AA/ME 507 Winter 2026
Homework 3
February 6, 2026
Due Date: February 20, 2026 (11:59 PM)
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Problem 1: Draining tank with two orifices Water drains from a vertical cylindrical tank through two circular holes of diameter d, one located at the bottom of the tank ($z = 0$) and the other at mid-height ($z = H/2$). The tank has diameter $R \gg d$ and the initial water level is H (see figure 1).

1. Find the time t^* it takes for the water level to reach $h(t^*) = \frac{H}{2}$.
2. Derive the governing relations between h and t for the two time intervals $t < t^*$ and $t > t^*$.

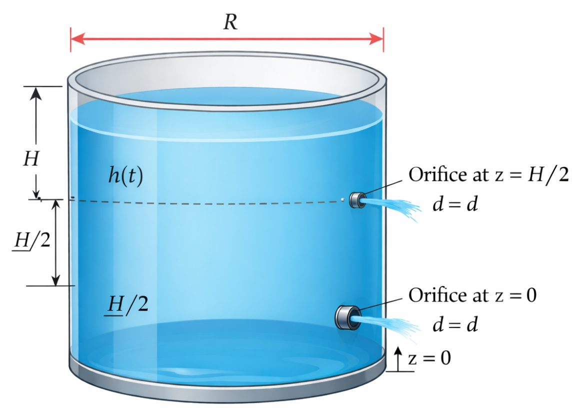

Figure 1: Draining tank with two orifices.

Problem 2: Navier-Stokes and irrotational flow Consider an incompressible, steady flow in spherical polar coordinates $\vec{v} = (u_r(r), 0, 0)$ (*i.e.* $\partial u_r/\partial t = \partial u_r/\partial \phi = \partial u_r/\partial \theta = 0$). The pressure at $r \to \infty$ is $p = p_\infty$.

1. Write down the simplified continuity and Navier-Stokes equations for this flow, assuming that $\vec{v} = (u_r(r), 0, 0)$ and viscous effects are not important.
2. Use the continuity equation to determine the velocity u_r up to a constant.
3. Plug $u_r(r)$ into the simplified Navier-Stokes equations to determine the pressure as a function of r.
4. Plug the $u_r(r)$ into the Bernoulli equation to determine the pressure as a function of r. Are you getting the same result as above? Why?
5. Redo the previous two parts including viscous effects, assuming a constant kinematic viscosity, ν. Are you you getting the same results for the pressure? Why?

Problem 3: The faucet jet - why does it reach constant diameter? Water flowing at velocity u_0 from a circular faucet of radius R_0 forms a free-falling steady jet that narrows and stretches as it accelerates under the action of gravity (Figure 2). Let's analyze the shape $r(z)$ of the jet.

1. Apply mass conservation to write an equation that relates the jet velocity v with its radius r.

2. Apply Bernoulli's equation to write a second equation that relates the jet velocity with its pressure and the position z. Eliminate the pressure from that equation by making use of the normal stress boundary condition at the interface between water and air. The surface tension is σ and the jet can be considered as a cylinder of radius $r(z)$ for the purpose of calculating its curvature.

3. Combine the two equations obtained in the previous parts and write them in non-dimensional form. Identify the Froude and Weber numbers of the flow.

4. Show that under these assumptions the jet keeps narrowing indefinitely as z increases (*Hint:* take the z derivative of your equation and show that $dr/dz = 0$ is not possible).

5. Let's see that equilibrium (i.e. $dr/dz = 0$) can be reached if we include viscous losses in the Bernoulli equation, e.g., $p(z) + \rho u(z)^2/2 - \rho g z = p(0) + \rho u_0^2/2 - \alpha \mu v z / r^2$, where α is a non-dimensional coefficient that depends on the shape of the velocity profile inside the jet.

   (a) Rewrite the equation of part 3 condidering viscous losses. Identify the Reynolds number.

   (b) Take the z derivative of that equation and determine the value of the asymptotic radius r_∞ for which $dr/dz = 0$.

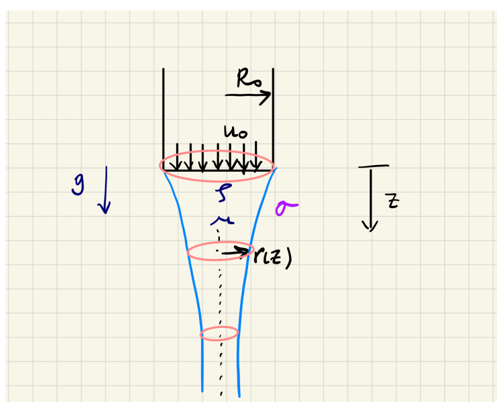

Figure 2: Jet coming out of a circular faucet

Problem 4: The Helmholtz decomposition and pressure in an incompressible flow. A smooth vector field, e.g., the velocity of a fluid, can be decomposed as the sum of a potential contribution and a divergence-free contribution as

$$\vec{v}^* = \vec{v} + \nabla \phi, \tag{1}$$

where $\nabla \cdot \vec{v} = 0$. Let us use this decomposition to explore the role of pressure gradient in the Navier-Stokes equations as an incompressibility enforcer.

