

# Complex Analysis

## Theory and Math

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# Presentation Outline

- 1 Introduction
- 2 Definitions
- 3 Bernoulli Differential equation
- 4 Exact Differential Equations
- 5 Practice Problems
- 6 Summary

## Overview

This lecture discusses two important types of first-order differential equations:

- Linear Differential Equations
- Exact Differential Equations

These equations have specific solution techniques based on their structure.

## Complex number

Any number of the form  $x + iy$  is called a complex number where  $x, y \in \mathbb{R}$ . A complex number defined by  $z = x + iy$

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## Conjugate complex number

Any number of the form  $x - iy$  is called a complex number where  $x, y \in \mathbb{R}$ . A complex number defined by  $\bar{z} = x - iy$ .

## Analytic Function

A complex function  $f(z)$  is said to be analytic at a point  $z_0$  if its derivative exists not only at  $z_0$  but also at each point  $z$  in some neighborhood of  $z_0$ .

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## Cauchy's Integral Formula

If  $f$  is analytic on a simply connected domain  $D$  and  $\gamma$  is a positively oriented, simple closed contour lying in  $D$ , then for any point  $z$  inside  $\gamma$ ,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$



# Theorem

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If  $f(z) = u(x, y) + iv(x, y)$  is analytic in a region  $R$ , and if  $u$  and  $v$  have continuous second-order partial derivatives in  $R$ , then both  $u$  and  $v$  are harmonic in  $R$ .

**Proof.** Since  $f$  is analytic in  $R$ , the real and imaginary parts  $u$  and  $v$  satisfy the Cauchy–Riemann equations:

$$u_x = v_y, \quad u_y = -v_x.$$

Because  $u$  and  $v$  have continuous second-order partial derivatives, we may differentiate the Cauchy–Riemann equations again.

Differentiate  $u_x = v_y$  with respect to  $x$ :

$$u_{xx} = v_{yx}.$$

Differentiate  $u_y = -v_x$  with respect to  $y$ :

$$u_{yy} = -v_{xy}.$$

Since mixed partial derivatives are equal ( $v_{xy} = v_{yx}$ ), we obtain

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Thus,

$$\Delta u = u_{xx} + u_{yy} = 0,$$

which shows that  $u$  is harmonic in  $R$ .

Similarly, differentiate  $u_x = v_y$  with respect to  $y$ :

$$u_{xy} = v_{yy}.$$

Differentiate  $u_y = -v_x$  with respect to  $x$ :

$$u_{yx} = -v_{xx}.$$

Again using  $u_{xy} = u_{yx}$ , we get

$$v_{yy} = -v_{xx},$$

or equivalently,

$$v_{xx} + v_{yy} = 0.$$

Thus,

$$\Delta v = v_{xx} + v_{yy} = 0,$$

so  $v$  is harmonic in  $R$ .

Hence, both  $u$  and  $v$  are harmonic functions in the region  $R$ .

Show that the function  $u = 2x - x^3 + 3xy^2$  is harmonic and also find the harmonic conjugate if  $f(z) = u + iv$  is analytic.

**Solution:**

Given that,  $u = 2x - x^3 + 3xy^2$

$$\therefore \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 = \Phi_1(x, y) \quad \dots\dots\dots(1)$$

$$\frac{\partial u}{\partial y} = 6xy = \Phi_2(x, y) \quad \dots\dots\dots(2)$$

$$\frac{\partial^2 u}{\partial x^2} = -6x \quad \dots\dots\dots(3)$$

$$\frac{\partial^2 u}{\partial y^2} = 6x \quad \dots\dots\dots(4)$$

Adding equation (3) & (4) we get,

## Example 2: Linear Equation(continued )

Previously :

$$\frac{d}{dx} \left( \frac{y}{x} \right) = x$$

Integrate both sides:

$$\frac{y}{x} = \frac{x^2}{2} + C$$

## Example 2: Linear Equation(continued )

Previously :

$$\frac{d}{dx} \left( \frac{y}{x} \right) = x$$

Integrate both sides:

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$\Rightarrow y = \frac{x^3}{2} + Cx$$

# Bernoulli Equation - Definition

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where  $n \neq 0, 1$ .

