INVERSE INITIAL BOUNDARY VALUE PROBLEM FOR A NON-LINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

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Abstract. In this article we are concerned with an inverse initial boundary value problem for a non-linear wave equation in space dimension $n \geq 2$. In particular we consider the so called interior determination problem. This non-linear wave equation has a trivial solution, i.e. zero solution. By linearizing this equation at the trivial solution, we have the usual linear wave equation with a time independent potential. For any small solution $u = u(t,x)$ of our non-linear wave equation which is the perturbation of linear wave equation with time-independent potential perturbed by a divergence with respect to (t,x) of a vector whose components are quadratics with respect to $\nabla_{t,x}u(t,x)$. By ignoring the terms with smallness $O(|\nabla_{t,x}u(t,x)|^3)$, we will show that we can uniquely determine the potential and the coefficients of these quadratics by many boundary measurements at the boundary of the spacial domain over finite time interval and the final overdetermination at $t = T$. In other words, our measurement is given by the so-called the input-output map (see (1.5)).

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. For $T > 0$, let $Q_T := (0,T) \times \Omega$ and denote its lateral boundary by $\partial Q_T := (0,T) \times \partial\Omega$.

Now consider the following initial boundary value problem (IBVP):

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) + a(x)u(t,x) = \nabla_{t,x} \cdot \overrightarrow{C}(t,x,\nabla_{t,x}u(t,x)), \ (t,x) \in Q_T, \\ u(0,x) = \epsilon\phi(x), \ \partial_t u(0,x) = \epsilon\psi(x), \quad x \in \Omega, \\ u(t,x) = \epsilon f(t,x), \quad (t,x) \in \partial Q_T, \end{cases} \tag{1.1}$$

where $\nabla_{t,x} := (\partial_0, \partial_1, \cdots, \partial_n)$, $\partial_0 = \partial_t$ and $\partial_j = \partial_{x_j}$ for $x = (x_1, \cdots, x_n)$. Here $\overrightarrow{C}(t,x,q)$ is given by

$$\overrightarrow{C}(t,x,q) := \overrightarrow{P}(t,x,q) + \overrightarrow{R}(t,x,q) \tag{1.2}$$

with $q := (q_0, \widetilde{q}) = (q_0, q_1, \cdots, q_n) \in \mathbb{C}^{1+n}$, we have

$$\overrightarrow{P}(t,x,q) := |q|^2 \vec{b}(t,x), \tag{1.3}$$

where $|q|^2$ mostly denotes $\sum_{j=0}^n q_j^2$, but

$$|q|^2 = \sum_{j=0}^n q_j \overline{q_j}$$

for estimates. This is because there will be cases when $q \in \mathbb{C}^{1+n}$. The meaning of this $|q|^2$ will be clear from the context.

Denote by $B^\infty(\partial Q_T)$ the Fréchet space obtained by completing $C^\infty(\partial Q_T) := \{f\big|_{\partial Q_T} : f \in C^\infty(\mathbb{R} \times \partial\Omega)\}$ with respect to the metric $d_\partial(\cdot,\cdot)$ induced by the countable norms

$$\sup_{t \in [0,T], 0 \leq k \leq \ell} \|\partial_t^k g(t,\cdot)\|_{C^\ell(\partial\Omega)}, \ \ell = 0, 1, \cdots.$$

Further, let $m \geq [n/2] + 3$ where $[n/2]$ is the largest integer not exceeding $n/2$ and $B_M := \{(\phi,\psi,f) \in C^\infty(\overline{\Omega})^2 \times B^\infty(\partial Q_T) : d_\partial(0,f) + d(0,\psi) + d(0,\phi) \leq M\}$ with the metric $d(\cdot,\cdot)$ in the Fréchet space $C^\infty(\overline{\Omega})$ induced by the countable number of norms $\|\cdot\|_{C^\ell(\overline{\Omega})}$, $\ell = 0, 1, \cdots$ and a fixed constant $M > 0$.

We assume that $a \in C^\infty(\overline{\Omega})$, $\vec{b}(t,x) := (b_0, b_1, b_2, \cdots, b_n) \in C^\infty_{(0)}\left((0,T]; C^\infty(\overline{\Omega})\right)$ is such that $\vec{b}(T,x) = 0$ in Ω and $\overrightarrow{R}(t,x,q) \in C^\infty_{(0)}((0,T]; C^\infty(\overline{\Omega} \times H))$ with $H := \{q = q_R + iq_I \in \mathbb{C}^{1+n} : q_R, q_I \in \mathbb{R}^{1+n}, |q| \leq h\}$ for some constant $h > 0$ satisfying the following estimate: there exists a constant $C > 0$ such that

$$|\partial_q^\alpha \nabla_{t,x}^\beta \overrightarrow{R}(t,x,q)| \leq C|q|^{3-|\alpha|} \text{ for multi-indices } \alpha \text{ with } |\alpha| \leq 3 \text{ and } \beta, \tag{1.4}$$

where $\partial_q = (\partial_{q_R}, \partial_{q_I})$ and $C^\infty_{(0)}((0,T]; E)$ is a set of a Fréchet space E valued C^∞ function over $[0,T]$ flat at $t = 0$.

Then, there exists $\epsilon_0 = \epsilon_0(h,T,m,M) > 0$ such that (1.1) has a unique solution $u \in X_m := X_m([0,T])$ for any $(\phi, \psi, f) \in B_M$ satisfying the compatibility condition of order $m-1$ and $0 < \epsilon < \epsilon_0$, where $X_m(I) := \cap_{j=0}^m C^j(I; W^{m-j,2}(\Omega))$ for a time interval I with the $L^2(\Omega)$ based Sobolev space $W^{m-j,2}(\Omega)$ of order $m - j$. We refer this by the *unique solvability* of (1.1). In Section 2, we will provide the proof of this together with the *ϵ-expansion* of solution to (1.1). Because of the presence of time-dependent coefficients in (1.1) and the space dimension $n \geq 2$, the proof of unique solvability and ϵ-expansion of solution to (1.1) does not follow from [6, 31] in a straight forward manner. The ϵ-expansion in [31], proved for one space dimension will work only for time-independent coefficient case. Hence in section 2, by using the ideas from [6, 31] and adding the several new arguments, we will prove the unique solvability together with the ϵ-expansion for solution to (1.1).

Based on the unique solvability of (1.1), define the *input-output map* $\Lambda^T_{\overrightarrow{C},a}$ by

$$\Lambda^T_{\overrightarrow{C},a}(\epsilon\phi, \epsilon\psi, \epsilon f) = \left(\left[\partial_\nu u^{\phi,\psi,f} + (0, \nu(x)) \cdot \overrightarrow{C}(t,x,\nabla_{t,x} u^{\phi,\psi,f})\right]\Big|_{\partial Q_T}, u^{\phi,\psi,f}|_{t=T}, \partial_t u^{\phi,\psi,f}|_{t=T} \right), \tag{1.5}$$

where $0 < \epsilon < \epsilon_0$, $u^{\phi,\psi,f}(t,x)$ for $(\phi,\psi,f) \in B_M$, is the solution to (1.1) and $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$ directed into the exterior of Ω.

The inverse problem we are going to consider is the uniqueness of identifying the potential $a = a(x)$, and the quadratic nonlinearity $\overrightarrow{P} = \overrightarrow{P}(t,x,q)$ of $\overrightarrow{C}$ from the input-output map $\Lambda^T_{\overrightarrow{C},a}$. More precisely it is to show the following:

$$\Lambda^T_{\overrightarrow{C}^{(1)},a_1} = \Lambda^T_{\overrightarrow{C}^{(2)},a_2} \Longrightarrow (a_1,\ \overrightarrow{P}^{(1)}) = (a_2,\ \overrightarrow{P}^{(2)}) \text{ with } \overrightarrow{P}^{(i)} = |q|^2 \vec{b}^{(i)},\ i = 1,2$$

where $\Lambda^T_{\overrightarrow{C}^{(i)},a_i}$, $i = 1,2$ are the input-output maps given by (1.5) for $\left(a, \overrightarrow{C}\right) = \left(a_i, \overrightarrow{C}^{(i)}\right)$, $i = 1,2$ and $(a_i, \overrightarrow{P}^{(i)})$, $i = 1,2$ are $(a, \overrightarrow{P})$ associated to $\left(a_i, \overrightarrow{C}^{(i)}\right)$ $i = 1,2$.

The non-linear wave equation of the form (1.1) with the assumptions $a(x) = 0$, $\overrightarrow{C}(t,x,q) = \overrightarrow{C}(x,\widetilde{q})$ arises as a model equation of a vibrating string with elasticity coefficient depending on strain and a model equation describing the anti-plane deformation of a uniformly thin piezoelectric material for the one spacial dimension ([32]), and as a model equation for non-linear Love waves for the two spacial dimension ([35]).

There are several earlier works on inverse problems for some non-linear wave equations in one space dimension. For example, Denisov [7] considered identifying a nonlinear potential depending on the space variable and the derivative of the solution with respect to the space variable and Grasselli [9] considered identifying the speed of a wave equation arising from the nonlinear vibration of elastic string with the nonlinearity given as the speed depending on the integration of the modulus of displacement over the string. Lorenzi-Paparoni in [27] considered identifying a nonlinear potential given as some first order derivative of a function depending on the solution of the equation arising from the theory of absorption.

Under the similar set up as our inverse problem except the space dimension and with the assumptions $a(x) = 0$, $\overrightarrow{C}(t,x,q) = \overrightarrow{C}(x,\widetilde{q})$ for the equation, authors in [31, 32] identified the time-independent coefficients by giving a reconstruction formula in one space dimension which also gives uniqueness. We are going to prove the uniqueness for our inverse problem when the space dimension $n \geq 2$ and coefficient of non-linearity $\vec{b}$ is time-dependent. Authors in [11] studied the inverse problems of determining the potential from the source to solution map for a non-linear wave equation in Riemannian geometry. Recently [12] considered the inverse problems for determining the coefficients of non-linearities appearing in a semilinear wave equation on Lorentzian manifold. We refer to [3, 14, 17, 23, 24, 25, 26, 44] for more works on inverse problems related to non-linear hyperbolic equations.

The physical meaning of our inverse problem can be considered as a problem to identify especially the higher order tensors in non-linear elasticity for its simplified model equation. In a smaller scale the higher order tensors become important. There is a recent uniqueness result proved in [5, 43] for some nonlinear isotropic elastic equation.

We also point out some related works for elliptic and parabolic equations. For elliptic equations, Kang-Nakamura in [16] studied the uniqueness for determining the non-linearity in conductivity equation. Our result can be viewed as a generalization of [16] for non-linear wave equation with constant conductivity and a potential. There are other works related to non-linear elliptic PDE, we refer to [1, 10, 13, 15, 22, 30, 39, 40, 41, 42]. Also, for nonlinear parabolic equations, we refer to [2, 4, 13, 21].

Now we state the main result.

Theorem 1.1. *For $i = 1, 2$, let*

$$\overrightarrow{C}^{(i)}(t,x,q) = q + \overrightarrow{P}^{(i)}(t,x,q) + \overrightarrow{R}^{(i)}(t,x,q) \text{ and } \overrightarrow{P}^{(i)}(t,x,q) = |q|^2\vec{b}^{(i)}(t,x)$$

with $\overrightarrow{P}^{(i)}$ and $\overrightarrow{R}^{(i)}$, $i = 1, 2$ satisfying the same conditions as for $\overrightarrow{P}$ and $\overrightarrow{R}$. Further let $u^{(i)} \in X_m$, $i = 1, 2$ be the solutions to the following IBVP:

$$\begin{cases} \partial_t^2 u^{(i)}(t,x) - \Delta u^{(i)}(t,x) + a_i(x)u^{(i)}(t,x) = \nabla_{t,x}\cdot \overrightarrow{C}^{(i)}(t,x,\nabla_{t,x}u^{(i)}(t,x)),\ (t,x)\in Q_T,\\ u^{(i)}(0,x) = \epsilon\phi(x),\ \partial_t u^{(i)}(0,x) = \epsilon\psi(x), \quad x\in\Omega,\\ u^{(i)}(t,x) = \epsilon f(t,x), \quad (t,x)\in\partial Q_T \end{cases} \tag{1.6}$$

with $(\phi,\psi,f) \in B_M$ satisfying the compatibility condition of order $m-1$ and any $0<\epsilon<\epsilon_0$. Let $\Lambda^T_{\overrightarrow{C}^{(1)},a_1}$ and $\Lambda^T_{\overrightarrow{C}^{(2)},a_2}$ be the input-output maps as defined in (1.5) corresponding to $u^{(1)}$ and $u^{(2)}$, respectively. Assume that T is larger than the diameter of Ω and

$$\Lambda^T_{\overrightarrow{C}^{(1)},a_1}(\epsilon\phi,\epsilon\psi,\epsilon f) = \Lambda^T_{\overrightarrow{C}^{(2)},a_2}(\epsilon\phi,\epsilon\psi,\epsilon f),\ (\phi,\psi,f)\in B_M,\ 0<\epsilon<\epsilon_0. \tag{1.7}$$

Then we have

$$a_1(x) = a_2(x),\ x\in\Omega \text{ and } \vec{b}^{(1)}(t,x) = \vec{b}^{(2)}(t,x),\ (t,x)\in Q_T.$$

Remark 1.2. Note that in Theorem 1.1 the coefficient of non-linearity $\vec{b}$ is time-dependent, hence our measurement is the input-output map which consists of the usual hyperbolic Dirichlet to Neumann map and the information of solution measured at the initial and final time. Due to this extra information of solutions, we can derive the integral identity given by (3.11) in Section 3, for any solution w to $\partial_t^2 w - \Delta w + a(x)w = 0$ and hence we can prove Lemma 3.1, which immediately gives the uniqueness of $\vec{b}(t,x)$. Recently there has been several works in the literature related to inverse problems for non-linear hyperbolic equations, but most of them showed the determination of time-independent coefficients of non-linearity from the boundary measurements. The determination of the time-dependent coefficients appearing in a non-linear hyperbolic partial differential equations from the boundary measurements has not been well studied in the prior works and this is the aim of the present work. To the best of our knowledge this is the first result which deals with the determination of the time-dependent coefficient of a quadratic non-linearity appearing in a non-linear hyperbolic partial differential equations from boundary measurements of the solution.

