International Mathematical Forum, Vol. 11, 2016, no. 6, 259 - 278
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http://dx.doi.org/10.12988/imf.2016.6223

A Historical Walk along the Idea of Curvature, from Newton to Gauss Passing from Euler

Gabriele Bardini and Gian Mario Gianella

Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy

Abstract

From Newton "Method of fluxions" to Gauss' Theorema Egregium, the notion of curvature notably evolved from its appliance to a curve in a plane, to the one about a surface in space. Euler was the first mathematician who tried to define the curvature of a surface, although employing the curvature of a curve by comparing a plane section of the surface and a circumference. Later, Gauss changed Euler's definition to overcome its limitations by defining it through the comparison between the linear element of the surface and the spherical one.

Keywords: Euler, curvature, surface

1 The curvature of a curve in a plane and the osculating circle

1671 Newton completes the "Method of fluxions" (published posthumously in 1736) [11]: in it, he derives a formula to determine the curvature of a curve at a point, using for the first time the cinematic notion of derivative (fluxion). He sets several axioms:

- The straight line has not curvature.
- The curvature of a circumference is a constant inversely proportional to its radius ($k = 1/R$).

- If a circumference is tangent to a curve at a point on the concave side, so that another one cannot be inserted in the contact angle (it will be the highest possible tangent circumference), then it will have the same curvature of the curve in the point of tangency.
- The curvature centre of a curve at a point is the centre of the circumference whose radius is perpendicular to the curve, with same curvature and tangent to the curve in the point. To find it, one has to look for the intersection of the normal to the curve, on the concave side, at the point and in another infinitesimally close. **Therefore, the curvature of a curve at a point will be the inverse of the radius of the highest tangent circumference in it**. The formula derivedd by Newton is

$$r = \frac{(1+z^2)^{3/2}}{\dot{z}} = \frac{1}{k} \quad ; \quad z = \frac{\dot{y}}{\dot{x}} \quad ; \quad \dot{x} = 1 \; , \tag{1}$$

where the apex $\cdot$ is the fluxion's (derivative) symbol.

1686 Leibniz, in the "Acta Eruditorum" [10], defines the osculating circle of a curve at a point as the circumference secant the curve in four points, to the limit coincident between them at the point of tangency, and whose curvature is equal to the one of the curve at the point.

1691 Jakob Bernoulli, in the "Acta Eruditorum" [2], rectifies Leibniz statement demonstrating that 3 secant points are sufficient to define the osculating circle. In the meantime, his brother Johann, in the "Lectiones Matematicae" [1], resultes a theorem to find the radius of curvature as

$$r = \frac{(dx^2+dy^2)^{3/2}}{-dxd^2y} = \frac{1}{k} \quad ; \quad d^2y = d(dy) \quad ; \quad dx^2 = 0 \; , \tag{2}$$

in which $d^2x = 0$ because of the cartesian coordinate frame used.

1758 Abraham G. Kästner, in his book "Anfangsgründe der Mathematik" ("Foundations of Mathematics") [9], defines the curvature of a curve at a point as the ratio of the angle $\Delta\Theta$ subtended between two tangents to the curve at two points, in the limit of their coincidence, and the length of the arc of the curve between them Δs. One gets that

$$\lim_{\Delta x \to 0} \frac{\Delta\Theta}{\Delta s} = \frac{d\Theta}{ds} \; . \tag{3}$$

In the limit of coincidence between the two extremes of the arc, $s = \dfrac{dy}{dx}$ is the derivative's value: therefore

$$ds = d\left(\frac{dy}{dx}\right) = \frac{d^2y\,dx - d^2x\,dy}{(dx)^2}. \tag{4}$$

If $d^2x = 0$ then $ds = \dfrac{d^2y\,dx}{(dx)^2}$: replacing in latter eq. (2), the new one is

$$r = -dx\frac{(1+s^2)^{3/2}}{ds} = \frac{1}{k}\ . \tag{5}$$

Leonhard Euler, Johann Bernoulli's doctoral student and entertainer of a long correspondence with Kästner, used eq. (5) in his 1763 paper "Recherches sur la courbure de surfaces" [5]. According to a recent demonstration, let $\boldsymbol{T}(s)$ be the unit vector tangent to the parametric curve $\alpha(s)$, parametrized through the arc length s: then $\boldsymbol{T}(s+ds) - \boldsymbol{T}(s) = \boldsymbol{N}d\Theta$, with $\boldsymbol{N}$ the orthogonal unit vector and $\boldsymbol{T}(s+ds)$ and $\boldsymbol{T}(s)$ tangent unit vectors to $\alpha(s)$ as shown in Fig. 1.

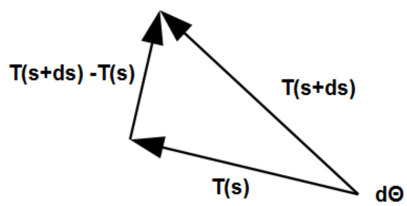

Figure 1: Vectors **T(s), T(s+ds), T(s+ds)-T(s)**

One has that

$$\boldsymbol{N} = \frac{\boldsymbol{T}(s+ds) - \boldsymbol{T}(s)}{d\Theta} = \boldsymbol{T}'\frac{ds}{d\Theta} = \frac{\boldsymbol{T}'}{k}\ , \tag{6}$$

thus $\boldsymbol{T}'(s) = k(s)\boldsymbol{N}(s)$. Moreover, the arc length s is the length, on the curve, between one inceptive fixed extreme A and a point P so that

$$s = \int_A^t \|\boldsymbol{\alpha}'(u)\|du \quad \Rightarrow \quad \frac{ds}{dt} = \|\boldsymbol{\alpha}'(u)\|\ . \tag{7}$$

Going ahead, one gets

$$\begin{aligned}
\boldsymbol{T}'(s) &= \frac{d\boldsymbol{T}(s(t))}{dt}\frac{dt}{ds} = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{\alpha}'(t)\|}\ , && \text{(8a)}\\
\boldsymbol{T}'(t) &= \boldsymbol{\alpha}'(t)\boldsymbol{T}'(s)\ , && \text{(8b)}\\
\boldsymbol{T}'(t) &= \|\boldsymbol{\alpha}'(t)\|k[s(t)]\boldsymbol{N}[s(t)] = \|\boldsymbol{\alpha}'(t)\|k(t)\boldsymbol{N}(t)\ , && \text{(8c)}\\
\boldsymbol{T}'(t)\cdot\boldsymbol{N}(t) &= \|\boldsymbol{\alpha}'(t)\|k(t)\boldsymbol{N}(t)\cdot\boldsymbol{N}(t) = \|\boldsymbol{\alpha}'(t)\|k(t) && \text{(8d)}
\end{aligned}$$

and, in conclusion

$$k(t) = \frac{\boldsymbol{T}'(t)\cdot\boldsymbol{N}(t)}{\|\boldsymbol{\alpha}'(t)\|}\ . \tag{9}$$

