Solving PDEs

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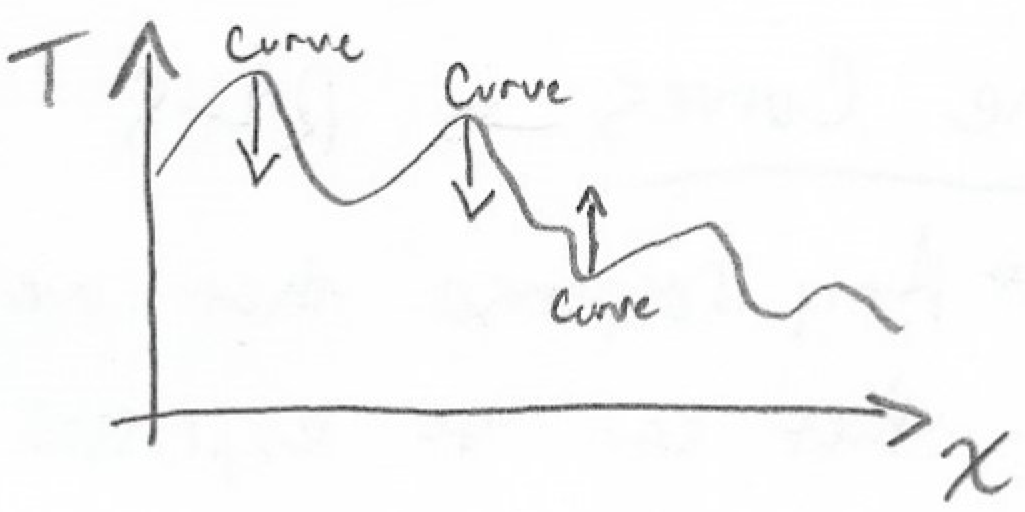
- The Heat Equation (Heat)



Heat: $$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t)$$

Change in temp. as time progresses

rate of change of the change in temp. for every step in space. X a const. α

E.g. How does the temp. distribution change as time progresses?

✱ → "Where there's curve in space, there's curve in time."

To solve: Remember: When you're solving a PDE, you're really finding a function that will satisfy several (3) constraints:

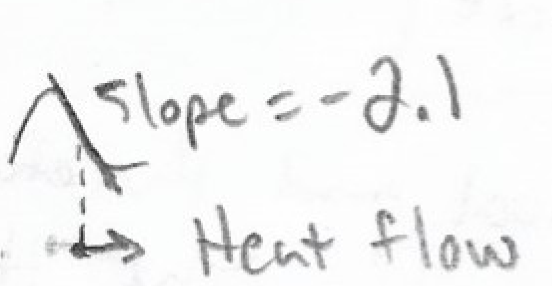
1. The PDE, $\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t)$

2. Boundary condition (BC):

$$\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(L,t) = 0 \quad \text{for all } t > 0,$$

this is saying that the BCs must be a constant value, as a const. produces a first derivative $= 0$

Remember: Elements change according to their neighbor, elements on either end of a 1D rod have no "neighbor" to their left/right, so they will tend toward their only neighbor w/in the body. Also, the slope (derivative) at a given point is proportional to the rate of heat flow at that point.



To model the restriction that no heat flows at either end, the slope must be zero for $t > 0$.

Physical intuition: We expect $T \to 0$ as $t \to \infty$

3. Initial Condition
① - Where are we starting?

Sine Curves → RHS

↳ Any response that we encounter in practice, is a function (signal) that can be expressed as a sum of sine waves.

↳ Easy to work with.

@ $t=0$ $\quad \frac{\partial T}{\partial x}(x,0) = \cos(x) \quad , \quad \frac{\partial^2 T}{\partial x^2}(x,0) = -\sin(x)$

Exponentials → LHS

↳ Each point (node) changes its temp at a rate proportional to the point itself, and, at every time step everything scales down by some factor.

*↳ When the rate of some value changes is proportional to that value itself, this should immediately tell us that this is an exponential.

* Keeping these two core concepts in mind (Sines and exponentials), we can assume a solution: $\boxed{T(x,t) = \sin(x)e^{-\alpha t}}$

↳ Sinusoid in space, exponential decay in time.

