ME-467: Turbulence
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5 May 2022 – Part II included

Course Project: Statistics of Turbulence and the Onset of Chaos.

This 10-page document contains the instructions for the project of the course **ME-467: Turbulence**, which determines the final grade. This project consists of two independent parts: In the first part you will be analyzing data from a modern wind tunnel experiment, and relating the findings to predictions of Kolmogorov's turbulence theory (hereafter called K41). In the second part you will study how chaos emerges from a deterministic evolution equation.

Material

To complete the assignment, you need the following files:

- `instructions.pdf`: This instruction sheet.
- `report.tex` (with files `titlepic.sty`, `fancyhdr.sty`, `figures/EPFL_LOGO.jpg`): Please use this template to prepare your report. Set your name in the `MyName` variable, so that it is displayed on every page.
- Six data files `u1.txt` to `u6.txt`: The ascii datafiles (around 11 million data points each) that contain all velocity data for the data analysis.

Rules / Honor Code

- **Individual report:** Each student has to write and submit her/his own report.
- **Collaborations:** We allow and encourage discussions and collaborations with your classmates. However, write the report individually, and at the end of the report cite all sources including a list of the people with whom you collaborated. Please limit collaborations to groups of at most 4 people.
- **Resources:** You are free to use any sources that you find useful, including books, scientific papers and internet resources. **The only exceptions are exercises/exams of ME-467 from previous years, which may not be consulted.**
- **Citing:** Please cite all literature and other sources used in preparing the report.
- **Page limits:** The report should be only as long as is needed to convey the message clearly, and the template gives recommendations for the maximal length of several sections. Please respect the page limits.
- **Software:** For the data analysis, please use `MATLAB` or `Python`. Note that you have to submit readable and commented scripts / code.

Submission

The deadline for the project is **May 22, 2022, 3PM**. Please submit electronically a `zip`-archive named `<Lastname>_<Firstname>.zip` including

- your report, as `.pdf` and `.tex` source code;
- the analysis scripts (readable ASCII source code with comments - no `MATLAB` live scripts or Jupyter notebooks or similar) you used for data analysis and production of figures;
- sources you have used to complete the assignment, if they are not available through the usual library channels;
- signed last page of the report certifying that you followed the Honor Code.

Please use the Moodle upload function to submit these files. Do **not** include the datafiles in your upload. **The upload closes at 3PM sharp. Late submissions will not be accepted.**

Tasks

In Part I 'Statistical Analysis of Turbulence' you will study fully developed turbulence based on experimental data obtained in a modern wind tunnel using state-of-the-art instrumentation. Specifically, you will

- **Task I.1:** Process and analyze experimental data using `Python` or `MATLAB`.
- **Task I.2:** Interpret the data in view of K41 turbulence theory and write a report presenting, discussing and interpreting the findings.

In Part II 'Nonlinear dynamics and the emergence of chaos' you will study deterministic chaos — the phenomenon necessitating a statistical description of turbulence despite its emergence from a deterministic evolution equation — for a simplified model system. Specifically, you will

- **Task II.1:** Both numerically (using again `Python` or `MATLAB`) and analytically investigate properties of the model system.
- **Task II.2:** Explain and discuss your findings in a written report.

Part I: Statistical Analysis of Turbulence

1 Data Analysis

Files `u1.txt` to `u6.txt` contain time series of the downstream velocity from an experiment in the Warhaft Wind and Turbulence Tunnel at Cornell University[1] in units of m/s. Figure 1 schematically shows the test section of the wind tunnel: Driven by a fan, flow of air passes through an active grid, and enters the test section which is of constant cross-sectional area. Six hot-wire anemometers A_1 to A_6 are placed evenly spaced on the centerline of the test section at downstream distance $d_1 = 1\,m$ to $d_6 = 6\,m$ from the grid, and have recorded time series `u1.txt` to `u6.txt`, respectively. The data were acquired at atmospheric pressure and room temperature, and sampled at frequency of $f = 20\,kHz$.

