Complex Analysis

Theory and Math

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Presentation Outline

Overview

This lecture discusses two important types of first-order differential equations:

- Linear Differential Equations
- Exact Differential Equations

These equations have specific solution techniques based on their structure.

Definitions

Complex number

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A variable which can take any complex number number is called complex variable. The complex variable of a complex number represented by $z = x + iy$.

Conjugate complex number

Any number of the form $x - iy$ is called a complex number where $x, y \in R$. A complex number defined by $\bar{z} = x - iy$.

Definitions

Analytic Function

A complex function $f(z)$ is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighborhood of z_0.

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Cauchy's Integral Formula

If f is analytic on a simply connected domain D and γ is a positively oriented, simple closed contour lying in D, then for any point z inside γ,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Theorem

I

f $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R, and if u and v have continuous second-order partial derivatives in R, then both u and v are harmonic in R.

Proof. Since f is analytic in R, the real and imaginary parts u and v satisfy the Cauchy–Riemann equations:

$$u_x = v_y, \qquad u_y = -v_x.$$

Because u and v have continuous second-order partial derivatives, we may differentiate the Cauchy–Riemann equations again.
Differentiate $u_x = v_y$ with respect to x:

$$u_{xx} = v_{yx}.$$

Differentiate $u_y = -v_x$ with respect to y:

$$u_{yy} = -v_{xy}.$$

Since mixed partial derivatives are equal ($v_{xy} = v_{yx}$), we obtain

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Thus,

$$\Delta u = u_{xx} + u_{yy} = 0,$$

which shows that u is harmonic in R.
Similarly, differentiate $u_x = v_y$ with respect to y:

$$u_{xy} = v_{yy}.$$

Differentiate $u_y = -v_x$ with respect to x:

$$u_{yx} = -v_{xx}.$$

Again using $u_{xy} = u_{yx}$, we get

$$v_{yy} = -v_{xx},$$

or equivalently,

$$v_{xx} + v_{yy} = 0.$$

Continued

Thus,

$$\Delta v = v_{xx} + v_{yy} = 0,$$

so v is harmonic in R.
Hence, both u and v are harmonic functions in the region R.

Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic and also find the harmonic conjugate if $f(z) = u + iv$ is analytic.

Solution:

Given that, $u = 2x - x^3 + 3xy^2$

$$\therefore \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 = \Phi_1(x, y) \qquad \text{........(1)}$$

$$\frac{\partial u}{\partial y} = 6xy = \Phi_2(x, y) \qquad \text{........(2)}$$

$$\frac{\partial^2 u}{\partial x^2} = -6x \qquad \text{........(3)}$$

$$\frac{\partial^2 u}{\partial y^2} = 6x \qquad \text{........(4)}$$

Adding equation (3) & (4) we get,

Example 2: Linear Equation(continued)

Previously :

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x$$

Integrate both sides:

$$\frac{y}{x} = \frac{x^2}{2} + C$$

Example 2: Linear Equation(continued)

Previously :

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x$$

Integrate both sides:

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$\Rightarrow y = \frac{x^3}{2} + Cx$$

Bernoulli Equation - Definition

Definition

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$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $n \neq 0, 1$.

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To solve it:

1. Divide both sides by y^n
2. Substitute $v = y^{1-n}$
3. Reduce to linear form
4. Solve using integrating factor

Example 1

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Solve:

$$\frac{dy}{dx} + y = y^2 \sin x$$

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Step 2: Divide both sides by y^2:

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \sin x$$

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Let $v = \frac{1}{y} \Rightarrow \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

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Let $v = \frac{1}{y} \Rightarrow \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$
Substitute:

$$-\frac{dv}{dx} + v = \sin x \Rightarrow \frac{dv}{dx} - v = -\sin x$$

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Integrating factor: $\mu(x) = e^{-x}$

Example 1(continued)

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$$ve^{-x} = \int -\sin xe^{-x}dx$$

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$$\frac{d}{dx}(ve^{-x}) = -\sin xe^{-x}$$

Integrate:

$$ve^{-x} = \int -\sin xe^{-x} dx$$

Result:

$$v = \frac{\sin x + \cos x}{2} + Ce^{x} \quad \Rightarrow \quad y = \frac{1}{\frac{\sin x + \cos x}{2} + Ce^{x}}$$

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Step 2: Divide both sides by y^3:

