Dimensional Analysis :

1) Buckingham π theorem :

→ No of π terms (n) = $K - R$
where the flow phenomenon depends on K-parameters with R-basic dimension

Steps

1) Identify K parameters
2) Express K parameters in terms of $(M, L, T) / (F, L, T)$
3) Identify R value
4) Select R out of K parameters.
   → Combined, these parameters must contain all basic dimensions
   → Each parameter must be independent
5) Select one additional parameter and start forming π terms

Frequently Used Non-dimensional Numbers:

→ Reynolds Number = $\dfrac{\rho V l}{\mu}$

→ Euler Number = $\dfrac{\Delta p}{\rho v^2}$

→ Froude Number = $\dfrac{V}{\sqrt{gl}}$

→ Weber Number = $\dfrac{\rho V^2 l}{\sigma}$

→ Mach Number = $\dfrac{V}{c}$

→ Strouhal Number = $\dfrac{\omega l}{V}$ (ω = freq)

→ Drag coeff, lift coeff = $\dfrac{F}{\frac{1}{2}\rho V^2 l^2}$

Similitude and Modelling.

$$\boxed{\pi_N^{(p)} = \pi_N^{(m)}}$$

Irrotational $\nabla \times \bar{V} = 0$, Incompressible $\nabla \cdot \bar{V} = 0$

$2\bar{\omega} = \Omega = \nabla \times \bar{V}$ | $\frac{\partial \phi}{\partial x} = u$ $\quad \frac{\partial \phi}{\partial y} = v$

$\frac{\partial \psi}{\partial y} = u \quad -\frac{\partial \psi}{\partial x} = v$

$\boxed{-\frac{\partial \psi}{\partial r} = V_\theta}$ $\quad \boxed{\frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_r}$ $\quad \nabla^2 \phi = 0 \quad \nabla^2 \psi = 0$

$d\phi = 0 \qquad u\,dx + v\,dy = 0$ → equipotential line

$v\,dx - u\,dy = 0$ → streamline

Circulation.

$$\boxed{\oint \vec{V} \cdot \vec{ds} = \Gamma}$$

$$\boxed{\Omega_z = \frac{\Gamma}{A}}$$

Basic Flows

1) Rectilinear Flows:

$$\vec{V} = u\hat{i} + v\hat{j}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$\boxed{\psi = uy - vx + c} \quad \text{— } ①$$

$$\boxed{\phi = ux + vy + c'} \quad \text{— } ②$$

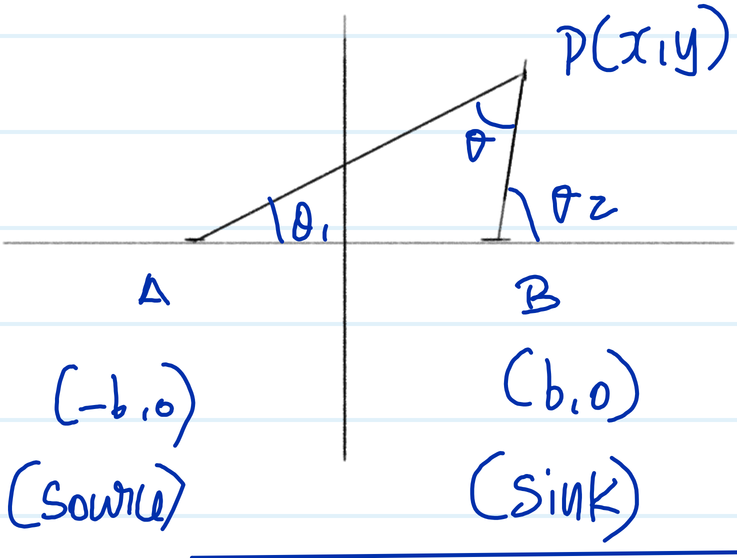
For a Source:

$$\boxed{\psi = \frac{\Lambda}{2\pi}\theta \quad \Big| \quad \phi = \frac{\Lambda}{2\pi}\ln r \quad \Big| \quad V_r = \frac{\Lambda}{2\pi r}}$$