To keep on with the demonstration, it is necessary to remember

$$\boldsymbol{T}'(t) = \frac{(x''(t), y''(t))}{\sqrt{x'^2+y'^2}} + \frac{d}{dt}\left(\frac{1}{\sqrt{x'^2+y'^2}}\right)(x'(t), y'(t)) \ , \tag{10a}$$

$$\boldsymbol{N}(t) = \frac{(-y'(t), x'(t))}{\sqrt{x'^2+y'^2}} \quad , \quad \|\boldsymbol{\alpha}'(t)\| = \sqrt{x'^2+y'^2}. \tag{10b}$$

Therefore, one finally gets

$$k(t) = \frac{-x''(t)y'(t) + x'(t)y''(t)}{(\sqrt{x'^2+y'^2})^{3/2}} = \frac{J\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}''(t)}{\|\boldsymbol{\alpha}'(t)\|^3} \ , \tag{11}$$

where $J\boldsymbol{\alpha}'(t)$ is the orthogonal complement of $\boldsymbol{\alpha}'(t)$.

Eq. (11) is an EXTRINSIC definition of curvature, obtained by immersing the curve in a larger set (plane), whose elements are used to define it. Now, one would have to verify two equivalences, the first one between eq. (11) and eq. (1), the second one between eq. (11) and eq. (5), in order to check the consistency between these different definitions of curvature of a curve at a point. Indeed, in the first case one can show that

$$\frac{-x''(t)y'(t)}{(x'^2+y'^2)^3} = \frac{x'^2\frac{d}{dt}(\frac{y'}{x'})}{x'^3(1+\frac{y'^2}{x'^2})^{\frac{3}{2}}} = \frac{z'}{x'(1+z^2)^{\frac{3}{2}}} = \frac{z'}{(1+z^2)^{\frac{3}{2}}} = k_N \ . \tag{12}$$

The value in eq. (11) is then equivalent to Newton's one given in eq. (1). In the second case, one can show that (from (4))

$$\begin{aligned} -ds &= -d(\frac{dy}{dx}) = \frac{d^2y dx - d^2x dy}{(dx)^2} = \\ &= \frac{(dt)^3}{(dx)^2}\left[\frac{-d}{dt}(\frac{dy}{dt})\frac{dx}{dt} + \frac{d}{dt}(\frac{dx}{dt})\frac{dy}{dt}\right] = \frac{dt}{x'^2}(y''x' - x''y'), \end{aligned} \tag{13}$$

$$s = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}. \tag{14}$$

Replacing the resulting values of s and ds in eq. (5), one finally gets

$$k_E = \frac{dt(y''x' - x''y')}{\frac{dx\, x'^2(x'^2+y'^2)^{\frac{3}{2}}}{x'^3}} = \frac{y''x' - x''y'}{\frac{x'^3(x'^2+y'^2)^{\frac{3}{2}}}{x'^3}} = k. \tag{15}$$

The value in eq. (11) is then also equivalent to Euler's one given in eq. (5). Lastly, one would have to verify the osculating circle properties by demonstrating the following theorem:

Let $\boldsymbol{\alpha}(t)$ be a differentiable plane curve, defined in an interval (a,b), $a < t_1 < t_2 < t_3 < b$, and let $C(t_{1,2,3})$ be a circle passing through the 3 distinct points $\boldsymbol{\alpha}(t_{1,2,3})$ not aligned. Lastly $k(t_0) \neq 0$ and $t_0 \in (a,b)$. Then, $C = \lim_{t_{1,2,3} \to t_0} C(t_{1,2,3})$ is the osculating circle of $\boldsymbol{\alpha}(t)$ in t_0.

Demonstration) Let $\boldsymbol{p}$ be the centre of $C(t_{1,2,3})$ and $f(t)$ be a function $(a,b) \to \Re$ such that $f(t) = \|\boldsymbol{\alpha}(t) - \boldsymbol{p}(t_{1,2,3})\|^2$, i.e. it restores the osculating radius' value in $\boldsymbol{\alpha}(t)$. Therefore, $f'(t) = 2\boldsymbol{\alpha}'(t) \cdot (\boldsymbol{\alpha}(t) - \boldsymbol{p})$ and $f''(t) = 2\boldsymbol{\alpha}(t) \cdot (\boldsymbol{\alpha}(t) - \boldsymbol{p}) + \|\boldsymbol{\alpha}'(t)\|^2$: since f(t) is a differentiable function and $f(t_1) = f(t_2) = f(t_3)$, then, according to Rolle's theorem, there are $u_1 \in (t_1, t_2), u_2 \in (t_2, t_3), v \in (u_1, u_2)$such that$f'(u_1) = f'(u_2) = f''(v) = 0$. Thus, $\lim_{t_{1,2,3} \to t_0} f'(t_0) = 2\boldsymbol{\alpha}'(t_0) \cdot (\boldsymbol{\alpha}(t_0 - \boldsymbol{p})) = 0$ and $\lim_{t_{1,2,3} \to t_0} f''(t_0) = 2\boldsymbol{\alpha}''(t_0) \cdot (\boldsymbol{\alpha}(t_0 - \boldsymbol{p})) + 2 + \|\boldsymbol{\alpha}'(t_0)\|^2 = 0$, are both verified in $\boldsymbol{\alpha}(t_0) - \boldsymbol{p} = \frac{-1}{k(t_0)} \frac{J\boldsymbol{\alpha}'(t_0)}{\|\boldsymbol{\alpha}'(t_0\|}$ (according to eq. (11)) : therefore C has a radius parallel to the orthogonal complement to the tangent unit vector $\frac{J\boldsymbol{\alpha}'(t_0)}{\|\boldsymbol{\alpha}'(t_0)\|}$, initial point $\boldsymbol{\alpha}(t_0)$, terminal point $\boldsymbol{p}$ and magnitude $1/k(t_0)$. **Then, $\boldsymbol{C}$ is the osculating circle of in $\boldsymbol{t_0}$.**

2 The curvature of a surface in space according to Euler

1763 Euler presents "Recherches sur la courbure des surfaces" (p. in 1767) [5].

In it, he states that one cannot determine the curvature of a surface in space by comparison (in analogy to the plane case) with a sphere whose curvature is measurable as equivalent to that of its great circles. This is not possible, because infinite different curves pass through every point on the surface: they are sections of the surface with a plan for every possible direction; in addition to that, for each of these curves, there are infinite possible planes of intersection (infinite planes pass through a straight line). Thus, endless determinations are necessary in order to discover an accurate value of k. **It is yet a comparison between curves, not between surfaces** . For this purpose, one considers only the infinite normal sections to the surface at a point, containing the normal to the surface at a point (as for the osculating circle whose point/centre radius lies on the normal to the curve at a point): then, all the considered curvature radii will be directed along the normal to the surface at a point. This set of curvature radii provides an accurate value of the curvature at a point. This definition is extrinsic as the previous one whence it is derived. Euler developed his theory essentially through 4 steps:

- the study of curvature radius of a section of a generic secant plane;

- the development of the special case of a normal section;
- the comparison between the curvatures of the normal sections through the mutual inclinations of the planes;
- the determination of the superficial curvature at a point.

