BOUNDARY DETERMINATION OF COEFFICIENTS APPEARING IN A PERTURBED WEIGHTED p-LAPLACE EQUATION

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ABSTRACT. We study an inverse boundary value problem associated with p-Laplacian which is further perturbed by a linear second order term, defined on a bounded set Ω in $\mathbb{R}^n, n \geq 2$. We recover the coefficients at the boundary from the boundary measurements which are given by the Dirichlet to Neumann map. Our approach relies on the appropriate asymptotic expansion of the solution and it allows one to recover the coefficients pointwise. Furthermore, by considering the localized Dirichlet-to-Neumann map around a boundary point, we provide a procedure to reconstruct the normal derivative of the coefficients at that boundary point.

1. INTRODUCTION

In this paper, we consider the following weighted p-Laplace equation which is perturbed by a linear term

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla u(x) + \gamma(x)|\nabla u(x)|^{p-2}\nabla u(x)) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where the domain $\Omega \subseteq \mathbb{R}^n, n \geq 2$ is a bounded open set with smooth boundary $\partial\Omega$. Throughout the paper, we assume that the exponent p satisfies $1 < p < \infty$. Furthermore, we consider the coefficients σ and γ are positive functions with the regularity $\sigma \in C^\infty(\overline{\Omega})$ and $\gamma \in L^\infty(\Omega)$ respectively. Moreover, the coefficients σ and γ are assumed to satisfy the following conditions:

$$\begin{cases} 0 < \lambda < \sigma(x) < \lambda^{-1}, \\ 0 < m_1 < \gamma(x) < m_1^{-1}, \end{cases}$$

where λ, and m_1 are positive constants. The Dirichlet data f is considered as follows: $f \in W^{1/2,2}(\partial\Omega)$ whenever $p \in (1, 2)$, and $f \in W^{1-1/p,p}(\partial\Omega)$ whenever $p \in (2, \infty)$.

The nonlinear Dirichlet to Neumann map (DN-map) is defined by

$$\Lambda_{\sigma,\gamma}(f) = (\sigma(x) + \gamma(x)|\nabla u|^{p-2})\partial_\nu u|_{\partial\Omega}, \tag{1.2}$$

where u is the unique solution of equation (1.1) with the boundary value f. Throughout the paper, we assume that ν is the outer unit normal to $\partial\Omega$.

In case of $p = 2$, the equation(1.1) becomes a linear conductivity equation and there are several results related to boundary determination of conductivity coefficients appearing in the linear equation. For instance, in [21, 34] the smooth conductivity and all its derivatives at the boundary are recovered from the boundary measurements of the solution. The authors in [21, 34] proved the uniqueness of conductivity together with its derivatives at the boundary of domain. Later Nachman in [26] recovered the coefficient at the boundary by assuming that $\gamma \in W^{1,m}(\Omega)$, $m > n$. Under the assumption $m > n/2$, Nachman in [26] also recovered the first normal derivative of $\gamma \in W^{2,m}(\Omega)$. Furthermore, Brown [7] described a procedure for recovering $\gamma|_{\partial\Omega}$ in a pointwise manner by assuming $\gamma \in L^\infty(\Omega)$. For the conductivity equation, Nakamura and Tanuma [27, 28] provided a formula for reconstructing the conductivity and its

higher order derivatives from the localized DN-map. Moreover, there are several local boundary determination results, for instance, one can refer to [1, 19, 29] and references therein.

In medical imaging, the inverse problems naturally occur to reconstruct an image of the internal structure of the human body based on the X-ray signal passing through the human body. More precisely, the conductivity distribution inside the body is reconstructed from the electrical measurements on the skin in electrical impedance tomography (EIT). For more details, one can refer to [3]. In fluid mechanics, inverse problems or parameter identification problems occur in the recovery of domain deformations and determining the obstacles. Similarly, in an inverse problem associated with the sand-piles model, the goal is to determine the properties of a physical system by interpreting observations of the system (see [2] and references therein).

Information of coefficients at the boundary can be used in several situations. Practically, boundary determination results can be used to calibrate a measurement device or measure errors in the devices and mathematically such results can be used in recovering the coefficients in the interior. As mentioned in (please refer to page 1 in [8]) that determination of coefficients at the boundary is first step in the interior determination of coefficients. For example, while proving the interior determination of conductivity in (please see page 157 in [33]) the boundary determination result from [21] is used. Boundary determination results are often useful in determining the coefficients of low regularity. For example while proving the interior unique determination of Lipschitz conductivity in [13] the boundary determination from [21] is used to prove the equality of DN map for the converted Schrödinger equation and from their conclusion for unique recovery of conductivity is concluded via the boundary determination result. We refer to page 2 in [13] for more details. The boundary identifiabilty result also arises in the *layer-stripping* method for solving the inverse conductivity problem [31]. Results related to determination of coefficients at boundary are also relevant in inverse problems in hybrid imaging as some of the results in hybrid imaging assume coefficients to be known on or close to the boundary. For more details, we refer to ([5, 6]) and references therein.

Motivated by these applications in image processing, fluid mechanics and modeling of sand-piles, Calderón-type inverse problems for the p-Laplace equation are studied by several authors. For instance, Salo and Zhong [30] studied the boundary uniqueness problem associated with p-Laplace equation for $n \geq 2$. Brander in [4] recovered the gradient of γ at the boundary using the Rellich's type identity. Furthermore, Cârstea and Kar in[9] considered the weighted perturbed p-Laplace equation and provided a procedure for reconstruction of the coefficients in the interior of the domain using the complex geometric optics solution for $n \geq 3$. However, up to our knowledge, the boundary determination problem for weighted perturbed p-Laplace equation concerning the coefficient γ has not been studied in the literature.

In this paper, we will give a procedure for recovery of the coefficients at boundary of the domain. To do this, we use the ϵ-expansion of solution which varies according to the choice of p. The approach of using the ϵ-expansion or linearization for solving the inverse problems for nonlinear elliptic equations have been used by several authors, see for example [9, 10, 11, 12, 15, 16, 17, 18, 20, 22, 23, 25]. Motivated by [9, 10, 18] our idea is to find an appropriate integral identity, from which reconstruction of the coefficient γ at the boundary of the domain is possible. Using the approach used in [10] we derived the integral identity by constructing a sequence of explicit functions which are solutions of conductivity equation. Such solutions can be used to extract the Taylor series of the coefficient γ at the boundary point from the knowledge of $\Lambda_{\sigma,\gamma}$.

We can now state our boundary determination results.

Theorem 1.1. *(Reconstruction of coefficients)*
Suppose $p > 1$ with $p \neq 2$ and $\Omega \subseteq \mathbb{R}^n, n \geq 2$ is a bounded domain with smooth boundary $\partial\Omega$. Furthermore, assume that the coefficients are continuous on $\overline{\Omega}$ and bounded below by a positive constant. Then the coefficients σ and γ in (1.1) can be reconstructed at the boundary from the knowledge of given DN-map $\Lambda_{\sigma,\gamma}$.

