Dynamics

$$\Sigma F = m\vec{a}_G$$

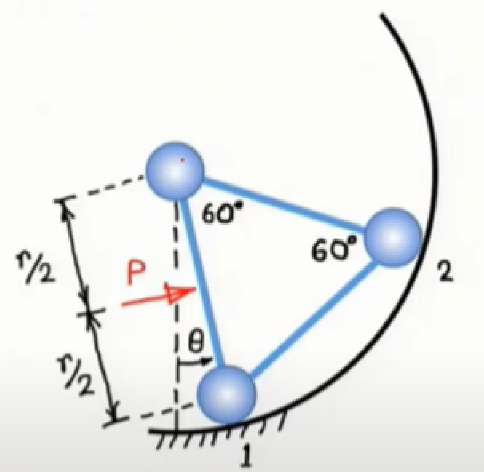
$$T = \frac{1}{2} m \bar{V}_G + \Sigma \frac{1}{2} m_i \rho_i^2$$

relative velocity w.r.t COM

$$\Delta T = W_{a-b}^{C+NC}$$

$$\Delta E = W_{a-b}^{NL}$$

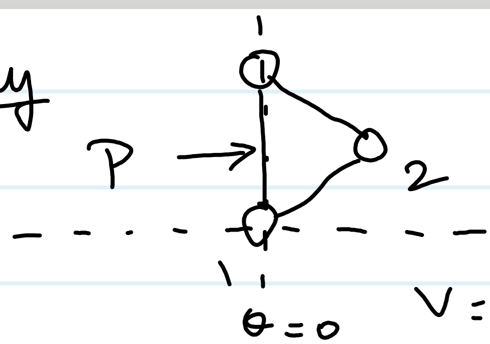
Three identical small spheres (treated as particles), each of mass m, are fixed by three identical light rigid rods as shown. The assembly moves on a smooth circular track in the vertical plane. If the unit starts from rest at θ=0 with a force of constant magnitude P always acting perpendicularly at the mid-point of a rod as shown, determine (a) the minimum force P_{min} which will bring the unit to rest at θ=60 deg, and (b) the common velocity v of spheres 1 and 2 when θ=60 deg if $P=2P_{min}$.



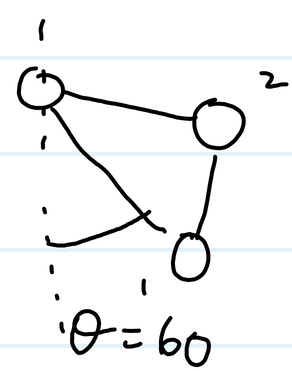
Source: Dynamics, Meriam and Kraige

Sol:

initially



V = 0



$$r(1-\cos\theta)$$

$$\Delta E = W_{a-b}^{NC}$$

(1) $\cancel{T_b}^{\,0} + V_b - (\cancel{T_a}^{\,0} + V_a) = W_{a-b}^{NC}$

$$\cancel{mgr(1-\cos 60^\circ)} + mgr(1-\cos(120))$$

$$- \cancel{mgr(1-\cos 60^\circ)}$$

$\cos(90+30)$

$\cos 90^\circ \sin 30^\circ -$

$\sin 90^\circ$

$\cos 30$

$- \frac{\sqrt{3}}{2}$

$$mgr\left(1-\cos\frac{2\pi}{3}\right) = \int_0^{\pi/3} P\frac{r}{2}\,d\theta$$

$$mg\cancel{r}\left(1-\cos\frac{2\pi}{3}\right) = \frac{P\cancel{r}}{2}\frac{\pi}{3}$$

$$\frac{3}{\cancel{2}}mg = \frac{P\pi}{\cancel{6}_{3}}$$

$$\boxed{P_{min} = \frac{9mg}{\pi}}$$

(2) $\frac{3}{2}mgr + mv^2 = \frac{\cancel{9}^{3}mg}{\cancel{\pi}}\frac{r}{\cancel{2}}\frac{\cancel{\pi}}{\cancel{3}}$