2.1 *Curvature radius of a section of a secant plane*

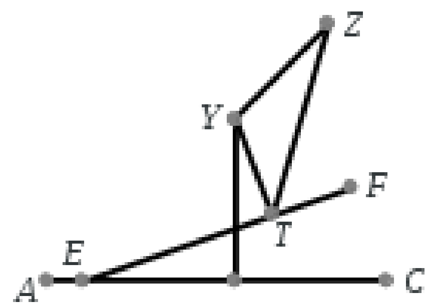

Figure 2: $Z = (x, y, z = f(x, y))$ belongs to the surface, EF segment of the straight line of intersection between the generic secant plane and xy, TY normal segment to EF, led from $Y = (x, y, 0)$ projection of Z on xy.

$$\begin{cases} p = \frac{dz}{dx} \\ q = \frac{dz}{dy} \end{cases} \quad \Rightarrow \quad \begin{cases} dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \\ dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy \end{cases} . \tag{16}$$

As shown in Fig. 2, the secant plane of the surface at Z is

$$z = \alpha y - \beta x + \gamma, \tag{17}$$

so that

$$dz = \alpha dy - \beta dx = q dy + p dx \tag{18}$$

and thus

$$\frac{dy}{dx} = \frac{\beta + p}{\alpha - q} . \tag{19}$$

Moreover, the straight line EF is $y = \frac{\beta}{\alpha}x - \frac{\gamma}{\alpha}$, thus $|AE| = \frac{\gamma}{\beta}$ (A is the axis' origin) and $\tan(F\hat{E}C) = \frac{\beta}{\alpha}$. EF is also normal to TZ, which is the intersection between the secant plane and another normal to it passing through YZ.
($\boldsymbol{ET = t, TZ = u}$) is the new couple of coordinates on the secant

plane, whose origin is T, whereby the curvature radius is determinable. $(AT = x, TY = y)$ are the old coordinates, so that

$$\begin{cases} t = \frac{x-|AE|}{\cos(F\hat{E}C)} + [y - (x - |AE|)\tan(F\hat{E}C)]\sin(F\hat{E}C) \\ u = [y - (x - |AE|)\tan(F\hat{E}C)]\cos(F\hat{E}C) \end{cases} \Rightarrow \quad (20a)$$

$$\begin{cases} t = \frac{\alpha x+\beta y}{\sqrt{\alpha^2+\beta^2}} - \frac{\alpha y}{\beta\sqrt{\alpha^2+\beta^2}} \\ u = \frac{\sqrt{\alpha^2+\beta^2+1}(\alpha x-\beta x+\gamma)}{\sqrt{\alpha^2+\beta^2}} \end{cases} . \quad (20b)$$

Setting

$$s = \frac{du}{dt} \qquad , \qquad r = -\frac{dt(1+s^2)^{\frac{3}{2}}}{ds} \quad (21)$$

and replacing u and t from eq. (20) into eq. (21), one obtains, after some algebraic and differential manipulations, the required formula

$$r = -\frac{(\alpha^2+\beta^2-2\alpha q+2\beta p+(\alpha p+\beta q)^2+p^2+q^2)^{\frac{3}{2}}}{\sqrt{\alpha^2+\beta^2+1}\left[(\alpha-q)^2\frac{\partial p}{\partial x}+(\beta+p)^2\frac{\partial q}{\partial y}+2(\alpha-q)(\beta+p)\frac{\partial p}{\partial y}\right]} . \quad (22)$$

2.2 *Special case of a normal section*

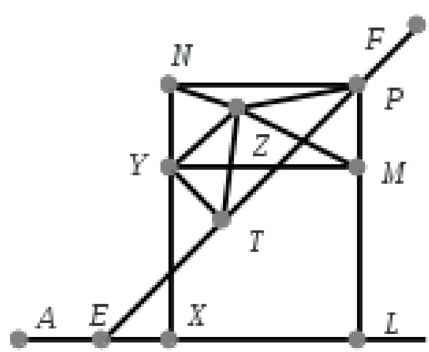

Figure 3: EF segment of the straight line of intersection between the normal section to the surface and xy, YZ is the z coordinate of $Z = (x, y, z = f(x, y))$, Z, Y and N belong to a secant plane λ perpendicular to xy and parallel to the ordinate axis, Z, Y and M belong to a secant plane μ perpendicular to xy and parallel to the abscissa axis.

M is the intersection between the normal to the surface section in Z and the x-axis, thus YM is the subnormal of Z in ν, while N is the intersection between the normal to the surface section in Z and the y-axis. Thus YN is the subnormal of Z in μ: thus, according to the subnormal equation

$$\begin{cases} YM = \frac{\partial z}{\partial x} z = pz \\ YN = \frac{\partial z}{\partial y} z = qz \end{cases} . \tag{23}$$

Through M one draws the straight line parallel to the y-axis and through N the one parallel to the x-axis, which intersect at the point P: then, the segment ZP will be normal to both considered sections and finally to the surface at the point Z. Indeed, in ν, the tangent to the section in Z is $(\frac{\partial z}{\partial x}, 0, 1)$ and thus the normal to the section in Z is $(1, 0, -\frac{\partial z}{\partial x})$, while in μ the tangent is $(0, -\frac{\partial z}{\partial y}, 1)$ and thus the normal is $(0, 1, -\frac{\partial z}{\partial y})$. Hence the vector passing through Z with direction ZP is $(1, 1, -\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y})$, which is also normal to the tangent to the surface at Z $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, 1)$. So $P \in EF$. As in eq. (17), $\zeta = P\hat{E}L \quad \Rightarrow \quad \tan(\zeta) = \frac{\beta}{\alpha}$: thus

$$YT = z[p\sin(\zeta) - q\cos(\zeta)] \Rightarrow \tag{24a}$$

$$\tan(Y\hat{T}Z) = \frac{1}{p\sin(\zeta) - q\cos(\zeta)}. \tag{24b}$$

Then,

$$\begin{cases} \alpha = \frac{\cos(\zeta)}{p\sin(\zeta) - q\cos(\zeta)} \\ \beta = \frac{\sin(\zeta)}{p\sin(\zeta) - q\cos(\zeta)} \end{cases} . \tag{25}$$

