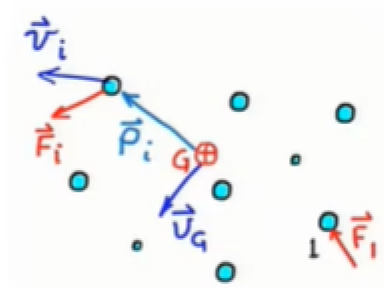
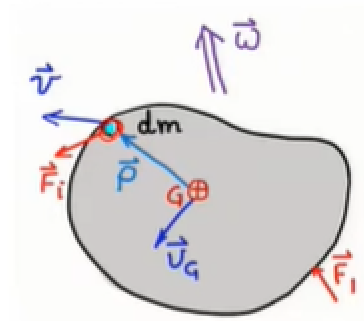
Equations of Rigid body motion using G

$\dot{\vec{G}} = \vec{F}$

$\dot{\vec{H}}_G = \vec{M}_G$





$$\vec{G} = \sum m_i \vec{r}_G = \int dm \vec{V}_G$$

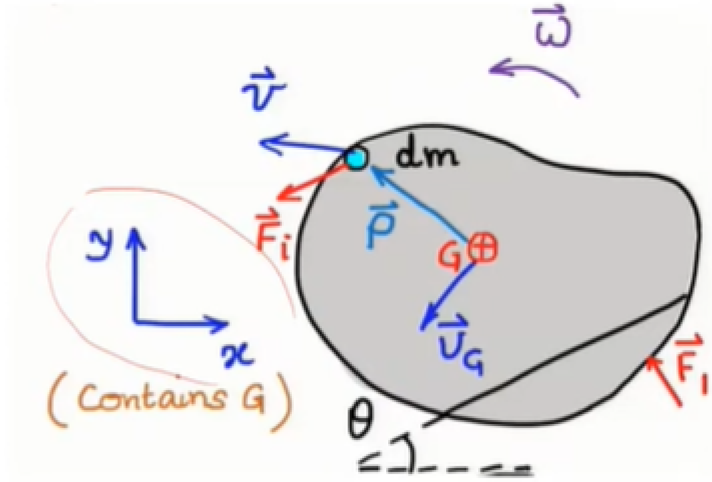
$$\boxed{\vec{G} = m\vec{V}_G} \Leftarrow ☆$$

$$\vec{\rho} \times dm\,\dot{\vec{\rho}}$$

$$\vec{H}_G = \int \vec{\rho} \times \dot{\vec{\rho}}\, dm = \int \vec{\rho} \times (\vec{\omega} \times \vec{\rho})\, dm$$

$$\boxed{\vec{H}_G = \int \left[(\vec{\rho} \cdot \vec{\rho})\vec{\omega} - (\vec{\rho} \cdot \vec{\omega})\vec{\rho}\right] dm} \quad ☆$$

Plane



$\vec{w} = \dot{\theta}\hat{k}$

$\vec{\rho} = \rho_x \hat{i} + \rho_y \hat{j}$

(primastic bodies)

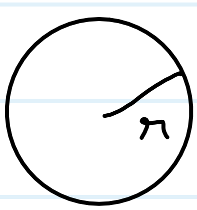
$$\vec{H}_G = \int (\rho_x^2 + \rho_y^2)\, dm\, \dot{\theta}\hat{k}$$

$$\boxed{\vec{H}_G = I_G \vec{w}} \quad ☆ \qquad I_G = (\rho_x^2 + \rho_y^2)\, dm$$

$$\boxed{\begin{aligned} m r_{Gx} &= F_x \\ m\ddot{r}_{Gy} &= F_y \\ I_G \alpha &= M_G \end{aligned}}$$

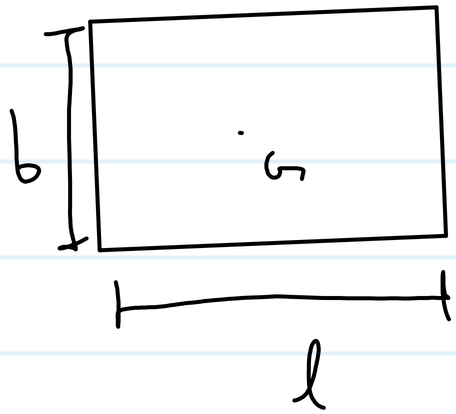
Plane prismatic rigid bodies ☆

Moment of Inertia

① 

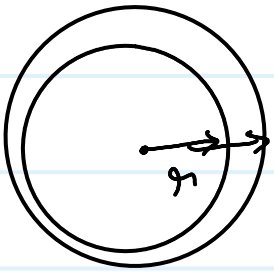
$I_G = \frac{1}{2} m r^2$

(2)



$$I_G = \frac{1}{12} m \left(l^2 + b^2\right)$$

(3)



$$I_G = mr^2$$

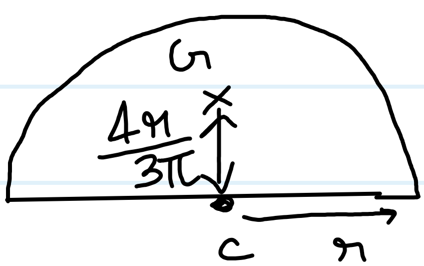
(4)

$$I_c = mr^2$$

$$I_G = I_c - md^2$$

(5)

$$I_c = \frac{1}{2} mr^2$$



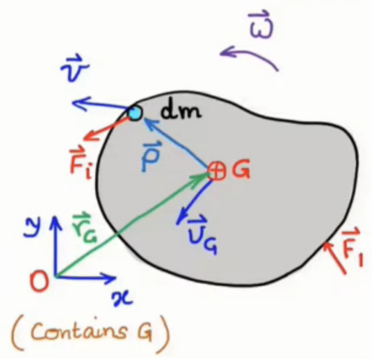
$$I_G = I_c - md^2$$

Plane kinetics: Equations of motion (using O)

$$\dot{\vec{G}} = \vec{F} \qquad \vec{G} = m\vec{v}_G$$

$$\dot{\vec{H}}_O = \vec{M}_O \qquad \vec{H}_O = \vec{r}_G \times \vec{G} + I_G\vec{\omega}$$

$$\boxed{\begin{aligned} &\dot{\vec{G}} = \vec{F} \\ &I_G\dot{\vec{\omega}} + \vec{r}_G \times \dot{\vec{G}} = \vec{M}_O \end{aligned}}$$

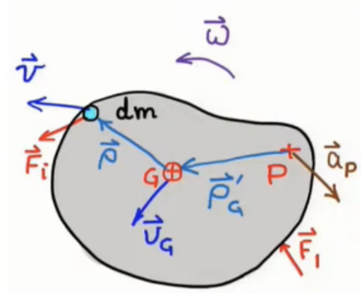


Plane kinetics: Equations of motion (using P)

$$\dot{\vec{G}} = \vec{F} \qquad \vec{G} = m\vec{v}_G$$

$$\dot{\vec{H}}_P^{rel} + \vec{\rho}_G{}' \times m\ddot{\vec{r}}_P = \vec{M}_P \qquad \vec{H}_P^{rel} = I_P\omega\hat{k}$$

$$\boxed{\begin{aligned} &\dot{\vec{G}} = \vec{F} \\ &I_P\dot{\vec{\omega}} + \vec{\rho}_G{}' \times m\ddot{\vec{r}}_P = \vec{M}_P \end{aligned}}$$



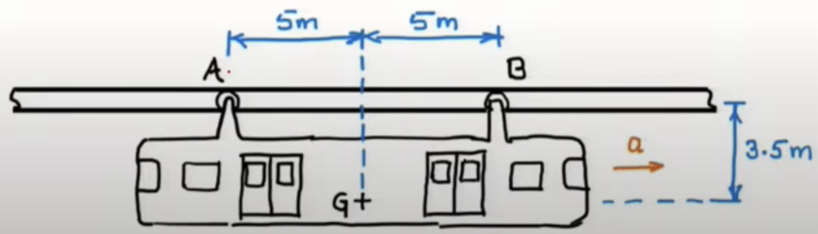
Final equations

$$\boxed{\begin{aligned} &\vec{F} = m\vec{a}_G \\ &I_P\vec{\alpha} + \vec{\rho}_G \times m\vec{a}_P = \vec{M}_P \end{aligned}}$$

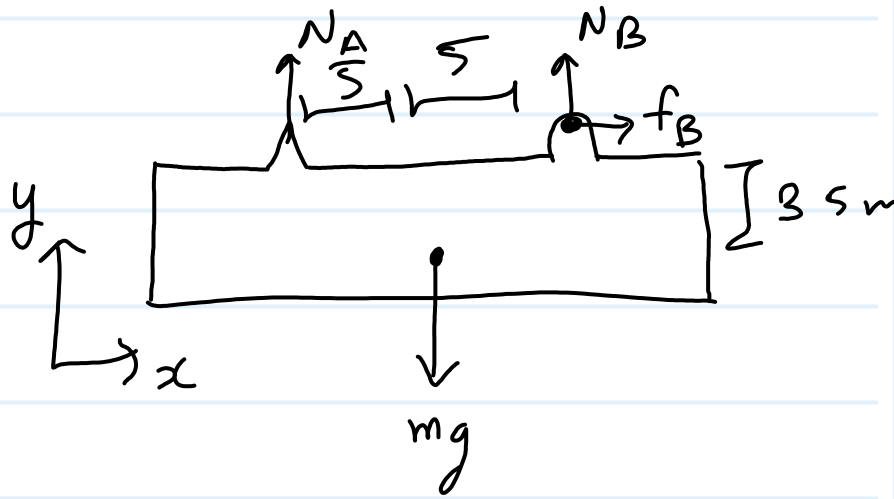
Problems :

Problem 1:

A passenger car of an overhead monorail system is driven by one of its two small wheels A or B. Determine which driving wheel will give greater acceleration without slip, and compute the maximum acceleration if the effective coefficient of friction is limited to 0.25 between the rail and the wheels. Neglect the small mass of the wheels.