$$\frac{3}{2}\cancel{m}gr = \cancel{m}v^2$$

$$\boxed{v = \sqrt{\frac{3gr}{2}}}$$

→ Impulse Momentum form:

$$\vec{r}_i = \vec{r}_G + \vec{\rho}_i$$

$$\vec{G}_i = m_i \dot{\vec{r}}_i \qquad \dot{\vec{r}}_i = \dot{\vec{r}}_G + \dot{\vec{\rho}}_i$$

$$\vec{G} = \sum m_i \dot{\vec{r}}_i$$

$$= \sum m_i \dot{\vec{r}}_G + \cancelto{0}{\sum m_i \dot{\vec{\rho}}_i} \quad \text{as com}$$

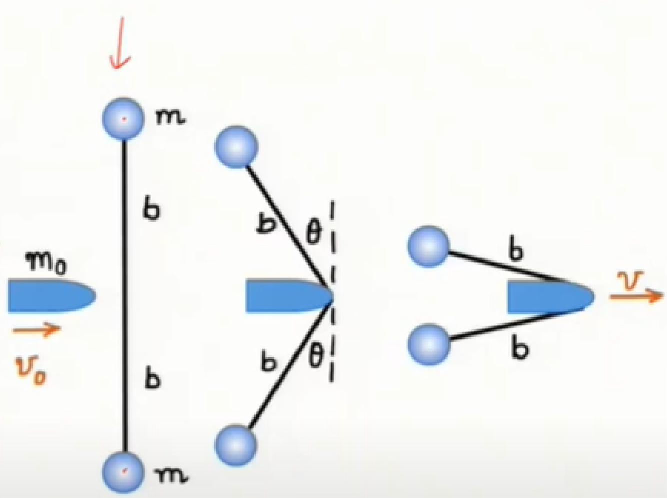
$$\boxed{\sum m_i \dot{\vec{\rho}}_i}$$

$$\boxed{\vec{G} = m\dot{\vec{r}}_G} \quad ★$$

$$\boxed{\Delta\vec{G} = \int_{t_1}^{t_2} \vec{F}\,dt} \quad \text{(Linear Impulse)} \; ★$$

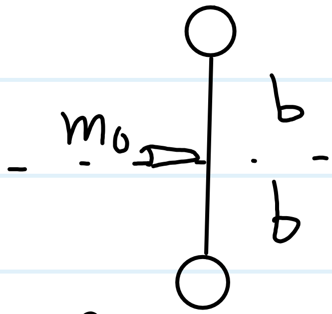
Problem 1:

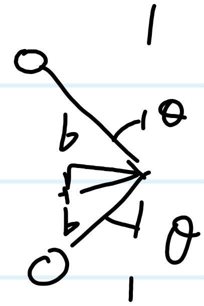
Two small spheres (treated as particles), each of mass m, are connected by a cord of length 2b (measured up to the sphere centers) and initially at rest on a smooth horizontal surface. A body of mass m_0 travelling in a straight line with velocity v_0 hits the cord perpendicularly in the middle causing deflection of the two parts as shown. Determine the velocity v of the mass m_0 as the two spheres near contact with θ approaching 90 deg. Also, find dθ/dt for this condition.



Source: Dynamics, Meriam and Kraige

Solution

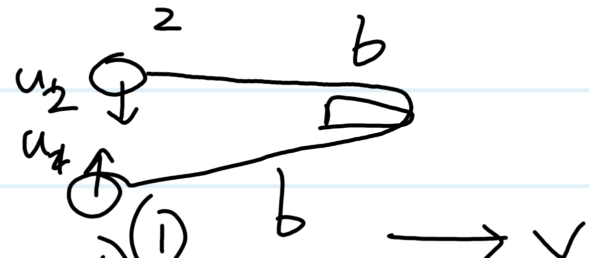




1) No external force: linear momentum conservation

2) Only intern. for Conser of Energy

$$\vec{G}_2 = \vec{G}_1$$



$$m_0 v\hat{i} + m(v\hat{i} + u_1\hat{j}) + m(v\hat{i} - u_2\hat{j}) = m_0 v_0 \hat{i}$$