Substituting the latter in (22), one obtains

$$r = \frac{-(1+p^2+q^2)^{\frac{3}{2}}(1+(psin\zeta - qcos\zeta)^2)}{\lambda\frac{\partial p}{\partial x} + \eta\frac{\partial q}{\partial y} + \psi\frac{\partial p}{\partial y}}, \tag{26a}$$

$$\begin{cases} \lambda = [(1+q^2)\cos(\zeta) - pq\sin(\zeta)]^2 \\ \eta = [(1+p^2)\sin(\zeta) - pq\cos(\zeta)]^2 \\ \psi = 2[(1+q^2)\cos(\zeta) - pq\sin(\zeta)][(1+p^2)\sin(\zeta) - pq\cos(\zeta)] \end{cases} \tag{26b}$$

2.3 *Comparison between curvature radii of the normal sections through planes' mutual inclinations*

Among the normal sections, the MAIN one is defined as the one whose intersection with xy passes through the projection Y of Z on xy. By construction, $YT = 0 \Rightarrow \begin{cases} \sin(\zeta) = \frac{q}{\sqrt{p^2+q^2}} \\ \cos(\zeta) = \frac{p}{\sqrt{p^2+q^2}} \end{cases}$. Then, considering (26a), one gets

$$r = \frac{-(1+p^2+q^2)^{\frac{3}{2}}(p^2+q^2)}{p^2\frac{\partial p}{\partial x} + q^2\frac{\partial q}{\partial y} + 2pq\frac{\partial p}{\partial y}}. \tag{27}$$

Between a generic normal section, intersecting the plane xy in EP, and the main one, containing Y, Z and P, there is an angle ϕ: by construction, it is equivalent to $Y\hat{R}S$ (YS normal to YP, R intersection of the normal plane to xy, passing through YS, with ZP).

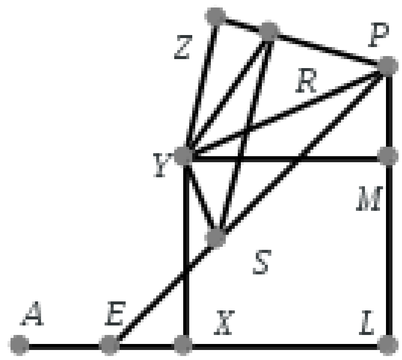

Figure 4: Mutual inclination between normal sections

$$\begin{aligned} YS &= YR\tan(\phi) = YP\sin(Y\hat{P}Z)\tan(\phi) = z\sqrt{p^2+q^2}\frac{YZ}{ZP}\tan(\phi) = \\ &= \frac{z\sqrt{p^2+q^2}}{z\sqrt{1+p^2+q^2}}\tan(\phi) = \frac{\sqrt{p^2+q^2}}{\sqrt{1+p^2+q^2}}\tan(\phi), \qquad (28a) \\ PS &= \sqrt{YP^2+YS^2} = \sqrt{\left(z\sqrt{p^2+q^2}\right)^2 + \left[\frac{z\sqrt{p^2+q^2}}{\sqrt{1+p^2+q^2}}\tan(\phi)\right]^2} = \\ &= z\sqrt{\frac{(p^2+q^2)(1+p^2+q^2)\tan^2(\phi)}{1+p^2+q^2}} \qquad (28b) \end{aligned}$$

From (28), it follows

$$\begin{aligned} \tan(E\hat{P}Y) &= \frac{YS}{YP} = \frac{\tan(\phi)}{\sqrt{1+p^2+q^2}} \Rightarrow \qquad (29a) \\ \tan(E\hat{P}M) &= \tan(Y\hat{P}M - E\hat{P}Y) = \frac{p\sqrt{1+p^2+q^2-q\tan(\phi)}}{q\sqrt{1+p^2+q^2+p\tan(\phi)}} = \\ &= \cot(P\hat{E}L) = \frac{\cos(\zeta)}{\sin(\zeta)}. \qquad (29b) \end{aligned}$$

Thus, one obtains

$$\begin{cases} \sin(\zeta) = \frac{q\sqrt{1+p^2+q^2+p\tan(\phi)}}{\sqrt{[1+p^2+q^2+p\tan^2(\phi)](p^2+q^2)}} \\ \cos(\zeta) = \frac{p\sqrt{1+p^2+q^2-q\tan(\phi)}}{\sqrt{[1+p^2+q^2+p\tan^2(\phi)](p^2+q^2)}} \end{cases}, \qquad (30)$$

and moreover, by substituting in (26a), one can write

$$r = \frac{V}{P(\cos(\phi))^2 + Q(\sin(\phi))^2 + 2R\cos(\phi)\sin(\phi)}, \tag{31a}$$

$$\begin{cases} u = \sqrt{(1+p^2+q^2}, V = -(1+p^2+q^2)(p^2+q^2)^3 \\ P = p^2\frac{\partial p}{\partial x} + q^2\frac{\partial q}{\partial y} + 2pq\partial p\partial y \\ Q = u^2(q^2\frac{\partial p}{\partial x} + p^2\frac{\partial q}{\partial y} - 2pq\partial p\partial y) \\ R = u[pq(-\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}) + (p^2+q^2)\partial p\partial y] \end{cases}. \tag{31b}$$

Finally, through further manipulations on (31a), one gets

$$\begin{cases} L = \frac{P+Q}{2V}, M = \frac{P}{2V}, N = \frac{2R-Q}{2V} \\ r = \frac{1}{L+M\cos(2\phi)+N\sin(2\phi)} \end{cases}. \tag{32}$$

The curvature radius depends on ϕ and no longer on ζ as in (26a).

2.4 *Determination of superficial curvature at a point*

In order to find the value of the superficial curvature at the point Z, one has to notice that L, M, N are functions of $p, q, \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial q}{\partial x}$ and $\frac{\partial q}{\partial y}$ calculated in Z. Thus, their values are common to all the normal sections passing through Z. Therefore r_Z depends ONLY on ϕ.