Sol :



i) Powered wheel B

$$ma_x = f_B$$

$$0 = N_A + N_B - mg$$

(using A) Angular momentum Equation

$$I_A \cancel{\alpha}^{0} + \rho_{AG} \times m\vec{a}_A = \vec{M}_A$$

$$(5\hat{i} - 3.5\hat{j}) \times ma_x\hat{i} = -5mg\hat{k} + (10)N_B\hat{k}$$

$$+3\,5ma_x = -5mg + 10N_B$$

$$N_B = \frac{1}{10}(3\,5ma_x + 5mg)$$

$$\frac{f_B}{N_B} \leq 0\,25$$

$$a_x^B \leq 1.34\,ms^{-2}$$

ii) Powered wheel a

$$ma_x = f_a$$

$$N_A + N_B - mg = 0$$

$$I_B \cancel{\alpha}^{0} + \vec{\rho}_{BG} \times m\vec{a}_B = \vec{M}_B$$

$$(-5\hat{i} - 3\,5\hat{j}) \times m\vec{a}_x\hat{i} = -10N_A\hat{k} + 5mg\hat{k}$$

$$3\,5ma_x = 5mg - 10N_A$$

$$N_A = \frac{1}{10}(5mg - 3\,5ma_x)$$

$$f_A = ma_x$$

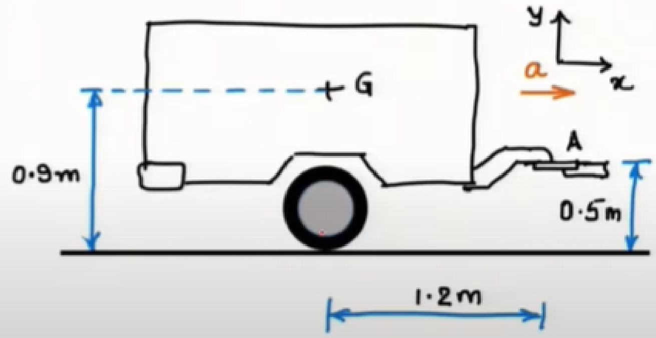
$$\frac{10\ ma_x}{5mg - 3\ 5\ ma_x} \le \dots$$

$$a_x \le 1\ 13\ m/s$$

B

Problem 2:

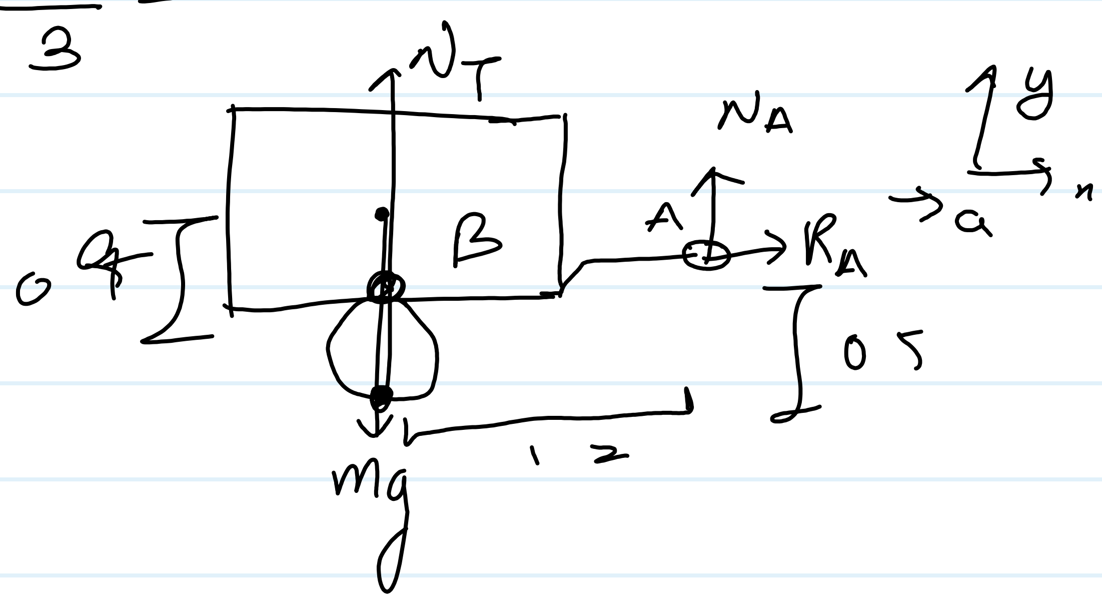
The loaded trailer, having a mass of 900 kg with G as the center of mass, is attached at A to a rear-bumper hitch. If the car and trailer reach a velocity of 60 km/h on a level road in a distance of 30 m starting from rest with constant acceleration, compute the vertical component of the force supported by the hitch at A during this motion. Neglect any resistance at the relatively light wheels.



Source: Dynamics, Meriam and Kraige

Sol^n

$$\frac{50}{3} = v \qquad a = 4.63\ m/s^2$$



$$N_T + N_A - mg = 0$$

$$R_A = ma$$

$$\vec{\rho}_{BG} \times m\vec{a}_B = \vec{M}_B = 1.2\hat{i} \times A_y\hat{j}$$

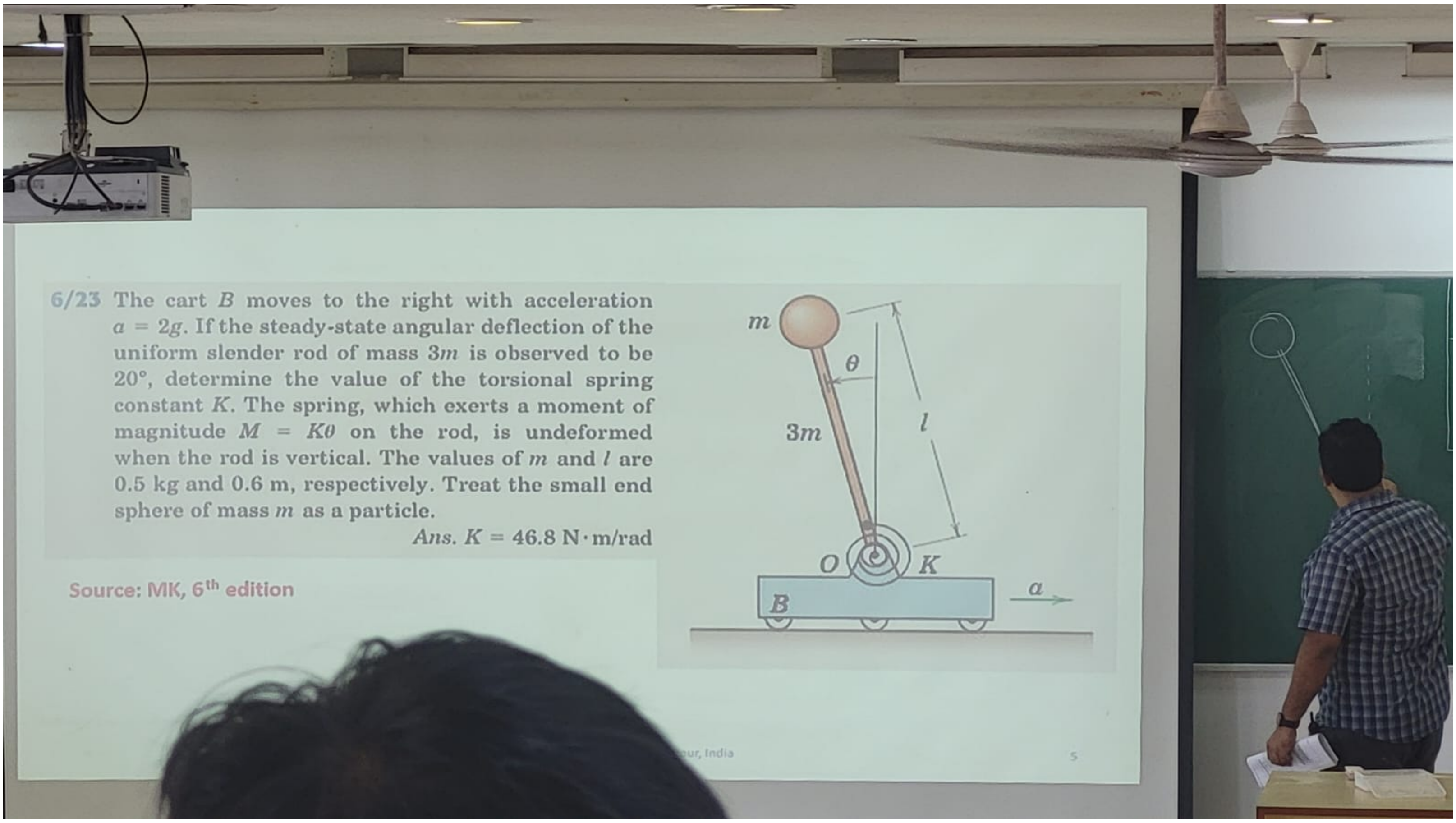
$$(0.4\hat{j}) \times (ma\hat{i}) = 1.2 A_y \hat{k}$$

