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Calculus (Tutorial # 7)

Integral Calculus in $\mathbb{R}$

1. True or False. Justify your answer.

   (a) Any continuous function on closed and bounded interval in $\mathbb{R}$ is integrable.

   (b) Any bounded function $f : [a, b] \to \mathbb{R}$ having only a finite number of point of discontinuity in $[a, b]$ is integrable.

   (c) Any monotone function on any interval $[a, b] \subseteq \mathbb{R}$ is integrable.

   (d) Improper integral of a continuous function on $[0, \infty)$ is convergent.

   (e) If $|f|$ is integrable on $[a, b]$ then f is also integrable.

   (f) Let $f : [0, 1] \to \mathbb{R}$ be a function such that $f(1/2) = 1$ and $f(x) = 0$ if $x \neq 1/2$. Then f is integrable on $[0, 1]$ and $\int_0^1 f(x)\, dx = 0$.

   (g) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Suppose $f(x) \geq 0$, for all $x \in [0, 1]$ and $\int_0^1 f(x)\, dx = 0$, then $f \equiv 0$.

   (h) Suppose $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_a^b f(x)\, dx = 0$, for any choices of $0 \leq a \leq b \leq 1$ then $f \equiv 0$.

   (i) If the improper integral $\int_1^\infty f(x)\, dx$ is convergent and $\lim_{x\to\infty} f(x) = L$, then $L = 0$.

   (j) $\int_0^\pi \sec^2 x\, dx = 0$.

   (k) The improper integral $\int_1^\infty \frac{\sin x}{x}\, dx$ is convergent but not absolutely convergent.

   (l) If $f : [a, b] \to \mathbb{R}$ is continuous, then $\int_a^b x f(x)\, dx = x \int_a^b f(x)\, dx$.

   (m) If $f : [a, b] \to \mathbb{R}$ is continuous with $f(x) \geq 0$ for all $x \in [a, b]$, then

   $$\int_a^b \sqrt{f(x)}\, dx = \sqrt{\int_a^b f(x)\, dx}$$

   .

   (n) If $f, g : [a, b] \to \mathbb{R}$ are continuous, then

   $$\int_a^b f(x)g(x)\, dx = \Big(\int_a^b f(x)\, dx\Big)\Big(\int_a^b g(x)\, dx\Big).$$

   (o) If $f : [a, b] \to \mathbb{R}$ is continuous, then $\frac{d}{dx}\int_a^b f(x)\, dx = f(x)$.

2. Sketch the region enclosed by the given curves and find its area.

   (a) $y = 12 - x^2$ and $y = x^2 - 6$.

   (b) $y = \sqrt{x-1}$ and $x - y = 1$.

   (c) $x = y^4$, $y = \sqrt{2-x}$ and $y = 0$

   (d) $y = 1/x$, $y = x$, $y = x/4$ and $x > 0$.

   (e) $y = \tan x$ and $y = 2\sin x$ for $-\pi/3 \leq x \leq \pi/3$.

   (f) $4x + y^2 = 12$ and $x = y$.

   (g) $y = e^x$, $y = xe^x$ and $x = 0$.

3. Prove that the improper integrals $I_1 := \displaystyle\int_0^\infty \frac{\cos x}{1+x}\,dx$ and $I_2 := \displaystyle\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$ are convergent and $I_1 = I_2$. Which of them is absolutely convergent?

4. Let $f(x) : [1, \infty) \to [0, \infty)$ be a decreasing function. Then prove that the improper integral $\displaystyle\int_1^\infty f(x)\,dx$ is convergent if and only if the series $\displaystyle\sum_{n=1}^\infty f(n)$ is convergent. (This is the so-called "**Cauchy integral test**"for convergent of series of non-negative terms.)

5. Given examples of continuous functions $f : [1, \infty) \to [0, \infty)$ satisfying the following.

   (a) $\sum\limits_{n=1}^\infty f(n)$ converges, but $\int_1^\infty f(x)dx$ diverges.

   (b) $\int_1^\infty f(x)dx$ converges, but $\sum\limits_{n=1}^\infty f(n)$ diverges.

6. Let f be a continuous function and g, h are differentiable functions on $\mathbb{R}$. Then find a formula for $\displaystyle\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt$.

7. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}dt = 2\sqrt{x}, \text{ for all } x > 0.$$

8. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases} \qquad \text{and define } \ g(x) := \int_0^x f(t)\,dx.$$

   (a) Find an expression for g similar to the one for $f(x)$.

   (b) Sketch the graph of f and g.

   (c) Where is f differentiable? Where is g differentiable?

9. If $f(x) = \displaystyle\int_0^{\sin x} \sqrt{1+t^2}\,dt$ and $g(y) = \int_3^y f(x)\,dx$, then find $g''(\pi/6)$.

10. (Gronwall's inequality) Let $f, g, h : [a, b] \to \mathbb{R}$ be non-negative continuous functions and

$$f(x) \leq g(x) + \int_a^x h(t)f(t)\, dt, \quad \text{for } \ x \in [a, b]. \tag{1}$$

Then the following inequality holds:

$$f(x) \leq g(x) + \int_a^x g(t)h(t) \exp\Big(\int_t^x h(s)\, ds\Big)\, dt, \ \text{for } x \in [a, b].$$

In particular, if $g \equiv 0$, then the function f satisfying (1) is identically zero.
(**Note:** This inequality is very important in differential equations.)

