

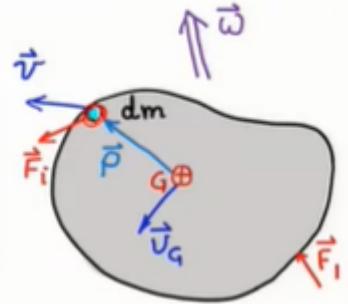
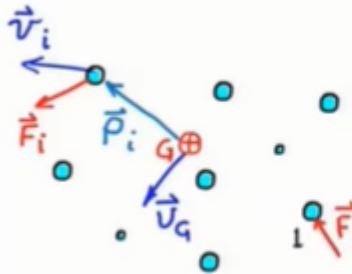
Equations of Rigid body motion using G

$$\dot{\overline{G}} = \overline{F}$$

$$\dot{\overline{H}_G} = \overline{M}_G$$

$$\overline{G} = \sum m_i \overline{r}_i = \int dm \overline{V}_m$$

$$\boxed{\overline{G} = m \overline{V}_m} \Leftrightarrow \star$$



$$\boxed{\overline{\vec{P}} \times dm \vec{p}}$$

$$\overline{H}_G = \int \overline{\vec{p}} \times \dot{\overline{\vec{p}}} dm = \int \overline{\vec{p}} \times (\overline{\omega} \times \overline{\vec{p}}) dm$$

$$\boxed{\overline{H}_G = \int [(\overline{\vec{p}} \cdot \overline{\vec{p}}) \overline{\omega} - (\overline{\vec{p}} \cdot \overline{\omega}) \overline{\vec{p}}] dm} \star$$

Plane

$$\vec{w} = \theta \hat{k}$$

$\vec{P} = p_x \hat{i} + p_y \hat{j}$
 (prismatic bodies)

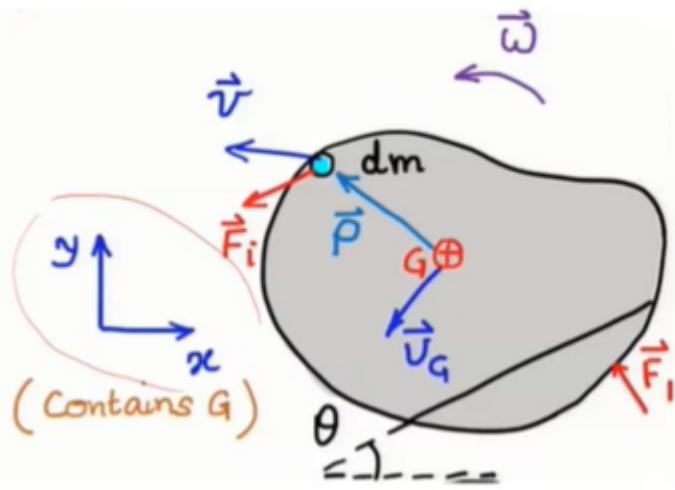
$$\vec{H}_G = \int (p_x^2 + p_y^2) dm \theta \hat{k}$$

$$\boxed{\vec{H}_G = I_G \vec{w}}$$

$$m g_{Gx} = F_x$$

$$m g_{Gy} = F_y$$

$$I_G \alpha = M_G$$

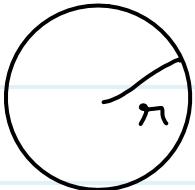


$$I_G = (p_x^2 + p_y^2) dm$$

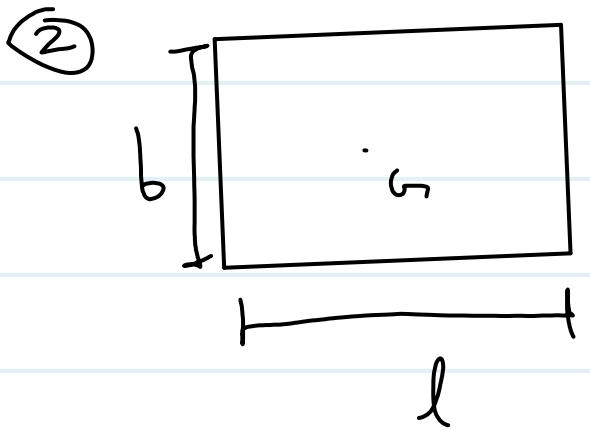
Plane prismatic rigid bodies *

Moment of Inertia

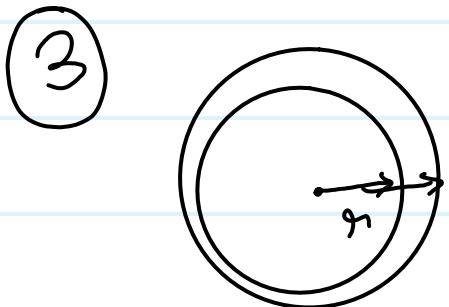
①



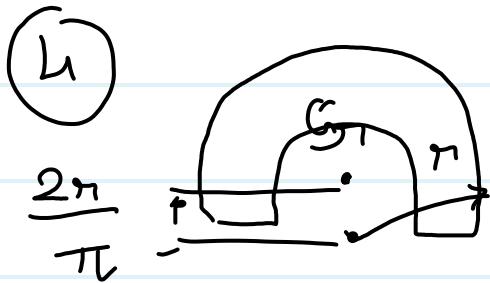
$$I_G = \frac{1}{2} m r^2$$



$$I_G = \frac{1}{12} m (l^2 + b^2)$$

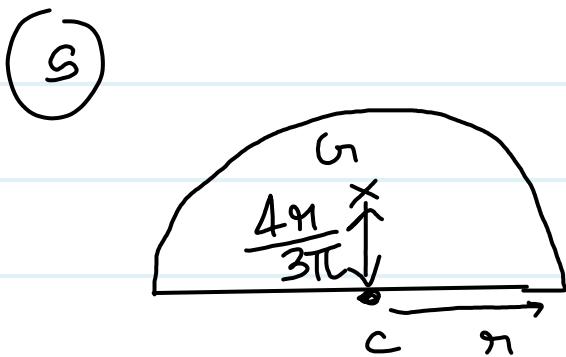


$$I_G = m r^2$$



$$I_c = m r^2$$

$$I_G = I_c - m d^2$$



$$I_c = \frac{1}{2} m r^2$$

$$I_G = I_c - m d^2$$

Plane kinetics: Equations of motion (using O)

$$\ddot{\vec{G}} = \vec{F}$$

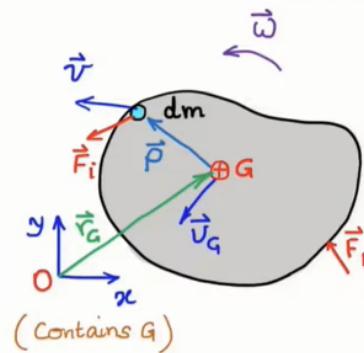
$$\dot{\vec{H}_o} = \vec{M}_o$$

$$\vec{G} = m\vec{v}_G$$

$$\vec{H}_o = \vec{r}_G \times \vec{G} + I_G \vec{\omega}$$

$$\boxed{\ddot{\vec{G}} = \vec{F}}$$

$$\boxed{I_G \ddot{\vec{\omega}} + \vec{r}_G \times \dot{\vec{G}} = \vec{M}_o}$$



Plane kinetics: Equations of motion (using P)

