Fuzzy Sets Approach in Mathematical Finance

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Outline

- Fuzzy Forecasts
- Random Coefficient Black-Scholes PDE
- A Black-Scholes model with fuzzy volatility
- Option Pricing with fuzzy volatility
- Asset-or-nothing Option Pricing based on Fuzziness

Volatility

Volatility

Asset prices change over time and move relatively slow when conditions are calm and they move fast when there is more news/uncertainty and trading.

The volatility of prices refers to the rate at which prices change.

Forecast of future prices and volatility is of interest in option pricing and in risk management.

Asset Price Dynamics, Volatility, and Prediction: Taylor (2005) .

Motivation

The following numerical example gives the motivation for using fuzzy coefficient time series models in forecasting. Usually point estimates of the parameters are used to obtain the minimum mean square error forecasts. The MMSE forecast of the future $(n+1)^{th}$ observation based on $y_1, \ldots, y_n$ is $y_{n+1} = \hat{\mu} + \hat{\phi}(y_n - \hat{\mu})$. Here the parameter variability is not taken into account.

Example

Consider the daily average number of truck manufacturing defects.

Table: Daily Average Number of Truck Manufacturing Defects

1.20	1.50	1.54	2.70	1.95	2.40	3.44	2.83	1.76
2.00	2.09	1.89	1.80	1.25	1.58	2.25	2.50	2.05
1.46	1.54	1.42	1.52	1.40	1.51	1.08	1.27	1.18
1.39	1.42	2.08	1.85	1.82	2.07	2.32	1.23	2.91
1.77	1.61	1.25	1.15	1.37	1.79	1.68	1.78	1.84

The fitted model using the first 44 observations is given as:

$$
\begin{aligned}
y_{n+1} - \hat{\mu} &= \hat{\phi}(y_n - \hat{\mu}) + a_t \\
\widehat{\mu} &= 1.769(0.124) \\
\widehat{\phi} &= 0.433(0.139) \\
\overline{\mu\left(\alpha\right)} &= \left[\mu_1\left(\alpha\right), \mu_2\left(\alpha\right)\right] \\
&= \left[1.769 - Z^*_{\frac{\alpha}{2}}(0.124), 1.769 + Z^*_{\frac{\alpha}{2}}(0.124)\right] \\
\overline{\phi\left(\alpha\right)} &= \left[\phi_1\left(\alpha\right), \phi_2\left(\alpha\right)\right] \\
&= \left[0.433 - Z^*_{\frac{\alpha}{2}}(0.139), 0.433 + Z^*_{\frac{\alpha}{2}}(0.139)\right]
\end{aligned}
$$

The fuzzy forecast value $\overline{y}_{44}(1) = \left[y_{45,1}, y_{45,2}\right]$ is:

$$\begin{aligned}
\overline{y}_{44}(1) &= \overline{\mu} + \overline{\phi}(y_{44} - \overline{\mu}) = \overline{\mu} + \overline{\phi}(y_{44} - \overline{\mu}) \\
&= [\mu_1(\alpha), \mu_2(\alpha)] + [\phi_1(\alpha), \phi_2(\alpha)]\,(1.78 - (\mu_1(\alpha),\ \mu_2(\alpha))) \\
&= [\mu_1(\alpha), \mu_2(\alpha)] \\
&\qquad + [\phi_1(\alpha), \phi_2(\alpha)]\,(1.78 - \mu_2(\alpha), 1.78 - \mu_1(\alpha))
\end{aligned}$$

$y_{45,1}$ and $y_{45,2}$ are given from above equation as

$$\begin{aligned}
y_{45,1} &= \mu_1(\alpha) + c(\alpha) \\
y_{45,2} &= \mu_2(\alpha) + d(\alpha)
\end{aligned}$$

where,

$$\begin{aligned}
c(\alpha) &= Min\left[\phi_1(\alpha)\,(1.78 - \mu_1(\alpha))\,, \phi_1(\alpha)\,(1.78 - \mu_2(\alpha))\,,\right. \\
&\qquad \left.\phi_2(\alpha)\,(1.78 - \mu_1(\alpha))\,, \phi_2(\alpha)\,(1.78 - \mu_2(\alpha))\right] \\
d(\alpha) &= Max\left[\phi_1(\alpha)\,(1.78 - \mu_1(\alpha))\,, \phi_1(\alpha)\,(1.78 - \mu_2(\alpha))\,,\right. \\
&\qquad \left.\phi_2(\alpha)\,(1.78 - \mu_1(\alpha))\,, \phi_2(\alpha)\,(1.78 - \mu_2(\alpha))\right]
\end{aligned}$$

Table: Fuzzy forecasts

α	$Z^*_{\frac{\alpha}{2}}$	μ_1	μ_2	ϕ_1	ϕ_2	$y_{45,1}$ Fuzzy	$y_{45,2}$ Fuzzy	MMSE Lower Forecast Interval	MMSE Upper Forecast Interval
0.025	2.2414	1.4911	2.0469	0.1214	0.7446	1.2923	2.2621	0.7035	2.8440
0.050	1.9600	1.5260	2.0120	0.1606	0.7054	1.3623	2.1912	0.8379	2.7096
0.075	1.7805	1.5482	1.9898	0.1855	0.6805	1.4055	2.1475	0.9236	2.6239
0.100	1.6449	1.5650	1.9730	0.2044	0.6616	1.4374	2.1152	0.9884	2.5592
0.125	1.5341	1.5788	1.9592	0.2198	0.6462	1.4629	2.0893	1.0412	2.5063
0.150	1.4395	1.5905	1.9475	0.2329	0.6331	1.4845	2.0675	1.0864	2.4611
0.175	1.3563	1.6008	1.9372	0.2445	0.6215	1.5031	2.0485	1.1261	2.4214
0.200	1.2816	1.6101	1.9279	0.2549	0.6111	1.5197	2.0318	1.1618	2.3857
0.225	1.2133	1.6185	1.9195	0.2643	0.6017	1.5346	2.0166	1.1944	2.3531
0.250	1.1503	1.6264	1.9116	0.2731	0.5929	1.5483	2.0027	1.2245	2.3230
0.275	1.0916	1.6336	1.9044	0.2813	0.5847	1.5609	1.9899	1.2525	2.2950
0.300	1.0364	1.6405	1.8975	0.2889	0.5771	1.5727	1.9780	1.2789	2.2687
0.325	0.9842	1.6470	1.8910	0.2962	0.5698	1.5837	1.9669	1.3038	2.2437
0.350	0.9346	1.6531	1.8849	0.3031	0.5629	1.5941	1.9563	1.3275	2.2200
0.375	0.8871	1.6590	1.8790	0.3097	0.5563	1.6039	1.9463	1.3502	2.1974
0.400	0.8416	1.6646	1.8734	0.3160	0.5500	1.6133	1.9368	1.3719	2.1756
0.425	0.7978	1.6701	1.8679	0.3221	0.5439	1.6223	1.9277	1.3928	2.1547
0.450	0.7554	1.6753	1.8627	0.3280	0.5380	1.6309	1.9190	1.4131	2.1345
0.475	0.7144	1.6804	1.8576	0.3337	0.5323	1.6391	1.9106	1.4327	2.1149
0.500	0.6745	1.6854	1.8526	0.3392	0.5268	1.6471	1.9025	1.4517	2.0958
0.525	0.6357	1.6902	1.8478	0.3446	0.5214	1.6548	1.8947	1.4702	2.0773
0.550	0.5978	1.6949	1.8431	0.3499	0.5161	1.6623	1.8871	1.4883	2.0592
0.575	0.5607	1.6995	1.8385	0.3551	0.5109	1.6696	1.8797	1.5060	2.0415
0.600	0.5244	1.7040	1.8340	0.3601	0.5059	1.6766	1.8725	1.5234	2.0242
0.625	0.4888	1.7084	1.8296	0.3651	0.5009	1.6835	1.8655	1.5404	2.0072
0.650	0.4538	1.7127	1.8253	0.3699	0.4961	1.6903	1.8586	1.5571	1.9904
0.675	0.4193	1.7170	1.8210	0.3747	0.4913	1.6969	1.8519	1.5736	1.9740
0.700	0.3853	1.7212	1.8168	0.3794	0.4866	1.7033	1.8454	1.5898	1.9578
0.725	0.3518	1.7254	1.8126	0.3841	0.4819	1.7097	1.8389	1.6058	1.9417
0.750	0.3186	1.7295	1.8085	0.3887	0.4773	1.7159	1.8326	1.6216	1.9259
0.775	0.2858	1.7336	1.8044	0.3933	0.4727	1.7220	1.8264	1.6373	1.9103
0.800	0.2533	1.7376	1.8004	0.3978	0.4682	1.7280	1.8203	1.6528	1.8947
0.825	0.2211	1.7416	1.7964	0.4023	0.4637	1.7340	1.8142	1.6682	1.8793
0.850	0.1891	1.7455	1.7925	0.4067	0.4593	1.7398	1.8083	1.6835	1.8641
0.875	0.1573	1.7495	1.7885	0.4111	0.4549	1.7456	1.8024	1.6986	1.8489
0.900	0.1257	1.7534	1.7846	0.4155	0.4505	1.7514	1.7966	1.7138	1.8338
0.925	0.0941	1.7573	1.7807	0.4199	0.4461	1.7570	1.7908	1.7288	1.8187
0.950	0.0627	1.7612	1.7768	0.4243	0.4417	1.7626	1.7851	1.7438	1.8037
0.975	0.0313	1.7651	1.7729	0.4286	0.4374	1.7682	1.7794	1.7588	1.7887
1.000	0.0000	1.7690	1.7690	0.4330	0.4330	1.7738	1.7738	1.7738	1.7738