$$-0.4ma = 1.2A_y$$

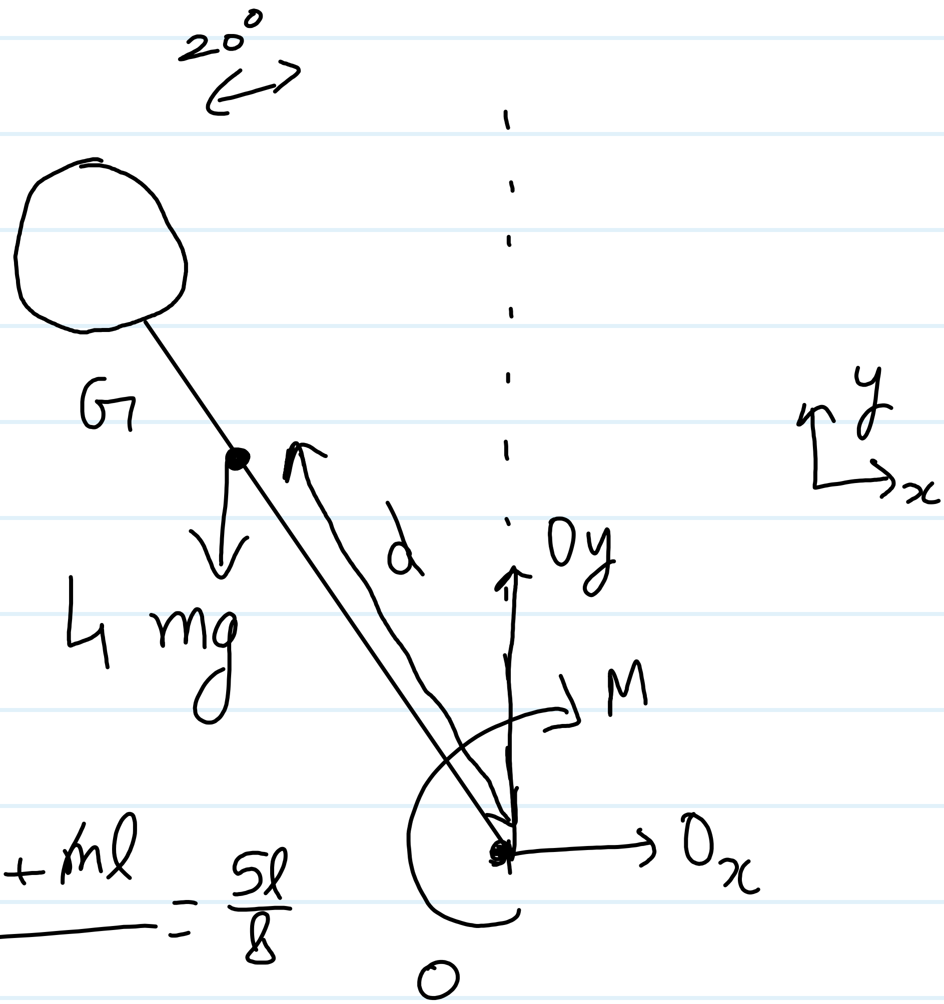
$$-\frac{\cancel{0.4}^{1}}{\cancel{1.2}_{3}} \times \cancel{900}^{300} \times 4.63 = A_y$$

$$1389\,N = A_y$$

$$\boxed{A_y = -1389\hat{k}}$$



Sol:



$$d = \frac{3\cancel{m}\left(\frac{l}{2}\right) + \cancel{m}l}{4\cancel{m}} = \frac{5l}{8}$$

$$I_p \alpha + \overrightarrow{PG} \times m\overrightarrow{a_p} = \overrightarrow{M_p}$$

<u>Using O:</u>

$$\cancel{I_0 \vec{\alpha}}^{\,0} + \overrightarrow{OG} \times m\overrightarrow{a_0} = \overrightarrow{M_0}$$

$$\overrightarrow{OG} = d(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\overrightarrow{a_0} = 2g\,\hat{i}$$

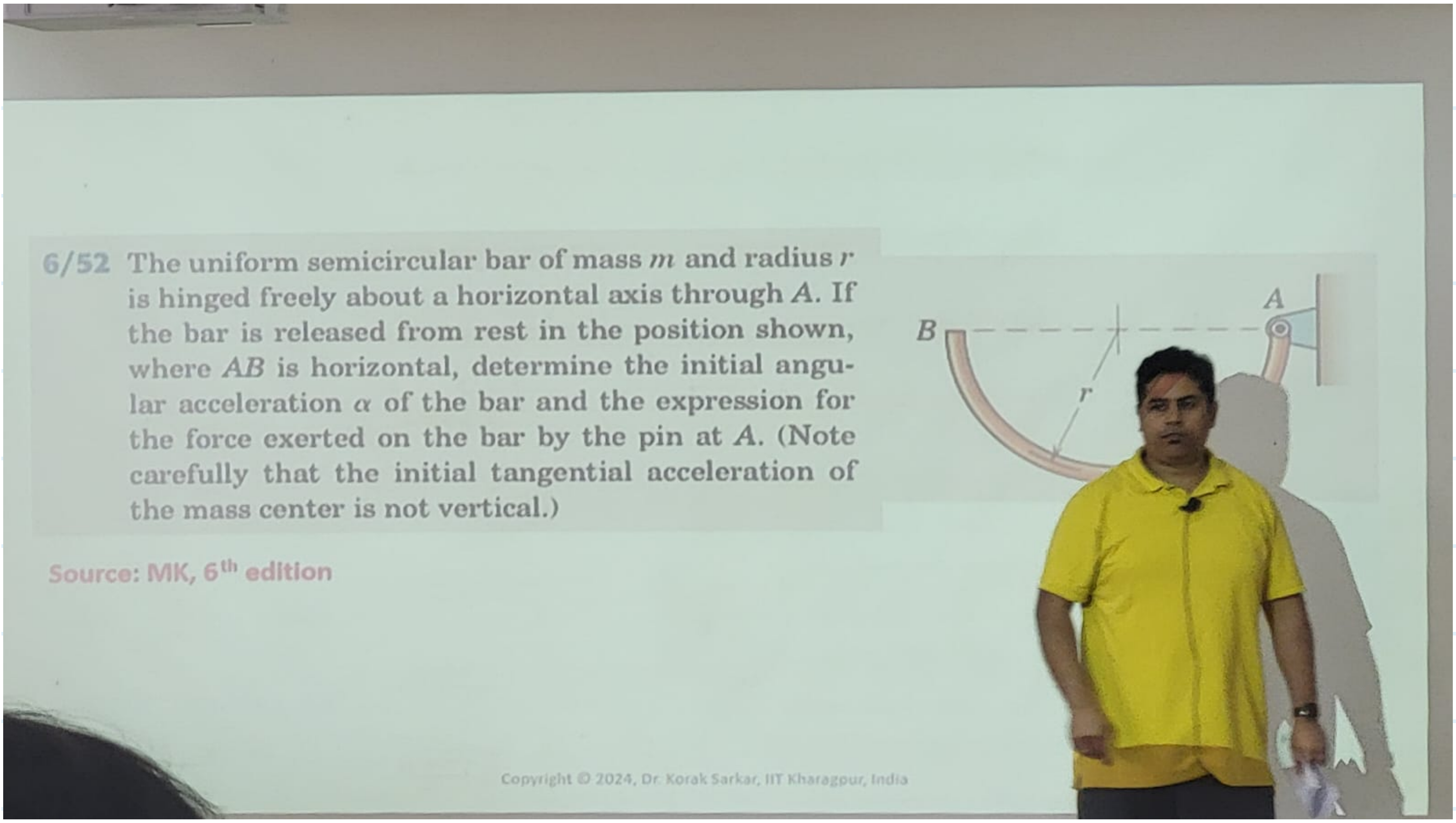
$$\overrightarrow{M_0} = (4mg\, d\sin\theta - M)\hat{k}$$

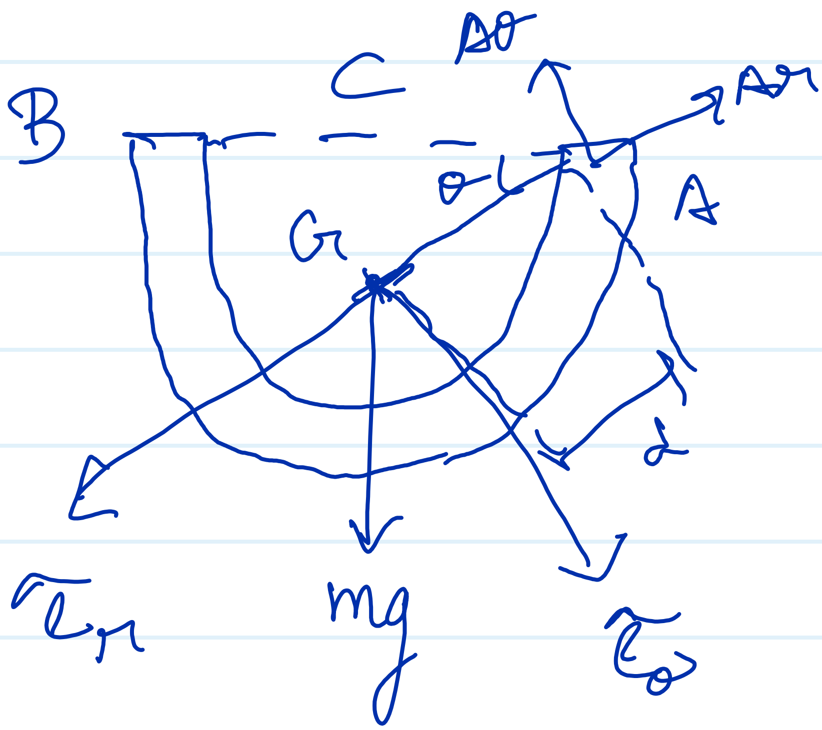
$$-2dg\cos\theta\,\hat{k} = (4mgd\sin\theta - M)\hat{k}$$