$$\ddot{\vec{G}} = \vec{F}$$

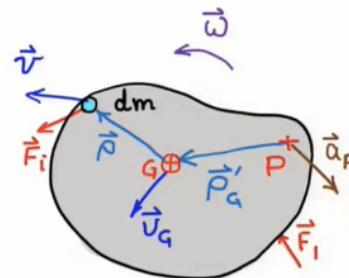
$$\dot{\vec{H}}_P^{rel} + \vec{r}'_G \times m \ddot{\vec{r}}_P = \vec{M}_P$$

$$\vec{G} = m\vec{v}_G$$

$$\vec{H}_P^{rel} = I_P \omega \hat{k}$$

$$\boxed{\ddot{\vec{G}} = \vec{F}}$$

$$\boxed{I_P \ddot{\vec{\omega}} + \vec{r}'_G \times m \ddot{\vec{r}}_P = \vec{M}_P}$$



Final equations

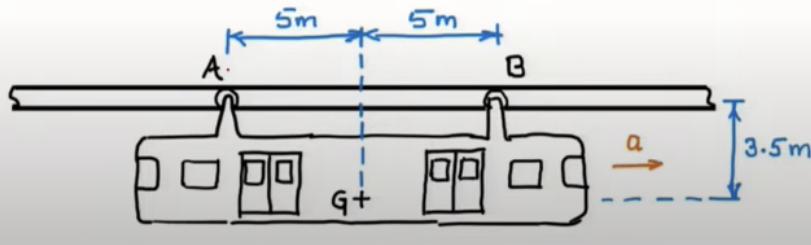
$$\vec{F} = m\vec{a}_G$$

$$I_P \vec{\alpha} + \vec{r}_G \times m \vec{a}_P = \vec{M}_P$$

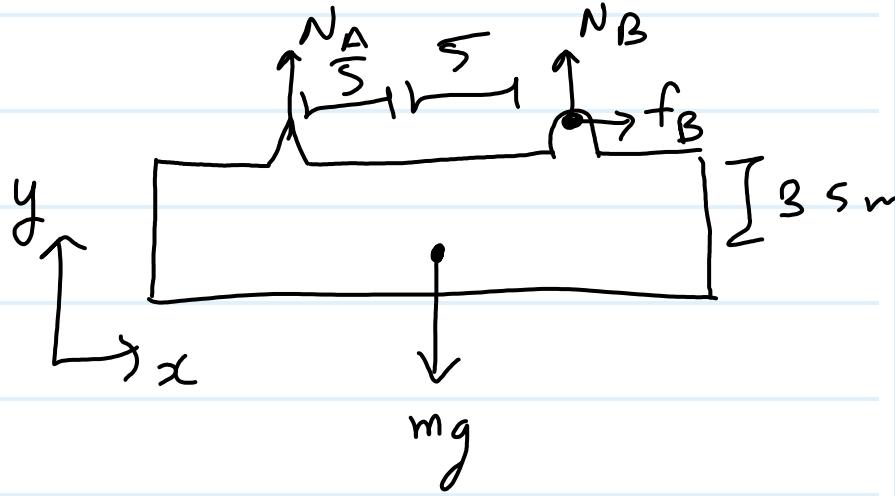
Problems :

Problem 1:

A passenger car of an overhead monorail system is driven by one of its two small wheels A or B. Determine which driving wheel will give greater acceleration without slip, and compute the maximum acceleration if the effective coefficient of friction is limited to 0.25 between the rail and the wheels. Neglect the small mass of the wheels.



Sol :



i) Powered wheel B

$$Ma_x = f_B$$

$$0 = N_A + N_B - mg$$

(using A) Angular momentum Equation

$$I_A \vec{\alpha}^0 + \int_{A(0)}^{A(t)} \vec{r}_A \times \vec{m} \vec{a}_A = \vec{M}_A$$

$$(S_i^1 - 3 S_j^1) \times m_a x^1 = - 5mg \hat{k} + (10) N_B \hat{k}$$

$$+ 3 S_{max} = - 5mg + 10N_B$$

$$N_B = \frac{1}{10}(3 S_{max} + 5mg)$$

$$\frac{f_B}{N_B} \leq 0.25$$

$$a_x^B \leq 1.34 m\bar{s}^2$$

ii) Powered wheel a

$$ma_x = f_a$$

$$N_A + N_B - 5mg = 0$$

$$\overrightarrow{F_B} + \overrightarrow{F_{Bu}} \times \overrightarrow{m a_B} = \overrightarrow{M_B}$$

$$(-5\hat{i} - 3\hat{j}) \times \vec{m a_x} = -10N_A\hat{k} + 5mg\hat{k}$$

$$3 S_{max} = 5mg - 10N_A$$

$$N_A = \frac{1}{10} (5mg - 3 S_{max})$$

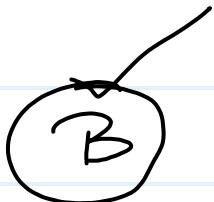
$$f_A = ma_x$$

10

 \max $\leq \dots$

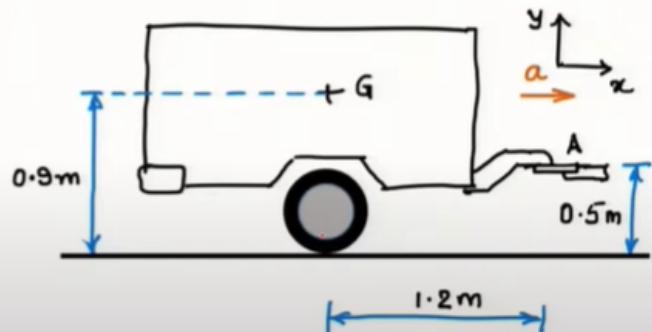
$$Smg - 3 S_{\max}$$

$$a_n \leq 113 \text{ m/s}^2$$



Problem 2:

The loaded trailer, having a mass of 900 kg with G as the center of mass, is attached at A to a rear-bumper hitch. If the car and trailer reach a velocity of 60 km/h on a level road in a distance of 30 m starting from rest with constant acceleration, compute the vertical component of the force supported by the hitch at A during this motion. Neglect any resistance at the relatively light wheels.