From the above expressions, we can build the *triangular shaped fuzzy number for the forecast* $\overline{y}_{44}(1)$*, as a function of* α*.* Consider the 90 percent MMSE prediction limits $y_{44}(1) \pm 1.645\widehat{\sigma} = 1.774 \pm 1.645(0.228)^{\frac{1}{2}} = 1.774 \pm 0.7855 = [0.9885, 2.5595]$. The corresponding fuzzy forecast interval is [1.4374, 2.1152]. Notice that the actual value of the 45th observation is 1.84.

Better Forecasts (Bayesian Interpretation)

It is of interest to note that the fuzzy forecast intervals with the fuzzy estimates of the parameters are generally narrower than the minimum mean square error forecast intervals.

Recently there has been a growing interest in option pricing with random/fuzzy volatility. Black and Scholes and Merton were the first to derive a closed form solution for European option prices, assuming that the underlying asset price follows a one-dimensional log-normal diffusion process.

Limitations

In the Black-scholes model the volatility has been considered constant, while in the Merton model the volatility is assumed to be a function of time. These assumptions have been widely criticized, since many empirical studies have shown strong evidence that the implied volatility surface displays smile or smirk (skew) effects. In most markets, the returns tend to be leptokurtic than the normal distribution as assumed in the B.S. model. Heston (1993),Duan (1995), Heston and Nandi (2000), Elliott et al. (2006),Barone-Adesi, Engle and Mancini (2008). Gong and Thavaneswaran (2010) B.S. model with GARCH volatility and demonstrated the superiority of the method over Duan (1995). Thavaneswaran (2010) option pricing for a Jump Diffusion model with GARCH volatility has been studied using the kurtosis dependent formula.

Coefficient Matching: Fuzzy Coefficient Black-Sholes PDE

Example

Consider the stock and bond model given by

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t \text{ and } d\beta_t = r(t, S_t)\beta_t dt,$$

where all of the model coefficients $\mu(t, S_t)$, $\sigma(t, S_t)$, and $r(t, S_t)$ are given by explicit function (possibly fuzzy estimates) of the current time and current stock price .

Example

Then the arbitrage price at time t of a European option with terminal time T and payout $h(S_T)$ is given by $f(t, S_t)$ where $f(t, x)$ is the solution of the PDE

$$f_t(t,x) = -\frac{1}{2}\sigma^2(t,x)f_{xx}(t,x) - r(t,x)xf(t,x) + r(t,x)f_x(t,x),$$
$$f(T,x) = h(x).$$

$f(t, x)$ and its derivatives may be used to provide explicit formulas for the portfolio weights a_t and b_t for the self-financing portfolio $a_tS_t + b_t\beta_t$ that replicates $h(S_T)$.

$$V_t = a_t S_t + b_t \beta_t$$

From the self-financing condition and the models for the stock and bond, we have

$$\begin{aligned} dV_t = a_t dS_t + b_t d\beta_t &= a_t\left(\mu(t, S_t)dt + \sigma(t, S_t)dW_t\right) + b_t r(t, S_t)\beta_t dt \\ &= \left(a_t \mu(t, S_t) + b_t r(t, S_t)\beta_t\right) dt + a_t \sigma(t, S_t) dW_t. \end{aligned}$$

From our assumption that $V_t = f(t, S_t)$ and the It$\hat{o'}s$ formula

$$dV_t = f_t(t, S_t)dt + \frac{1}{2}f_{xx}(t, S_t)\sigma^2(t, S_t)dt + f_x(t, S_t)dS_t$$

$$= \left(f_t(t, S_t) + \frac{1}{2}f_{xx}(t, S_t)\sigma^2(t, S_t) + f_x(t, S_t)\mu(t, S_t)\right)dt + f_x(t, S_t)\sigma(t, S_t)dW_t.$$

The size of the stock portion of our replicating portfolio is:

$$a_t = f_x(t, S_t).$$

$$\mu(t,S_t)f_x(t,S_t)+r(t,S_t)b_t\beta_t$$
$$=f_t(t,S_t)+\frac{1}{2}f_{xx}(t,S_t)\sigma^2(t,S_t)+f_x(t,S_t)\mu(t,S_t).$$

The $\mu(t,S_t)f_x(t,S_t)$ terms cancel, and the bond portion b_t is

$$b_t=\frac{1}{r(t,S_t)\beta_t}\left(f_t(t,S_t)+\frac{1}{2}f_{xx}(t,S_t)\sigma^2(t,S_t)\right).$$

Because V_t is equal to both $f(t,S_t)$ and $a_tS_t+b_t\beta_t$, the values for a_t and b_t give us a PDE for $f(t,S_t)$:

$$f(t,S_t)=V_t=a_tS_t+b_t\beta_t$$
$$=f_x(t,S_t)S_t+\frac{1}{r(t,S_t)\beta_t}\left(f_t(t,S_t)+\frac{1}{2}f_{xx}(t,S_t)\sigma^2(t,S_t)\right)\beta_t.$$

Now, when we cancel β_t from the last term and replace S_t by x, we arrive at the general *Black-Scholes PDE* and its terminal boundary condition.
The portfolio weights a_t and b_t for the self-financing portfolio $a_t S_t + b_t \beta_t$ that replicates $h(S_T)$ are explicitly given above.
Moreover the same argument holds when we use *fuzzy* estimates for the unknown volatility parameter in the model.

