k} \left(\int_{\mathbb{R}} \partial_{\tau} \exp \left(\int_{\tau}^{\infty} \widetilde{\omega} \cdot \mathcal{A}(k + s(1, -\omega)) \mathrm{d}s \right) \mathrm{d}\tau \right) \mathrm{d}k \\
&= -\sqrt{2} \int_{(1,-\omega)^{\perp}} e^{-i\xi \cdot k} \left(1 - \exp \left(\int_{\mathbb{R}} \widetilde{\omega} \cdot \mathcal{A}(k + s(1, -\omega)) \mathrm{d}s \right) \right) \mathrm{d}k \\
&= C \mathcal{F}_{(1,-\omega)^{\perp}} \left(1 - \exp \left(\int_{\mathbb{R}} \widetilde{\omega} \cdot \mathcal{A}(k + s(1, -\omega)) \mathrm{d}s \right) \right) (\xi) \text{ for some constant } C.
\end{aligned}$$

Since the integral $J = 0$, we have that

$$\mathcal{F}_{(1,-\omega)^{\perp}} \left(1 - \exp \left(\int_{\mathbb{R}} \widetilde{\omega} \cdot \mathcal{A}(k + s(1, -\omega)) \mathrm{d}s \right) \right) (\xi) = 0, \;\; k \in (1, -\omega)^{\perp}.$$

This gives us

$$\exp \left(\int_{\mathbb{R}} \widetilde{\omega} \cdot \mathcal{A}(k + s(1, -\omega)) \mathrm{d}s \right) = 1, \;\text{ for all }\;\; k \in (1, -\omega)^{\perp} \text{ and all } \omega \text{ with } |\omega - \omega_0| < \varepsilon.$$

Thus we deduce that

$$\int_{\mathbb{R}} \widetilde{\omega} \cdot \mathcal{A}(t + s, x - s\omega) \mathrm{d}s = 0, \quad (t, x) \in (1, -\omega)^{\perp} \text{ and for all } \omega \text{ with } |\omega - \omega_0| < \varepsilon. \tag{3.6.1}$$

Now we show that the orthogonality condition $(t, x) \in (1, -\omega)^{\perp}$, can be removed using a change of variables as used in [64].

Consider any $(t,x) \in \mathbb{R}^{1+n}$. Then

$$\left(\frac{t+x\cdot\omega}{2}, x+\frac{(t-x\cdot\omega)\omega}{2}\right)$$

is a point on $(1,-\omega)^{\perp}$, and we have

$$\int_{\mathbb{R}} \widetilde{\omega}\cdot\mathcal{A}\left(\frac{t+x\cdot\omega}{2}+\widetilde{s}, x+\frac{(t-x\cdot\omega)\omega}{2}-\widetilde{s}\omega\right)\mathrm{d}\widetilde{s}=0.$$

Consider the following change of variable in the above integral:

$$s=\frac{x\cdot\omega-t}{2}+\widetilde{s}.$$

Then we have

$$\int_{\mathbb{R}} \widetilde{\omega}\cdot\mathcal{A}\left(t+s, x-s\omega\right)\mathrm{d}s=0.$$

Therefore, we have that

$$\int_{\mathbb{R}}(1,-\omega)\cdot\mathcal{A}(t+s,x-s\omega)\mathrm{d}s=0 \text{ for all } (t,x)\in\mathbb{R}^{1+n} \text{ and for all } \omega \text{ with } |\omega-\omega_0|<\varepsilon.$$

To conclude the uniqueness result for $\mathcal{A}$, we prove the following lemma.

Lemma 3.6.1. *Let $n \geq 3$ and $F=(F_0,F_1,\cdots,F_n)$ be a real-valued vector field whose components are $C_c^\infty(R^{1+n})$ functions. Suppose*

$$LF(t,x,\omega):=\int_{\mathbb{R}}(1,\omega)\cdot F(t+s,x+s\omega)\mathrm{d}s=0$$

for all $\omega\in\mathbb{S}^{n-1}$ near a fixed $\omega_0\in\mathbb{S}^{n-1}$ and for all $(t,x)\in\mathbb{R}^{1+n}$. Then there exist a $\Phi\in C_c^\infty(\mathbb{R}^{1+n})$ such that $F(t,x)=\nabla_{t,x}\Phi(t,x)$.

Proof. The proof follows the analysis similar to the ones used in [48, 68], where support theorems involving light ray transforms have been proved.

Denote $\omega=(\omega^1,\cdots,\omega^n)\in\mathbb{S}^{n-1}$. We write

$$LF(t,x,\omega)=\int_{\mathbb{R}}\sum_{i=0}^{n}\omega^i F_i(t+s,x+s\omega)\mathrm{d}s,\ \text{where } \omega^0=1.$$

Let $\eta = (\eta_0, \eta_1, \cdots, \eta_n) \in \mathbb{R}^{n+1}$ be arbitrary. We have

$$(\eta \cdot \nabla_{t,x}) LF(t,x,\omega) = \int\limits_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j \partial_j F_i(t+s, x+s\omega) \mathrm{d}s. \tag{3.6.2}$$

By fundamental theorem of calculus, we have

$$\int\limits_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}s} (\eta \cdot F)(t+s, x+s\omega) \mathrm{d}s = 0.$$

But

$$\frac{\mathrm{d}}{\mathrm{d}s} (\eta \cdot F)(t+s, x+s\omega) = \sum_{i,j=0}^{n} \omega^i \eta_j \partial_i F_j (t+s, x+s\omega).$$

with $\partial_0 = \partial_t$ and $\partial_j = \partial_{x_j}$ for $j = 1, 2, \cdots, n$. Therefore

$$\int\limits_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j \partial_i F_j (t+s, x+s\omega) \mathrm{d}s = 0. \tag{3.6.3}$$

Subtracting (3.6.3) from (3.6.2), we get,

$$(\eta \cdot \nabla_{t,x}) LF(t,x,\omega) = \int\limits_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j \left(\partial_j F_i - \partial_i F_j\right)(t+s, x+s\omega) \mathrm{d}s.$$

Since $LF(t,x,\omega) = 0$ for all ω near ω_0, and for all $(t,x) \in \mathbb{R}^{1+n}$, we have,

$$Ih(t,x,\omega,\eta) := \int\limits_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j h_{ij}(t+s, x+s\omega) \mathrm{d}s = 0. \tag{3.6.4}$$

Next we will show that the $(n+1)$ dimensional Fourier transform $\widehat{h}_{ij}(\zeta) = 0$ for all space-like vectors ζ near the set $\{\zeta : \ \zeta \cdot (1, \omega) = 0, \omega \text{ near } \omega_0\}$.

We have

$$\omega^i \eta_j \widehat{h}_{ij}(\zeta) = \int\limits_{\mathbb{R}^{1+n}} e^{-\mathrm{i}(t,x)\cdot\zeta} \omega^i \eta_j h_{ij}(t,x) \mathrm{d}t\mathrm{d}x,$$

where ω, η are fixed and $\zeta \in (1,\omega)^{\perp}$. Decomposing

$$\mathbb{R}^{1+n} = \mathbb{R}(1,\omega) + k, \text{ where } k \in (1,\omega)^{\perp},$$

we get,

$$\omega^i \eta_j \widehat{h}_{ij}(\zeta) = \sqrt{2} \int\limits_{(1,\omega)^\perp} e^{-\mathrm{i}k\cdot\zeta} \int\limits_{\mathbb{R}} \omega^i \eta_j h_{ij}(k + s(1,\omega))\, \mathrm{d}s\mathrm{d}k.$$

Using (3.6.4), we get that,

$$\sum_{i,j=0}^{n} \omega^i \eta_j \widehat{h}_{ij}(\zeta) = 0, \text{ for all } \zeta \in (1,\omega)^\perp, \text{ for all } \eta \in \mathbb{R}^{1+n}, \text{ and for all } \omega \text{ near } \omega_0 \text{ with } \omega^0 = 1. \tag{3.6.5}$$

Let us take for η the standard basis vectors in $\mathbb{R}^{1+n}$.

Now $\{e_j, 1 \le j \le n\}$ be the standard basis of $\mathbb{R}^n$. Let $\omega_0 = e_1$, and assume that $\zeta^0 = (0, e_2)$. Then this is a space-like vector that satisfies the condition $\zeta^0 \in (1, \omega_0)^\perp$. We will show that $\widehat{h}_{ij}(\zeta^0) = 0$ for all $0 \le i, j \le n$. Consider the collection of the following unit vectors for $3 \le k \le n$:

$$\omega_k(\alpha) = \cos(\alpha) e_1 + \sin(\alpha) e_k.$$

Note that for each $3 \le k \le n, \omega_k(\alpha)$ is near ω_0 for α near 0. Also $\zeta^0 \in (1, \omega_k(\alpha))^\perp$ for all such α and for all $3 \le k \le n$. Let η be the collection of standard basis vectors in $\mathbb{R}^{1+n}$.

Substituting the above vectors $\omega_k(\alpha)$ and η, we get the following equations:

$$\widehat{h}_{0j}(\zeta^0) + \cos(\alpha)\widehat{h}_{1j}(\zeta^0) + \sin(\alpha)\widehat{h}_{kj}(\zeta^0) = 0 \text{ for all } 3 \le k \le n, 0 \le j \le n \text{ and } \alpha \text{ near } 0.$$

From this we have that

$$\widehat{h}_{0j}(\zeta^0) = \widehat{h}_{1j}(\zeta^0) = \widehat{h}_{kj}(\zeta^0) = 0 \text{ for all } 3 \le k \le n \text{ and } 0 \le j \le n.$$

Using the fact that

$$\widehat{h}_{ij}(\zeta^0) = -\widehat{h}_{ji}(\zeta^0) \text{ for all } 0 \le i, j \le n,$$

we get

$$\widehat{h}_{ij}(\zeta^0) = 0, \text{ for all } 0 \le i, j \le n \text{ with } \zeta^0 = (0, e_2). \tag{3.6.6}$$

Now our goal is to show that the Fourier transform $\widehat{h}_{ij}(\zeta)$ vanishes for all light-like vectors ζ in a small enough neighborhood of $(0, e_2)$. To show this, assume $\zeta = (\tau, \xi)$ be a space-like vector close to $(0, e_2)$. Without loss of generality, assume that ζ is of the form

$$\zeta = (\tau, \xi) \text{ where } |\xi| = 1 \text{ and } |\tau| < 1.$$

For instance, we can take ζ as

$$\zeta = (-\sin\varphi, \xi),$$

where φ is close to 0 and $\xi = (\xi_1, \cdots, \xi_n)$ is written in spherical coordinates as

$$\begin{aligned}
&\xi_1 = \sin\varphi_1 \cos\varphi_2 \\
&\xi_2 = \cos\varphi_1 \\
&\xi_3 = \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\
&\vdots \\
&\xi_{n-1} = \sin\varphi_1 \cdots \sin\varphi_{n-2} \cos\varphi_{n-1} \\
&\xi_{n-1} = \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\varphi_{n-1}.
\end{aligned}$$

Note that if $\varphi_1, \cdots, \varphi_{n-1}$ are close to 0, then ξ is close to e_2.

Let

$$\omega_\varphi = \cos(\varphi)e_1 + \sin(\varphi)e_2.$$

Also let

$$\omega_\varphi^k(\alpha) = \cos(\alpha)\cos(\varphi)e_1 + \sin(\varphi)e_2 + \sin(\alpha)\cos(\varphi)e_k \text{ for } k \geq 3.$$

Since φ is close to 0, and letting α close enough to 0, we have that ω_φ and ω_φ^k are close to ω_0.