$$M - 4mgd\sin\theta = 2gd\cos$$

$$K = \frac{4mgd\sin\theta + 2gd\cos\theta}{\theta}$$

$$\boxed{K = 46\,82 \text{ Nm/rad}}$$





$$\vec{F} = m\vec{a}_G$$

$$I_P\alpha + \vec{\rho}_G \times m\vec{a}_P = M_P$$

$$CG = \frac{2r}{\pi}$$

$$AC = r$$

$$d = r\sqrt{\frac{4}{\pi^2}+1}$$

$$\vec{a}_G = (\cancel{\ddot{d}}^{\,0} - d\dot{\theta}^2)\,\hat{e}_r + (d\ddot{\theta} + \cancel{2\dot{d}\dot{\theta}}^{\,0})\,\hat{e}_\theta$$

$$\vec{a_G} = -d\dot{\theta}^2 \hat{e}_r + d\ddot{\theta}\,\hat{e}_\theta$$

$$\vec{F} = m\vec{a_G}$$

$$(mg\sin\theta - A_r)\hat{e}_r + (mg\cos\theta - A_\theta)\hat{e}_\theta$$

$$= -md\dot{\theta}^2\hat{e}_r + md\ddot{\theta}\hat{e}_\theta$$

$$\boxed{\begin{aligned} A_r - mg\sin\theta &= md\dot{\theta}^2 \\ mg\cos\theta - A_\theta &= md\theta \end{aligned}}$$

$$\boxed{\dot{\theta} = 0 \,\big|_{t=0}}$$

at $t = 0$

$$A_r = mg\sin\theta$$

About A :

$$I_A\ddot{\theta} + \vec{AG} \times m\cancel{a_A}^{\,0} = \vec{M_A}$$

$$I_A\ddot{\theta} = \vec{M_A}$$

$$I_c = mr^2 \qquad I_A = I_c + mr^2$$

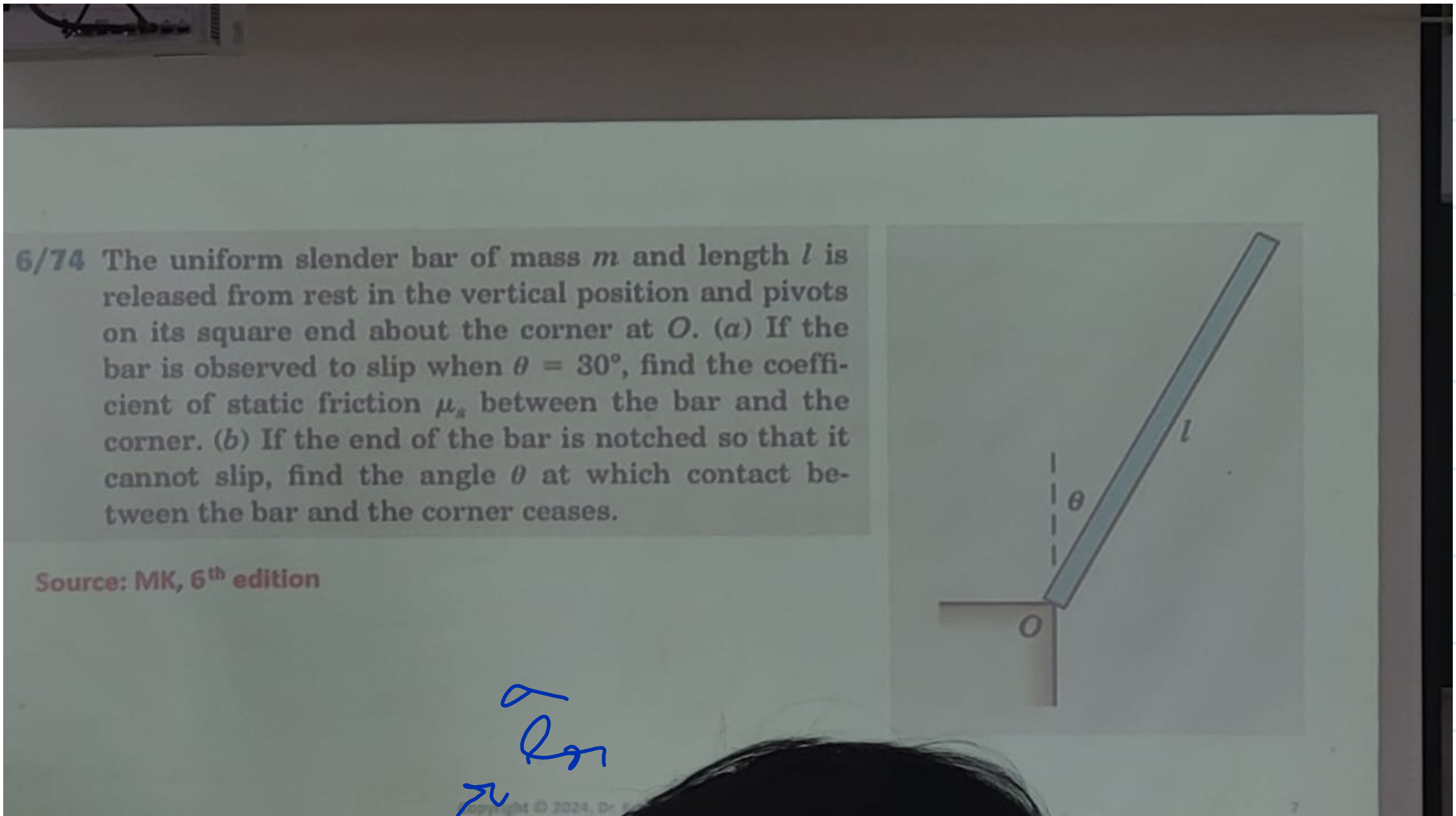
$$I_A = 2mr^2$$

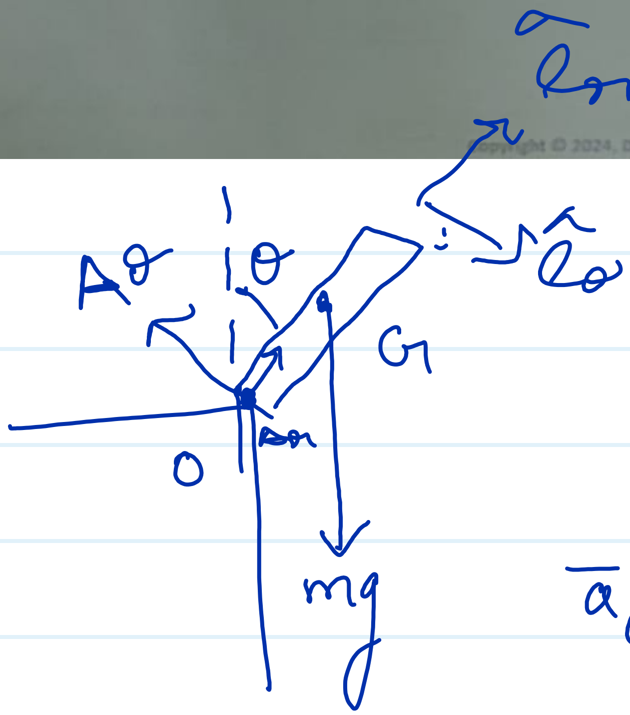
$$2mr\ddot{\theta} = mg\cos\theta\, d \qquad \ddot{\theta} = \frac{g\cos\theta}{2}\sqrt{1+\frac{4}{r^2}}$$

at $\theta = 0$

$$\boxed{\ddot{\theta} = \frac{mg}{2}\sqrt{\frac{4}{r^2}+1}}$$

$$mg\cos\theta - \frac{1}{2}mg\cos\theta \qquad \frac{d^2}{r^2}$$





$$\bar{a} = (\cancel{\ddot{r}} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + \cancel{2\dot{r}\dot{\theta}})\hat{e}_\theta$$

$$r = \frac{l}{2} \quad \dot{r} = 0 \quad \ddot{r} = 0$$

$$\bar{a}_G = (-r\dot{\theta}^2)\hat{e}_r + \left(\frac{l}{2}\ddot{\theta}\right)\hat{e}_\theta$$

$$(A_r - mg\cos\theta) = -m\frac{l}{2}\dot{\theta}^2$$

$$(mg\sin\theta - A_\theta) = \frac{l}{2}m\ddot{\theta}$$

$$I_0 \alpha + \overline{OG} \times m\vec{a_0} \quad M_0$$

$$\frac{ml^2}{3}\ddot{\theta}(\hat{l}_z) + \quad 0 \qquad = M_0$$

$$\frac{ml^{\cancel{2}}}{3}\alpha = mg\sin\theta \frac{\cancel{l}}{2}$$

$$\alpha = \frac{3g\sin\theta}{2l}$$

$$\ddot{\theta} = \frac{3g\sin\theta}{2l}$$

$$\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = \frac{3}{2}\frac{g\sin\theta}{l}$$

constant.

