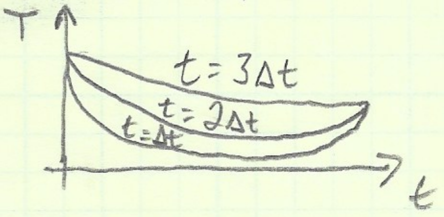
What is a parabolic equation?

See. Chapra, p. 815

↳ Determines how an unknown varies in both space and time.

↳ It's called parabolic because the function has the form of a parabola, the curvature changes as we step in time.



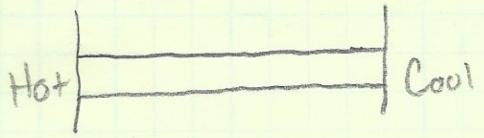
↳ Manifested by the presence of both spatial and temporal derivatives in the heat conduction equation.

↳ The solution will consist of a series of spatial distributions corresponding to the state of the element at various times.

30.1 The Heat Conduction Equation

see Chapra p. 840

Ex. 1-D case; A thin rod insulated at all points except the ends.



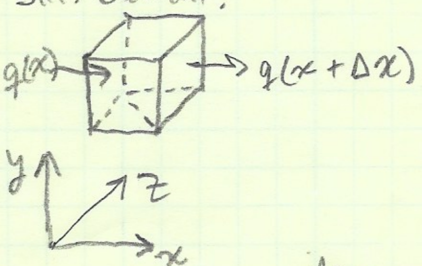
Derivation: Conservation of heat principle is used to develop a heat balance for the differential element:

$$\begin{pmatrix}\text{rate of}\\ \text{heat in}\end{pmatrix} = \begin{pmatrix}\text{rate of}\\ \text{heat out}\end{pmatrix} + \begin{pmatrix}\text{rate of}\\ \text{heat stored}\end{pmatrix} \qquad (1)$$

or

$$\text{input} - \text{outputs} = \text{stored} \qquad (2)$$

Diff. element:



, where $q(x)$ is the heat input in the x-direction, to obtain the rate we multiply by the area of the face on which it enters and the duration (time)

Here the heat output is a function of the change in direction in which the heat traveled

$$q(x)\overbrace{\Delta y \Delta z}^{\text{Area}}\overbrace{\Delta t}^{\text{Duration}} - q(x+\Delta x)\Delta y \Delta z \Delta t = \overbrace{\Delta x \Delta y \Delta z}^{\text{Volume}} \rho C \Delta T, \qquad (3)$$

ρ = density of material

C = heat capacity of the material [cal/(g·°C)]

ΔT = change in temp. across the body in the x-direction

Dividing (3) by the volume of the element and Δt gives:
$(\Delta x \Delta y \Delta z)$

$$\underbrace{\frac{q(x) - q(x+\Delta x)}{\Delta x}} = \rho C \underbrace{\frac{\Delta T}{\Delta t}} \tag{4}$$

the Spatial rate of change of heat input
-or-
a change in heat input after a step in space

change in temp. after a change in time.

Now, to make this more general, we encode the limit to the ratios after smaller and smaller changes to the input with ∂ (del):

$$-\frac{\partial q}{\partial x} = \rho C \frac{\partial T}{\partial t} \tag{5}$$

The link between flux and temperature is provided by <u>Fourier's</u> law of heat conduction:

$$q_i = -K \rho C \frac{\partial T}{\partial i}, \text{ where} \qquad \text{see Chapra, p. 821} \tag{29.4}$$

q_i = heat flux in the direction of the i dimension
(flow of energy per unit of area per unit of time)

K = coefficient of thermal diffusivity

Taking the derivative of (29.4) to substitute it into (5) gives:

$$\frac{\partial q}{\partial i} = -K \rho C \frac{\partial^2 T}{\partial i^2}, \text{ then (6) into (5)} \tag{6}$$

$$\boxed{K \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \Leftarrow \text{Heat conduction equation}} \tag{30.1}$$

↳ Contains a second derivative in space and a first derivative in time.

<u>Summary</u>: The rate at which temp. at a point changes w/respect to time is equal to:
- The second derivative of that point w/respect to space.
  ↳ "Where there's curvature in space, there's change in time."

→ The LHS is telling us how the temp. distribution curves as we move across space.