Now, let A be an orthogonal transformation such that $Ae_2 = \xi$, where ξ is as above. With this A, consider the vectors

$$\omega_\zeta = A\omega_\varphi \text{ and } \omega_\zeta^k = A\omega_\varphi^k \text{ for } k \geq 3.$$

We have that

$$\zeta \in (1, \omega_\zeta)^\perp.$$

To see this, note that $\zeta = (-\sin\varphi, \xi)$ and we have

$$-\sin\varphi + \langle Ae_2, A\omega_\varphi\rangle = -\sin\varphi + \langle e_2, \omega_\varphi\rangle = 0.$$

Also similarly, we have that,

$$\zeta \in (1, \omega_\zeta^k)^\perp \text{ for all } k \geq 3.$$

Using the standard basis vectors for η_j and the vectors $(1, \omega_\zeta^k)$ in (3.6.5), we get, for all $k \geq 3$,

$$\widehat{h}_{0j}(\zeta)+\sin\varphi\left(\sum_{i=1}^{n} a_{i2}\widehat{h}_{ij}(\zeta)\right)+\cos\alpha\cos\varphi\left(\sum_{i=1}^{n} a_{i1}\widehat{h}_{ij}(\zeta)\right)+\sin\alpha\cos\varphi\left(\sum_{i=1}^{n} a_{ik}\widehat{h}_{ij}(\zeta)\right)=0$$

This then implies that

$$\widehat{h}_{0j}(\zeta) + \sin\varphi\left(\sum_{i=1}^{n} a_{i2}\widehat{h}_{ij}(\zeta)\right) = 0$$
$$\cos\varphi\left(\sum_{i=1}^{n} a_{i1}\widehat{h}_{ij}(\zeta)\right) = 0$$
$$\cos\varphi\left(\sum_{i=1}^{n} a_{ik}\widehat{h}_{ij}(\zeta)\right) = 0$$

Letting $j = 0$, since $\widehat{h}_{00}(\zeta) = 0$, we have,

$$\sum_{i=1}^{n} a_{ij}\widehat{h}_{i0}(\zeta) = 0 \text{ for all } 1 \leq j \leq n.$$

Since A is an invertible matrix, we have that $\widehat{h}_{i0}(\zeta) = 0$ for all $1 \leq i \leq n$. Now proceeding similarly and using the fact that h_{ij} is alternating, we have that $\widehat{h}_{ij}(\zeta) = 0$ for all $0 \leq i, j \leq n$. The same argument as above also works for $r\zeta$, where ζ is as above and $r > 0$.

Since the support of all h_{ij} is a compact subset of $\mathbb{R}^{1+n}$, by Paley-Wiener theorem, we have $\widehat{h}_{ij}(\zeta) = 0\ \forall\ i, j = 0, 1, 2, ..., n$. Hence, by Fourier inversion formula, we see that $h_{ij}(t, x) = 0\ \forall\ (t, x) \in \mathbb{R}^{1+n}$, this gives us $\nabla_{t,x}F(t, x) = 0\ \forall\ (t, x) \in \mathbb{R}^{1+n}$.

Using Poincaré lemma, we have that there exists a $\Phi(t,x) \in C_c^\infty(\mathbb{R}^{1+n})$ such that $F = \nabla_{t,x}\Phi$.

This then gives that

$$\mathcal{A}(t,x) := \left(\mathcal{A}^{(2)} - \mathcal{A}^{(1)}\right)(t,x) := (A_0, A_1, \cdots, A_n) := \nabla_{t,x}\Phi(t,x).$$

$\square$

3.6.2 Recovery of potential q

In Section 3.6.1, we showed that there exist a Φ such that $(\mathcal{A}_2 - \mathcal{A}_1)(t,x) = \nabla_{t,x}\Phi(t,x)$. After replacing the pair $(\mathcal{A}^{(1)}, q_1)$ by $(\mathcal{A}^{(3)}, q_3)$ where $\mathcal{A}^{(3)} = \mathcal{A}^{(1)} + \nabla_{t,x}\Phi$ and $q_3 = q_1$, we conclude that $\mathcal{A}^{(3)} = \mathcal{A}^{(2)}$. Therefore substituting (3.4.4) for u_2 and (3.4.1) for v, and letting $h \to 0^+$ in

$$\int\limits_Q q(t,x)u_2(t,x)v(t,x)\mathrm{d}x\mathrm{d}t = \int\limits_\Omega \partial_t u(T,x)v(T,x)\mathrm{d}x - \int\limits_{\Sigma\setminus G} \partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t,$$

we get,

$$\int\limits_{\mathbb{R}^{1+n}} q(t,x)e^{-\mathrm{i}\xi\cdot(t,x)}\mathrm{d}x\mathrm{d}t = 0 \text{ for all } \xi \in (1,-\omega)^\perp \text{ and } \omega \text{ near } \omega_0.$$

The set of all ξ such that $\xi \in (1,-\omega)^\perp$ for ω near ω_0 forms an open cone and since $q \in L^\infty(Q)$ has compact support, using Paley-Wiener theorem we conclude that $q_1(t,x) = q_2(t,x)$ for all $(t,x) \in Q$. This completes the proof of Theorem 3.2.3.

Chapter 4

Inverse Boundary Value Problem for a Non-linear Hyperbolic Partial Differential Equations

4.1 Introduction and statement of the main result

In this chapter, we consider the inverse boundary value problem for a non-linear wave equation of divergence form with space dimension $n \geq 3$. We will prove the unique determination of non-linearity from the boundary measurements so-called the hyperbolic Dirichlet to Neumann map.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. For $T > 0$, let $Q_T := (0, T) \times \Omega$ and denote its lateral boundary by $\partial Q_T := (0, T) \times \partial\Omega$, and also denote $[\partial Q_T] := [0, T] \times \partial\Omega$. We will simply write $Q = Q_T$, $\partial Q = \partial Q_T$ for $T = \infty$.

Consider the following initial boundary value problem (IBVP):

$$\begin{cases} \partial_t^2 u(t,x) - \nabla_x \cdot \vec{C}(x, \nabla_x u(t,x)) = 0, \ (t,x) \in Q_T, \\ u(0,x) = \partial_t u(0,x) = 0, \ x \in \Omega, \\ u(t,x) = \epsilon f(t,x), \ (t,x) \in \partial Q_T, \end{cases} \tag{4.1.1}$$

where $\nabla_x := (\partial_1, \partial_2, \cdots, \partial_n)$, $\partial_j = \partial_{x_j}$ for $x = (x_1, x_2, \cdots, x_n)$. Here $\vec{C}(x, q)$ is given by

$$\vec{C}(x,q) := \gamma(x)q + \vec{P}(x,q) + \vec{R}(x,q) \tag{4.1.2}$$

for vector $q := (q_1, q_2, \cdots, q_n) \in \mathbb{R}^n$, $C^\infty(\overline{\Omega}) \ni \gamma(x) \geq C > 0$ for some constant C,

$$\vec{P}(x,q) := \left(\sum_{k,l=1}^{n} c_{kl}^1 q_k q_l, \sum_{k,l=1}^{n} c_{kl}^2 q_k q_l, \sum_{k,l=1}^{n} c_{kl}^3 q_k q_l, \cdots, \sum_{k,l=1}^{n} c_{kl}^n q_k q_l \right), \tag{4.1.3}$$

each $c_{kl}^i \in C^\infty(\overline{\Omega})$ and $\vec{R}(x,q) \in C^\infty(\overline{\Omega} \times H)$ with $H := \{q \in \mathbb{R}^n : |q| \leq h\}$ for some constant $h > 0$, satisfies the following estimate: there exists a constant $C > 0$ such that

$$\left| \partial_q^\alpha \partial_x^\beta \vec{R}(x,q) \right| \leq C|q|^{3-|\alpha|} \text{ for all multi-indices } \alpha,\ \beta \text{ with } |\alpha| \leq 3. \tag{4.1.4}$$

Denote by $B^\infty(\partial Q_T)$ the completion of $C_0^\infty(\partial Q_T)$ with respect to the topology of the Fréchet space $C^\infty([\partial Q_T])$ with metric $d(\cdot,\cdot)$. Let $m \geq [n/2] + 3$ with the largest integer $[n/2]$ not exceeding $n/2$ and $B_M := \{f \in B^\infty(\partial Q_T) :\ d(0,f) \leq M\}$ with a fixed constant $M > 0$, then there exists $\epsilon_0 = \epsilon_0(h, m, T, M) > 0$ such that (4.1.1) has a unique solution $u \in X_m := \bigcap_{j=0}^m C^j\left([0,T]; H^{m-j}(\Omega)\right)$ for any $f \in B_M$ and $0 < \epsilon < \epsilon_0$. We refer this by the *unique solvability* of (4.1.1). We will provide some arguments about this together with the ϵ-expansion of the solution u to the IBVP (4.1.1) in Section 4.2.

Based on this, we define the Dirichlet to Neumann (DN) map $\Lambda_{\vec{C}}^T$ by

$$\Lambda_{\vec{C}}^T(\epsilon f) = \nu(x) \cdot \vec{C}(x, \nabla_x u^f)|_{\partial Q_T},\ \ f \in B_M,\ \ 0 < \epsilon < \epsilon_0 \tag{4.1.5}$$

where $u^f(t,x)$ is the solution to (4.1.1) and $\nu(x)$ is the outer unit normal vector of $\partial\Omega$ at $x \in \partial\Omega$.

The inverse problem we are going to consider is the uniqueness of identifying $\gamma = \gamma(x)$ and $\vec{P} = \vec{P}(x,q)$ from the DN map $\Lambda_{\vec{C}}^T$. More precisely it is to show that if the DN maps $\Lambda_{\vec{C}_i}^T$, $i = 1,2$ given by (4.1.5) for $\vec{C} = \vec{C}_i$, $i = 1,2$ are the same, then $(\gamma_i,\ \vec{P}_i)$, $i = 1,2$ are the same, where $(\gamma_i, \vec{P}_i)$, $i = 1,2$ are $(\gamma, \vec{P})$ associated to $\vec{C}_i$, $i = 1,2$.

The non-linear wave equation of the form (4.1.1) arises as a model equation of a vibrating string with elasticity coefficient depending on strain and a model equation describing the anti-plane deformation of a uniformly thin piezoelectric material for the one spatial dimension ([46]), and as a model equation for non-linear Love waves for the two spatial dimension ([62]).

There are several works on inverse problems for non-linear wave equations. For example, Denisov [14], Grasselli [21] and Lorenzi-Paparoni[42] considered the inverse problems related to non-linear wave equations, but non-linearity in their works is in lower order terms. Under the same set up as our inverse problem except the space dimension, Nakamura-Watanabe in [45] identified $(\gamma, \vec{P})$ by giving a reconstruction formula in one space dimension which also gives uniqueness. We are going to prove the uniqueness for our inverse problem when the space dimension $n \geq 3$.

We will also mention about some related works for elliptic and parabolic equations. For elliptic equations, Kang-Nakamura in [31] studied the uniqueness for determining the non-linearity in conductivity equation. Our result can be viewed as a generalization of [31] for non-linear wave equation. There are other works related to non-linear elliptic PDE, we refer to [23, 28, 30, 43, 69–71]. For parabolic equations, we refer to [11, 28, 34].

In order to state our main result, we need to define the filling time T^*.