For a Sink:

$$\boxed{\psi = -\frac{\Lambda}{2\pi}\theta \quad \Big| \quad \phi = \frac{\Lambda}{2\pi}\ln r \quad \Big| \quad V_r = \frac{-\Lambda}{2\pi r}}$$

Source and Sink:



$$\psi_A = \frac{\Lambda}{2\pi}\theta_1$$

$$\psi_B = \frac{-\Lambda}{2\pi}\theta_2$$

$$\therefore \psi_{A-B} = \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)$$

$$\boxed{\psi_{A-B} = \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)}$$

Doublet.

$b \to 0 \quad \Lambda \to \infty \qquad 2\Lambda b = \text{constant} = \chi$

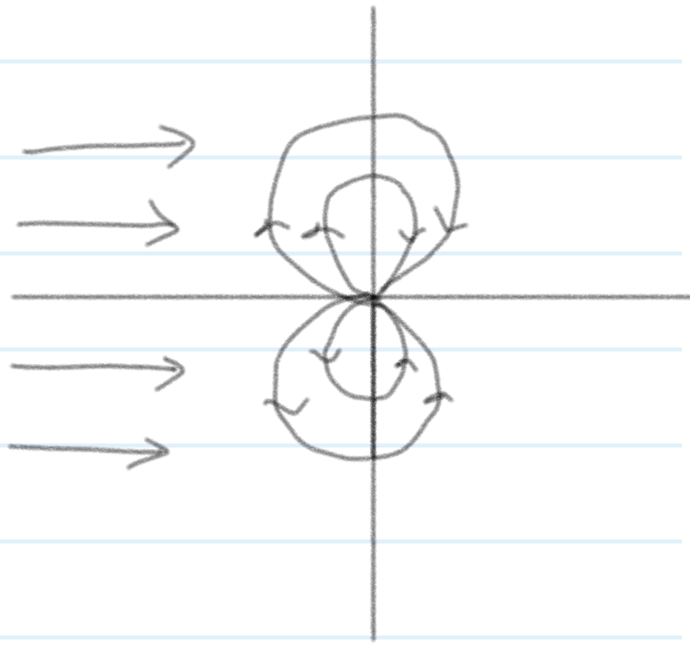
$$\psi = \chi\left(\frac{y}{x^2+y^2}\right) \qquad \psi = \chi\,\frac{\sin\theta}{r}$$