The Black-Sholes PDE when Dealing with Dividends

Suppose that the stock pays a dividend $D_t = k(t, S_t)$ which is a function of the stock price S_t.

$$V_t = a_t S_t + b_t \beta_t.$$

and

$$dV_t = a_t dS_t + b_t d\beta_t - D_t dt,$$

respectively, with the terminal boundary condition

$$f(T, x) = h(x) \text{ for all } x \in \mathbb{R}.$$

Using cm argument, we arrive at the *Black-Scholes PDE*:

$$f_t(t, x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t, x) - rxf_x(t, x) + rf(t, x) - D_t,$$

with its terminal boundary condition

$$f(T, x) = h(x) \text{ for all } x \in \mathbb{R}.$$

For the B.S. model

$$dS_t \quad = \quad \mu S_t dt + \theta S_t dW_t$$

there exist a unique martingale measure Q given by Girsanov's theorem

$$\frac{dQ}{dP} = \exp\left[\frac{r-\mu}{\theta} W_T - \frac{(r-\mu)^2}{2\theta^2} T\right]$$

Solving the Black-Scholes PDE, the arbitrage price of the European call option at time t with current stock price S

$$C_{BS} = S\Phi\left[\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\theta^2}{2}\right)\tau}{\theta\sqrt{\tau}}\right] - Ke^{-r\tau}\Phi\left[\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{\theta^2}{2}\right)\tau}{\theta\sqrt{\tau}}\right]$$

where Φ denotes the cumulative distribution function of a standard normal variable, K is the strike price, T is the expiry date and $\tau = T - t$ is the residual time.

The amount of stock a_t that we hold in the replicating portfolio at time t is

$$a_t = \Phi \left[\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\theta^2}{2}\right)\tau}{\theta\sqrt{\tau}} \right]$$

$$b_t\beta_t = -Ke^{-r\tau}\Phi \left[\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\theta^2}{2}\right)\tau}{\theta\sqrt{\tau}} \right].$$

Option Pricing with Fuzzy Volatility

- Some fuzzy coefficient volatility models are introduced,
- applied to analytical approximation of option pricing.

Viewing the option pricing formula as a moment of a truncated lognormal distribution, kurtosis dependent option pricing formula is derived for various class of fuzzy volatility models.

Fuzzy Numbers

Fuzzy numbers represent an important class of possibilistic distributions and generalize the real numbers. Using Zadeh's well known extension principle the operations with real numbers can be extended to fuzzy numbers and the operations with fuzzy numbers have good arithmetical properties. Carlson and Fuller(2001), Thiagarajah et al.(2007),Thavaneswaran et al.(2009),Paseka et al. (2011) and risk aversion Georgescu(2009).

Carlsson and Fuller (2001).

Definition

A fuzzy set A in $X \subset \Re$, where $\Re$ is the set of real numbers, is a set of ordered pairs $A = \{(x,\ \mu(x)) : x\ \in X\}$, where $\mu(x)$ is the membership function (or grade of membership, or degree of compatibility) or degree of truth of $x \in\ X$ which maps $x \in\ X$ on the real interval [0, 1].

Definition

A fuzzy set A in $\Re^n$ is said to be a convex fuzzy set if its γ-level sets $A(\gamma)$ are (crisp) convex sets for all $\gamma \in [0,1]$. Alternatively, a fuzzy set A in $\Re^n$ is a convex fuzzy set if and only if for all $x_1, x_2 \in \Re^n$ and $0 \leq \lambda \leq 1$,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \quad \geq \quad \mathrm{Min}(\mu_A(x_1), \mu_A(x_2))$$

Definition

A fuzzy number $\tilde{A} \in \mathcal{F}$is called a trapezoidal fuzzy number (Tr.F.N.) with core $[a, b]$, left width γ and right width β if its membership function has the following form:

$$\mu(x) = \begin{cases} 1 - \dfrac{a-t}{\gamma} & \text{if } a - \gamma \leq t \leq a \\ 1 & \text{if } a \leq t \leq b \\ 1 - \dfrac{t-b}{\beta} & \text{if } b \leq t \leq b + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation $\tilde{A} = (a, b, \gamma, \beta)$.

Definition

It can easily be shown that

$$A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [a - (1-\alpha)\gamma, b + (1-\alpha)\beta] \quad \forall\, \alpha \in [0, 1].$$

The support of $\tilde{A}$ is $(a - \gamma, b + \beta)$.

Definition

A fuzzy number A with shape functions g and h defined by

$$g(x) = \left(\frac{x-a}{b-a}\right)^m$$

$$h(x) = \left(\frac{d-x}{d-c}\right)^n.$$

where g is a real valued, increasing and right continuous function, h is a real valued, decreasing and left continuous function, and a, b, c, d are real numbers such that $a < b < c < d$.

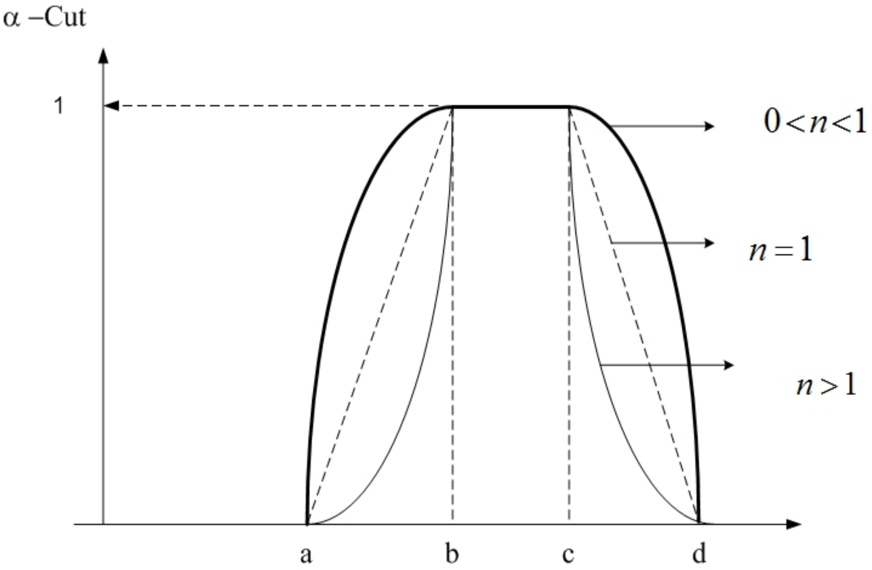

Figure: Nonlinear Membership Function.

If m *and* $n = 1$, we simply write $A = [a, b, c, d]$, which is known as a trapezoidal fuzzy number (Linear type fuzzy number). If m or $n \neq 1$, a fuzzy number $A^{*} = [a, b, c, d]_{m,n}$ is a modification of a trapezoidal fuzzy number (nonlinear type fuzzy number) $A = [a, b, c, d]$.