Definition 4.1.1. Let E^T be the maximal subdomain of Ω such that any solution $v = v(t,x)$ of $\partial_t^2 v - \nabla_x \cdot (\gamma \nabla_x v) = 0$ in Q_T will become zero in this subdomain at $t = 2T$ if the Cauchy data of v on ∂Q_{2T} are zero. We call E^T the influence domain. Then define the filling time T^* by $T^* = \inf\{T > 0 : \, E^T = \Omega\}$.

By the Holmgren-John-Tataru unique continuation property of solutions of the above equation for v in Definition 4.1.1, there exists a finite filling time T^* (see [32] and the references there in for further details). Based on this we have the following main theorem.

Theorem 4.1.2. *For $i = 1, 2$, let*

$$\vec{P}^{(i)}(x,q) := \left(\sum_{k,l=1}^{n} c_{kl}^{1(i)} q_k q_l, \sum_{k,l=1}^{n} c_{kl}^{2(i)} q_k q_l, \sum_{k,l=1}^{n} c_{kl}^{3(i)} q_k q_l, \cdots, \sum_{k,l=1}^{n} c_{kl}^{n(i)} q_k q_l \right)$$

and $\vec{C}^{(i)}(x,q) = \gamma_i(x)p + \vec{P}^{(i)}(x,q) + \vec{R}^{(i)}(x,q)$ *with* γ_i, $\vec{P}^{(i)}$ *and* $\vec{R}^{(i)}$, $i = 1,2$ *satisfying the same conditions as for* γ, $\vec{P}$ *and* $\vec{R}$. *Further let* $u^{(i)}$, $i = 1,2$ *be the solutions to the following IBVP:*

$$\begin{cases} \partial_t^2 u^{(i)}(t,x) - \nabla_x \cdot \vec{C}^{(i)}(x, \nabla_x u^{(i)}(t,x)) = 0, \ (t,x) \in Q_T, \\ u^{(i)}(0,x) = \partial_t u^{(i)}(0,x) = 0, \ x \in \Omega, \\ u^{(i)} = \epsilon f(t,x), \ (t,x) \in \partial Q_T \end{cases} \tag{4.1.6}$$

with any $0 < \epsilon < \epsilon_0$. *Assume* $T > 2T^*$ *and let* $\Lambda^T_{\vec{C}^{(1)}}$ *and* $\Lambda^T_{\vec{C}^{(2)}}$ *be the DN maps as defined in* (4.1.5) *corresponding to* $u^{(1)}$ *and* $u^{(2)}$ *respectively. Assume that*

$$\Lambda^T_{\vec{C}^{(1)}}(\epsilon f) = \Lambda^T_{\vec{C}^{(2)}}(\epsilon f), \ f \in B_M, \ 0 < \epsilon < \epsilon_0. \tag{4.1.7}$$

Then we have

$$\gamma_1(x) = \gamma_2(x), \ c^{j(1)}_{kl}(x) = c^{j(2)}_{kl}(x), \ x \in \Omega, \ 1 \le j,k,l \le n.$$

The most difficult part of proving Theorem 4.1.2 is showing the uniqueness of identifying the quadratic nonlinear part $\vec{P}(x,q)$. The key ingredients for showing this are to use the control with delay in time (see (4.4.6)) and the special polarization for the difference of the quadratic nonlinear part with integration with respect to the delay time (see (4.4.7), (4.4.11)) coming from two $\vec{C}^{(i)}(x,q)$, $i = 1,2$ so that via the Laplace tranform with respect to t, we can relate the problem of identifying the quadratic part to that for a nonlinear elliptic equation. The reduced problem is almost the same as the one considered in [31].

The chapter is organized as follows. In §4.2, we will introduce the ϵ-expansion of the IBVP to analyze the hyperbolic DN map. As a consequence, we will show that the hyperbolic DN map determines the hyperbolic DN map associated with the equation $\partial_t^2 v - \nabla_x \cdot (\gamma \nabla_x v) = 0$ in $(0,T) \times \Omega$. This immediately implies the uniqueness of identifying γ. §4.4 is devoted to proving the uniqueness of identifying $\vec{P}(x,q)$.

4.2 ϵ-expansion of the solution to the IBVP

Theorem 4.2.1. *Let $m \geq [n/2] + 3$ with the largest integer $[n/2]$ not exceeding $n/2$ and $f \in B_M$ with a fixed constant $M > 0$, then for given $T > 0$, there exists $\epsilon_0 = \epsilon_0(h, m, M) > 0$ such that for any $0 < \epsilon < \epsilon_0$, (4.1.1) has a unique solution $u \in X_m$ where h and B_M, X_m were defined in Section 4.1 right after (4.1.3) and the paragraph after (4.1.4), respectively. Moreover, it admits an expansion which we call ϵ-expansion:*

$$u = \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3); \quad (\epsilon \to 0) \tag{4.2.1}$$

where u_1 is a solution to

$$\begin{cases} \partial_t^2 u_1(t,x) - \nabla_x \cdot (\gamma(x)\nabla_x u_1(t,x)) = 0, \ (t,x) \in Q_T \\ u_1(0,x) = \partial_t u_1(0,x) = 0, \ x \in \Omega \\ u_1 = f(t,x), \ (t,x) \in \partial Q_T \end{cases} \tag{4.2.2}$$

and u_2 is a solution to

$$\begin{cases} \partial_t^2 u_2(t,x) - \nabla_x \cdot (\gamma(x)\nabla_x u_2(t,x)) = \nabla_x \cdot \vec{P}(x, \nabla_x u_1), \ (t,x) \in Q_T \\ u_2(0,x) = \partial_t u_2(0,x) = 0, \ x \in \Omega \\ u_2 = 0, \ (t,x) \in \partial Q_T. \end{cases} \tag{4.2.3}$$

Remark 4.2.2. For the well-posedness of initial boundary value problems (4.2.2) and (4.2.3), we refer to Theorem 2.45 of [32].

Remark 4.2.3. $w_\epsilon = O(\epsilon^3)$ means that $\sup_{0 \leq t \leq T} \sum_{k=0}^{m} ||w_\epsilon(t)||_{m-k}^2 = O(\epsilon^3)$, where $||.||_k$ is the norm of the usual Sobolev space $W^{k,2}(\Omega)$.

Proof. We look for a solution $u(t,x)$ to (4.1.1) of the form

$$u(t,x) = \epsilon \left\{ u_1(t,x) + \epsilon \left(u_2(t,x) + w(t,x) \right) \right\}, \quad w = O(\epsilon) \ (\epsilon \to 0). \tag{4.2.4}$$

Then, $w(t,x)$ has to satisfy

$$\begin{cases} \partial_t^2 w(t,x) - \nabla_x\left(\gamma(x)\nabla_x w(t,x)\right) = \epsilon^{-2}\nabla_x\cdot R(x,\epsilon\nabla_x u_1 + \epsilon^2\nabla_x u_2 + \epsilon^2\nabla_x w) \\ \quad +\epsilon\sum_{j=1}^n \partial_j\left(\sum_{k,l=1}^n c_{kl}^j(x)\left(\partial_k u_1\partial_l u_2 + \partial_l u_1\partial_k u_2\right)(t,x)\right) \\ \quad +\epsilon\sum_{j=1}^n \partial_j\left(\sum_{k,l=1}^n c_{kl}^j(x)\left(\partial_k u_1\partial_l w + \partial_l u_1\partial_k w\right)(t,x)\right) \\ \quad +\epsilon^2\sum_{j=1}^n \partial_j\left(\sum_{k,l=1}^n c_{kl}^j\left(\partial_k u_2\partial_l u_2 + \partial_k u_2\partial_l w + \partial_l u_2\partial_k w + \partial_k w\partial_l w\right)\right) \text{ in } Q_T, \\ w(0,x) = \partial_t w(0,x) = 0,\ x\in\Omega \text{ and } w|_{\partial Q_T} = 0. \end{cases}$$

By the mean value theorem, we have

$$\begin{aligned} R(x,\epsilon\nabla_x u_1 + \epsilon^2\nabla_x u_2 + \epsilon^2\nabla_x w) &= R(x,\epsilon\nabla_x u_1 + \epsilon^2\nabla_x u_2) \\ &\quad + \int_0^1 \frac{d}{d\theta}R(x,\epsilon\nabla_x u_1 + \epsilon^2\nabla_x u_2 + \theta\epsilon^2\nabla_x w)d\theta \\ &= R(x,\epsilon\nabla_x u_1 + \epsilon^2\nabla_x u_2) + \epsilon^3 K(x,\epsilon\nabla_x w;\epsilon)\nabla_x w \end{aligned}$$

where

$$\epsilon K(x,\epsilon\nabla_x w;\epsilon) := \int_0^1 D_q R(x,\epsilon\nabla_x u_1 + \epsilon^2\nabla_x u_2 + \theta\epsilon^2\nabla_x w)d\theta$$

with $D_q R(x,q) = ((\partial_{q_i}R_j))_{1\le i,j\le n}$ and $K_{ij} = \partial_{q_i}R_j$. After using this in previous equation, we get

$$\begin{cases} \partial_t^2 w - \nabla_x\left(\gamma(x)\nabla_x w\right) = \epsilon\sum_{j=1}^n \partial_j\left(\sum_{k,l=1}^n c_{kl}^j(x)\left(\partial_k u_1\partial_l u_2 + \partial_l u_1\partial_k u_2\right)\right) \\ +\epsilon\sum_{j=1}^n\sum_{k,l=1}^n \partial_j\left(c_{kl}^j(x)\partial_k u_1\right)\partial_l w + \epsilon\sum_{j=1}^n\sum_{k,l=1}^n\left(c_{kl}^j(x)\partial_k u_1\right)\partial_{jl}^2 w \\ +\epsilon\sum_{j=1}^n\sum_{k,l=1}^n \partial_j\left(c_{kl}^j(x)\partial_l u_1\right)\partial_k w + \epsilon\sum_{j=1}^n\sum_{k,l=1}^n\left(c_{kl}^j(x)\partial_k u_1\right)\partial_{jk}^2 w \\ +\epsilon^2\sum_{j=1}^n \partial_j\left(\sum_{k,l=1}^n c_{kl}^j(x)\partial_k u_2\partial_l u_2\right) + \epsilon^2\sum_{j=1}^n\sum_{k,l=1}^n \partial_j\left(c_{kl}^j(x)\partial_k u_2\right)\partial_l w \\ +\epsilon^2\sum_{j=1}^n\sum_{k,l=1}^n\left(c_{kl}^j(x)\partial_k u_2\right)\partial_{jl}^2 w + \epsilon^2\sum_{j=1}^n\sum_{k,l=1}^n \partial_j\left(c_{kl}^j(x)\partial_l u_2\right)\partial_k w \\ +\epsilon^2\sum_{j=1}^n\sum_{k,l=1}^n\left(c_{kl}^j\partial_k u_2\right)\partial_{jk}^2 w + \epsilon^2\sum_{j=1}^n\sum_{k,l=1}^n c_{kl}^j\left(\partial_{jk}^2 w\partial_l w + \partial_k w\partial_{jl}^2 w\right) \\ +\epsilon^2\sum_{j=1}^n\sum_{k,l=1}^n \partial_j c_{kl}^j\partial_k w\partial_l w + \epsilon\sum_{i=1}^n\sum_{j=1}^n \partial_i K_{ij}(x,\epsilon\nabla_x w;\epsilon)\partial_j w \\ +\epsilon^2\sum_{i=1}^n\sum_{j,l=1}^n \partial_{q_l}K_{ij}(x,\epsilon\nabla_x w;\epsilon)\partial_j w\partial_{il}^2 w \\ +\epsilon\sum_{i=1}^n\sum_{j=1}^n K_{ij}(x,\epsilon\nabla_x w;\epsilon)\partial_{ij}^2 w + \epsilon^{-2}\nabla_x\cdot R(x,\epsilon\nabla_x u_1 + \epsilon\nabla_x u_2),\ \text{in } Q_T, \\ w(0,x) = \partial_t w(0,x) = 0,\ x\in\Omega \text{ and } w|_{\partial Q_T} = 0. \end{cases}$$