$$(m + 2m)v = m_0 v_0 \qquad u_1 = u_2 = u = b\dot{\theta}$$

$$v = \frac{m_0}{m_0 + 2m} v_0$$

$$\boxed{\Delta E = 0}$$

$$\frac{1}{2} m_0 v^2 + 2\left[\frac{1}{2} m (v^2 + u^2)\right] = \frac{1}{2} m_0 v_0^2$$

$$(m_0 + 2m)v^2 + 2mu^2 = m_0 v_0^2$$

$$\frac{m_0^2 v_0^2}{m_0 + 2m} + 2mb^2\dot{\theta}^2 = m_0 v_0^2$$

$$2mb^2\dot{\theta}^2 = \left[m_0 - \frac{m_0^2}{m_0+2m}\right] v_0^2$$

$$\dot{\theta}^2 = \frac{\left(\frac{m_0}{m_0+2m}\right) v_0^2}{b^2}$$

$$\boxed{\dot{\theta} = \left(\sqrt{\frac{m_0}{m_0+2m}}\right) \frac{v_0}{b}}$$

→ Angular-Impulse Momentum

Recap

$$\boxed{\begin{aligned} \vec{F} &= m\dot{\vec{r}}_G \\ \Delta E &= W^{nc}_{a-b} \\ G &= m\dot{\vec{r}}_G \end{aligned}}$$

$$\vec{r}_1 \times \left(m_1 \ddot{\vec{r}}_1 = \vec{F}_1 + \Sigma f_{1j} \right)$$

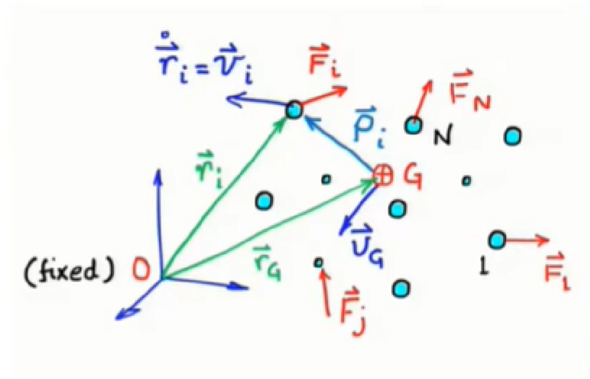
$$\Sigma\left(\vec{r}_i \times m_i \ddot{\vec{r}}_i\right) = \Sigma \vec{r}_i \times \vec{F}_i + \Sigma\left(\vec{r}_i \times \Sigma f_{ij}\right)$$

$$\boxed{\frac{d\dot{\vec{H}}_0}{dt} = \vec{M}_0} \longrightarrow \text{External Moment}$$

$$\boxed{\Delta \vec{H}_0 = \int_{t_1}^{t_2} M_0 \, dt}$$

Angular Momentum (about O)

$$\vec{H}_O = \sum \vec{r}_o \times m\dot{\vec{r}}_i$$



$$\boxed{\begin{aligned} \dot{\vec{r}}_i &= \dot{\vec{r}}_G + \dot{\vec{\rho}}_i \\ \sum m_i \vec{\rho}_i &= 0 \\ \sum m_i \dot{\vec{\rho}}_i &= 0 \end{aligned}}$$

$$\vec{H}_o = \sum (\vec{r}_G + \vec{\rho}_i) \times m(\dot{\vec{r}}_G + \dot{\vec{\rho}}_i)$$

$$\sum \vec{r}_G \times m_i \dot{\vec{r}}_G + \cancel{\sum (\vec{r}_G \times m\dot{\vec{\rho}}_i)}_{0} + \cancel{\sum \vec{\rho}_i \times m\dot{\vec{r}}_G}_{0} + \sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

$$\vec{H}_O = \vec{r}_G \times m\dot{\vec{r}}_G + \underbrace{\sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i}_{H_G^{rel}}$$