Theorem 1.2. *(Reconstruction of normal derivatives of coefficients at the boundary)*
Assume that $p \in (1, \infty) \backslash \{2\}$ and for $n \geq 2$, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial \Omega$. Suppose $\sigma \in C^\infty(\overline{\Omega})$ and $D_{x'}^{\alpha'} D_{x_n}^{\alpha_n} \gamma$ is continuous around x_0 for any multi-index (α', α_n) such that $|\alpha'| + 2\alpha_n \leq 2$, where $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$. Then the normal derivative of the coefficients can be evaluated at the boundary point of Ω from the knowledge of given DN-map $\Lambda_{\sigma, \gamma}$ associated to (1.1).

Remark 1.3. *Theorems 1.1 and 1.2 deals with the recovery of coefficients appearing in (1.1) from the knowledge of DN-map while the interior determination for the same problem has been studied in [10]. Now due to the regularity assumptions on the coefficients, one can see that the boundary determination results stated in Theorems 1.1 and 1.2 follows from [10] however the approach used in [10] works in dimension $n \geq 3$ and does not work when $n = 2$. The approach used in [10, Proposition 3.1] is based on the use of complex geometric optics (CGO) solutions of linear conductivity equation, initially constructed in Sylvester and Uhlmann [33] to prove the interior determination of the conductivity from the boundary measurements for a linear conductivity equation. The approach of using CGO solutions work only in dimension $n \geq 3$ as the argument of the proof in [10, 33] required to choose three mutually orthogonal vectors in $\mathbb{R}^n$ which holds true whenever $n \geq 3$. On the other hand, the approach used in our paper is based on construction of special solutions for linear conductivity equation which are concentrated near a boundary point at which we are interested to reconstruct the coefficients. This approach has been used in prior works (see [7, 21, 27] and references therein) for the linear conductivity equation which also work in dimension $n = 2$ as well. Using the ε-expansion together with these special solutions, we have given the reconstruction at the boundary for the coefficients associated to the linear part as well as the nonlinear part. Moreover, in Section 4, we have also reconstructed the normal derivative of the coefficient γ at the boundary for $n \geq 2$ using approximate solutions of the linear conductivity equation and the knowledge of reconstructed γ at the boundary.*

The paper is organized as follows: we state the well-posedness results of (1.1) along with corresponding ε-expansion of the solution in section 2 which further includes the epsilon expansion of Dirichlet to Neumann map. In section 3, we provide a detailed proof for the reconstruction of the coefficient at the boundary of the domain (cf. Theorem 1.1). We construct a few approximate solutions using the localized Dirichlet data to extract the normal derivative of γ and obtain a reconstruction formula in Section 4.

2. ε-EXPANSION OF THE SOLUTION FOR PERTURBED p-LAPLACIAN PROBLEM

In this section, we state the existence of the solution and corresponding energy estimates for the boundary value problem (BVP) (1.1). Based on the range of p, we have weak solution and strong solution of the BVP (1.1) corresponding to sufficiently small boundary data. We remark that such boundary data are considered since we require the resulting estimates while analyzing the boundary determination problem. More precisely, we shall use the weak solution and strong solution of BVP 1.1 while reconstructing the scalar coefficient γ at the boundary. The following Proposition 2.1 can be proved by defining suitable energy functionals and subsequently, considering the minimizing sequence (for details, we refer to [10, Section 2]).

Proposition 2.1 (Weak solution).
(i) *Assume that $1 < p < 2$ and $f \in W^{1/2,2}(\partial \Omega)$. Then the BVP (1.1) has a unique solution*
$$u \in W^{1/2,2}(\Omega) \cap W^{1,p}(\Omega),$$
and it satisfies the following estimate
$$\|u\|_{W^{1,2}(\Omega)} \leq C \|f\|_{W^{1/2,2}(\partial \Omega)},$$
where positive constant C does not depend on the boundary data f.

(ii) *Assume that $2 < p < \infty$ and $f \in W^{1-1/p,p}(\partial\Omega)$. Then the BVP (1.1) has a unique solution*
$$u \in W^{1/2,2}(\Omega) \cap W^{1,p}(\Omega),$$

along with the stability estimate
$$\|u\|_{W^{1,p}(\Omega)} \le C\|f\|_{W^{1-1/p,p}(\partial\Omega)},$$

where $C = C(\Omega) > 0$.

Proposition 2.1 can be further improved in case of $p > 2$. In fact, there exists a strong solution for the BVP (1.1). By defining an appropriate operator and using the Sobolev embedding results and finally, demonstrating the fixed point of contraction mapping, the existence of the strong solution can be established. For detailed proof of the following Proposition 2.2, we refer to [10, Theorem 2.2].

Proposition 2.2 (Strong solution). *Assume that $p > 2$ and $m \in \mathbb{R}$ such that $m > n$. Furthermore, let $f \in W^{2-1/m,m}(\partial\Omega)$ and there holds*
$$\|f\|_{W^{2-1/m,m}(\partial\Omega)} \le M,$$

for some $M > 0$. Then there exists a unique solution for BVP (1.1)
$$u \in W^{2,m}(\Omega).$$

Moreover, the solution u satisfies
$$\|u\|_{W^{2,m}(\Omega)} \le C\|f\|_{W^{2-1/m,m}(\partial\Omega)}, \tag{2.1}$$

where constant $C > 0$ is depending on the choice of p, domain Ω and coefficients σ, and γ.

Based on the well-posedness results, described in Proposition 2.1 and Proposition 2.2, we define the nonlinear DN-map in a weak sense as follows:
$$\langle \Lambda_{\sigma,\gamma}(f), w|_{\partial\Omega}\rangle = \int_{\Omega} (\sigma(x) + \gamma(x)|\nabla u|^{p-2})\nabla u \cdot \overline{\nabla w}\, dx, \tag{2.2}$$

where $w \in W^{2,m}(\Omega) \cap W^{1,2}(\Omega)$. Afterwards, we look into the ε-expansion of the solution for the BVP (1.1). Since the expansion depends on the range of p, we carry out the analysis via two cases.

2.1. **Case-1:** $p > 2$. We look for ε-expansion of the solution of (1.1) for $p > 2$ based on the existence of strong solution mentioned in Proposition 2.2.