Thus, finally we have derived the following initial boundary value problem for w:

$$\begin{cases} \partial_t^2 w - \nabla_x \cdot (\gamma(x)\nabla_x w) = \epsilon \sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x) \left(\partial_k u_1 \partial_l u_2 + \partial_l u_1 \partial_k u_2 \right) \right) \\ +\epsilon \sum_{j=1}^n \sum_{k,l=1}^n \partial_j \left(c_{kl}^j(x) \partial_k u_1 \right) \partial_l w + \epsilon \sum_{j=1}^n \sum_{k,l=1}^n \left(c_{kl}^j(x) \partial_k u_1 \right) \partial_{jl}^2 w \\ +\epsilon \sum_{j=1}^n \sum_{k,l=1}^n \partial_j \left(c_{kl}^j(x) \partial_l u_1 \right) \partial_k w + \epsilon \sum_{j=1}^n \sum_{k,l=1}^n \left(c_{kl}^j(x) \partial_l u_1 \right) \partial_{jk}^2 w \\ +\epsilon^2 \sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x) \partial_k u_2 \partial_l u_2 \right) + \epsilon^2 \sum_{j=1}^n \sum_{k,l=1}^n \partial_j \left(c_{kl}^j(x) \partial_k u_2 \right) \partial_l w \\ +\epsilon^2 \sum_{j=1}^n \sum_{k,l=1}^n \left(c_{kl}^j(x) \partial_k u_2 \right) \partial_{jl}^2 w + \epsilon^2 \sum_{j=1}^n \sum_{k,l=1}^n \partial_j \left(c_{kl}^j(x) \partial_l u_2 \right) \partial_k w \\ +\epsilon^2 \sum_{j=1}^n \sum_{k,l=1}^n \left(c_{kl}^j \partial_k u_2 \right) \partial_{jk}^2 w + \epsilon^2 \sum_{j=1}^n \sum_{k,l=1}^n c_{kl}^j \left(\partial_{jk}^2 w \partial_l w + \partial_k w \partial_{jl}^2 w \right) \\ +\epsilon^2 \sum_{j=1}^n \sum_{k,l=1}^n \partial_j c_{kl}^j \partial_k w \partial_l w + \epsilon^{-2} \nabla_x \cdot R(x, \epsilon \nabla_x u_1 + \epsilon \nabla_x u_2) \\ +\epsilon \sum_{i=1}^n \sum_{j=1}^n \partial_i K_{ij}(x, \epsilon \nabla_x w; \epsilon) \partial_j w + \epsilon \sum_{i=1}^n \sum_{j=1}^n K_{ij}(x, \epsilon \nabla_x w; \epsilon) \partial_{ij}^2 w \\ +\epsilon^2 \sum_{i=1}^n \sum_{j,l=1}^n \partial_{q_l} K_{ij}(x, \epsilon \nabla_x w; \epsilon) \partial_j w \partial_{il}^2 w, \text{ in } Q_T, \\ w(0,x) = \partial_t w(0,x) = 0, \ x \in \Omega \text{ and } w|_{\partial Q_T} = 0. \end{cases} \tag{4.2.5}$$

In order to simplify the description of the above equation for w, let us introduce the following notations:

$$\begin{cases} A(w(t))w = \nabla_x \cdot (\gamma(x)\nabla_x w(t,x)) + \epsilon \Gamma(x, \nabla_x w; \epsilon) \cdot \partial_x^2 w, \\ \epsilon \Gamma(x, \nabla_x w; \epsilon) := \epsilon \left(\sum_{k=1}^n c_{kl}^j(x) \partial_k u_1 \right)_{1 \le j,l \le n} + \epsilon \left(\sum_{l=1}^n c_{kl}^j(x) \partial_l u_1 \right)_{1 \le j,k \le n} \\ +\epsilon^2 \left(\sum_{k=1}^n c_{kl}^j(x) \partial_k u_2 \right)_{1 \le j,l \le n} + \epsilon^2 \left(\sum_{l=1}^n c_{kl}^j(x) \partial_l u_2 \right)_{1 \le j,k \le n} \\ +\epsilon^2 \left(\sum_{k=1}^n c_{kl}^j(x) \partial_k w \right)_{1 \le j,l \le n} + \epsilon^2 \left(\sum_{l=1}^n c_{kl}^j(x) \partial_l w \right)_{1 \le j,k \le n} \\ +\left(K_{ij}(x, \epsilon \nabla_x w; \epsilon) \right)_{1 \le i,j \le n} + \left(\sum_{j=1}^n \partial_{q_l} K_{ij}(x, \epsilon \nabla_x w; \epsilon) \partial_j w \right)_{1 \le i,l \le n}, \\ \epsilon \vec{G}(x, \nabla_x w; \epsilon) := \epsilon \left(\sum_{j=1}^n \sum_{k=1}^n \partial_j \left(c_{kl}^j(x) \partial_k u_1 \right) \right)_{1 \le l \le n} \\ +\epsilon \left(\sum_{j=1}^n \sum_{l=1}^n \partial_j \left(c_{kl}^j(x) \partial_k u_1 \right) \right)_{1 \le k \le n} + \epsilon^2 \left(\sum_{j=1}^n \sum_{k=1}^n \partial_j \left(c_{kl}^j(x) \partial_k u_2 \right) \right)_{1 \le l \le n} \\ +\epsilon^2 \left(\sum_{j=1}^n \sum_{l=1}^n \partial_j \left(c_{kl}^j(x) \partial_k u_2 \right) \right)_{1 \le k \le n} + \epsilon^2 \left(\sum_{j=1}^n \sum_{k=1}^n \partial_j c_{kl}^j(x) \partial_k w \right)_{1 \le l \le n} \\ +\epsilon \left(\sum_{i=1}^n \partial_i K_{ij}(x, \epsilon \nabla_x w; \epsilon) \right)_{1 \le j \le n}, \\ \epsilon F(x, \nabla_x u_1, \nabla_x u_2; \epsilon) := \epsilon \sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x) \left(\partial_k u_1 \partial_l u_2 + \partial_l u_1 \partial_k u_2 \right) \right) \\ +\epsilon^2 \sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x) \partial_k u_2 \partial_l u_2 \right) + \epsilon^{-2} \nabla_x \cdot R(x, \epsilon \nabla_x u_1 + \epsilon^2 \nabla_x u_2). \end{cases} \tag{4.2.6}$$

Here $\partial_x^2 w = (\partial_{ij} w)_{1 \le i,j \le n}$ and $\Gamma(x, \nabla_x w; \epsilon) \cdot \partial_x^2 w$ denotes the real inner product of $\Gamma(x, \nabla_x w; \epsilon)$ and $\partial_x^2 w$. Using Equation (4.2.6) in (4.2.5), we get

$$\begin{cases} \partial_t^2 w - A(w(t))w - \epsilon \vec{G}(x, \epsilon \nabla_x w; \epsilon) \cdot \nabla_x w = \epsilon F(x, \nabla_x u_1, \nabla_x u_2; \epsilon) \text{ in } Q_T, \\ w(0,x) = \partial_t w(0,x) = 0, \ x \in \Omega \text{ and } w|_{\partial Q_T} = 0. \end{cases} \tag{4.2.7}$$

Justification of ϵ-expansion is given by the following lemma.

Lemma 4.2.4. *Let $m \ge [n/2] + 3$ and $f \in B_M$. Then, for given $T > 0$ there exists $\epsilon_0 = \epsilon_0(h, m, M) > 0$ and $w = w(t, x; \epsilon) \in X_m$ for $0 < \epsilon < \epsilon_0$ such that each $w = w(\cdot, \cdot; \epsilon)$ is the unique solution to the initial boundary value problem (4.2.7) with the estimate*

$$|||w|||_m := \left\{ \sum_{j=0}^{m} \sum_{k=0}^{m-j} \sup_{t \in [0,T]} \|\partial_t^k w(t, \cdot; \epsilon)\|^2_{H^{m-j-k}(\Omega)} \right\}^{1/2} = O(\epsilon) \text{ as } \epsilon \to 0. \tag{4.2.8}$$

Here h and B_M, X_m were defined in Section 1 right after (4.1.3) and the paragraph after (4.1.4), respectively.

Remark 4.2.5. The above Lemma 4.2.4 can be proved along the same line as the proof for the case $n = 1$ which is given in [45]. For $n = 1$, [45] proved Lemma 4.2.4 by adapting argument of [13]. In order to apply the same proof given in [45] for our case, we have used the same notations as in [45] for (4.2.7). But for the sake of completeness, we will provide the details here.

Proof. For $M > 0$, we define $Z(M)$ as the set of U satisfying

$$\begin{cases} U \in \cap_{k=0}^m W^{k,\infty}([0,T]; W^{m-k,2}(\Omega)) \\ \partial_t^k U(0,x) = 0; \text{ for } (k = 0, 1, 2, \cdots\cdots, m-1) \ x \in \Omega \\ U(t,x) = 0, \quad (t,x) \in \partial Q_T \\ ess\ sup \sum_{k=0}^m ||\partial_t^k U(t)||_{m-k} \le M^2. \end{cases} \tag{4.2.9}$$

Now for $U \in Z(M)$, consider the following semilinear wave equation

$$\begin{cases} \partial_t^2 w - A(U(t))w = \epsilon \mathcal{F}(t,x,\nabla_x w;\epsilon) & (t,x) \in Q_T \\ w(0,x) = \partial_t w(0,x) = 0, \quad x \in \Omega \\ w(t,x) = 0, \quad (t,x) \in \partial Q_T \end{cases} \tag{4.2.10}$$

where

$$\begin{cases} \mathcal{F}(t,x,h;\epsilon) := F(x,\nabla_x u_1, \nabla_x u_2;\epsilon) + \vec{G}(x,h;\epsilon)\cdot h \\ A(U(t))w := \nabla_x \cdot (\gamma(x)\nabla_x w(t,x)) + \epsilon\Gamma(x,\nabla_x U;\epsilon)\cdot \partial_x^2 w. \end{cases} \tag{4.2.11}$$

We will first state the unique solvability of the semi-linear equation (4.2.10) in the form of following proposition.

Proposition 4.2.6. *[45] Let $m \geq 3$ be an integer. Then, there exists $\epsilon_1 > 0$ such that the initial boundary value problem (4.2.10) has a unique solution*

$$w \in \cap_{k=0}^{m+1} C^{m+1-j}\left([0,T]; W^{j,2}(\Omega)\right)$$

for each $0 < \epsilon < \epsilon_1$. Moreover w satisfies the estimate

$$\sum_{k=0}^{m} ||\partial_t^k w(t)||_{m-k}^2 \leq \epsilon C e^{K_\epsilon T} \quad (t \in [0,T]), \tag{4.2.12}$$

where C is a positive constant and K_ϵ is a positive constant depending on ϵ, which is uniformly bounded for ϵ.

Remark 4.2.7. The Proposition 4.2.6 can be proved by using the iteration method together with energy estimates for the linear hyperbolic PDE of second order (see [13]).