Crank-Nicolson method

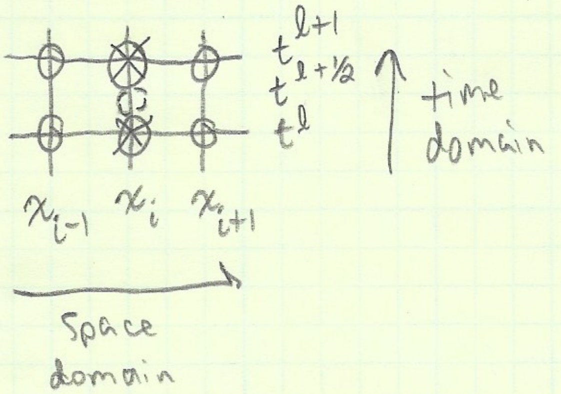
Solving Parabolic PDEs (e.g. heat conduction eq.) using the finite difference Crank-Nicolson method:

See Chapra, p. 849

↳ this method is second-order accurate in both space and time

Consider the grid of computational molecules for the Crank-Nicolson method:

Fig. 30.9



X grid point involved in time difference

O grid point involved in space difference

the temporal first derivative is approx. at $t^{l+1/2}$ by

$$\frac{\partial T}{\partial t} \cong \frac{T_i^{l+1} - T_i^{l}}{\Delta t} \qquad (30.12)$$

the spatial second derivative is determined at the midpoint by averaging the difference approxs. at t^l and t^{l+1} of the time increment

cancels with the ½ fraction

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{1}{2}\left[\underbrace{\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2}} + \underbrace{\frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}}\right] \qquad (30.13)$$

Because we're taking the average of two computational molecules

approxs the 2nd derivative of Temp. w/respect to the step in space at the beginning of the time increment t^l

same, but at the end of of the time increment t^{l+1}

Sub. (30.12) and (30.13) into (30.1):

$$\frac{K}{2}\left[\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right] = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

Collect terms:

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^{l} + 2(1-\lambda)T_i^{l} + \lambda T_{i+1}^{l} \quad (30.14)$$

where, $\lambda = \frac{K\Delta t}{(\Delta x)^2}$, λ is known as a (computational) stability criterion

Boundary conditions:

why is this plus 1?? → It doesn't make much sense why they would use this superscript; why not $f_0(t^l)$? Oh! I think it's meant to express that BC temp. changes w/time

$T_0^{l+1} = f_0(t^{l+1})$, first ~~interior~~ node, this is a function describing how the boundary temp. changes w/time. (30.14.1)

$= 100°C$?

$T_{m+1}^{l+1} = f_{m+1}(t^{l+1})$, last ~~interior~~ node (30.14.2)

$= 50°C$?

(30.14.1) and (30.14.2) are sub. into (30.14) to obtain the eqs. for the first and last nodes, respectively.

↳ Generating eqs. (30.15) and (30.16), see Chapra, p.850

Evaluate (30.15) numerically for node 1 ($i=1$)

$$2.04174\,T_1^{l+1} - 0.02087\,T_2^{l+1} = 0.02087(100°C) + 2(1-0.02087)\,\cancel{T_1^{l}}^{\,0} + 0.02087\,\cancel{T_2^{l}}^{\,0} + 0.02087(100°C)$$

↳ [The problem statement says the BCs are fixed for all times at

$$T(0) = 100°C$$
$$T(L) = 50°C$$

$$\therefore T_0^{l} = T_0^{l+1} = T_0^{l+2} = T_0^{l+n} = 100°C$$

– and, at $t=0$, the temp of the rod $= 0$]

at time $= 0$ ($l=0$)

$$\Rightarrow 2.04174\,T_1^{1} - 0.02087\,T_2^{1} = 4.174$$

This is the first equation in the system of equations

l = time Evaluate (30.15) numerically for node 1 ($i=1$) 6/23/21 16:00 4/6
i = space

$$-\cancel{\lambda T_{i-1}^{l+1}} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \cancel{\lambda T_{i-1}^{l}} + 2(1-\lambda)T_i^l + \lambda T_{i+1}^l \quad (30.14)$$

Boundary Condition ($i=0$)

$$T_0^{l+1} = f_0(t^{l+1}) \quad (30.14.1)$$

→ this is a function describing how the boundary temp. changes w/time.

Derive the equation for the first interior node ($i=1$):

$$-\lambda T_0^{l+1} + 2(1+\lambda)T_1^{l+1} - \lambda T_2^{l+1} = \lambda T_0^l + 2(1-\lambda)T_1^l + \lambda T_2^l$$

Recall: The problem statement says the BCs are fixed for all times at $T(0) = 100°C$
$T(L) = 50°C$

$$\therefore T_{0\,(l=0)}^{l} = T_0^{l+1} = T_0^{l+2} = T_0^{l+n} = 100°C$$

And, at $t=0$, the temp of the rod $= 0$ (Initial Condition)

$$\Rightarrow 2(1+\lambda)T_1^{l+1} - \lambda T_2^{l+1} = \lambda T_0^l + \cancel{2(1-\lambda)T_1^{l=0}} + \cancel{\lambda T_2^{l=0}} + \lambda T_0^{l+1}$$

$$= \lambda(T_0^l + T_0^{l+1})$$

@ $t=n$ ($l=l$)

$$2(1+\lambda)T_1^{l+1} - \lambda T_2^{l+1} = \lambda f_0(t^l) + 2(1-\lambda)T_1^l + \lambda T_2^l + \lambda f_0(t^{l+1})$$