1. Take the non-dimensional Navier-Stokes equation *without the pressure term* at a time (t), i.e.,

$$\partial_t \vec{v}(t) = -[\vec{v}(t) \cdot \nabla]\vec{v}(t) + Re^{-1}\nabla^2 \vec{v}(t), \tag{2}$$

   and integrate it over an infinitesimally small interval of time to calculate $\vec{v}^*(t+dt) \approx \vec{v}(t) + \partial_t \vec{v}(t) dt$ as a function of $\vec{v}(t)$.

2. Assuming that $\nabla \cdot \vec{v}(t) = 0$, use the previous result to find $\nabla \cdot \vec{v}^*(t+dt)$. Reason if $\vec{v}^*(t+dt)$ will be divergence-free or not.

3. Apply the following Helmholtz decomposition to $\vec{v}^*(t+dt)$,

$$\vec{v}(t+dt) = \vec{v}^*(t+dt) - dt\nabla\phi, \tag{3}$$

   and enforce $\nabla \cdot \vec{v}(t+dt) = 0$, to find an equation for ϕ.

4. Combine equations 2 and 3 to write $\vec{v}(t+dt)$ as a function of $\vec{v}(t)$ and ϕ

5. Compare your previous result for $\vec{v}(t+dt)$ with the result you would obtain by integrating the *complete* Navier-Stokes equation

$$\partial_t \vec{v}(t) = -\nabla p(t) - [\vec{v}(t) \cdot \nabla]\vec{v}(t) + Re^{-1}\nabla^2 \vec{v}(t),$$

6. Can you establish a parallelism between ϕ and the pressure p?

Problem 5: What Causes The Cheerios Effect? The Cheerios effect is a colloquial name for the phenomenon of floating objects appearing to feel attractive or repelling forces with the wall of the fluid container and with one another, as long as the Bond number based on their size is small. The example which gives the effect its name is the observation that pieces of breakfast cereal (for example, Cheerios) floating on the surface of a bowl will appear to stick to the side of the bowl or form rafts of multiple objects.
Before considering the phenomenon of wall attraction for an air bubble, let us study why an air bubble can trapped at the surface of a water container. Refer to the diagram in Figure 3, where the air-water surface is assumed to remain flat.

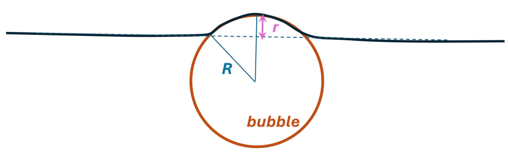

Figure 3: Bubble diagram, flat surface

1. Write down the equilibrium of forces for the bubble assuming that the interface remains flat and the bubble barely protrudes ($r/R \ll 1$, $\alpha \ll 1$).

2. Find an expression for r/R as a function of the Bond number. Justify that the bubble remains trapped when the Bond number is small, $Bo \ll 1$.

Now that we have established that the air bubble remains trapped under the water surface, let us see that buoyancy attracts it to the wall of the container.

3. Use the normal stress balance boundary condition to show that the height of the water interface $h(x)$ is governed by the equation
$$\sigma \frac{d^2h}{dx^2} = \rho g h$$
under the assumption that container is 2D and the water surface curvature is small so that $1/R \approx dh/dx$. Here, σ is the surface tension coefficient, ρ the water density (assume air has zero density), and g the gravitational acceleration.

4. Calculate $h(x)$ if the contact angle at the container wall is θ as shown in Figure 4.

5. Write a free body diagram for the bubble showing it will move towards the wall if $\theta < \pi/2$.

6. Assuming the bubble experiences Stokes' drag with $D = 6\pi\mu D v$, calculate its velocity v as a function of x (*Hint*: you may find the trigonometric expression $\sin 2\theta = \frac{2\cot\theta}{1+\cot^2\theta}$ useful). Demonstrate that the bubble accelerates as it gets closer to the wall.

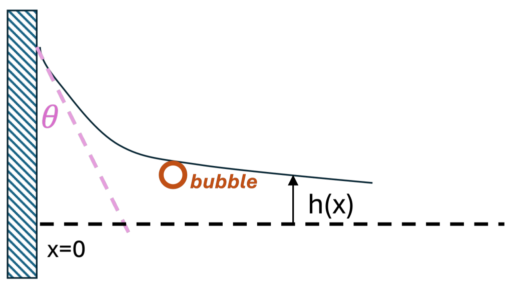

Figure 4: Bubble in inclined surface