Kinematics: Reference Frames, Flow Fields, and Reynolds Transport Theorem (RTT)

AA 507 (Incompressible Fluid Mechanics) Winter 2026, UW

Assignment: Homework 2

(Ref file: HW2.pdf)

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```
clear % Clears workspace
clc % Clears command window
close all % Closes all figures
```

Problem 1. Data Collapse for Laminar Pipe Flow (Buckingham-Π)

In practice, fluid flow in pipes is commonly encountered in systems. For example, the flow of liquid propellants through the network of ducts and lines found in a rocket engine.

"In fully developed laminar flow, each fluid particle moves at a constant axial velocity along a streamline and the velocity profile *u(r)* remains unchanged in the flow direction." [1] See Fig. 8-11.

```
imshow('Fig8-11_LaminarFlow.png')
```

- [*Newtonian fluid* - viscosity is constant; shear stress is proportional to shear rate.]

Why aerospace engineers care about the Newtonian assumption:

Area	Newtonian assumption works	Non-Newtonian risk
Aerodynamics	External flow	Icing fluids, contaminants
Propulsion	Gas dynamics	Slurries, gels
Thermal	Convective models	Phase-changing coatings
CFD	Standard solvers	Needs constitutive modeling

1. Raw plot

```
%% HW2 Problem 1 — Raw plot: Δp vs Q grouped by constant (D, mu, rho, L)
% 1) Reads pipe_lab.csv
% 2) Plots one curve per unique set of (D, mu, rho, L)

clear; clc; close all;

%% ---- 1) Read the raw data file ----
% Assumes pipe_lab.csv is in the current folder. Otherwise, set full path
here:
fname = "pipe_lab.csv";
T = readtable(fname);

% Expected columns (per HW PDF):
% D_mm, mu_mPa_s, rho_kg_m3, L_m, Q_mL_s, dp_kPa
% If your file has different headers, adjust these names accordingly.
D   = T{:,2};  % [mm]
mu  = T{:,3};  % [mPa*s]
rho = T{:,4};  % [kg/m^3]
L   = T{:,5};  % [m]
Q   = T{:,6};  % [mL/s]
dp  = T{:,7};  % [kPa]

%% ---- 2) Group by constant parameters and plot Q vs Δp ----
% If any of these are floating with tiny roundoff, rounding makes grouping
robust.
Dg   = round(D,   6);
mug  = round(mu,  6);
rhog = round(rho, 6);
Lg   = round(L,   6);
```