Check that this solution "fits" the Heat Equation

$$\frac{\partial T}{\partial t} \stackrel{?}{=} \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T(x,t) = \sin(x)e^{-\alpha t}$$

$\frac{\partial}{\partial t}$ ↙ $\qquad$ ↘ $\frac{\partial}{\partial x}$

$$\cos(x)e^{-\alpha t}$$

↘ $\frac{\partial}{\partial x}$

$$\boxed{-\alpha \sin(x) e^{-\alpha t}} \stackrel{?}{=} \alpha \boxed{-\sin(x) e^{-\alpha t}}$$

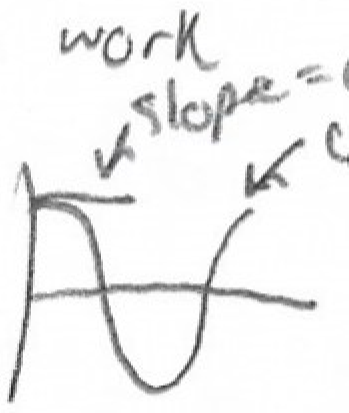
→ Both sides are ineed equal and therefore "fit" the Heat Equation.

(2) ~~BCs~~
Now that we have a function that makes the PDE true, we can move on to the next constraint to satisfy: BCs

Question we have to answer: How to apply the BCs that make the slope of either end equal to zero.

↳ Cosine wave starts at a local maximum, so this will work

$$\cos(x)e^{-\alpha t} \xrightarrow{\partial/\partial x} -\sin(x)e^{-\alpha t} \xrightarrow{\partial/\partial x} -\cos(x)e^{-\alpha t}$$



for the left side...

What about the right??

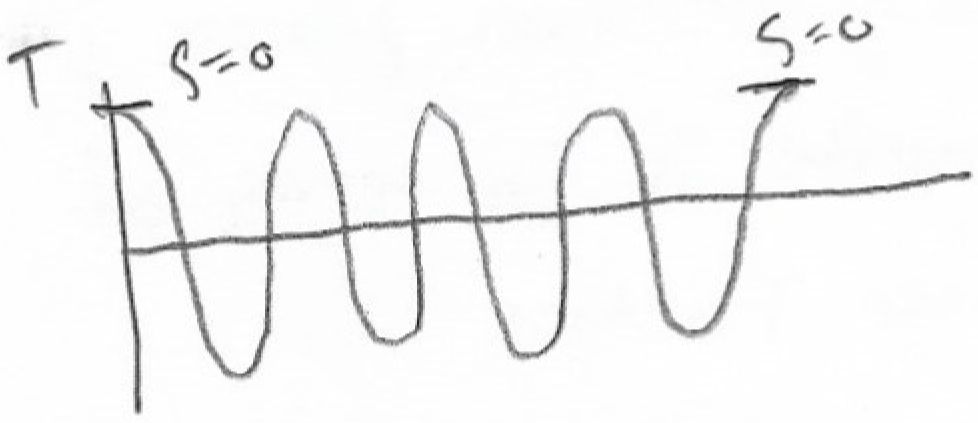
↳ we adjust the frequency, ω

$\cos(\omega \cdot x)$

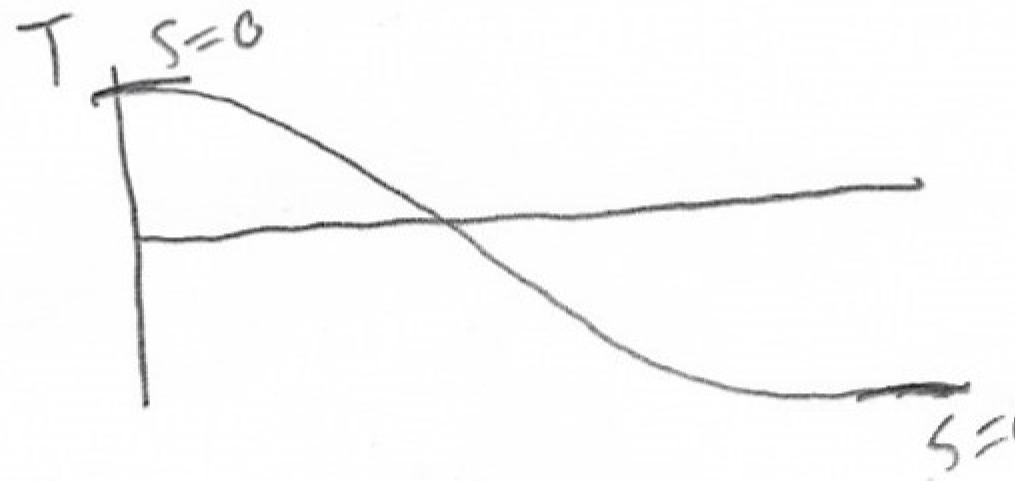
[Recall: The frequency is simply a measure of "how often" a wave moves across a certain distance.]

by adjusting ω, we can 'force' the function to end at a maximum (or min), making the slope $= 0$

↑ High freq. curves



↓ Low freq. curves



because the frequency affects the "sharpness" of the curvature, so does the derivative of the function.

$$\cos(\omega \cdot x) \xrightarrow{\partial/\partial x} -\omega \sin(\omega \cdot x) \xrightarrow{\partial/\partial x} -\omega^2 \cos(\omega \cdot x)$$

chain rule...