The proof of Theorem 1.1 will be done in two steps. Namely we first show that from

$$\Lambda^T_{\overrightarrow{C}^{(1)},a_1}(\epsilon\phi,\epsilon\psi,\epsilon f) = \Lambda^T_{\overrightarrow{C}^{(2)},a_2}(\epsilon\phi,\epsilon\psi,\epsilon f),\ \ (\phi,\psi,f)\in B_M$$

we can have $a = a_1 = a_2$ and a can be reconstructed from one of $\Lambda^T_{a_i}$ which is the linearization of input-output map $\Lambda_{\vec{C}^{(i)},a_i}$ defined in (1.5) and given by

$$\Lambda^T_{a_i}(\phi,\psi,f) = \left(\nu(x)\cdot\nabla_x u_1^{(i)\phi,\psi,f}\Big|_{\partial Q_T}, u_1^{(i)}\Big|_{t=T}, \partial_t u_1^{(i)}\Big|_{t=T}\right),\ (\phi,\psi,f)\in B_M$$

where $u_1^{(i)\phi,\psi,f} \in X_m$ is the solution to the initial boundary value problem:

$$\begin{cases} \partial_t^2 v(t,x) - \Delta v(t,x) + a_i(x)v(t,x) = 0,\ (t,x)\in Q_T,\\ v(0,x) = \phi(x),\ \partial_t v(0,x) = \psi(x),\ x\in\Omega,\\ v(t,x) = f(t,x),\ (t,x)\in\partial Q_T. \end{cases} \tag{1.8}$$

Using the reconstruction for $a(x)$ and varying the initial and Dirichlet data for (1.8), we can know the solution $v(t,x)$ of (1.8) in Q_T. Now using the uniqueness for a_i in Ω, we have the corresponding solutions to the linearized problem (1.8) are equal. This will help us to derive an integral identity involving $\vec{b}$. Finally using (1.7) and the special solutions for the linearized equation (1.8), we prove the unique identification of $\vec{b}(t,x)$. We remark here that since our arguments for identifying $\vec{b}$ require the reconstruction for the lower order coefficient therefore we have assumed that a is time-independent.

The rest of this paper is organized as follows. In Section 2, we will introduce the ϵ-expansion of the IBVP and analyze the input-output map in ϵ-expansion. As a consequence, we will show that the input-output map determines the input-output map Λ^T_a associated with the equation $\partial_t^2 v - \Delta_x v + av = 0$ in Q_T. This immediately implies the uniqueness of identifying a. Section 3 is devoted to proving the uniqueness of identifying $\vec{b}(t,x)$, the coefficient of quadratic non-linearity with respect to $\nabla_{t,x}u$.

2. ϵ-expansion of the solution to the IBVP and input-output map in ϵ-expansion

In this section, we provide the ϵ-expansion, which will also provide the proof of the unique solvability of (1.1). For the case when space dimension is one and the coefficients are time independent, there is a brief proof of the verification of the ϵ-expansion given in [31]. Although in principle the idea of proof is the same as in [31], it becomes more complicated and needs to add further arguments for the case when the space dimension becomes higher and the coefficients are time-dependent. Hence we will give the full proof verifying the ϵ-expansion.

Theorem 2.1. *Let $m \geq [n/2]+3$ and $(\phi,\psi,f)\in B_M$ with a fixed constant $M>0$, then for given $T>0$, there exists $\epsilon_0 = \epsilon_0(h,m,M) > 0$ such that for any $0<\epsilon<\epsilon_0$, (1.1) has a unique solution $u\in X_m$, where h,B_M and X_m were defined in Section 1. Moreover, it admits an expansion which we call ϵ-expansion:*

$$u = \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3),\quad \epsilon\to 0, \tag{2.1}$$

where u_1 is a solution to

$$\begin{cases} \partial_t^2 u_1(t,x) - \Delta u_1(t,x) + a(x)u_1(t,x) = 0,\ (t,x)\in Q_T,\\ u_1(0,x) = \phi(x),\ \partial_t u_1(0,x) = \psi(x),\ x\in\Omega,\\ u_1(t,x) = f(t,x),\ (t,x)\in\partial Q_T, \end{cases} \tag{2.2}$$

and u_2 is a solution to

$$\begin{cases} \partial_t^2 u_2(t,x) - \Delta u_2(t,x) + a(x)u_2(t,x) = \nabla_{t,x}\cdot\left(\vec{b}(t,x)|\nabla_{t,x}u_1(t,x)|^2\right),\ (t,x)\in Q_T,\\ u_2(0,x) = \partial_t u_2(0,x) = 0,\ x\in\Omega,\\ u_2(t,x) = 0,\ (t,x)\in\partial Q_T \end{cases} \tag{2.3}$$

and $O(\epsilon^3)$ means the following:

$$w(t,x) = O(\epsilon^3) \Longleftrightarrow \|w\|_{X_m} := \sup_{0\le t\le T}\left(\sum_{k=0}^{m}\|\overset{(k)}{w}(t,.)\|_{m-k}^2\right)^{1/2} = O(\epsilon^3),$$

where $\overset{(k)}{w} := \frac{\partial^k w}{\partial t^k}$ *and* $\|\cdot\|_k$ *is the norm of the* $L^2(\Omega)$ *based Sobolev space* $W^{k,2}(\Omega)$ *of order* k*.*

Remark 2.2.

(1) For the well-posedness of initial boundary value problem (2.2) we have the following. Let $m \in \mathbb{N}$ and let $\phi \in W^{m,2}(\Omega)$, $\psi \in W^{m-1,2}(\Omega)$, $f \in \widetilde{X}_m$ satisfy the compatibility condition of order $m-1$. Then there exists a unique solution $u_1 \in X_m$ to (2.2) with the estimate

$$\|u_1\|_{X_m} \le C(\|\phi\|_{W^{m,2}(\Omega)} + \|\psi\|_{W^{m-1,2}(\Omega)} + \|f\|_{\widetilde{X}_m}),$$

where $C>0$ is a general constant and $\|f\|_{\widetilde{X}_m} := \sup_{0\le t\le T}\left(\sum_{j=0}^{m}\|\overset{(j)}{f}(t,\cdot)\|^2_{W^{m-j,2}(\partial\Omega)}\right)^{1/2}$ is the norm of f in the space $\widetilde{X}_m := \cap_{j=0}^{m} C^j([0,T];W^{m-j,2}(\partial\Omega))$. This can be proved by starting from $m=1$ given in Theorem 2.45 of [20] and argue as in the arguments given from (2.35) in Subsection 2.1 to the end of this subsection.

(2) For the well-posedness of initial boundary value problem (2.3) with a general inhomogeneous term $F = F(t,x)$ instead of $\nabla_{t,x}\cdot\overrightarrow{P}(t,x,\nabla_{t,x}u_1)$, we have the following. Let $m\in\mathbb{N}$ and let $F \in X_{m-1}$ satisfy the compatibility condition of order $m-1$. Then there exists a unique solution $u_2 \in X_m$ to (2.3) with the estimate

$$\|u_2\|_{X_m} \le C\|F\|_{X_{m-1}},$$

where $C>0$ is a general constant. This can be proved by referring [6] and [8].

(3) The compatibility condition of order $m-1$ given in (1) and (2) are that considered independently from (1.1). Nevertheless, in relation with (1.1), if we want to have the solution u of (1.1) to be in X_m, then the compatibility conditions for both (2.2) and (2.3) are of the same order $m-1$ with $m \ge [n/2]+3$. This is due to the assumption we made for $\vec{b}$ and $\overrightarrow{R}$.

Our strategy for the proof of Theorem 2.1 is as follows:

- We look for a solution $u(t,x)$ to (1.1) of the form
$$u(t,x) := \epsilon\left\{u_1(t,x) + \epsilon\left(u_2(t,x) + w(t,x)\right)\right\}, \tag{2.4}$$
where u_1, u_2 are the solutions to the initial boundary value problems (2.2) and (2.3), respectively, and derive the equation for w, which has the form
$$\partial_t^2 w - B(w(t))w = \epsilon\mathcal{F}(t,x,\nabla_{t,x}w;\epsilon)$$
(see (2.10) and (2.25)).
- For a given function $U(t)$, we prove the unique solvability of the semilinear wave equation of the form:
$$\partial_t^2 w_{sem} - B(U(t))w_{sem} = \epsilon\mathcal{F}(t,x,\nabla_{t,x}w_{sem};\epsilon)$$
with zero initial and boundary data.
- We prove that the map $T(U) = w_{sem}$ is a contraction mapping.

We first derive the equation for w. Direct computations show that $w(t,x)$ has to satisfy

$$\begin{cases} \partial_t^2 w - \Delta w + a(x)w = \epsilon^{-2}\nabla_{t,x}\cdot \overrightarrow{R}(t,x,\epsilon\nabla_{t,x}u_1+\epsilon^2\nabla_{t,x}u_2+\epsilon^2\nabla_{t,x}w) \\ \quad +2\epsilon\nabla_{t,x}\cdot\left(\nabla_{t,x}u_1\cdot\nabla_{t,x}u_2\,\vec{b}\right)+2\epsilon\nabla_{t,x}\cdot\left(\nabla_{t,x}u_1\cdot\nabla_{t,x}w\,\vec{b}\right) \\ \quad +\epsilon^2\nabla_{t,x}\cdot\left(\left(|\nabla_{t,x}u_2|^2+2\nabla_{t,x}u_2\cdot\nabla_{t,x}w+|\nabla_{t,x}w|^2\right)\vec{b}\right),\ \ (t,x)\in Q_T, \\ w(0,x)=\partial_t w(0,x)=0,\ \ x\in\Omega, \\ w(t,x)=0,\ \ (t,x)\in\partial Q_T. \end{cases} \tag{2.5}$$

By the mean value theorem, we have

$$\begin{aligned} \overrightarrow{R}(t,x,\epsilon\nabla_{t,x}u_1+\epsilon^2\nabla_{t,x}u_2+\epsilon^2\nabla_{t,x}w) &= \overrightarrow{R}(t,x,\epsilon\nabla_{t,x}u_1+\epsilon^2\nabla_{t,x}u_2) \\ &\quad + \int_0^1 \frac{d}{d\theta}\overrightarrow{R}(t,x,\epsilon\nabla_{t,x}u_1+\epsilon^2\nabla_{t,x}u_2+\theta\epsilon^2\nabla_{t,x}w)d\theta \\ &= \overrightarrow{R}(t,x,\epsilon\nabla_{t,x}u_1+\epsilon^2\nabla_{t,x}u_2)+\epsilon^3 K(t,x,\epsilon\nabla_{t,x}w;\epsilon)\nabla_{t,x}w, \end{aligned}$$

where

$$\epsilon K(t,x,\epsilon\nabla_{t,x}w;\epsilon) := \int_0^1 \nabla_q\overrightarrow{R}(t,x,\epsilon\nabla_{t,x}u_1+\epsilon^2\nabla_{t,x}u_2+\theta\epsilon^2\nabla_{t,x}w)d\theta$$

with $\nabla_q\overrightarrow{R}(x,q)=\left(\partial_{q_j}R_i\right)_{0\le i,j\le n}$ and $K=(K_{ij})$ with $K_{ij}=\partial_{q_j}R_i$.