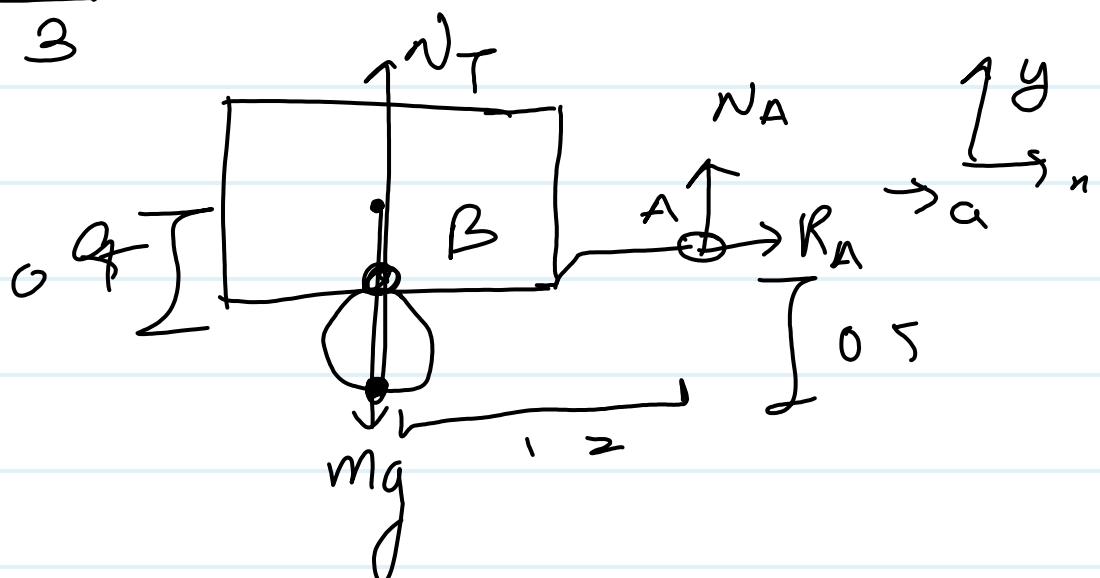


Source: Dynamics, Meriam and Kraige

Solⁿ

$$\frac{50}{3} = \checkmark$$

$$a = 4.63 \text{ m/s}^2$$



$$N_T + N_A - mg = 0$$

$$R_A = ma$$

$$\vec{F}_{B\text{ext}} \times \vec{m}\vec{a}_B = \overline{\pi}_{Bz} \hat{i} \cdot 2\hat{i} \times A_y \hat{j}$$

$$(6 \text{ kg}) \times (ma \hat{i}) = 12 A_y \hat{k}$$

$$- 0.4 ma = 1.2 A_y$$

$$- \frac{0.4}{T^2} \times \frac{300}{900} \times 4.63 = A_y$$

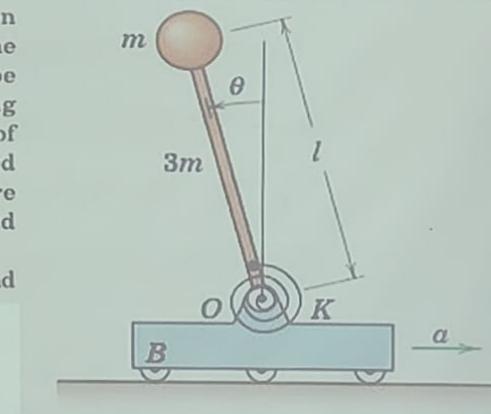
$$1389 \text{ N} = A_y$$

$$A_y = -1389 \text{ N}$$

6/23 The cart B moves to the right with acceleration $a = 2g$. If the steady-state angular deflection of the uniform slender rod of mass $3m$ is observed to be 20° , determine the value of the torsional spring constant K . The spring, which exerts a moment of magnitude $M = K\theta$ on the rod, is undeformed when the rod is vertical. The values of m and l are 0.5 kg and 0.6 m , respectively. Treat the small end sphere of mass m as a particle.

$$\text{Ans. } K = 46.8 \text{ N}\cdot\text{m}/\text{rad}$$

Source: MK, 6th edition



Sol:

Diagram showing the free body diagram of the rod. The center of mass of the rod is at distance d from the pivot O. The forces acting on the rod are the weight mg at the center of mass and the reaction force G at the pivot O. A coordinate system is established at O with the x-axis pointing to the right and the y-axis pointing upwards. A clockwise moment M is applied at O.

$$d = \frac{3m \left(\frac{l}{2}\right) + ml}{a m} = \frac{5l}{8}$$



$$\overrightarrow{I_p \alpha} + \overrightarrow{P G} \times \overrightarrow{m \vec{a}_p} = \overrightarrow{M_p}$$

Using 0:

$$\overrightarrow{I_0 \alpha^0} + \overrightarrow{O G} \times \overrightarrow{m \vec{a}_0} = \overrightarrow{M_0}$$

$$\overrightarrow{O G} = d (-\sin \theta \hat{i} + \cos \theta \hat{j})$$

$$\overrightarrow{a_0} = 2g \hat{i}$$

$$\overrightarrow{M_0} = (Lmg d \sin \theta - M) \hat{k}$$

$$- 2dg \cos \theta \hat{k} = (Lmg d \sin \theta - M) \hat{k}$$

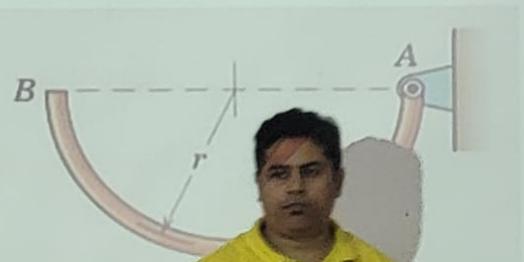
$$M - Lmg d \sin \theta = 2g d \cos \theta$$

$$K = \frac{Lmg d \sin \theta + 2g d \cos \theta}{\theta}$$

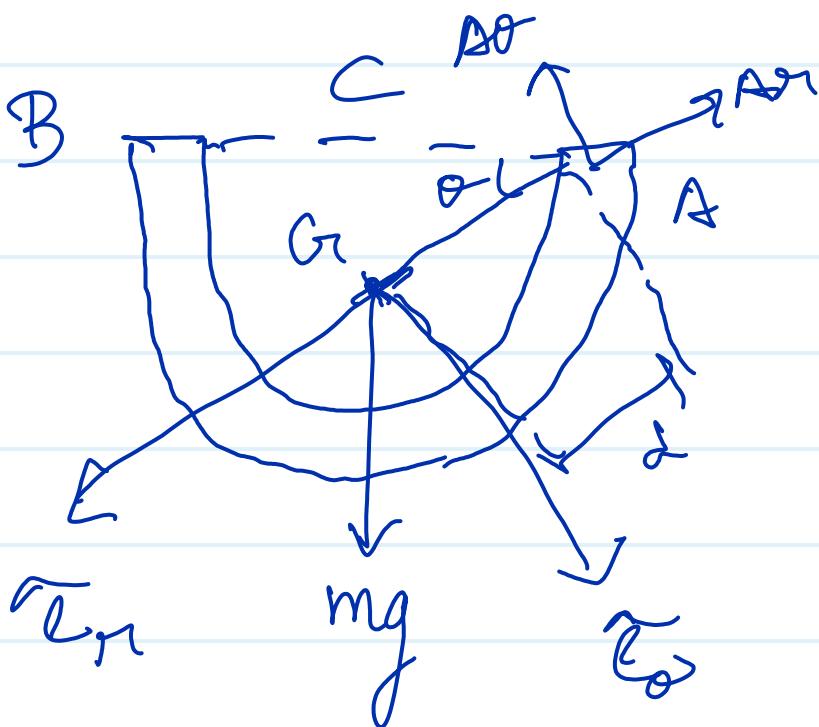
$K = 4682 \text{ Nm/rad}$

6/52 The uniform semicircular bar of mass m and radius r is hinged freely about a horizontal axis through A . If the bar is released from rest in the position shown, where AB is horizontal, determine the initial angular acceleration α of the bar and the expression for the force exerted on the bar by the pin at A . (Note carefully that the initial tangential acceleration of the mass center is not vertical.)