A has the following α-level sets (α-level sets),
$A(\alpha) = [a(\alpha), b(\alpha)]$, $a(\alpha),\ b(\alpha)\ \in\ \Re,\ \alpha \in\ [0,1]$ and

$$A(\alpha) \;\; = \;\; [g^{-1}(\alpha), h^{-1}(\alpha)], \qquad A_1 = [b, c], \qquad A_0 = [a, d].$$

If $A = [a, b, c, d]_{m,n}$ then, for all $\alpha \in [0,1]$,

$$A(\alpha) \;\; = \;\; [a + \alpha^{\frac{1}{m}}(b-a), d - \alpha^{\frac{1}{n}}(d-c)].$$

Then for an increasing function $g(x)$ and a fuzzy number A whose $\alpha-$level sets is $[A_1(\alpha), A_2(\alpha)]$ then we have the following

$$(g(A))_\alpha = \{g(x)\,|x \in (A)_\alpha\} = \{g(x)\,|A_1(\alpha) \leq x \leq A_2(\alpha)\}$$
$$= \{g(A_1(\alpha)), g(A_2(\alpha))\}$$

A Trangular F.N., A, having the parametric form $A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)], \forall\, \alpha \in\ (0, 1]$.
A parabolic fuzzy number with the membership function

$g_A(x) = 1 - \left(\dfrac{x - a}{a}\right)^2$ with α-cuts of e^A,

$\left(e^A\right)_\alpha = \left[e^{\left(1-\sqrt{(1-\alpha)}\right)a}, e^{\left(1+\sqrt{(1-\alpha)}\right)a}\right]$,

An adaptive fuzzy number $A = [a_1, a_2, a_3, a_4]_n$ with parametric form $A(\alpha) = [a_1 + \alpha^{\frac{1}{n}}(a_2 - a_1), a_4 - \alpha^{\frac{1}{n}}(a_4 - a_3)]$.

Let $\mathfrak{F}$ be the class of all fuzzy numbers. The first order f-WPM (or weighted possibilistic mean) of $A \in \mathfrak{F}$ is given by

$$M_f(A) = \int_0^1 f(\alpha)\frac{(a_1(\alpha) + a_2(\alpha))}{2} d\alpha$$

where $f(\alpha)$ is a weight function such that $\int_0^1 f(\alpha)d\alpha = 1$. Similarly, the centered WPV (or weighted possibilistic variance) of $A \in \mathfrak{F}$ is

$$Var_f(A) = \frac{1}{2}\int_0^1 f(\alpha)\left[(a_1(\alpha) - M_f(A))^2 + (a_2(\alpha) - M_f(A))^2\right] d\alpha$$

and for any positive integer r, the f-WPM(weighted possibilistic moment) of order r about the weighted possibilistic mean value of A is defined as

$$E_r(A) = \frac{1}{2}\int_0^1 f(\alpha)\left[(a_1(\alpha) - M_f(A))^r + (a_2(\alpha) - M_f(A))^r\right] d\alpha$$

The f-weighted possibilistic skewness of fuzzy number A is defined as $S_f(A) = \dfrac{E_3(A)}{\left(\sqrt{E_2(A)}\right)^3}$ and similarly, the f-weighted possibilistic kurtosis of A is defined as $K_f(A) = \dfrac{E_4(A)}{(E_2(A))^2}$.

According to the fundamental theorem of asset pricing, an arbitrage-free price C_t of an option at time t is given by the conditional expectation of the discounted payoff under an equivalent martingale measure Q,

$$C_t = E_Q[e^{-r(T-t)}g(S_t) \mid F_t]$$

Let X be a log-normal variable with parameters μ and θ^2 and K is a constant. We have the following Lemma:

Lemma

(a) $EMax[\exp(\mu+\theta Z)-K,0]=\exp\,(\mu+\frac{\theta^2}{2})\Phi(d)-K\Phi(d-\theta)$,

where $d=\dfrac{ln\frac{1}{K}+\mu+\theta^2}{\theta}$.

(b) *Under the risk neutral measure Q such that* $\mu+\dfrac{\theta^2}{2}=0$, $EMax[\exp(\mu+\theta Z)-K,0]=\Phi(d)-K\Phi(d-\theta)$.

If the conditional distribution of ln $[(\frac{S_{t+1}}{S_t})|F_t^s]$ is normal with parameters μ_t and θ_t^2 where F_t^s is the σ-fields generated by $S_1, S_2, \ldots, S_t$. Then, the payoff for one period returns of the underlying asset can be obtained from the following Lemma.

Lemma

(a) $E[S_{t+1}|F_t^s] = S_t \exp(\mu_t + \frac{\theta_t^2}{2})$.

(b) $Var[S_{t+1}|F_t^s] = S_t^2 \exp(2\mu_t + \theta_t^2(\theta_t^2 - 1))$.

(c)

$$E[Max(S_{t+1} - K_t, 0)|F_t^s] = S_t \exp(\mu_t + \frac{\theta_t^2}{2})\Phi(d_t) - K_t\Phi(d_t - \theta_t)$$

where $d_t = [\frac{ln\frac{1}{K_t} + \mu_t + \theta_t^2}{\theta_t}]$

For the model $dS_t = \theta_t S_t dW_t$ with time varying volatility (θ is a deterministic function of t), under no-arbitrage opportunities, the payoff of the call option is given by

$$E[S_t - K_+] = S\Phi(d) - K\Phi(d - \psi))$$

where $\psi^2 = \int_0^t \theta_s^2 ds$ and $S_t - K_+ = \text{Max}[S_t - K, 0]$.
Proof. By applying the Ito's formula to $\ln S_t$, we have

$$dlnS_t \quad = \quad \frac{dS_t}{S_t} \; - \; \frac{1}{2}\frac{\theta_t^2}{S_t^2}S_t^2 dt \quad = \quad -\frac{1}{2}\theta_t^2 dt \; + \; \theta_t dW_t$$

$$X = \ln S_t - \ln S = -\frac{1}{2}\psi^2 + \int_0^t \theta_t dW_t$$

where $\psi^2 = \int_0^t \theta_s^2 ds$. Since $E(X) = -\frac{1}{2}\psi^2$, Var($X$)=$\psi^2$. The marginal distribution of S_t is lognormal and the result follows.

