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THE UNSATISFACTORY STORY OF CURVATURE

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1. The Preliminaries. The fact that plane loci are sometimes straight and sometimes are not would seem to be about as obvious as any property of such figures; it is therefore very surprising that this was not placed in the foreground by early writers who dealt with the matter. Euclid defines a straight line as one which lies evenly with the points on itself, which meant that the end points would completely cover those which lay in between or completely hide the latter points, and countless variations of this idea were evolved by the early writers of geometry, but the dynamic idea that a straight line was generated by a point which always moved in the same direction was not, as far as I know, stressed by anyone. Of course whoever admits direction as a primary idea of geometry is involving himself in a sea of trouble, and this may be the reason for its avoidance, but the total neglect of anything so fundamental is certainly striking.

The Greek writers were familiar with all sorts of specific curved loci, witness Proclus' famous summary of the history of geometry, and he says that Aristotle recognized three kinds of loci "*C'est aussi la raison pour laquelle il-y-a trois mouvements, l'une en direction de la ligne droite, l'autre circulaire, et le troisième mixte*" [1]. Proclus himself expands this idea in [1] p. 234, "*Tandis que d'autres ont affectuès la section au moyen des quadratices de Hippias et de Nicomède lesquels avaient aussi fait usage de lignes mixtes quadratices.*" The Greeks were familiar with curved loci, but singularly slow in pointing out the distinguishing characteristics.

The fundamental idea lay very close at hand. Some curves were straight, others were curved. Whatever curvature might be, a circle was everywhere equally curved. The greater the curvature, the less the radius, so it would be natural to take as the curvature a quantity inversely porportional to the radius of the circle, and for any curve, the curvature of the circle lying nearest to it. All this is so simple and natural, but it was strangely slow in coming to be realized. I am sure that a complete account can be found; it certainly is not in [2]. I write in the hope that someone will succeed where I have failed to find it.

The question of curvature is intimately connected with that of the centre of curvature, and Apollonius is perfectly aware that from certain points but one normal can be drawn to a conic, and the careful discussion of this question was earnestly pursued, but the writer did not go on to other loci, and it is curious that neither he nor Euclid in discussing optics took the quite obvious steps in this connection. On the contrary, the attention was diverted to the analogous problem of horn angles. A horn or cornicular angle is roughly the figure formed by the circumference of a circle and its tangent; a good discussion by Heath is found in [**3**]. We learn by Euclid III, **16** that the angle of the semicircle is greater and the remaining angle is less than any acute rectilineal angle. As long as a secant cuts a circumference twice, the angle which it makes with the curve is less than the horn angle, but the latter, though it can be indefinitely increased, by decreasing the radius of the circle, is nevertheless smaller than a right angle. This is very much contrary to the notion that the two have a definite ratio. Vieta in [**4**] takes up the question of whether a circumference and its tangent really can be said to make an angle, and points out that in any case this must be a different sort of object from other figures which cannot pass from greater to less without passing through equality. Wallis in [**5**] insists, in a withering attack on Clavius, that a horn angle is not an angle in Euclid's sense as it is not an inclination between two loci. A new word is needed "*Cui respondeo, mihi cum Clavio hactem convenire (et convenisse semper). Quod quem ille vocat Angulum Contactus, nil aliud est quod ego voco Gradum Curvitatis. Sunt utque curvitatum gradum semper proportionales longitudinibus Diametrorum Chordarium arcuum similibus.*"

I will not go further into the complicated question of horn angles but return to the larger question of definition of curvature. The first writer to give a hint of the definition of curvature was the fourteenth century writer Nicolas Oresme, whose work was called to my attention by Carl Boyer. We find Oresme saying in [**6**] "*Nunc restat de Curvitate dicendum.*" He assumes the existence of something which he calls *Curvitas*, and if we have two curves touching the same line at the same point, and on the same side, the smaller curve will have the greater curvature. He further states that the curvature of a circle is "*Uniformis*" and on p. 219 "*Sit circulus major cuius semidyameter sit AB et circulus minor cuius semidyameter sit AC, tunc si sit semidyameter AB duplo ad semidyanmeter AC, curvitas minoris circuli erit duplo intensior curvitate majoris, et ita proportionibus et curvitatibus.*" We could not have a clearer proof that Oresme conceived the curvature of a circle as inversely proportional to the radius; how did he find this out?

We apparently have to wait nearly three hundred years before finding anything further on the subject of curvature when we turn to the work of Kepler [**7**]. He undertakes to find the image of a certain brilliant point, the generalized problem of Al Hazen. He shows the standard approach but adds that instead of dealing with the curve itself it would be wiser to deal with its circle of curvature "*At verior ratio jubet invenire circulum, qui continett rationem curvitatis.*" The ratio he connects with a certain circle, the circle of curvature. His editor, Frischauf in a note on p. 403 to [7] adds "*Primum hic occurit circulus quem dicunt osculatorem, quem posteriores mathematici tum add lineas tum ad superficies curvas maximo commodos adhibebunt, cujus inventio Leibnitio huc usque tributa ets.*" We shall presently see that it was certainly a mistake to associate Leibniz with the invention of the circle of curvature, but we have a glimpse of the fact that the problem of the rightful ascription was not too simple.