Let us consider the following BVP
$$\begin{cases} \nabla \cdot (\sigma(x)\nabla u_\varepsilon(x) + \gamma(x)|\nabla u_\varepsilon(x)|^{p-2}\nabla u_\varepsilon(x)) = 0, & x \in \Omega, \\ u_\varepsilon(x) = \varepsilon f(x), & x \in \partial\Omega. \end{cases} \tag{2.3}$$

From the stability estimate (2.1), we obtain
$$\|\varepsilon^{-1}u_\varepsilon\|_{W^{2,m}(\Omega)} \le C\|f\|_{W^{2-1/m,m}(\partial\Omega)}. \tag{2.4}$$

As a consequence, we consider the following ε-expansion
$$u_\varepsilon = \varepsilon u_0 + \varepsilon^{p-1}v, \tag{2.5}$$

where $u_0 \in W^{2,m}(\Omega)$ and $v \in W^{2,m}(\Omega)$ will be determined by substituting (2.5) in the BVP (2.3). We obtain that u_0 satisfies the following BVP
$$\begin{cases} \nabla \cdot (\sigma(x)\nabla u_0(x)) = 0, & x \in \Omega, \\ u_0(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{2.6}$$

and eventually, v satisfies the BVP

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla v) + \varepsilon^{1-p}\nabla \cdot (\gamma(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = 0, & x \in \Omega, \\ v(x) = 0 & x \in \partial\Omega. \end{cases} \tag{2.7}$$

2.2. **Case-2:** $1 < p < 2$. For $p \in (1,2)$, ε-expansion of the solution of BVP (1.1) is a consequence of existence of a weak solution of (1.1). Since $p \in (1,2)$, then we get a sublinear term instead of a nonlinear term. Therefore, for arbitrary small $\varepsilon > 0$, we will take the Dirichlet boundary data as $\varepsilon^{-1}f$ in (1.1).

Let us consider

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla u_\varepsilon + \gamma(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = 0, & x \in \Omega, \\ u_\varepsilon(x) = \varepsilon^{-1}f(x), & x \in \partial\Omega. \end{cases} \tag{2.8}$$

The well-posedness of above BVP (2.8) is assured by the Proposition 2.1 along with the stability estimate

$$\|\varepsilon u_\varepsilon\|_{W^{1,2}(\Omega)} \le C\|f\|_{W^{1/2,2}(\partial\Omega)}. \tag{2.9}$$

By taking into account the estimate (2.9), we consider the following expansion:

$$u_\varepsilon = \varepsilon^{-1}u_0 + \varepsilon^{1-p}v, \tag{2.10}$$

where $u_0 \in W^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$ will be determined by substituting (2.10) in the BVP (2.8). It is straightforward to observe that u_0 satisfies the following BVP

$$\begin{cases} \nabla \cdot (\sigma\nabla u_0) = 0, & x \in \Omega, \\ u_0(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{2.11}$$

and v satisfies the BVP

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla v) + \varepsilon^{p-1}\nabla \cdot (\gamma(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.12}$$

2.3. **ε-expansion of DN-map.** In order to recover the parameter γ at the boundary, we require the ε-expansion of the DN-map described in (2.2). We carry out the following analysis motivated by Cârstea et al. [10] in which the reconstruction of γ in the interior of domain in $\mathbb{R}^n$ ($n \ge 3$) is described. The following estimate will be instrumental for our analysis: for $p > 2$,

$$\left| |\eta|^{p-2}\eta - |\beta|^{p-2}\beta \right| \le C(|\eta| + |\beta|)^{p-2}|\eta - \beta|, \tag{2.13}$$

where $C > 0$ and $\eta, \beta \in \mathbb{C}^n$ (for detailed proof, one can look into [30]). Let us consider $w \in W^{2,m}(\Omega) \cap W^{1,2}(\Omega)$. Using the estimate (2.13), we obtain

$$\left| \int_\Omega \gamma(x)(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - \varepsilon^{p-1}|\nabla u_0|^{p-2}\nabla u_0) \cdot \overline{\nabla w}\, dx \right|$$
$$\le C\varepsilon^{2p-3} \int_\Omega (|\varepsilon^{-1}\nabla u_\varepsilon|^{p-2} + |\nabla u_0|^{p-2})|\nabla v||\nabla w|\, dx = \mathcal{O}(\varepsilon^{2p-3}). \tag{2.14}$$

With the help of expansion (2.5) and above estimate (2.14), we get

$$\langle \Lambda_{\sigma,\gamma}(\varepsilon f), w|_{\partial\Omega} \rangle = \int_\Omega (\sigma(x) + \gamma(x)|\nabla u_\varepsilon|^{p-2})\nabla u_\varepsilon \cdot \overline{\nabla w}\, dx$$

$$= \varepsilon \int_\Omega \sigma(x)\nabla u_0 \cdot \overline{\nabla w}\, dx + \varepsilon^{p-1} \int_\Omega \sigma(x)\nabla v \cdot \overline{\nabla w}\, dx$$

$$+ \varepsilon^{p-1} \int_\Omega \gamma(x)|\nabla u_0|^{p-2}\nabla u_0 \cdot \overline{\nabla w}\, dx + \mathcal{O}(\varepsilon^{2p-3}).$$

Subsequently, by considering the limit ε tending to zero, we have

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle \Lambda_{\sigma,\gamma}(\varepsilon f), w\big|_{\partial\Omega} \rangle &= \int_{\Omega} \sigma(x) \nabla u_0 \cdot \overline{\nabla w} \ dx \\
&= \langle \Lambda_{\sigma}(f), w\big|_{\partial\Omega} \rangle,
\end{aligned} \tag{2.15}
$$

where $\Lambda_{\sigma}(f)$ is the DN-map associated to the Dirichlet BVP (2.6).

For $p \in (1,2)$, in order to recover the parameter γ at the boundary, we need to consider the ε-expansion of the DN-map using u_{ε} described in (2.10). As in the earlier case, we start with $w \in W^{1,2}(\Omega)$. Using the estimate (2.13), we obtain

$$
\begin{aligned}
\Big| \int_{\Omega} \gamma(x)(|\nabla u_{\varepsilon}|^{p-2}\nabla u_{\varepsilon} &- \varepsilon^{1-p}|\nabla u_0|^{p-2}\nabla u_0) \cdot \overline{\nabla w} \ dx \Big| \\
&\leq C\varepsilon^{3-2p} \int_{\Omega} (|\varepsilon \nabla u_{\varepsilon}|^{p-2} + |\nabla u_0|^{p-2})|\nabla v||\nabla w| \ dx = \mathcal{O}(\varepsilon^{3-2p}).
\end{aligned} \tag{2.16}
$$

With the help of expansion (2.10) and above estimate (2.16), we get

$$
\begin{aligned}
\langle \Lambda_{\sigma,\gamma}(\varepsilon^{-1}f), w\big|_{\partial\Omega} \rangle &= \int_{\Omega} (\sigma(x) + \gamma(x)|\nabla u_{\varepsilon}|^{p-2})\nabla u_{\varepsilon} \cdot \overline{\nabla w} \ dx \\
&= \varepsilon^{-1} \int_{\Omega} \sigma(x)\nabla u_0 \cdot \overline{\nabla w} \ dx \ + \ \varepsilon^{1-p} \int_{\Omega} \sigma(x)\nabla v \cdot \overline{\nabla w} \ dx \\
&\quad + \varepsilon^{1-p} \int_{\Omega} \gamma(x)|\nabla u_0|^{p-2}\nabla u_0 \cdot \overline{\nabla w} \ dx \ + \ \mathcal{O}(\varepsilon^{3-2p}).
\end{aligned}
$$

By considering the DN-map $\Lambda_{\sigma}(f)$ associated with the Dirichlet BVP (2.11), we have

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \varepsilon \langle \Lambda_{\sigma,\gamma}(\varepsilon^{-1}f), w\big|_{\partial\Omega} \rangle &= \int_{\Omega} \sigma(x)\nabla u_0 \cdot \overline{\nabla w} \ dx \\
&= \langle \Lambda_{\sigma}(f), w\big|_{\partial\Omega} \rangle.
\end{aligned} \tag{2.17}
$$

Hence we conclude that the DN-map $\Lambda_{\sigma}(f)$ is known from the knowledge of given DN-map $\Lambda_{\sigma,\gamma}$, for all values of p under consideration.