Introduce the following notations:

$$\begin{cases} \epsilon F(t,x,\nabla_{t,x}u_1,\nabla_{t,x}u_2;\epsilon) := 2\epsilon\nabla_{t,x}\cdot((\nabla_{t,x}u_1\cdot\nabla_{t,x}u_2)\vec{b})+\epsilon^2\nabla_{t,x}\cdot(|\nabla_{t,x}u_2|^2\vec{b}) \\ \qquad\qquad +\epsilon^{-2}\nabla_{t,x}\cdot\overrightarrow{R}(t,x,\epsilon\nabla_{t,x}u_1+\epsilon^2\nabla_{t,x}u_2), \\ \epsilon\Gamma(t,x,\nabla_{t,x}w;\epsilon) := 2\epsilon(\vec{b}\otimes\nabla_{t,x}u_1)+2\epsilon^2(\vec{b}\otimes\nabla_{t,x}w)+2\epsilon^2(\vec{b}\otimes\nabla_{t,x}u_2) \\ \qquad\qquad +\epsilon K(t,x,\epsilon\nabla_{t,x}w;\epsilon)+\epsilon^2\mathcal{K}(t,x,\epsilon\nabla_{t,x}w;\epsilon)\nabla_{t,x}w, \\ \epsilon\overrightarrow{G}(t,x,\nabla_{t,x}w;\epsilon)\cdot\nabla_{t,x}z := 2\epsilon(\nabla_{t,x}^2u_1)\cdot(\vec{b}\otimes\nabla_{t,x}z)+2\epsilon(\nabla_{t,x}\cdot\vec{b})(\nabla_{t,x}u_1\cdot\nabla_{t,x}z) \\ \qquad\qquad +\epsilon^2(\nabla_{t,x}\cdot\vec{b})(\nabla_{t,x}w\cdot\nabla_{t,x}z)+2\epsilon^2(\nabla_{t,x}^2u_2)\cdot(\vec{b}\otimes\nabla_{t,x}z) \\ \qquad\qquad +\epsilon^2(\nabla_{t,x}\cdot\vec{b})(\nabla_{t,x}u_2\cdot\nabla_{t,x}z)+\epsilon(\nabla_{t,x}\cdot K)\cdot\nabla_{t,x}z, \\ B(w)z := \Delta z - a(x)z+\epsilon\Gamma(t,x,\nabla_{t,x}w;\epsilon)\cdot\nabla_{t,x}^2 z, \end{cases} \tag{2.6}$$

where "$\cdot$"=real inner product, $\otimes$ = tensor product, $\nabla_{t,x}^2 w$ = Hessian of w, the j-th component of $(\nabla_{t,x}\cdot K)$ is $\sum_{i=0}^n \partial_i K_{ij}$ and the (i,j)-component of $\mathcal{K}\nabla_{t,x}w$ is $\sum_{l=0}^n \partial_{q_j}K_{il}\partial_l w$. Notice here that ∂_i in $\sum_{i=0}^n \partial_i K_{ij}$ is just acting to the x_i variable of $K_{ij}(t,x,q;\epsilon)$. Also $\Gamma(t,x,\nabla_{t,x};\epsilon)\cdot\nabla_{t,x}^2 w$ is the inner product of the two matrices $\Gamma(t,x,\nabla_{t,x};\epsilon)$ and $\nabla_{t,x}^2 w$. Then (2.5) can be written in the following form:

$$\begin{cases} \partial_t^2 w - B(w)w - \epsilon\overrightarrow{G}(t,x,\epsilon\nabla_{t,x}w;\epsilon)\cdot\nabla_{t,x}w = \epsilon F(t,x,\nabla_{t,x}u_1,\nabla_{t,x}u_2;\epsilon)\ \text{ in } Q_T, \\ w(0,x)=\partial_t w(0,x)=0 \text{ in } \Omega \text{ and } w(t,x)=0 \text{ on } \partial Q_T. \end{cases} \tag{2.7}$$

Now to complete the proof of Theorem 2.1 it is enough to prove the following.

Theorem 2.3. *Let $m\ge[n/2]+3$ and $(\phi,\psi,f)\in B_M$ satisfying the compatibility condition of order $m-1$ for (1.1). Then, for given $T>0$ there exists $\epsilon_0=\epsilon_0(h,m,M)>0$ and $w=w(t,x;\epsilon)\in X_m$ for $0<\epsilon<\epsilon_0$ such that each $w=w(\cdot,\cdot;\epsilon)$ is the unique solution to the initial boundary value problem (2.7) with the estimate*

$$\|w\|_{X_m}=O(\epsilon)\ \text{ as } \epsilon\to 0. \tag{2.8}$$

In order to prove this, let $Z(M)$ with $M > 0$ be the set of U satisfying

$$\begin{cases} U \in X_m = X_m([0,T]), \\ U(0,x) = \partial_t U(0,x) = 0,\ x \in \Omega, \\ U(t,x) = 0,\ (t,x) \in \partial Q_T, \\ \|U\|_{X_m} \le M. \end{cases} \tag{2.9}$$

Then based on the aforementioned strategy of proof of Theorem 2.1, we consider for $U \in Z(M)$ the following semilinear wave equation corresponding to the equation (2.7)

$$\begin{cases} \partial_t^2 w_{sem} - B(U) w_{sem} = \epsilon \mathcal{F}(t,x,\nabla_{t,x} w_{sem};\epsilon),\ (t,x) \in Q_T, \\ w_{sem}(0,x) = \partial_t w_{sem}(0,x) = 0,\ x \in \Omega, \\ w_{sem}(t,x) = 0,\ (t,x) \in \partial Q_T, \end{cases} \tag{2.10}$$

where

$$\mathcal{F}(t,x,h;\epsilon) := F(t,x,\nabla_{t,x}u_1,\nabla_{t,x}u_2;\epsilon) + \overrightarrow{G}(t,x,h;\epsilon)\cdot h \tag{2.11}$$

2.1. **Unique solvabilty for the semilinear wave equation** (2.10)**.**

In this subsection, we give a proof of the following unique solvability for the semilinear wave equation (2.10).

Proposition 2.4. *Let $m \ge [n/2]+3$ be an integer. Then, there exists $\epsilon_1 > 0$ such that the initial boundary value problem* (2.10) *has a unique solution $w_{sem} \in Z(M)$ for each $0 < \epsilon < \epsilon_1$ with the estimate*

$$\|w_{sem}\|_{X_m} \le \epsilon C e^{KT},\ 0 < \epsilon < \epsilon_1, \tag{2.12}$$

where C and K are positive constants depending on M and ϵ_1.

It is convenient to introduce the following notations for the proof of Proposition 2.4. We first introduce the notation $\widetilde{B}(U)$. From (2.6) we have

$$\begin{aligned} B(U)w &= \Delta w - aw + \epsilon\Gamma(t,x,\nabla_{t,x}U;\epsilon)\cdot\nabla_{t,x}^2 w \\ &= \Delta w - aw + \epsilon\Gamma_{00}(t,x,\nabla_{t,x}U;\epsilon)\partial_t^2 w \\ &\quad + \epsilon\sum_{j=1}^{n}\big(\Gamma_{0j}(t,x,\nabla_{t,x}U;\epsilon) + \Gamma_{j0}(t,x,\nabla_{t,x}U;\epsilon)\big)\partial_{tx_j}^2 w \\ &\quad + \epsilon\sum_{1\le i\le n,\ 1\le j\le n}\Gamma_{ij}(t,x,\nabla_{t,x}U;\epsilon)\partial_{x_i x_j}^2 w \\ &=: \epsilon\Gamma_{00}(t,x,\nabla_{t,x}U;\epsilon)\partial_t^2 w + \widetilde{B}(U)w, \end{aligned}$$

where Γ_{ij} stands for (i,j) component of the matrix Γ and $\partial_{tx_j} := \partial_t\partial_{x_j} = \partial_0\partial_j$. Note that the indices $0j$, $j0$ of Γ_{0j}, Γ_{j0} correspond to $\partial_{tx_j} = \partial_0\partial_j = \partial_j\partial_0$. We further introduce the notations $A_U(t)$, L and some other notations. Namely, denote by

$$A_U(t) := \Big(1 - \epsilon\Gamma_{00}(t,x,\nabla_{t,x}U,\epsilon)\Big)^{-1}\widetilde{B}(U(t))w \tag{2.13}$$

then

$$Lw := \Big(1 - \epsilon\Gamma_{00}(t,x,\nabla_{t,x}U,\epsilon)\Big)^{-1}\Big(\partial_t^2 w - B(U(t))w\Big) = \partial_t^2 w - A_U(t)w. \tag{2.14}$$

Also let $\|\cdot\|_m$ be the norm of the space $W^{m,2}(\Omega)$ and let $W_0^{1,2}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_0}$ with the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω. Further we write $\partial_t u = \dot{u}$ and $\partial_t^m u = \overset{(m)}{u}$.

We first prove several lemmas which will lead us to give the proof of Proposition 2.4.

Lemma 2.5. *Let U satisfy (2.9) and restrict ε to vary in $[0,\varepsilon_0]$ for a fixed small $\varepsilon_0>0$. Then $A_U(t)$ has the following properties.*

(1) There is a constant $\nu>0$ such that

$$\|v\|_{k+1}\le \nu(\|v\|_{k-1}+\|A_U(t)v\|_{k-1}),\quad k=0,\cdots,m-2, \tag{2.15}$$

for any $v\in W_0^{1,2}(\Omega)\cap W^{k+1,2}(\Omega)$ and $t\in[0,T]$.

(2) The coercivity holds for A_U. That is there are positive constants χ, λ such that

$$-\Big\langle A_U(t)v,v\Big\rangle+\chi\|v\|_0^2\ge\lambda\|v\|_1^2,\ \ t\in[0,T],\ \text{real valued } v\in W_0^{1,2}(\Omega) \tag{2.16}$$

with the continuous extension of $L^2(\Omega)$ inner product giving the pairing $\Big\langle\cdot,\cdot\Big\rangle$ between $W_0^{1,2}(\Omega)$ and its dual space $W^{-1,2}(\Omega)$.

(3) There is a continuous function $\sigma:[0,\infty)\times[0,\infty)\to[0,\infty)$ such that for every $M>0$ and every U, $\overline{U}\in W^{1,2}(\Omega)$ with $[U(t)]_1$, $[\overline{U}(t)]_1\le M$, we have

$$\|(A_U-A_{\overline{U}})w\|_0\le\epsilon\sigma(M,\epsilon)\cdot\|\nabla_{t,x}(U(t)-\overline{U}(t))\|_0,\ \ t\in[0,T] \tag{2.17}$$

for $w\in Z(M)$.