Rectilinear + doublet

$$\psi = -uy + \frac{\chi\sin\theta}{r}$$

$$\psi = -ur\sin\theta + \frac{\chi\sin\theta}{r}$$

$$\psi = -r\sin\theta\left(u - \frac{\chi}{r^2}\right)$$



D $\psi = 0$ when

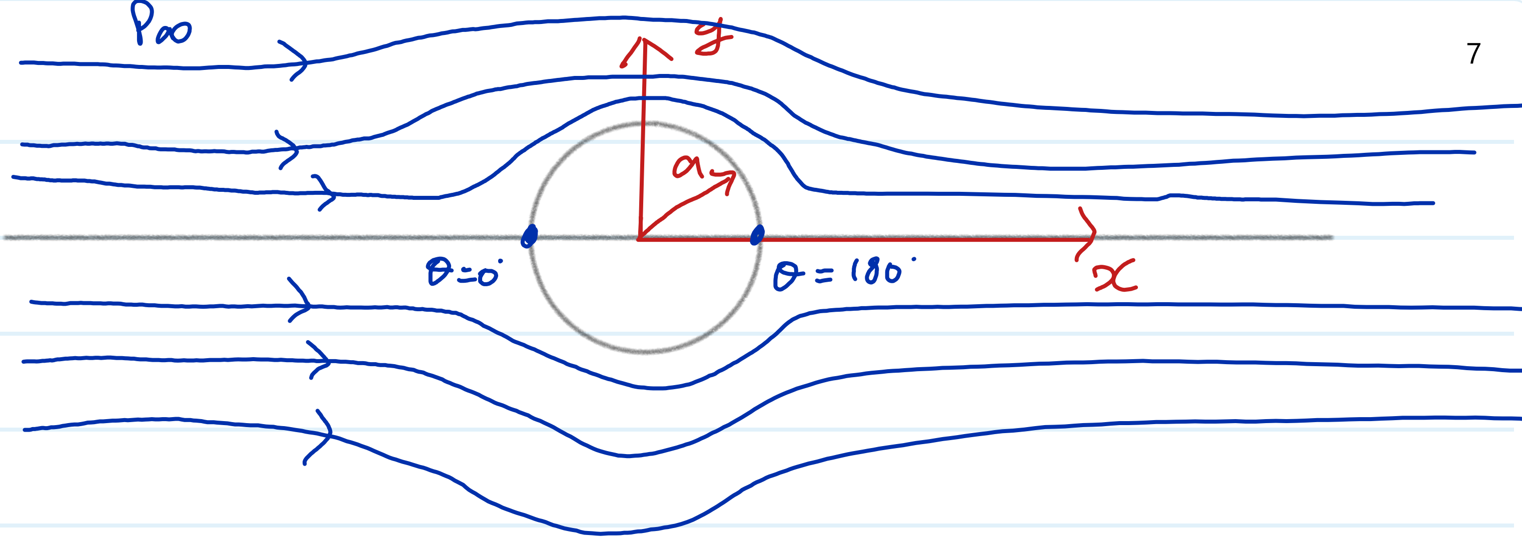
$$\frac{\chi}{r} = ur \qquad \sin\theta\,(0 \text{ or } 180^\circ)$$

$$\Downarrow$$

$$\boxed{a = \sqrt{\frac{\chi}{u}}}$$

$$\psi = -u\sin\theta\left(r - \frac{\chi}{ur}\right)$$

$$\boxed{\psi = -u\sin\theta\left(r - \frac{a^2}{r}\right)}$$



Stagnation point at $\theta = 0, 180$, $r = a$

$$V_\theta = -u\sin\theta\left(1 + \frac{a^2}{r^2}\right) \qquad V_r = u\cos\theta\left(r - \frac{a^2}{r}\right)$$

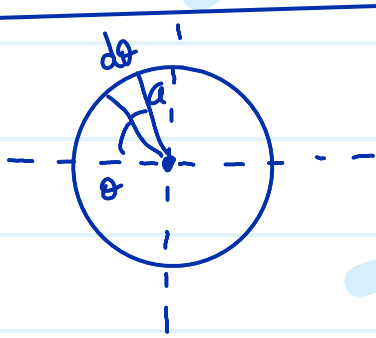
$\boxed{r = a}$ $\quad V_r = 0$

$\boxed{V_\theta = -2u\sin\theta}$ θ 20, 180 stagnat.

$$P_\infty + \frac{1}{2}\rho u^2 = P + \frac{1}{2}\rho\left(-2u\sin\theta\right)^2$$

$$\boxed{C_p = \frac{P - P_\infty}{\frac{1}{2}\rho u^2} = 1 - 4\sin^2\theta}$$

pressure co-eff



$$F_x = \int_0^{2\pi} P\cos\theta \, a\,d\theta = 0$$

$$F_y = -\int_0^{2\pi} P\sin\theta \, a\,d\theta = 0$$

Vortex Flow

$V_r = 0 \qquad V_\theta = \dfrac{C_1}{r}$

$\omega_z = 0 \qquad \Omega_{r\theta} = 0$

$$\Gamma = \oint_0^{2\pi} V_\theta \, r \, d\theta = \frac{C}{\cancel{r}} \cancel{r} \, 2\pi$$

$$\boxed{C = \frac{\Gamma}{2\pi}} \qquad \boxed{V_\theta = \frac{\Gamma}{2\pi r}}$$

$$V_\theta = \frac{\partial \psi}{\partial r} \qquad \boxed{\psi = \frac{\Gamma}{2\pi} \ln(r)}$$