$$\frac{1}{2}\dot{\theta}^2 = -\frac{3}{2}\frac{g}{l}\cos\theta + \frac{3g}{2l}$$

$$\boxed{\dot{\theta} = \sqrt{\frac{3g}{l}(1-\cos\theta)}}$$

$$mg\sin\theta - A_\theta = \frac{m}{2}\,\frac{3g\sin\theta}{2}$$

$$mg\sin\theta - \frac{3mg\sin\theta}{4} = A_\theta$$

$$\boxed{\frac{1}{4}mg\sin\theta = A_\theta}$$

$$A_r = mg\cos\theta - \frac{m\cancel{l}}{2}\,\frac{3g}{\cancel{l}}(1-\cos\theta)$$

$$A_r = mg\cos\theta - \frac{3}{2}mg + \frac{3mg}{2}\cos\theta$$

$$A_r = \frac{5}{2}mg\cos\theta - \frac{3}{2}mg$$

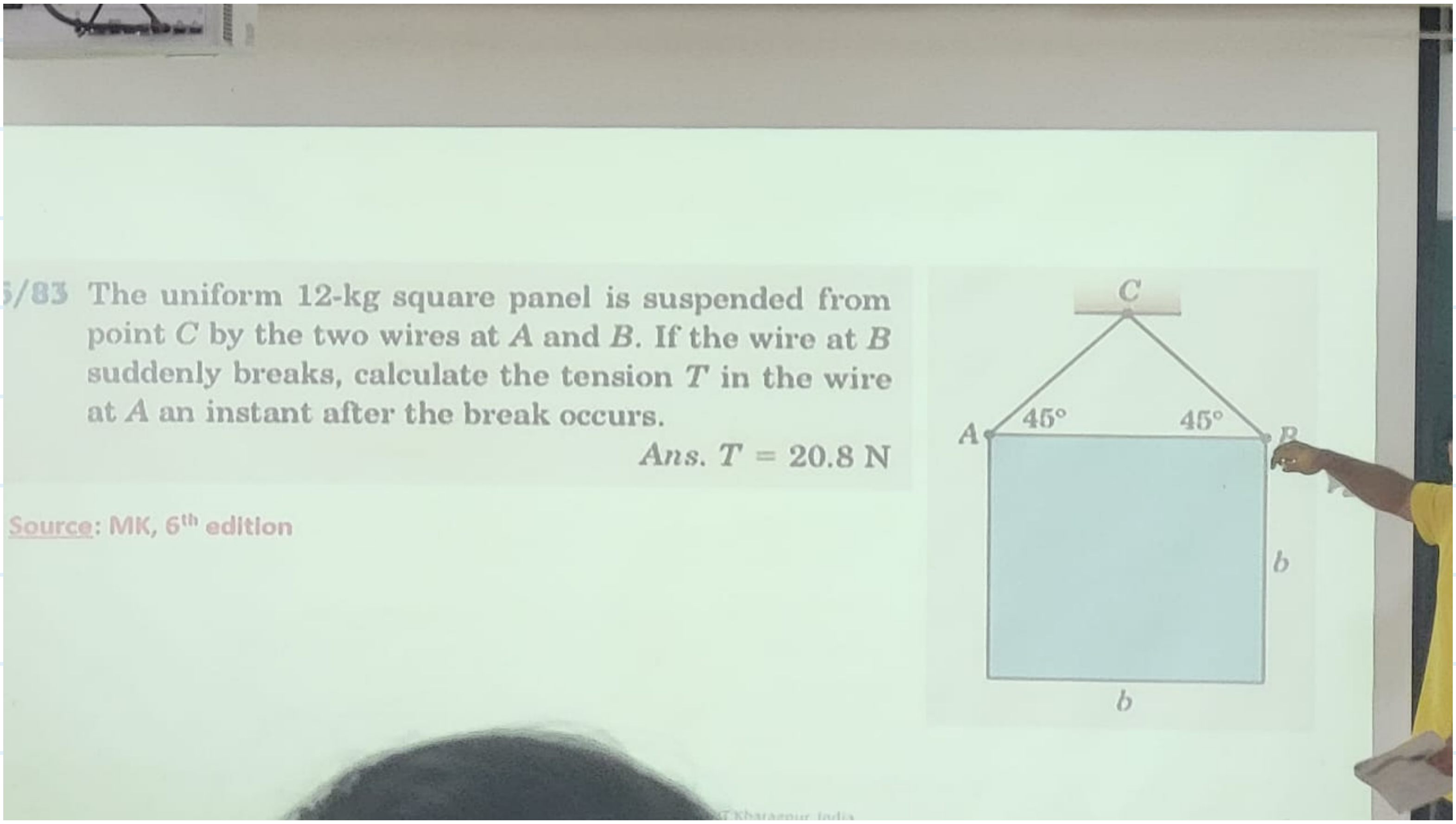
$$\boxed{A_r = \frac{mg}{2}\left[5\cos\theta - 3\right]}$$

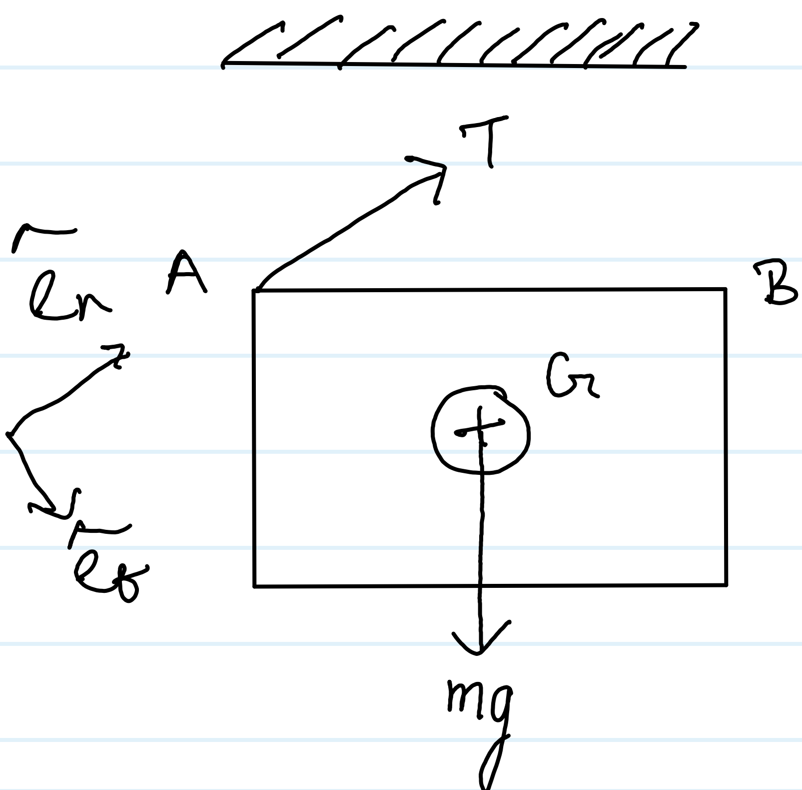
a) $\theta = 30^\circ$ $\quad \mu = \dfrac{A_\theta}{A_r}$

$$\mu = \frac{\sin\theta}{2\left[5\cos\theta - 3\right]} = \frac{\frac{1}{\cancel{2}}}{2\left[\frac{5\sqrt{3}}{\cancel{2}} - \frac{6}{\cancel{2}}\right]}$$