Source: MK, 6th edition



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$$\vec{F}_{mag}$$

$$I_p \ddot{\alpha} + \bar{r} \bar{\alpha} \times \bar{m} \bar{a}_p = M_p$$

$$CG = \frac{2r}{\pi}$$

$$AC = r$$

$$d = r \sqrt{\frac{4}{\pi^2} + 1}$$

$$\vec{a}_m = (\ddot{\alpha} - d\dot{\alpha}^2) \hat{e}_r + (d\ddot{\alpha} + 2\dot{\alpha}\dot{\theta}) \hat{e}_\theta$$

$$\vec{a}_{cr} = -\ddot{\theta}^2 \hat{e}_r + \ddot{\theta} \hat{e}_\theta$$

$$\vec{F} = m \vec{a}_{cr}$$

$$(mg \sin \theta - A_r) \hat{e}_r + (mg \cos \theta - A_\theta) \hat{e}_\theta$$

$$= -m \ddot{\theta}^2 \hat{e}_r + m \dot{\theta} \hat{e}_\theta$$

$$A_r - mg \sin \theta = m \ddot{\theta}^2$$

$$\dot{\theta} = 0 \quad t = 0$$

$$mg \cos \theta - A_\theta = m \dot{\theta}$$

$$\text{at } t = 0$$

$$A_r = mg \sin \theta$$

About A :

$$I_A \ddot{\theta} + \vec{A} \times \vec{m}_A \hat{x}^0 = \vec{M}_A$$

$$I_A \ddot{\theta} = \frac{1}{M_A}$$

$$I_c = mr^2$$

$$I_A = I_c + mr^2$$

$$I_S = 2mr^2$$

$$2mr\ddot{\theta} = mg \cos\theta d \quad \ddot{\theta} = \frac{g \cos\theta}{2} \sqrt{1 + \frac{4}{\pi^2} \frac{d^2}{r^2}}$$

at $\theta = 0$

$$\ddot{\theta} = \frac{mg}{2} \sqrt{\frac{4}{\pi^2} \frac{d^2}{r^2}}$$

$$mg \cos\theta - \frac{1}{2} mg \cos\theta \frac{d^2}{r^2}$$

6/74 The uniform slender bar of mass m and length l is released from rest in the vertical position and pivots on its square end about the corner at O . (a) If the bar is observed to slip when $\theta = 30^\circ$, find the coefficient of static friction μ_s between the bar and the corner. (b) If the end of the bar is notched so that it cannot slip, find the angle θ at which contact ceases.

Source: MK, 6th edition

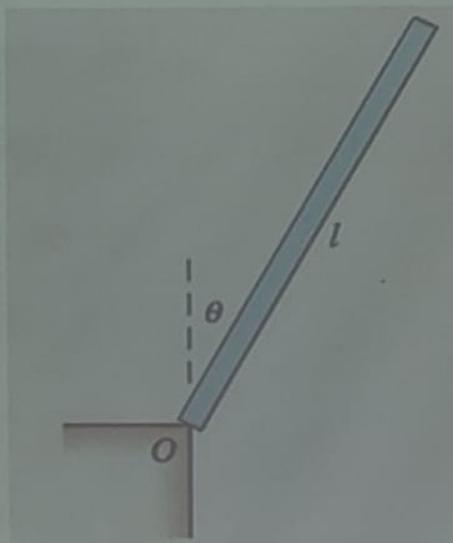


Diagram showing the bar pivoted at O with angle θ from the vertical. Gravity mg acts at center of mass G . Normal force $A\dot{\theta}$ acts at the pivot. Friction force \bar{f}_n acts perpendicular to the bar. Angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ are shown.

$$\bar{a} = (\bar{f}_n - \bar{r}\dot{\theta}^2)\hat{e}_r + (\ddot{\theta} + 2\dot{\theta}\dot{\phi})\hat{e}_\theta$$

$$r = \frac{l}{2} \quad \dot{\theta} \geq 0 \quad \ddot{\theta} = \alpha$$

$$\bar{a}_G = (-\bar{r}\dot{\theta}^2)\hat{e}_r + \left(\frac{l\alpha}{2}\right)\hat{e}_\theta$$

$$(A\dot{\theta} - mg \cos\theta) = -m \frac{l}{2} \dot{\theta}^2$$

$$(mg \sin\theta - A\dot{\theta}) = \frac{l}{2} m \ddot{\theta}$$

$$I_0 \alpha + \overrightarrow{O}G \times \overrightarrow{ma_0} = M_0$$

15

$$\frac{ml^2}{3} \ddot{\theta} + O = M_0$$

$$\frac{ml}{3} \ddot{\alpha} = mg \sin \frac{\theta}{2}$$

$$\ddot{\alpha} = \frac{3g \sin \theta}{2l}$$

$$\ddot{\theta} = \frac{3g \sin \theta}{2l}$$

$$\frac{d}{d\theta} \left(\frac{1}{2} \dot{\theta}^2 \right) = \frac{3}{2} g \sin \theta \quad \text{constant.}$$

$$\frac{1}{2} \dot{\theta}^2 = -\frac{3}{2} \frac{g}{l} \cos \theta + \frac{3g}{2l}$$

