Dimensional Analysis and Vector Calculus

AA 507 Winter 2026, UW

Assignment: Homework 1

(Ref file: HW1_2026.pdf)

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```
clear all;
```

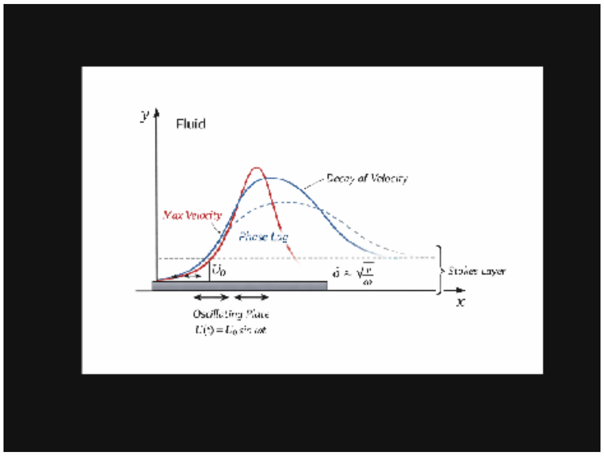
Problem 1. Dimensional analysis: oscillating Stokes layer

Background

The Stokes oscillating layer (or oscillatory boundary layer) describes the thin layer of viscous fluid near a surface that moves back and forth due to oscillatory motion, such as a large flat plate oscillating siunusoidally in its own plane with a given velocity, $U(t)$. Ref figure.

Where $U(t) = U_0 cos(\omega t)$

```
imshow("Stokes oscillatory boundary layer diagram.21JAN2026.png")
```



Exponential Decay: The fluid velocity oscillates with the same frequency as the wall, but its amplitude decays exponentially as distance from the wall increases. [1]

Applications

Heat Transfer Enhancement

Compact heat exchangers need effective mixing, which occurs when the oscillation frequency is tuned so δ matches the thermal diffusion length. [2]

Vibration-Induced Drag & Noise

Used in early-stage aeroelastic modeling to estimate viscous energy losses. The Stokes layer determines aerodynamic damping, noise generation, and flutter stability. [3]

Analysis

Assumptions:

- Flow is laminar and two dimensional
- Compressibility and gravity is neglected

The characteristic thickness δ of the oscillatory boundary layer is the distance from the wall where the velocity amplitude has decayed to 1/*e* of its wall value.

1. Dimensional analysis for δ

Using the Buckingham- $\prod$ theorem --

<u>Step 1</u>

List parameter in functional form:

$\delta = f(\omega, \rho, \mu)$

this functional relationship posits the idea that the boundary layer thickness (dependent variable) is a function of independent variables: angular frequency, and the fluid's density and viscosity.

$n = 4$ (four total parameters)

<u>Step 2</u>

Primary dimensions

- $\delta : [L^1]$
- $\omega : [T^{-}1]$
- $\rho : [M^1 L^{-3}]$
- $\mu : [M^1 L^{-1} T^{-1}]$

→ 3 primary dimensions

<u>Step 3</u>

Set the reduction *j* as the number of primary dimensions (initial guess)

Reduction: $j = 3$

If correct, number of $\prod$'s: $k = n - j = 4 - 3 = 1$

<u>Step 4</u>

Choose *j* repeating parameters.

Repeating parameters: ω, ρ, μ

- Note: Following the guidelines for picking repeating parameters, we cannot pick the dependent variable δ

<u>Step 5</u>

Construct *k* Π's, manipulate as necessary.

The dependent Π_1 is generated by assuming a dimensionless product:

$$\Pi_1 = \delta \omega^a \rho^b \mu^c$$

Force Π_1 to be dimensionless by setting the exponent of each primary dimension to zero. Then determine the constant exponents by equating the exponents of each primary dimension:

```
%% Buckingham Pi via Symbolic Math Toolbox
% Variables: delta, omega, rho, mu
% Dimensions (M, L, T):
%   delta : L^1
%   omega : T^-1
%   rho   : M^1 L^-3
```

```
%   mu    : M^1 L^-1 T^-1

clear; clc;

syms a b c real           % unknown exponents for omega, rho, mu
syms delta omega rho mu    % symbols for readability (not strictly needed)

% Dimension exponents for each variable in [M L T] order
% Rows correspond to variables: [delta; omega; rho; mu]
D = [ 0  1  0;   % delta
      0  0 -1;   % omega
      1 -3  0;   % rho
      1 -1 -1];  % mu

% Build the dimension balance for Pi = delta^1 * omega^a * rho^b * mu^c
% Total exponents in [M L T] must be zero:
%   1*D(delta,:) + a*D(omega,:) + b*D(rho,:) + c*D(mu,:) = [0 0 0]
eqM = D(1,1) + a*D(2,1) + b*D(3,1) + c*D(4,1) == 0;
eqL = D(1,2) + a*D(2,2) + b*D(3,2) + c*D(4,2) == 0;
eqT = D(1,3) + a*D(2,3) + b*D(3,3) + c*D(4,3) == 0;

sol = solve([eqM, eqL, eqT], [a, b, c], 'ReturnConditions', true);

a_sol = simplify(sol.a);
b_sol = simplify(sol.b);
c_sol = simplify(sol.c);

fprintf('Solved exponents:\n');
```

```
Solved exponents:
```

```
fprintf('a = %s\n', string(a_sol));
```

```
a = 1/2
```

```
fprintf('b = %s\n', string(b_sol));
```

```
b = 1/2
```

```
fprintf('c = %s\n', string(c_sol));
```

```
c = -1/2
```

```


% Construct the Pi group
Pi = simplify(delta * omega^a_sol * rho^b_sol * mu^c_sol);
disp('Buckingham-Pi group Pi =');
```

