$$\rho \frac{D\bar{V}}{Dt} = \rho \vec{f_b} + \mu \nabla^2 \bar{V} - \nabla p$$

as $\nabla \bar{V} = 0$

incompressible

Exact Solutions for the eqn will arise for different flows.

Parallel flows:

→ Let's choose x to be direction of flow

$u \neq 0 \qquad v = w = 0$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{\,0} + \cancel{\frac{\partial w}{\partial z}}^{\,0} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = 0 \implies u = f(y, z, t)$$

$$\rho\left(\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}}^{0} + \cancel{v\frac{\partial u}{\partial y}}^{0} + \cancel{w\frac{\partial w}{\partial z}}^{0}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\cancel{\rho\left(\frac{\partial v}{\partial t} + \cdots\right)}^{0} = -\frac{\partial p}{\partial y} + \cancel{\mu\left(\partial^2 \cdots\right)}^{0}$$

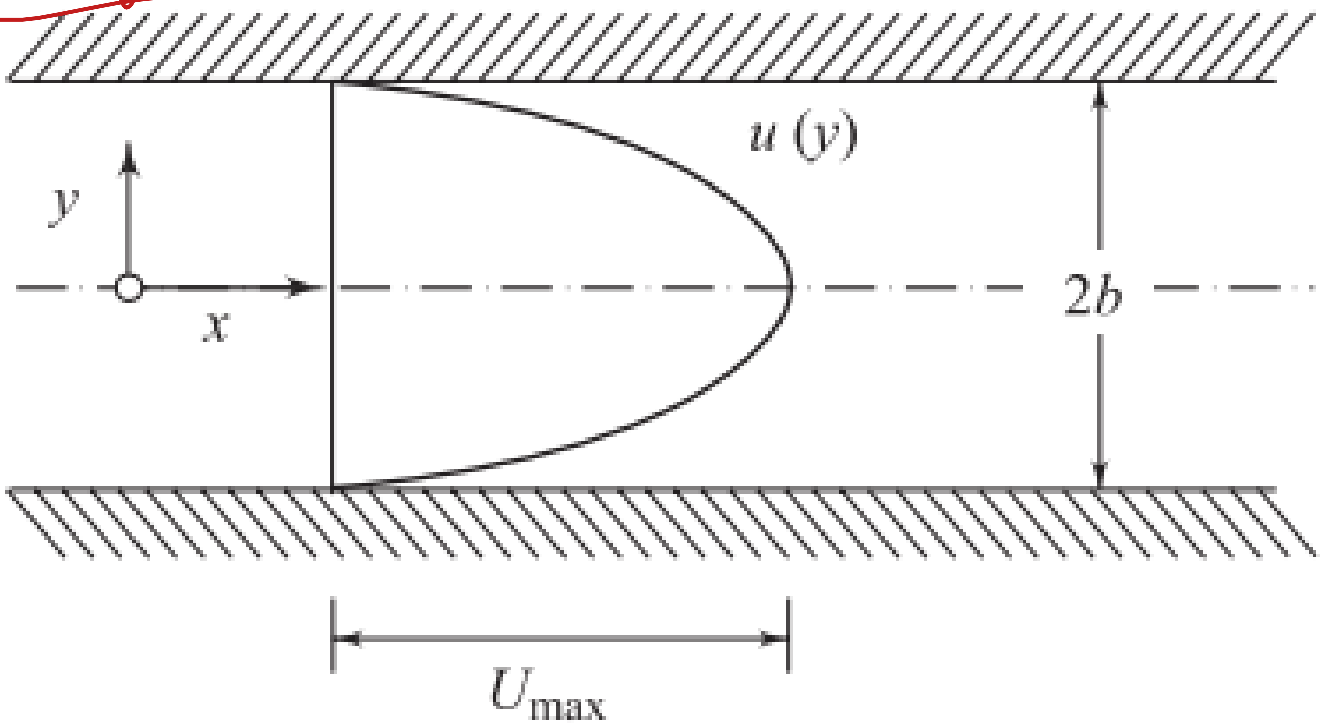
$$\boxed{\frac{\partial p}{\partial y} = 0 \quad \Big| \quad \frac{\partial p}{\partial z} = 0}$$

$$P = p(x)$$

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Parallel flow in Straight Channel :

steady



$$\cancel{\frac{\partial u}{\partial t}} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial^2 y} + \cancel{\frac{\partial^2 u}{\partial^2 z}}\right]$$

steady $\qquad\qquad$ $u = u(y)$

$$\frac{dP}{dx} = \mu \frac{d^2 u}{dy^2}$$

at $y = b \quad u = 0 \qquad y = -b \quad u = 0$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + c_1$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + c_1 y + c_2$$