3. Proof for the Theorem 1.1

In this section, we provide the reconstruction procedure of the coefficients σ, and γ. In order to prove the Theorem 1.1, we shall consider two separate cases based on the range of p.

3.1. **Reconstruction of σ.** For the reconstruction of σ from the knowledge of the DN-map $\Lambda_{\sigma}(f)$ associated to BVP (2.6), one can refer to [26, Theorem 5.1].

3.2. **Reconstruction of γ.** Using the ε-expansion of the DN-map, we will give a procedure for the recovery of the parameter γ at the boundary $\partial\Omega$. For this, our idea is to find a suitable integral identity which is known from the knowledge of the DN-map. By the asymptotic expansion of the solution along with appropriate choice of sequence of solutions of conductivity equation, the desired reconstruction of γ at the boundary of Ω is carried out.

3.2.1. *Case: $p > 2$.* Let us consider $w \in W^{2,m}(\Omega)$ such that it satisfies

$$
\nabla \cdot (\sigma \nabla w) = 0 \quad \text{on } \Omega. \tag{3.1}
$$

By multiplying equation (2.3) with w and integrating the resulting equation over Ω, we get

$$
\int_{\Omega} w(\nabla \cdot (\sigma\nabla u_{\varepsilon} + \gamma|\nabla u_{\varepsilon}|^{p-2}\nabla u_{\varepsilon})) \ dx = 0
$$

and it further yields the following equation by using the integration by parts

$$\int_\Omega \nabla w \cdot (\sigma \nabla u_\varepsilon + \gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx = \int_{\partial\Omega} w(\sigma + \gamma |\nabla u_\varepsilon|^{p-2}) \partial_\nu u_\varepsilon \, dS.$$

Using (3.1), the above expression becomes

$$\int_\Omega \nabla w \cdot (\gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx = \int_{\partial\Omega} w(\sigma + \gamma |\nabla u_\varepsilon|^{p-2}) \partial_\nu u_\varepsilon \, dS - \varepsilon \langle \Lambda_\sigma(f), w|_{\partial\Omega} \rangle. \qquad (3.2)$$

We observe that the right hand side terms of (3.2) are known by the prescribed DN-maps $\Lambda_{\sigma,\gamma}$ and Λ_σ respectively. As a result, the left hand side of (3.2) is known to us. Let us define

$$K^* := \int_\Omega \nabla w \cdot (\gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx.$$

The term K^* is known from the knowledge of the DN-map associated with perturbed p-Laplacian problem.

We can further rewrite K^* as

$$K^* = \int_\Omega \nabla w \cdot (\gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx - \varepsilon^{p-1} \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx$$
$$+ \varepsilon^{p-1} \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx.$$

By taking into account the estimate (2.14), we deduce that

$$K^* = \varepsilon^{p-1} \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx + \mathcal{O}(\varepsilon^{2p-3}).$$

Further, we define $I(u_0, w)$ as follows:

$$I(u_0, w) := \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx. \qquad (3.3)$$

As a consequence, we have

$$K^* = \varepsilon^{p-1} I(u_0, w) + \mathcal{O}(\varepsilon^{2p-3}), \qquad (3.4)$$

where $w \in W^{2,m}(\Omega)$ and u_0 satisfies the BVP (2.6). Furthermore, we observe that

$$I(u_0, w) = \lim_{\varepsilon \to 0^+} \varepsilon^{1-p} \left(K^* - \mathcal{O}(\varepsilon^{2p-3}) \right)$$

and subsequently, $I(u_0, w)$ is known to us.

Let us choose the functions $u_1, u_2, u_3 \in C^\infty(\Omega)$ such that

$$\nabla \cdot (\sigma \nabla u_j) = 0, \quad j = 1, 2, 3. \qquad (3.5)$$

In addition, we assume that u_1 is real valued and ∇u_1 does not have any zeros. Since $I(u_0, w)$ is known for any $w \in W^{2,m}(\Omega)$. In particular $I(u_0, u_3)$ is known. By defining J_1 and J_2 as follows:

$$J_1(u_1, u_2, u_3) := \frac{2}{p-2} \partial_{\bar{z}} [I(u_1 + z\overline{u_2}, u_3)] \Big|_{z=0}$$
$$= \int_\Omega \gamma |\nabla u_1|^{p-4} (\nabla u_1 \cdot \nabla u_2)(\nabla u_1 \cdot \nabla u_3) \, dx,$$

$$J_2(u_1, u_2, u_3) := \frac{2}{p-2} \partial_z [I(u_1 + zu_2, u_3)] \Big|_{z=0}$$
$$= \int_\Omega \gamma |\nabla u_1|^{p-4} [(\overline{\nabla u_1} \cdot \nabla u_2)(\nabla u_1 \cdot \nabla u_3) + \frac{2}{p-2} |\nabla u_1|^2 (\nabla u_2 \cdot \nabla u_3)] \, dx,$$

we conclude that the functionals $J_1(u_1, u_2, u_3)$ and $J_2(u_1, u_2, u_3)$ will be known to us. Furthermore, we define

$$\beta_{u_1}(x) := \gamma |\nabla u_1(x)|^{p-2} \tag{3.6}$$

and

$$K(u_1, u_2, u_3) := \frac{p-2}{2} [J_2(u_1, u_2, u_3) - J_1(u_1, u_2, u_3)].$$

Hence K can be represented as

$$K(u_1, u_2, u_3) = \int_\Omega \beta_{u_1}(\nabla u_2 \cdot \nabla u_3) \ dx.$$

For each choice of u_2 and u_3 satisfying (3.5), the term $K(u_1, u_2, u_3)$ is known. Since, σ is real valued, therefore by choosing $u_3 = \overline{u_2}$, we find that

$$K(u_1, u_2, \overline{u_2}) = \int_\Omega \beta_{u_1} \nabla u_2 \cdot \overline{\nabla u_2} \ dx, \tag{3.7}$$

is known to us.