$$\dot{\theta} = \sqrt{\frac{3g}{l} (1 - \cos \theta)}$$

$$mg \sin\theta - A\theta = m \frac{3g \sin\theta}{2}$$

$$mg \sin\theta - \frac{3mg \sin\theta}{2} = A\theta$$

$$\boxed{\frac{1}{2} mg \sin\theta = A\theta}$$

$$A_r = mg \cos\theta - m \frac{3}{2} \frac{3g}{2} (\theta - \cos\theta)$$

$$A_r = mg \cos\theta - \frac{3}{2} mg + \frac{3mg \cos\theta}{2}$$

$$A_r = \frac{5}{2} mg \cos\theta - \frac{3}{2} mg$$

$$\boxed{A_r = \frac{mg}{2} [5 \cos\theta - 3]}$$

2) $\theta = 30^\circ$ $\mu = \frac{A\theta}{A_r}$

$\sin\theta$

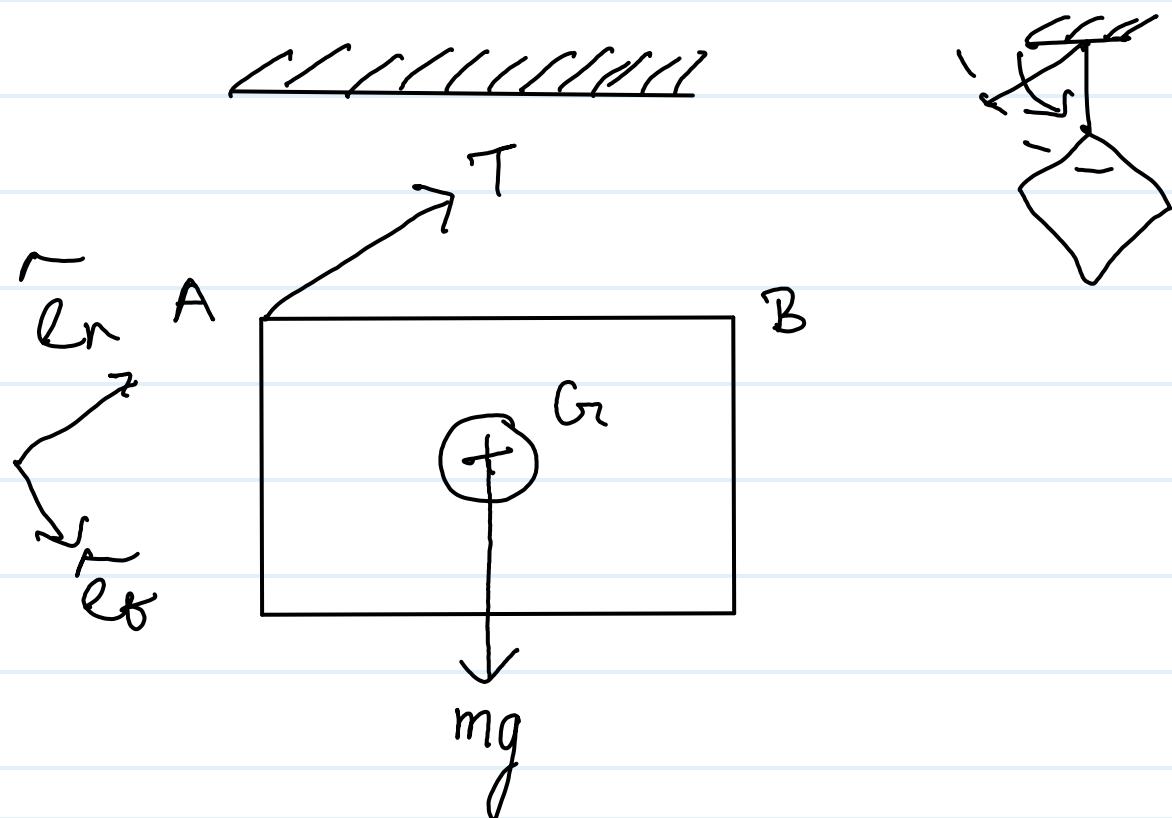
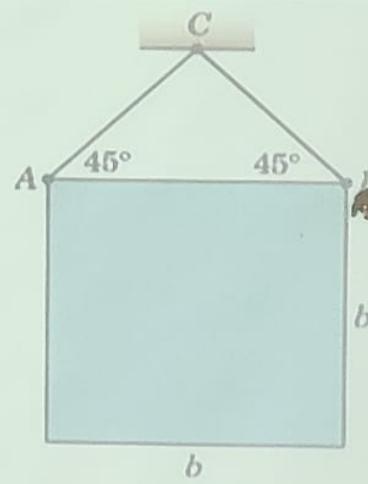
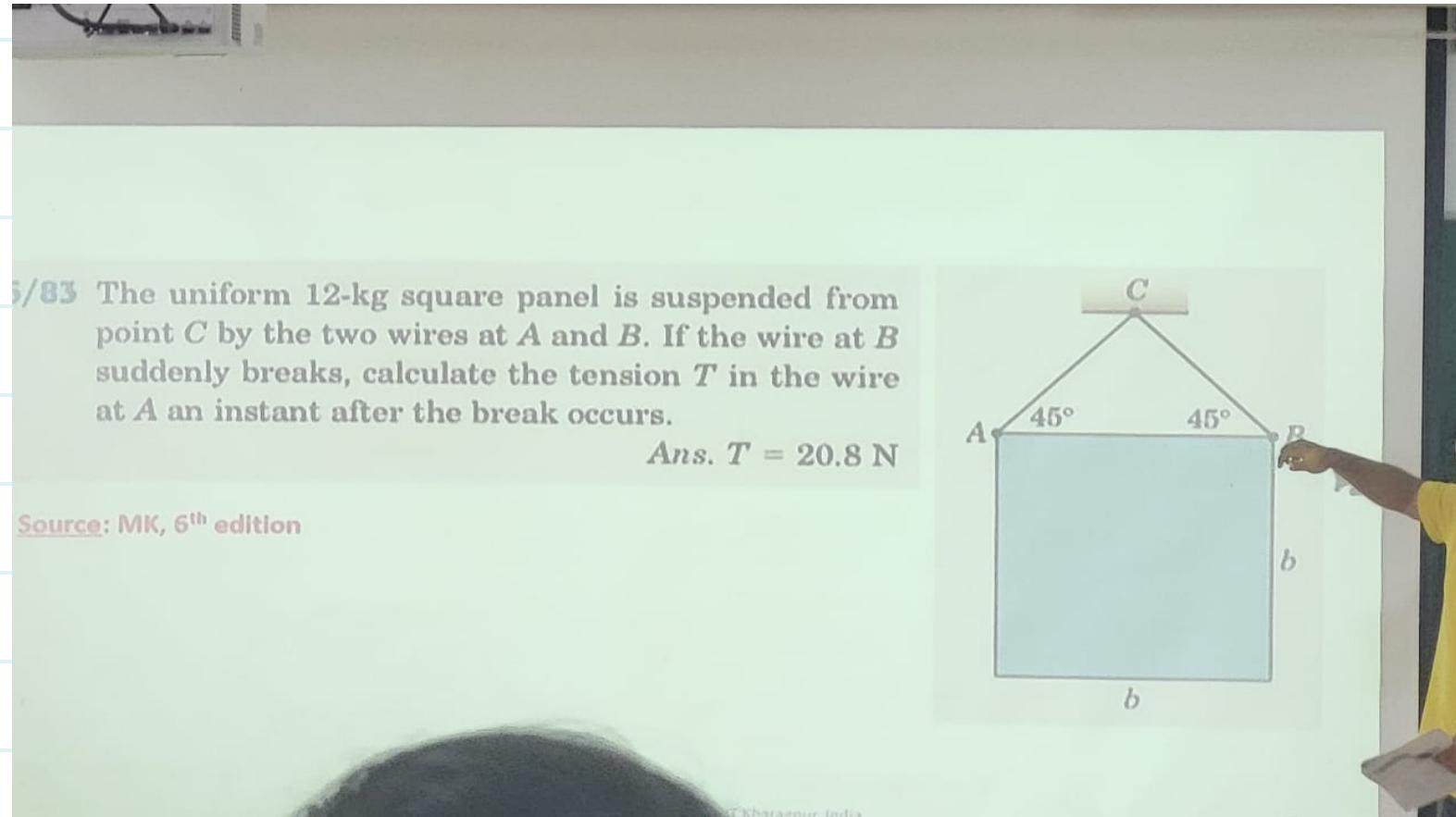
$$\mu = \frac{1}{2 [5 \cos 30^\circ - 3]} = \frac{\frac{1}{\sqrt{2}}}{2 \left[\frac{5\sqrt{3}}{2} - \frac{3}{2} \right]}$$