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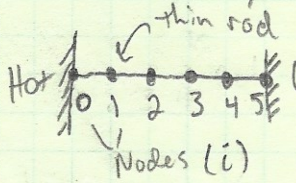

Let's setup the linear system of eqs.

for 6 nodes, 5 elements (incl. BCs) at $t = t_1$ ($l = 1$),
Since $L = 10$, $dx = 2$ $\Rightarrow N_{el} = 10/2 = 5$; # elements

Note: Because the BCs are fixed for all time, we only "care" about the temp. distribution w/in the rod, i.e. at nodes 1:4

Using Eq. (30.14), evaluate T_i^1 (time 1, node i) @ $l = t' = 0.1s$, for $i = 1:4$

$$i = 1: \quad -\lambda T_0^2 + 2(1+\lambda)T_1^2 - \lambda T_2^2 = \lambda T_0^1 + 2(1-\lambda)T_1^1 + \lambda T_2^1$$

↳ Bring T_1 and T_2 to LHS, T_0 to RHS, collect terms...

$$2(1+\lambda)T_1^2 - \lambda T_2^2 - 2(1-\lambda)T_1^1 - \lambda T_2^1 = \underbrace{\lambda T_0^1 + \lambda T_0^2}_{\text{Left node, BC}}$$

$$i = 2: \quad -\lambda T_1^2 + 2(1+\lambda)T_2^2 - \lambda T_3^2 = \lambda T_1^1 + 2(1-\lambda)T_2^1 + \lambda T_3^1$$

$$2(1+\lambda)T_2^2 - \lambda T_3^2 - 2(1-\lambda)T_2^1 - \lambda T_3^1 = \underbrace{\lambda T_1^1 + \lambda T_1^2}_{\text{Left node}}$$

$$i = 3: \quad -\lambda T_2^2 + 2(1+\lambda)T_3^2 - \lambda T_4^2 = \lambda T_2^1 + 2(1-\lambda)T_3^1 + \lambda T_4^1$$

$$2(1+\lambda)T_3^2 - \lambda T_4^2 - 2(1-\lambda)T_3^1 - \lambda T_4^1 = \lambda T_2^1 + \lambda T_2^2$$

$$i = 4: \quad -\lambda T_3^2 + 2(1+\lambda)T_4^2 - \lambda T_5^2 = \lambda T_3^1 + 2(1-\lambda)T_4^1 + \lambda T_5^1$$

$$-\lambda T_3^2 + 2(1+\lambda)T_4^2 - \lambda T_3^1 - 2(1-\lambda)T_4^1 = \underbrace{\lambda T_5^1 + \lambda T_5^2}_{\text{Right node, BC}}$$

@ $l = 1$ (time $= t^1$), system of eqs.: 5/16

BC

Unknowns

$$i=1: \quad 2(1+\lambda)T_1^2 - \lambda T_2^2 - 2(1-\lambda)T_1^1 - \lambda T_2^1 = \lambda T_0^1 + \lambda T_0^2 \qquad \cancel{6}\ 4$$

$$i=2: \quad 2(1+\lambda)T_2^2 - \lambda T_3^2 - 2(1-\lambda)T_2^1 - \lambda T_3^1 = \lambda T_1^1 + \lambda T_1^2 \qquad 2$$

$$i=3: \quad 2(1+\lambda)T_3^2 - \lambda T_4^2 - 2(1-\lambda)T_3^1 - \lambda T_4^1 = \lambda T_2^1 + \lambda T_2^2 \qquad 2$$

$$i=4: \quad -\lambda T_3^2 + 2(1+\lambda)T_4^2 - \lambda T_3^1 - 2(1-\lambda)T_4^1 = \lambda T_5^1 + \lambda T_5^2 \qquad 0$$

BC

↳ 6 unknowns, 4 eqs. ??

↳ the problem is that I'm starting my analysis at t^1, when I should be starting at t^0; starting at t^0 would make the internal nodes $=0$, since this is the initial condition.

→ Okay, so let's do that.