Proof. First of all we note that for (1), (2), the terms in $A_U(t)$ which have ∂_t do not contribute because v is independent of t. Then by using the standard elliptic regularity argument, we can have the properties (1) and (2). As for the property (3), we divide $(A_U-A_{\overline{U}})w$ into three parts. That is by using the definitions of $A_U(t)$, we start estimating $\|(A_U-A_{\overline{U}})w\|_0$ as follows.

$$\begin{aligned}\|(A_U-A_{\overline{U}})w\|_0\ &=\left\|\left[\Big(1-\epsilon\Gamma_{00}(\nabla_{t,x}U)\Big)^{-1}\widetilde{B}(U)-\Big(1-\epsilon\Gamma_{00}(\nabla_{t,x}\overline{U})\Big)^{-1}\widetilde{B}(\overline{U})\right]w\right\|_0\\ &\le H_1+H_2+H_3,\end{aligned} \tag{2.18}$$

where we have suppressed the variables t,x,ϵ in $\Gamma_{00}(t,x,\nabla_{t,x}U,\epsilon)$ and $H_j,\ j=1,2,3$ are defined as

$$\begin{aligned}&H_1:=\|(1-\epsilon\Gamma_{00}(\nabla_{t,x}U))^{-1}(1-\epsilon\Gamma_{00}(\nabla_{t,x}\overline{U}))^{-1}\big(\widetilde{B}(U))-\widetilde{B}(\overline{U})\big)w\|_0,\\ &H_2:=\epsilon\|(1-\epsilon\Gamma_{00}(\nabla_{t,x}U))^{-1}(1-\epsilon\Gamma_{00}(\nabla_{t,x}\overline{U}))^{-1}\Gamma_{00}(\nabla_{t,x}U)\big(\widetilde{B}(\overline{U})-\widetilde{B}(U)\big)w\|_0,\\ &H_3:=\epsilon\|(1-\epsilon\Gamma_{00}(\nabla_{t,x}U))^{-1}(1-\epsilon\Gamma_{00}(\nabla_{t,x}\overline{U}))^{-1}\big(\Gamma_{00}(\nabla_{t,x}U)-\Gamma_{00}(\nabla_{t,x}\overline{U})\big)\widetilde{B}(U)w\|_0.\end{aligned} \tag{2.19}$$

In order to estimate $H_j,\ j=1,2,3$ we introduce the following notation. That is for any matrix $Q=(Q_{ij})$, we define a matrix $Q^\natural=(Q^\natural_{ij})$ with $Q^\natural_{ij}$ defined as

$$Q^\natural_{ij}=\begin{cases}0, & i=0,j=0,\\ Q_{ij}, & \text{otherwise.}\end{cases} \tag{2.20}$$

We will only give how to estimate H_1 because H_2,H_3 can be estimated similarly. By the definition of $\widetilde{B}(U)$ we have

$$\begin{aligned}&\|\big(\widetilde{B}(U)-\widetilde{B}(\overline{U})w\|_0=2\epsilon\|(\overrightarrow{b}\otimes\nabla_{t,x}U)^\natural\cdot\nabla^2_{t,x}w-(\overrightarrow{b}\otimes\nabla_{t,x}\overline{U})^\natural\cdot\nabla^2_{t,x}w\|_0\\ &\quad+\epsilon\|\big(K^\natural(\epsilon\nabla_{t,x}U)-\big(K^\natural(\epsilon\nabla_{t,x}\overline{U})\big)\nabla^2_{t,x}w\|_0+\epsilon^2\|\big((\mathcal{K}(\nabla_{t,x}U)\nabla_{t,x}U)^\natural-(\mathcal{K}(\nabla_{t,x}\overline{U})\nabla_{t,x}U))^\natural\big)\nabla^2_{t,x}w\|_0\\ &\quad=:I_1+I_2+I_3,\end{aligned} \tag{2.21}$$

where

$$I_1:=2\epsilon\|(\overrightarrow{b}\otimes\nabla_{t,x}U)^\natural\cdot\nabla^2_{t,x}w-(\overrightarrow{b}\otimes\nabla_{t,x}\overline{U})^\natural\cdot\nabla^2_{t,x}w\|_0\le 2C_M\epsilon(\|\partial_t w\|_1+\|w\|_2)\|\overrightarrow{b}\|_{L^\infty(\Omega)}\|\nabla_{t,x}(U-\overline{U})\|_0 \tag{2.22}$$

and due to (1.4) we have

$$I_2 := \epsilon\|\big(K^\natural(\epsilon\nabla_{t,x}U) - K^\natural(\epsilon\nabla_{t,x}\overline{U})\big)\nabla^2_{t,x}w\|_0$$
$$= \epsilon\left(\int_\Omega \Big|\int_0^1 \{\nabla_q\overrightarrow{R}(\epsilon^2\theta\nabla_{t,x}U)^\natural - \nabla_q\overrightarrow{R}(\epsilon^2\theta\nabla_{t,x}\overline{U})^\natural\}\,d\theta\,\nabla^2_{t,x}w\Big|^2\,dx\right)^{1/2} \le C'_M\epsilon^3\|\nabla_{t,x}(U-\overline{U})\|_0^2, \tag{2.23}$$

for some constant $C'_M > 0$ depending only on M. And the estimate for I_3 can be obtained similar to that of I_2. We have

$$I_3 := \epsilon^2\|\big((\mathcal{K}(\nabla_{t,x}U)\nabla_{t,x}U)^\natural - (\mathcal{K}(\nabla_{t,x}\overline{U})\nabla_{t,x}U)U)^\natural\big)w\|_0 \le C'_M\epsilon^3\|\nabla_{t,x}(U-\overline{U})\|_0$$

for some constant $C'_M > 0$ depending only on M.

Now we recall the following estimate similar to the one given in Theorem 7.2 of [29] as a lemma.

Lemma 2.6. *Let $m \ge [n/2]+3$ and $\kappa > 0$ be the Sobolev embedding $W^{[n/2]+1,2}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ constant. For a given C^{m-1} function $f(t,x;z)$ on $\mathcal{Q} := \{(t,x,z) \in [0,T]\times\Omega\times\mathbf{C} : |z| \le \kappa M\}$, we have*

$$\|f(t,\cdot;z)\|_{m-1} \le C_{m-1}M_{m-1}\Big\{1 + \Big(1 + \|z(t)\|_{m-2}^{m-2}\Big)\|z(t)\|_{m-1}\Big\},\ \ t\in[0,T] \tag{2.24}$$

for each integer $m \ge [n/2]+3$, where

$$M_{m-1} := \max_{|\beta|\le m-1}\sup_{\mathcal{Q}}\left|\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial z}\right)^\beta f(t,x;z)\right|$$

and a general constant C_{m-1} depending on $m-1$.

Proof. The inequality given in Theorem 7.2 of [29] is for the case $\Omega = \mathbb{R}^n$. Nevertheless its argument of proof can be carry over to have (2.24) by noticing the following fact due to the existence of the extension operator $\mathcal{E} : W^{s,2}(\Omega) \to W^{s,2}(\mathbb{R}^n)$ for $s \ge 0$ coming from the C^∞ smoothness of $\partial\Omega$. $W^{s,2}(\Omega) = H^s(\Omega)$ for $s \ge 0$ with equivalence of norms of these spaces, where $H^s(\Omega) := \{\phi|_\Omega : \phi \in H^s(\mathbb{R}^n)\}$ with the norm $\|\varphi\|_{H^s(\Omega)} := \min\{\|\phi\|_{H^s(\mathbb{R}^n)} : \phi|_\Omega = \varphi,\ \phi\in H^s(\mathbb{R}^n)\}$ and $H^s(\mathbb{R}^n) := \{\phi \in L^2(\mathbb{R}^n) : \|\phi\|_{H^s(\mathbb{R}^n)} := \big(\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\widehat{\phi}(\xi)|^2d\xi\big)^{1/2} < \infty\}$ with the Fourier transform $\widehat{\phi}(\xi)$ of ϕ (see page 77 of [28]). We emphasize here that the norm of $\|\varphi\|_{H^s(\Omega)}$ is given as the minimum of $\|\phi\|_{H^s(\mathbb{R}^n)}$. □

By Lemma 2.6, we have $\|\big(1-\epsilon\Gamma_{00}(\nabla_{t,x}U)\big)^{-1}\|_{L^\infty(\Omega)} \le C''_M$ with some constant $C''_M > 0$ depending only on M. Summing up these estimate we have

$$H_1 \le \epsilon\sigma_1(M,\epsilon)\|\nabla_{t,x}(U-\overline{U})\|_0,$$

where $\sigma_1(M,\epsilon)$ is defined likewise $\sigma(M,\epsilon)$. This finishes the proof of Lemma 2.5.

□

The following Lemma follows from an estimate similar to (2.24).

Lemma 2.7. *Let*

$$\widetilde{\mathcal{F}}(t,x,\nabla_{t,x}w) := \Big(1-\epsilon\Gamma_{00}(t,x,\nabla_{t,x}U,\epsilon)\Big)^{-1}\mathcal{F}\left(t,x,\nabla_{t,x}w\right). \tag{2.25}$$

Assume that $w \in Z_M$. If $m \ge [n/2]+3$, then we have

$$[\widetilde{\mathcal{F}}(\cdot,t,\nabla_{t,x}w;\epsilon)]_{m-1} \le C(1+[w(t)]_m^{m-1}),\ \ t\in[0,T],$$

where

$$[w(t)]_m^2 := \sum_{j=0}^m \|\partial_t^j w(t)\|_{m-j}^2 \tag{2.26}$$

and $C > 0$ is a general constant depending only on M.

Proof. By an argument similar to deriving (2.24), we have

$$[\widetilde{\mathcal{F}}(\cdot, t, z; \epsilon)]_{m-1} \leq C_{m-1} M_{m-1} \left\{ 1 + \left(1 + [z(t)]_{m-2}^{m-2} \right) [z(t)]_{m-1} \right\}$$

with constants C_{m-1}, M_{m-1} as in (2.24). This is because the space X_r with non-negative integer r has the property

$$z \in X_r \Longrightarrow \partial_{t,x}^{\alpha} z \in X_{r-|\alpha|}$$

for any multi-index α such that $|\alpha| \leq r$. Observe that

$$\begin{aligned} 1 + \left(1 + [z(t)]_{m-2}^{m-2} \right) [z(t)]_{m-1} &\leq 1 + \left(1 + [z(t)]_{m-1} \right)^{m-2} \left(1 + [z(t)]_{m-1} \right) \\ &= 1 + \left(1 + [z(t)]_{m-1} \right)^{m-1} \\ &\leq C \left(1 + [z(t)]_{m-1}^{m-1} \right) \end{aligned}$$

with a general constant $C > 0$. Then by taking $z = \nabla_{t,x} w$, we obtain the desired estimate. □

Lemma 2.8. *For $S \in X_{m-1}$ consider the following initial boundary value problem*

$$\begin{cases} L[v] = S \ \text{in } Q_T, \\ v(0, x) = 0, \ \partial_t v(0, x) = 0, \quad x \in \Omega, \\ v(t, x) = 0, \ (t, x) \in \partial Q_T. \end{cases} \tag{2.27}$$

If $(0, 0, S)$ satisfies the compatibility condition of order $m - 1$, there exists a unique solution $v \in X_m$ to (2.27) *with the energy estimate*

$$[v(t)]_m^2 \leq C_m \int_0^T [S(t)]_{m-1}^2 \, dt, \ t \in [0, T], \tag{2.28}$$

where $C_m > 0$ is a general constant depending on m.

Proof. By using Lemma 2.5 and handling the terms of L with mixed derivatives $\partial_t \partial_{x_j}$, $1 \leq j \leq n$ by integration by parts using the boundary condition to derive an energy estimate, it follows from the standard argument that there exists a unique solution to (2.27) which satisfies the energy estimate (2.28) (see [6], [8] and [45]). Since we will have a similar situation to estimate the solution w of (2.10), the details about how to handle the mixed derivatives $\partial_t \partial_{x_j}$, $1 \leq j \leq n$ can be seen in the proof of Proposition 2.4. □

Proof of Proposition 2.4. First we prove the existence of a solution. We simply write (2.10) as

$$\begin{cases} L[w] = \epsilon \widetilde{\mathcal{F}}(t, x, \nabla_{t,x} w), \ (t, x) \in Q_T, \\ w(0, x) = \partial_t w(0, x) = 0, \quad x \in \Omega, \\ w(t, x) = 0, \ (t, x) \in \partial Q_T. \end{cases} \tag{2.29}$$

In order to solve (2.29), we define a series of functions $\{w_j\}$ by

$$\begin{aligned} L[w_1] &= \epsilon \widetilde{\mathcal{F}}(t, x, 0), \\ L[w_2] &= \epsilon \widetilde{\mathcal{F}} \left(t, x, \nabla_{t,x} w_1 \right), \\ &\ \ \vdots \\ L[w_j] &= \epsilon \widetilde{\mathcal{F}} \left(t, x, \nabla_{t,x} w_{j-1} \right), \ \ j = 3, 4, \cdots. \end{aligned}$$

We first prove that $w_j \in X_m$ for each j is bounded for any small enough $\epsilon > 0$. By (2.28) and Lemma 2.7, if $\sup_{t\in[0,t]}[w_{j-1}(t)]_m \le M$, then we have

$$\begin{aligned}[w_j(t)]_m^2 &\le \epsilon^2 C \int_0^T [\widetilde{\mathcal{F}}\,(t,x,\nabla_{t,x}w_{j-1})]_{m-1}^2\,dt \\ &\le \epsilon^2 C \int_0^T \left(1+[w_{j-1}(t)]_m^{m-1}\right)^2\,dt,\ \ t\in[0,T]\end{aligned}$$

with some general constant $C>0$ which may differ by lines and may depend on m. By (2.28) and

$$\widetilde{\mathcal{F}}(t,x,0) = \Big(1-\epsilon\Gamma_{00}(t,x,\nabla_{t,x}U;\epsilon)\Big)^{-1} F(x,\nabla_{t,x}u_1,\nabla_{t,x}u_2;\epsilon),$$

we have $\|w_1\|_{X_m} = \sup_{t\in[0,T]}[w_1(t)]_m \le M$ if we take ϵ small enough. Then, we further take ϵ small enough if necessary, so that it satisfies

$$\epsilon \le \min\left\{\frac{1}{\sqrt{CT}}\frac{M}{1+M^{m-1}},\ \frac{1}{2}\right\}.$$

Then it is easy to see by induction on $j\ge 2$ that

$$\|w_1\|_{X_m} \le M,\ \ j\ge 2.$$

Next we prove that $\{w_j(t)\}_{j=1,2,\cdots}$ is a Cauchy sequence. Notice that

$$L[w_{j+1}-w_j] = \epsilon\Big\{\widetilde{\mathcal{F}}\,(t,x,\nabla_{t,x}w_j) - \widetilde{\mathcal{F}}(t,x,\nabla_{t,x}w_{j-1})\Big\},$$

and $\{w_j\}$ is bounded. Then by (2.28) and applying Lemma 2.7 to

$$\begin{aligned}&\widetilde{\mathcal{F}}(t,x,\nabla_{t,x}w_j) - \widetilde{\mathcal{F}}(t,x,\nabla_{t,x}w_{j-1}) = \\ &\qquad \left\{\int_0^1 \nabla_q\widetilde{\mathcal{F}}(t,x,\nabla_{t,x}w_{j-1}+\theta\nabla_{t,x}(w_j-w_{j-1})\,d\theta\right\}\cdot\nabla_{t,x}(w_j-w_{j-1}),\end{aligned}$$

we have

$$\Big[w_{j+1}(t)-w_j(t)\Big]_m^2 \le \epsilon^2 C\int_0^t \Big[w_j(s)-w_{j-1}(s)\Big]_m^2\,ds,\ \ t\in[0,T].$$

By the choice of ϵ, this immediately implies that $\{w_j(t)\}$ is a Cauchy sequence with respect to the norm $\sup_{t\in[0,T]}[\,\cdot\,]_m$. If we denote the limit of this Cauchy sequence by $w(t)$, then the standard regularity argument gives us that $w\in X_m$ and it is a solution to (2.29).