11. The **error function** defined below is used in probability, statistics and engineering.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

(a) Show that $\displaystyle\int_a^b e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$

(b) Show that the function $y = e^{x^2} \operatorname{erf}(x)$ satisfies the differential equation

$$\frac{dy}{dx} = 2xy + \frac{2}{\sqrt{\pi}}.$$

12. Let p and q be positive real numbers such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Suppose f and g be two integrable functions on $[a, b]$ such that both $\displaystyle\int_a^b |f(x)|^p\, dx$ and $\displaystyle\int_a^b |g(x)|^q\, dx$, are finite. Then prove the "**Hölder's inequality**"

$$\int_a^b \Big|f(x)g(x)\Big|\, dx \leq \Big(\int_a^b |f(x)|^p\, dx\Big)^{1/p} \Big(\int_a^b |g(x)|^q\, dx\Big)^{1/q}$$

by completing the following steps.

(a) Show that $|uv| \leq \dfrac{|u|^p}{p} + \dfrac{|v|^q}{q}$ for all $u, v \in \mathbb{R}$ and equality hold if and only if $|u|^p = |v|^q$.

(b) If $\displaystyle\int_a^b |f(x)|^p\, dx = 1 = \int_a^b |g(x)|^q\, dx$, then $\displaystyle\int_a^b \Big|f(x)g(x)\Big|\, dx \leq 1$.

(c) Define $\alpha := \displaystyle\Big(\int_a^b |f(x)|^p\, dx\Big)^{1/p}$ and $\beta := \displaystyle\Big(\int_a^b |g(x)|^q\, dx\Big)^{1/q}$ then $\displaystyle\int_a^b \Big|\frac{f(x)}{\alpha}\Big|^p dx = 1 = \int_a^b \Big|\frac{g(x)}{\beta}\Big|^q\, dx.$

(d) To complete the proof of the "Hölder's inequality" apply part (c) to the functions $\dfrac{f}{\alpha}$ and $\dfrac{g}{\beta}$.

13. A function f is defined by

$$f(x) = \int_0^{\pi} \cos t \ \cos(x - t) \ dt, \quad 0 \le x \le 2\pi.$$

Then find the minimum value of f.

14. Determine whether each of the following improper integral is convergent or divergent.

(i) $\displaystyle\int_0^{\pi} \frac{\sin^2 x}{x} \ dx$ (ii) $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}} \ dx$ (iii) $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3} \ dx$

(iv) $\displaystyle\int_0^{\infty} e^{-x^2} \ dx$ (v) $\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}} \ dx$ (vi) $\displaystyle\int_0^2 x^2 \log x \ dx$

(vii) $\displaystyle\int_1^{\infty} \frac{e^{\sqrt{x}}}{\sqrt{x}} \ dx$ (viii) $\displaystyle\int_1^{\infty} \frac{\log x}{x} \ dx$ (ix) $\displaystyle\int_{-\infty}^{\infty} x^3 e^{-x^4} \ dx$

(x) $\displaystyle\int_0^{\infty} \frac{1}{x^2 + 3x + 2} \ dx$ (xi) $\displaystyle\int_0^{\pi/2} \sec x \ dx$ (xii) $\displaystyle\int_1^{\infty} \frac{1 + e^{-x}}{x} \ dx$

15. Find the values of p for which the improper integral converges and evaluate the integral for those values of p.

(i) $\displaystyle\int_0^1 \frac{1}{x^p} \ dx$ (ii) $\displaystyle\int_e^{\infty} \frac{1}{x(\log x)^p} \ dx$ (iii) $\displaystyle\int_0^1 x^p \log x \ dx$

(iv) $\displaystyle\int_0^{\infty} x^p e^{-x} \ dx.$ (v) $\displaystyle\int_0^{\infty} e^{px} \cos x \ dx$ (vi) $\displaystyle\int_0^{\infty} x^p \ dx$

(vii) $\displaystyle\int_0^1 (\log x)^p \ dx$ (viii) $\displaystyle\int_0^1 (1 - x^2)^p \ dx.$

16. (a) If $f : [0, \infty) \to \mathbb{R}$ be a continuous function and if there are constants M and a such that $0 \le |f(t)| \le Me^{at}$ for $t \ge 0$, then show that the improper integral

$$F(s) := \int_0^{\infty} f(t)e^{-st}dt \tag{2}$$

is convergent for each $s > a$.

(b) Suppose the improper integral $\displaystyle\int_0^{\infty} f(x) \ dx$, is absolutely convergent. Then the function F define by Equation (2) in part (a) is well-defined for each $s \ge 0$.

Remark: $F(s)$ defined by (2) is called the ***Laplace transform*** of f at s.

(c) Assume the improper integrals $\displaystyle\int_0^{\infty} f(x) \ dx$ and $\displaystyle\int_0^{\infty} f'(x) \ dx$, are absolutely convergent and suppose $F(s)$ and $G(s)$ denote the Laplace transform of f and f' respectively. Then show that $G(s) = sF(s) - F(0)$, for $s \ge 0$.

17. If $f : [0, 1] \to \mathbb{R}$ is continuous, then show that

$$\lim_{n\to\infty} \frac{1}{n} \left\{ f\left(\frac{0}{n}\right) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n}{n}\right) \right\} = \int_0^1 f(x)dx.$$

Use this to evaluate the following limits.

(a) $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right].$

(b) $\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)$

(c) $\lim_{n\to\infty} \frac{1}{n}(e^{3/n} + e^{6/n} + \cdots + e^{3n/n})$