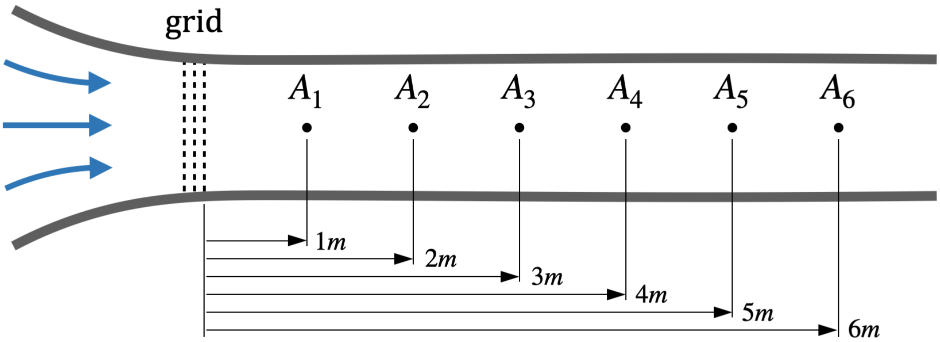

Figure 1: Schematic of the test section of the wind tunnel and position of hot-wire anemometers A_1 to A_6 with distances measured from the grid.

1.1 Velocity Signal in the Spatial Domain

The statistical turbulence theory studied in this project is formulated mainly in terms of the spatial structure of the flow field. Thus, interpret the time series as a spatial measurement of the total streamwise velocity signal, $u_{\text{tot}}(x)$, via Taylor's frozen flow hypothesis, with x being the streamwise coordinate. For this:

- Plot u_{tot} against x for each anemometer (Plot A). The plotting ranges must be chosen such that similarities and differences between the datasets can be identified visually.
- Calculate the mean velocity, U, for each anemometer, and fill the relevant row in the results table (see the report document).
- Comment on the observed dependence of U on the downstream distance, d. Explain why (or why not) this dependence is physically reasonable.
- Calculate the turbulence intensity,

$$I \equiv \frac{\sqrt{\langle u^2 \rangle}}{U}, \tag{1}$$

for each anemometer, where u is the deviation of the total velocity form its mean value:

$$u \equiv u_{\text{tot}} - U. \tag{2}$$

Fill the relevant row in the results table.

[1]Yoon and Warhaft, J. Fluid Mech. **215**, 601–38 (1990), DOI: 10.1017/S0022112090002786

- Comment on the trend in values of I as a function of the downstream distance, d. How is this trend related to the properties of the signal observed in Plot A?
- To what extent are spatial interpretations of the time series acquired by each anemometer appropriate? [Extra: Can you quantify possible errors?]

1.2 Correlation Length of the Velocity Signal

For short distances the velocity signal is correlated, whereas for larger distances velocity values can be considered statistically independent. For the velocity fluctuations, $u(x)$, define the autocorrelation, $C(l)$, as

$$C(l) \equiv \frac{\langle u(x+l)u(x)\rangle}{\langle u^2(x)\rangle}. \tag{3}$$

The correlation length, L_C, is the length over which the fluctuations are correlated. Define L_C as the length at which $C(l)$ has dropped to $1/e$.

- Plot $C(l)$ against l for each anemometer, and mark the corresponding correlation length on each plot (Plot B). Fill in the relevant row in the results table.
- Comment on the observed variation of L_C with the downstream distance, d. Explain whether or not this trend is consistent with your intuition into the physics of the flow.

The correlation length approximates the integral scale given by the *integral* (hence the name)

$$L_{\text{int}} \equiv \int_0^\infty C(l)\mathrm{d}l. \tag{4}$$

- Compute L_{int} for each dataset, and compare to the estimate L_C. Fill in the relevant row in the results table.

From here on, only use L_C as estimate for the integral scale.

1.3 Energy Spectrum of the Flow

Essential information about the turbulent flow can be extracted by analyzing the spectral energy density $E(k)$. In the experiment, velocity data is not available over an infinitely extended spatial domain, but only for a finite downstream distance L. The spectral energy density is then

$$\tilde{E}(k) \equiv \frac{1}{2}\left|\frac{1}{\sqrt{2\pi L}}\int_0^L u(x)e^{-ikx}\mathrm{d}x\right|^2 \quad ; \qquad k \in \mathbb{R}. \tag{5}$$

- Calculate the energy spectrum for $k > 0$, which is $E(k) \equiv \tilde{E}(k) + \tilde{E}(-k)$. Plot $E(k)$ against k in log-log scale for all probe distances d_1 to d_6 <u>in a single figure</u> (Plot C). Use appropriate smoothing to minimize the noise.
- In order to confirm that your normalization is correct, use Parseval's theorem,

  $$\frac{1}{2}\langle u^2\rangle = \int_0^\infty E(k)\mathrm{d}k, \tag{6}$$

  and report the relative error for the dataset recorded by the first anemometer.
- Indicate predictions of the K41 theory on Plot C. Does the data follow the theory predictions?
- Estimate the integral length scale, $L_{\text{int},E}$, and Kolmogorov length scale, η_E, by inspecting each spectrum. Mark the corresponding length scales on each curve in Plot C, and report the estimated values in the relevant rows in the results table.