Next we prove the estimate (2.12). Differentiating (2.29), $m-1$ times with respect to t yields

$$\overset{(m+1)}{w}(t) - A_U(t)\overset{(m-1)}{w}(t) = \sum_{k=1}^{m-1}\binom{m-1}{k}\overset{(k)}{A_U}(t)\overset{(m-1-k)}{w}(t) + \epsilon\partial_t^{m-1}\widetilde{\mathcal{F}}. \tag{2.30}$$

By taking the $\Big\langle\cdot,\cdot\Big\rangle$ product of this identity with $2\overset{(m)}{w}(t)$ and integrating by parts, we have the identity

$$\begin{aligned}\Big\|\overset{(m)}{w}(t)\Big\|_0^2 \ &-\Big\langle A_U(t)\overset{(m-1)}{w}(t),\overset{(m-1)}{w}(t)\Big\rangle = -\int_0^t\Big\langle \dot{A}_U(\tau)\overset{(m-1)}{w}(\tau),\overset{(m-1)}{w}(\tau)\Big\rangle\,d\tau \\ &+\int_0^t \mathbf{A}(U(\tau);\overset{(m-1)}{w}(\tau),\overset{(m)}{w}(\tau))\,d\tau \\ &+2\epsilon\int_0^t\left\langle \partial_t^{m-1}\widetilde{\mathcal{F}} + \textstyle\sum_{k=1}^{m-1}\dbinom{m-1}{k}\overset{(k)}{A_U}(\tau)\overset{(m-1-k)}{w}(\tau),\overset{(m)}{w}(\tau)\right\rangle\,d\tau,\ \ t\in[0,T].\end{aligned} \tag{2.31}$$

Here $\mathbf{A}(U(\tau);V(\tau),W(\tau))$ is defined by

$$\mathbf{A}(U(\tau);V(\tau),W(\tau)) := \Big\langle A_U(\tau)V(\tau),W(\tau)\Big\rangle - \Big\langle A_U(\tau)W(\tau),V(\tau)\Big\rangle.$$

and we have used the following identity obtained by integration by parts

$$\begin{aligned} 2\int_0^t \Big\langle A_U(\tau)W(\tau),\dot W(\tau)\Big\rangle\,d\tau &= \Big\langle A_U(\tau)W(t),W(t)\Big\rangle \\ &\quad -\int_0^t \Big\langle \dot A_U(\tau))W(\tau),W(\tau)\Big\rangle\,d\tau + \int_0^t \mathbf{A}(U(\tau);W(\tau),\dot W(\tau))\,d\tau. \end{aligned} \tag{2.32}$$

Now we show the inequality

$$\Big\|\overset{(m)}{w}(t)\Big\|_0^2 + \Big\|\overset{(m-1)}{w}(t)\Big\|_1^2 \le \epsilon^2 C + K_\epsilon \int_0^t \sum_{k=0}^m \Big\|\overset{(k)}{w}(\tau)\Big\|_{m-k}^2\,d\tau \tag{2.33}$$

for any $t\in[0,T]$ with some general constant $C>0$ and a constant $K_\epsilon>0$ bounded with respect to ϵ. To prove this we give the estimates for

(i) $\int_0^t \mathbf{A}(U(\tau);W(\tau),\dot W(\tau))\,d\tau$ with $W(\tau)=\overset{(m-1)}{w}(\tau)$,

(ii) a quadratic term $|\nabla_{t,x}w(t)|^2$ contained in $\mathcal{F}$.

We first deal with (i). Write $A_U(t)$ in the form

$$A_U(t)\cdot = \widehat A_U(t) + \overrightarrow{\ell}\cdot\nabla_x$$

with

$$\widehat A_U(t) := (\overrightarrow{M}\cdot\nabla_x)\partial_t\,\cdot + \nabla_x\cdot(N\nabla_x\cdot),$$

where $\overrightarrow{M}=\overrightarrow{M}(t,x,\nabla_{t,x}U;\epsilon)$ and $\overrightarrow{\ell}=\overrightarrow{\ell}(t,x,\nabla_{t,x}U;\epsilon)$ are real vectors and $N=N(t,x,\nabla_{t,x}U;\epsilon)$ is a positive matrix. Then by integrating by parts, we can have the estimate

$$\Big|\int_0^t \mathbf{A}(U(\tau);W(\tau),\dot W(\tau))\,d\tau\Big| \le C\left\{\left(\|\dot W(t)\|_0^2+\|W(t)\|_1^2\right) + \int_0^t \left(\|\dot W(\tau)\|_0^2+\|W(\tau)\|_1^2\right)d\tau\right\} \tag{2.34}$$

with a general constant $C>0$. In fact by defining $\widehat{\mathbf{A}}(U(t);W(t),\dot W(t))$ similarly as $\mathbf{A}(U(t);W(t),\dot W(t))$, we have

$$\int_0^t \widehat{\mathbf{A}}(U(\tau);W(\tau),\dot W(\tau)) = J_1+J_2$$

with

$$\begin{aligned} J_1 &:= \int_0^t\int_\Omega \Big\{\nabla_x\cdot(N\nabla_x W(\tau))\dot W(\tau) - \nabla_x\cdot(N\nabla_x\dot W(\tau))W(\tau)\Big\}\,dx\,d\tau = 0,\\ J_2 &:= \int_0^t\int_\Omega \Big\{(\overrightarrow{M}\cdot\nabla_x)\dot W(\tau))\dot W(\tau) - (\overrightarrow{M}\cdot\nabla_x)\ddot W(\tau))W(\tau)\Big\}\,dx\,d\tau\\ &= \int_\Omega \Big\{(\nabla_x\cdot\overrightarrow{M})W(t)\dot W(t) + \dot W(t)(\overrightarrow{M}\cdot\nabla_x)W(t)\Big\}\,dx\\ &\quad -\int_0^t \Big\{(\nabla_x\cdot\overrightarrow{M})W(\tau)\dot W(\tau) + (\nabla_x\cdot\partial_\tau\overrightarrow{M})W(\tau)\dot W(\tau)\Big\}\,dx\,d\tau. \end{aligned}$$

Further it is easy to see that $\big|\int_0^t \mathbf{A}(U(\tau);W(\tau),\dot W(\tau))\,d\tau\big|$ coming from $\overrightarrow{\ell}\cdot\nabla_x$ can be absorbed into the second term of the right hand side of (2.34). Hence taking these into account we can have (2.34).

Next we deal with (ii). Let $\epsilon > 0$ be small enough such that $\|\partial_t w(t)\|_{m-1} < 1$ and $\|w(t)\|_m < 1$ for any $t \in [0,T]$. Then from the Sobolev embedding theorem that the quadratic term $|\nabla_{t,x} w(t)|^2$ contained in $\mathcal{F}$ is estimated as follows.

$$
\begin{aligned}
\int_\Omega \left|\nabla_{t,x} w(\tau)\right|^2 \left|\overset{(m)}{w}(\tau)\right| dx &\le \sup_\Omega |\nabla_{t,x} w(\tau)| \cdot \left\|\overset{(m)}{w}(\tau)\right\|_0 \cdot \left\|\nabla_{t,x} w(\tau)\right\|_0 \\
&\le C\left\|\nabla_{t,x} w(\tau)\right\|_{m-1} \cdot \left\|\overset{(m)}{w}(\tau)\right\|_0 \cdot \left\|\nabla_{t,x} w(\tau)\right\|_0 \\
&\le C\left\{\left\|\overset{(1)}{w}(\tau)\right\|_{m-1} + \left\|w(\tau)\right\|_m\right\} \cdot \left\|\overset{(m)}{w}(\tau)\right\|_0 \cdot \left\{\left\|\overset{(1)}{w}(\tau)\right\|_0 + \left\|\nabla_x w(\tau)\right\|_0\right\} \\
&\le C\left(\left\|w(\tau)\right\|_m^2 + \left\|\overset{(1)}{w}(\tau)\right\|_{m-1}^2 + \left\|\overset{(m)}{w}(\tau)\right\|_0^2\right),
\end{aligned}
$$

for any $\tau \in [0,T]$. Consequently, using identity (2.31), it follows from Lemma 2.5 and a straightforward computation (see, e.g., [6, Theorem 3.1 pp. 274-277]) that we have (2.33) for sufficiently small $\epsilon > 0$.

To finish the proof we want to derive the estimate

$$
\sum_{k=0}^m \left\|\overset{(k)}{w}(t)\right\|_{m-k}^2 \le \epsilon^2 C + K_\epsilon \int_0^t \sum_{k=0}^m \left\|\overset{(k)}{w}(\tau)\right\|_{m-k}^2 d\tau,\ t \in [0,T] \tag{2.35}
$$

with a general constant $C > 0$ and a constant $K_\epsilon > 0$ bounded with respect to ϵ. Once we have this estimate, Gronwall's inequality allows us to prove estimate (2.12), which implies that solutions are unique. In order to see (2.35), we prove by induction on $\ell = 0, 1, \cdots, m-1$ the following estimate

$$
\sum_{k=\ell}^m \left\|\overset{(k)}{w}(t)\right\|_{m-k}^2 \le \epsilon^2 C + K_\epsilon \int_0^t \sum_{k=0}^m \left\|\overset{(k)}{w}(\tau)\right\|_{m-k}^2 d\tau,\ t \in [0,T] \tag{2.36}
$$

with another general constant $C > 0$ and another constant K_ϵ. By (2.33), we have already proven (2.36) for $\ell = m-1$. Assume (2.36) holds for some $1 \le \ell \le m-2$. Then we want to show that (2.36) holds for $\ell - 1$. We first have from (2.30), the following identity

$$
\begin{aligned}
-A_U(t)\overset{(\ell-1)}{w}(t) = &-\overset{(\ell+1)}{w}(t) + \\
&\sum_{i=1}^{\ell-1} \int_0^t \binom{\ell-1}{i} \left(\overset{(i)}{A_U}(\tau)\overset{(\ell-i)}{w}(\tau) + \overset{(i+1)}{A_U}(\tau)\overset{(\ell-i-1)}{w}(\tau)\right) d\tau + \epsilon\partial_t^{l-1}\widetilde{\mathcal{F}}(t),\ t \in [0,T].
\end{aligned} \tag{2.37}
$$

