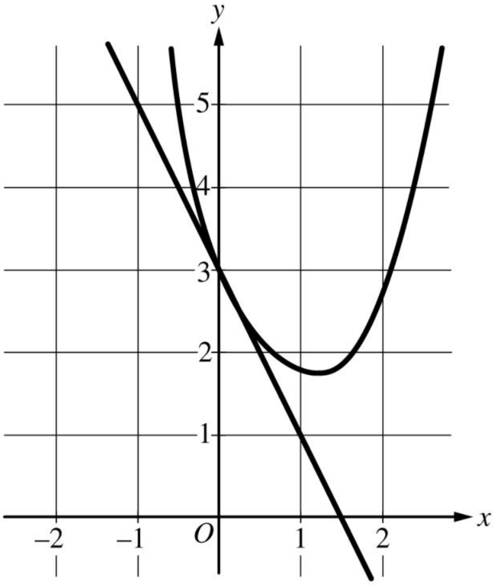
Example 1 2019 BC FR Q6



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

Example 2 2018 BC FR Q6

The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $\left|P_4(2) - f(2)\right|$.

Example 3 2017 BC FR Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

Example 4 2016 BC FR Q6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

Example 5 2017 BC FR Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Example 6 2015 BC FR Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

Example 7 2012 BC FR Q4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

Question 1 2013 BC Q12

For which of the following does $\lim_{x\to\infty} f(x) = 0$?

I. $f(x) = \dfrac{\ln x}{x^{99}}$

II. $f(x) = \dfrac{e^x}{\ln x}$

III. $f(x) = \dfrac{x^{99}}{e^x}$

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and III only

Question 2 2013 BC Q14

If a and b are positive constants, then $\lim_{x\to\infty} \dfrac{\ln(bx+1)}{\ln(ax^2+3)} =$

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{1}{2}ab$ (D) 2 (E) ∞

Question 3 2013 BC Q17

If $\lim_{h\to 0} \dfrac{\arcsin(a+h) - \arcsin(a)}{h} = 2$, which of the following could be the value of a ?

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\sqrt{3}$ (D) $\frac{1}{2}$ (E) 2

Question 4 2015 BC Q12

$\lim_{x \to 0} \frac{e^x - 1}{x}$ is

(A) ∞ (B) $e - 1$ (C) 1 (D) 0 (E) e^x

Question 5 2014 AB Q24

Let f be the function defined by $f(x) = \frac{(3x + 8)(5 - 4x)}{(2x + 1)^2}$. Which of the following is a horizontal asymptote to the graph of f?

(A) $y = -6$

(B) $y = -3$

(C) $y = -\frac{1}{2}$

(D) $y = 0$

(E) $y = \frac{3}{2}$

Question 6 2015 BC Q22

What are the equations of the horizontal asymptotes of the graph of $y = \frac{2x}{\sqrt{x^2 - 1}}$?

(A) $y = 0$ only

(B) $y = 1$ only

(C) $y = 2$ only

(D) $y = -2$ and $y = 2$ only

(E) $y = -1$ and $y = 1$ only

Question 7 2015 AB Q11

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 - 2x - 15} \text{ is}$$

(A) 0 (B) $\frac{3}{5}$ (C) $\frac{3}{4}$ (D) 1 (E) nonexistent

Question 8 2016 AB Q25

$$\lim_{h \to 0} \frac{e^{-1-h} - e^{-1}}{h} \text{ is}$$

(A) -1 (B) $\frac{-1}{e}$ (C) 0 (D) $\frac{1}{e}$ (E) nonexistent

Question 9 2015 AB Q86

The vertical line $x = 2$ is an asymptote for the graph of the function f. Which of the following statements must be false?

(A) $\lim_{x\to 2} f(x) = 0$

(B) $\lim_{x\to 2} f(x) = -\infty$

(C) $\lim_{x\to 2} f(x) = \infty$

(D) $\lim_{x\to \infty} f(x) = 2$

(E) $\lim_{x\to \infty} f(x) = \infty$

Question 10 2016 AB Q7

For which of the following pairs of functions f and g is $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ infinite?

(A) $f(x) = x^2 + 2x$ and $g(x) = x^2 + \ln x$

(B) $f(x) = 3x^3$ and $g(x) = x^4$

(C) $f(x) = 3^x$ and $g(x) = x^3$

(D) $f(x) = 3e^x + x^3$ and $g(x) = 2e^x + x^2$

(E) $f(x) = \ln(3x)$ and $g(x) = \ln(2x)$

Question 11 2016 AB Q82

If f is a continuous function such that $f(2) = 6,$ which of the following statements must be true?