$r'(\phi) = \frac{2[M_Z\sin(2\phi)-N_Z\cos(2\phi)]}{[L_Z+M_Z\cos(2\phi)+N_Z\sin(2\phi)]^2} = 0$ if $\phi = \arctan\left(\frac{N_Z}{2M_Z} + l\frac{\pi}{2}\right), l \in \mathbb{Z} \Rightarrow$

$$\phi_{MAX} = \arctan\left(\frac{N_Z}{2M_Z}\right), \phi_{min} = \arctan\left(\frac{N_Z}{2M_Z} + \frac{\pi}{2}\right). \tag{33}$$

Other extreme points are coincident to those or opposite on the same direction (on the same normal section): thus **the directions of maximum and minimum curvature are normal to each other.** From (33), one gets

$$\begin{cases} \cos(2\phi_{MAX}) = \frac{M_Z{}^2-N_Z{}^2}{M_Z{}^2+N_Z{}^2}, \sin(2\phi_{MAX}) = \frac{2M_ZN_Z}{M_Z{}^2+N_Z{}^2} \\ \cos(2\phi_{min}) = \frac{N_Z{}^2-M_Z{}^2}{M_Z{}^2+N_Z{}^2}, \quad \sin(2\phi_{min}) = \frac{-2M_ZN_Z}{M_Z{}^2+N_Z{}^2} \end{cases}, \tag{34a}$$

$$f = r_{Z,MAX} \quad , \quad g = r_{Z,min}, \tag{34b}$$

$$\begin{cases} f = \frac{M_Z{}^2+N_Z{}^2}{L_Z(M_Z{}^2+N_Z{}^2)^2+M_Z(M_Z{}^2-N_Z{}^2)^2+2M_ZN_Z{}^2} \\ g = \frac{M_Z{}^2+N_Z{}^2}{L_Z(M_Z{}^2+N_Z{}^2)^2-M_Z(M_Z{}^2-N_Z{}^2)^2-2M_ZN_Z{}^2} \end{cases}. \tag{34c}$$

Now, one introduces a new frame of reference on the tangent plane of the surface at Z: on it, one rotates the coordinate axes to put them onto the directions of extreme curvature radii. $\phi'_{MAX} = 0, \phi'_{min} = \frac{\pi}{2} \Rightarrow N_Z = 0$. Thus

(34c) becomes

$$\begin{cases} f = \frac{M_Z{}^2}{L_Z M_Z{}^2 + M_Z{}^3} \\ g = \frac{M_Z{}^2}{L_Z M_Z{}^2 - M_Z{}^3} \end{cases} \Rightarrow \begin{cases} L = \frac{f+g}{2fg} \\ M = \frac{f-g}{2fg} \end{cases} . \tag{35}$$

By substituting in (32), one obtains

$$r_Z = \frac{2fg}{(f+g)+(g-f)\cos(2\phi)}. \tag{36}$$

The value of the curvature radius depends on its maximum f, on its minimum g and on ϕ.

3 The development of a surface in space on a plane

1772 Euler publishes "De solidis quorum superficiem in planum explicare licet" [4].

Starting from the property, known in elementary geometry, of cones and cilinders (not of spheres) to be unfolded ("developed") on a plane without distortions, Euler wants to establish sufficient conditions under which a generic surface has this feature. He expands its analysis in three steps:

I. the search for a solution obtained from analytical principles

II. the search for a solution obtained from geometrical principles

III. the use of the second one in the first one.

The second part of this article and the third one will not be dealt there because they are irrelevant for paper's purpose.

3.1 *General definition of surface development obtained from analytical principles*

Let Z be a point of a surface in space whose coordinates are $AX = x$, $XY = y$, $YZ = z$ and let V a point in the development plane such that $Z \equiv V$. Because V is a point on a plane, it has a couple a orthogonal coordinates $OT = t$, $TV = u$ and Euler writes "it is evident the triad of the previous coordinates (x, y, z) somehow must depend on the couple (t, u)" (introducing in mathematics history the idea of *parametric surfaces*). The differential equations for

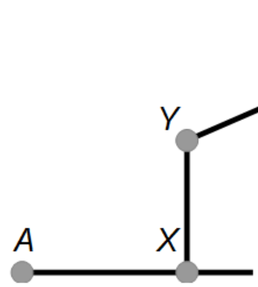

Figure 5: Coordinates of Z point belonging to the surface

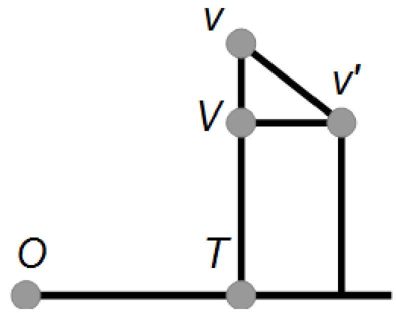

Figure 6: Development on a plane

$x(t,u), y(t,u), z(t,u)$ are

$$\begin{cases} dx = ldt + \lambda du, dy = mdt + \mu du, dz = ndt + \nu du \\ l = \dfrac{\partial x}{\partial t}, m = \dfrac{\partial y}{\partial t}, n = \dfrac{\partial z}{\partial t}, \lambda = \dfrac{\partial x}{\partial u}, \mu = \dfrac{\partial y}{\partial u}, \nu = \dfrac{\partial z}{\partial u} \end{cases} \tag{37}$$

Let v, v' two points indefinitely near to V such that their coordinates are $(t, u + du), (t + dt, u)$: because the development is an isometry, the triangle Vvv' must be equivalent to the triangle Zzz', with z and z' respectively counterimages of v and v' on the surface. Thus

$$Zz = Vv = du, Zz' = Vv' = dt, zz' = vv' = \sqrt{du^2 + dt^2} \tag{38}$$

It follows $z = Z + du$, $z' = Z + dt$ and then

$$z = (x + \lambda du, y + \mu du, z + \mu du), z' = (x + ldt, y + mdt, z + ndt) \tag{39}$$

From (39), using Euclidean formula of distance beetwen two points on a plane

$$Zz = du\sqrt{\lambda^2 + \mu^2 + \nu^2}, Zz' = dt\sqrt{l^2 + m^2 + n^2}, \tag{40a}$$

$$zz' = \sqrt{du^2(\lambda^2 + \mu^2 + \nu^2) + dt^2(l^2 + m^2 + n^2) - 2dudt(\lambda l + \mu m + \nu n)} \tag{40b}$$

One finally discovers the six conditions under which a surface in space is developable on a plane by comparison the equations (38) with the equations (40) and from the irrelevance of derivation order in partial derivatives in (37):

$$\begin{cases} \dfrac{\partial l}{\partial u} = \dfrac{\partial \lambda}{\partial t}, \dfrac{\partial m}{\partial u} = \dfrac{\partial \mu}{\partial t}, \dfrac{\partial n}{\partial u} = \dfrac{\partial \nu}{\partial t}, \\ \\ \lambda^2 + \mu^2 + \nu^2 = 1, l^2 + m^2 + n^2 = 1, \lambda l + \mu m + \nu n = 0 \end{cases} \tag{41}$$

TWO SIMPLE EXAMPLES:

- **The cylindrical surface**
  Let $\xi : (t,u) \to (\cos(t), \sin(t), u)$ be the parametrization of a cylindrical surface: in this case all the six conditions are satisfied.