linear momentum about com

$$\Rightarrow \vec{H}_O = \underbrace{\vec{r}_G \times m\dot{\vec{r}}_G}_{\text{Angular momentum due to motion of center of mass}} + \underbrace{\sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i}_{\text{Angular momentum due to relative motion about center of mass } \vec{H}_G^{rel}}$$

$$H_G = \sum \vec{\rho}_i \times m_i \dot{\vec{r}}_i$$

$$H_G = \sum \vec{\rho}_i \times m\dot{\vec{r}}_G + \sum \vec{\rho}_i \times m\dot{\vec{\rho}}_i$$

$$\boxed{H_G = \sum \vec{\rho}_i \times m\dot{\vec{\rho}}_i}$$

Angular Momentum (about G)

$$\vec{H}_G = \Sigma\left(\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i\right)$$

For rigid frame

$$\dot{\vec{\rho}}_i = \frac{\partial \vec{\rho}_i}{dt} + \vec{\omega} \times \vec{\rho}_i$$

$$\dot{\vec{\rho}}_i = \vec{\omega} \times \rho_i$$

Rigid frames

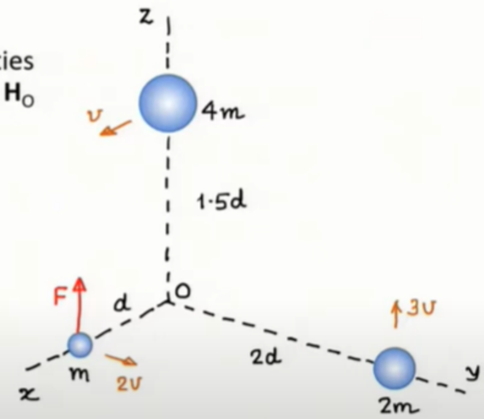
$$\vec{H}_G = \Sigma\left(\vec{\rho}_i \times m_i \omega \times \vec{\rho}_i\right)$$

$$\vec{H}_G = \Sigma \vec{\rho}_i \times m_i \dot{\vec{r}}_i$$

$$\boxed{\dot{\vec{H}}_G = \Sigma \vec{\rho}_i \times \vec{F}_i = \vec{M}_G}$$

Problem 1:

The system of three particles has the indicated masses, velocities and external force as shown. Determine $\mathbf{r}_G$, $d\mathbf{r}_G/dt$, $d^2\mathbf{r}_G/dt^2$, T, $\mathbf{H}_O$ and $d\mathbf{H}_O/dt$ for this system.



Source: Dynamics, Meriam and Kraige

solution

$$\vec{r}_G = \frac{\Sigma m_i r_i}{\Sigma m_i}$$

$$\vec{r}_G = \frac{4m(1.5d\hat{k}) + 2m(2d\hat{j}) + m(d\hat{i})}{4m+2m+m}$$

$$= \frac{6md\hat{k} + 4md\hat{j} + md\hat{i}}{7m}$$

$$\boxed{\bar{r}_G = \left(\frac{d}{7}\right)\hat{i} + \left(\frac{4d}{7}\right)\hat{j} + \left(\frac{6d}{7}\right)\hat{k}}$$

$$\frac{d\bar{r}_G}{dt} = \frac{\Sigma m_i \vec{v}_i}{\Sigma m_i} = \frac{4m(v\hat{i}) + 2m(3v\hat{k}) + m(2v\hat{j})}{7m}$$

$$\boxed{\frac{d\bar{r}_G}{dt} = \left(\frac{4}{7}v\right)\hat{i} + \left(\frac{6}{7}v\right)\hat{k} + \left(\frac{2}{7}\right)v\hat{j}}$$