(A) $\lim_{x \to 1} f(2x) = 3$

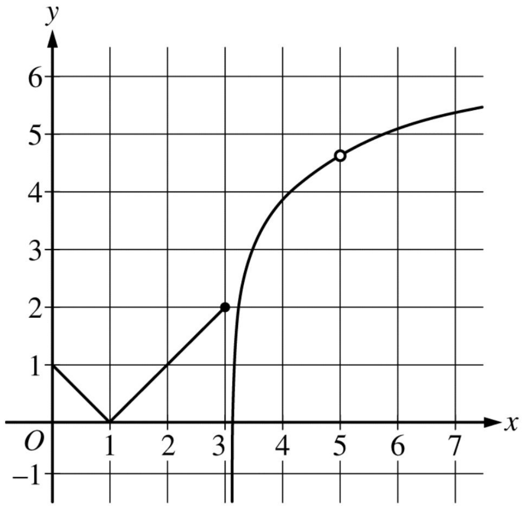
(B) $\lim_{x \to 2} f(2x) = 12$

(C) $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 6$

(D) $\lim_{x \to 2} f\left(x^2\right) = 36$

(E) $\lim_{x \to 2} (f(x))^2 = 36$

Question 12 2016 AB Q76



Graph of f

The graph of a function f is shown above. Which of the following limits does not exist?

(A) $\lim_{x\to 1^-} f(x)$ (B) $\lim_{x\to 1} f(x)$ (C) $\lim_{x\to 3^-} f(x)$ (D) $\lim_{x\to 3} f(x)$ (E) $\lim_{x\to 5} f(x)$

Question 13 2016 BC Q1

$$\lim_{x\to 1} \frac{x^2-1}{\sin(\pi x)} \text{ is}$$

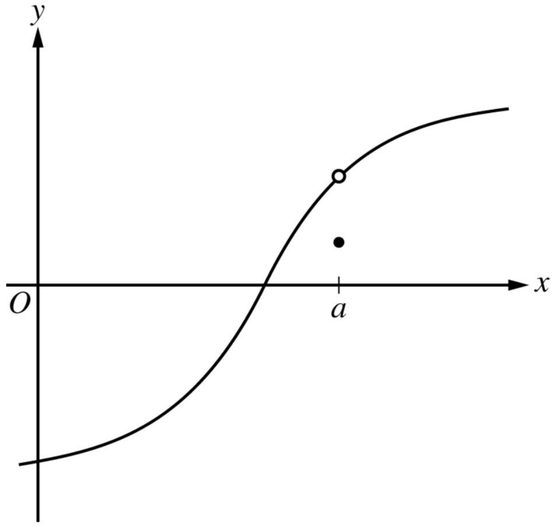
(A) -2 (B) $-\frac{2}{\pi}$ (C) 0 (D) $\frac{2}{\pi}$ (E) nonexistent

Question 14 2017 AB Q21

If $f(x) = \ln x$, then $\lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$ is

(A) $\frac{1}{3}$ (B) e^3 (C) $\ln 3$ (D) nonexistent

Question 15 2017 BC Q6



Graph of f

The graph of $y = f(x)$ is shown above. Which of the following is true?

(A) $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

(B) $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

(C) $\lim_{x \to a} f(x) \neq f(a)$

(D) $\lim_{x \to a} f(x)$ does not exist.

Question 16 2017 BC Q18

Which of the following limits are equal to -1 ?

I. $\lim_{x \to 0^-} \dfrac{|x|}{x}$

II. $\lim_{x \to 3} \dfrac{x^2 - 7x + 12}{3 - x}$

III. $\lim_{x \to \infty} \dfrac{1 - x}{1 + x}$

(A) I only (B) I and III only (C) II and III only (D) I, II, and III

Question 17 2017 BC Q78

The continuous function f is positive and has domain $x > 0$. If the asymptotes of the graph of f are $x = 0$ and $y = 2$, which of the following statements must be true?

(A) $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 2} f(x) = \infty$

(B) $\lim_{x \to 0^+} f(x) = 2$ and $\lim_{x \to \infty} f(x) = 0$

(C) $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 2$

(D) $\lim_{x \to 2} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 2$

Question 18 2017 AB Q26

$$\lim_{x \to \infty} \frac{\ln\left(e^{3x} + x\right)}{x} =$$

(A) 0 (B) 1 (C) 3 (D) ∞

Question 19 2017 AB Q29

The graph of which of the following functions has exactly one horizontal asymptote and no vertical asymptotes?

(A) $y = \dfrac{1}{x^2 + 1}$

(B) $y = \dfrac{1}{x^3 + 1}$

(C) $y = \dfrac{1}{e^x - 1}$

(D) $y = \dfrac{1}{e^x + 1}$

Question 20 2017 AB Q90

For any function f, which of the following statements must be true?

I. If f is defined at $x = a$, then $\lim_{x \to a} f(x) = f(a)$.

II. If f is continuous at $x = a$, then $\lim_{x \to a} f(x) = f(a)$.

III. If f is differentiable at $x = a$, then $\lim_{x \to a} f(x) = f(a)$.

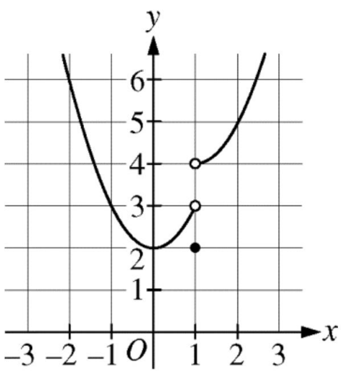
(A) III only

(B) I and II only

(C) II and III only

(D) I, II, and III

Question 21 2016 BC Q88



Graph of f

The graph of the function f is shown in the figure above. The value of $\lim_{x \to 0} f\left(1 - x^2\right)$ is

(A) 1 (B) 2 (C) 3 (D) 4 (E) nonexistent