AA/ME 507 Winter 2026
Homework 1
January 21, 2026
Due Date: January 30, 2026 (11:59 PM)
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Problem 1. "Mini-Lab" Data Collapse for Laminar Pipe Flow (Buckingham Π) . You are performing an experimental characterization of a Newtonian fluid that flows steadily through a straight, circular tube of length L and inner diameter D. In each run, you measure:

$$Q \quad \text{(volumetric flow rate)}, \qquad \Delta p \quad \text{(pressure drop over length } L\text{)},$$

and you are given fluid properties (ρ, μ) and tube geometry (D, L). All runs are in the laminar regime. The full dataset is provided as a CSV file named `pipe_lab.csv`. Columns are:

$$(D \text{ in mm}), \quad (\mu \text{ in mPa} \cdot \text{s}), \quad (\rho \text{ in kg/m}^3), \quad (L \text{ in m}), \quad (Q \text{ in mL/s}), \quad (\Delta p \text{ in kPa}).$$

1. **Raw plot.** Plot Δp versus Q for all runs on the same axes. Comment on why the curves do *not* collapse, making it impossible to elucidate a clear functional dependence.

2. **Buckingham Π analysis.** Assume the pressure drop depends on

$$\Delta p = F(Q, D, L, \mu, \rho),$$

   where μ is the dynamic viscosity and ρ is density.

   (a) How many dimensionless groups do you expect?
   (b) Construct a valid set of Π groups (any correct set earns full credit).
   (c) Rewrite your result in a convenient form using the mean speed

$$U = \frac{4Q}{\pi D^2}.$$

3. **Choose variables that should collapse the data.** Define the Darcy friction factor

$$f \equiv \frac{\Delta p}{(L/D)\,(\rho U^2/2)} = \frac{2\,\Delta p\, D}{\rho U^2 L},$$

   and the Reynolds number $Re \equiv \frac{\rho U D}{\mu}$. Compute (f, Re) for every run and plot f versus Re on log–log axes.

4. **Compare to laminar theory.** For fully developed laminar flow in a circular tube, the theoretical collapse is $f = \frac{64}{\mathrm{Re}}$. Overlay the curve $64/\mathrm{Re}$ on your plot and comment on agreement/disagreement (scatter, bias, and any outliers).

5. **Inference.** Use the collapse to estimate an "effective" viscosity μ_{fit} for each fluid condition (e.g. by fitting $f\mathrm{Re} \approx 64$) and compare to the tabulated μ.

Notes and unit reminders Use SI units in computations:

$$D\,[\text{m}], \quad \mu\,[\text{Pa} \cdot \text{s}], \quad Q\,[\text{m}^3/\text{s}], \quad \Delta p\,[\text{Pa}].$$

Conversions: 1 mm $= 10^{-3}$ m, 1 mPa $\cdot$ s $= 10^{-3}$ Pa $\cdot$ s, 1 mL/s $= 10^{-6}$ m^3/s, 1 kPa $= 10^3$ Pa.

Problem 2. Passive scalar transport from Reynolds Transport Theorem A fluid has density $\rho(\mathbf{x}, t)$ and velocity field $\mathbf{u}(\mathbf{x}, t)$. A passive scalar $c(\mathbf{x}, t)$ is carried by the flow, where c is the *mass fraction* of a dilute dye (mass of dye per mass of mixture). Assume:

- No sources/sinks (no reaction): dye is conserved.
- Molecular diffusion relative to the bulk flow is described by Fick's law $\mathbf{j} = -\rho D \nabla c$, where $\mathbf{j}$ is the diffusive mass flux of dye (mass per area per time) *relative to the mass-averaged velocity* $\mathbf{u}$, and D is the (possibly space- and time-dependent) diffusivity.

1. Starting from an integral conservation statement on an *arbitrary fixed control volume* V with outward normal $\mathbf{n}$, write an equation for the time rate of change of dye mass in V in terms of the net dye flux through ∂V. Be explicit about the *total* dye flux (advective + diffusive).
2. Use Reynolds Transport Theorem for a fixed control volume and the divergence theorem to derive the local (differential) equation for $c(\mathbf{x}, t)$.
3. Simplify your result for an incompressible flow with constant ρ and constant D.

Problem 3: Streamlines, Pathlines, Streaklines Consider the unsteady 2D velocity field

$$u(x, y, t) = -y, \qquad v(x, y, t) = x + \beta t,$$

where β is a constant.

1. Show that the flow is incompressible.
2. For fixed $t = t_0$, find the equation of the streamlines and describe their geometry.
3. Find the pathline $(x(t), y(t))$ of a fluid particle starting at $(x(0), y(0)) = (x_0, y_0)$.
4. A dye source continuously injects dye at the fixed point (x_s, y_s) for $t \geq 0$. Derive a parametric representation of the streakline observed at time $t = t_0$.
5. State when streamlines, pathlines, and streaklines coincide.

Problem 4: Local Deformation — Strain-Rate and Rotation Tensors Consider the 2D velocity field

$$u(x, y, t) = a\,x - \Omega\,y, \qquad v(x, y, t) = \Omega\,x - a\,y + \beta t,$$

where a, Ω, β are constants.

1. Show that the flow is incompressible.
2. Compute the velocity gradient tensor $\nabla \mathbf{u}$.
3. Decompose $\nabla \mathbf{u}$ into its symmetric and antisymmetric parts
$$\mathbf{D} = \tfrac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right), \qquad \mathbf{W} = \tfrac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^T\right).$$
4. Compute the scalar vorticity ω_z and relate it to $\mathbf{W}$.
5. Find the principal strain rates and principal directions (eigenvalues/eigenvectors of $\mathbf{D}$).

6. For a material line element $\delta\mathbf{x} = (\delta x, \delta y)^T$, show

$$\frac{d}{dt}\,|\delta\mathbf{x}|^2 = 2\,\delta\mathbf{x}^T\mathbf{D}\,\delta\mathbf{x},$$

and evaluate this expression explicitly. Identify orientations that instantaneously stretch vs. compress.

Problem 5: Polar Coordinates Practice Consider the *steady* 2D velocity field (for $r > 0$) in polar coordinates:

$$u_r(r,\theta) = -\frac{A}{r}, \qquad u_\theta(r,\theta) = \frac{B}{r},$$

where $A > 0$ and B are constants.

1. Verify incompressibility: $\nabla \cdot \mathbf{u} = 0$ for $r > 0$.

2. Derive and solve the streamline equation $r(\theta)$ using tangency in polar coordinates. Classify the geometry and discuss the special cases $A = 0$ and $B = 0$.

3. Because the flow is steady, pathlines coincide with streamlines. Starting from $(r(0), \theta(0)) = (r_0, \theta_0)$, find $r(t)$ and $\theta(t)$, and eliminate t to recover the same $r(\theta)$.

4. Compute the scalar vorticity ω_z for $r > 0$.

5. Compute the polar strain-rate components

$$D_{rr} = \frac{\partial u_r}{\partial r}, \qquad D_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad D_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),$$

and interpret what kinds of deformation occur.