Lemma

Let A be a fuzzy number with finite possibilistic moments of order four. Let $M_f(A)$ and $Var_f(A)$ are the mean and variance of A. Let $h \in C^1$ and $g \in C^2$ then

(a) $M_f[h(A)] = h(M_f(A))$
$Var_f[h(A)] = [h'(M_f(A))]^2 Var_f(A)$

(b) $M_f[g(A)] = g(M_f(A)) + \frac{1}{2}g''(M_f(A))Var_f(A)$

$Var_f[g(A)] = [g'(M_f(A))]^2 Var_f(A) + \frac{1}{4}(g''(M_f(A)))^2 E_4(A) + \frac{1}{2}g'(M_f(A))g''(M_f(A))E_3(A)$

(c) $M_f[h((A - M_f(A))^2)] = h(Var_f(A))$
$Var_f[h((A - M_f(A))^2)] = [f'(Var_f(A))]^2[K_f(A) - 1][Var_f(A)]^2$
where $K_f(A) = \dfrac{E_4(A)}{(Var_f(A))^2}$ *and $E_k(A)$ is the kth central moment of A.*

Lemma

When θ is assumed to be a fuzzy number $\tilde{\Theta}$ having possibilistic mean $M_f(\tilde{\Theta})$ and possibilistic variance $Var_f(\tilde{\Theta})$ then

$$
\begin{aligned}
M_f Max[\exp(\mu + \tilde{\Theta} Z) - K, 0] = {} & h[(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})] \\
& + \frac{1}{2} h''[(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})] \\
& [(Var_f(\tilde{\Theta}))^2 K_f(\tilde{\Theta}) + 4 M_f(\tilde{\Theta})(Var_f(\tilde{\Theta}))^{3/2} S_f(\tilde{\Theta}) \\
& + 4(M_f(\tilde{\Theta}))^2 Var_f(\tilde{\Theta}) - (Var_f(\tilde{\Theta}))^2] \\
& - K[g((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})) + \frac{1}{2} h''((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})) \\
& ((Var_f(\tilde{\Theta}))^2 K_f(\tilde{\Theta}) + 4 M_f(\tilde{\Theta})(Var_f(\tilde{\Theta}))^{3/2} S_f(\tilde{\Theta}) \\
& + 4(M_f(\tilde{\Theta}))^2 Var_f(\tilde{\Theta}) - (Var_f(\tilde{\Theta}))^2)]
\end{aligned}
$$

where $K_f(\tilde{\Theta})$ and $S_f(\tilde{\Theta})$ are the kurtosis and skewness of $\tilde{\Theta}$, respectively, and

$$h((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})) = \exp\left(\mu + \frac{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}{2}\right) \times \Phi\left(\frac{\ln\frac{1}{K} + \mu + (M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}{\sqrt{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}}\right),$$

$$g((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})) = \Phi\left(\frac{\ln\frac{1}{K} - \mu + (M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}{\sqrt{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}}\right)$$

$$
\begin{aligned}
&h''((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})) \\
&= \frac{1}{4}\Phi\left(\frac{\ln\frac{1}{K} + (M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}{\sqrt{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}}\right)\exp\left(\mu + \frac{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}{2}\right) \\
&+ \frac{1}{2\sqrt{2\pi}}[(\frac{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}) - \ln\frac{1}{K} - \mu}{((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))\sqrt{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}}) \\
&(\frac{((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))^2 + (\ln\frac{1}{K} + \mu)^2}{2((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))^2}) \\
&+ (\frac{3(\ln\frac{1}{K} + \mu) - ((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))}{2((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))^2\sqrt{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}})] \\
&\exp(-\frac{((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}) + \ln\frac{1}{K} + \mu)^2}{2((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))})\exp(\mu + \frac{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}{2}),
\end{aligned}
$$

$$
\begin{aligned}
&g''((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})) \\
&= \frac{1}{\sqrt{2\pi}}[(\frac{((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))^2 - (\ln\frac{1}{K} + \mu)^2}{2((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))^2}) \\
&(\frac{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}) + \ln\frac{1}{K} + \mu}{2((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))\sqrt{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}}) \\
&+ (\frac{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}) + 3(\ln\frac{1}{K} + \mu)}{4((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))^2\sqrt{(M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta})}})] \\
&\exp(-\frac{((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}) - \ln\frac{1}{K} + \mu)^2}{2((M_f(\tilde{\Theta}))^2 + Var_f(\tilde{\Theta}))})
\end{aligned}
$$

Proof. When θ is assumed to be a fuzzy number $\tilde{\Theta}$ having possibilistic mean $M_f(\tilde{\Theta})$ and possibilistic variance $Var_f(\tilde{\Theta})$ then by the lemma

$$\begin{aligned} M_f Max[\exp(\mu + \tilde{\Theta} Z) - K, 0] = M_f[\exp(\mu + \frac{\tilde{\Theta}^2}{2})\Phi(d)|\tilde{\Theta}] \\ - K M_f[\Phi(d - \tilde{\Theta})|\tilde{\Theta}] \end{aligned}$$

Applying the Taylor series expansion

$$\begin{aligned} h(\tilde{\Theta}^2) = h[M_f(\tilde{\Theta}^2)] + h'[M_f(\tilde{\Theta}^2)][\tilde{\Theta}^2 - M_f(\tilde{\Theta}^2)] \\ + \frac{1}{2} h''[M_f(\tilde{\Theta}^2)][\tilde{\Theta}^2 - M_f(\tilde{\Theta}^2)]^2 + R''', \end{aligned}$$

to $h(\tilde{\Theta}^2) = \exp(\mu + \frac{\tilde{\Theta}^2}{2})\Phi(d)$ and $h(\tilde{\Theta}^2) = \Phi(d - \tilde{\Theta})$, and then taking expectation for both $h(\tilde{\Theta}^2)$ and $g(\tilde{\Theta}^2)$, to find $h''[M_f(\tilde{\Theta}^2)]$ and $g''[M_f(\tilde{\Theta}^2)]$.

If the conditional distribution of ln $[(\frac{S_{t+1}}{S_t})|F_t^s]$ is normal with mean μ_t and standard deviation θ_t which is assumed to be a fuzzy number $\tilde{\Theta}$ having possibilistic mean $M_f(\tilde{\Theta}_t)$ and possibilistic variance $Var_f(\tilde{\Theta}_t)$ and F_t^s is the σ-fields generated by $S_1, S_2, \ldots, S_t$. Then, the payoff for one period returns of the underlying asset can be obtained from the following Lemma.

Lemma

(a) $M_f[S_{t+1}|F_t^s] = S_t \exp(\mu_t + \frac{Var_f(\tilde{\Theta}_t)}{2})$.

(b) $Var_f[S_{t+1}|F_t^s] = S_t^2 \exp(2\mu_t + Var_f(\tilde{\Theta}_t)(Var_f(\tilde{\Theta}_t) - 1))$.

(c) $M_f[Max(S_{t+1} - K_t, 0)|F_t^s] =$

$$S_t \exp(\mu_t + \frac{Var_f(\tilde{\Theta}_t)}{2})\Phi(d_t) - K_t\Phi(d_t - \tilde{\Theta}_t)$$

$$where\ d_t = [\frac{ln\frac{1}{K_t} + \mu_t + Var_f(\tilde{\Theta}_t)}{\tilde{\Theta}_t}]$$