$$\boxed{\mu = \frac{1}{(10\sqrt{3} - 12)}}$$

b) $F_r = 0$ $\cos\theta = \frac{3}{5}$ $\theta = 53^\circ$





$$\vec{a}_A = a_t \hat{e}_t + \cancel{\frac{v^2}{\rho}}^{\,0} \hat{e}_n$$

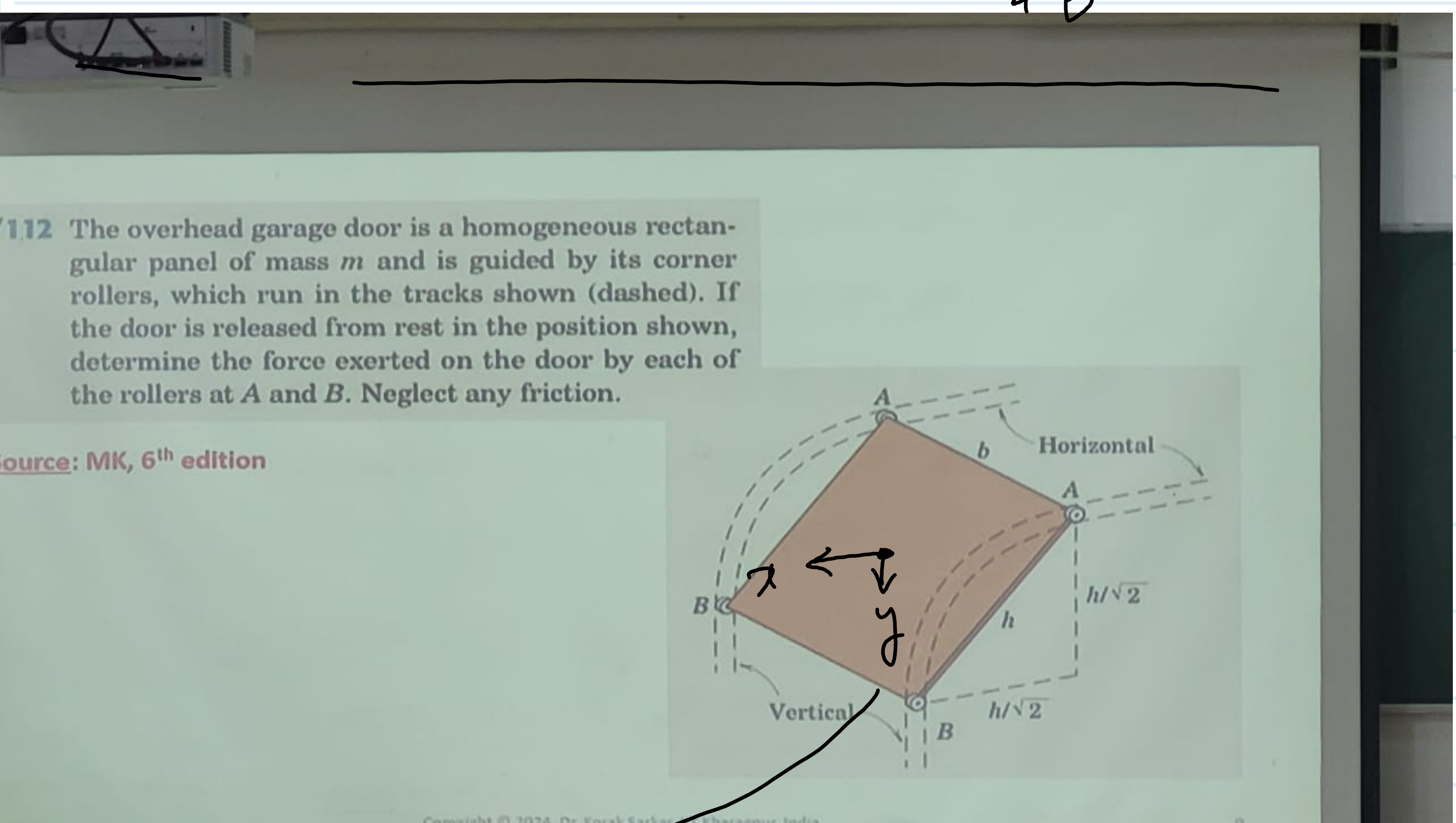
$$I_G = \frac{1}{6} m b^2$$

$$I_A = \frac{2}{3} m b^2$$

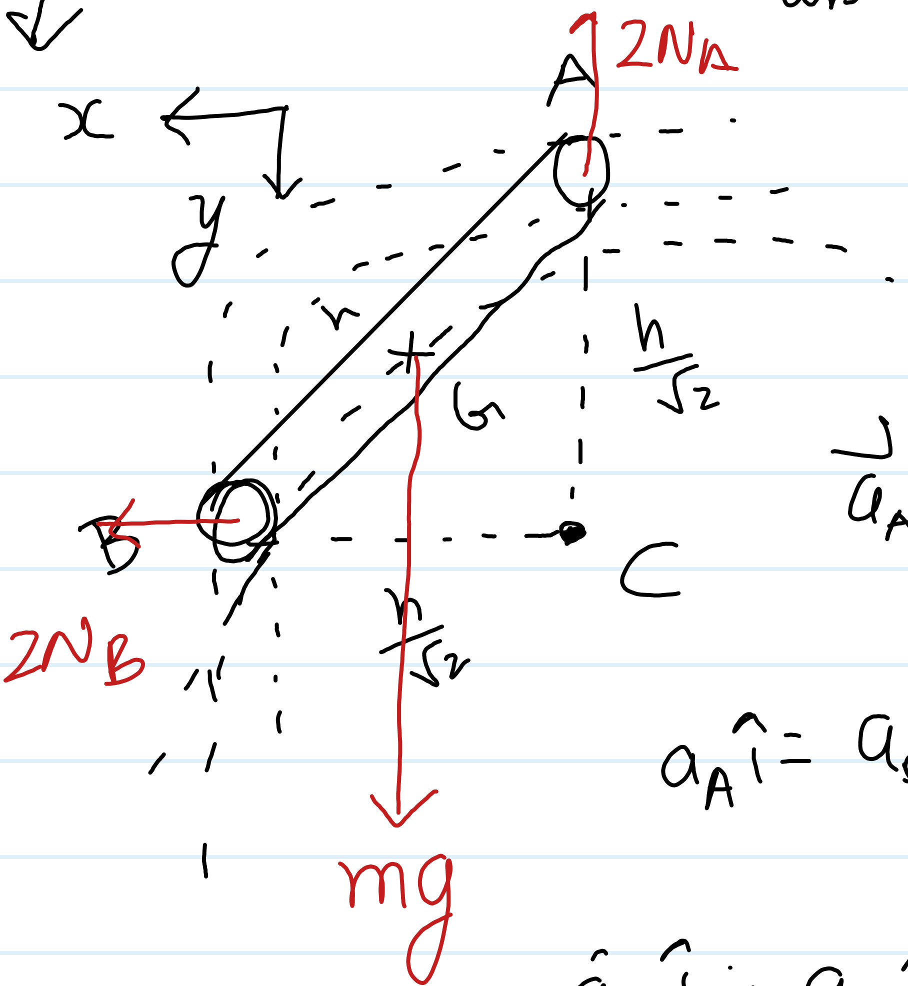
$$I_A \alpha + \vec{AG} \times \cancel{m\vec{a}_A}^{\,0} = \vec{M}_A$$

$$\frac{2}{3}\cancel{m} b^{\cancel{2}} \alpha = -\cancel{m} g \frac{\cancel{b}}{2}$$

$$\alpha = -\frac{3g}{4b}$$



→ C.O M must be contained in the plane → main assumption of plane kinematics



$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{BA}$$

$$a_A \hat{i} = a_B \hat{j} + \left(\alpha \hat{k} \times \frac{h}{\sqrt{2}}(-\hat{i} - \hat{j})\right)$$

$$a_A \hat{i} = a_B \hat{j} + \left(-\frac{\alpha h}{\sqrt{2}} \hat{j} + \frac{\alpha h}{\sqrt{2}} \hat{i}\right)$$

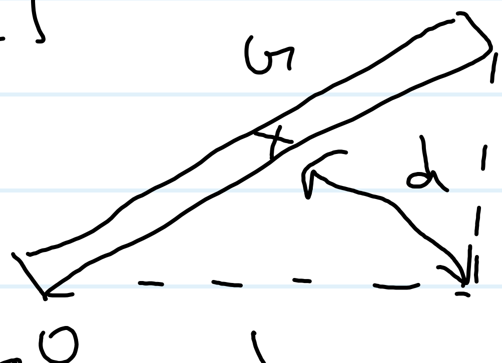
$$\frac{\alpha h}{\sqrt{2}} = a_A \qquad a_B = \frac{\alpha h}{\sqrt{2}}$$

$$\vec{a}_G = \vec{a}_A + \left(\alpha \times \vec{AG}\right)$$

$$\vec{a}_G = \frac{\alpha h}{\sqrt{2}}\hat{i} + \left(\alpha\hat{k} \times \frac{h}{2\sqrt{2}}(\hat{i}+\hat{j})\right)$$

$$\vec{a}_G = \frac{2\alpha h}{2\sqrt{2}}\hat{i} + \frac{\alpha h}{2\sqrt{2}}\hat{j} - \frac{\alpha h}{2\sqrt{2}}\hat{i}$$

$$\vec{a}_G = \frac{\alpha h}{2\sqrt{2}}\hat{i} + \frac{\alpha h}{2\sqrt{2}}\hat{j}$$



(using c)

$$I_c\vec{\alpha} + \vec{CG} \times \cancel{m\vec{a}_c} = \vec{M}_c$$

$$I_c = \frac{mh^2}{3}$$

$$\frac{\cancel{m}h^{\cancel{2}}}{3}\alpha = \cancel{m}g\frac{\cancel{h}}{2\sqrt{2}}$$

$$\boxed{\alpha = \frac{3g}{2\sqrt{2}h}}$$

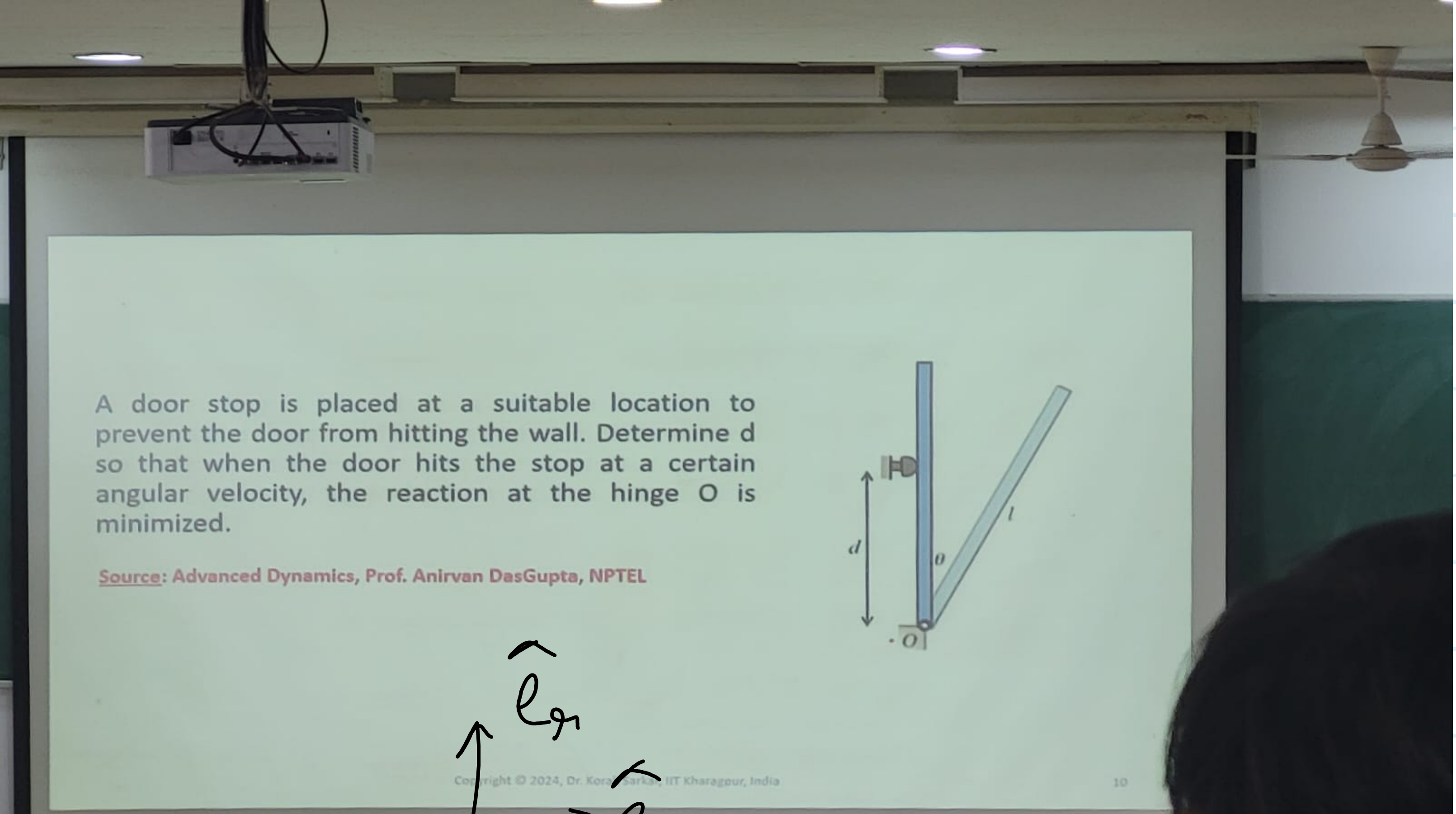
$$\vec{a}_G = \frac{3g}{2\sqrt{2}\cancel{K}} \frac{\cancel{K}}{2\sqrt{2}} (\hat{i}+\hat{j})$$