$$0 = \frac{1}{2\mu}\frac{dp}{dx}b^2 + c_1 b + c_2 \qquad \frac{1}{2\mu}\frac{dp}{dx}b^2 - c_1 b + c_2 = 0$$

$$-\frac{1}{2\mu}\frac{dp}{dx}b^2 = c_2$$

$$c_1 = 0$$

$$\boxed{u = -\frac{1}{2\mu}\frac{dp}{dx}(b^2 - y^2)}$$

$$Re = \frac{\rho l V}{\mu}$$

D) Max Vel

$$\boxed{u_{max} = -\frac{b^2}{2\mu}\frac{dp}{dx}}$$

$$\boxed{Re = \frac{(2b)U_{av}}{\nu}}$$

2) Average Vel

$Re \le 2300$
↳ laminar

$$u_{av} = \frac{Q}{2b} = \frac{\text{flowrate}}{\text{flowarea}}$$

$$= \frac{1}{2b}\int_{-b}^{b} u\,dy$$

$$\boxed{u_{av} = \frac{2}{3}u_{max}}$$

3) Shear Stress at wall

$$\tau_b = \mu\left(\frac{\partial u}{\partial y}\right)_b = \frac{2\mu b^2}{b\,2\mu}\frac{dP}{dx} = -\frac{2\mu}{b}u_{max}$$

$$\boxed{\tau_b = -\frac{2\mu}{b}u_{max}}$$

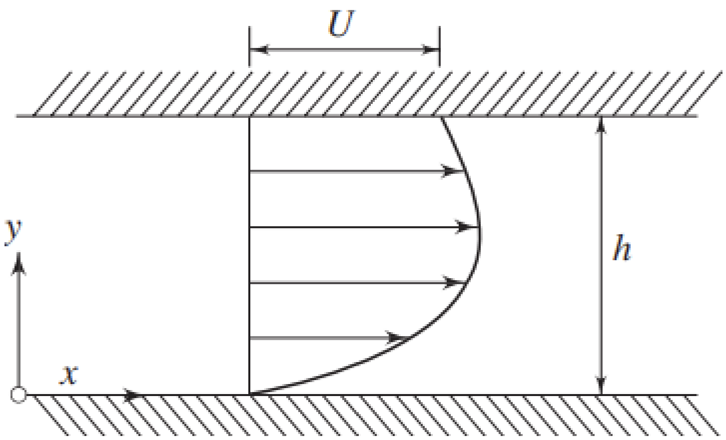
$$\boxed{Re = \frac{\rho l v}{\mu}}$$

4) $$C_f = \frac{|\tau_b|}{\frac{1}{2}\rho v_{av}^2} = \frac{12}{\frac{\rho v_{av}(2b)}{\mu}} = \boxed{\frac{12}{Re} = C_f}$$

Couette Flow

governing equation : $\dfrac{dp}{dx} = \mu \dfrac{d^2u}{dy^2}$

$y=0 \quad u=0 \qquad\qquad y=h \quad u=U$



$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

$y = \quad u=0 \qquad C_2 = 0$

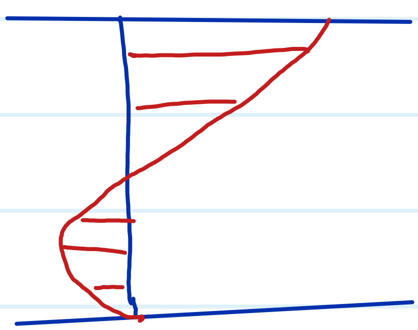
$$C_1 = \frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx}h$$

$$u = \frac{y}{h}U - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}\left(1-\frac{y}{h}\right)$$

$$\boxed{\frac{u}{U} = \frac{y}{h} + P\,\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad P = -\frac{h^2}{2\mu U}\left(\frac{dp}{dx}\right)}$$

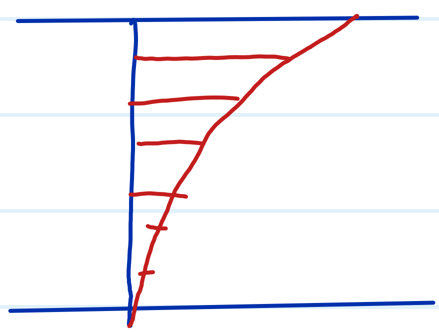
acording to the value of P, the flow changes