Using the Proposition 4.2.6, we have for any small enough $\epsilon > 0$, there exists a unique solution $w \in Z(M)$ to (4.2.10). Thus, the map $T : Z(M) \to Z(M)$ given by $T(U) = w$ is well-defined, where w is the solution to (4.2.10). Now the idea is to use the fixed poitn arguments to prove that for any $\epsilon > 0$ small enough, there exists a unique solution w to the initial boundary value problem (4.2.7). In order to apply fixed point argument, we first need to show that $T : Z(M) \to Z(M)$ is a contraction mapping for which we proceed as follows:

Let $T(U_i) = w_i$ for $i = 1, 2$, where w_i is the solution to semi-linear wave equation (4.2.10) for $U = U_i$. Let $W = w_1 - w_2$ and $V = U_1 - U_2$, then W will satisfies the following initial boundary value problem

$$\begin{cases} \partial_t^2 W(t) - A(U(t))W(t) = \{A(U_1(t)) - A(U_2(t))\}\, w_2 \\ \qquad +\epsilon \vec{G}(x, \nabla_x w_1; \epsilon) \cdot \nabla_x W(t) \\ \qquad +\epsilon \Big\{ \vec{G}(x, \nabla_x w_1; \epsilon) - \vec{G}(x, \nabla_x w_2; \epsilon) \Big\} \cdot \nabla_x w_2(t), \ (t, x) \in Q_T \\ W(0, x) = \partial_t W(0, x) = 0, \ x \in \Omega \\ W(t, x) = 0, \ (t, x) \in \partial Q_T. \end{cases} \tag{4.2.13}$$

Now multiplying Equation (4.2.13) by $2\partial_t W$ and integrating over $[0, t]$, we get

$$\begin{aligned} ||\partial_t W(t)||^2 + \Big\langle A(U_1(t))W(t), W(t) \Big\rangle &= \int_0^t \Big\langle \partial_t A(U_1(\tau))W(\tau), W(\tau) \Big\rangle d\tau \\ &+ 2 \int_0^t \Big\langle \Big[A(U_1(\tau)) - A(U_2(\tau)) \Big] w_2(\tau), \partial_t W(\tau) \Big\rangle d\tau \\ &+ 2\epsilon \int_0^t \Big\langle \vec{G}(x, \nabla_x w_1; \epsilon) \cdot \nabla_x W(\tau), \partial_t W(\tau) \Big\rangle d\tau \\ &+ 2\epsilon \int_0^t \Big\langle \Big\{ \vec{G}(x, \nabla_x w_1; \epsilon) - \vec{G}(x, \nabla_x w_2; \epsilon) \Big\} \cdot \nabla_x w_2(\tau), \partial W(\tau) \Big\rangle d\tau. \end{aligned}$$

Using the expressions for $\vec{G}$ from Equation (4.2.6), we have

$$\begin{aligned} \vec{G}(x, \nabla_x w_1; \epsilon) - \vec{G}(x, \nabla_x w_2; \epsilon) &= \epsilon \Big(\sum_{j=1}^{n} \sum_{k=1}^{n} \partial_j c_{kl}^j(x) \partial_k W \Big)_{1 \le l \le n} \\ &+ \Big(\sum_{i=1}^{n} \partial_i K_{ij}(x, \epsilon \nabla_x w_1; \epsilon) - \sum_{i=1}^{n} \partial_i K_{ij}(x, \epsilon \nabla_x w_2; \epsilon \Big) \end{aligned}$$

Now using the estimate given by Equation (4.1.4), we get

$$\begin{aligned} ||\partial_t W(t)||^2 + ||W(t)||_1^2 \le \epsilon^2 C \sup_{t \in [0, T]} ||V(t)||_1 \\ + K_\epsilon \int_0^t \left(||\partial_t W(\tau)||^2 + ||W(\tau)||_1^2 \right) d\tau. \end{aligned}$$

We equip $Z(M)$ with the complete metric ρ defined by

$$\rho(f,g) := \max_{t\in[0,T]} \left\{||f(t)-g(t)||_1^2 + ||\partial_t f(t) - \partial_t g(t)||^2\right\}^{\frac{1}{2}}.$$

Finally using Grownwall's inequality, we have

$$\rho(T(U_1), T(U_2) \leq \epsilon^2 C e^{K_\epsilon T} \rho(U_1, U_2)$$

which is contraction for $\epsilon > 0$ small enough. Therefore, for each $\epsilon > 0$ small enough, we have there exists $w \in Z(M)$ such that $T(w) = w$ and it will satisfies the estimate (4.2.8) which follows from (4.2.12). Hence, Lemma 4.2.4 is proved. $\square$

Now Theorem 4.2.1 follows from Lemma 4.2.4. $\square$

4.3 Analysis of DN map in ϵ-expansion

To prove the theorem, we will use the ϵ-expansion of the solution to (4.1.6) which is given by

$$u^{(i)f}(t,x) = \epsilon u_1^{(i)f}(t,x) + \epsilon^2 u_2^{(i)f}(t,x) + O(\epsilon^3). \tag{4.3.1}$$

By the straight forward calculation, we have the followings:

$$\begin{aligned}
&\partial_t^2 u^{(i)f} = \epsilon \partial_t^2 u_1^{(i)f}(t,x) + \epsilon^2 \partial_t^2 u_2^{(i)f}(t,x) + O(\epsilon^3),\\
&\nabla_x u^{(i)f} = \epsilon \nabla_x u_1^{(i)f}(t,x) + \epsilon^2 \nabla_x u_2^{(i)f}(t,x) + O(\epsilon^3),\\
&\vec{C}^{(i)}(x, \nabla_x u^{(i)f}) = \gamma_i(x) \nabla_x u^{(i)f}(t,x)\\
&\quad + \Big(\sum_{k,l=1}^n c_{kl}^{1(i)} \partial_k u^{(i)f} \partial_l u^{(i)f}, \sum_{k,l=1}^n c_{kl}^{2(i)} \partial_k u^{(i)f} \partial_l u^{(i)f}, \cdots, \sum_{k,l=1}^n c_{kl}^{n(i)} \partial_k u^{(i)f} \partial_l u^{(i)f}\Big)\\
&\quad = \epsilon \gamma_i(x) \nabla_x u_1^{(i)f} + \epsilon^2 \gamma_i(x) \nabla_x u_2^{(i)f} + \epsilon^2 \left(\sum_{k,l=1}^n c_{kl}^{j(i)} \partial_k u_1^{(i)f} \partial_l u_1^{(i)f}\right)_{1\leq j\leq n} + O(\epsilon^3),\\
&\nabla_x \cdot \vec{C}^{(i)}(x, \nabla_x u^{(i)f}) = \epsilon \nabla_x \cdot (\gamma_i(x) \nabla_x u_1^{(i)f}) + \epsilon^2 \nabla_x \cdot (\gamma_i(x) \nabla_x u_2^{(i)f})\\
&\quad + \epsilon^2 \sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^{j(i)} \partial_k u_1^{(i)f} \partial_l u_1^{(i)f}\right) + O(\epsilon^3).
\end{aligned}$$

Substitute (4.3.1) into (4.1.6), and arrange the terms into ascending order of power of ϵ by using the above calculations. Then setting the coefficients of ϵ and ϵ^2 equal zero, we have the following equations for $u_1^{(i)} = u_1^{(i)f}$ and $u_2^{(i)} = u_2^{(i)f}$:

$$\begin{cases} \partial_t^2 u_1^{(i)}(t,x) - \nabla_x \cdot (\gamma_i(x)\nabla_x u_1^{(i)}(t,x)) = 0, & (t,x) \in Q_T \\ u_1^{(i)}(0,x) = \partial_t u_1^{(i)}(0,x) = 0, & x \in \Omega \\ u_1^{(i)} = f(t,x), & (t,x) \in \partial Q_T, \end{cases} \tag{4.3.2}$$

$$\begin{cases} \partial_t^2 u_2^{(i)}(t,x) - \nabla_x \cdot (\gamma_i(x)\nabla_x u_2^{(i)}(t,x)) = \nabla_x \cdot \vec{P}^{(i)}(x, \nabla_x u_1^{(i)f}), & (t,x) \in Q_T \\ u_2^{(i)}(0,x) = \partial_t u_2^{(i)}(0,x) = 0, & x \in \Omega \\ u_2^{(i)} = 0, & (t,x) \in \partial Q_T. \end{cases} \tag{4.3.3}$$

By using the ϵ-expansion (4.3.1) of solution to Equation (4.1.5), we have the ϵ-expansion of the DN map:

$$\begin{aligned} \Lambda_{\vec{C}^{(i)}}(\epsilon f) &= \epsilon \left(\gamma_i(x)\partial_\nu u_1^{(i)}(t,x) \right) \Big|_{\partial Q_T} \\ &\quad + \epsilon^2 \left(\gamma_i(x)\partial_\nu u_2^{(i)}(t,x) + \nu(x) \cdot \vec{P}^{(i)}(x, \nabla_x u_1^{(i)f}) \right) \Big|_{\partial Q_T} + O(\epsilon^3) \\ &= \epsilon g_1^{(i)} + \epsilon^2 g_2^{(i)} + O(\epsilon^3). \end{aligned} \tag{4.3.4}$$

This gives us

$$\Lambda_{\gamma_i}^T(f) = \gamma_i(x)\partial_\nu u_1^{(i)f}|_{\partial Q_T} = g_1^{(i)}(t,x)|_{\partial Q_T}, \ (t,x) \in \partial Q_T. \tag{4.3.5}$$

where each $\Lambda_{\gamma_i}^T$ is the DN map associated to the initial boundary value problem (4.3.2) defined by

$$\Lambda_{\gamma_i}^T(f) = \gamma_i(x)\partial_\nu u_1^{(i)f}\big|_{\partial Q_T}, \ f \in C_0^\infty(\partial Q_T). \tag{4.3.6}$$

Therefore we have shown the following implication:

$$\begin{aligned} \Lambda_{\vec{C}^{(1)}}^T(\epsilon f) = \Lambda_{\vec{C}^{(2)}}^T(\epsilon f), \ f \in B_M, \ 0 < \epsilon < \epsilon_0 \\ \Longrightarrow \Lambda_{\gamma_1}^T = \Lambda_{\gamma_2}^T \qquad\qquad . \end{aligned} \tag{4.3.7}$$

4.4 Proof of Theorem 4.1.2

4.4.1 Proof for uniqueness of γ

By knowing each $\Lambda^T_{\vec{C}^{(i)}}(\epsilon f)$, $i=1,2$ for any $f \in B_M$, $0<\epsilon<\epsilon_0$, we do know each $\Lambda^T_{\gamma_i}$, $i=1,2$ from (4.3.5). Then, recalling $T>2T^*$, we can reconstruct each γ_i, $i=1,2$ from $\Lambda^T_{\gamma_i}$, $i=1,2$ by the boundary control method (see[5]). By (4.3.7) the reconstructed γ_i, $i=1,2$ in Ω are the same. We denote this common γ_i, $i=1,2$ by γ, i.e.

$$\gamma=\gamma_1=\gamma_2 \text{ in } \Omega. \tag{4.4.1}$$

Together with this and the given Dirichlet data f is the same for $u_1^{(i)}$, $i=1,2$, we do know

$$u_1^{(1)}=u_1^{(2)} \text{ in } Q_T.$$

Due to the fact that γ is independent of t, this implies

$$u_1^{(1)}=u_1^{(2)} \text{ in } Q.$$

We denote this common solution by $u_1=u_1^f$, i.e.