Let $x_0 \in \partial\Omega$ be arbitrary, but fixed. From the knowledge of $K(u_1, u_2, \overline{u_2})$ in (3.7), we will recover β_{u_1} at x_0. By using (3.6), we recover $\gamma(x_0)$ from $\beta_{u_1}(x_0)$. Following the similar approach presented in [30, 7], we can construct a sequence of solutions $\{u_M\}$ satisfying the conductivity equation (3.5) with coefficient frozen at the boundary point x_0. Suppose ρ is a C^1 function and we choose ζ as a unit tangent vector to $\partial\Omega$ at x_0 such that

$$\zeta \cdot \nabla \rho(x_0) = 0, \qquad |\zeta| = |\nabla \rho(x_0)|. \tag{3.8}$$

Furthermore, we choose a cut-off function $\eta \in C^\infty(\mathbb{R}^n)$ with $0 \le \eta(x - x_0) \le 1$ such that

$$\eta(x - x_0) = \begin{cases} 1, & |x - x_0| \le \frac{1}{2}, \\ 0, & |x - x_0| \ge 1. \end{cases}$$

For large positive numbers M and N, we define $\eta_M(x) = \eta(Mx)$ and $N = N(M)$ with $\frac{M}{N} = \mathcal{O}(1)$ as $M \to \infty$. Subsequently, since η is supported in the ball $B(x_0, 1)$, following the similar approach in [30, Lemma 2.2], we can deduce

$$M^{n-1}N \int_\Omega \eta(M(x - x_0)) \exp(-2N\rho(x)) \ dx \longrightarrow \frac{1}{2} \int_{\mathbb{R}^{n-1}} \eta(x', x_0) \ dx' \quad \text{as } M \to \infty. \tag{3.9}$$

For each $M \in \mathbb{N}$ and $M \ge 2$, we define a solution of (3.5) as

$$u_M := C_{M,N}(w_0 + w_1), \tag{3.10}$$

where $C_{M,N}$ is a scaling constant and it is described by

$$C_{M,N} = \sqrt{\frac{M^{n-1}N^{-1}}{c_\eta}}$$

in which c_η is given by

$$c_\eta := \frac{1}{2} \int_{\mathbb{R}^{n-1}} \eta(x', x_0) \ dx'.$$

In (3.10), we consider w_0 as an approximate solution of (3.5) and it assumes the following form

$$w_0(x) = \eta(M(x - x_0)) \exp\left(N(i\zeta \cdot x - \rho(x))\right), \qquad x \in \mathbb{R}^n.$$

The above form of w_0 is justified thanks to (3.8). As a consequence, we observe that w_1 satisfies

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla w_1(x)) = -\nabla \cdot (\sigma(x)\nabla w_0(x)), & x \in \Omega, \\ w_1(x) = 0, & x \in \partial\Omega, \end{cases}$$

in addition with [30]

$$\int_\Omega |\nabla w_0|^2 \, dx = \mathcal{O}(M^{1-n}N), \qquad \int_\Omega |\nabla w_1|^2 \, dx = o(M^{1-n}N) \quad \text{as } M \to \infty. \tag{3.11}$$

Note that $w_0 \in C_c^1(\mathbb{R}^n)$ is vanishing outside a small ball $B(x_0, 1/M)$. For this choice of sequence $\{u_M\}$ in (3.10) we get

$$\int_\Omega \beta_{u_1} \nabla u_M \cdot \overline{\nabla u_M} \, dx = \frac{M^{n-1}N^{-1}}{c_\eta} \int_\Omega \beta_{u_1}(|\nabla w_0|^2 + \nabla w_0 \cdot \overline{\nabla w_1} + \overline{\nabla w_0} \cdot \nabla w_1 + |\nabla w_1|^2) \, dx. \tag{3.12}$$

We observe that

$$\left| \int_\Omega \beta_{u_1}(\nabla w_0 \cdot \overline{\nabla w_1} + \overline{\nabla w_0} \cdot \nabla w_1 + |\nabla w_1|^2) \, dx \right|$$
$$\leq C(\|\nabla w_0\|_{L^2(\Omega)} + \|\nabla w_1\|_{L^2(\Omega)})\|\nabla w_1\|_{L^2(\Omega)} = o(M^{1-n}N),$$

where we have used the Cauchy-Schwarz inequality and estimates (3.11).

For the first term on right hand side of (3.12), we have

$$\nabla w_0 = N(i\zeta - \nabla\rho)\eta_M \exp\big(N(i\zeta \cdot x - \rho(x))\big) + M(\nabla\eta)(M\cdot)\exp\big(N(i\zeta \cdot x - \rho(x))\big)$$
$$=: T_1 + T_2,$$

and consequently,

$$\int_\Omega \beta_{u_1}(x)|\nabla w_0|^2 \, dx = \beta_{u_1}(x_0)\int_\Omega |\nabla w_0|^2 \, dx + \int_\Omega (\beta_{u_1}(x) - \beta_{u_1}(x_0))|\nabla w_0|^2 \, dx$$
$$= \beta_{u_1}(x_0)\int_\Omega |T_1|^2 \, dx + \beta_{u_1}(x_0)\int_\Omega (T_1 \cdot \overline{T_2} + \overline{T_1} \cdot T_2 + |T_2|^2) \, dx$$
$$+ \int_\Omega (\beta_{u_1}(x) - \beta_{u_1}(x_0))|\nabla w_0|^2 \, dx$$
$$= \beta_{u_1}(x_0)\int_\Omega |T_1|^2 \, dx + o(M^{1-n}N),$$

where we have used the fact that $\|T_1\|_{L^2(\Omega)}^2 = \mathcal{O}(M^{1-n}N)$, and $\|T_2\|_{L^2(\Omega)}^2 = o(M^{1-n}N)$ as $M \to \infty$, and the continuity of β_{u_1} at x_0. Moreover, we have

$$M^{n-1}N^{-1}\int_\Omega |T_1|^2 \, dx = M^{1-n}N\int_\Omega \eta(M(x - x_0))^2 \exp(-2N\rho)(1 + |\nabla\rho|^2) \, dx$$
$$= M^{1-n}N\bigg(2\int_\Omega \eta(M(x - x_0))^2 \exp(-2N\rho) \, dx$$
$$+ \int_\Omega \eta(M(x - x_0))^2 \exp(-2N\rho)(|\nabla\rho|^2 - |\nabla\rho(x_0)|^2) \, dx\bigg),$$

where the first term and second term in the right hand side converges to c_η (due to (3.9)) and $\mathcal{O}(1)$ (since $x \mapsto |\nabla\rho|^2$ is continuous at x_0) respectively as $M \to \infty$. Hence we obtain

$$\frac{M^{n-1}N^{-1}}{c_\eta}\int_\Omega \beta_{u_1}(x)|\nabla w_0|^2 \, dx \longrightarrow \beta_{u_1}(x_0) \quad \text{as } M \to \infty. \tag{3.13}$$

As a consequence, we conclude that from (3.12)

$$\int_\Omega \beta_{u_1}(x)\nabla u_M \cdot \overline{\nabla u_M} \, dx \longrightarrow \beta_{u_1}(x_0) \quad \text{as } M \to \infty.$$

In (3.7) replacing u_2 by u_M, we are able to find $\beta_{u_1}(x_0)$. Hence $\gamma(x_0)$ is known from (3.6). Since $x_0 \in \partial\Omega$ is arbitrary, one can recover γ at the boundary of the domain.