Rectilinear, Doublet, Vortex.

$$\psi = -u\sin\theta\, r + \frac{ua^2}{r} + \frac{\Gamma}{2\pi}\ln r$$

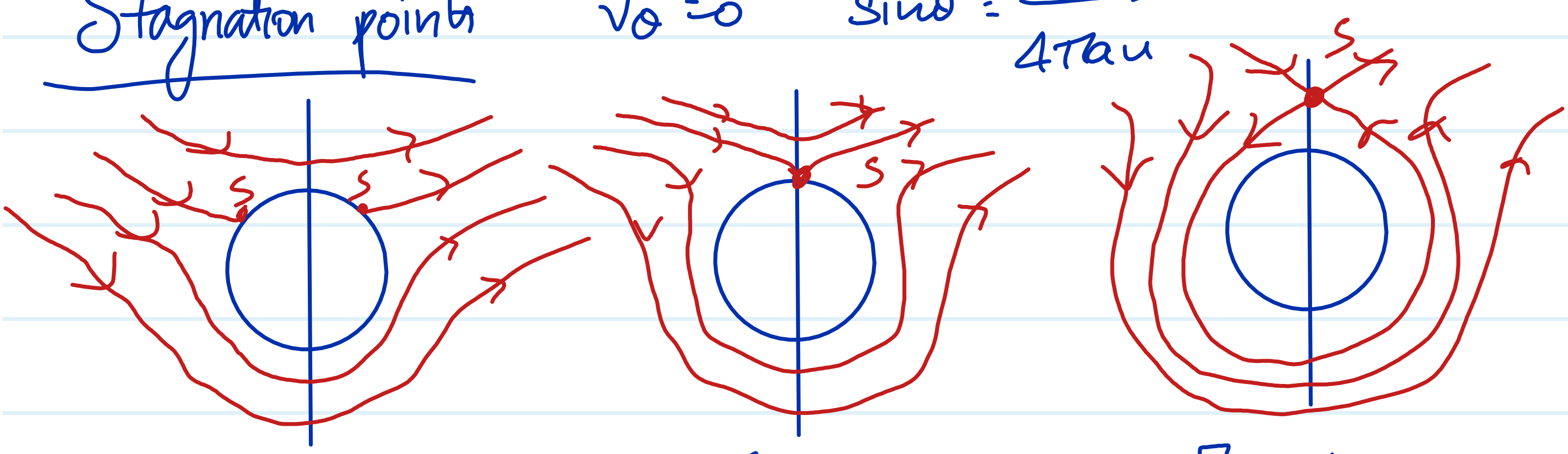
$$\boxed{\frac{\partial\psi}{\partial r} : V_\theta = -u\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi r}}$$

$$\boxed{\frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad V_r = u\cos\theta\left(1 - \frac{a^2}{r^2}\right)}$$

$$\boxed{\boxed{r = a} \quad V_r = 0 \quad V_\theta = -2u\sin\theta + \frac{\Gamma}{2\pi a}}$$