$$u_1=u_1^f=u_1^{(1)}=u_1^{(2)} \text{ in } Q. \tag{4.4.2}$$

4.4.2 Proof for uniqueness of $c^j_{kl}(x)$

We abuse the notations to denote $c^j_{kl}(x):=c^{j(1)}_{kl}(x)-c^{j(2)}_{kl}(x)$ so that $\vec{P}(x,q):=\vec{P}^{(1)}(x,q)-\vec{P}^{(2)}(x,q)=\left(\sum_{k,l=1}^n c^j_{kl}q_kq_l\right)_{1\le j\le n}$ and $u_2^f(t,x)=u_2^{(1)f}(t,x)-u_2^{(2)f}(t,x)$. Then, from (4.3.2) and (4.3.3), $u_1=u_1^f(t,x)$ and $u_2=u_2^f(t,x)$ are the solutions to the following initial boundary value problems:

$$\begin{cases} \partial_t^2 u_1(t,x)-\nabla_x\cdot(\gamma(x)\nabla_x u_1(t,x))=0, \ (t,x)\in Q \\ u_1(0,x)=\partial_t u_1(0,x)=0, \ x\in\Omega, \\ u_1(t,x)=f(t,x), \ (t,x)\in\partial Q, \end{cases} \tag{4.4.3}$$

and

$$\begin{cases} \partial_t^2 u_2 - \nabla_x \cdot (\gamma(x)\nabla_x u_2) = \nabla_x \cdot (\vec{P}(x, \nabla_x u_1(t,x))), \ (t,x) \in Q \\ u_2(0,x) = \partial_t u_2(0,x) = 0, \ x \in \Omega, \\ u_2 = 0, \ (t,x) \in \partial Q, \end{cases} \tag{4.4.4}$$

respectively. We emphasize here that u_1, $u_2 \in C^\infty([0,\infty) \times \overline{\Omega})$ are the unique solutions to (4.4.3) and (4.4.4), respectively.

From the equality of DN map in (4.3.4), we have

$$[\gamma(x)\partial_\nu u_2^f(t,x) + \nu(x) \cdot \vec{P}(x, \nabla_x u_1^f(t,x))]|_{\partial Q} = 0. \tag{4.4.5}$$

Consider the even extension of $f(t,\cdot)$ with respect to t, so that the extended $f(t,\cdot)$ is defined on $\mathbb{R} \times \overline{\Omega}$. By abusing the notation, we denote this extended $f(t,\cdot)$ by the same notation. We also define Y_s for any fixed $s \in \mathbb{R}$ by

$$Y_s f(t,.) := f(t-s,\cdot), \quad t \in \mathbb{R}. \tag{4.4.6}$$

This is a control with delay time s. Then we have $u^{Y_s f}(t,x) = u^f(t-s,x)$. By abusing the notation once again, let $u_2 = u_2(s,x;t)$ denote

$$u_2 := u_2^{f+Y_s g} - u_2^{f-Y_s g}, \ f, g \in C_0^\infty(\mathbb{R} \times \Omega), \tag{4.4.7}$$

where g is defined as the even extension of $g \in C_0^\infty(\partial Q_T)$ likewise f. Using this u_2 we can polarize $\vec{P}(x,q)$.

It is easy to see that u_2 is the solution to the following initial boundary value problem:

$$\begin{cases} \partial_s^2 u_2(s,x) - \nabla_x \cdot (\gamma(x)\nabla_x u_2(s,x)) \\ \quad = 2\sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x)\partial_k u_1^f(s,x)\partial_l u_1^g(t-s,x) \right) \\ \qquad +2\sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x)\partial_k u_1^g(t-s,x)\partial_l u_1^f(s,x) \right), \ t, s \in \mathbb{R}, \\ u_2(0,x) = \partial_s u_2(0,x) = 0, \ x \in \Omega, \\ u_2(s,x) = 0, \ t, s \in \mathbb{R}, x \in \partial\Omega. \end{cases} \tag{4.4.8}$$

Also the equality of DN map gives

$$\gamma(x)\partial_\nu u_2(s,x) + 2\sum_{j=1}^{n}\nu_j(x)\left(\sum_{k,l=1}^{n} c_{kl}^{j}(x)\partial_k u_1^f(s,x)\partial_l u_1^g(t-s,x)\right)$$
$$+ 2\sum_{j=1}^{n}\nu_j(x)\left(\sum_{k,l=1}^{n} c_{kl}^{j}(x)\partial_k u_1^g(t-s,x)\partial_l u_1^f(s,x)\right) = 0$$

Next let

$$u_2^{(-1)}(t,x) = \int_0^t u_2(s,x)ds \in C^\infty(\mathbb{R}\times\overline{\Omega}).$$

Then, $u_2^{(-1)}$ is the solution to the following initial boundary value problem:

$$\begin{cases} \partial_t^2 u_2^{(-1)}(t,x) - \nabla_x\cdot(\gamma(x)\nabla_x u_2^{(-1)}(t,x)) \\ \quad = 2\sum_{j=1}^{n}\partial_j\left(\sum_{k,l=1}^{n} c_{kl}^{j}(x)\partial_k u_1^f * \partial_l u_1^g(t,x)\right) \\ \qquad +2\sum_{j=1}^{n}\partial_j\left(\sum_{k,l=1}^{n} c_{kl}^{j}(x)\partial_k u_1^g * \partial_l u_1^f(t,x)\right), \quad (t,x)\in\mathbb{R}\times\Omega, \\ u_2^{(-1)}(0,x) = \partial_t u_2^{(-1)}(0,x) = 0, \quad x\in\Omega, \\ u_2^{(-1)}(t,x) = 0, \quad (t,x)\in\mathbb{R}\times\partial\Omega, \end{cases} \tag{4.4.9}$$

where $\partial_k u_1^f * \partial_l u_1^g(t,x)$ is defined by

$$\partial_k u_1^f * \partial_l u_1^g(t,x) := \int_0^t \partial_k u_1^f(s,x)\partial_l u_1^g(t-s,x)ds.$$

Also the corresponding Neumann data is

$$\begin{aligned} &\gamma(x)\partial_\nu u_2^{(-1)}(t,x) + 2\sum_{j=1}^{n}\nu_j(x)\left(\sum_{k,l=1}^{n} c_{kl}^{j}(x)\partial_k u_1^f * \partial_l u_1^g(t,x)\right) \\ &+ 2\sum_{j=1}^{n}\nu_j(x)\left(\sum_{k,l=1}^{n} c_{kl}^{j}(x)\partial_k u_1^g * \partial_l u_1^f(t,x)\right) = 0, \quad (t,x)\in\mathbb{R}\times\partial\Omega. \end{aligned} \tag{4.4.10}$$

Now idea is to use the Laplace transform on both side of (4.4.9), in order to convert it to an elliptic equation and then by using complex geometric optics solutions argument, we will have the required uniqueness. Let

$$\widehat{u^{(-1)}}(\tau, x) := \int_0^\infty e^{-\tau t} u^{(-1)}(t, x)dt, \ \tau > 0. \tag{4.4.11}$$

We note that this is well-defined, for justification of this fact, we refer to Lemmas 4.5.1 and 4.5.2 in appendix §4.5. Further, for each fixed $\tau > 0$, $\widehat{u_2^{(-1)}}(\tau, x)$ is solution to the following boundary value problem:

$$\begin{cases} \tau^2 \widehat{u_2^{(-1)}}(\tau, x) - \nabla_x \cdot (\gamma(x) \nabla_x \widehat{u_2^{(-1)}})(\tau, x) \\ \qquad = 2 \sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x) \partial_k \widehat{u_1^f}(\tau, x) \partial_l \widehat{u_1^g}(\tau, x) \right) \\ \qquad\quad + 2 \sum_{j=1}^n \partial_j \left(\sum_{k,l=1}^n c_{kl}^j(x) \partial_k \widehat{u_1^g}(\tau, x) \partial_l \widehat{u_1^f}(\tau, x) \right), \ x \in \Omega, \\ \widehat{u_2^{(-1)}}(\tau, x) = 0, \ x \in \partial\Omega. \end{cases} \tag{4.4.12}$$

From (4.4.10) we have

$$\begin{aligned} &\gamma(x) \partial_\nu \widehat{u_2^{(-1)}}(\tau, x) + 2 \sum_{j=1}^n \nu_j(x) \left(\sum_{k,l=1}^n c_{kl}^j(x) \partial_k \widehat{u_1^f}(\tau, x) \partial_l \widehat{u_1^g}(\tau, x) \right) \\ &+ 2 \sum_{j=1}^n \nu_j(x) \left(\sum_{k,l=1}^n c_{kl}^j(x) \partial_k \widehat{u_1^g}(\tau, x) \partial_l \widehat{u_1^f}(\tau, x) \right) = 0, \quad x \in \partial\Omega. \end{aligned} \tag{4.4.13}$$

Next take a solution $\widehat{w}(\tau, x)$ of the following equation

$$\tau^2 v - \nabla_x \cdot (\gamma(x) \nabla_x v) = 0, \quad x \in \Omega. \tag{4.4.14}$$

Multiplying the first equation of (4.4.12) by $\widehat{w}(\tau, x)$ and integrating over Ω, we have

$$\begin{aligned} &\int_\Omega \tau^2 \widehat{u_2^{(-1)}}(\tau, x) \widehat{w}(\tau, x) dx - \int_\Omega \nabla_x \cdot (\gamma(x) \nabla_x \widehat{u_2^{(-1)}})(\tau, x) \widehat{w}(\tau, x) dx \\ &\quad = 2 \sum_{j=1}^n \sum_{k,l=1}^n \int_\Omega \partial_j \left(c_{kl}^j(x) \partial_k \widehat{u_1^f}(\tau, x) \partial_l \widehat{u_1^g}(\tau, x) \right) \widehat{w}(\tau, x) dx \\ &\qquad + 2 \sum_{j=1}^n \sum_{k,l=1}^n \int_\Omega \partial_j \left(c_{kl}^j(x) \partial_k \widehat{u_1^g}(\tau, x) \partial_l \widehat{u_1^f}(\tau, x) \right) \widehat{w}(\tau, x) dx \end{aligned}$$

Then, integration by parts and using the fact that $\widehat{w}(\tau, x)$ is a solution of (4.4.14), we have

$$
\begin{aligned}
&-\int_{\partial\Omega} \gamma(x)\partial_\nu \widehat{u_2^{(-1)}}(\tau,x)\widehat{w}(\tau,x)dS_x + \int_{\partial\Omega} \gamma(x)\widehat{u_2^{(-1)}}(\tau,x)\partial_\nu\widehat{w}(\tau,x)dS_x \\
&= -2\sum_{j=1}^{n}\sum_{k,l=1}^{n}\int_{\Omega}\left(c_{kl}^{j}(x)\partial_k\widehat{u_1^f}(\tau,x)\partial_l\widehat{u_1^g}(\tau,x)\right)\partial_j\widehat{w}(\tau,x)dx \\
&\quad -2\sum_{j=1}^{n}\sum_{k,l=1}^{n}\int_{\Omega}\left(c_{kl}^{j}(x)\partial_k\widehat{u_1^g}(\tau,x)\partial_l\widehat{u_1^f}(\tau,x)\right)\partial_j\widehat{w}(\tau,x)dx \\
&\quad +2\sum_{j=1}^{n}\sum_{k,l=1}^{n}\int_{\partial\Omega}\nu_j(x)\left(c_{kl}^{j}(x)\partial_k\widehat{u_1^f}(\tau,x)\partial_l\widehat{u_1^g}(\tau,x)\right)\widehat{w}(\tau,x)dS_x \\
&\quad +2\sum_{j=1}^{n}\sum_{k,l=1}^{n}\int_{\partial\Omega}\nu_j(x)\left(c_{kl}^{j}(x)\partial_k\widehat{u_1^g}(\tau,x)\partial_l\widehat{u_1^f}(\tau,x)\right)\widehat{w}(\tau,x)dS_x
\end{aligned}
$$

Using (4.4.13), this implies

$$
\sum_{j=1}^{n}\sum_{k,l=1}^{n}\int_{\Omega} c_{kl}^{j}(x)\left\{\partial_k\widehat{u_1^f}(\tau,x)\partial_l\widehat{u_1^g}(\tau,x)+\partial_k\widehat{u_1^g}(\tau,x)\partial_l\widehat{u_1^f}(\tau,x)\right\}\partial_j\widehat{w}(\tau,x)dx = 0, \tag{4.4.15}
$$

where $\widehat{u_1^f}$ and $\widehat{u_1^g}$ are solution to the following equation

$$
\tau^2 u - \nabla_x\cdot(\gamma(x)\nabla_x u) = 0 \ \text{ in } \ \Omega \tag{4.4.16}
$$

with the Dirichlet data on $\partial\Omega$ equal to f and g respectively.

