Coupled SDEs

Consider

$$d\theta_t = -\theta_t dt + dW_t,$$
$$d\mathscr{Y}(\theta_t, t) = \left(-\theta_t \partial_\eta \mathscr{Y} + \partial_\eta^2 \mathscr{Y}\right) dt + (\partial_\eta \mathscr{Y}) dW_t,$$

where W_t is a Wiener process and $\mathscr{Y}(\theta_t, t)$ a function that is twice differentiable in θ_t.

To determine $Y_t = \mathscr{Y}(\theta_t, t)$ and its derivatives, requires

$$F_Y(\mathscr{Y}(\eta, t), t) = G(\eta),$$

where $G(\eta)$ is the cumulative normal distribution (CDF) or equivalently the distribution of θ_t. Solving for the inverse CDF of F_Y gives

$$\mathscr{Y}(\eta, t) = F_Y^{-1}(G(\eta), t),$$

as a function of η. Differentiating $F_Y(\mathscr{Y}(\eta, t), t) = G(\eta)$ then gives

$$\frac{\partial \mathscr{Y}_t}{\partial \eta} = \frac{g(\eta)}{f_Y(\mathscr{Y}(\eta, t), t)},$$

where $g(\eta)$ is the normal distribution, which differentiated again respect to η gives

$$\frac{\partial^2 \mathscr{Y}_t}{\partial \eta^2} = -\eta \frac{g}{f_Y} - \frac{g}{f_Y^2} \frac{\partial \mathscr{Y}_t}{\partial \eta} = -\frac{g}{f_Y}\left(\eta + \frac{g}{f_Y^2}\right),$$

an expression that is a function of g, f_Y and η only.

Substituting for the derivatives in terms of η, g, f_Y then gives

$$d\theta_t = -\theta_t dt + dW_t,$$
$$d\mathscr{Y}(\theta_t, t) = \left(\frac{g}{f_Y}\right)\left(\left[-2\theta_t + \frac{g}{f_Y^2}\right] dt + dW_t\right),$$

evolution equations for $\mathscr{Y}_t, \theta_t$ which depend on their global distribution.