$$\boxed{\mu = \frac{1}{(10\sqrt{3} - 12)}}$$

b)

$$F_{\text{fr}} = 0$$

$$\cos \theta = \frac{3}{5} \quad \theta = 53^\circ$$



$$\vec{a}_A = \vec{a}_t \hat{e}_t + \cancel{\vec{a}_r} \hat{e}_r^0$$

$$I_G = \frac{1}{6}mb^2$$

$$I_A = \frac{2}{3}mb^2$$

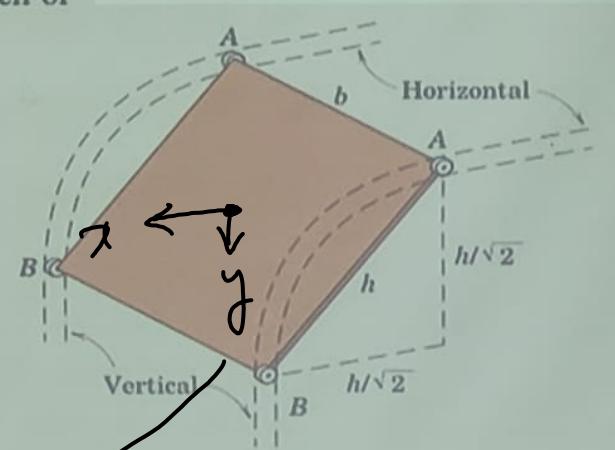
$$I_A \ddot{\alpha} + \bar{A} \bar{b} \times \cancel{m\vec{a}_A} = \bar{M}_A$$

$$\frac{2}{3}mb^2\ddot{\alpha} = -mg \frac{b}{2}$$

$$\ddot{\alpha} = -\frac{3g}{4b}$$

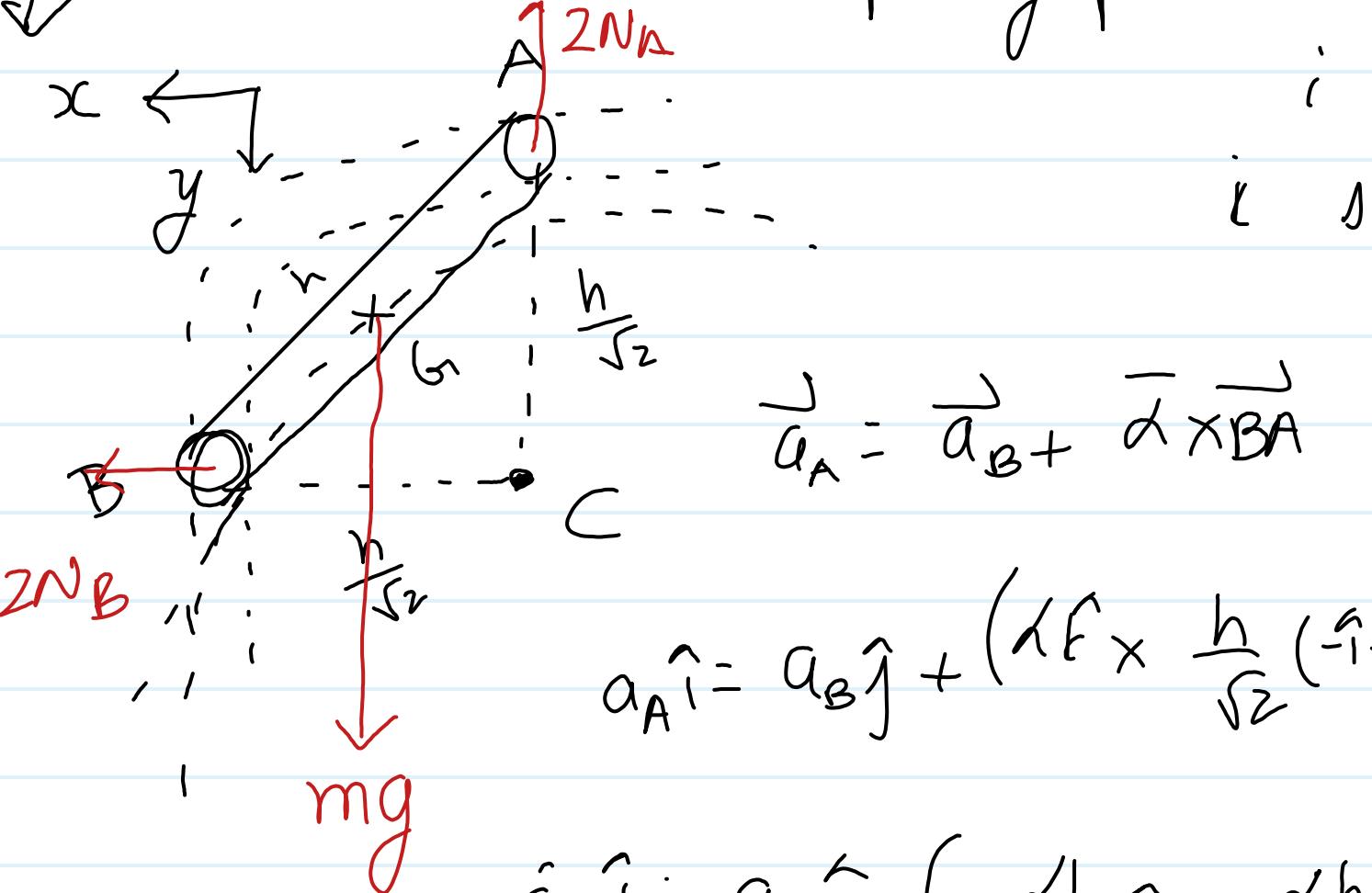
- 112 The overhead garage door is a homogeneous rectangular panel of mass m and is guided by its corner rollers, which run in the tracks shown (dashed). If the door is released from rest in the position shown, determine the force exerted on the door by each of the rollers at A and B . Neglect any friction.

source: MK, 6th edition



→ C.O.M must be contained in
the plane ↴

main assumption of plane kinematics



$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{BA}$$

$$\hat{a}_A = \hat{a}_B + \left(\vec{\alpha} \times \frac{h}{\sqrt{2}} (-\hat{i} - \hat{j}) \right)$$

$$\hat{a}_A = \hat{a}_B + \left(-\frac{\alpha h}{\sqrt{2}} \hat{j} + \frac{\alpha h}{\sqrt{2}} \hat{i} \right)$$

$$\frac{\alpha h}{\sqrt{2}} = a_A \quad a_B = \frac{\alpha h}{\sqrt{2}}$$

$$\vec{a}_G = \vec{a}_A + \left(\vec{\alpha} \times \vec{r}_{AG} \right)$$

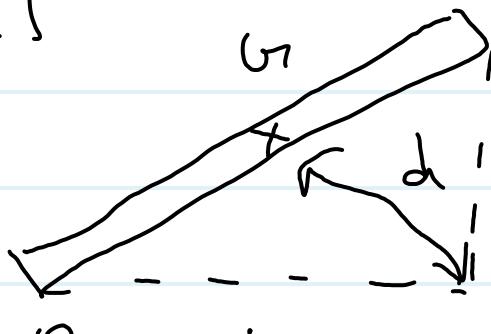
$$\vec{a}_G = \frac{\alpha h}{\sqrt{2}} \hat{i} + (\alpha \hat{k} \times \frac{h}{2\sqrt{2}} (\hat{i} + \hat{j}))$$

$$\vec{a}_G = \frac{2\alpha h}{2\sqrt{2}} \hat{i} + \frac{\alpha h}{2\sqrt{2}} \hat{j} - \frac{\alpha h}{2\sqrt{2}} \hat{k}$$

$$\vec{a}_G = \frac{\alpha h}{2\sqrt{2}} \hat{i} + \frac{\alpha h}{2\sqrt{2}} \hat{j}$$