1.4 The Dissipation Rate and Different Reynolds Numbers

The Taylor Reynolds number is

$$Re_\lambda \equiv \frac{\sqrt{\langle u^2 \rangle}\lambda}{\nu}, \tag{7}$$

where ν and λ denote the kinematic viscosity of the fluid and the Taylor length scale, respectively. $\lambda = \sqrt{15\nu \langle u^2 \rangle / \epsilon}$ depends on the energy dissipation rate ϵ that can be estimated from the velocity fluctuations at the integral scale where energy is injected into the energy cascade:

$$\epsilon = \frac{1}{2} \frac{\sqrt{\langle u^2 \rangle^3}}{L_C}. \tag{8}$$

- Based on the analysis above estimate the energy dissipation rate, ϵ, at each downstream probe distance, d, and fill the relevant row in the results table.
- Estimate the Taylor Reynolds number, Re_λ, at each downstream probe distance, d, and fill the relevant row in the results table.
- Estimate the (outer-scale) Reynolds number, Re, of the flow at each downstream probe distance, d, and fill the relevant row in the results table.
- Describe the trends in the change of the energy dissipation rate and Reynolds numbers with probes' downstream position, d. Explain how the physics of the flow supports (or contradicts) the observed dependence of ϵ on d. Are the trends in ϵ, Re_λ and Re consistent with the trend in turbulence intensity, I, calculated in Section 1.1?

1.5 Turbulence Decay

In this section we examine predictions of the law of decay of turbulence. You are familiar with the case of unforced turbulence decaying in time. In a wind tunnel, however, turbulence decays as it moves downstream from the grid. Consequently, the downstream distance, d, and time, t, can be linked by the mean flow speed, U:

$$t = \frac{d}{U}. \tag{9}$$

Substituting t from this equation into the formulation you are familiar with, the theory predicts:

$$l_0 \propto (d - d_0)^{1/(1-h)} \tag{10}$$

$$u_0 \propto (d - d_0)^{h/(1-h)} \tag{11}$$

$$Re \propto (d - d_0)^{(1+h)/(1-h)} \tag{12}$$

$$E \propto (d - d_0)^{2h/(1-h)} \tag{13}$$

where d_0 is the position of the so-called virtual spatial origin, and h is the scaling exponent of the large-scale scaling $u_l \sim Cl^h$ for $l \to \infty$. l_0 and u_0 represent the integral length scale and the velocity at the integral length scale, respectively.

- The overall kinetic energy per unit mass of the flow is

  $$\mathcal{E} = \frac{3}{2} \langle u^2 \rangle. \tag{14}$$

  Explain the 3/2 prefactor, calculate $\mathcal{E}$ for each dataset and fill the relevant row in the results table.

- From values of $\mathcal{E}$ find the best d_0 and h that fit the scaling (13) using appropriate numerical fitting methods.

- Instead of automatic curve fitting, one can use a graphical approach to identify power laws and determine their exponents:
  - Show that for the correct value of d_0 the log-log plot of $\mathcal{E}$ against $(d - d_0)$ should follow a straight line.
  - Vary d_0 in a physically justified range. For each value of d_0, plot $\mathcal{E}$ against $(d - d_0)$, determine the exponent $\mathcal{E} \propto (d - d_0)^q$ and include the fit in the plot (Plot D).
  - From the plotted data and fits, determine the best-fit values of d_0 and $h = q/(q+2)$.

- A third method for determining h is the following:
  - It cane be shown easily that the kinetic energy content of the flow, $\mathcal{E}$, scales with the integral length scale, l_0, as
  $$\mathcal{E} \propto l_0^{2h}. \tag{15}$$
  In order to estimate h from the scaling relation (15), plot $\mathcal{E}$ against L_C (from Section 1.2) such that a constant exponent can be identified graphically (Plot E).
  - Add theoretical predictions $2h = -3$ (Saffman's decay), $2h = -5$ (Loitsyanskii's decay), and $2h = -2$ (self-similar decay) to your plot. Which, if any, decay scenario provides the best-matching description of your data (by a visual inspection)?
  - Now assume h is fixed to the best-matching theoretical prediction, and find the best fit d_0 in $\mathcal{E} = C(d - d_0)^{2h/(1-h)}$.