@ $l = 0$ (time $= t^0$), system of eqs.:

IC IC BC

Unknowns

$$i=1: \quad 2(1+\lambda)T_1^1 - \lambda T_2^1 - 2(1-\lambda)\cancelto{0}{T_1^0} - \lambda \cancelto{0}{T_2^0} = \lambda T_0^0 + \lambda T_0^1 \qquad 2$$

$$i=2: \quad 2(1+\lambda)T_2^1 - \lambda T_3^1 - 2(1-\lambda)\cancelto{0}{T_2^0} - \lambda \cancelto{0}{T_3^0} = \lambda \cancelto{0}{T_1^0} + \lambda T_1^1$$

$$-\lambda T_1^1 + 2(1+\lambda)T_2^1 - \lambda T_3^1 = 0 \qquad 1$$

IC IC IC

$$i=3: \quad 2(1+\lambda)T_3^1 - \lambda T_4^1 - 2(1-\lambda)\cancelto{0}{T_3^0} - \lambda \cancelto{0}{T_4^0} = \lambda \cancelto{0}{T_2^0} + \lambda T_2^1$$

$$-\lambda T_2^1 + 2(1+\lambda)T_3^1 - \lambda T_4^1 = 0 \qquad 1$$

IC BC

$$i=4: \quad -\lambda T_3^1 + 2(1+\lambda)T_4^1 - \lambda \cancelto{0}{T_3^0} - 2(1-\lambda)\cancelto{0}{T_4^0} = \lambda T_5^0 + \lambda T_5^1 \qquad 4$$

↳ 4 unknowns, 4 eqs.

↳ So now I see where the tridiagonal pattern is coming from.

↳ But this pattern will not hold for $t = t^1$, right?

5''/6

let's re-write the system for $l=1$ ($time = t^1$); noting that all terms with T^1 were determined from the previous system ($t = t^0$):

$$i=1: \quad 2(1+\lambda)T_1^2 - \lambda T_2^2 = \underbrace{\lambda T_0^1 + \lambda T_0^2}_{BCs} + \underbrace{2(1-\lambda)T_1^1 + \lambda T_2^1}_{\text{determined @ previous iteration}}$$

$$i=2: \quad -\lambda T_1^2 + 2(1+\lambda)T_2^2 - \lambda T_3^2 = \underbrace{\lambda T_1^1 + 2(1-\lambda)T_2^1 - \lambda T_3^1}_{\text{all known (scaler)}}$$

$$i=3: \quad -\lambda T_2^2 + 2(1+\lambda)T_3^2 - \lambda T_4^2 = \lambda T_2^1 + 2(1-\lambda)T_3^1 + \lambda T_4^1$$

$$i=4: \quad -\lambda T_3^2 + 2(1+\lambda)T_4^2 = \underbrace{\lambda T_5^1 + \lambda T_5^2}_{BC} + \lambda T_3^1 + 2(1-\lambda)T_4^1$$

↳ Actually, the tridiagonal pattern holds...

↳ Okay, so essentially, the RHS vector is the heat flow from the neighboring nodes (which the quantity is known/calc. from previous loop), we'll call this vector $\{T_{neighbors}\}$

↳ The general matrix form can be expressed as follows:

$$[\lambda]\{T_{node,time}\} = \{T_{neighbors}\}$$

↳ Alright! I love it when things make sense!

↳ Now I need to write a MATLAB program to solve this for all time.

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$5'''/6$

$$
\begin{array}{c c}
i & \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & [\lambda] = \begin{bmatrix} 2(1+\lambda) & -\lambda & 0 & 0 \\ -\lambda & 2(1+\lambda) & -\lambda & 0 \\ 0 & -\lambda & 2(1+\lambda) & -\lambda \\ 0 & 0 & -\lambda & 2(1+\lambda) \end{bmatrix}
\end{array}
$$

$$
\{T_{node,\,time}\} = \begin{Bmatrix} T_1^{t+1} \\ T_2^{t+1} \\ T_3^{t+1} \\ T_4^{t+1} \end{Bmatrix}
$$

Unknowns → quantities for the next moment in time

BCs

Know that $t_1 = 0$

$$
\{T_{neighbors}\} = \begin{Bmatrix} \lambda T_0^t + \lambda T_0^{t+1} + 2(1-\lambda)T_1^t + \lambda T_2^t \\ \lambda T_1^t + 2(1-\lambda)T_2^t - \lambda T_3^t \\ \lambda T_2^t + 2(1-\lambda)T_3^t + \lambda T_4^t \\ \lambda T_3^t + 2(1-\lambda)T_4^t + \lambda T_5^t + \lambda T_5^{t+1} \end{Bmatrix}
$$

↑ (RHS temp. vector)

BCs

All quantities known for the current moment in time, with the exception of BCs.

MATLAB syntax:

$$
\{T_{node,\,time}\} = [\lambda] \backslash \{T_{neighbors}\}
$$

or

$\{T_{rhs}\}$ ← Right hand side

What does the solution look like?

$$\{T_{time,node}\} = [\lambda] \backslash \{T_{neighbors}\}$$

↳ This vector output is only for one time step, and all nodes. I need to create a matrix that will hold all time steps for all nodes in the system.

$$[T_{solution}] = \begin{bmatrix} \text{transpose}\{T_{t^1, nodes}\} \\ \text{transpose}\{T_{t^2, nodes}\} \\ \vdots \\ \text{transpose}\{T_{t^f, nodes}\} \end{bmatrix}$$