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To solve it:

- 1 Divide both sides by  $y^n$
- 2 Substitute  $v = y^{1-n}$
- 3 Reduce to linear form
- 4 Solve using integrating factor



# Example 1

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$$\frac{dy}{dx} + y = y^2 \sin x$$

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Substitute:

$$-\frac{dv}{dx} + v = \sin x \Rightarrow \frac{dv}{dx} - v = -\sin x$$

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Integrating factor:  $\mu(x) = e^{-x}$

## Example 1(continued )

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Multiply through:

$$\frac{d}{dx}(ve^{-x}) = -\sin x e^{-x}$$

Integrate:

$$ve^{-x} = \int -\sin x e^{-x} dx$$

Result:

$$v = \frac{\sin x + \cos x}{2} + Ce^x \Rightarrow y = \frac{1}{\frac{\sin x + \cos x}{2} + Ce^x}$$

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$$\text{Let } v = y^{-2} \Rightarrow \frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$$

## Example 2(continued )

Solve for  $\frac{dy}{dx}$ :

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} = -\frac{1}{2} \frac{dv}{dx}$$

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Substitute:

$$-\frac{1}{2} \frac{dv}{dx} + \frac{2}{x} v = x^2 \Rightarrow \frac{dv}{dx} - \frac{4}{x} v = -2x^2$$



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$$\mu(x) = e^{\int -\frac{4}{x} dx} = x^{-4}$$

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$$x^{-4} \frac{dv}{dx} - \frac{4}{x^5} v = -2x^{-2} \Rightarrow \frac{d}{dx}(vx^{-4}) = -2x^{-2}$$

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$$vx^{-4} = \int -2x^{-2} dx = 2x^{-1} + C \Rightarrow v = x^4(2x^{-1} + C) = 2x^3 + Cx^4$$

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$$\text{Recall } v = y^{-2} \Rightarrow y = \frac{1}{\sqrt{2x^3 + Cx^4}}$$

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$$\frac{dy}{dx} - 3y = -2y^{1/2}$$

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$$\text{Let } v = y^{1-1/2} = y^{1/2} \Rightarrow \frac{dv}{dx} = \frac{1}{2}y^{-1/2} \frac{dy}{dx}$$



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$$\Rightarrow y^{-1/2} \frac{dy}{dx} = 2 \frac{dv}{dx}$$

## Example 3(continued )

Substitute:

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$$\text{Recall } v = y^{1/2} \Rightarrow y = \left( \frac{2}{3} + Ce^{\frac{3}{2}x} \right)^2$$

# Definition of Exact Equations

## General Form

A differential equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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is said to be **exact** if

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## Concept

If exact, then there exists a function  $\phi(x, y)$  such that:

$$d\phi = M dx + N dy$$

and the solution is given by  $\phi(x, y) = C$ .



# Example 1: Exact Equation

**Solve:**

$$(2xy + y^2) dx + (x^2 + 2xy) dy = 0$$

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Here,

$$M = 2xy + y^2, \quad N = x^2 + 2xy$$

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Compute partial derivatives:

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y$$

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Compute partial derivatives:

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , the equation is **exact**.

# Solution of Example 1

Integrate  $M$  with respect to  $x$ :

$$\phi(x, y) = \int M \, dx = \int (2xy + y^2) \, dx = x^2y + xy^2 + h(y)$$

# Solution of Example 1

Integrate  $M$  with respect to  $x$ :

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Differentiate  $\phi$  with respect to  $y$ :

$$\frac{\partial \phi}{\partial y} = x^2 + 2xy + h'(y)$$

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Compare with  $N = x^2 + 2xy$ , so  $h'(y) = 0 \Rightarrow h(y) = C$ .

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Compare with  $N = x^2 + 2xy$ , so  $h'(y) = 0 \Rightarrow h(y) = C$ . Hence,

$$\phi(x, y) = x^2y + xy^2 = C$$

**This is the required solution.**



## Example 2: Exact Equation

**Solve:**

$$(3x^2y - y^3) dx + (x^3 - 3xy^2) dy = 0$$

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$$M = 3x^2y - y^3, \quad N = x^3 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 3x^2 - 3y^2, \quad \frac{\partial N}{\partial x} = 3x^2 - 3y^2$$

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Since these are equal, the equation is exact.

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Integrate  $M$  with respect to  $x$ :

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Differentiate with respect to  $y$ :

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## Example 2: Exact Equation[continued]

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## Example 2: Exact Equation[continued]

Differentiate with respect to  $y$ :

$$\frac{\partial \phi}{\partial y} = x^3 - 3xy^2 + h'(y)$$

Compare with  $N$ :  $h'(y) = 0 \Rightarrow h(y) = 0$

$$\Rightarrow x^3y - xy^3 = C$$



# Try Yourself

- 1 Solve:  $\frac{dy}{dx} + 2y = e^{-x}$
- 2 Test for exactness and solve:  $(2x + y) dx + (x + 2y) dy = 0$
- 3 Solve:  $(y - 2x) dx + (x - 2y) dy = 0$

# Summary

- Linear differential equations use the integrating factor method.
- Exact equations satisfy  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ .
- Both types have structured solution techniques.
- These methods are essential for solving real-world physical, chemical, and engineering problems.