From Rao, F.13 Solution of Differential Equations in MATLAB
p.1056-1058

Example:

Consider the solution of the differential equation; with

$c = 0.1$ $m =$
$K = 10.0$

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} + Ky = 0;$$

Initial conditions,

$y(0) = 1$ (initial displacement)

$\frac{dy}{dt}(0) = 0$ (initial velocity)

Note: An n-th order ODE is to be converted into a system of n-first-order ODEs before using MATLAB functions.

The equation of motion can be written as a set of two first-order diff. equations by introducing

$$y_1 = y$$

$$y_2 = \frac{dy}{dt} = \frac{dy_1}{dt}$$

as

$$\underbrace{\frac{d\vec{y}}{dt}}_{\text{Vector for eq. of motion}} = \underbrace{\vec{f}}_{\text{function Vector}} = \begin{Bmatrix} f_1(t, \vec{y}) \\ f_2(t, \vec{y}) \end{Bmatrix} = \begin{Bmatrix} y_2 \\ -cy_2 - Ky_1 \end{Bmatrix}$$

Note the sign convention, since they are now on the right hand side of the equation

What I'm essentially doing is:

$$\frac{d^2y}{dt^2} = y_2$$

$$c\frac{dy}{dt} + Ky = cy_2 + Ky_1$$

With initial conditions

$$\vec{y}(0) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

Step 2: Spring-mass-damper system equation of motion:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + Ky = 0 \quad ; \qquad (1)$$

Step 3: From Chapra, p. 671-672

Higher-order equations can be reduced to a system of first-order equations

Define a new variable, y_2

$$y_2 = \frac{dy}{dt} \quad (2)$$

, which itself can be differentiated to yield

$$\frac{dy_2}{dt} = \frac{d^2y}{dt^2} \qquad (3)$$

Now, (2) and (3) can be substituted into (1)

$$\Rightarrow m\frac{dy_2}{dt} + cy_2 + Ky_1 = 0 \qquad (4)$$

$$\Rightarrow \frac{dy_2}{dt} = \frac{-cy_2 - Ky_1}{m} \qquad (5)$$

Thus, (2) and (5) are a pair of first-order equations that are equiavelent to the original second-order equation.

$$\vec{f} = \begin{Bmatrix} f_1(t, y) \\ f_2(t, y) \end{Bmatrix} = \begin{Bmatrix} y_2 \\ \frac{-cy_2 - Ky_1}{m} \end{Bmatrix}$$

function vector system

Now, create an anonymous function

F = @(t, y) [y(2) ;(- c*y(2) - K * y(1))/m]

independent (t), dependent (y), column vector ([…])

Initial conditions

$y(0) = 0.539657$; $yd(0) = 1.0$; $ydd(0) = 0$

in vector terms:

$$y(0) = \begin{pmatrix} 0.539657 \\ 1.0 \end{pmatrix}; \quad y0 = [0.539657; 1.0]$$

Declare and set constants

c = 1000.0

K = 26519.2

m = 450.0

To gather stats on how "hard" the ODE solver works in MATLAB; use odeset

ex. opts = odeset('stats','on')

~~How to~~ find acceleration?

$$\frac{d^2 y_2}{dt^2} = \frac{-c\frac{dy_2}{dt} - K\frac{dy_1}{dt}}{m} = \frac{-c\left[\frac{-cy_2 - Ky_1}{m}\right] - Ky_2}{m}$$

$$= \frac{\frac{c^2 y_2 + Ky_1}{m} - Ky_2}{m}$$

$$\Rightarrow = \frac{c^2 y_2 + Ky_1 - Ky_2}{m}$$

Numerical Methods

MATLAB ode23 function

ode23 - solves nonstiff diff. eq. - low order method

(An ordinary diff. eq. problem is stiff if the solution being sought is varying slowly, but there are nearby solutions that vary rapidly, so the numerical method must take small steps to obtain satisfactory results.)

ode23 implements a combination of second- and third-order Runge-Kutta methods

From Chapra, p. 702, Sect. 25.3.1, p. 702

Second-Order Runge-Kutta Methods

$$\underbrace{y_{i+1}}_{\text{New Value}} = \underbrace{y_i}_{\text{old value}} + \underbrace{(\underbrace{a_1 K_1 + a_2 K_2}_{\text{slope}})\underbrace{h}_{\text{Step size}}}_{\text{increment function}}$$