```
G = findgroups(Dg, mug, rhog, Lg); % Assign group numbers
nG = max(G); % Total number of unique groups
```

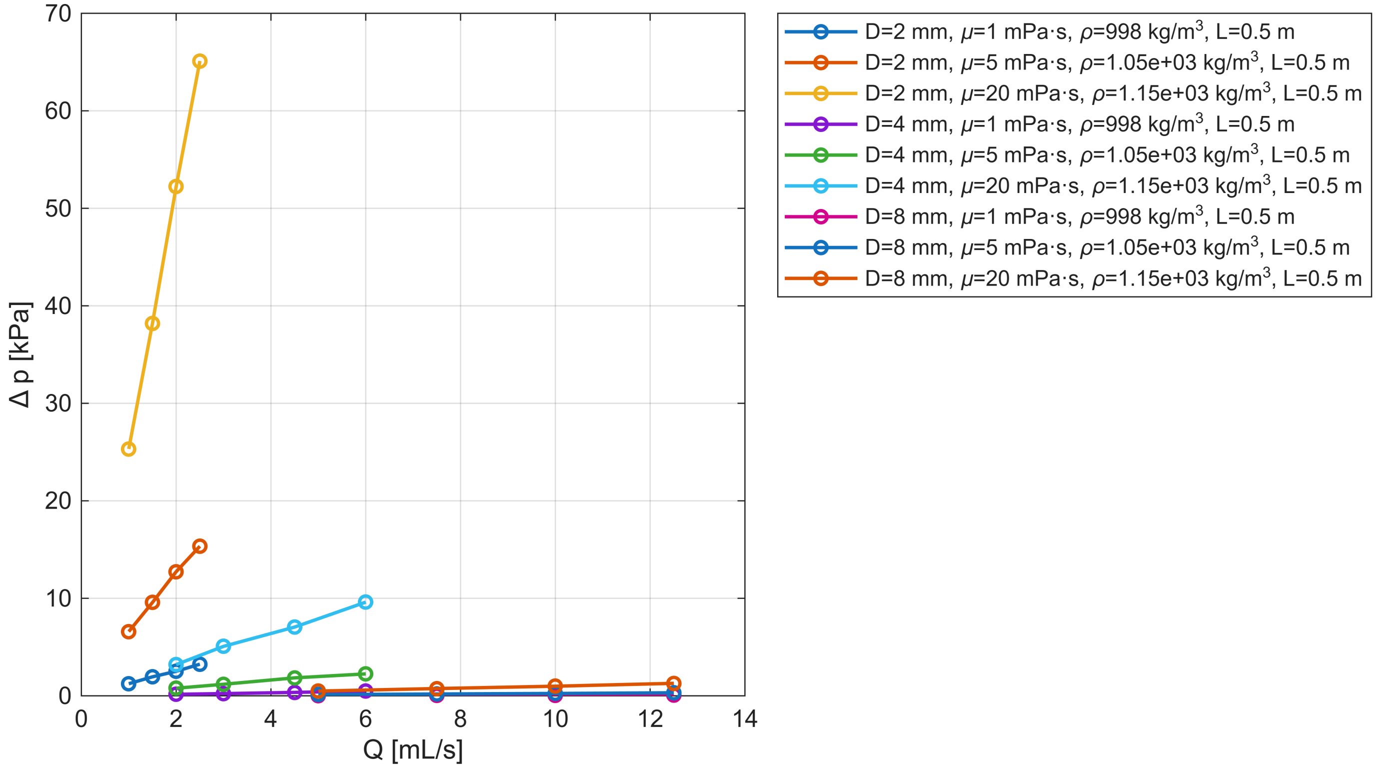
```

figure('Color','w','Visible', 'off');
%figure('Color','w');
```

```
hold on; grid on; box on;

% Plot each group as its own curve
for k = 1:nG
    idx = (G == k);

    % Sort by Q so curves don't zig-zag
    [Qk, order] = sort(Q(idx));
    dpk = dp(idx);
    dpk = dpk(order);

    plot(Qk, dpk, '-o', 'LineWidth', 1.4, 'MarkerSize', 5);
end

xlabel('Q [mL/s]');
ylabel('\Delta p [kPa]');
t = title('\Delta p vs Q (grouped by constant D, \mu, \rho, L)');
t.Units = 'normalized';
t.Position(2) = 1.00;
t.Position(1) = 3.00;

% Build a readable legend label for each curve
labels = strings(nG,1);
for k = 1:nG
    idx = (G == k);
    % pull representative values from first row in group
    i0 = find(idx, 1, 'first');
    labels(k) = sprintf('D=%.3g mm, \\mu=%.3g mPa·s, \\rho=%.3g kg/m^3,
L=%.3g m', ...
        D(i0), mu(i0), rho(i0), L(i0));
end
legend(labels, 'Location', 'bestoutside');
```

```
%% Optional: if you want to also save the figure
%exportgraphics(gcf, "dp_vs_Q_grouped.png", "Resolution", 600);
```

```
%imshow('dp_vs_Q_grouped.png');
```

```
fig = gcf;
set(fig,'Units','inches');
set(fig,'Position',[0.5 0.5 8.5 4.8]);   % wide figure
```