Next by using the coercivity of A_U given in Lemma 2.5, we have the following regularity estimate

$$
\|z\|_{1+r} \lesssim \|z\|_1 + \|g\|_{-1+r},\ r = 0, 1, \cdots \tag{2.38}
$$

for any solution $z \in W_0^{1,2}(\Omega) \cap W^{1+r,2}(\Omega)$ satisfying $-A_U(t)z = g \in W^{-1+r,2}(\Omega)$ in Ω, where the notation $\lesssim$ stands for $\le$ modulo multiplication by a positive general constant (see Chapter 20, (114) in [45]). By using (2.38) with $r = m - \ell$ and

$$
\overset{(\alpha)}{w}(t) = \int_0^t \frac{(t-\tau)^{s-1}}{(s-1)!}\overset{(\alpha+s)}{w}(\tau)\, d\tau,\ \alpha, s \in \mathbb{Z},\ \alpha \ge 0,\ s \ge 1,
$$

we have from (2.37) the following estimate

$$\begin{cases} \|\overset{(\ell-1)}{w}(t)\|^2_{m-(\ell-1)} \lesssim \\ \quad \|\overset{(\ell+1)}{w}\|^2_{m-(\ell+1)} + \|\int\limits_0^t \frac{(t-\tau)^{m-\ell-1}}{(m-\ell-1)!}\overset{(m-1)}{w}(\tau)\,d\tau\|^2_1 + \|\big((\overrightarrow{M}\cdot\nabla_x)\partial_t + \overrightarrow{\ell}\cdot\nabla_x\big)\overset{(\ell-1)}{w}(t)\|^2_{m-(\ell+1)} \\ \quad + \sum_{i=1}^{\ell-1}\left\{\left(\int\limits_0^t \frac{(t-\tau)^{i-1}}{(i-1)!}\|\overset{(\ell-1)}{w}(\tau)\|_{m-(\ell-1)}\,d\tau\right)^2 + \left(\int\limits_0^t \frac{(t-\tau)^{i}}{i!}\|\overset{(\ell-1)}{w}(\tau)\|_{m-(\ell-1)}\,d\tau\right)^2\right\} + \epsilon^2 C. \end{cases} \tag{2.39}$$

Here note that

$$\|\int\limits_0^t \frac{(t-\tau)^{m-\ell-1}}{(m-\ell-1)!}\overset{(m-1)}{w}(\tau)\,d\tau\|^2_1 \leq \Big(\int\limits_0^t \frac{(t-\tau)^{m-\ell-1}}{(m-\ell-1)!}\|\overset{(m-1)}{w}(\tau)\|_1\,d\tau\Big)^2,$$

$$\|\big((\overrightarrow{M}\cdot\nabla_x)\partial_t + \overrightarrow{\ell}\cdot\nabla_x\big)\overset{(\ell-1)}{w}(t)\|^2_{m-(\ell+1)} \lesssim \|\overset{(\ell)}{w}(t)\|^2_{m-\ell} + \left(\int\limits_0^t \|\overset{(\ell)}{w}(\tau)\|_{m-\ell}\,d\tau\right)^2.$$

Then together with (2.36) for $k = \ell+1$ in its right hand side and a direct computation, we have (2.36) for $m = \ell - 1$. Thus Proposition 2.4 is proved.

2.2. **Proofs of Theorem 2.1 and 2.3.** Theorem 2.1 immediately follows from Theorem 2.3. Hence it is enough to give the proof of Theorem 2.3. Now using Proposition 2.4, we have for any small enough $\epsilon > 0$, there exists a unique solution $w \in Z(M)$ to (2.10). Thus, the map $T : Z(M) \to Z(M)$ given by $T(U) = w$ is well-defined, where w is the solution to (2.10). Now the idea is to use the fixed point argument to prove that for any $\epsilon > 0$ small enough, there exists a unique solution w to the initial boundary value problem (2.7). More precisely we will prove that $T : Z(M) \to Z(M)$ is a contraction mapping. To begin with let $T(U_i) = w_i$ for $i = 1, 2$, where w_i is the solution to semi-linear wave equation (2.10) for $U = U_i$. Let $W = w_1 - w_2$ and $V = U_1 - U_2$, then W will satisfy the following initial boundary value problem

$$\begin{cases} \partial_t^2 W(t) - A_{U_1}(t)W(t) = \{A_{U_1}(t) - A_{U_2}(t)\}\,w_2(t) + \epsilon\Big(1-\epsilon\Gamma_{00}(t,x,\nabla_{t,x}U,\epsilon)\Big)^{-1}\overrightarrow{G}(t,x,\nabla_{t,x}w_1;\epsilon)\cdot\nabla_{t,x}W(t), \\ \qquad +\epsilon\Big(1-\epsilon\Gamma_{00}(t,x,\nabla_{t,x}U,\epsilon)\Big)^{-1}\Big\{\overrightarrow{G}(t,x,\nabla_{t,x}w_1;\epsilon) - \overrightarrow{G}(t,x,\nabla_{t,x}w_2;\epsilon)\Big\}\cdot\nabla_{t,x}w_2(t),\ (t,x)\in Q_T, \\ W(0,x) = \partial_t W(0,x) = 0,\ x\in\Omega, \\ W(t,x) = 0,\ (t,x)\in\partial Q_T. \end{cases} \tag{2.40}$$

Here we have suppressed x variable if it is clear from the context. Now let $\overrightarrow{\mathcal{G}} := \Big(1-\epsilon\Gamma_{00}(t,x,\nabla_{t,x}U,\epsilon)\Big)^{-1}\overrightarrow{G}$. Multiply (2.40) by $2\partial_t W$ and integrate over $[0,t]\times\Omega$, we have

$$\begin{aligned} ||\dot{W}(t)||_0^2 - \Big\langle A_{U_1}(t)W(t), W(t)\Big\rangle &= -\int\limits_0^t \Big\langle \Big(\dot{A}_{U_1}(\tau)\Big)W(\tau), W(\tau)\Big\rangle d\tau + \int\limits_0^t \mathbf{A}(U_1(\tau); W(\tau), \dot{W}(\tau))\,d\tau \\ &+ 2\int\limits_0^t \Big\langle \Big[A_{U_1}(\tau) - A_{U_2}(\tau)\Big]w_2(\tau), \dot{W}(\tau)\Big\rangle d\tau + 2\epsilon\int\limits_0^t \Big\langle \overrightarrow{\mathcal{G}}(t,x,\nabla_{t,x}w_1;\epsilon)\cdot\nabla_{t,x}W(\tau), \dot{W}(\tau)\Big\rangle d\tau \\ &+ 2\epsilon\int\limits_0^t \Big\langle \Big\{\overrightarrow{\mathcal{G}}(t,x,\nabla_{t,x}w_1;\epsilon) - \overrightarrow{\mathcal{G}}(t,x,\nabla_{t,x}w_2;\epsilon)\Big\}\cdot\nabla_{t,x}w_2(\tau), \dot{W}(\tau)\Big\rangle d\tau, \end{aligned}$$

where $\dot{A}_{U_1}(\tau) := \partial_\tau A_{U_1}(\tau)$ and $\dot{W}(\tau) := \partial_\tau W(\tau)$.

Using the expression $\overrightarrow{G}$ given in (2.6), we have

$$\epsilon\left(\overrightarrow{G}(t,x,\nabla_{t,x}w_1;\epsilon)-\overrightarrow{G}(t,x,\nabla_{t,x}w_2;\epsilon)\right)=\epsilon\left((\nabla_{t,x}\cdot K)(t,x,\epsilon\nabla_{t,x}w_1;\epsilon)-(\nabla_{t,x}\cdot K)(t,x,\epsilon\nabla_{t,x}w_2;\epsilon)\right).$$

Hence by using (1.4), the coercivity (2.16) and estimate (2.17) given in Lemma 2.5, we have

$$\|\dot{W}(t)\|_0^2+\|W(t)\|_1^2\le\epsilon^2C\sup_{t\in[0,T]}\left\{\|\partial_tV(t)\|_0^2+\|V(t)\|_1^2\right\}+K_\epsilon\int_0^t\left(\|\dot{W}(\tau)\|_0^2+\|W(\tau)\|_1^2\right)d\tau$$

with a constant $K_\epsilon>0$ bounded with respect to ϵ and a general constant $C>0$.

Now we equip $Z(M)$ with the metric ρ defined by

$$\rho(f,g):=\max_{t\in[0,T]}\left\{||f(t)-g(t)||_1^2+||\partial_tf(t)-\partial_tg(t)||_0^2\right\}^{\frac{1}{2}}.$$

Finally using Grownwall's inequality, we have

$$\rho\left(T(U_1),T(U_2)\right)\le\epsilon Ce^{K_\epsilon T}\rho(U_1,U_2)$$

with another general constant $C>0$. This implies that T is a contraction mapping for $\epsilon>0$ small enough. Therefore, for each $\epsilon>0$ small enough, there exists $w\in Z(M)$ such that $T(w)=w$ and it will satisfies the estimate (2.8) which follows from (2.12). Hence, Theorem 2.3 is proved. ☐

2.3. **Analysis of input-output map in ϵ-expansion.**

Let Λ_a denote the input-output map corresponding to (2.2). Using the ϵ-expansion of the solution to (1.1) we show that Λ_a can be reconstructed from $\Lambda_{\overrightarrow{C},a}$. In particular, we prove the following lemma.

Lemma 2.9. *For $m\ge[n/2]+3$ and $(\phi,\psi,f)\in B_M$ satisfying the compatibility conditions of order $m-1$, we have*

$$\lim_{\epsilon\to0}\frac{1}{\epsilon}\left\|\Lambda_{\overrightarrow{C},a}(\epsilon\phi,\epsilon\psi,\epsilon f)-\epsilon\Lambda_a(\phi,\psi,f)\right\|_{\widetilde{X}_m}=0, \tag{2.41}$$

and

$$\lim_{\epsilon\to0}\frac{1}{\epsilon^2}\left\|\Lambda_{\overrightarrow{C},a}(\epsilon\phi,\epsilon\psi,\epsilon f)-\epsilon\Lambda_a(\phi,\psi,f)-\epsilon^2\left(\partial_\nu u_2+(0,\nu(x))\cdot\vec{b}(t,x)|\nabla_{t,x}u_1(t,x)|^2\right)\right\|_{\widetilde{X}_m}=0, \tag{2.42}$$

where $\widetilde{X}_m$ is the one defined in Remark 2.2, (i).

Proof. Using the ϵ-expansion of the solution u given in Theorem 2.1, we have the ϵ-expansion of $\Lambda_{\overrightarrow{C},a}(\epsilon\phi,\epsilon\psi,\epsilon f)$ is given by

$$\begin{aligned}\Lambda_{\overrightarrow{C},a}(\epsilon\phi,\epsilon\psi,\epsilon f)&=\left(\left[\partial_\nu u^{\phi,\psi,f}+(0,\nu(x))\cdot\overrightarrow{C}(t,x,\nabla_{t,x}^{\phi,\psi,f}\right]\Big|_{\partial Q_T},u^{\phi,\psi,f}|_{t=T},\partial_tu^{\phi,\psi,f}|_{t=T}\right)\\&=\epsilon\left(\partial_\nu u_1,u_1|_{t=T},\partial_tu_1|_{t=T}\right)\big|_{\partial Q_T}+\epsilon^2\left(\partial_\nu u_2+(0,\nu(x))\cdot\vec{b}(t,x)|\nabla_{t,x}u_1|^2\right)\Big|_{\partial Q_T}+O(\epsilon^3),\ (\epsilon\to0).\end{aligned}$$

Using this we can have (2.41) and (2.42). ☐

3. Proof of Theorem 1.1

To prove the theorem, we will use the ϵ-expansion of the solution $u^{(i)}$ to (1.6). Following Theorem 2.1 we have the ϵ-expansion of the solution $u^{(i)}$ to (1.6) is given by

$$u^{(i)}(t,x)=\epsilon u_1^{(i)}(t,x)+\epsilon^2u_2^{(i)}(t,x)+O(\epsilon^3). \tag{3.1}$$

By the straight forward calculations, we have

$$\begin{cases} \partial_t^2 u^{(i)} = \epsilon \partial_t^2 u_1^{(i)} + \epsilon^2 \partial_t^2 u_2^{(i)} + O(\epsilon^3), \\ a_i u^{(i)} = \epsilon a_i u_1^{(i)} + \epsilon^2 a_i u_2^{(i)} + O(\epsilon^3), \\ \nabla_{t,x} u^{(i)} = \epsilon \nabla_{t,x} u_1^{(i)} + \epsilon^2 \nabla_{t,x} u_2^{(i)} + O(\epsilon^3), \\ \Delta u^{(i)} = \epsilon \Delta u_1^{(i)} + \epsilon^2 \Delta u_2^{(i)} + O(\epsilon^3), \\ \overrightarrow{C}^{(i)}\left(t, x, \nabla_{t,x} u^{(i)}\right) = \epsilon^2 |\nabla_{t,x} u_1^{(i)}|^2 \vec{b}^{(i)} + O(\epsilon^3), \\ \nabla_{t,x} \cdot \overrightarrow{C}^{(i)}\left(t,x, \nabla_{t,x} u^{(i)}\right) = \epsilon^2 \nabla_{t,x} \cdot \left(\left| \nabla_{t,x} u_1^{(i)} \right|^2 \vec{b} \right) + O(\epsilon^3). \end{cases}$$