3.2.2. *Case:* $1 < p < 2$. Let w be any function such that $w \in W^{1,2}(\Omega)$ and it satisfies

$$\nabla \cdot (\sigma \nabla w) = 0 \quad \text{in } \Omega. \tag{3.14}$$

We multiply the equation (2.8) by w and integrate the resulting equation over Ω to obtain

$$\int_\Omega w \Big(\nabla \cdot (\sigma \nabla u_\varepsilon + \gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \Big) \, dx = 0. \tag{3.15}$$

The above equation reduces to

$$\int_\Omega \nabla w \cdot (\sigma \nabla u_\varepsilon + \gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx = \int_{\partial\Omega} w(\sigma + \gamma |\nabla u_\varepsilon|^{p-2}) \partial_\nu u_\varepsilon \, dS,$$

where we have applied integration by parts. With the help of (3.14) we get

$$\int_\Omega \nabla w \cdot (\gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx = \int_{\partial\Omega} w(\sigma + \gamma |\nabla u_\varepsilon|^{p-2}) \partial_\nu u_\varepsilon \, dS - \varepsilon^{-1} \langle \Lambda_\sigma(f), w|_{\partial\Omega} \rangle. \tag{3.16}$$

Since the DN-maps $\Lambda_{\sigma,\gamma,p}$ and Λ_σ defined in a weak sense in (2.2) are given to us, therefore the right hand side of (3.16) is also known.

Let us define

$$L^* := \int_\Omega \nabla w \cdot (\sigma \nabla u_\varepsilon + \gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx.$$

From the knowledge of the DN-map associated with perturbed problem, we know the integral L^*. Moreover, L^* can be represented as follows

$$L^* = \int_\Omega \nabla w \cdot (\gamma |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) \, dx - \varepsilon^{1-p} \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx$$

$$+ \varepsilon^{1-p} \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx$$

$$= \varepsilon^{1-p} \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx + \mathcal{O}(\varepsilon^{3-2p})$$

$$= \varepsilon^{1-p} I(u_0, w) + \mathcal{O}(\varepsilon^{3-2p}),$$

where the estimate (2.16) is used and $I(u_0, w)$ is defined in (3.3). Hence

$$I(u_0, w) = \int_\Omega \nabla w \cdot (\gamma |\nabla u_0|^{p-2} \nabla u_0) \, dx \tag{3.17}$$

is known to us. As we have described a reconstruction procedure of the coefficient at the boundary using $I(u_0, w)$ in case of $p > 2$, in a similar way, we can recover the coefficient γ at the boundary $\partial\Omega$ from (3.17) for $1 < p < 2$.

4. Proof for the Theorem 1.2

In this section, we present a procedure for the reconstruction of normal derivative of γ at the boundary point x_0 of Ω under certain regularity assumption on γ. Unlike in [26, 28], we do not require the values of γ in the neighbourhood of x_0. However, we require certain regularity assumption on γ around x_0. We also use the fact that γ is known at x_0 as described in section 3.

4.1. **Reconstruction of normal derivative of σ.** Let $\alpha \in \mathbb{R}^n \backslash \{0\}$. The reconstruction of $\partial_\alpha \sigma$ from the knowledge of the DN-map $\Lambda_\sigma(f)$ associated to BVP (2.6), can be done using [27, 28].

4.2. Reconstruction of normal derivative of γ. For every $x \in \mathbb{R}^n$, we write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. Without loss of generality, we assume that $x_0 = 0$ and $\partial\Omega$ is flat around x_0. More precisely, there exists $\delta > 0$ such that

$$\Omega \cap B_\delta(x_0) = \{x \in B_\delta(x_0) : x_n > 0\}. \tag{4.1}$$

In fact, using the boundary normal coordinates [24], we can locally transform the coefficients to the form (4.1). Let $\zeta = (\zeta', 0)$ be an unit tangent vector to $\partial\Omega$ at x_0.

Motivated by [30, 28], we wish to choose the approximate solutions Φ_N and Ψ_N having oscillating boundary data supported in a neighbourhood of x_0. For $N \in \mathbb{N}$ and $x \in \partial\Omega$, we consider

$$\Phi_N(x) := \exp\left(iNx' \cdot \zeta' - Nx_n\right)\eta(\sqrt{N}x'),$$
$$\Psi_N(x) := \exp\left(i\frac{N}{2}x' \cdot \zeta' - \frac{N}{2}x_n\right)\eta(\sqrt{N}x'); \tag{4.2}$$

corresponding to the boundary data

$$f_N(x') = \exp\left(iNx' \cdot \zeta'\right)\eta(\sqrt{N}x'), \qquad g_N(x') = \exp\left(i\frac{N}{2}x' \cdot \zeta'\right)\eta(\sqrt{N}x'); \tag{4.3}$$

respectively, where $\eta \in C^2$ is a cutoff function supported in a unit ball such that

$$0 \leq \eta(x') \leq 1, \quad \int_{\mathbb{R}^{n-1}} \eta^2(x') \, dx' = 1. \tag{4.4}$$

We have observed that (cf. (3.7))

$$K(u_1, u_2, u_3) = \int \beta_{u_1}(x)\nabla u_2 \cdot \nabla u_3 \, dx$$

is known for any u_1, u_2 and u_3 satisfying the conductivity equation (3.5). Let us denote $h := \beta_{u_1}$. To prove the Theorem 1.2, we use the following proposition:

Proposition 4.1. *Let $\partial\Omega$ be locally C^2 at x_0. Suppose $D_{x'}^{\alpha'} D_{x_n}^{\alpha_n}\gamma$ is continuous around x_0 for any multi-index (α', α_n) such that $|\alpha'| + 2\alpha_n \leq 2$. Then*

$$\lim_{N \to \infty} N^{\frac{n-1}{2}}\left(2K(u_1, u_2, \overline{u_2}) - K(u_1, u_3, \overline{u_3})\right) = \frac{1}{2}\partial_{x_n}h(x_0) + \frac{3}{2}h(x_0)\int_{\mathbb{R}^{n-1}} |\nabla_{x'}\eta|^2 \, dx'.$$

Proof. Let $\xi \in C^\infty[0, \infty)$ be a cutoff function which satisfies $0 \leq \xi \leq 1$ with

$$\xi(x_n) = \begin{cases} 1, & 0 \leq x_n \leq \frac{1}{2}, \\ 0, & x_n \geq 1. \end{cases}$$

For each $N \in \mathbb{N}$, we define $\xi_N(x_n) := \xi(\sqrt{N}x_n)$. Our computation is heavily relying on the choice of approximate solutions of conductivity equation (3.5). As a consequence, we consider

$$u_2 = \xi_N\Psi_N + s_N, \qquad u_3 = \xi_N\Phi_N + r_N, \tag{4.5}$$

where Φ_N and Ψ_N are defined in (4.2), and s_N satisfies

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla s_N) = -\nabla \cdot (\sigma(x)\nabla(\xi_N\Psi_N)), & x \in \Omega, \\ s_N = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

and similarly, r_N satisfies

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla r_N) = -\nabla \cdot (\sigma(x)\nabla(\xi_N\Phi_N)), & x \in \Omega, \\ r_N = 0, & x \in \partial\Omega. \end{cases} \tag{4.7}$$