```
set(fig,'PaperPositionMode','auto');     % keep on-screen size for printing/
export
```

<u>Discussion</u>

The curves do not collapse since the experiment's input parameters (independent variables - D, μ, ρ, and *L*) have not been properly *nondimensionalized.* In other words, it is not possible to gain insight about the relationships between the parameters from this raw plot.

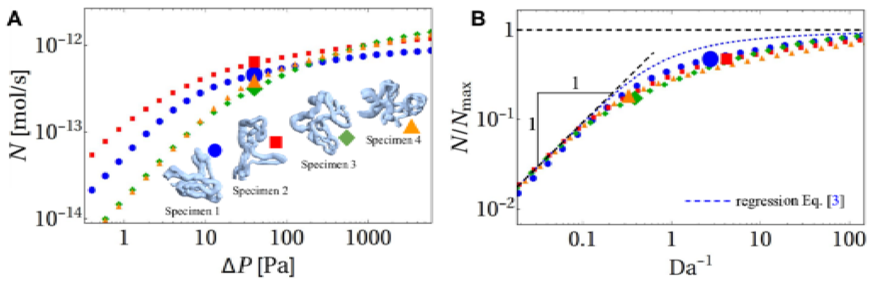
- [Data collapse - is a way of establishing *scaling*; scaling may be accomplished through **nondimensionalization.** The law of dimensional homogeneity guarantees that every additive term in an equation has the same dimensions. [1]]

When properly nondimensionalized, the data from many experiments will produce a family of curves (or a single curve) that reveals the underlying physics or makes a universal law more apparent. As an illustration of this claim, the two plots shown in the figure below show "raw" computational data (A) and collapse when plotted using suitable dimensionless variables (B) - as presented by Erlich, et.al. in their work on *Physical and geometric determinants of transport in feto-placental microvascular networks*. [2]

```
figure; % Force a new figure window to open
imshow('Computational-data-A-collapse-B.png')
```

```
Warning: PNG library warning:
iCCP: profile 'icc': 0h: PCS illuminant is not D50
```

```
Warning: PNG library warning:
iCCP: profile 'icc': 0h: PCS illuminant is not D50
```



2. Buckingham Π analysis

Step 1:

Pressure drop:

$$\Delta p = f(D, L, \mu, \rho) \rightarrow n = 5$$

Note: Only one dependent parameter is analyzed here, Δp.

Step 2:

Parameter	Δp	Q	D	L	μ	ρ
Primary dimensions	$M^1L^{-1}T^{-2}$	L^3T^1	L^1	L^1	$M^1L^{-1}T^{-1}$	M^1L^{-3}

Step 3:

As an initial guess, set the reduction j as the number of primary dimensions --

$$j = 3$$

(a) If this initial guess is *correct*, the number of Π's, or dimensionless groups expected is k:

$k = n - j = 5 - 3 = \mathbf{2}$

Step 4:

Choose j repeating parameters --

- **Hint of the day:** A *wise* choice of repeating parameters for most fluid flow problems is a **length**, a **velocity**, and a **mass** or **density**.

```
% Call the 'buckinghamPi' sub-routine;
% it computes the Buckingham Pi groups (one set for 1 dependent variable)
% based off the input of repeating parameters.

params = ["dP", "D", "L", "mu", "rho"];
%depParams = ["dP", "Q"];
dimNames = ["M", "L", "T"];
```

```
dimMatrix = [1 -1 -2;   % dP
             0  1  0;   % D
             0  1  0;   % L
             1 -1 -1;   % mu
             1 -3  0];  % rho
```

```
results = buckinghamPi(params, dimNames, dimMatrix);
```

```
--- Buckingham Pi Setup ---
n = 5 parameters
j = 3 primary dimensions: M, L, T
k = n - j = 2 expected Pi group(s)

Parameters and their dimension exponents (rows):
Param    M    L    T
dP    1    -1    -2
D    0    1    0
L    0    1    0
mu    1    -1    -1
rho    1    -3    0

Choose 3 repeating parameters.
Rule of thumb (fluids): pick ones that span M, L, T and are independent.
(A wise selection for fluid flow problems is a length, a velocity, and a mass or density.)Available: dP, D


Repeating parameters chosen: L, mu, rho
Non-repeating parameters:     dP, D

--- Results ---
Computed 2 Pi group(s):

Pi_1 =
```

$$\frac{L^2 \, \mathrm{dP} \, \rho}{\mu^2}$$

```
Exponents on repeating parameters [L, mu, rho]:
```

$$\begin{pmatrix} 2 & -2 & 1 \end{pmatrix}$$

```
Pi_2 =
```

$$\frac{\mathrm{D}}{L}$$

```
Exponents on repeating parameters [L, mu, rho]:
```

$$\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$$

```
disp(results.exponents);
```

PiGroup	L	mu	rho
1	2	-2	1
2	-1	0	0

```
disp(results.piGroups);
```

$$\begin{pmatrix} \frac{L^2\,\mathrm{dP}\,\rho}{\mu^2} & \frac{\mathrm{D}}{L} \end{pmatrix}$$

Step 5:

Construct k Π's, and manipulate as necessary.