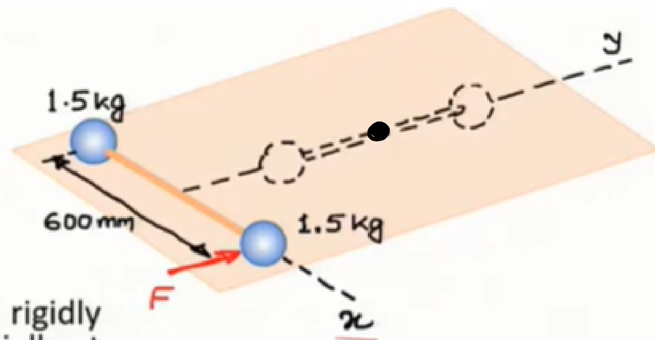
$$\vec{F} = m\ddot{\vec{r}}_G$$

$$F = 7m\ddot{\vec{r}}_G \quad \boxed{\ddot{\vec{r}}_G = \left(\frac{F}{7m}\right)\hat{k}}$$

$$\boxed{T = 13mv^2}$$

$$\vec{H}_O = \sum \vec{r}_i \times m_i \vec{v}_i$$
$$= mvd\left[12\hat{i} + \hat{j} + 2\hat{k}\right)$$

Problem 2:



Source: Dynamics, Meriam and Kraige

Two spheres (treated as particles) of mass 1.5 kg each are rigidly connected to a 600 mm rigid rod of negligible mass and initially at rest on a smooth horizontal surface oriented along the x-axis as shown. A force F along the y-axis direction is suddenly applied that imparts an impulse of 10 Ns during a negligibly short period of time. As the spheres pass the y-axis aligned dashed configuration, calculate the velocity of each sphere.

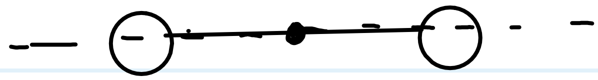
$$\Delta G = \vec{G}_2 - \cancel{\vec{G}_1}^{\,0} = \int_{t_1}^{t_2} \vec{F}\,dt$$

$$mv_x\hat{i} + mv_y\hat{j} = 10\hat{j}$$

$$v_x = 0 \qquad v_y = \frac{10}{3} = 3.333\,ms^{-1}$$

$$\Delta \vec{H}_G = \vec{H}_{G_2} - \vec{H}_{G_1} = \int_{t_1}^{t_2} M_G \, dt$$

$$\vec{H}_{G_2} = \sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$



$$= \sum \vec{\rho}_i \times m_i (\vec{\omega} \times \vec{\rho}_i)$$

$$\dot{\vec{\rho}}_i = \cancel{\frac{d\vec{\rho}_i}{dt}}^{\,0} + (\vec{\omega} \times \vec{\rho}_i)$$

$$= (0.3\hat{j}) \times (1.5)(\omega\hat{k} \times 0.3$$

$$\omega = \frac{100}{9} \text{ rad/s}$$

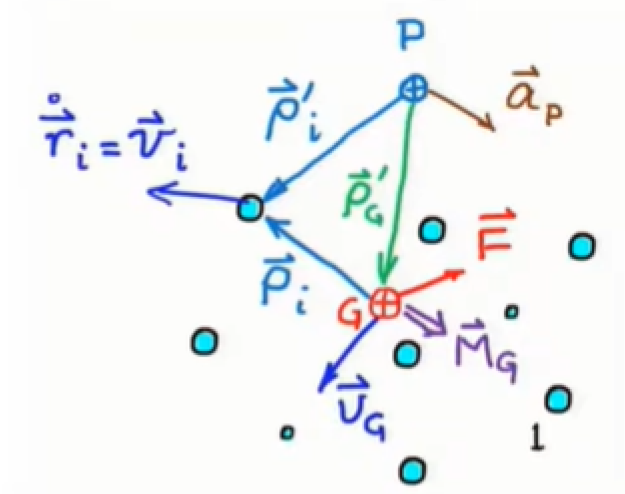
$$\vec{v}_1 = \vec{v}_G + (\vec{\omega} \times \vec{\rho}_i) + \cancel{v_{rel}}^{\,0}$$

$$\vec{v}_1 = \frac{10}{3}\hat{j} + \left(\frac{100}{9}\hat{k} \times (0.3\hat{i})\right)$$

$$\boxed{v_1 = -\frac{10}{3}\hat{i} + \frac{10}{3}\hat{j} \quad \text{m/s}}$$

Angular Momentum (about P)