- **The spherical surface**
  Let $\gamma : (t,u) \to (\sin t \cos u, \sin t \sin u, \cos t)$ be the parametrization of a unit spherical surface with radius r: in this case, condition $\lambda_\gamma^2 + \mu_\gamma^2 + \nu_\gamma^2 = \sin^2 t \neq 1$ if $t \neq \frac{\pi}{2} \vee \frac{3\pi}{2}$ is not satisfied and thus γ is not developable.

In this work, Euler starts to use *intrinsic* (not dependent on shape) properties of surfaces, manages to determine a general definition of linear element of a surface in space ((40b)) but only for the developable ones.

4 The curvature of a surface in space according to Gauss

1827 Gauss publishes "Disquisitiones generales circa superficies curvas" [6].

In this paper he investigates the surface curvature by the comparison between the studied surface and the spherical one. His assumption is then different from the Eulerian one, because Euler used the notion of curvature of a curve in a plane on the sections of the studied surface. To this purpose, he introduces a correspondence between points of the surface in space and points of unitary radius sphere: for a point $P(x,y,z)$ of the surface σ, there is a point $P'(X,Y,Z)$ of unitary sphere Σ whose coordinates are equivalent to the direction cosines of the unit vector **OP** (it is still an extrinsic approach). The MEASURE OF CURVATURE k of the surface at a point is the ratio among the spherical surface element $d\Sigma$, centred in P', and the surface element $d\sigma$, centred in P (local measure): its SIGN will depend on the projection of $d\sigma$ on $d\Sigma$;. If it maintains its orientation in relation to the normal outgoing from the external face of $d\Sigma$, then k is positive, otherwise it is negative.
Let $zd\sigma$ and $zd\Sigma$ be the projections of surface elements on xy: then $\frac{zd\Sigma}{zd\sigma} = \frac{d\Sigma}{d\sigma} = k$ and the ratio among projections is still k. Let $A, B, C \in d\sigma$, projected on xy, whose projections are $A_{xy} = (x,y), B_{xy} = (x+dx, y+dy), C_{xy} = (x+\delta x, y+\delta y)$. Let $A', B', C' \in d\Sigma$, projected on xy, whose projections are $A'_{xy} = (X,Y), B'_{xy} = (X+dX, Y+dY), C'_{xy} = (X+\delta X, Y+\delta Y)$. Let A', B', C' be the projections respectively of A, B, C on $d\Sigma$. Since the cross product magnitude is equivalent to parallelogram area subtended by two vectors, the value of surface elements will be the cross product magnitude of distance vectors of two points

in respect to the origin (P or P'). Finally,

$$\begin{cases} zd\Sigma = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ dX & dY & 0 \\ \delta X & \delta Y & 0 \end{vmatrix} = dX\delta Y - dY\delta X \\ zd\sigma = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ dx & dx & 0 \\ \delta x & \delta y & 0 \end{vmatrix} = dx\delta y - dy\delta x \end{cases} \Rightarrow \boldsymbol{k} = \frac{\boldsymbol{dX\delta Y - dY\delta X}}{\boldsymbol{dx\delta y - dy\delta x}}. \quad (42)$$

5 The determination of curvature measure

In order to result a formula for curvature from the differential relation (42), three different ways to define a surface in space (thus three different differential relations $(x, y) \leftrightarrow (X, Y)$) are studied:

I. $W(x, y, z) = 0$ IMPLICIT,

II. $(p, q) \to (x(p, q), y(p, q), z(p, q))$ PARAMETRIC,

III. $z = z(x, y)$ EXPLICIT.

Let (dx, dy, dz) be the distance vector between two points on σ at limit coincident: it is therefore normal to (X, Y, Z) (unit vector of direction cosines in (x, y, z)) so that subsequently

$$\begin{cases} Xdx + Ydy + Zdz = 0 \\ X^2 + Y^2 + Z^2 = 1 \end{cases} . \quad (43)$$

The first one will not be dealt here because it is irrelevant for paper's purpose.

5.1 *Determination of k by third definition of surface*

Gauss first investigats the third definition $z = z(x, y)$, reformulated in order to develop the reasoning:

$$S(x, y, z) = z(x, y) - z = 0 \Rightarrow ds = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz = 0. \quad (44)$$

Thus, from (43) one derives

$$\begin{cases} \frac{\partial S}{\partial x} = \frac{\partial z}{\partial x} = t \\ \frac{\partial S}{\partial y} = \frac{\partial z}{\partial y} = u \\ \frac{\partial S}{\partial z} = -(\frac{\partial z}{\partial z}) = -1 \end{cases} \Rightarrow \begin{cases} X = \frac{-t}{\sqrt{1+t^2+u^2}} \\ Y = \frac{-u}{\sqrt{1+t^2+u^2}} \\ Z = \frac{1}{\sqrt{1+t^2+u^2}} \end{cases} , \quad (45)$$

so that

$$\begin{cases} dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy, dY = \frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy \\ \delta X = \frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y, \delta Y = \frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial y}\delta y \end{cases} \Rightarrow \tag{46a}$$

$$dX\delta Y - dY\delta X = \left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right)(dx\delta y - dy\delta x) \Rightarrow \tag{46b}$$

$$k = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}. \tag{46c}$$

Setting

$$T = \frac{\partial^2 z}{\partial x^2} = \frac{\partial t}{\partial x} \quad , U = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial x} \quad , V = \frac{\partial^2 z}{\partial y^2} = \frac{\partial u}{\partial y}, \tag{47}$$

and after several algebraic and differential manipulations performed on (45) and (47), one finally finds the curvature formula

$$k = \frac{TV - U^2}{(1 + t^2 + u^2)^2}. \tag{48}$$

Gauss keep on writing that, if xy is tangent to the surface at the studied point $\boldsymbol{z_0}$, then in $I^{\epsilon}_{z_0}$ $\nabla z|_{z_0} = (t|_{z_0}, u_{z_0}) = \mathbf{0}$: therefore $k_{z_0} = T_{z_0}V_{z_0} - U^2_{z_0}$. Moreover, if one puts the origin axis in $\boldsymbol{z_0}$, then (Ω negligible, H Hessian of z)

$$\begin{aligned} z &= z|_{z_0} + \nabla z|_{z_0} \cdot ((x, y) - \boldsymbol{z_0}) + \frac{((x, y) - \boldsymbol{z_0}) \cdot H|_{z_0} \cdot ((x, y) - \boldsymbol{z_0})^T}{2} + \Omega = \\ &= \frac{T_{z_0}x^2}{2} + U_{z_0}xy + \frac{V_{z_0}y^2}{2} + \Omega. \end{aligned} \tag{49}$$