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Let $v = y^{-2} \Rightarrow \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

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Solve for $\frac{dy}{dx}$:

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

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Substitute:

$$-\frac{1}{2}\frac{dv}{dx} + \frac{2}{x}v = x^2 \Rightarrow \frac{dv}{dx} - \frac{4}{x}v = -2x^2$$

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Multiply:

$$x^{-4}\frac{dv}{dx} - \frac{4}{x^5}v = -2x^{-2} \Rightarrow \frac{d}{dx}(vx^{-4}) = -2x^{-2}$$

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Integrate:

$$vx^{-4} = \int -2x^{-2}dx = 2x^{-1} + C \Rightarrow v = x^4(2x^{-1} + C) = 2x^3 + Cx^4$$

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Recall $v = y^{-2} \Rightarrow y = \frac{1}{\sqrt{2x^3+Cx^4}}$

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Solve:

$$\frac{dy}{dx} - 3y = -2y^{1/2}$$

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Step 1: Identify form:

$$P(x) = -3, \quad Q(x) = -2, \quad n = \frac{1}{2}$$

Step 2: Divide by $y^{1/2}$:

$$y^{-1/2}\frac{dy}{dx} - 3y^{1/2} = -2$$

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Let $v = y^{1-1/2} = y^{1/2} \Rightarrow \frac{dv}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$

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Let $v = y^{1-1/2} = y^{1/2} \Rightarrow \frac{dv}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$

$$\Rightarrow y^{-1/2}\frac{dy}{dx} = 2\frac{dv}{dx}$$

Example 3(continued)

Substitute:

$$2\frac{dv}{dx} - 3v = -2 \Rightarrow \frac{dv}{dx} - \frac{3}{2}v = -1$$

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$$\frac{d}{dx}\left(ve^{-\frac{3}{2}x}\right) = -e^{-\frac{3}{2}x}$$

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$$ve^{-\frac{3}{2}x} = \int -e^{-\frac{3}{2}x}dx = \frac{2}{3}e^{-\frac{3}{2}x} + C \Rightarrow v = \frac{2}{3} + Ce^{\frac{3}{2}x}$$

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Recall $v = y^{1/2} \Rightarrow y = \left(\frac{2}{3} + Ce^{\frac{3}{2}x}\right)^2$

Definition of Exact Equations

General Form

A differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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Concept

If exact, then there exists a function $\phi(x, y)$ such that:

$$d\phi = M\,dx + N\,dy$$

and the solution is given by $\phi(x, y) = C$.

Example 1: Exact Equation

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$$(2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$$

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Compute partial derivatives:

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y$$

Solve:

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Compute partial derivatives:

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is **exact**.

Solution of Example 1

Integrate M with respect to x:

$$\phi(x, y) = \int M\, dx = \int (2xy + y^2)\, dx = x^2y + xy^2 + h(y)$$

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Compare with $N = x^2 + 2xy$, so $h'(y) = 0 \Rightarrow h(y) = C$.

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Compare with $N = x^2 + 2xy$, so $h'(y) = 0 \Rightarrow h(y) = C$.Hence,

$$\phi(x,y) = x^2y + xy^2 = C$$

This is the required solution.

Example 2: Exact Equation

Solve:

$$(3x^2y - y^3)\,dx + (x^3 - 3xy^2)\,dy = 0$$

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$$M = 3x^2y - y^3, \quad N = x^3 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 3x^2 - 3y^2, \quad \frac{\partial N}{\partial x} = 3x^2 - 3y^2$$

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Differentiate with respect to y:

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Example 2: Exact Equation[continued]

Differentiate with respect to y:

$$\frac{\partial \phi}{\partial y} = x^3 - 3xy^2 + h'(y)$$

Compare with N: $h'(y) = 0 \Rightarrow h(y) = 0$

$$\Rightarrow x^3y - xy^3 = C$$

Try Yourself

1. Solve: $\dfrac{dy}{dx} + 2y = e^{-x}$
2. Test for exactness and solve: $(2x + y)\,dx + (x + 2y)\,dy = 0$
3. Solve: $(y - 2x)\,dx + (x - 2y)\,dy = 0$

Summary

- Linear differential equations use the integrating factor method.
- Exact equations satisfy $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- Both types have structured solution techniques.
- These methods are essential for solving real-world physical, chemical, and engineering problems.