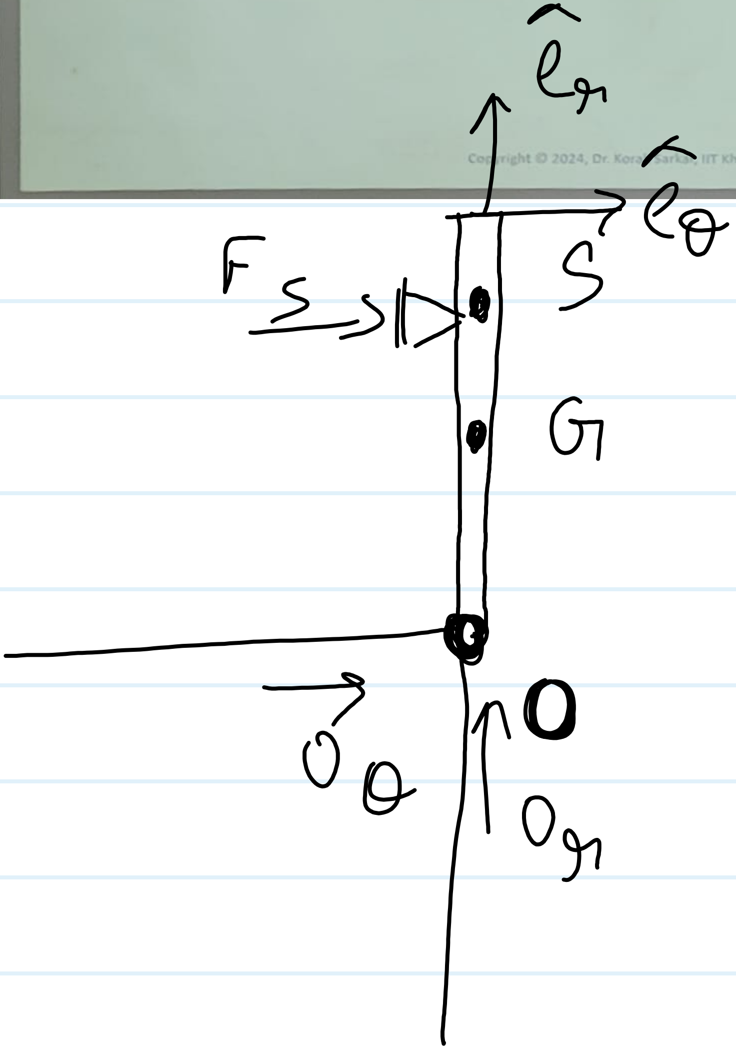
$$\boxed{\bar{a}_G = \frac{3g}{8}(\hat{i}+\hat{j})}$$

$$\vec{F} = m\vec{a}_G$$

$$2N_B\hat{i} + (2N_A - mg)\hat{j} = \frac{3mg}{8}(\hat{i}+\hat{j})$$

$$\boxed{N_B = \frac{3mg}{16}} \qquad \boxed{N_A = \frac{11mg}{16}}$$





$$\vec{a}_G = -\frac{l}{2}\dot{\theta}^2\hat{e}_r + \frac{l}{2}\theta\hat{e}_\theta$$

$$O_r = m\frac{l}{2}\ddot{\theta}^2$$

$$O_\theta + F_s = m\frac{l}{2}\theta$$

<u>usingS</u>

$$I_s \bar{\alpha} + \bar{S}_G \times m\bar{a}_s = \vec{M}_s$$

$$I_s \ddot{\theta} + \left(d - \frac{l}{2}\right)\left(-\hat{e}_r\right) \times m\left(-d\dot{\theta}^2 \hat{e}_r + d\right.$$

$$= -d\, O_\theta\, \hat{e}_z$$

$$\left[I_s - md\left(d - \frac{l}{2}\right)\right]\ddot{\theta} = -d\, O_\theta$$

$\boxed{O_\theta = 0}$ then

$$I_s = m\left(d - \frac{l}{2}\right)$$

$$\frac{1}{12} ml^2 + m\left(d - \frac{l}{2}\right)^2$$

$$\frac{1}{12} ml^2 + md^2 + \frac{3ml^2}{12} - mdl$$

$$= md^2 + \frac{mdl}{2}$$

$\frac{1}{3} \cancel{m} l^{\cancel{2}} = 3 \frac{\cancel{m} d \cancel{l}}{2}$

$$\boxed{d = \frac{2}{9} l}$$

Work Energy.

$T = \frac{1}{2} m \vec{V}_G \cdot \vec{V}_G + \Sigma \left(\frac{1}{2} m_i \dot{\vec{\rho}}_i \cdot \dot{\vec{\rho}}_i \right)$ (system of particles)

Rigid bodies: $\dot{\vec{\rho}}_i = \bar{\omega} \times \vec{\rho}_i$

$$T = \int \frac{1}{2} \vec{r}_G \vec{r}_G \, dm + \int \frac{1}{2} (\bar{\omega} \times \bar{p})(\bar{\omega} \times \bar{p}) \, dm$$

$$T = \frac{1}{2} m \dot{\vec{r}}_G \vec{r}_G + \frac{1}{2} \left[\int \vec{p} \times (\bar{\omega} \times \bar{p}) \, dm \right] \vec{\omega}$$

$$\boxed{T = \underbrace{\frac{1}{2} m \vec{V}_G \cdot \vec{V}_G}_{\text{Translation}} + \underbrace{\frac{1}{2} \bar{\omega} \cdot I_G \bar{\omega}}_{\text{Rotation}}}$$

$$\boxed{P = \vec{F} \cdot \dot{\bar{r}}_G + \vec{M}_G \cdot \bar{\omega}}$$

(power)

$$\frac{dT}{dt} = P$$

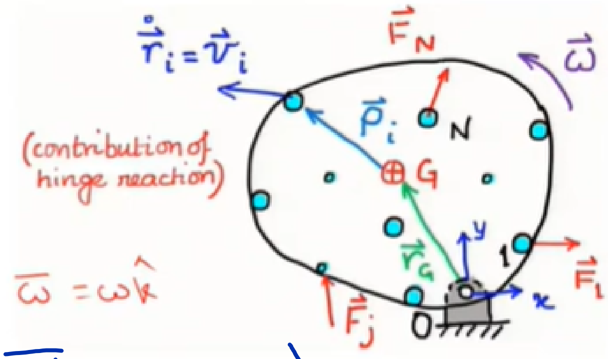
$$\frac{d}{dt}\left[\frac{1}{2} m \vec{V}_G \vec{V}_G + \frac{1}{2} \bar{\omega} \cdot I_G \bar{\omega}\right] = \vec{F} \cdot \dot{\bar{r}}_G + \bar{M}_G \bar{\omega}$$

$$\boxed{\Delta T = T_2 - T_1 = \int_1^2 \vec{F}\, d\bar{r} + \int_{t_1}^{t_2} \vec{M}_G\, \bar{\omega}\, dt}$$

planar rotation about fixed axis

$$\frac{d}{dt}\left[\frac{1}{2}m\overline{V}_G\cdot\overline{V}_G+\frac{1}{2}\overline{\omega}\cdot I_G\overline{\omega}\right]$$