- Compare values of d_0 and h determined in all three approaches. What are the advantages and disadvantages of each method?

- The energy spectrum $E(k)$ is predicted to scale as $E(k) \sim k^{-(1+2h)}$ for length scales sufficiently larger than the integral length scale. Plot all energy spectrum curves on the same figure once again, and this time draw the slope $-(1+2h)$ over the energy containing range (Plot F). Comment on how well the predicted slope matches the curve.

- Interpret the location of the virtual origin d_0. Is the virtual origin only a fitting parameter or does it mark a location where the flow properties change physically? Discuss the location and a potential physical significance of d_0.

- Based on your previous considerations, do you expect d_0 to be different when you do the same calculation for another scaling relation, e.g. (10) or (12)? Briefly explain why (or why not). [You may test your hypothesis by analyzing the other scaling relations using the different curve-fitting methods used above, but such an analysis is not required.]

1.6 Velocity Increments

Consider the longitudinal velocity increment

$$\delta u_{||}(x, l) \equiv u(x + l) - u(x). \tag{16}$$

- For $l \in \{1\,mm,\ 1\,cm,\ 10\,cm,\ 10\,m\}$ and only for the dataset acquired at distance $d_1 = 1\,m$, plot $\delta u_{||}$ against x for suitable x- and $\delta u_{||}$-ranges (Plot G).

- Where do these l-values lie relative to the turbulent length scales you determined from the energy spectrum?

- Describe the differences in the signal between small and large length scales. Explain whether or not your observations can reveal the trend in each of the autocorrelation curves you plotted in Section 1.2.

- For each l, the probability distribution of $\delta u_{||}$ has a bell-shaped curve. To investigate how close such a curve is to the Gaussian distribution, compare the flatness of the velocity increment signal, defined as follows, with that of the Gaussian distribution:

$$f(l) \equiv \frac{\left\langle \delta u_{||}^4(x,l) \right\rangle}{\left\langle \delta u_{||}^2(x,l) \right\rangle^2}. \tag{17}$$

  For this:
  - Plot the flatness as a function of l only for the dataset acquired by the probe located at distance $d_1 = 1\,m$ (Plot H).
  - Analytically calculate the flatness of the Gaussian distribution, and mark its value on the figure.
  - Describe the convergence behaviour of $f(l)$ for large l. What can be the reason why the flatness grows by decreasing l?

1.7 Structure Functions and Energy Dissipation

The n^{th}-order structure function is defined as

$$S_n(l) \equiv \left\langle \delta u_{||}^n(x,l) \right\rangle \; ; \qquad n \in \mathbb{N}. \tag{18}$$

- Only for the dataset acquired at distance $d_1 = 1\,m$ plot $S_2(l)$ and $S_3(l)$ versus l in log-log scale for suitable l-ranges (Plots I and J).

- K41 predicts a scaling of S_2 as a function of l. How is the scaling of $S_2(l)$ related to scaling of the spectral energy spectrum $E(k)$? Draw the expected slope on the plot, and compare ranges over which a clear scaling is observed for $E(k)$ and $S_2(l)$.

- According to K41, the third order structure function follows the four-fifth law. Draw the predicted slope on the plot and discuss to which extent your data supports the K41 prediction.

- How can you estimate the energy dissipation rate ϵ from $S_2(l)$ and $S_3(l)$? Discuss how the estimates compare to the estimates based on integral-scale quantities in Section 1.4 (Note: According to K. Sreenivasan, the prefactor of the second order structure function is $C_2 \approx 2.2$ with $S_p(l) = C_p \epsilon^{p/3} l^{p/3}$.)

2 Interpretation

Use the template `report.tex` to prepare your report. Fill in the following sections:

2.1 Introduction

Review the predictions of Kolmogorov theory for the decay of unforced turbulence. What does the theory predict and how does K41 theory yield these prediction? Are all hypotheses underlying K41 theory (theory for forced turbulence) strictly valid for decaying turbulence? If not, what conditions need to be (at least approximately) satisfied for K41 to be applicable? (Limit: 1 page)

2.2 Data Analysis

Present and discuss the data analysis from **Task I.1**. There are no page limits here. If you want to add additional plots to help explain your reasoning, that is possible but all the explicitly mentioned plots should be presented and described.