(using c)

$$I_c \vec{\alpha} + \vec{CG} \times m \vec{a}_c = M_c$$



$$I_c = \frac{mh^2}{3}$$

$$\frac{mh^2}{3} \alpha = mg \frac{h}{2\sqrt{2}}$$

$$\alpha = \frac{3g}{2\sqrt{2}h}$$

$$\vec{a}_n = \frac{3g}{2\sqrt{2}K} \frac{K}{2\sqrt{2}} (\hat{i} + \hat{j})$$

$$\vec{a}_n = \frac{3g}{8} (\hat{i} + \hat{j})$$

$$\vec{F} = m\vec{a}_n$$

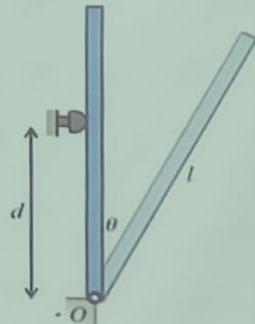
$$2N_B \hat{i} + (2N_A - mg) \hat{j} = \frac{3mg}{8} (\hat{i} + \hat{j})$$

$$N_B = \frac{3mg}{16}$$

$$N_A = \frac{11mg}{16}$$

A door stop is placed at a suitable location to prevent the door from hitting the wall. Determine d so that when the door hits the stop at a certain angular velocity, the reaction at the hinge O is minimized.

Source: Advanced Dynamics, Prof. Anirvan DasGupta, NPTEL



$$\vec{a}_h = -\frac{l}{2} \dot{\theta}^2 \hat{e}_r + \frac{l}{2} \ddot{\theta} \hat{e}_\theta$$

$$O_m = m \frac{l}{2} \ddot{\theta}^2$$

$$\partial_\theta + F_S = m \frac{l}{2} \ddot{\theta}$$

using S

$$I_s \ddot{\theta} + \bar{S} \bar{r} \times \bar{m} \bar{a}_s = \bar{M}_s$$

$$I_s \ddot{\theta} + \left(d - \frac{l}{2} \right) \left(-\hat{e}_y \right) \times m \left(-d \theta^2 \hat{e}_t + d \right) = -d \dot{\theta} \hat{e}_z$$

$$\left[I_s - m d \left(d - \frac{l}{2} \right) \right] \ddot{\theta} = -d \dot{\theta}$$

$\boxed{\dot{\theta} = 0}$ then

$$I_s = m \left(d - \frac{l}{2} \right)$$

$$\frac{1}{12} m l^2 + m \left(d - \frac{l}{2} \right)^2$$

$$\frac{1}{12} m l^2 + m d^2 + \frac{3}{12} m l^2 - m d l$$

$$= m d^2 + m d l$$

$$\frac{1}{3} m l^2 - \frac{3 m d \dot{\theta}}{2}$$

$$d = \frac{2}{9} l$$

Work Energy .

$$T = \frac{1}{2} m \vec{V}_G \cdot \vec{V}_G + \sum \left(\frac{1}{2} m_i \vec{P}_i \cdot \vec{P}_i \right) \quad (\text{Synth by point})$$

Rigid bodies: $\vec{P}_i = \bar{\omega} \times \vec{r}_i$

$$T = \int \frac{1}{2} \vec{r}_G \cdot \vec{r}_G dm + \int \frac{1}{2} (\bar{\omega} \times \vec{P}) \cdot (\bar{\omega} \times \vec{P}) dm$$

$$T = \frac{1}{2} m \vec{r}_G \cdot \vec{r}_G + \frac{1}{2} \left[\int \vec{P} \times (\bar{\omega} \times \vec{P}) dm \right] \bar{\omega}$$

$$T = \frac{1}{2} m \vec{v}_G \cdot \vec{v}_G + \frac{1}{2} \bar{\omega} \cdot I_G \bar{\omega}$$

Translation

Rotation

$$P = \vec{F} \cdot \dot{\vec{r}}_G + \vec{M}_G \cdot \bar{\omega}$$

(power)

$$\frac{dT}{dt} = P$$

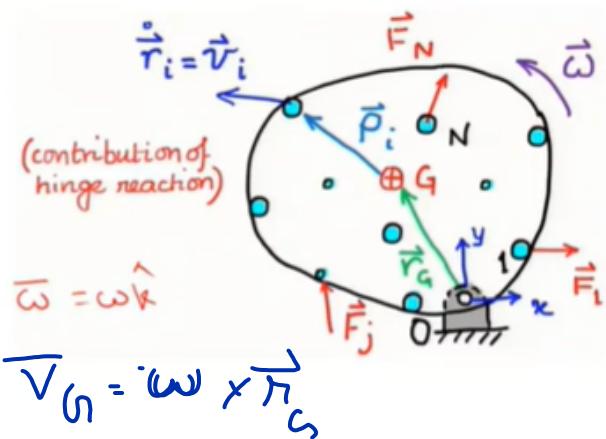
$$\frac{d}{dt} \left[\frac{1}{2} m \vec{v}_G \cdot \vec{v}_G + \frac{1}{2} \bar{\omega} \cdot I_G \bar{\omega} \right] = \vec{F} \cdot \dot{\vec{r}}_G + \vec{M}_G \cdot \bar{\omega}$$

$$\Delta T = T_2 - T_1 = \int_{t_1}^{t_2} \vec{F} \cdot \dot{\vec{r}}_G dt + \int_{t_1}^{t_2} \vec{M}_G \cdot \bar{\omega} dt$$

planar rotation about fixed axis

$$\frac{d}{dt} \left[\frac{1}{2} m \vec{V}_G \cdot \vec{V}_G + \frac{1}{2} \bar{\omega} \cdot I_{G\bar{\omega}} \right]$$

$$= \vec{F} \cdot \vec{\pi}_G + \vec{M}_G \cdot \bar{\omega}$$



$$\frac{d}{dt} \left[\frac{1}{2} I_0 \omega^2 \right] = \bar{\omega} \cdot (\vec{r}_G \times \vec{F}) + \vec{M}_G \cdot \bar{\omega} = M_{0\omega}$$

Total KE in fixed axis rotation.