Now for a fixed $\tau > 0$, we want to use complex geometric optics (CGO) solutions for $\widehat{u_1^f}(\cdot,\tau)$ and $\widehat{u_1^g}(\cdot,\tau)$ to derive $c_{kl}^j(x) = 0$, for all $1 \le j,k,l \le n$ and $x \in \Omega$. For that we need to take some special $f = \chi(t)\widetilde{f}$, $g = \chi(t)\widetilde{g}$ with $\chi(t) \in C_0^\infty((0,T))$ such that $\mu(\tau) := \int_0^\infty e^{-\tau t}\chi(t)\,dt > 0$ and $\widetilde{f},\ \widetilde{g} \in C^\infty(\partial\Omega)$.

Let u be the solution to

$$
\begin{cases}
\partial_t^2 u - \nabla\cdot(\gamma\nabla u) = 0 \text{ in } Q_T, \\
u = \chi(t)\widetilde{f}(x), \ \text{ on } \partial Q_T, \\
u = 0, \ \text{ at } t = 0.
\end{cases} \tag{4.4.17}
$$

Consider the even extension of this u corresponding to the even extension of the above $f = \chi(t)\widetilde{f}$. By abusing the notation, we denote the extended u by the same notation u. Further extend this u to the whole $\mathbb{R} \times \Omega$ and denote the extended one by the same notation u. Then $\widehat{u} = \widehat{u}(\tau, \cdot)$ is the solution to the following boundary value problem:

$$\begin{cases} \tau^2 \widehat{u} - \nabla \cdot (\gamma \nabla \widehat{u}) = 0, \text{ in } \Omega, \\ \widehat{u} = \mu(\tau)\widetilde{f}, \text{ on } \partial\Omega. \end{cases} \tag{4.4.18}$$

If we fix $\tau > 0$ and note that $\mu(\tau) > 0$, $\mu(\tau)\widetilde{f}$ can be taken arbitrarily due to the freedom of choosing $\widetilde{f}$. Hence we can just look for some special solution $\widehat{u}$ of $\tau^2\widehat{u} - \nabla \cdot (\gamma \nabla_x \widehat{u}) = 0$ in $\mathbb{R}^n$ which would be the CGO solution. One might concern about the freedom of choosing $f = \chi(t)\widetilde{f}$ in the argument given here, because the original f should be coming from that in (4.1.1). As already mentioned before on the unique solvability of (4.1.1), recall that B_M can be any fixed single $f \in C_0^\infty(\partial Q_T)$ and can consider the ϵ-expansion of the solution u to (4.1.1) with this Dirichlet data ϵf, $0 < \epsilon < \epsilon_0$ to derive (4.4.17).

To have the CGO solutions of (4.4.16) with fixed $\tau > 0$, we will make use of following theorem given by Sylvester and Uhlmann.

Theorem 4.4.1. *[72] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $s > n/2$, there exists a constant C such that if*

$$\zeta \cdot \zeta = 0, \quad \zeta \in \mathbb{C}^n$$

and $|\zeta|$ is large, then there exists $u(x, \zeta)$ solving to (4.4.16) such that

$$u(x, \zeta) := e^{\zeta \cdot x} m(x) \left(1 + R(x, \zeta)\right),$$

where $m(x) = \gamma^{-\frac{1}{2}}(x)$ and

$$\|R\|_{H^s(\Omega)} \leq \frac{C}{|\zeta|}.$$

Using the above theorem, we have expressions for $\widehat{u_1^f}$ and $\widehat{u_1^g}$ given by

$$\begin{cases} \widehat{u_1^f}(x, \zeta^1) := e^{\zeta^1 \cdot x} m(x) \left(1 + R_1(x, \zeta^1)\right), \\ \widehat{u_1^g}(x, \zeta^2) := e^{\zeta^2 \cdot x} m(x)(1 + R_2(x, \zeta^2)), \end{cases} \tag{4.4.19}$$

where $m(x) = \gamma(x)^{-\frac{1}{2}}$ and R_i satisfies the following estimate

$$\|R_i\|_{H^s(\Omega)} \leq \frac{C}{|\zeta_i|} \quad \text{for i=1,2.} \tag{4.4.20}$$

Let $n \geq 3$ and use the CGO solutions. Let $a \in \mathbb{R}^n$ and choose unit vectors $\xi, \eta \in \mathbb{R}^n$ such that

$$a \cdot \xi = a \cdot \eta = \xi \cdot \eta = 0.$$

Now choose $r, s > 0$ so that

$$r^2 = \frac{|a|^2}{4} + s^2.$$

Define $\zeta^1, \zeta^2 \in \mathbb{C}^n$ by

$$\zeta^1 := r\eta + i(\frac{a}{2} + s\xi), \quad \zeta^2 = -r\eta + i(\frac{a}{2} - s\xi). \tag{4.4.21}$$

Now if we denote $\rho = \eta + i\xi$, we have

$$\zeta^i \cdot \zeta^i = 0, \quad \lim_{s\to\infty} \frac{\zeta^1}{s} = \rho, \; \lim_{s\to\infty} \frac{\zeta^2}{s} = -\rho.$$

By using the expression for the solutions $\widehat{u_1^f}$ and $\widehat{u_1^g}$ of the form given in (4.4.19) with values of ζ^i given by (4.4.21) in (4.4.15) and dividing by s^2 and taking $s \to \infty$, we get

$$\int_\Omega \sum_{j=1}^n \sum_{k,l}^n \rho_k \rho_l c_{kl}^j(x) m^2(x) \partial_j \widehat{w}(x) e^{ia\cdot x} dx = 0, \; a \in \mathbb{R}^n. \tag{4.4.22}$$

This gives us

$$\sum_{j=1}^n \left[\sum_{k,l=1}^n \rho_k \rho_l c_{kl}^j(x) m^2(x) \right] \partial_j \widehat{w}(x) = 0, \quad x \in \Omega. \tag{4.4.23}$$

To prove that $c_{kl}^j(x) = 0$ in Ω, we will follow the argument given in [31] and make use of the following lemma similar to Lemma 3.1 of [31] in our setup.

Lemma 4.4.2. *Suppose $n \geq 2$. For each fix τ, There exist solutions $v_j(\tau, x) \in H^2(\Omega)$, $1 \leq j \leq n$ of*

$$\tau^2 v - \nabla_x \cdot (\gamma(x) \nabla_x v) = 0, \; x \in \Omega$$

such that

$$\det(\frac{\partial v_j}{\partial x_i}) \neq 0, \;\; a.e.\ x \in \Omega.$$

Proof. Consider the set V defined by

$$V := \{\rho \in \mathbb{C}^n : \rho \cdot \rho = 0, \;\; |\rho| = \sqrt{2}\}. \tag{4.4.24}$$

Let $\rho^1, \rho^2, \cdots, \rho^n \in V$ be n-linearly independent vectors over $\mathbb{C}$. Now let

$$v_j(\tau, x) := m(x)e^{r\rho^j \cdot x}(1 + R_j(x)), \;\; x \in \Omega, \, r > 0, \;\; j = 1, 2, \cdots, n \tag{4.4.25}$$

as before. Then, we have

$$\det \begin{bmatrix} \nabla_x v_1 \\ \nabla_x v_2 \\ \nabla_x v_3 \\ \vdots \\ \nabla_x v_n \end{bmatrix} = r^n m^n(x) e^{r(\rho^1+\rho^2+\cdots+\rho^n)\cdot x} \det \begin{bmatrix} \rho^1(1+R_1) + O(t^{-1}) \\ \rho^2(1+R_2) + O(t^{-1}) \\ \rho^3(1+R_3) + O(t^{-1}) \\ \vdots \\ \rho^n(1+R_n) + O(t^{-1}) \end{bmatrix}$$

Now since $||R_j||_{L^\infty(\Omega)} \leq \frac{C}{r}$, we have

$$\det \begin{bmatrix} \nabla_x v_1 \\ \nabla_x v_2 \\ \nabla_x v_3 \\ \vdots \\ \nabla_x v_n \end{bmatrix} \neq 0 \text{ for sufficiently large } r.$$

□

Using Lemma 4.4.2, there exist solutions $\{\widehat{w}_j\}_{1\leq j\leq n}$ of (4.4.16) in such way that $\{\nabla_x \widehat{w}_j\}_{1\leq j\leq n}$ are linearily independent. Taking these choices of $\{\widehat{w}_j\}_{1\leq j\leq n}$ in (4.4.23), we have

$$\sum_{k,l=1}^{n} \rho_k \rho_l c^j_{kl}(x) = 0, \;\; x \in \Omega, \;\; j = 1, 2, \cdots, n. \tag{4.4.26}$$

Now since (4.4.26) holds for all $\rho \in V$, where V is defined as in (4.4.24). Now using $\rho = ze_k + \sqrt{-1}ze_l$ for $z \in \mathbb{R}$ with $|z| = 1$ and $k \neq l$, where $e_j's$ for $1 \leq j \leq n$

are standard basis for $\mathbb{R}^n$, in (4.4.26), we have

$$z^2 c^j_{kk}(x) - z^2 c^j_{ll}(x) + iz^2 c^j_{kl}(x) = 0,\ x \in \Omega,\ k \neq l.$$

From here, we will have

$$c^j_{kk}(x) = c^j_{ll}(x),\ c^j_{kl}(x) = 0 \text{ for all } j \text{ and } k \neq l. \tag{4.4.27}$$

Now using (4.4.27) in (4.4.15), we have

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\Omega} c^j_{kk}(x)\partial_k \widehat{u^f_1}(\tau,x)\partial_k \widehat{u^g_1}(\tau,x)\partial_j \widehat{w}(\tau,x)dx = 0 \tag{4.4.28}$$

holds for all $\widehat{u^f_1}, \widehat{u^g_1}$ and $\widehat{w}$ satisfying (4.4.16). Now using CGO solutions for $\widehat{u^g_1}$ and $\widehat{w}$, we have

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\rho_j\rho_k c^j_{kk}(x)\partial_k \widehat{u^f_1} = 0. \tag{4.4.29}$$

Note that (4.4.29) holds for any $f \in C^\infty_0(\mathbb{R}\times\partial\Omega)$. Hence by taking n different f and using Lemma 4.4.2, we have

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\rho_j\rho_k c^j_{kk}(x) = 0$$

Now take same $\rho \in V$ as defined before, we have

$$c^j_{kk}(x) = 0 \text{ for all } 1 \leq j,k \leq n \text{ and } x \in \Omega.$$

Hence combining this and (4.4.27), we have

$$c^j_{kl}(x) = 0 \text{ in } \Omega \text{ for all } 1 \leq j,k,l \leq n.$$

This completes the proof.