```
Buckingham-Pi group Pi =
```

```
pretty(Pi)
```

```
delta sqrt(omega) sqrt(rho)
---------------------------
          sqrt(mu)
```

```
% Optional: express Pi in a nicer combined form
Pi_nice = simplify(delta * sqrt(omega*rho/mu));
disp('Equivalent simplified form:');
```

```
Equivalent simplified form:
```

```
pretty(Pi_nice)
```

```
          / omega rho \
delta sqrt| --------- |
          \     mu    /
```

(a) Thus, the only minimal independent group is:

$$\Pi_1 = \delta\left(\frac{\omega\rho}{\mu}\right)^{1/2}$$

<u>Step 6</u>

(b) Re-writing the result in the form:

$$\frac{\delta}{Ł_*} = \Phi(\Pi)$$

```
%% Dimensional analysis – Part (2)
% Writing the result as delta / L* = Phi(Pi)

clear; clc;

syms delta omega nu real positive

% Define characteristic length scale
Lstar = sqrt(nu/omega);

% Define the Buckingham Pi group
Pi = simplify(delta / Lstar);

disp('Characteristic length scale L* =');
```

```
Characteristic length scale L* =
```

```
pretty(Lstar)
```

```
  sqrt(nu)
-----------
sqrt(omega)
```

```
disp('Buckingham Pi group Pi = delta / L* =');
```

```
Buckingham Pi group Pi = delta / L* =
```

```
pretty(Pi)
```

```
delta sqrt(omega)
-----------------
     sqrt(nu)
```

```


% Express the dimensional-analysis result symbolically
Phi = sym('C');   % unknown O(1) constant from dimensional analysis
delta_result = simplify(Phi * Lstar);

disp('Final dimensional-analysis result:');
```

```
Final dimensional-analysis result:
```

```
disp('delta = Phi * L* =');
```

```
delta = Phi * L* =
```

```
pretty(delta_result)
```

```
 C sqrt(nu)
-----------
sqrt(omega)
```

This is what the code is communicating mathematically:

- Characteristic length scale: $L^* = \sqrt{\frac{\nu}{\omega}}$
- **Single Π-group:** $\Pi = \frac{\delta}{L^*} = \delta\sqrt{\frac{\omega}{\nu}}$
- Since there is only one Π-group, the functional relation reduces to a constant: $\frac{\delta}{L^*} = \Phi = C$
- Therefore, the Stokes layer thickness scales as: $\delta = C\sqrt{\frac{\nu}{\omega}}$

2. Identify the controlling parameter

[Note: When the problem asks for the controlling parameter, it is asking:

- Which dimensionless combination of variables governs the physics of the problem -- i.e., determines the regime and behavior of the flow?]

Let $\nu = \mu/\rho$ be the kinematic viscosity.

Equivalently,

$$\delta = C\sqrt{\frac{\nu}{\omega}}$$

Rewriting the Π-group to:

$$\Pi^2 = \frac{\omega\delta^2}{\nu}$$ ← **is the controlling parameter.**

One sentence interpretation of what the parameter compares:

$\omega\delta^2/\nu$ compares the oscillation timescale $(1/\omega)$ to the viscous diffusion timescale across the oscillatory boundary layer $\delta(\delta^2/\nu)$ - i.e., how far momentum can diffuse in one cycle.

Bonus

If we retain the plate velocity, then our oscillating boundary layer becomes:

$$\delta = f(U_0, \omega, \rho, \mu)$$

- Using the same characteristic length: $L^* = \sqrt{(\nu/\omega)}$ (the viscous penetration depth per oscillation)

Π-group 1: nondimensional thickness

$$\Pi_1 = \frac{\delta}{\sqrt{\nu/\omega}}$$ ← This measures how thick the layer is compared to the viscous diffusion length.

Π-group 2: nondimensional velocity amplitude

We nondimensionalize U_0 using the velocity scale associated with L^*:

$$U^* \sim \frac{L^*}{T_{osc}} = \sqrt{\nu\omega}$$

So:

$$\Pi_2 = \frac{U_0}{\sqrt{\nu\omega}}$$ ← This is essentially a **Reynolds number** based on the Stokes layer thickness; it measures the oscillation strength.

- Large Π_2 indicates possible flow seperation, streaming, turbulence.

Buckingham-Π form:

$$\frac{\delta}{\sqrt{\nu/\omega}} = \Phi\left(\frac{U_0}{\sqrt{\nu\omega}}\right)$$

The velocity amplitude now controls whether the Stokes solution is valid.

References

[1] https://en.wikipedia.org/wiki/Stokes_problem

[2] **S. T. Raml, *Oscillatory Flow Mechanism for Enhanced Heat Transfer*** — Master's thesis demonstrating how oscillating boundary motion can improve convective heat transfer by inducing unsteady boundary–layer motion.

[3] **Stokes Problem overview (classic fluid dynamics text references)** — Standard fluid mechanics sources (e.g., Batchelor, *An Introduction to Fluid Dynamics*, and Landau & Lifshitz, *Fluid Mechanics*) establish the fundamental oscillatory boundary layer solutions from which damping and drag models are derived.