Because we are adding an ω to the RHS, to balance the equation we must also add it to the left - LHS (exponential decay part).

↳ Our revised solution becomes: $T(x,t) = \cos(\omega \cdot x)e^{-\alpha \omega^2 t}$

Physical intuition: A temp. function with sharper curves decays much quicker than one w/out as many.
→ It does so quadratically.

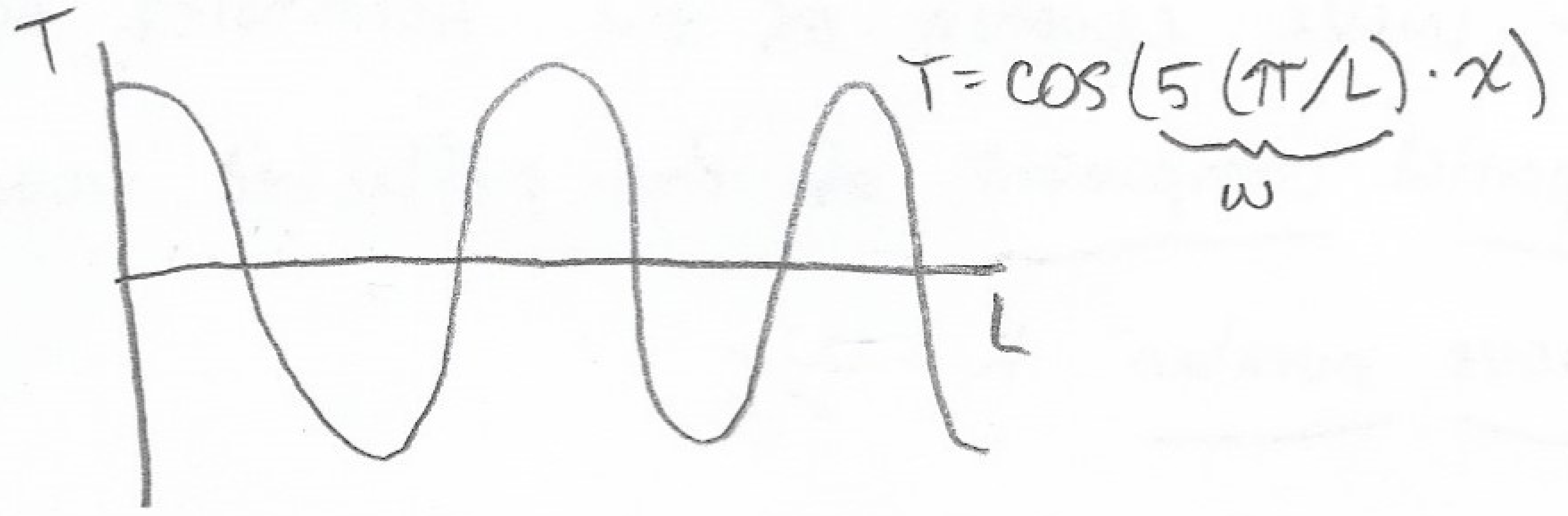
② , cont. The lowest frequency to make both ends

* have a slope of 0, if $x_{max} = L$, $\omega = \pi/L$

↳ when $x = L$, the input into the cosine expression becomes π → this is half the period of a cosine wave

↳ to find other omega values that satisfy, we move through harmonics to find the base # frequencies.

E.g.



* ↳ this gives us an infinite family of functions that satisfy both the PDE and the BCs.

I.e. there are multiple functions that fit the Heat Equation PDE and the BCs.

③ Linearity

Remember this: Everything stems from this basic observation:

↳ A function that looks like a sine curve in space and an exponential decay in time fits this equation, relating second derivatives in space with first derivatives in time.