If one applies an axis rotation of Θ such that $\tan(2\Theta) = \frac{2U_{z_0}}{T_{z_0} - V_{z_0}}$, then

$$z = \frac{T'_{z_0}}{2} + \frac{V'_{z_0}}{2} + \Omega \Rightarrow \tag{50a}$$

$$k'_{z_0} = T'_{z_0}V'_{z_0}. \tag{50b}$$

From (50a) one understands (remembering (11)) that T'_{z_0} and V'_{z_0} are the curvature values in $\boldsymbol{z_0}$ along the surface restrictions in respect to new axes x', y':

$$\begin{aligned} y' &= 0 \Rightarrow z \approx \frac{T'_{z_0}}{2}x'^2 \Rightarrow \tau(x') = (x', \frac{T'_{z_0}}{2}x'^2) \Rightarrow k'_{z_0,y'} = T'_{z_0}, \\ x' &= 0 \Rightarrow z \approx \frac{V'_{z_0}}{2}y'^2 \Rightarrow \upsilon(y') = (y', \frac{V'_{z_0}}{2}y'^2) \Rightarrow k'_{z_0,x'} = V'_{z_0}. \end{aligned}$$

One considers a surface restriction in respect to a generic direction subtending an angle ρ with new abscissas: by using polar coordinates one can write

$$z \approx \left[\frac{T'_{z_0}}{2}\cos^2(\rho)+\frac{V'_{z_0}}{2}\sin^2(\rho)\right]r^2 \Rightarrow \iota(r)=\left(r,\left[\frac{T'_{z_0}}{2}\cos^2(\rho)+\frac{V'_{z_0}}{2}\sin^2(\rho)\right]r^2\right),$$

and thus

$$\begin{aligned} k'_{z_0,\rho} &= T'_{z_0}\cos^2(\rho)+V'_{z_0}\sin^2(\rho) \Rightarrow \qquad (51)\\ \frac{dk'_{z_0,\rho}}{d\rho} &= (V'_{z_0}-T'_{z_0})\sin(2\rho)=0 \quad if\rho=0,\frac{\pi}{2} \Rightarrow k'_{z_0,\rho}(0)=T'_{z_0}, k'_{z_0,\rho}(\frac{\pi}{2})=V'_{z_0}. \end{aligned}$$

According to what found, T'_{z_0} and V'_{z_0} are the extreme values related to the extreme points (respectively minimum and maximum) of angle ρ. Gauss itself writes: "These conclusions contain almost all what that the famous Euler realized about the curvature of curve surfaces". In fact, according to (36), one gets

$$\begin{aligned} r_Z &= \frac{2fg}{(f+g)+(g-f)\cos(2\phi)}=\frac{fg}{f\sin^2(\phi)+g\cos^2(\phi)}=\frac{1}{\frac{\sin^2(\phi)}{g}+\frac{\cos^2(\phi)}{f}} \Rightarrow\\ k_z &= \left(\frac{\sin^2(\phi)}{g}+\frac{\cos^2(\phi)}{f}\right)=k'_{MAX}\sin^2(\phi)+k'_{min}\cos^2(\phi). \qquad (52) \end{aligned}$$

Since ϕ has the same definition of ρ, (36) is then equivalent to (51).
He concludes this part of his work:
"THE MEASURE OF CURVATURE AT ANY POINT WHATEVER OF THE SURFACE IS EQUAL TO A FRACTION WHOSE NUMERATOR IS UNITY, AND WHOSE DENOMINATOR IS THE PRODUCT OF THE TWO EXTREME RADII OF CURVATURE OF THE SECTIONS BY NORMAL PLANES".
Checking the Gaussian reasoning through a modern perspective, one notices that the angle Θ such that $\tan(2\Theta)=\frac{2U_{z_0}}{T_{z_0}-V_{z_0}}$ (used for the rotation applied on the tangent plane of the surface at z_0) is the one relative to a rotation matrix between the canonical basis and Hessian unit eigenvectors one. Indeed, $H(z)|_{z_0}=\begin{bmatrix}T_{z_0} & U_{z_0}\\ U_{z_0} & V_{z_=}\end{bmatrix}$, $det(H(z)|_{z_0}-\lambda I)=0 \quad \Leftrightarrow \quad \lambda_{1,2}=\frac{T_{z_0}+V_{z_0}\pm\sqrt{(T_{z_0}-V_{z_0})^2+4U_{z_0}^2}}{2}$
and the eigenvectors are
$\begin{cases} \boldsymbol{v_1}=\left(\frac{T_{z_0}-V_{z_0}-\sqrt{(T_{z_0}+V_{z_0})^2+4U_{z_0}^2}}{2U_{z_0}},1\right)=(v_{11},v_{12})\\ \boldsymbol{v_2}=\left(\frac{T_{z_0}-V_{z_0}+\sqrt{(T_{z_0}+V_{z_0})^2+4U_{z_0}^2}}{2U_{z_0}},1\right)=(v_{21},v_{22}) \end{cases}$. The matrix equation for

the change of basis $\begin{bmatrix} \frac{v_{11}}{\sqrt{v_{11}^2+v_{12}^2}} & \frac{v_{12}}{\sqrt{v_{11}^2+v_{12}^2}} \\ \frac{v_{21}}{\sqrt{v_{21}^2+v_{12}^2}} & \frac{v_{22}}{\sqrt{v_{21}^2+v_{22}^2}} \end{bmatrix} = \begin{bmatrix} \cos(\Xi) & -\sin(\Xi) \\ \sin(\Xi) & \cos(\Xi) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is solved by the solution Θ given above.

5.2 *Determination of k by second definition of surface*

One introduces the variables

$$a = \frac{\partial x}{\partial p}, a' = \frac{\partial x}{\partial q}, \alpha = \frac{\partial^2 x}{\partial p^2}, \alpha' = \frac{\partial^2 x}{\partial p \partial q}, \alpha'' = \frac{\partial^2 x}{\partial q^2}, \tag{53a}$$

$$b = \frac{\partial y}{\partial p}, b' = \frac{\partial y}{\partial q}, \beta = \frac{\partial^2 y}{\partial p^2}, \beta' = \frac{\partial^2 y}{\partial p \partial q}, \beta'' = \frac{\partial^2 y}{\partial q^2}, \tag{53b}$$

$$c = \frac{\partial z}{\partial p}, c' = \frac{\partial z}{\partial q}, \gamma = \frac{\partial^2 z}{\partial p^2}, \gamma' = \frac{\partial^2 z}{\partial p \partial q}, \gamma'' = \frac{\partial^2 z}{\partial q^2}, \tag{53c}$$

$$A = bc' - cb', B = ca' - ac', C = ab' - ba', \tag{53d}$$

$$D = \alpha A + \beta B + \gamma C, D' = \alpha' A + \beta' B + \gamma' C, D'' = \alpha'' A + \beta'' B + \gamma'' C \tag{53e}$$

a, b, c, a', b', c' in (53) are respectively equivalent to $l, m, n, \lambda, \mu, \nu$ in (37).

