Department of Mathematics, IIT Jammu, India
Instructors: Rahul Kitture and Manmohan Vashisth.

November 30, 2020

Calculus (Tutorial # 2)

Sequences

1. Let $a_n = \dfrac{3n+4}{n+5}$, for $n \in \mathbb{N}$ then $\lim_{n\to\infty} a_n = 3$. Given an $\epsilon > 0$, find $N_\epsilon \in \mathbb{N}$ such that $|a_n - 3| < \epsilon$ for all $n \geq N_\epsilon$.

2. (a) Suppose $a \in \mathbb{Q}$, then can you find a sequence $\{a_n\}$ of irrational numbers such that $\lim_{n\to\infty} a_n = a$?

   (b) Suppose $a \in \mathbb{R} \setminus \mathbb{Q}$, then can you find a sequence $\{a_n\} \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} a_n = a$?

3. Let $b_n \geq 0$ and $b_n \to 0$ as $n \to \infty$. Let $\{a_n\} \subseteq \mathbb{R}$ be a sequence such that $|a_n - a| \leq b_n$ for all $n \geq N \in \mathbb{N}$. Then using the definition of convergence of sequences, show that $a_n \to a$ as $n \to \infty$.

4. Let $\{a_n\} \subseteq \mathbb{R}$ be such that $\alpha \leq a_n \leq \beta$ and $\{a_n\}$ is converges to a then show that $\alpha \leq a \leq \beta$.

5. Let $\{a_n\} \subseteq \mathbb{R}$ is such that for each $\epsilon > 0$, $|a_n - a| < 5\epsilon$, for all $n \geq N$ where N does not depend on ϵ. Then characterize such $\{a_n\}$.

6. Let $\{a_n\} \subseteq \mathbb{R}$ be a convergent sequence. Then using the definition of convergence of sequences show that $\{a_n^k\}$ for some fixed $k \in \mathbb{N}$ is also convergent. How about $\{a_n^n\}$?

7. Find all the possible conditions on $\{a_n\} \subseteq \mathbb{R}$, for which $\{a_n\}$ is convergent if and only if $\{|a_n|\}$ is convergent.

8. Let $\{a_n\} \subseteq \mathbb{R}$ be such that $\lim_{k\to\infty} a_{2k} = a$ and $\lim_{k\to\infty} a_{2k-1} = a$. Then is it true that $\{a_n\}$ is convergent?

9. Let $\{a_n\}$ be a sequence such that $|a_n| \leq \dfrac{1+n}{1+n+7n^2}$, for all $n \in \mathbb{N}$. Check whether $\{a_n\}$ is a Cauchy sequence or not.

10. Let $\{a_n\}$ be a sequence such that $|a_n - a_{n-1}| \leq c|a_{n-1} - a_{n-2}|$, for some constant $0 < c < 1$ and for all $n \geq 2$. Show that $\{a_n\}$ is convergent.

11. Let $\{a_n\}$ be a sequence of positive real numbers. Assume that $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = l$, with $l < 1$. Prove that $a_n \to 0$ as $n \to \infty$.

12. Let $\{a_n\}$ be a bounded sequence. Assume that $a_{n+1} \geq a_n - 2^{-n}$. Check the convergence of $\{a_n\}$.

13. Let $a_1 = \sqrt{101}$ and define $a_n = \sqrt{101 + a_{n-1}}$ for $n \in \mathbb{N}$. Then check the convergence of $\{a_n\}$ and also find the limit if it converges.

14. Let $0 < b_1 < a_1$. Define the two sequences $\{a_n\}$ and $\{b_n\}$ as follow:

$$a_{n+1} := \frac{a_n + b_n}{2} \text{ and } b_{n+1} := \sqrt{a_n b_n}, \text{ for } n \in \mathbb{N}.$$

Then check the convergence of $\{a_n\}$ and $\{b_n\}$. Also find the limit if they converge.

15. If $\{a_n\}$ is a bounded sequence of real numbers and $\liminf_{n\to\infty} a_n = l$, prove that there is a subsequence of $\{a_n\}$ which converge to l. Also, prove that no subsequence of $\{a_n\}$ can converge to a limit less than l.

16. Find the lim sup and lim inf if they exist for the sequences given below. Also check the convergence of them and find the limit if they converge.

    (a) $a_n = \sqrt{n^2 + 5n} - n$

    (b) $a_1 = 3$ and $a_{n+1} = 1 - \dfrac{1}{a_n}$

    (c) $a_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n}$

    (d) $\left\{2, -1, \frac{3}{2}, -\frac{1}{2}, \frac{4}{3}, -\frac{1}{3}, \ldots\right\}$

    (e) $a_n = \sin\left(\dfrac{n\pi}{7}\right)$

    (f) $\{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\}$

    (g) $a_n = \left(1 + \dfrac{1}{n}\right)^n$

    (h) $x_1 = 1$, $x_2 = 2$ and $x_{n+2} = \dfrac{1}{2}(x_{n+1} + x_n)$ for $n \geq 1$.

    (i) $a_n = \dfrac{\{n\pi\}}{n}$ where $\{n\pi\}$ is the fractional part of $n\pi$.

    (j) $a_n = \frac{\sin n}{n}$ for $n \in \mathbb{N}$.