$\dot{\vec{H}}_G = \vec{M}_G \qquad \dot{\vec{H}}_O = \vec{M}_O$



$\vec{F}, \vec{M}_G$: equivalent force system at G

$$\vec{H}_G = \sum \vec{\rho}_i \times m \dot{\vec{r}}_i$$

$$= \sum \vec{\rho}_i \times m \dot{\vec{\rho}}_i = \vec{H}_G^{rel}$$

$$\vec{H}_O = \sum \vec{r}_i \times m_i \dot{\vec{r}}_i$$

P ↳ neither a fixed point, not COM
P can move

$$\boxed{\sum m_i \dot{\vec{r}}_i = \vec{G} = M \vec{r}_G}$$

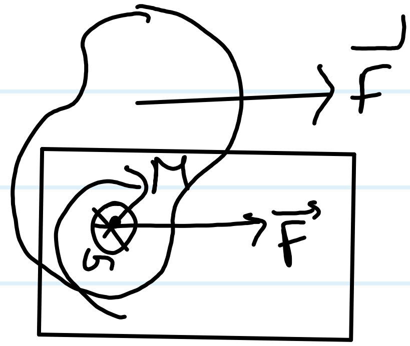
$$\vec{H}_P = \sum \vec{\rho}_i' \times m_i \dot{\vec{r}}_i$$

$$= \sum (\vec{\rho}_G' + \vec{\rho}_i) \times m_i \dot{\vec{r}}_i$$

$$= \sum \vec{\rho}_G' \times m_i \dot{\vec{r}}_i + \sum \vec{\rho}_i \times m_i \dot{\vec{r}}_i$$

$$\boxed{\vec{H}_P = \vec{\rho}_G' \times m \vec{V}_G + \vec{H}_G}$$ ☆

$$\vec{M}_P = \vec{\rho}_G' \times \vec{F} + \vec{M}_G$$



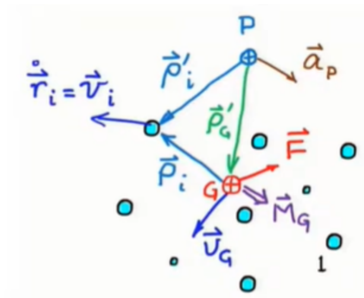
$$\boxed{\vec{M}_P = \vec{\rho}_G' \times m\vec{a}_G + \dot{\vec{H}}_G}$$ ☆

Relative Angular Momentum (about P)

$$\vec{H}_P^{rel} = \sum \vec{\rho}_i' \times m_i \dot{\vec{\rho}}_i' \qquad \vec{\rho}_i' = \vec{\rho}_G' + \vec{\rho}_i$$

$$= \sum \left(\vec{\rho}_G' + \vec{\rho}_i\right) \times m_i \left(\dot{\vec{\rho}}_G' + \dot{\vec{\rho}}_i\right)$$

$$\boxed{\vec{H}_P^{rel} = \vec{\rho}_G' \times m\dot{\vec{\rho}}_G' + \vec{H}_G^{rel}}$$



$\vec{F}, \vec{M}_G$: equivalent force system at G

$$\boxed{\begin{aligned} \vec{H}_P^{rel} &= \vec{H}_G^{rel} + \vec{\rho}_G' \times m\vec{V}_G^{rel} \\ \vec{H}_P &= \vec{H}_G + \vec{\rho}_G' \times m\vec{V}_G \end{aligned}}$$ ☆

$$\vec{H}_P^{rel} = \sum \vec{\rho}_i\,' \times m_i \dot{\vec{\rho}}_i\,' = \sum (\vec{\rho}_G\,' + \vec{\rho}_i) \times m_i (\dot{\vec{\rho}}_G\,' + \dot{\vec{\rho}}_i)$$

$$= \underbrace{\vec{\rho}_G\,' \times m \dot{\vec{\rho}}_G\,'}_{\text{Angular momentum due to motion of center of mass}} + \underbrace{\sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i}_{\text{Angular momentum due to relative motion about center of mass } \vec{H}_G^{rel}} \quad \left(\sum m_i \vec{\rho}_i = 0\right) \;\; ☆$$