P = pressure gradient



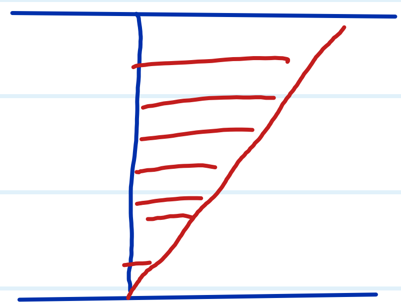
$P < -1$

$\frac{dP}{dx} >> 0$



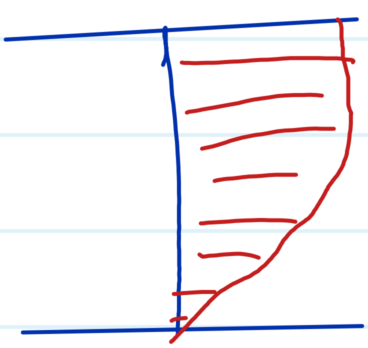
$-1 < P < 0$

$\frac{dP}{dx} > 0$



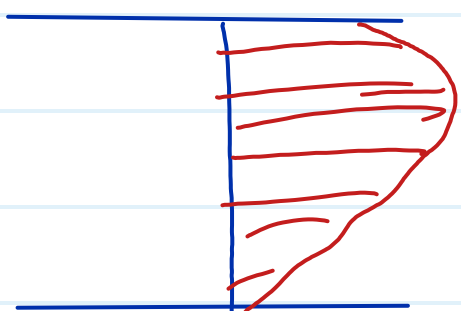
$P = 0$

$\frac{dP}{dx} = 0$



$\frac{dP}{dx} < 0$

$P > 0$



$\frac{dP}{dx} << 0$

$P > 1$

$$\frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1-\frac{y}{h}\right)$$

$$P = -\frac{h^2}{2\mu U}\frac{dP}{dx}$$

$$\frac{du}{dy} = \frac{U}{h} + \frac{PU}{h}\left(1-2\frac{y}{h}\right)$$

max & min vel

$$\frac{du}{dy} = 0$$

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P}$$

$P > 1$

$\frac{y}{h} < h$

max vel at location below the moving plate

$P = 1$

$y = h$

max vel at $y = h$

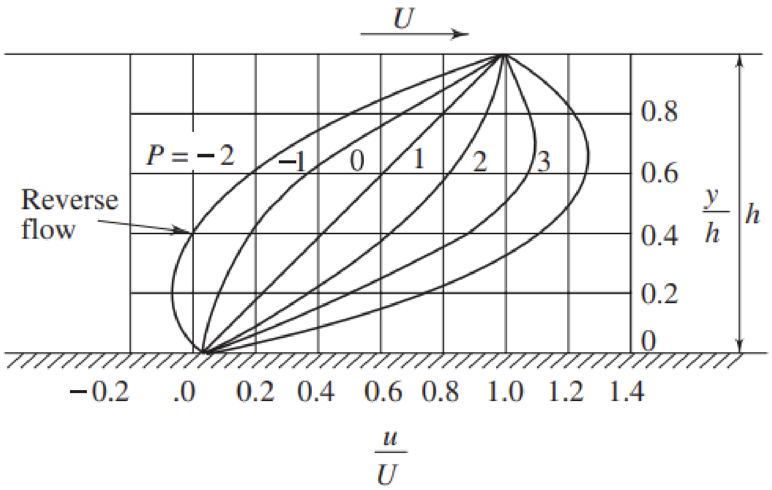
$P = -1$

$\frac{y}{h} = 0$

$P < -1$

$$U_{max} = \frac{U(1+P)^2}{4P} \qquad P \geq 1$$

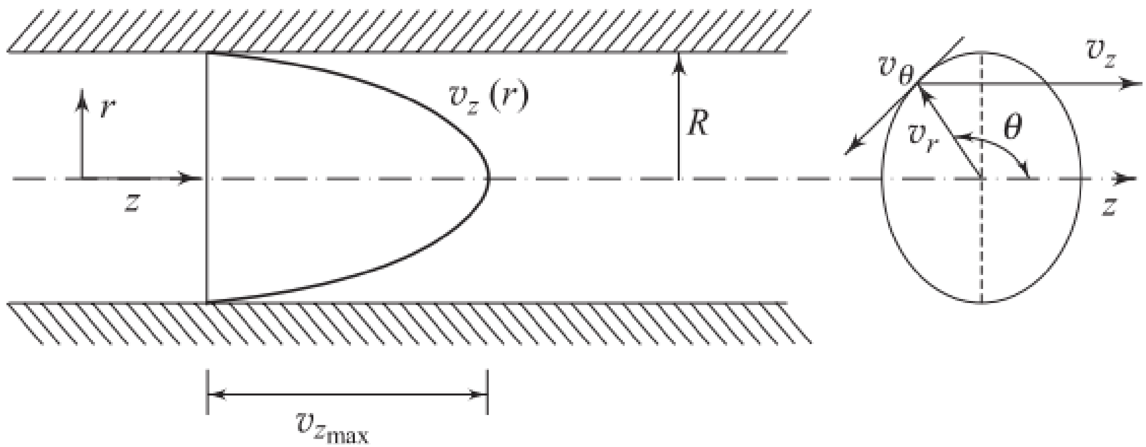
$$U_{min} = \frac{U(1+P)^2}{4P} \qquad P \leq 1$$