Where, $K_1 = f(x_i, y_i)$

$K_2 = f(x_i + p_1 h,\ y_i + q_{11} K_1 h)$

Using Ralston's Method ($a_2 = 2/3$).

$$y_{i+1} = y_i + \left(\tfrac{1}{3}K_1 + \tfrac{2}{3}K_2\right)h$$

where

$K_1 = f(x_i, y_i)$

$K_2 = f(x_i + \tfrac{3}{4}h,\ y_i + \tfrac{3}{4}K_1 h)$

$C = 1000.0$
$K = 26519.2$
$m = 450.0$

So, for the second function, $\dfrac{-Cx(2) - Kx(1)}{m}$

the third term, initial conditions:

$x(2) = y_2 = 0.9150$, $t_2 = 0.0050$

$x(1) = 0.5421$

$$K_1 = \frac{-C(0.9150) - K(0.5421)}{m} = -33.98$$

$$K_2 = -33.98 - \frac{C\left(0.4150 \times .0025 \times \frac{3}{4}\right) - K\left(0.5421 \times 0.0025 \times 3/4\right)}{m}$$

$$= -34.04$$

$\therefore$ The avg. slope is computed by:

$$\phi = \frac{1}{3}(-33.98) + \frac{2}{3}(-34.04)$$

$$= -34.02$$

$$\therefore x(2)_3 = 0.4150 - 34.02\,(0.0025)$$

$$\boxed{x(2)_3 = 0.82995}$$

MATLAB output: 0.8301

Or

$\varepsilon_t = 0.02\%$

Not bad!

So far I have calculated displacement and velocity, now I just need acceleration.

I can use the gradient function, from Chapra, see pp. 360–363

the gradient is defined mathematically as,

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

; this vector is also referred to as "del f"
→ represents the directional derivative of $f(x, y)$

$$\therefore\ xdd = \nabla xd = \frac{\partial xd}{\partial t} \quad ; \quad x = x_0 + \frac{\partial f}{\partial x}h$$

See 14.2.2 Steepest Ascent Method

~~Starting point $xd_2 = 0.4150$, $t_2 = 0.0050$~~

~~$\nabla f_{2\text{-}1} = -0.085\hat{\imath} + 0.0025\hat{\jmath}$~~

~~In the domain of: 2 to 1~~

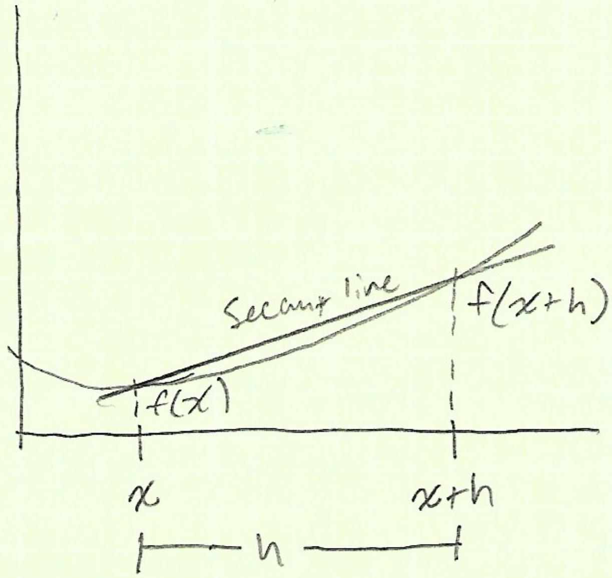
~~the coords of any point along the h axis~~

~~$x = 0.4150 - 0.085h$~~

~~$t = 0.0050 + 0.0025h$~~

Gradient, or Numerical differentiation

See, Chapra, Part 6
p.564



$$\therefore \quad xdd_1 = \frac{0.9150 - 1}{0.0025} = -34.0021487 \checkmark$$

$$xdd_2 = \frac{0.83012 - 0.9150}{0.0025} = -33.9479 \checkmark$$