We consider the following fuzzy coefficient Black Scholes model

$$dS_t = \mu S_t dt + \tilde{\Theta} S_t dW_t$$

If we assume that $h, g \in C^2$ (up to second derivative) and $\tilde{\Theta}$ be a fuzzy number with finite fourth possibilistic moments, then the call price:

$$\begin{aligned} C(S_t) =& S_0 M_f[\Phi(d)|\tilde{\Theta}] - Ke^{-r\tau} M_f[\Phi(d - \tilde{\Theta}\sqrt{\tau})|\tilde{\Theta}] \\ =& S_0 \left(h[M_f(\tilde{\Theta}^2)] + \frac{1}{2} h''[M_f(\tilde{\Theta}^2)] Var_f(\tilde{\Theta}^2) \right) \\ & - Ke^{-r\tau} \left(g[M_f(\tilde{\Theta}^2)] + \frac{1}{2} g''[M_f(\tilde{\Theta}^2)] Var_f(\tilde{\Theta}^2) \right) \end{aligned}$$

where

$$h[M_f(\tilde{\Theta}^2)] = \Phi(d) = \Phi \left(\frac{\ln \frac{S_0}{K} + r\tau + \frac{1}{2} M_f(\tilde{\Theta}^2)\tau}{\sqrt{\tau}\sqrt{M_f(\tilde{\Theta}^2)}} \right)$$

$$h''[M_f(\tilde{\Theta}^2)] = \frac{1}{\sqrt{2\pi}}[\left(\frac{M_f(\tilde{\Theta}^2)\tau - 2(\ln\frac{S_0}{K} + r\tau)}{4M_f(\tilde{\Theta}^2)\sqrt{M_f(\tilde{\Theta}^2)}\sqrt{\tau}}\right)\left(\frac{4(\ln\frac{S_0}{K} + r\tau)^2 - [M_f(\tilde{\Theta}^2)]^2\tau^2}{8\tau[M_f(\tilde{\Theta}^2)]^2}\right)$$
$$+\left(\frac{6(\ln\frac{S_0}{K} + r\tau) - (M_f(\tilde{\Theta}^2))\tau}{8\sqrt{\tau}[M_f(\tilde{\Theta}^2)]^2\sqrt{M_f(\tilde{\Theta}^2)}}\right)]\exp\left(-\frac{(\ln\frac{S_0}{K} + r\tau + \frac{1}{2}\tau M_f(\tilde{\Theta}^2))^2}{2\tau M_f(\tilde{\Theta}^2)}\right)$$

$$g''[M_f(\tilde{\Theta}^2)] = \frac{1}{\sqrt{2\pi}}[\left(\frac{-M_f(\tilde{\Theta}^2)\tau - 2(\ln\frac{S_0}{K} + r\tau)}{4M_f(\tilde{\Theta}^2)\sqrt{M_f(\tilde{\Theta}^2)}\sqrt{\tau}}\right)\left(\frac{4(\ln\frac{S_0}{K} + r\tau)^2 - [M_f(\tilde{\Theta}^2)]^2\tau^2}{8\tau[M_f(\tilde{\Theta}^2)]^2}\right)$$
$$+\left(\frac{6(\ln\frac{S_0}{K} + r\tau) + (M_f(\tilde{\Theta}^2))\tau}{8\sqrt{\tau}[M_f(\tilde{\Theta}^2)]^2\sqrt{M_f(\tilde{\Theta}^2)}}\right)]\exp\left(-\frac{(\ln\frac{S_0}{K} + r\tau - \frac{1}{2}\tau M_f(\tilde{\Theta}^2))^2}{2\tau M_f(\tilde{\Theta}^2)}\right)$$

Thus we assume that the initial stock price is an adaptive fuzzy number of the form $\tilde{S_0} = [S_1, S_2, S_3, S_4]_n$. In a similar manner $e^{\tilde{r}\tau} = [e^{\tilde{r_4}\tau}, e^{\tilde{r_3}\tau}, e^{\tilde{r_2}\tau}, e^{\tilde{r_1}\tau}]_n$ for the discount factor and of the form $\tilde{\Theta} = [\Theta_1, \Theta_2, \Theta_3, \Theta_4]_n$ for the volatility can also be modeled.

We take an example of the Black-Scholes formula for a dividend paying stock with exercise price K.

$$\widehat{FCOV} = \tilde{S}_0 e^{-\delta\tau} M_f[\Phi(d_1)] - K e^{-\tilde{\delta}\tau} M_f[\Phi(d_2)] \tag{1}$$

where

$$d_1 = \frac{\ln\frac{\tilde{S}_0}{K} + (\tilde{r} - \delta + \frac{\tilde{\Theta}^2}{2})\tau}{\tilde{\Theta}\sqrt{\tau}} \tag{2}$$

$$d_2 = d_1 - \tilde{\Theta}\sqrt{\tau} \tag{3}$$

$M_f(A)$ is the possibilistic mean value of variable A. We assume that the stock pays dividends continuously at a known rate δ.

The α-cuts of the fuzzy call option value $\widehat{FCOV}$ are as follows:

$$\widehat{FCOV}(\alpha) = [FCOV_1(\alpha), FCOV_2(\alpha)]$$

where

$$FCOV_1(\alpha) = [e^{-\delta\tau} M_f[\Phi(d_1)]S_1 - KM_f[\Phi(d_2)]e^{-r_1\tau}]$$
$$+ [KM_f[\Phi(d_2)](e^{-r_1\tau} - e^{-r_2\tau}) + (S_2 - S_1)e^{-\delta\tau} M_f[\Phi(d_1)]]\alpha^{1/n}$$

$$FCOV_2(\alpha) = [e^{-\delta\tau} M_f[\Phi(d_1)]S_4 - KM_f[\Phi(d_2)]e^{-r_4\tau}]$$
$$- [KM_f[\Phi(d_2)](e^{-r_3\tau} - e^{-r_4\tau}) + (S_4 - S_3)e^{-\delta\tau} M_f[\Phi(d_1)]]\alpha^{1/n}$$

The membership function of the fuzzy call option is given below

$$\mu_{\tilde{C}} = \left[\frac{C - (e^{-\delta\tau} M_f[\Phi(d_1)]S_1 - KM_f[\Phi(d_2)]e^{-r_1\tau})}{(KM_f[\Phi(d_2)](e^{-r_1\tau} - e^{-r_2\tau}) + (S_2 - S_1)e^{-\delta\tau} M_f[\Phi(d_1)])}\right]^n$$

$$\forall\ C \in \begin{pmatrix} S_1 e^{-\delta\tau} M_f[\Phi(d_1)] - KM_f[\Phi(d_2)]e^{-r_1\tau}, \\ S_2 e^{-\delta\tau} M_f[\Phi(d_1)] - KM_f[\Phi(d_2)]e^{-r_2\tau} \end{pmatrix}$$

$$\mu_{\tilde{C}} = 1 \qquad \forall\ C \in \begin{pmatrix} S_2 e^{-\delta\tau} M_f[\Phi(d_1)] - KM_f[\Phi(d_2)]e^{-r_2\tau}, \\ S_3 e^{-\delta\tau} M_f[\Phi(d_1)] - KM_f[\Phi(d_2)]e^{-r_3\tau} \end{pmatrix}$$

$$\mu_{\tilde{C}} = \left[\frac{(e^{-\delta\tau} M_f[\Phi(d_1)]S_4 - KM_f[\Phi(d_2)]e^{-r_4\tau}) - C}{KM_f[\Phi(d_2)](e^{-r_3\tau} - e^{-r_4\tau}) + (S_4 - S_3)e^{-\delta\tau} M_f[\Phi(d_1)]}\right]^n$$

$$\forall\ C \in \begin{pmatrix} S_3 e^{-\delta\tau} M_f[\Phi(d_1)] - KM_f[\Phi(d_2)]e^{-r_3\tau}, \\ S_4 e^{-\delta\tau} M_f[\Phi(d_1)] - KM_f[\Phi(d_2)]e^{-r_4\tau} \end{pmatrix}$$

$$\mu_{\tilde{C}} = 0 \qquad \text{otherwise}$$

In the following example we consider the fuzzy call price on a stock option using the adaptive fuzzy numbers. Consider a European call option on a stock with the following assumptions. The current stock price, the stock price volatility and the risk-free interest rate are all taken as adaptive fuzzy numbers. $\tilde{S}_0 = [158, 160, 162, 164]_n$, $\tilde{r} = [0.03, 0.04, 0.05, 0.06]_n$, $\tilde{\Theta} = [0.10, 0.20, 0.30, 0.40]_n$, $\tau = 2$, $\delta = 0.03$, $K = 140$. Thus, $M_f[\Phi(d_1)] = 0.7569$, and $M_f[\Phi(d_2)] = 0.6450$.