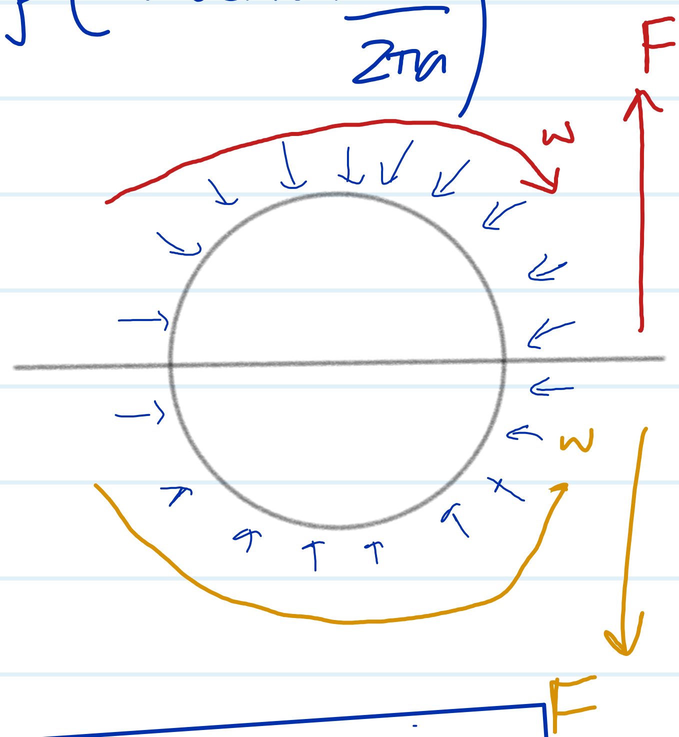
Stagnation points

$V_\theta = 0 \quad \sin\theta = \frac{\Gamma}{4\pi a u}$



$\frac{\Gamma}{4\pi a u} < 1$ $\quad$ $\frac{\Gamma}{4\pi a u} = 1$ $\quad$ $\frac{\Gamma}{4\pi a u} > 1$

$$P_\infty + \frac{1}{2}\rho u^2 = P + \frac{1}{2}\rho\left(-2u\sin\theta + \frac{\Gamma}{2\pi a}\right)^2$$



$F_x = 0$

$$\boxed{F_y = -\rho u \Gamma}$$

Γ = anticlockwise (neg)

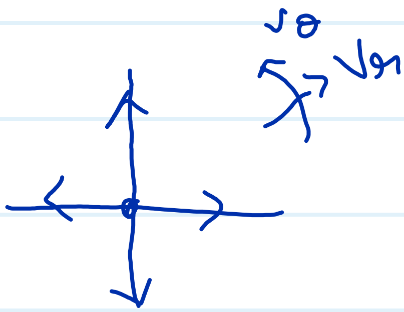
2) Source / Sink

$$\dot m = 2\pi r V_r \rho$$

$\dot m$ = mass flow rate

Λ = volume flow rate = $\dfrac{\dot m}{\rho}$

K = strength = $\dfrac{\dot m}{2\pi\rho} = \dfrac{\Lambda}{2\pi}$



$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \cancel{v_\theta}^{\,0} = -\frac{\partial\psi}{\partial r}$$

$$\boxed{\psi = \frac{\dot m}{2\pi\rho}\theta + C_1 = \frac{\Lambda}{2\pi}\theta + C_1}$$

$$v_r = \frac{\partial\phi}{\partial r} \qquad \cancel{v_\theta}^{\,0} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

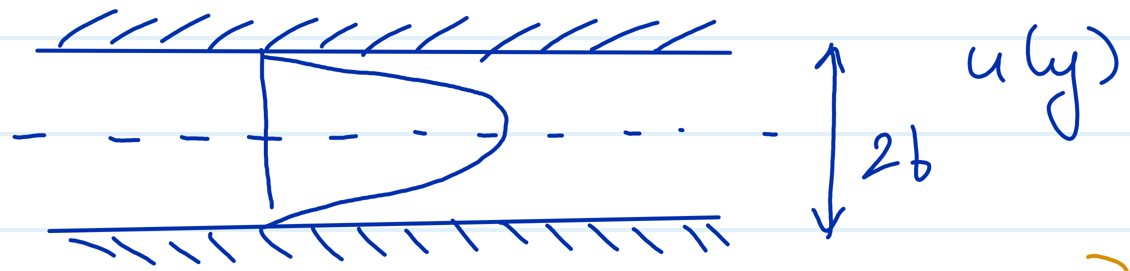
$$\boxed{\phi = \frac{\Lambda}{2\pi}\ln\left(\frac{r}{C_2}\right)}$$

Parallel flows :

→ governing equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

⇒ Straight channel - Parallel flow



→ $\frac{\partial u}{\partial t} = 0 \quad \frac{\partial^2 u}{\partial z^2} = 0$

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 u}{dy^2}$$

$$\boxed{u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + C_1 y + C_2}$$

→ $y = b \quad u = 0$

$y = -b \quad u = 0$

① Velocity Eqn

$$\boxed{u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(b^2 - y^2\right)}$$