Substitute (3.1) into (1.6), and arrange the terms into ascending order of power of ϵ by using the above calculations. Further setting the coefficients of ϵ and ϵ^2 equal to zero. Then we have the following equations for $u_1^{(i)}$ and $u_2^{(i)}$:

$$\begin{cases} \partial_t^2 u_1^{(i)}(t,x) - \Delta u_1^{(i)}(t,x) + a_i(x) u_1^{(i)}(t,x) = 0, \quad (t,x) \in Q_T, \\ u_1^{(i)}(0,x) = \phi(x), \ \partial_t u_1^{(i)}(0,x) = \psi(x), \quad x \in \Omega, \\ u_1^{(i)}(t,x) = f(t,x), \quad (t,x) \in \partial Q_T, \end{cases} \tag{3.2}$$

$$\begin{cases} \partial_t^2 u_2^{(i)}(t,x) - \Delta u_2^{(i)}(t,x) + a_i(x) u_2^{(i)}(t,x) = \nabla_{t,x} \cdot \left(\left| \nabla_{t,x} u_1^{(i)}(t,x) \right|^2 \vec{b}(t,x) \right), \ (t,x) \in Q_T, \\ u_2^{(i)}(0,x) = \partial_t u_2^{(i)}(0,x) = 0, \ x \in \Omega, \\ u_2^{(i)}(t,x) = 0, \ (t,x) \in \partial Q_T. \end{cases} \tag{3.3}$$

3.1. **Proof of the uniqueness for a.**

By knowing $\Lambda^T_{\overrightarrow{C}^{(i)},a_i}(\epsilon\phi, \epsilon\psi, \epsilon f)$, $i = 1, 2$ for any $(\phi, \psi, f) \in B_M$, and $0 < \epsilon < \epsilon_0$, we do know $\Lambda^T_{a_i}$, $i = 1, 2$ (see Lemma 2.9) and $\Lambda_{\overrightarrow{C}^{(1)},a_1} = \Lambda_{\overrightarrow{C}^{(2)},a_2}$ gives $\Lambda_{a_1} = \Lambda_{a_2}$. Therefore, using the arguments from [33] we can reconstruct $a_i(x)$ from Λ_{a_i}, and from [34], we have $a_1 = a_2$ in Ω. We denote this common a_i, $i = 1, 2$ by a, i.e.

$$a = a_1 = a_2 \ \text{ in } \ \Omega. \tag{3.4}$$

Before closing this subsection, we give some by products of (3.4). Since the given data (ϕ, ψ, f) is the same for $u_1^{(i)}$, $i = 1, 2$, therefore we do know $u_1^{(1)} = u_1^{(2)}$ in Q_T and we denote this common solution by $u_1 = u_1^{\phi,\psi,f}$, i.e.

$$u_1 = u_1^{\phi,\psi,f} = u_1^{(1)} = u_1^{(2)} \text{ in } Q_T. \tag{3.5}$$

3.2. **Proof of the uniqueness for $\vec{b}(t,x)$.**

Now we abuse the notations to denote $\vec{b}(t,x) := \vec{b}^{(1)}(t,x) - \vec{b}^{(2)}(t,x)$ so that $\overrightarrow{P}(t,x,q) := \overrightarrow{P}^{(1)}(t,x,q) - \overrightarrow{P}^{(2)}(t,x,q) = |q|^2 \vec{b}(t,x)$. Also we denote the solutions of (3.3) by $u_2^{(i)\phi,\psi,f}$, $i = 1, 2$ with $u_1^{(i)} = u_1^{\phi,\psi,f}$, $i = 1, 2$ and define $u_2^{\phi,\psi,f}(t,x) := u_2^{(1)\phi,\psi,f}(t,x) - u_2^{(2)\phi,\psi,f}(t,x)$. Then, from (3.2) and (3.3), $u_1(t,x) := u_1^{\phi,\psi,f}(t,x) \in X_m$ and $u_2(t,x) := u_2^{\phi,\psi,f}(t,x) \in X_m$ are the unique solutions to the following initial boundary value problems:

$$\begin{cases} \partial_t^2 u_1(t,x) - \Delta u_1(t,x) + a(x) u_1(t,x) = 0, \ (t,x) \in Q_T, \\ u_1(0,x) = \phi(x), \ \partial_t u_1(0,x) = \psi(x), \ x \in \Omega, \\ u_1(t,x) = f(t,x), \ (t,x) \in \partial Q_T \end{cases} \tag{3.6}$$

and

$$\begin{cases} \partial_t^2 u_2(t,x) - \Delta u_2(t,x) + a(x)u_2(t,x) = \nabla_{t,x} \cdot \left(\left| \nabla_{t,x} u_1(t,x) \right|^2 \vec{b}(t,x) \right), \ (t,x) \in Q_T, \\ u_2(0,x) = \partial_t u_2(0,x) = 0, \ x \in \Omega, \\ u_2(t,x) = 0, \ (t,x) \in \partial Q_T, \end{cases} \tag{3.7}$$

respectively.

From (2.42) and Lemma 2.9, we have that

$$u_2|_{t=T} = \partial_t u_2|_{t=T} = \left[\partial_\nu u_2(t,x) + (0,\nu(x)) \cdot \vec{b}(t,x) |\nabla_{t,x} u_1^{\phi,\psi,f}(t,x)|^2 \right] \Big|_{\partial Q_T} = 0, \tag{3.8}$$

where $\partial_\nu u_2$ is the Neumann derivative of u_2 given by $\partial_\nu u_2 = \nu \cdot \nabla_x u_2$ and $\nu(x)$ stands for the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Now let w be any solution to the following equation

$$\partial_t^2 w(t,x) - \Delta w(t,x) + a(x)w(t,x) = 0, \ (t,x) \in Q_T. \tag{3.9}$$

Multiplying (3.7) by w and integrating over Q_T, we have

$$\int_{Q_T} \left(\partial_t^2 u_2(t,x) - \Delta u_2(t,x) + a(x)u_2(t,x) \right) w(t,x) dx dt = \int_{Q_T} \nabla_{t,x} \cdot \left(\left| \nabla_{t,x} u_1(t,x) \right|^2 \vec{b}(t,x) \right) w(t,x) dx dt.$$

Now using the integration by parts and using (3.8), we have

$$\int_{Q_T} \vec{b}(t,x) \cdot \nabla_{t,x} w(t,x) |\nabla_{t,x} u_1(t,x)|^2 dx dt = 0 \tag{3.10}$$

holds for all solutions u_1 of (3.6) and solutions w of (3.9). We remark here that w only needs to satisfy (3.9) is the advantage coming from taking the input-output map as our measurement.

Now let $u_1^{\phi_1 \pm \phi_2, \psi_1 \pm \psi_2, f_1 \pm f_2}$ be solutions to (3.6) when $\phi = \phi_1 \pm \phi_2, \psi = \psi_1 \pm \psi_2$ and $f = f_1 \pm f_2$, respectively. Use the two sets of solution $u_1^{\phi,\psi,f} = u_1^{\phi_1 \pm \phi_2, \psi_1 \pm \psi_2, f_1 \pm f_2}$ in (3.10) and subtract the two sets of equations. Then we have

$$\int_{\mathbb{R}^{1+n}} \beta_w(t,x) \nabla_{t,x} u_1^{\phi_1,\psi_1,f_1} \cdot \nabla_{t,x} u_1^{\phi_2,\psi_2,f_2}(t,x) dx dt = 0, \ (\phi_j, \psi_j, f_j) \in B_M, \ j = 1,2, \tag{3.11}$$

where $\beta_w(t,x) = \chi_{Q_T} \vec{b}(t,x) \cdot \nabla_{t,x} w(t,x)$ with the characteristic function χ_{Q_T} of Q_T. In deriving the above identity, we have used the fact that $u_1^{\phi_1 \pm \phi_2, \psi_1 \pm \psi_2, f_1 \pm f_2} = u_1^{\phi_1,\psi_1,f_1} \pm u_1^{\phi_2,\psi_2,f_2}$.

Since the principal term in (2.7) has the coefficients which contains the functions involving the solution u_1 to (2.2), therefore to make the coefficients to be real-valued, we use the real-valued semi-classical solutions for $u_1^{\phi_i,\psi_i,f_i}$, $i = 1,2$ in (3.11). Now from [18, 19], we can have the real-valued semi-classical solutions $u_1^{\phi_i,\psi_i,f_i}$, $i = 1,2$ given as

$$u_1^{\phi_1,\psi_1,f_1} = e^{-(t+x\cdot\omega)/h} \left(\varphi(x+t\omega) + hR_1(t,x) \right),$$
$$u_1^{\phi_2,\psi_2,f_2} = e^{(t+x\cdot\omega)/h} \left(\varphi(x+t\omega) + hR_2(t,x) \right),$$

where $\omega \in \mathbb{S}^{n-1}$, $0 < h \le h_0$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $R_i(t,x) = R_i(t,x;h)$, $i = 1,2$ satisfy the estimate

$$\|R_i\|_{L^2(\overline{Q}_T)} + \|h\nabla_{t,x} R_i\|_{L^2(\overline{Q}_T)} \le C, \ \ i = 1,2, \ 0 < h \le h_0 \tag{3.12}$$

here the constant $C>0$ depends only on Ω, T, a and we have suppressed h for each $R_i(t,x)$ for simplicity. Using these choices for $u_1^{\phi_1,\psi_1,f_1}$ and $u_1^{\phi_2,\psi_2,f_2}$ in (3.11), we have

$$\begin{aligned}
&-\frac{2}{h^2}\int\limits_{\mathbb{R}^{1+n}}\beta_w\varphi^2(x+t\omega)dxdt+\frac{1}{h}\int\limits_{\mathbb{R}^{1+n}}\beta_w\varphi(x+t\omega)(1,\omega)\cdot\left(h\nabla_{t,x}R_1-h\nabla_{t,x}R_2\right)dxdt\\
&-\frac{2}{h}\int\limits_{\mathbb{R}^{1+n}}\beta_w\varphi(x+t\omega)\left(R_1+R_2\right)dxdt+\int\limits_{\mathbb{R}^{1+n}}\beta_w\left(|\nabla_{t,x}\varphi|^2-2R_1R_2+h\nabla_{t,x}\varphi\cdot\left(\nabla_{t,x}R_1+\nabla_{t,x}R_2\right)\right)dxdt\\
&+\int\limits_{\mathbb{R}^{1+n}}\beta_w(1,\omega)\cdot\left(R_2\nabla_{t,x}\varphi-R_1\nabla_{t,x}\varphi+\varphi\nabla_{t,x}R_1-\phi\nabla_{t,x}R_2-hR_1\nabla_{t,x}R_2+hR_2\nabla_{t,x}R_1\right)dxdt=0
\end{aligned}$$

for any solution w of (3.9), $\omega\in\mathbb{S}^{n-1}$ and $\varphi\in C_0^\infty(\mathbb{R}^n)$. Now multiplying by h^2 and taking $h\to 0$, we get

$$\int\limits_{\mathbb{R}^{1+n}}\beta_w(t,x)\varphi^2(x+t\omega)dxdt=0,\ \text{for all } \omega\in\mathbb{S}^{n-1} \text{ and for all } \varphi\in C_0^\infty(\mathbb{R}^n).$$

After substituting $x+t\omega=y$, we get

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\varphi^2(y)\beta_w(t,y-t\omega)dtdy=0,\quad \varphi\in C_0^\infty(\mathbb{R}^n),\ \omega\in\mathbb{S}^{n-1}.$$

Thus finally we have

$$\int\limits_{\mathbb{R}}\beta_w(t,y-t\omega)dt=0,\ (t,y)\in\mathbb{R}^{1+n},\ \omega\in\mathbb{S}^{n-1}. \tag{3.13}$$

For each $\omega\in\mathbb{S}^{n-1}$, take $(r,y)\in\mathbb{R}\times\mathbb{R}^n$ such that $2r+y\cdot\omega=0$. Then $\ell:=(r,y+r\omega)\in(1,\omega)^{\perp}$. Hence by the change of variable $t=r+s$, we have

$$\int_{\mathbb{R}}\beta_w(\ell+s(1,\omega))\,ds=0,\ \ell\in(1,\omega)^{\perp},\ \omega\in\mathbb{S}^{n-1}. \tag{3.14}$$