```
clipboard('copy', latex(results.piGroups))
```

$$\Pi_1 = \left(\frac{D^2\,\mathrm{dP}\,\rho}{\mu^2}\right) \qquad \text{(5-1)}$$

For laminar flow, the pressure drop in a pipe is [1]:

$$\Delta P = P_1 - P_2 = \frac{32\mu L V_{avg}}{D^2} \qquad \text{(5-2)}$$

Sub (5-2) into (5-1):

$$\Pi_{1,modified} = \left(\frac{32 L V_{avg}\rho}{\mu}\right) \qquad \text{(5-3)}$$

Manipulate further, multiply by taking the inverse of the fraction then multiplying by a pure dimensionless constant 2 and replace *characteristic length* with *diameter* to obtain:

$$\Pi_{1,modified} = \left(\frac{64}{Re}\right) = f \qquad \text{(5-4)}$$

- f is known as the friction factor for laminar flow in a circular pipe; this *dimensionless parameter* shows this factor to be a function of the **Reynolds number** only (independent of pipe roughness). [1]

$$\Pi_2 = \left(\frac{L}{D}\right) = AR \qquad \text{(5-5)}$$

The established nondimensional parameters are the **friction factor** (5-4) and the **aspect ratio** (5-5).

<u>Step 6</u>:

Write the functional relationship

$\Delta P = f(f, AR)$

Problem 2. Passive scalar transport from Reynolds Transport Theorem

- [A passive scalar is a scalar quantity that is transported by advection (the transfer of a substance, or matter, by bulk motion of a fluid) and diffusion in a flow but does not influence the velocity field or fluid dynamics.]

1. Time rate of change of dye mass

Fundamentals

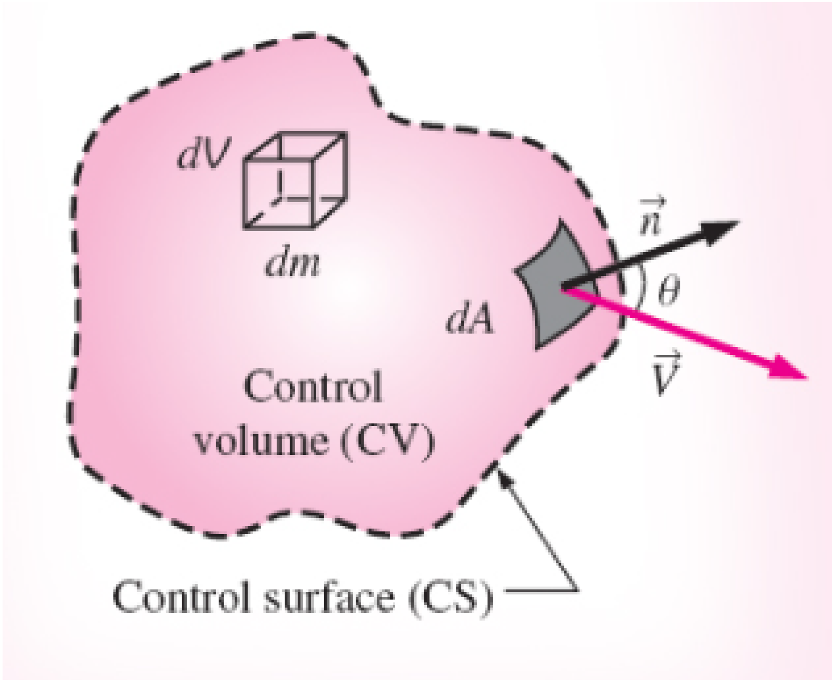
<u>a. Conservation of Mass Principle</u>

$m_{in} - m_{out} = \Delta_{CV}$

The net mass transfer to or from a control volume during a time interval Δt *is equal to the net change in the total mass within the control volume during* Δt. [1]

General Conservation of Mass