Hagen Poiseuille Flow

flow inside a circular tube

$V_z \neq 0 \qquad V_r = 0 \qquad V_\theta = 0$



continuity Eqn.

$$\cancel{\frac{\partial V_r}{\partial r}}^{0} + \cancel{\frac{V_r}{r}}^{0} + \frac{\partial V_z}{\partial z} = 0$$

rotational Symmt

$$\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$$

$$\frac{\partial V_z}{\partial z} = 0$$

rotational Symm

$$\left(\frac{\partial}{\partial \theta}(\text{anything})\right) = 0$$

$$V_z = f(r, \theta)$$

Navier Stokes $\quad v_r = 0 \quad v_\theta = 0 \quad \frac{\partial v_z}{\partial z} = 0 \quad \frac{\partial}{\partial \theta}(\;) = 0$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right)$$

Steady flow $\quad \frac{\partial v_z}{\partial t} = 0$

$$\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

$r = 0 \quad v_z$ is finite

$r = R \quad v_z = 0$

$$r\frac{\partial^2 v_z}{\partial r^2} + \frac{\partial v_z}{\partial r} = \frac{1}{\mu}\frac{\partial p}{\partial z} r$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z} r$$

$$r\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\frac{\partial p}{\partial z} r^2 + A$$

$$\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\frac{\partial p}{\partial z} r + \frac{A}{r}$$

$$\boxed{v_z = \frac{1}{4\mu}\frac{\partial p}{\partial z} r^2 + A\ln r + B}$$

$r = 0 \quad v_z = \text{finite} \Rightarrow A = 0$

$r = R \quad v_z = 0 \qquad B = -\frac{1}{4\mu}\frac{\partial p}{\partial z} R^2$

$$\boxed{V_{max} = \frac{R^2}{4\mu}\left(-\frac{\partial p}{\partial z}\right)} \quad ☆$$

$$V_{av} = \frac{Q}{\pi R^2} = \frac{\int_0^{2\pi} 2\pi r\, v_z(r)\, dr}{\pi R^2}$$

$$\boxed{V_{z_{max}} = 2 V_{z_{av}}} \quad ☆$$

$$\boxed{Q = -\frac{\pi R^4}{128\mu}\left(\frac{\partial p}{\partial z}\right)} \quad ☆$$

$$\tau\Big|_r = \mu \frac{dv_z}{dr} \qquad \frac{\partial v_r}{dr} = \frac{1}{2\mu}\left(\frac{\partial p}{dz}\right) r$$

$$\boxed{\tau\Big|_r = \frac{1}{2}\left(\frac{dp}{dz}\right) r}$$ ☆

$$\boxed{\tau\Big|_R = \frac{1}{2}\left(\frac{dp}{dz}\right) R}$$ ☆

over a pipe of length l

$$F_s = \tau_{max}\, 2\pi R l$$

$$F_s = -\pi R^2 \text{(pressure drop between lengths)}$$

negative sign means fo

$$V_{z(av)} = -\frac{1}{8\mu}\left(\frac{dp}{dz}\right) R^2$$

$$h_l = \frac{\text{presure drop}}{\rho g}$$

$$= \frac{-\frac{8\mu V_{z(av)}}{R^2}}{\rho g} = \frac{32\mu (V_z)_{av}^2 l}{D^2 \rho g} \frac{1}{(V_z)_{av}}$$

but

$$h_f = \frac{f l (V_z)_{av}^2}{2g}$$ (Darcy - Weisbach form)

f = Darcy friction factor $= \frac{64}{Re}$ $\quad Re = \frac{\rho (V_z)_{av} D}{\mu}$

skin friction coeff.

$$C_f = \frac{|\tau_R|}{\frac{1}{2}\rho (v_z)_{av}^2} = \frac{16}{Re}$$

Fanning's friction factor.

$$\boxed{f = 4 C_f}$$

laminar $Re \leq 2300$ turbulent $Re \geq 4000$