Based on this we will prove $\beta_w(t,y)=0$ in $\mathbb{R}^{1+n}$ by using the Fourier-slice theorem (see for example in [38]). We start by considering

$$\widehat{\beta_w}(\zeta):=\int\limits_{\mathbb{R}^{1+n}}e^{i\zeta\cdot(t,x)}\beta_w(t,x)\,dxdt.$$

Using the decomposition, $\mathbb{R}^{1+n}=\mathbb{R}(1,\omega)\oplus\ell$ and Fubini's theorem, we have

$$\widehat{\beta_w}(\zeta)=\sqrt{2}\int\limits_{(1,\omega)^{\perp}}\int\limits_{\mathbb{R}}\beta_w(\ell+s(1,\omega))e^{-\mathrm{i}(\ell+s(1,\omega))\cdot\zeta}\,ds\,d\ell.$$

By (3.13) and $\zeta\in(1,\omega)^{\perp}$ implies

$$\widehat{\beta_w}(\zeta)=\sqrt{2}\int\limits_{(1,\omega)^{\perp}}\int\limits_{\mathbb{R}}\beta_w(s(1,\omega)+\ell)e^{-i\ell\cdot\zeta}\,ds\,d\ell=0.$$

Hence $\widehat{\beta_w}(\zeta)=0$ for all $\zeta\in(1,\omega)^{\perp}$ and $\omega\in\mathbb{S}^{n-1}$. Now since $\cup_{\omega\in\mathbb{S}^{n-1}}(1,\omega)^{\perp}=\{(t,x):|t|\le|x|\}$, we have $\widehat{\beta_w}(\zeta)=0$ for all space-like vectors ζ, hence using the Paley-wiener theorem, we have $\widehat{\beta_w}(\zeta)=0$ for all $\zeta\in\mathbb{R}^{1+n}$. Thus we have $\beta_w(t,x)=0$ for all $(t,x)\in\mathbb{R}^{1+n}$ and w solutions to (3.9) which gives us $\vec{b}(t,x)\cdot\nabla_{t,x}w(t,x)=0$ in Q_T for all solution w of (3.9). Now to prove that $\vec{b}(t,x)=0$ in Q_T, we use the following lemma.

Lemma 3.1. *Suppose $n \geq 2$ and $N > \frac{1+n}{2} + 2$. There exists solutions $v_j \in H^N(Q_T)$, $0 \leq j \leq n$ such that*

$$det\left(\frac{\partial v_j}{\partial x_i}\right)_{0\leq i,j\leq n} \neq 0 \text{ a.e. in } Q_T$$

Proof. Let us choose $\omega_j \in \mathbb{S}^{n-1}$ for $0 \leq j \leq n$ such that $(1,\omega_0),(1,\omega_1),(1,\omega_2),\cdots,(1,\omega_n)$ are linearly independent. This can be done for example we can choose $\omega_0 = \frac{1}{\sqrt{n}}(1,1,\cdots,1)$ and $\omega_j = e_j$ for $1 \leq j \leq n$ where e_j represent the standard basis of $\mathbb{R}^n$. Then it is easy to see that $(1,\omega_0),(1,\omega_1),\cdots,(1,\omega_n)$ are linearly independent. Next extending $a(x)$ to a function in $C_0^\infty(\mathbb{R}^n)$, we choose the WKB solutions $v_j(t,x)$ for $0 \leq j \leq n$ of $Lw := \partial_t^2 w(t,x) - \Delta w(t,x) + a(x)w(t,x) = 0$ in $\mathbb{R}^{1+n}$ which take the following form

$$v_j(t,x) = e^{i\lambda(t+x\cdot\omega_j)}\sum_{k=0}^{N}\frac{A_{jk}(t,x)}{(2i\lambda)^k} + R_j(t,x) \text{ with } N > \frac{1+n}{2}+2,\ \lambda \gg 1 \tag{3.15}$$

(see for example [36]). Observe that

$$Lv_j = e^{i\lambda(t+x\cdot\omega)}\left[(2i\lambda\mathcal{L}+L)\left(A_{j0}(t,x) + \frac{A_{j1}(t,x)}{2i\lambda} + \frac{A_{j2}(t,x)}{(2i\lambda)^2} + \cdots + \frac{A_{jN}(t,x)}{(2i\lambda)^N} + e^{-i\lambda(t+x\cdot\omega)}R_j(t,x)\right)\right]$$

where $\mathcal{L} := \partial_t - \omega\cdot\nabla_x$ is the transport operator. By equating the terms with same power of $2i\lambda$, we have

$$\begin{aligned} 2i\lambda\mathcal{L}A_{j0} + (\mathcal{L}A_{j1} + LA_{j0}) + \frac{1}{2i\lambda}(\mathcal{L}A_{j2} + LA_{j1}) + \cdots + \frac{1}{(2i\lambda)^{N-1}}(\mathcal{L}A_{jN} + LA_{j,N-1}) \\ + \frac{1}{(2i\lambda)^N}LA_{jN} + e^{-i\lambda(t+x\cdot\omega_j)}LR_j = 0. \end{aligned}$$

Then we have the transport equations for $A_{jk},\ 0 \leq k \leq N$ given as

$$\mathcal{L}A_{j0} = 0 \tag{3.16}$$

and for $1 \leq k \leq N$

$$\begin{cases}\mathcal{L}A_{jk} = -LA_{j,k-1},\\ A_{jk}(0,x) = 0.\end{cases} \tag{3.17}$$

We take $A_{j0} = 1$ for the first equation in (3.16). After finding A_{jk} for $0 \leq k \leq N$, we take R_j as the solution to

$$\begin{cases} LR_j(t,x) = -e^{i\lambda(t+x\cdot\omega)}\frac{1}{(2i\lambda)^N}LA_{jN}(t,x) \text{ for } (t,x) \in \mathbb{R}^{1+n}\\ R_j(t,x) = \partial_t R_j(t,x) = 0 \text{ at } t = 0.\end{cases}$$

Now solving this Cauchy problem for R_j, we get that $R_j \in H^N(\mathbb{R}^{1+n})$. Hence restricting these solutions to Q_T and using the Sobolev embedding theorem, we have R_j will satisfy the following estimate $\|\nabla_{t,x}R_j\|_{L^\infty(Q_T)} \leq C$ for some constant C independent of λ.

Now consider the matrix

$$\begin{aligned} A(t,x,\lambda) &:= \left(\left(\frac{\partial v_j}{\partial x_i}\right)\right)_{0\leq i,j\leq n} \\ &= \begin{bmatrix} i\lambda e^{i\lambda(t+x\cdot\omega_0)} + \partial_t\widetilde{R}_0 & i\lambda w_{01}e^{i\lambda(t+x\cdot\omega_0)} + \partial_1\widetilde{R}_0 & \cdots & i\lambda w_{0n}e^{i\lambda(t+x\cdot\omega_0)} + \partial_n\widetilde{R}_0 \\ i\lambda e^{i\lambda(t+x\cdot\omega_1)} + \partial_t\widetilde{R}_1 & i\lambda w_{11}e^{i\lambda(t+x\cdot\omega_1)} + \partial_1\widetilde{R}_1 & \cdots & i\lambda w_{1n}e^{i\lambda(t+x\cdot\omega_1)} + \partial_n\widetilde{R}_1 \\ \vdots & \vdots & \cdots & \vdots \\ i\lambda e^{i\lambda(t+x\cdot\omega_n)} + \partial_t\widetilde{R}_n & i\lambda w_{n1}e^{i\lambda(t+x\cdot\omega_n)} + \partial_1\widetilde{R}_n & \cdots & i\lambda w_{nn}e^{i\lambda(t+x\cdot\omega_n)} + \partial_n\widetilde{R}_n \end{bmatrix}, \end{aligned}$$

where ω_{ij} denote the $j'th$ component in $\omega_i \in \mathbb{S}^{n-1}$ and $\widetilde{R}_j(t,x) = e^{i\lambda(t+x\cdot\omega)}\sum_{k=1}^{N}\frac{A_{jk}(t,x)}{(2i\lambda)^k} + R_j(t,x)$. Let us denote by $\alpha_j := e^{i\lambda(t+x\cdot\omega_j)}$ for $0 \le j \le n$, then matrix $A(t,x,\lambda)$ becomes

$$A(t,x,\lambda) = \begin{bmatrix} i\lambda\alpha_0 + \partial_t\widetilde{R}_0 & i\lambda\alpha_0\omega_{01} + \partial_1\widetilde{R}_0 & \cdots & i\lambda\alpha_0\omega_{0n} + \partial_n\widetilde{R}_0 \\ i\lambda\alpha_1 + \partial_t\widetilde{R}_1 & i\lambda\alpha_1\omega_{11} + \partial_1\widetilde{R}_1 & \cdots & i\lambda\alpha_1\omega_{1n} + \partial_n\widetilde{R}_1 \\ \vdots & \vdots & \cdots & \vdots \\ i\lambda\alpha_n + \partial_t\widetilde{R}_n & i\lambda\alpha_n\omega_{n1} + \partial_1\widetilde{R}_n & \cdots & i\lambda\alpha_n\omega_{nn} + \partial_n\widetilde{R}_n \end{bmatrix}. \tag{3.18}$$

Next we want to show that $\mathrm{Det}A(t,x,\lambda) \neq 0$ almost everywhere in Q_T for $\lambda \gg 1$.
Using the fact that $\|\nabla_{t,x}\widetilde{R}_j\|_{L^\infty(Q_T)} \le C$ for some constant $C > 0$ independent of λ, we have

$$\lim_{\lambda\to\infty}\left\| \frac{\mathrm{Det}\, A(t,x,\lambda)}{\lambda^{1+n}} - \mathrm{Det}\begin{bmatrix} i\alpha_0 & i\alpha_0\omega_{01} & \cdots & i\alpha_0\omega_{0n} \\ i\alpha_1 & i\alpha_1\omega_{11} & \cdots & i\alpha_1\omega_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ i\alpha_n & i\alpha_n\omega_{21} & \cdots & i\alpha_n\omega_{nn} \end{bmatrix} \right\|_{L^2(Q_T)} = 0.$$

Therefore we have that

$$\frac{\mathrm{Det}\, A(t,x,\lambda)}{\lambda^{1+n}} \to \mathrm{Det}\begin{bmatrix} i\alpha_0 & i\alpha_0\omega_{01} & \cdots & i\alpha_0\omega_{0n} \\ i\alpha_1 & i\alpha_1\omega_{11} & \cdots & i\alpha_1\omega_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ i\alpha_n & i\alpha_n\omega_{21} & \cdots & i\alpha_n\omega_{nn} \end{bmatrix} \neq 0 \text{ as } \lambda \to \infty \text{ in } L^2(Q_T).$$

Thus we can find a subsquence of $\frac{\mathrm{Det}\, A(t,x,\lambda)}{\lambda^{1+n}}$ and still denote the same such that

$$\lim_{\lambda\to\infty}\frac{\mathrm{Det}\, A(t,x,\lambda)}{\lambda^{1+n}} = \mathrm{Det}\begin{bmatrix} i\alpha_0 & i\alpha_0\omega_{01} & \cdots & i\alpha_0\omega_{0n} \\ i\alpha_1 & i\alpha_1\omega_{11} & \cdots & i\alpha_1\omega_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ i\alpha_n & i\alpha_n\omega_{n1} & \cdots & i\alpha_n\omega_{nn} \end{bmatrix} \neq 0 \text{ pointwise for a.e. } (t,x) \in Q_T.$$

Hence we conclude that $\mathrm{Det}\, A(t,x,\lambda) \neq 0$ for $\lambda \gg 1$, a.e. $(t,x) \in Q_T$. Thus we have that $\nabla_{t,x}v_0, \nabla_{t,x}v_1, \cdots, \nabla_{t,x}v_n$ are linearly independent a.e. in Q_T. This completes the proof of Lemma 3.1. $\square$

Recall that

$$\vec{b}(t,x)\cdot\nabla_{t,x}w(t,x) = 0 \text{ for a.e. } (t,x) \in Q_T \text{ and any solution } w \text{ to (3.9)}.$$

Now using Lemma 3.1, we can choose $w_0, w_1, \cdots w_n$ solutions to (3.9) such that $\nabla_{t,x}w_0, \nabla_{t,x}w_1, \cdots, \nabla_{t,x}w_n$ are linearly independent for a.e. in Q_T. Using these choices of w_j for $0 \le j \le n$ in (3.2), we get $\vec{b}(t,x) = 0$, for a.e. $(t,x) \in Q_T$ but $\vec{b} \in C^\infty(Q_T)$ therefore we have $\vec{b}(t,x) = 0$ for all $(t,x) \in Q_T$. Hence $\vec{b}^{(1)} = \vec{b}^{(2)}$ in Q_T. This completes the proof of uniqueness for $\vec{b}$.

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