```
imshow('element for cons of mass.png')
```



$$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho \, (\vec{V} \cdot \vec{n}) \, dA = 0 \qquad (2\text{-}1)$$

"... *the time rate of change of mass within the control volume plus the net mass flow rate through the control surface is equal to zero."*

<u>b. Total Dye Mass</u>:

Since:

$$c = \frac{M_{dye}}{M_{mixture}}$$

then,

$$M_{dye} = cM_{mixture} = c\rho_{mixture}V$$

Generally:

$$M_{dye}(t) = \int_V \rho(x,t)c(x,t)\,dV \qquad (2\text{-}2)$$

<u>c. Time Rate of Change of Dye Mass</u>:

The dye is conserved, so expanding on the *general conservation of mass* expression (2-1) above, gives:

$$\frac{d}{dt}\int_{\mathrm{V}}\rho c\, dV + \oint_{\partial V}(\rho c \vec{u} + \mathbf{j})\cdot \vec{n}\, dA = 0 \qquad \text{(2-3)}$$

- The left integral represents the time rate of change of mass within the CV (i.e., change in a CV).
- The right integral represents the total flux through the boundary surface (i.e., change in a system).
- Where $(\rho c \vec{u})\vec{n}$ is the **advective/convective** dye flux and $\mathbf{j}\vec{n}$ is the **diffusive dye flux.**

2. Reynolds Transport Theorem and Divergence Theorem for the Differential Equation

d. Reynolds Transport Theorem

For the fixed CV, RTT reduces to:

$$\frac{d}{dt}\int_{\mathrm{V}}\rho c\, dV = \int_{V}\frac{\partial(\rho c)}{\partial t}dV$$

Since the CV is not moving or deforming with time, the time derivative can be moved inside since it is irrelevant whether we differentiate or integrate first. [1] It is expressed as a partial derivative since density and the dye mass ratio are space-dependent (functions of *x* and *t*).

e. Divergence Theorem on the Surface Flux

$$\oint_{\partial V}\mathbf{G}\cdot\mathbf{n}\, dA = \int_{V}\nabla\cdot\mathbf{G}\, dV$$

Let:

$$\mathbf{G} = (\rho c \vec{u} + \mathbf{j})$$

$$\oint_{\partial V}(\rho c \vec{u} + \mathbf{j})\cdot \vec{n}\, dA = \int_{V}\nabla\cdot(\rho c \vec{u} + \mathbf{j})\, dV$$

f. Combine Them

$$\frac{\partial(\rho c)}{t} + \nabla\cdot(\rho c \vec{u} - \rho D \nabla c) = 0 \qquad \text{(2-4)}$$

3. Incompressible Flow and Constant Density and Diffusivity

Incompressible flow:

$\frac{\partial\rho}{\partial t} = 0$ (density does not change with time)

and

$$\nabla\cdot\vec{u} = 0$$

Expand $\nabla \cdot (c\vec{u})$

$$\nabla \cdot (c\vec{u}) = \vec{u} \cdot c + c(\nabla \cdot \vec{u}) = \vec{u} \cdot c$$

(2-4) becomes:

$$\frac{\partial c}{t} + \vec{u} \cdot \nabla c = D\nabla^2 c \qquad \text{(2-5)}$$

- This is the simplified passive-scalar transport equation.

Applications

[pending]

References

[1] Cengel, Y. (2018). *Fluid Mechanics: Fundamentals and Applications* (4th ed.). New York: McGraw-Hill.

[2] Erlich, Alexander & Pearce, Philip & Plitman, Romina & Jensen, Oliver & Chernyavsky, Igor. (2018). Physical and geometric determinants of transport in feto-placental microvascular networks. 10.48550/arXiv.1809.00749.