② Max Velocity:

$$\boxed{u_{max} = -\frac{b^2}{2\mu}\frac{\partial p}{\partial x}}$$

③ Avg Velocity

$$\boxed{u_{av} = \frac{2}{3} u_{max}}$$

④ Shear stress

$$\tau = \mu \frac{\partial u}{\partial y}$$

$$\boxed{\tau\Big|_{wall} = -\frac{2\mu}{b} u_{max}.}$$

5) Skin friction coefficient

$$C_f = \frac{|\tau|_b}{\frac{1}{2}\rho V_a^2}$$

$$\boxed{C_f = \frac{12}{Re}}$$

6) Reynold's Number

$$Re = \frac{\rho l V}{\mu} = \frac{\rho(2b)U_{avg}}{\mu}$$

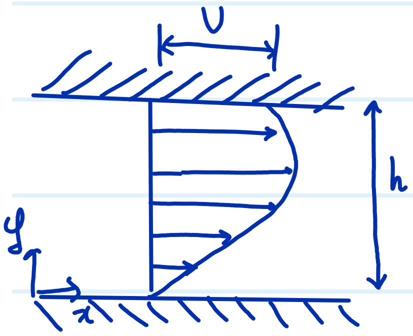
$$\boxed{Re = \frac{\rho(2b)U_{avg}}{\mu}}$$

laminar

$\Rightarrow \quad Re \leq 2300$

Couette Flow :

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{dy^2}$$



$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + C_1 y + C_2$$

$$y=0,\ u=0 \iff C_2 = 0$$

$$y=h,\ u=U \iff C_1 = \frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx} h$$

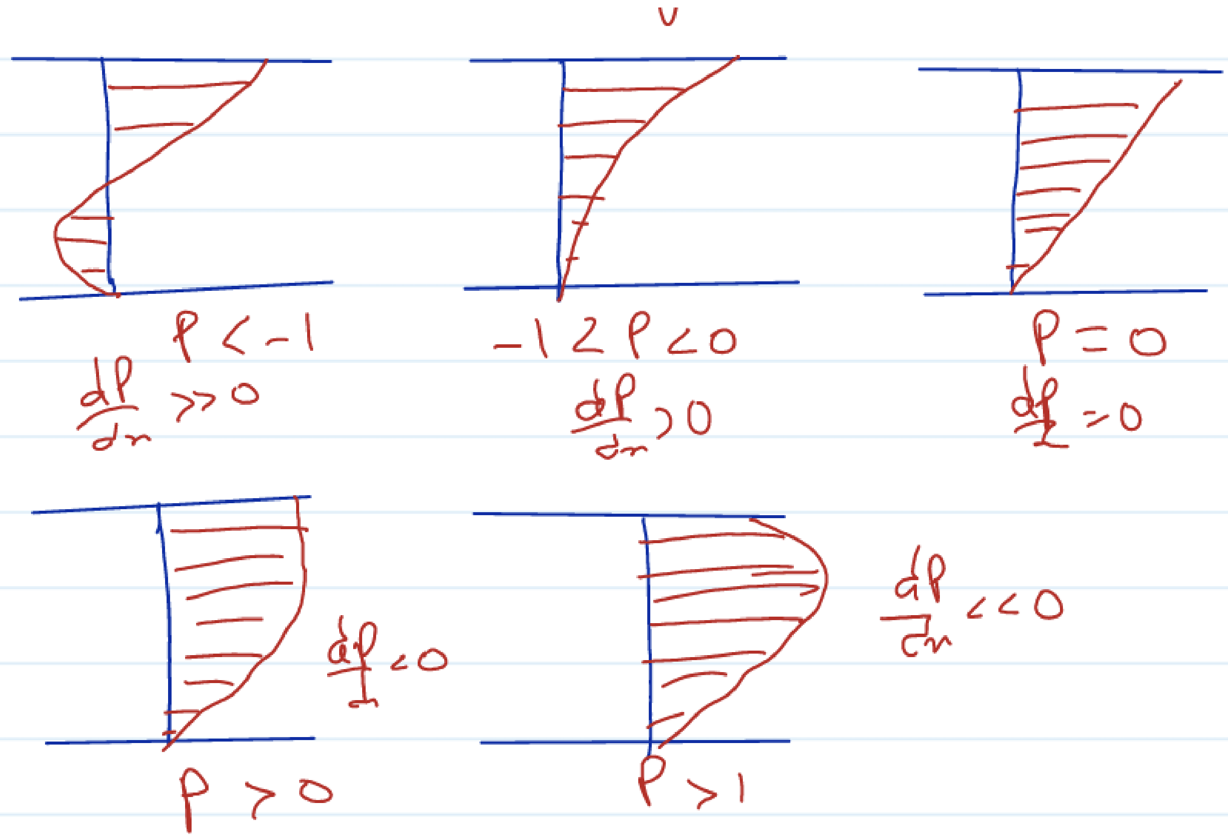
$$u = \frac{U}{2\mu U}\frac{dp}{dx} y^2 + \left(\frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx} h\right) y$$