Relative Angular Momentum (about P)

$$\vec{H}_P^{rel} = \sum \vec{\rho}_i\,' \times m_i \dot{\vec{\rho}}_i\,' \qquad \dot{\vec{\rho}}_i\,' = (\dot{\vec{r}}_i - \dot{\vec{r}}_P)$$

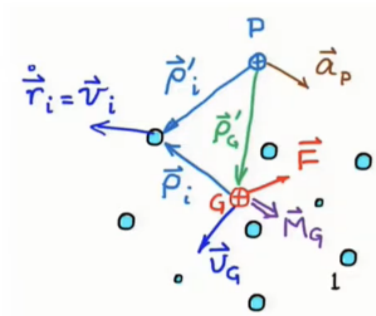
Differentiating with respect to time

$$\dot{\vec{H}}_P^{rel} = \sum \vec{\rho}_i\,' \times m_i (\ddot{\vec{r}}_i - \ddot{\vec{r}}_P)$$

$$\Rightarrow \dot{\vec{H}}_P^{rel} = \underline{\sum \vec{\rho}_i\,' \times \vec{F}_i} - \underline{\sum m_i \vec{\rho}_i\,' \times \ddot{\vec{r}}_P} \qquad \left(\sum_i \vec{\rho}_i\,' \times \sum_j f_{ij} = 0\right)$$

$$\Rightarrow \dot{\vec{H}}_P^{rel} = \vec{M}_P - m \vec{\rho}_G\,' \times \ddot{\vec{r}}_P \qquad \left(\sum m_i \vec{\rho}_i\,' = m \vec{\rho}_G\,'\right)$$

$$\Rightarrow \boxed{\dot{\vec{H}}_P^{rel} + \vec{\rho}_G\,' \times m \ddot{\vec{r}}_P = \vec{M}_P} \;\; ☆$$



$$\dot{\vec{H}}_P^{rel} + \vec{\rho}_G\,' \times m \vec{a}_P = \vec{M}_P$$

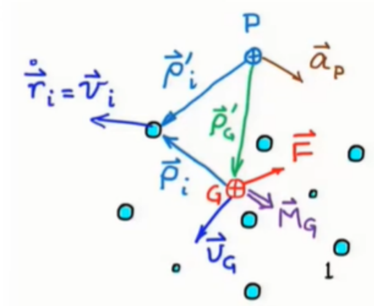
$$\dot{\vec{H}}_P^{rel} = \vec{M}_P \rightarrow$$

- $\vec{a}_P = 0$ case(1)
- $\vec{\rho}_G\,' = 0$ case(2)
- $\vec{\rho}_G\,' \times \ddot{\vec{r}}_P = 0$ P accelerates towards G

Equations of motion

$$\boxed{\begin{aligned} \dot{\vec{G}} &= \vec{F} \\ \dot{\vec{H}}_P^{rel} + \vec{\rho}_G' \times m\ddot{\vec{r}}_P &= \vec{M}_P \end{aligned}}$$

$$\vec{G} = m\vec{v}_G \qquad \vec{H}_P^{rel} = \vec{\rho}_G' \times m\dot{\vec{\rho}}_G' + \underbrace{\sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i}_{\vec{H}_G = \vec{H}_G^{rel}}$$



P = O (fixed)

$$\dot{\vec{H}}_O = \vec{M}_O$$

$$\vec{H}_O = \vec{r}_G \times \vec{G} + \vec{H}_G$$

P = G

$$\dot{\vec{H}}_G = \vec{M}_G$$

$$\vec{H}_G = \sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

P accelerates towards G

$$\dot{\vec{H}}_P^{rel} = \vec{M}_P$$

$$\vec{H}_P^{rel} = \vec{\rho}_G' \times m\dot{\vec{\rho}}_G' + \vec{H}_G$$

$$\vec{H} = \vec{\rho}_G \times m\dot{\bar{\rho}}_G + \sum \bar{\rho}_i \times m_i \dot{\bar{\rho}}_i$$