$$
\begin{aligned}
FCOV_1(\alpha) &= [e^{-\delta\tau} M_f[\Phi(d_1)]S_1 - KM_f[\Phi(d_2)]e^{-r_1\tau}] \\
&+ [KM_f[\Phi(d_2)](e^{-r_1\tau} - e^{-r_2\tau}) + (S_2 - S_1)e^{-\delta\tau} M_f[\Phi(d_1)]]\alpha^{1/n} \\
&= 27.5845 + 3.1096\alpha^{1/n}
\end{aligned}
$$

$$
\begin{aligned}
FCOV_2(\alpha) &= [e^{-\delta\tau} M_f[\Phi(d_1)]S_4 - KM_f[\Phi(d_2)]e^{-r_4\tau}] \\
&- [KM_f[\Phi(d_2)](e^{-r_3\tau} - e^{-r_4\tau}) + (S_4 - S_3)e^{-\delta\tau} M_f[\Phi(d_1)]]\alpha^{1/n} \\
&= 36.8138 - 3.0435\alpha^{1/n}
\end{aligned}
$$

we obtain the fuzzy call option values for various levels of α and *n* and are presented below. Therefore, a fuzzy adaptive Black-Scholes model is sufficiently flexible and can be easily adjusted or tuned for optimal solution.

Table: Fuzzy option values for different values of α and n

α	$\text{FCOV}_1(\alpha)$ $n = 0.2$	$\text{FCOV}_2(\alpha)$ $n = 0.2$	$\text{FCOV}_1(\alpha)$ $n = 1$	$\text{FCOV}_2(\alpha)$ $n = 1$	$\text{FCOV}_1(\alpha)$ $n = 5$	$\text{FCOV}_2(\alpha)$ $n = 5$
0	27.58	36.81	27.58	36.81	27.58	36.81
0.1	27.58	36.81	27.90	36.51	29.55	34.89
0.2	27.59	36.81	28.21	36.21	29.84	34.61
0.3	27.59	36.81	28.52	35.90	30.03	34.42
0.4	27.62	36.78	28.83	35.60	30.17	34.28
0.5	27.68	36.72	29.14	35.29	30.29	34.16
0.6	27.83	36.58	29.45	34.99	30.39	34.07
0.7	28.11	36.30	29.76	34.68	30.48	33.98
0.8	28.60	35.82	30.07	34.38	30.56	33.90
0.9	29.42	35.02	30.38	34.07	30.63	33.83
1.0	30.69	33.77	30.69	33.77	30.69	33.77

Asset-or-nothing Option with fuzziness

Consider the case when the claim pays off the amount $C_2 = (S_T)^\nu I_{\{a \le S_T \le b\}}$. The time-$t$ price is given by

$$
\begin{aligned}
&\mathbb{E}_Q\left[e^{-r(T-t)}(S_T)^\nu I_{\{a\le S_T^\nu \le b\}}|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[e^{-r(T-t)}(S_t)^\nu e^{\nu(r-\sigma^2/2)(T-t)+\nu\sigma(W_T-W_t)} I_{\{a\le S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W_T-W_t)}\le b\}}|\mathcal{F}_t\right]\\
&= (S_t)^\nu e^{\left((\nu-1)r-\frac{\nu\sigma^2}{2}\right)(T-t)}\mathbb{E}\left[e^{\nu X} I_{\left\{\ln\left(\frac{a}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)\le X\le \ln\left(\frac{b}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)\right\}}\right]\\
&= (S_t)^\nu e^{\left((\nu-1)r-\frac{\nu\sigma^2}{2}\right)(T-t)} e^{\frac{1}{2}\nu^2\sigma^2(T-t)}\mathbb{P}\left(\ln\left(\frac{a}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)\right.\\
&\left.\le X+\nu\sigma^2(T-t)\le \ln\left(\frac{b}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)\right)\\
&= (S_t)^\nu e^{\left((\nu-1)r+\frac{1}{2}\nu(\nu-1)\right)(T-t)}\mathbb{P}\left(\frac{\ln\left(\frac{a}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}-\nu\sigma\sqrt{T-t}\right.\\
&\left.\le Z\le \frac{\ln\left(\frac{b}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}-\nu\sigma\sqrt{T-t}\right)\\
&= (S_t)^\nu e^{\left((\nu-1)r+\frac{1}{2}\nu(\nu-1)\sigma^2\right)(T-t)}\left(\Phi(d_\nu(b))-\Phi(d_\nu(a))\right),
\end{aligned}
$$

where

$$d_\nu(u) = \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \nu\sigma\sqrt{T-t}.$$

Note that when $\nu = 2$, the time-t price is

$$(S_t)^\nu e^{(r+\sigma^2)(T-t)}(S_t)^2\Phi(d_2(b)) - \Phi(d_2(a)),$$

where

$$d_2(u) = \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - 2\sigma\sqrt{T-t}.$$

For the asset-or-nothing option stated above, fuzzy concept is taken into account in a model. Define a fuzzy set as trapezoidal fuzzy number with core $[S_a, S_b]$, left width α and right width β. Consider the membership function related to asset price which follows the trapezoid function. By introducing the fuzzy concept into a binary function an investor may have more opportunities to think about his decision in some aspect such as risk.

$$\phi_S = \begin{cases} 1 - \left(\dfrac{S_a - S(T)}{\alpha}\right) & \text{if} \quad S_a - \alpha \leq S(T) \leq S_a \\ 1 & \text{if} \quad S_a \leq S(T) \leq S_b \\ 1 - \left(\dfrac{S(T) - S_b}{\beta}\right) & \text{if} \quad S_b \leq S(T) \leq S_b + \beta \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$\alpha \geq 0$, $\beta \geq 0$.

The most possible values of the underlying asset price at the maturity date lie in the interval $[S_a, S_b]$, and $S_b + \beta$ is the upward potential and $S_a + \alpha$ is the downward potential for the values of the underlying-assets price. For fixing parameter values of α, β, S_a, and S_b there are many ways to be considered. For example, when the investor can not predict how the underlying asset price changes at the maturity date, in other words, when he becomes confident that the asset price is fluctuated greatly he will take the range of sufficiently large width so that the premium values become high. On the other hand, when much fluctuation is not observed the width will become small, so that S_a gets equal to S_b, resulting in triangular fuzzy numbers. For each of three sets the corresponding payoff is obtained by multiplying its grade of membership function, ϕ_S.