$$\boxed{\frac{u}{U} = \frac{y}{h} + P \frac{y}{h}\left(1 - \frac{y}{h}\right)}$$

$$\boxed{P = -\frac{h^2}{2\mu U}\left(\frac{\partial P}{\partial x}\right)}$$

according to the value of P, flow changes

P = pressure gradient



$$\frac{u}{U} = \frac{y}{h} + \frac{Py}{h}\left(1 - \frac{y}{h}\right)$$

$$P = -\frac{h^2}{2\mu U}\frac{\partial p}{\partial x}$$

$$\frac{du}{dy} = \frac{U}{h} + \frac{PU}{h}\left[1 - \frac{2y}{h}\right]$$

$$\frac{du}{dy} = 0 \quad \text{min, max.}$$

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P}$$

$$\boxed{u_{max} = \frac{U(1+P)^2}{4P} \quad P \geq 1}$$

$$\boxed{V_{min} = \frac{U(1+P)^2}{4P} \quad P \leq 1}$$

Hagen Poiseulle Flow

$V_z \neq 0 \quad V_r = 0 \quad V_\theta = 0$

$V_{av} = \frac{Q}{\pi R^2}$

① Velocity

$$\boxed{V_z = \frac{1}{4\mu} \frac{\partial P}{\partial z} (r^2 - R^2)}$$

② Max Velocity

$$\boxed{V_{max} = \frac{R^2}{4\mu} \left(\frac{-\partial P}{\partial z} \right)}$$

③ Avg Velocity

$$\boxed{V_{avg} = \frac{V_{z,max}}{2}}$$

④ Flow Rate:

$$\boxed{Q = \frac{-\pi D^4}{128\mu}\left(\frac{dP}{dz}\right)}$$

⑤ Shear Stress

$$\boxed{\tau|_r = \frac{1}{2}\left(\frac{dP}{dz}\right) r}$$

Wall Stress

$$\boxed{\tau|_R = \frac{1}{2}\left(\frac{dP}{dz}\right) R}$$

⑥ Over a pipe of length l.

$$F_s = -\pi R^2 \,(\text{Pressure drop})$$

$$F_s = \tau_{max}\, 2\pi R l$$

$$\text{headloss} = \frac{\text{pressure drop}}{\rho g} = \frac{32\mu (v_z)_{av}^2 l}{D^2 \rho g}\,\frac{1}{(v_z)_{av}}$$

according (Darcy-Weisbach Eq.) $h_l = \dfrac{f\, l\, (v_z)_{av}^2}{2g}$

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f = Darcy friction factor = $\frac{64}{Re}$

C_f = Fanning's friction factor = $\frac{16}{Re}$

$$f = 4C_f$$

8 Reynolds Number

$$\boxed{Re = \frac{\rho (v_z)_{av} D}{\mu}}$$

laminar

$Re \leq 2300$

turbulent

$Re \geq 4000$