$$\frac{d}{dt} \left[\frac{1}{2} I_0 \omega^2 \right] = M_{0\omega}$$

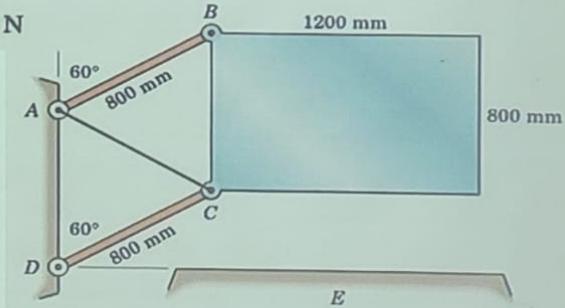
$$\Delta T = T_2 - T_1 = \int_{\theta_1}^{\theta_2} M_{0\omega} d\theta$$

$$[\omega = \dot{\theta}]$$

6/119 The uniform rectangular plate has a mass of 300 kg and is supported in the vertical plane by the two parallel links of negligible mass and by the cable AC . If the cable suddenly breaks, determine the angular velocity ω of the links an instant before the plate strikes the horizontal surface E . Also find the force in member DC at the same instant.

$$\text{Ans. } \omega = 3.50 \text{ rad/s}, F_{DC} = 1472 \text{ N}$$

Source: Dynamics, Meriam & Kraige, 6th edition

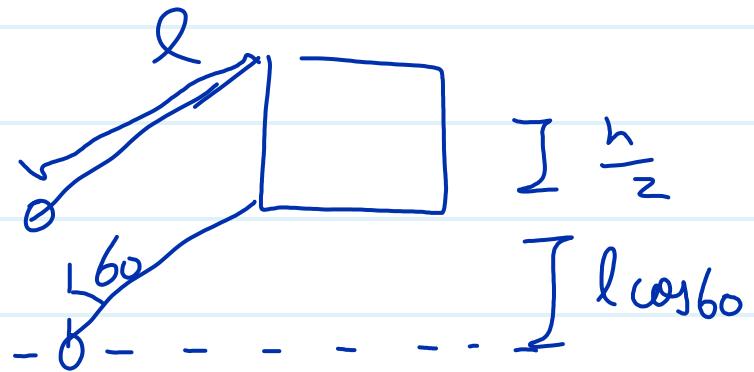


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12

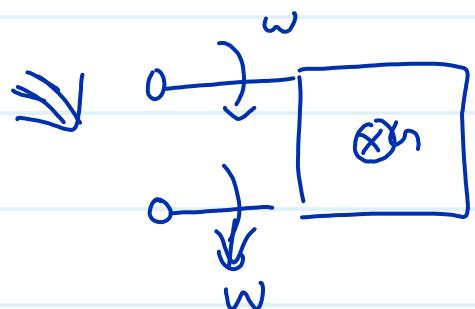
Energy conservation

$$T_1 + V_1 = T_2 + V_2$$



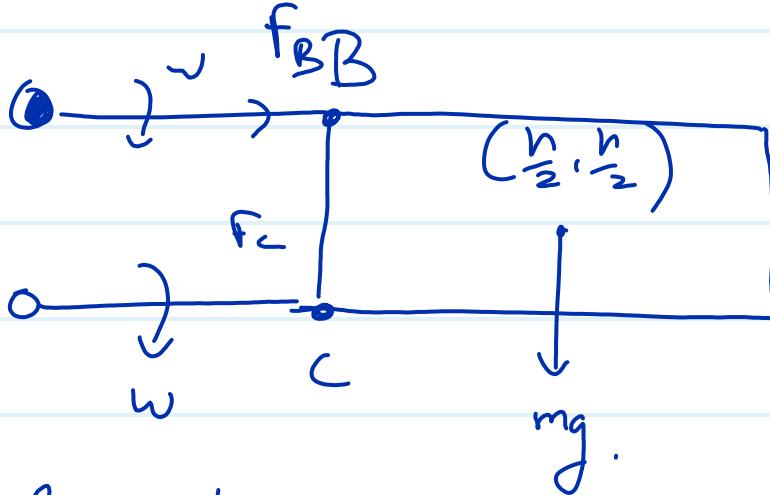
$$0 + mg \left(l \cos 60^\circ + \frac{h}{2} \right)$$

$$= \frac{1}{2} m v_G^2 + mg \frac{h}{2}$$



$$g l \cos 60^\circ = \frac{1}{2} v_G^2$$

$$\sqrt{gl} = \sqrt{G}$$



$$\sqrt{C} = l\omega = \sqrt{G}$$

$$l\omega = \sqrt{gl}$$

$$\omega = \sqrt{\frac{g}{l}}$$

About B

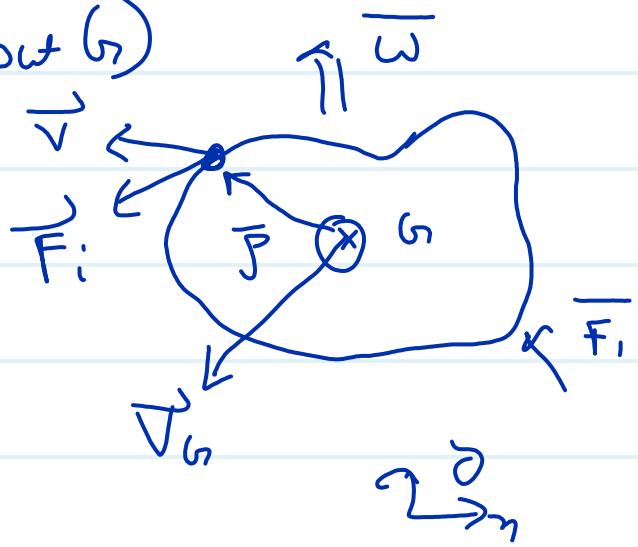
~~$$\vec{I}_{pd} \ddot{\vec{x}}^0 + \vec{B}_{Gn} \times \vec{m}\vec{a}_{Gn} = \vec{M}_B$$~~

$$\left(\frac{h}{2} \hat{i} - \frac{h}{2} \hat{j} \right)$$

Equations of motion (about G)

$$\overrightarrow{G} = \overrightarrow{F}$$

$$\dot{\overrightarrow{H}_G} = \overrightarrow{M_G}$$



$$\overrightarrow{G} = m\vec{V}_G$$

$$\overrightarrow{H}_G = I_{Gzz}\vec{\omega} = I_{Gzz}\dot{\theta}\hat{k} \text{ (planar)}$$

$$\frac{d}{dt} \left[\underbrace{\frac{1}{2} m \vec{V}_G \vec{V}_G}_{T_T} \right] = \vec{F} \cdot \frac{d\vec{V}_G}{dt}$$

$$\frac{d}{dt} \left[\underbrace{\frac{1}{2} I_{Gzz}\dot{\theta}^2}_{T_R} \right] = M_G \dot{\theta}$$

$$\Delta T_T = T_{T_2} - T_{T_1} = \int_1^2 \vec{F} \cdot d\vec{V}_G$$

$$\Delta T_R = T_{R_2} - T_{R_1} = \int_{\theta_1}^{\theta_2} M_G d\theta$$

Impulse-Momentum

$$\Delta \vec{p} = \int_{t_1}^{t_2} \vec{F} dt$$

$$\Delta \vec{P}_b = \int_{t_1}^{t_2} \vec{M}_b dt$$