Taking into account (4.5), we have

$$2K(u_1, u_2, \overline{u_2}) - K(u_1, u_3, \overline{u_3})$$

$$= \int_\Omega h(x) \Big\{ 2\nabla(\xi_N\Psi_N + s_N) \cdot \overline{\nabla(\xi_N\Psi_N + s_N)} - \nabla(\xi_N\Phi_N + r_N) \cdot \overline{\nabla(\xi_N\Phi_N + r_N)} \Big\} \, dx$$

$$= \int_\Omega h(x) \Big\{ 2\nabla(\xi_N\Psi_N) \cdot \overline{\nabla(\xi_N\Psi_N)} - \nabla(\xi_N\Phi_N) \cdot \overline{\nabla(\xi_N\Phi_N)} \Big\} \, dx$$

$$+ 2\int_\Omega h(x)\nabla s_N \cdot \overline{\nabla(\xi_N\Psi_N)} \, dx - \int_\Omega h(x)\nabla r_N \cdot \overline{\nabla(\xi_N\Phi_N)} \, dx$$

$$+ 2\int_\Omega h(x)\nabla(\xi_N\Psi_N) \cdot \overline{\nabla s_N} \, dx - \int_\Omega h(x)\nabla(\xi_N\Psi_N) \cdot \overline{\nabla r_N} \, dx$$

$$+ 2\int_\Omega h(x)\nabla s_N \cdot \overline{\nabla s_N} \, dx - \int_\Omega h(x)\nabla r_N \cdot \overline{\nabla r_N} \, dx$$

$$=: T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7,$$

where T_2 is given by

$$T_2 = 2\int_\Omega h(x)\nabla s_N \cdot \overline{\nabla(\xi_N\Psi_N)} \, dx,$$

and using the arguments from [27, 28] we have $T_2 = \mathcal{O}(N^{-\frac{n-1}{2}})$ as $N \to \infty$. Following the similar argument in [19], we can demonstrate that $N \to \infty$,

$$T_3 = -\int_\Omega h(x)\nabla r_N \cdot \overline{\nabla(\xi_N\Phi_N)} \, dx = \mathcal{O}(N^{-\frac{n-1}{2}}),$$

$$T_4 = 2\int_\Omega h(x)\nabla(\xi_N\Psi_N) \cdot \overline{\nabla s_N} \, dx = \mathcal{O}(N^{-\frac{n-1}{2}}),$$

$$T_5 = -\int_\Omega h(x)\nabla(\xi_N\Psi_N) \cdot \overline{\nabla r_N} \, dx = \mathcal{O}(N^{-\frac{n-1}{2}}),$$

and using the equation (4.6)-(4.7) and estimates given for T_2 and T_3 one can show that as $N \to \infty$,

$$T_6 = 2\int_\Omega h(x)\nabla s_N \cdot \overline{\nabla s_N} \, dx = \mathcal{O}(N^{-\frac{n-1}{2}}),$$

$$T_7 = -\int_\Omega h(x)\nabla r_N \cdot \overline{\nabla r_N} \, dx = \mathcal{O}(N^{-\frac{n-1}{2}}).$$

We set the scaling transformation

$$y_i = \sqrt{N}x_i, \quad i = 1, 2, \cdots, n-1, \qquad y_n = Nx_n. \tag{4.8}$$

Taking into account the supports of $\xi_N\Psi_N(y)$ and $\xi_N\Phi_N(y)$, we define

$$\Omega_N = \Big\{ x : |x'| \le \frac{1}{\sqrt{N}}, 0 \le x_n \le \frac{1}{2\sqrt{N}} \Big\}, \quad \Omega'_N = \Big\{ x : |x'| \le \frac{1}{\sqrt{N}}, \frac{1}{2\sqrt{N}} \le x_n \le \frac{1}{\sqrt{N}} \Big\}$$

and hence, we can rewrite T_1 as

$$T_1 = \int_{\Omega_N} h(x) \Big\{ 2\nabla\Psi_N \cdot \overline{\nabla\Psi_N} - \nabla\Phi_N \cdot \overline{\nabla\Phi_N} \Big\} \, dx$$

$$+ \int_{\Omega'_N} h(x) \Big\{ 2\nabla(\xi_N\Psi_N) \cdot \overline{\nabla(\xi_N\Psi_N)} - \nabla(\xi_N\Phi_N) \cdot \overline{\nabla(\xi_N\Phi_N)} \Big\} \, dx$$

$$= T_{11} + T_{12},$$

where T_{11} and T_{12} are given by

$$T_{11} := \int_{\Omega_N} h(x) \Big\{ 2\nabla\Psi_N \cdot \overline{\nabla\Psi_N} - \nabla\phi_N \cdot \overline{\nabla\Phi_N} \Big\} \, dx,$$

$$T_{12} := \int_{\Omega'_N} h(x)\left\{2\nabla(\xi_N\Psi_N)\cdot\overline{\nabla(\xi_N\Psi_N)} - \nabla(\xi_N\Phi_N)\cdot\overline{\nabla(\xi_N\Phi_N)}\right\}dx.$$

T_{11} can be further represented by

$$T_{11} = \int_{\Omega_N} h(x)\left\{2\nabla\Psi_N\cdot\overline{\nabla\Psi_N} - \nabla\Phi_N\cdot\overline{\nabla\Phi_N}\right\}dx$$

$$= N^2\int_{\Omega_N} h(x',x_n)\eta^2(\sqrt{N}x')\left(\exp(-Nx_n) - 2\exp(-2Nx_n)\right)dx$$

$$+ N\int_{\Omega_N} h(x',x_n)|(\nabla_{y'}\eta)(\sqrt{N}x')|^2\left(2\exp(-Nx_n) - \exp(-2Nx_n)\right)dx,$$

and using the scaling transformation introduced in (4.8), we get

$$N^{\frac{n-1}{2}}T_{11} = N\int_0^{\sqrt{N}/2}\int_{|y'|\le 1} h\left(\frac{y'}{\sqrt{N}}, \frac{y_n}{N}\right)\eta^2(y')\left(\exp(-y_n) - 2\exp(-2y_n)\right)dy'dy_n$$

$$+ \int_0^{\sqrt{N}/2}\int_{|y'|\le 1} h\left(\frac{y'}{\sqrt{N}}, \frac{y_n}{N}\right)|(\nabla_{y'}\eta)(y')|^2\left(2\exp(-y_n) - \exp(-2y_n)\right)dy'dy_n.$$

$$(4.9)$$

From the regularity assumption on h, we can expand h around $x_0 = 0$. More precisely, we can write

$$h\left(\frac{y'}{\sqrt{N}}, \frac{y_n}{N}\right) = h(x_0) + \frac{1}{\sqrt{N}}\sum_{|\alpha'|=1}\partial_{x'^{\alpha'}}^{|\alpha'|}h(x_0)y'^{\alpha'} + \frac{1}{N}\partial_{x_n}h(x_0)y_n$$

$$+ \frac{1}{N}\sum_{|\alpha'|=2}\frac{1}{\alpha!}\partial_{x'^{\alpha'}}^{|\alpha'|}h(x_0)y'^{\alpha'} + o(N^{-1}) \qquad \text{as } N\to\infty.$$

$$(4.10)$$

It is observed that as $N\to\infty$,

$$\int_0^{\sqrt{N}/2}(\exp-y_n - 2\exp-2y_n)\,dy_n = \mathcal{O}\left(\exp(-\sqrt{N}/2)\right),$$

$$\int_0^{\sqrt{N}/2}\left(2\exp(-y_n) - \exp(-2y_n)\right)dy_n = \frac{3}{2} + \mathcal{O}\left(\exp(-\sqrt{N}/2)\right),$$

$$\int_0^{\sqrt{N}/2}y_n\left(\exp(-y_n) - 2\exp(-2y_n)\right)dy_n = \frac{1}{2} + \mathcal{O}(\sqrt{N}e^{-\sqrt{N}/2}).$$