In this case, and the underlying asset $S(T)$ moves between $S_a - \alpha$ and $S_b + \beta$. Then, the present value of option may be computed as a difference between the present value of S(T) which exceeds $S_a - \alpha$ and that of $S(T)$ which is above $S_b + \beta$.

$$\text{payoff} = \phi_S \times S(T) = \begin{cases} S(T) - \left(\dfrac{S_a S(T) - S(T)^2}{\alpha}\right) & \text{if} \quad S_a - \alpha \leq S(T) \leq S_a \\ S(T) & \text{if} \quad S_a \leq S(T) \leq S_b \\ S(T) - \left(\dfrac{S(T)^2 - S_b S(T)}{\beta}\right) & \text{if} \quad S_b \leq S(T) \leq S_b + \beta \\ 0 & \text{otherwise} \end{cases}$$

Then, the values of asset-or-nothing option with fuzzy nature are showed as follows

$$C = C_1 + C_1 - C_2 - C_3$$

where

$$\begin{aligned} C_1 &= e^{-r(T-t)} E^{\#}\left[S(T)\right], \;\; S_a - \alpha < S(T) < S_b + \beta \\ C_2 &= e^{-r(T-t)} \left(\frac{S_a E^{\#}\left[S(T)\right] - E^{\#}\left[S(T)^2\right]}{\alpha}\right), \;\; S_a - \alpha < S(T) < S_a \\ C_3 &= e^{-r(T-t)} \left(\frac{S_a E^{\#}\left[S(T)^2\right] - S_b E^{\#}\left[S(T)\right]}{\beta}\right), \;\; S_b < S(T) < S_b + \beta \end{aligned}$$

with appropriate boundary conditions, where $E\#$ indicates the conditional expectation with respect to risk-neutral probability. Also, From (1), the value of $S(\tau)$ after $\tau = T - t$ period passed is expressed by Ito's Lemma as

Payoff

$$S(T) \quad = \quad S_t \left[\phi\left(d(b)\right) - \phi\left(d(a)\right)\right]$$

where,

$$d(u) \quad = \quad \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}$$

Payoff

$$S^{\upsilon}(T) \quad = \quad (S_t)^{\upsilon}\, e^{\left((\upsilon-1)r + \frac{1}{2}\upsilon(\upsilon-1)\sigma^2\right)(T-t)} \left[\phi\left(d_{\upsilon}(b)\right) - \phi\left(d_{\upsilon}(a)\right)\right]$$

where

$$d_{\upsilon}(u) \quad = \quad \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} - \upsilon\sigma\sqrt{T-t}$$

$$
\begin{aligned}
C_1 &= e^{-r(T-t)} E^{\#}\left[S(T)\right] = e^{-r(T-t)} S_t \left[\phi\left(d(b)\right) - \phi\left(d(a)\right)\right] \\
&= e^{-r(T-t)} S_t \left(\begin{array}{c} \phi\left(\dfrac{\ln\left(\dfrac{b}{S_t}\right) - \left(r - \dfrac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t} \right) - \\ \phi\left(\dfrac{\ln\left(\dfrac{a}{S_t}\right) - \left(r - \dfrac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}) \right) \end{array} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 &= e^{-r(T-t)} \left(\frac{S_a E^{\#}\left[S(T)\right]}{\alpha} - \frac{E^{\#}\left[S(T)^2\right]}{\alpha} \right) \\
&= e^{-r(T-t)} \left(\frac{S_a S(T)}{\alpha} - \frac{S^2(T)}{\alpha} \right) \\
&= e^{-r(T-t)} \left(\frac{S_a S_t \left[\phi\left(d(b)\right) - \phi\left(d(a)\right)\right]}{\alpha} - \frac{(S_t)^2 e^{\left((2-1)r + \frac{1}{2}2(2-1)\sigma^2\right)(T-t)} \left[\phi\left(d_2(b)\right) - \right.}{\alpha} \right. \\
&= e^{-r(T-t)} \left(\frac{S_a S_t \left[\phi\left(d(b)\right) - \phi\left(d(a)\right)\right]}{\alpha} - \frac{S_t^2 e^{\left(r+\sigma^2\right)(T-t)} \left[\phi\left(d_2(b)\right) - \phi\left(d_2(a)\right)\right]}{\alpha} \right)
\end{aligned}
$$

$$\phi_S = \begin{cases} 1 - \left(\dfrac{S_a - S(T)}{\alpha}\right)^n & \text{if} \quad S_a - \alpha \le S(T) \le S_a \\ 1 & \text{if} \quad S_a \le S(T) \le S_b \\ 1 - \left(\dfrac{S(T) - S_b}{\beta}\right)^n & \text{if} \quad S_b \le S(T) \le S_b + \beta \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The payoff is given by

$$\text{payoff} = \phi_S \times S(T) = \begin{cases} S(T) - \left(\dfrac{S_a S(T) - S(T)^2}{\alpha}\right)^n & \text{if} \quad S_a - \alpha \le S(T) \le S_a \\ S(T) & \text{if} \quad S_a \le S(T) \le S_b \\ S(T) - \left(\dfrac{S(T)^2 - S_b S(T)}{\beta}\right)^n & \text{if} \quad S_b \le S(T) \le S_b + \beta \\ 0 & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} C_1 &= e^{-r(T-t)} E^{\#}\left[S(T)\right], \quad S_b - \alpha < S(T) < S_b + \beta \\ &= e^{-r(T-t)} S(T) = S_t\left[\phi\left(d(b)\right) - \phi\left(d(a)\right)\right] \end{aligned}$$

and

$$C_2 = e^{-r(T-t)} E^{\#}\left(\frac{S_a\left[S(T)\right] - S(T)^2}{\alpha}\right)^n$$

Given

$$g(x) = \frac{2(a-x)(x-2a)}{3a}$$

$$\begin{aligned} \text{Payoff (Parabolic)} &= \phi_S S(T) = S(T)\left(\frac{2(a-S(T))(S(T)-2a)}{3a}\right) \\ &= 2S^2(T) - \frac{4a}{3}S(T) - \frac{2}{3}\left(\frac{S^3(T)}{a}\right) \end{aligned}$$

Because $S^{\upsilon}(T) = (S_t)^{\upsilon}\, e^{\left((\upsilon-1)r+\frac{1}{2}\upsilon(\upsilon-1)\sigma^2\right)(T-t)}\left[\phi\left(d_{\upsilon}(b)\right) - \phi\left(d_{\upsilon}(a)\right)\right]$. We have

$$\begin{aligned} S^3(T) &= S_t^3 e^{\left(2r+3\sigma^2\right)(T-t)}\left[\phi\left(d_3(b)\right) - \phi\left(d_3(a)\right)\right] \\ S^2(T) &= S_t^2 e^{\left(r+\sigma^2\right)(T-t)}\left[\phi\left(d_2(b)\right) - \phi\left(d_2(a)\right)\right] \\ S(T) &= S_t\left[\phi\left(d_1(b)\right) - \phi\left(d_1(a)\right)\right] \end{aligned}$$

We have,

$$\begin{aligned} \text{Payoff (Parabolic)} &= 2\left(S_t^2 e^{\left(r+\sigma^2\right)(T-t)}\left[\phi\left(d_2(b)\right) - \phi\left(d_2(a)\right)\right]\right) - \frac{4a}{3}\left(S_t\left[\phi\left(d_1(b)\right) - \phi\left(\right.\right.\right. \\ &\quad -\frac{2}{3}\left(\frac{S_t^3 e^{\left(2r+3\sigma^2\right)(T-t)}\left[\phi\left(d_3(b)\right) - \phi\left(d_3(a)\right)\right]}{a}\right) \end{aligned}$$

where

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