2.3 Discussion

In light of the statistical analysis you have performed on the velocity data from an actual decaying turbulent flow:

- Discuss whether K41 provides an accurate description of the observed data.
- Does the data suggest a specific decay scenario? If yes, how strong is the evidence?
- Which of the discrepancies between the observations and K41 are caused by the experimental setup or the design of the experiment, that could be avoided in an ideal turbulence experiment; and which could be inherent features of turbulence that the K41 theory fails to describe properly?

(Limit: 1 page) [Please respect this page limit. In case you want to perform additional data analysis to support your conclusions, you are of course allowed to do so but please put this material in an appendix.]

Hints / Suggestions

The following ideas might be helpful:

- The purpose of this project is to develop deeper understanding of turbulence and not to assess your programming skills. Therefore, you are allowed (and encouraged) to employ built-in functions for calculating statistical quantities whenever such a function or library is available, but be careful! You always need to consult the documentation of the function or library you use to make sure it returns exactly the intended quantity.
- When developing your analysis code, use only part of the full data (to speed up the computation).
- When making a plot or graph, it is usually sufficient to evaluate a function at around 100 points (to speed up plotting).
- Label the axes of your plots in the right physical units. Deliberately choose axis ranges and a linear/logarithmic scaling of the axes to support your analysis.
- When giving the results of calculations, whenever possible include the uncertainty in the result.
- When discussing data compared to theoretical predictions, add the predictions to the plot.
- If you make an assumption, check its validity to the extent possible.

Part II: The Baker's Map, Chaos and Fractals

3 Analysis of the Dynamics

The generalized Baker's Map is a great, yet simple, example to study how chaos emerges in deterministic systems, and to visualize and characterize properties of strange attractors. The map is defined as the discrete transformation of the unit square $[0,1] \times [0,1]$. In each iteration, any point (x_n, y_n) within the unit square is mapped onto its iterate (x_{n+1}, y_{n+1}) as follows

$$x_{n+1} = \begin{cases} \alpha_1 x_n & \text{if } y_n < \beta \\ (1-\alpha_2) + \alpha_2 x_n & \text{if } y_n \geq \beta \end{cases} \tag{19}$$

$$y_{n+1} = \begin{cases} y_n/\beta & \text{if } y_n < \beta \\ (y_n - \beta)/(1-\beta) & \text{if } y_n \geq \beta \end{cases} \tag{20}$$

where α_1, α_2 and β with $\alpha_1 + \alpha_2 \leq 1$ and $0 < \beta < 1$ are positive real-valued parameters.

The map is a slight generalization of a map whose action corresponds to how a baker stretches and folds dough when kneading, hence the name of the map. The unit square is divided into two parts for $y < \beta$ and $y \geq \beta$, that are horizontally compressed by a factor α_1 and α_2 respectively, then stretched by a factor $1/\beta$ (the lower piece) and $1/(1-\beta)$ (the upper piece) and finally mapped back in the unit square (Figure 2).

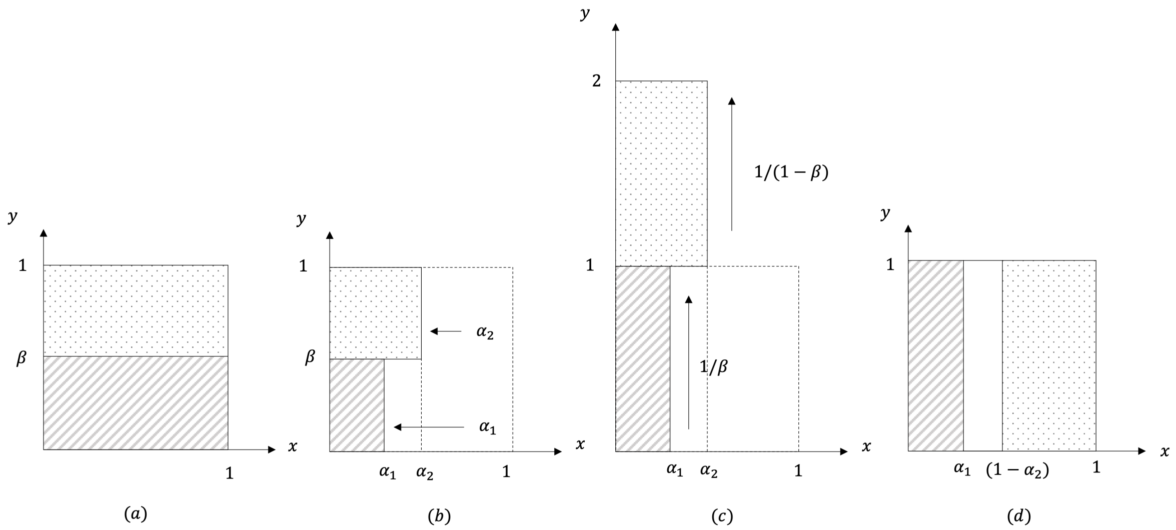

Figure 2: Schematic of the action of the generalized Baker's map on the unit square in the case $\beta = 0.5$.