With the help of above estimates and expansion (4.10), and using the properties of η introduced in (4.4), we end up with

$$N^{\frac{n-1}{2}}\int_{\Omega_N} h(x)\left\{2\nabla\Psi_N\cdot\overline{\nabla\Psi_N} - \nabla\Phi_N\cdot\overline{\nabla\Phi_N}\right\}dx$$

$$= \frac{1}{2}\partial_{x_n}h(x_0) + \frac{3}{2}h(x_0)\int_{\mathbb{R}^{n-1}}|(\nabla_{x'}\eta)|^2\,dx' + o(1) \qquad \text{as } N\to\infty.$$

$$(4.11)$$

We see that

$$\nabla(\xi_N\Psi_N) = \exp\left(i\frac{N}{2}x'\cdot\zeta' - \frac{N}{2}x_n\right)\left[\begin{matrix}\left\{i\frac{N}{2}\zeta'\eta(\sqrt{N}x') + \sqrt{N}(\nabla_{y'}\eta)(\sqrt{N}x')\right\}\xi(\sqrt{N}x_n) \\ \left\{-\frac{N}{2}\xi(\sqrt{N}x_n) + \sqrt{N}(\nabla_{x_n}\xi)(\sqrt{N}x_n)\right\}\eta(\sqrt{N}x')\end{matrix}\right],$$

and

$$\nabla(\xi_N \Phi_N) = \exp\left(iNx' \cdot \zeta' - Nx_n\right) \begin{bmatrix} \left\{iN\zeta'\eta(\sqrt{N}x') + \sqrt{N}(\nabla_{y'}\eta)(\sqrt{N}x')\right\}\xi(\sqrt{N}x_n) \\ \left\{-N\xi(\sqrt{N}x_n) + \sqrt{N}(\nabla_{x_n}\xi)(\sqrt{N}x_n)\right\}\eta(\sqrt{N}x') \end{bmatrix}.$$

As a consequence, we have

$$T_{12} = \int_{\Omega'_N} h(x) \left\{ 2\nabla(\xi_N \Psi_N) \cdot \overline{\nabla(\xi_N \Psi_N)} - \nabla(\xi_N \Phi_N) \cdot \overline{\nabla(\xi_N \Phi_N)} \right\} dx$$

$$= N^2 \int_{\Omega'_N} h(x', x_n)\xi^2(\sqrt{N}x_n)\eta^2(\sqrt{N}x')\{\exp(-Nx_n) - 2\exp(-2Nx_n)\} \, dx$$

$$+ N \int_{\Omega'_N} h(x', x_n)\xi^2(\sqrt{N}x_n)|\nabla_{y'}\eta|^2(\sqrt{N}x')\{2\exp(-Nx_n) - \exp(-2Nx_n)\} \, dx$$

$$- 2N^{3/2} \int_{\Omega'_N} h(x)\xi(\sqrt{N}x_n)\eta(\sqrt{N}x')(\nabla_{x_n}\xi)(\sqrt{N}x_n)\{\exp(-Nx_n) - \exp(-2Nx_n)\} \, dx$$

$$+ N \int_{\Omega'_N} h(x', x_n)\eta^2(\sqrt{N}x')|(\nabla_{x_n}\xi)(\sqrt{N}x_n)|^2\{2\exp(-Nx_n) - \exp(-2Nx_n)\} \, dx$$

$$=: I + II + III + IV.$$

Note that

$$I = N^2 \int_{\Omega'_N} h(x', x_n)\xi^2(\sqrt{N}x_n)\eta^2(\sqrt{N}x')\{\exp(-Nx_n) - 2\exp(-2Nx_n)\} \, dx$$

$$= \frac{N}{N^{\frac{n-1}{2}}} \int_{\frac{\sqrt{N}}{2}}^{\sqrt{N}} \int_{|y'|\leq 1} h\left(\frac{y'}{\sqrt{N}}, \frac{y_n}{N}\right) \xi^2\left(\frac{y_n}{\sqrt{N}}\right) \eta^2(y')\{\exp(-y_n) - 2\exp(-2y_n)\} \, dy' dy_n,$$

where we have used the scaling transformation (4.8). Taking into account the inequality

$$\int_{\frac{\sqrt{N}}{2}}^{\sqrt{N}} \{\exp(-y_n) - 2\exp(-2y_n)\} \, dy_n = \mathcal{O}\left(\exp(-\sqrt{N}/2)\right) \quad \text{as } N \to \infty,$$

we can deduce that

$$|I| \leq C\exp(-\sqrt{N}/2) \quad \text{as } N \to \infty,$$

for some constant C. Following the similar argument as above, we see that

$$|II| + |III| + |IV| \leq C\exp(-\sqrt{N}/2) \quad \text{as } N \to \infty.$$

Subsequently, we can conclude that

$$T_{12} = \mathcal{O}\left(\exp(-\sqrt{N}/2)\right) \quad \text{as } N \to \infty. \tag{4.12}$$

Hence combining the above estimates (4.11) and (4.12), (4.9) reduces to

$$\lim_{N \to \infty} N^{\frac{n-1}{2}}\left(2K(u_1, u_2, \overline{u_2}) - K(u_1, u_3, \overline{u_3})\right)$$
$$= \frac{1}{2}\partial_{x_n}h(x_0) + \frac{3}{2}h(x_0) \int_{\mathbb{R}^{n-1}} |(\nabla_{x'}\eta)|^2 \, dx'. \tag{4.13}$$

Hence the Proposition 4.1 follows. $\qquad\square$

In section 3, we have recovered h at the boundary of Ω, hence $h(x_0)$ is known to us. The cutoff function η introduced in (4.4) is arbitrary, the right hand side term

$$h(x_0) \int_{\mathbb{R}^{n-1}} |\nabla_{x'}\eta(x')|^2 \, dx'$$

of (4.13) is known. Hence, the normal derivative $\partial_{x_n} h(x_0)$ is determined. As a consequence, $\partial_{x_n} \gamma(x_0)$ can be reconstructed and hence the Theorem 1.2 follows.

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