$$=\overline{F}\cdot\dot{\overline{r}}_G+\overline{M}_G\cdot\overline{\omega}$$

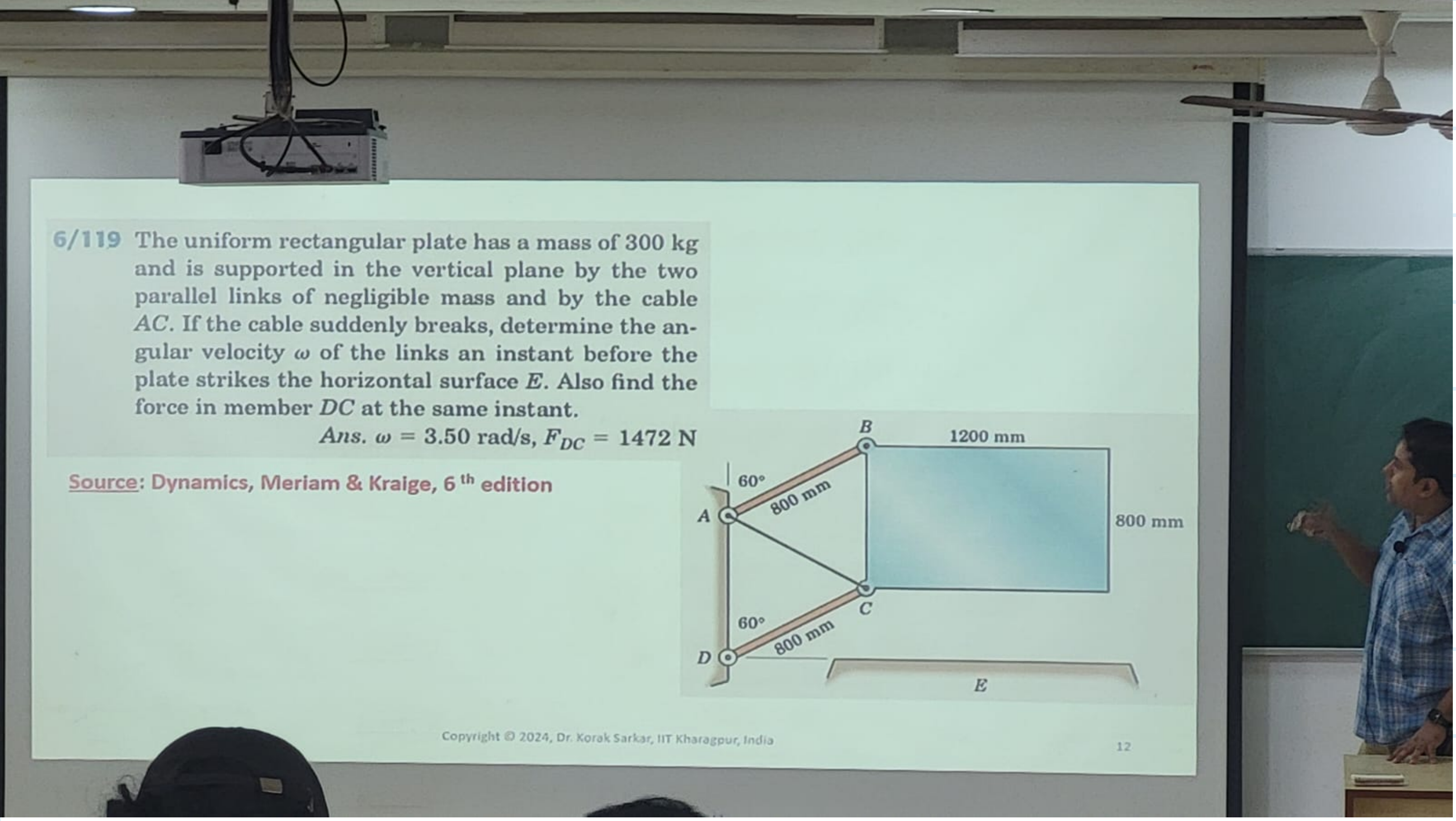


$$\overline{V}_G=\overline{\omega}\times\overline{r}_G$$

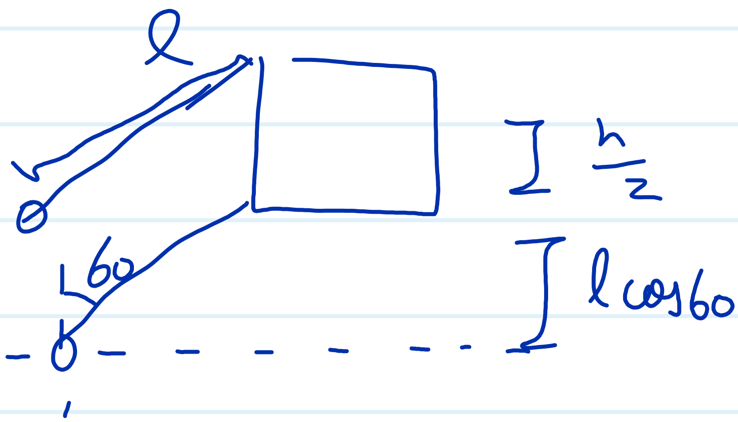
$$\frac{d}{dt}\left[\underbrace{\frac{1}{2}I_o\omega^2}_{\text{Total KE in fixed axis rotation.}}\right]=\overline{\omega}\cdot(\overline{r}_G\times\overline{F})+\overline{M}_G\cdot\overline{\omega}=M_o\omega$$

$$\boxed{\frac{d}{dt}\left[\frac{1}{2}I_o\omega^2\right]=M_o\omega}$$

$$\boxed{\Delta T=T_2-T_1=\int_{\theta_1}^{\theta_2}M_o\,d\theta}\qquad[\omega=\dot{\theta}]$$



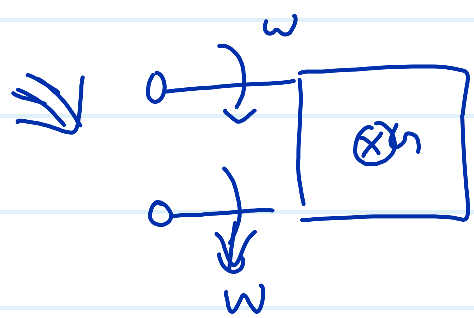
Energy conservation



$T_1 + V_1 = T_2 + V_2$

$0 + mg\left(l\cos 60^\circ + \frac{h}{2}\right)$

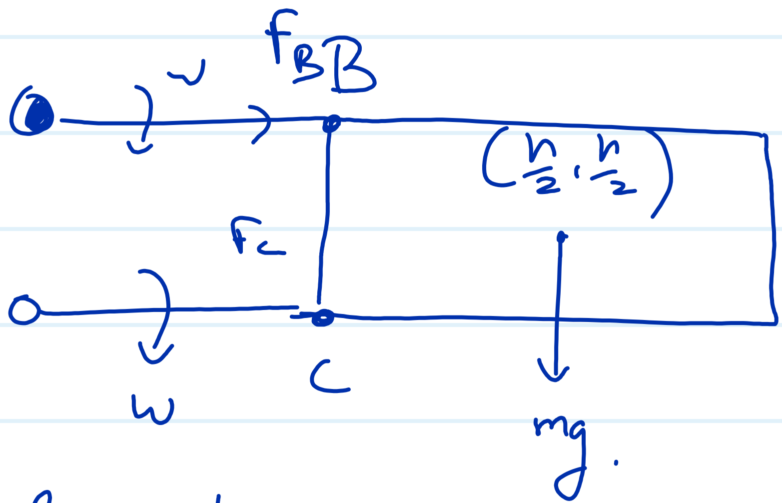
$= \frac{1}{2} m V_G^2 + mg\frac{h}{2}$



$g\, l\cos 60^\circ = \frac{1}{2} V_G^2$

$\sqrt{gl} = V_G$

$(0, h)$



$V_c = l\omega = V_G$

$l\omega = \sqrt{gl}$

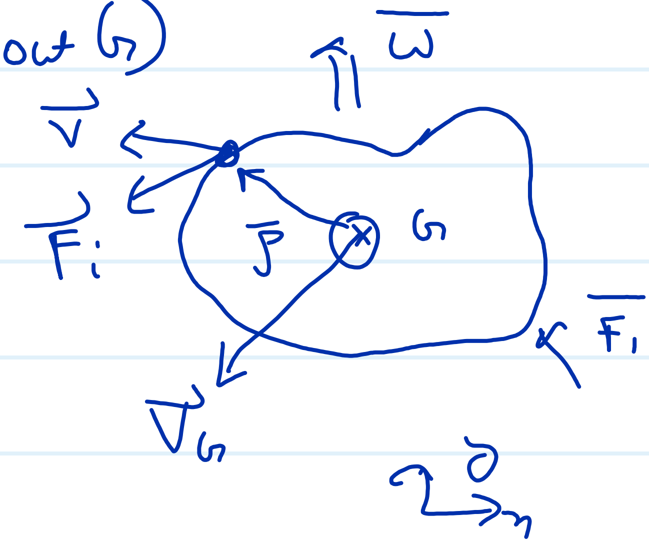
$\omega = \sqrt{\frac{g}{l}}$

<u>About B</u>

$\cancel{I_p\alpha}^{\,0} + \vec{BG} \times m\vec{a}_G = \vec{M}_B$

$\left(\frac{h}{2}\hat{i} - \frac{h}{2}\hat{j}\right)$

Equations of motion (about G)



$$\dot{\overline{G}} = \overline{F}$$
$$\dot{\overline{H}}_G = \overline{M}_G$$

$$\overrightarrow{G} = m\overrightarrow{V}_G$$
$$\overrightarrow{H}_G = I_G \vec{\omega} = I_{Gzz}\dot{\theta}\hat{k} \text{ (planar)}$$

$$\frac{d}{dt}\Big[\underbrace{\tfrac{1}{2} m \overrightarrow{V}_G \overrightarrow{V}_G}_{T_T}\Big] = \overrightarrow{F} \frac{d\overrightarrow{r}_G}{dt}$$

$$\frac{d}{dt}\Big[\underbrace{\tfrac{1}{2} I_{Gzz}\dot{\theta}^2}_{T_R}\Big] = M_G\dot{\theta}$$

$$\boxed{\begin{aligned} \Delta T_T &= T_{T_2} - T_{T_1} = \int_1^2 \overrightarrow{F} \cdot d\overrightarrow{r}_G \\ \Delta T_R &= T_{R_2} - T_{R_1} = \int_{\theta_1}^{\theta_2} M_G \, d\theta \end{aligned}}$$

Impulse-Momentum

$$\Delta G = \int_{t_1}^{t_2} \vec{F}\, dt \qquad \Delta \vec{H}_G = \int_{t_1}^{t_2} \vec{M}_G\, dt$$