It is necessary to perform several algebraic and differential manipulations in order to reformulate (48) with the variables just written: at last it will be

$$k = \frac{DD'' - D'^2}{(A^2 + B^2 + C^2)^2}. \tag{54}$$

To keep on, one has to introduce the new variables

$$m = a\alpha + b\beta + c\gamma, m' = a\alpha' + b\beta' + c\gamma', m'' = a\alpha'' + b\beta'' + c\gamma'', \tag{55a}$$

$$n = a'\alpha + b'\beta + c'\gamma, n' = a'\alpha' + b'\beta' + c'\gamma', n'' = a'\alpha'' + b'\beta'' + c'\gamma'' \tag{55b}$$

and

$$E = a^2 + b^2 + c^2, F = aa' + bb' + cc', G = a'^2 + b'^2 + c'^2. \tag{56}$$

Remebering (37),

$$E = l^2 + m^2 + n^2, F = l\lambda + m\mu + n\nu, G = \lambda^2 + \mu^2 + \nu^2 \tag{57}$$

By performing another long sequence of algebraic and differential manipulations (54) is reformulated, through the variables (55) and (56), in

$$\begin{aligned} k &= \frac{\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2}{EG - F^2} + \\ &+ \frac{(n'^2 - nn'')E + (nm'' - 2m'n' + mn'')F + (m'^2 - mm'')G}{EG - F^2}. \quad (58) \end{aligned}$$

At last, (58) is reformulated only through E, F, G and their partial derivatives with respect to p and q. According to eqs (55), (56) and (53), one gets

$$n = \frac{\partial F}{\partial p} - \frac{1}{2}\frac{\partial E}{\partial q}, n' = \frac{1}{2}\frac{\partial G}{\partial p}, n'' = \frac{1}{2}\frac{\partial G}{\partial q}, \quad (59a)$$

$$m = \frac{\partial E}{\partial p} - \frac{1}{2}\frac{\partial E}{\partial q}, m' = \frac{1}{2}\frac{\partial G}{\partial p}, m'' = \frac{\partial F}{\partial q} - \frac{1}{2}\frac{\partial G}{\partial p}, \quad (59b)$$

$$\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2 = -\frac{1}{2}\frac{\partial^2 E}{\partial q^2} + \frac{\partial^2 F}{\partial p \partial q} - \frac{1}{2}\frac{\partial^2 G}{\partial p^2}, \quad (59c)$$

and thus it becomes

$$\begin{aligned} k &= \frac{E\left[\frac{\partial E}{\partial q}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} + \left(\frac{\partial G}{\partial p}\right)^2\right]}{4(EG - F^2)^2} + \\ &+ \frac{F\left[\frac{\partial E}{\partial p}\frac{\partial E}{\partial q} - \frac{\partial E}{\partial q}\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial q}\frac{\partial F}{\partial q} + 4\frac{\partial F}{\partial p}\frac{\partial F}{\partial q} - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial p}\right]}{4(EG - F^2)^2} + \\ &+ \frac{G\left[\frac{\partial E}{\partial p}\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial p}\frac{\partial F}{\partial q} + \left(\frac{\partial E}{\partial q}\right)^2\right]}{4(EG - F^2)^2} - \frac{\left[\frac{\partial^2 E}{\partial q^2} - 2\frac{\partial^2 F}{\partial p \partial q} + \frac{\partial^2 G}{\partial p^2}\right]}{2(EG - F^2)}. \quad (60) \end{aligned}$$

5.3 *The "Theorema Egregium"*

The relevance of expressing k as a function of only E, F, G (from (56)) resides in the formula of linear element of curved surface. For a parametric surface,

$$\begin{cases} dx = adp + a'dq \\ dy = bdp + b'dq \\ dz = cdp + c'dq \end{cases} \Rightarrow \quad (61)$$

$$\sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{E(dp)^2 + 2Fdpdq + G(dq)^2}. \quad (62)$$

Formula (62) is almost equivalent to (40b) (the difference of a minus is due to the fact the second one is a distance and the first a vector sum).

Gauss writes: "The analysis developed shows us that for finding the measure of curvature there is no need of finite formulae, which express the coordinates x, y, z as functions of the indeterminates p and q; but that the general expression for the magnitude of any linear element is sufficient". Let $\sigma(x, y, z)$ e $\sigma'(x', y', z')$ be two surfaces in space such that the first can be DEVELOPED UPON the second: therefore, "to each point of the former surface, determined by the coordinates x, y, z will correspond a definite point of the later one, whose coordinates are x', y', z'. Evidently, x', y', z' can also be regarded as functions of the indeterminates p and q and thus for the element $\sqrt{(dx')^2 + (dy')^2 + (dz')^2}$ we shall have an expression of the form $\sqrt{E'(dp)^2 + 2F'(dpdq)^2 + G'(dq)^2}$". He concludes: "But from the very notion of the development of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Hence we shall have identically $E = E', F = F', G = G'$. Thus the previous formula leads of itself to the remarkable theorem ("Egregium theorema"): **If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged**".

Overcoming Euler's work, Gauss manages to define a measurable feature also for spherical surfaces and all those not developable on a plane, not only the equivalence between local plane metric and developable surfaces' ones.

According to modern terminology, the importance of this theorem is related to the possibility to express the second fundamental form of a surface entirely through the variables of the first one, i.e. the Gaussian curvature k of a surface in space depends only on its local metric and not on its shape (how it is parametrized in the external frame of reference from where it is observed). Finally one deduces the passage from an EXTRINSIC definition of curvature to an INTRINSIC one, because the local metric, not dependent on shape (and its operator), does not change if surfaces are locally homeomorph between them.

TWO SIMPLE EXAMPLES

- **The plane and the cylindrical surface**
  Let $\mu : (p, q) \rightarrow (\cos(p), \sin(p), q)$ be the local parametrization of a cylindrical surface. In this case $E = G = 1$ and $F = 0$ for the plane (p, q) and for the cylindrical surface: thus μ is a locally isometric to a plane.

- **The spherical surface and the plane**
  Let $\gamma : (\theta, \zeta) \rightarrow (r \sin\theta \cos\zeta, r \sin\theta \sin\zeta, r \cos\theta)$ be the local parametrization of a spherical surface with radius r. Its Gaussian curvature is $k = r^{-2}$, while the plane's one is 0. The two surfaces are then not locally isometric.

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Received: January 3, 2016; Published: February 26, 2016