4.5 Appendix

Lemma 4.5.1. *Suppose u_1 is a solution to*

$$\begin{cases} \partial_t^2 u_1(t,x) - \nabla_x \cdot (\gamma(x)\nabla_x u_1(t,x)) = 0; \quad (t,x) \in Q := (0,\infty)\times\Omega \\ u_1(0,x) = \partial_t u_1(0,x) = 0; \quad x \in \Omega \\ u_1(t,x) = f(t,x), \quad (t,x) \in (0,\infty)\times\partial\Omega \end{cases} \tag{4.5.1}$$

where $\gamma \in C^\infty(\Omega)$ such that $\gamma(x) \geq c > 0$. Now if $f \in C_c^\infty((0,\infty)\times\partial\Omega)$ then, there exists $M > 0$ independent of t such that

$$\sup_{t\in[0,\infty)} \left(\|\partial_t^k u_1(t,.)\|_{L^2(\Omega)} + \|\partial_t^l \partial_j u_1(t,\cdot)\|_{L^2(\Omega)} + \|\partial_t^l \partial_{ij}^2 u_1(t,\cdot)\|_{L^2(\Omega)} \right) \leq M \tag{4.5.2}$$

holds for any $k \geq 1$, $l \geq 0$ and for all $1 \leq i,j \leq n$.

Proof. Multiplying first equation in (4.5.1), by $2\partial_t u_1$ and integrating over $(0,t)\times\Omega$, we get

$$\begin{aligned} \int_\Omega \left(|\partial_t u_1(t,x)|^2 + |\nabla_x u_1(t,x)|^2 \right) dx &= 2\int_0^t \int_{\partial\Omega} \gamma(x)\partial_\nu u_1(s,x)\partial_t u_1(s,x) dS_x ds \\ &\leq 2C \int_0^t \|\partial_\nu u_1(s,.)\|_{L^2(\partial\Omega)} \|\partial_t u_1(s,.)\|_{L^2(\partial\Omega)} ds \end{aligned} \tag{4.5.3}$$

Now since $f \in C_c^\infty((0,\infty)\times\partial\Omega)$, therefore we have,

$$\int_\Omega \left(|\partial_t u_1(t,x)|^2 + |\nabla_x u_1(t,x)|^2 \right) dx \leq 2C \int_0^\infty \|\partial_\nu u_1(s,.)\|_{L^2(\partial\Omega)} \|\partial_t u_1(s,.)\|_{L^2(\partial\Omega)} ds$$

holds for all $t \in [0,\infty)$. Finally we have

$$\sup_{t\in[0,\infty)} \left(\int_\Omega \left(|\partial_t u_1(t,x)|^2 + |\nabla_x u_1(t,x)|^2 \right) dx \right) \leq \widetilde{C}$$

where $\widetilde{C} > 0$ is independent of t. Now define $\widetilde{u}_1$ by

$$\widetilde{u}_1(t,x) = \partial_t u_1(t,x)$$

then, $\widetilde{u}_1$ satisfies the following IBVP

$$\begin{cases} \partial_t^2\widetilde{u}_1(t,x) - \nabla_x \cdot (\gamma(x)\nabla_x\widetilde{u}_1(t,x)) = 0; \quad (t,x) \in Q := (0,\infty)\times\Omega \\ \widetilde{u}_1(0,x) = \partial_t\widetilde{u}_1(0,x) = 0; \quad x \in \Omega \\ \widetilde{u}_1(t,x) = \widetilde{f}(t,x) : \partial_t f(t,x), \quad (t,x) \in (0,\infty)\times\partial\Omega. \end{cases} \tag{4.5.4}$$

Again we have $\widetilde{f} \in C_c^\infty\left((0,\infty)\times\partial\Omega\right)$, therefore, by using the same argument as above, we get

$$\sup_{t\in[0,\infty)}\left(\int_\Omega \left(|\partial_t\widetilde{u}_1(t,x)|^2 + |\nabla_x\widetilde{u}_1(t,x)|^2\right)dx\right) \leq \widetilde{\widetilde{C}} \text{ for some } \widetilde{\widetilde{C}}.$$

Now by repeating the same process and using (4.5.1), we have the estimate (4.5.2) holds which is what we wanted to prove. □

Lemma 4.5.2. *Let $F \in C^\infty\left([0,\infty)\times\Omega\right)$ such that $\sup_{t\in[0,\infty)}\|\partial_t^k F(t,.)\|_{L^2(\Omega)} \leq M$ for $k \geq 0$ and $M > 0$ is independent of t and u_2 is a solution to the following IBVP*

$$\begin{cases} \partial_t^2 u_2(t,x) - \nabla_x \cdot (\gamma(x)\nabla_x u_2(t,x)) = F(t,x); \quad (t,x) \in ([0,\infty)\times\Omega) \\ u_2(0,x) = \partial_t u_2(0,x) = 0; \quad x \in \Omega \\ u_2(t,x) = 0; \quad (t,x) \in [0,\infty)\times\partial\Omega. \end{cases} \tag{4.5.5}$$

Then,

$$||\partial_t^k u_2(t,\cdot)||_{H^2(\Omega)} = O(|t|), \quad \text{for } k \geq 1 \tag{4.5.6}$$

holds. Hence, the Laplace transform of the solution u_2 to (4.5.5) defined by

$$\widehat{u}_2(\tau,x) := \int_0^\infty e^{-\tau t}u_2(t,x)dt; \quad \tau > 0 \tag{4.5.7}$$

is well-defined for all $x \in \bar{\Omega}$.

Proof. This can be proved by using very standard idea used for deriving the energy estimates for wave equation. Multiplying (4.5.5) by $2\partial_t u_2$ and integrating over

$[0,t]$, we get

$$\begin{aligned}\int\limits_{\Omega}\left(|\partial_t u_2(t,x)|^2+\gamma(x)|\nabla_x u_2(t,x)|^2\right)dx &= 2\int\limits_0^t\int\limits_{\Omega}F(s,x)\partial_t u_2(s,x)dxds\\ &\leq 2\int\limits_0^t\|F(s,\cdot)\|_{L^2(\Omega)}\|\partial_t u(s,\cdot)\|_{L^2(\Omega)}\end{aligned}\tag{4.5.8}$$

where in the last step, we have used the Cauchy-Schwartz inequality. Now denote

$$\begin{aligned}&E_{u_2}(t):=\int\limits_{\Omega}\left(|\partial_t u_2(t,x)|^2+\gamma(x)|\nabla_x u_2(t,x)|^2\right)dx \text{ and}\\ &\phi(t)=\max_{0\leq t'\leq t}E_{u_2}(t)\end{aligned}$$

Using these notations and properties of F, in (4.5.8), we get

$$E_{u_2}(t)\leq 2C\int\limits_0^t E_{u_2}(s)^{\frac{1}{2}}ds$$

where $C>0$ is independent of t. This, gives

$$\phi(t)\leq 2Ct^2.$$

Thus, finally we have

$$E_{u_2}(t)=O(|t|^2).\tag{4.5.9}$$

Now define $\widetilde{u}_2$ by

$$\widetilde{u}_2(t,x)=\partial_t u_2(t,x)$$

then $\widetilde{u}_2$ satisfies the following IBVP

$$\begin{cases}\partial_t^2\widetilde{u}_2(t,x)-\nabla_x\cdot(\gamma(x)\nabla_x\widetilde{u}_2(t,x))=\partial_t F(t,x); \quad (t,x)\in([0,\infty)\times\Omega)\\ \widetilde{u}_2(0,x)=\partial_t\widetilde{u}_2(0,x)=0; \quad x\in\Omega\\ \widetilde{u}_2(t,x)=0; \quad (t,x)\in[0,\infty)\times\partial\Omega.\end{cases}$$

Then, using the same argument as used in deriving the estimate (4.5.6), we will have the required estimate. Next we will show that the Laplace transform defined by (4.5.7) makes sense for all $x \in \bar{\Omega}$, for this enough to prove that the integral

$$\int_{\Omega} \Big| \int_{0}^{\infty} e^{-\tau t} u_2(t,x)dt \Big| dx$$

is finite. To prove this, we proceed as follows

$$\begin{aligned}\int_{\Omega} \Big| \int_{0}^{\infty} e^{-\tau t} u_2(t,x)dt \Big| dx &\leq \int_{\Omega} \int_{0}^{\infty} e^{-\tau t} |u_2(t,x)| dt dx = \int_{0}^{\infty} \int_{\Omega} e^{-\tau t} |u_2(t,x)| dx dt \\ &\leq |\Omega| \int_{0}^{\infty} e^{-\tau t} \|u_2(t,\cdot)\|_{L^2(\Omega)} dt\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Now using (4.5.6), we have

$$\int_{\Omega} \Big| \int_{0}^{\infty} e^{-\tau t} u_2(t,x)dt \Big| dx < \infty, \text{ for all } x \in \Omega.$$

Therefore, we have

$$\widehat{u}_2(\tau,x) = \int_{0}^{\infty} e^{-\tau t} u_2(t,x)dt$$

is well-defined for all $x \in \Omega$. $\square$

Chapter 5

Conclusion

In this chapter, we will give a brief summary of the problems considered in this thesis.

In chapter 2, we studied an inverse problem for a wave equation with source and receiver at distinct points. Rakesh in [52] considered the inverse problem of unique determination of the density coefficient appearing in the wave equation when source and receiver are at the same point whereas our resut deals with the case when the source and receiver are at distinct points. In order to prove our uniqueness result, we derive an inetgral identity using the solution to an adjoint problem and then using the prolate spheroidal coordinates, we end up with a Gronwall type inequality, which gives the required uniqueness results.

In chapter 3, we studied the inverse problem of unique determination of time-dependent vector and scalar potentials appearing in the wave equation in space dimensions $n \geq 3$ from partial boundary data. Salazar in [64] proved the unique determination of these potentials from the knowledge of Dirichlet to Neumann data measured on the full boundary and for all time. Recently Kian in [33] addressed the unique determination of the damping coefficient and the scalar potential (both time-dependent) from partial boundary data. Our work extends the result of Salazar, in the sense that the boundary measurements in our work are assumed to be known only on a part of the boundary, and also we work in a finite-time domain. It is an extension of the aforementioned work of Kian as well, since we consider the full up to first order time-dependent perturbations, whereas Kian assumes that only a damping coefficient and zeroth order time-dependent potential are present in the PDE. In our uniqueness result, one has to deal with the inversion of light ray transform with partial data whereas in Kian's case one needs to invert the Fourier transform of bounded functions with compact support.

In chapter 4, we addressed the inverse problem of determining the linear part and quadratic non-linear part with respect to the spacial gradient of an unknown appearing in a non-linear wave equation of divergence form from many boundary measurements, that is the hyperbolic Dirichlet to Neumann map. We proved the uniqueness result for determining both the linear part and quadratic non-linear part from the hyperbolic Dirichlet to Neumann map when the spacial dimension is larger than or equal to 3. We used the idea of using the control with delay in time at boundary and the special polarization for the difference of quadratic non-linear part and finally using the Laplace Transform with respect to t, we related the problem of identifying the quadratic part to that for a non-linear elliptic equation which is similar the problem considered in [31].

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