A great step in advance is due to Huygens. I mention especially his *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demon strationes geometricae* of 1673 [8]. Kepler starts curiously with the involutes of curves. We take a curve all of whose tangents are on one side. To this is attached a flexible string which is pulled taut and then unwound. The curve from which the string springs is called the evolute. He assumes that the string will always remain tangent to this evolute; the locus of a point fixed in the string will be the involute. He begins with a careful proof that the string will always be normal to the involute. Then comes the curious theorem that two curves with tangents on one side which have a common point cannot have the same set of normals. After this we see that if all tangents to one curve are normals to another they will be an involute and evolute as defined above. He next passes to certain specific curves, notably the cycloid, and in theorem XI he undertakes the problem of finding the radius of curvature of any given curve, defined as the distance up the normal from the foot to the point of contact with the evolute. This is defined as the limit of the intersection with an infinitely near normal. In our notation this would involve Δx and $\Delta[y(dy/dx)]$. He says in regard to this "*Illas vero dari in omnibus curvis geometricis.*" He indicates the method of doing this in general, but his ignorance of the calculus prevented him from attaining a satisfactory result in every case.

2. **Sir Isaac Newton.** The first writer to handle the question of curvature of a plane curve in what we should call today a thoroughly satisfactory manner was Sir Isaac Newton, no less. Huygens wrote the *Horologium oscillatorium* in 1673. Newton started thinking about the Calculus in around 1665 but published nothing on the subject till his letter to Collins of 1669; his ideas were well developed in 1671. The first systematic account appeared in [9] with the date of 1736. There is a long appendix due to Colson himself, but the bulk of the work purports to be a direct translation of Newton's own Latin. I shall return later to the possible relation of this to the *Horologium* of Huygens.

I begin by quoting Newton's own words, pp. 59–61 of [9]:

"The same Circle has everywhere the same curvature, and in different Circles it is reciprocally proportional to their Diameters.

"If a Circle touches any Curve on its concave side, in any given point, and if it be of such magnitude that no other tangent Circle be inscribed in the angle of contact of that Point, that Circle will be of the same Curvature as the Curve is of, in the Point of Contact.

"Therefore the Centre of Curvature to any Point of the Curve is the Centre of the Circle equally curved, and thus the Radius or Semi-diameter of Curvature is Part of the Perpendicular to the Curve which is terminated at the Centre.

"And the Proportion of Curvature at different Points will be known from the Proportion of Curvature of aequi-curve Circles, or from the reciprocal Proportion of the Radii of Curvature."

A further explanation comes presently:

"But there are several Symptoms or Properties of this Point C which may be used in its Determination:

1) That it is the Concourse of Perpendiculars to that Arc on each side at an infinitely little distance from DC.
2) If DC be conceived to move while it insists perpendicularly to the Curve, that point of it C (if you accept the motion of approaching to, or receding from the point of insistance C) will be the least moved, but will be its Centre of Motion.
3) If a Circle be described with the Centre C, and the Distance DC, no other Circle can be described that can lie between it and the Circle of Contact.
4) Lastly if the Centre H or h of any other touching Circle approaches by degrees to C the Centre of this, till at last it coincides with it, then any of the points in which the Circle shall cut the Curve, will coincide with the Point of Contact at D.
5) And each of these Properties may supply the means of solving this Problem in different ways. But we shall make choice of the first, as being the most simple."

What all this amounts to is the following. Newton assumes, as does Huygens, that if a point not an inflection is fixed on a curve, and a second point approaches it from either side, the intersection of the normals approaches a definite limiting position. This is the centre of curvature, the centre of a circle having the same curvature, and no other tangent circle can lie between this and the curve. All of these properties can be proved if one of them is assumed. He seeks the intersection of nearby normals, and if we take x as an independent variable so that $\dot{x}=1$ and $z=\dot{y}/\dot{x}$ he proves very simply

$$DC = \frac{(1+z^2)^{3/2}}{\dot{z}}.$$

This is very satisfactory and may be said to close the question of the discovery of curyature, but there remains a troublesome question of priority. Cantor in his *Geschichte der Mathematik*, Vol. 3, p. 17, raises the question whether between 1671 and 1736 Newton did not see Huygens' work of 1673 and find therein an excellent opportunity to make use of his own vastly superior methods. Newton made reference to flexible strings and pendulums and the preoccupation of both mathematicians with the cycloid troubles Cantor. On the other hand, it should be noticed that this was a time when a good many geometers were occupied with the cycloid where the determination of the radius was particularly easy. Newton pointed out various methods of finding the radius of curvature, and chooses the best. I cannot feel that we are justified in accusing Newton of plagiarism; it was not in his nature.

There remains the question of Leibniz. In [10] he discusses the angles of mutually tangent curves, and the contact with the "*Circulus osculans*" as if this were an already familiar figure. But the date is late, 1686, and he erroneously says that the osculating circle has four coincident intersections, a mistake promptly pointed out by James Bernoulli.

This represents the limit of my most incomplete knowledge. Where did Oresme get the idea that the curvature of a circle was inversely proportional to the radius? Where did Kepler find the circle of curvature? There are various intersecting questions still to be answered. More power to the persevering man who will answer them.

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A CANONICAL BASIS FOR THE IDEALS OF A POLYNOMIAL DOMAIN

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1. Introduction. This paper is concerned with ideals in the domain of polynomials with integral coefficients. To put the problem in a more general perspective, I begin with compiling some basic algebraic notions and facts. For details the reader may consult Van der Waerden's *Modern Algebra*, Vols. 1 and 2, henceforth quoted as MAI and MAII.*

* Page numbers refer to the English edition, 1949.