3.1 Implementation of the Map and (Numerical) Observations

Use `MATLAB` or `Python` to follow many points under the action of the given map. Starting from an initial condition where points are distributed uniformly, similar to the one in Figure 3, numerically explore the dynamics of the Baker's map for different parameters:

- Plot and describe the evolution of the system at different time iterations in the most generic case: $\alpha_1 \neq \alpha_2$ and $\beta \neq 0.5$.
- Describe the changes in the attractor when varying the parameters α_1 and α_2. Support your analysis with appropriate plots.
- Qualitatively describe how the emergence of chaos can be recognized in the system.

3.2 Strange Attractor and Fractal Dimensions

The long-term dynamics of many chaotic systems often (not always!) lands on attractive sets characterized by a fractal dimension, strange attractors.

- In the special case of $\alpha_1 = \alpha_2$ compute the box counting dimension D_0 of the attractor analytically. Also compute the box counting dimension numerically. Compare analytical and numerical results. Comment on the source of potential discrepancies. (Reminder: In order to observe fractal structures you need a sufficient number of realizations)
- What do you observe when $\alpha_1 + \alpha_2 = 2\alpha_1 = 1$?

3.3 Chaos and Lyapunov Exponents

The chaotic behaviour of a system can be quantified via its Lyapunov exponents, characterizing the rate of divergence of infinitesimally close trajectories.

- In the special case $\alpha_1 = \alpha_2$ and $\beta = 1/2$ compute analytically the Lyapunov exponents of the system and compare them to the numerical estimates.
- Provide a graphical interpretation of the Lyapunov exponents for this specific system.
- What else can be deduced from the specific values of the Lyapunov exponents for the chosen parameters? Do they indicate any noteworthy property of the map?

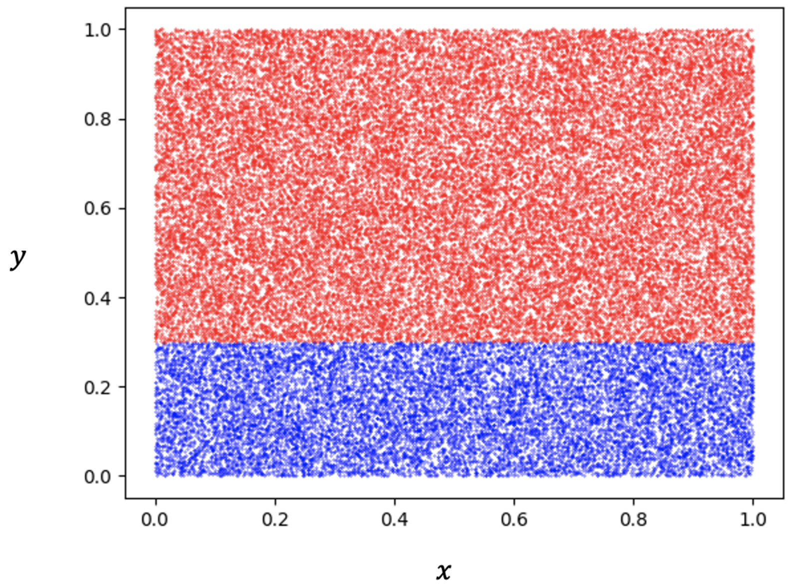

Figure 3: Initial condition for the iteration of the Baker's map for $\beta < 1/2$. At $t = 0$ points are uniformly distributed in the unit square $[0, 1] \times [0, 1]$ and coloured to visually highlight the partition of the domain: blue for points with $y < \beta$ and red for the others.

4 Discussion of the results

As for the first part of this project, use the template to prepare your report. Discuss your analysis including numerical and analytical components using appropriate visualizations and calculations. No extended introduction or detailed final discussion (as in part I) is needed - you can keep these parts very short!