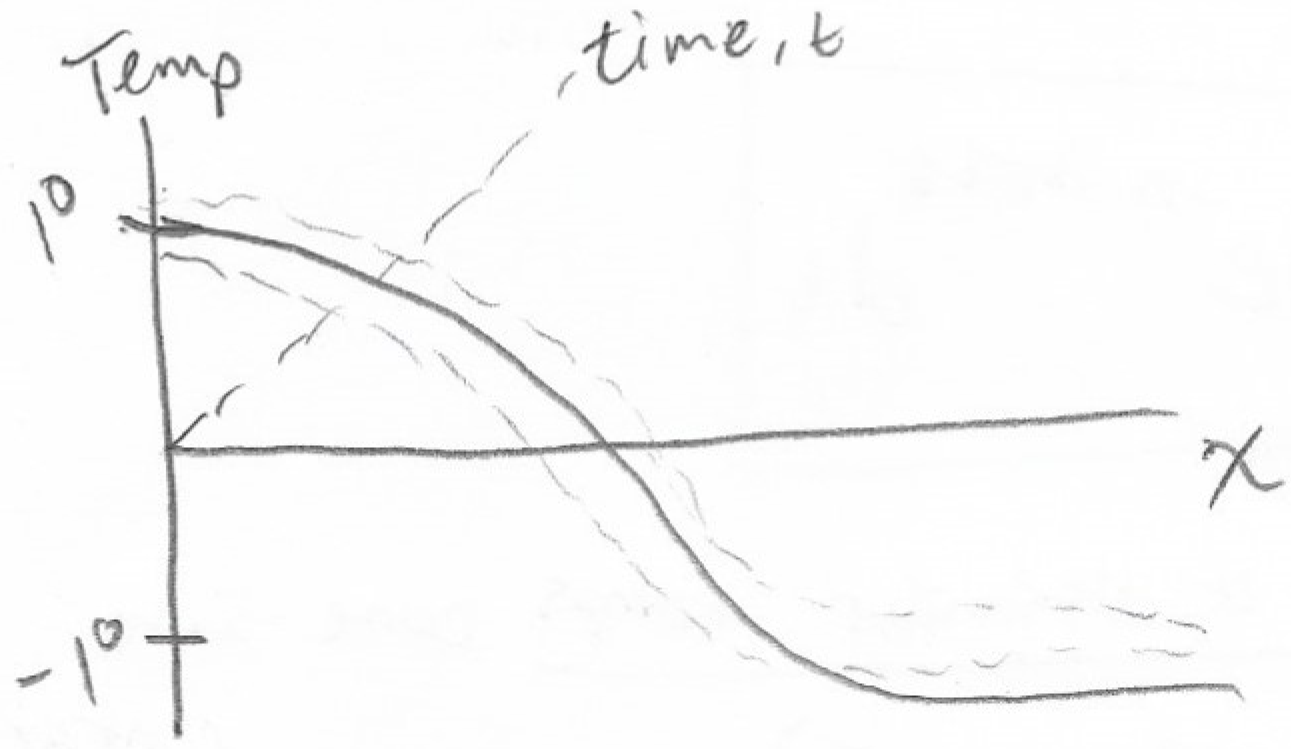
↳ $$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t)$$

↳ A linear solution simply means, that if we know two solutions, we add them up and the sum becomes a new solution.

↳ Cosine waves - we take the family of solutions, scale them, and add them to create a custom solution - which is a combination of cosine waves.

Fourier asked (boldly): How can I represent a function as a sum of sine waves?

Heat equation solutions: - take these, associated with the cosine waves, and add them all up - you will arrive at an exact solution - which describes how the function (temp. distribution + decay) will evolve over time.



$$T(x,t) = \frac{4}{\pi}\left(\frac{\cos(1\pi x)}{1}e^{-\alpha 1^2 t} - \frac{\cos(3\pi x)}{3}e^{-\alpha 3^2 t} + \frac{\cos(5\pi x)}{5}e^{-\alpha 5^2 t} \ldots\right.$$

* The key challenge is to find the coefficients.
  ↳ Pairs of vectors rotating in opposite directions.

  ↳ This is the broader (and what makes computations 'easier') case - rotating vectors composed of a real and imaginary part.

→ The heart + soul of the Fourier Series is the complex exponential,

$$e^{it}$$

Frequency $\omega = 1$; $\boxed{e^{2\pi i t}}$ ← Formula

↑

1 cycle/sec rotation.

* The Knobs that we get to adjust, are both the size and direction of the vectors, by multiplying each one by some constant. C_n.

$\omega = 1$

Formula: $c_1 e^{\omega 2\pi i t} = \underbrace{0.30 e^{(\pi/4)i}}_{C_1} e^{1 \cdot 2\pi i t}$

$$\boxed{\text{* General formula for } \quad C_n = \int_0^1 f(t) e^{-n \cdot 2\pi i t} \, dt}$$

Recall the step function when two rods of dissimilar temps come into contact. (see plot on pg. 5)

↳ To find the coefficients, we need to compute this integral: (over and over again until our function is refined)

$$C_n = \int_0^1 \text{step}(t) e^{-n \cdot 2\pi i t} \, dt = \int_0^{0.5} 1 \cdot e^{-n \cdot 2\pi i t} \, dt + \int_{0.5}^{1} -1 \cdot e^{-n \cdot 2\pi i t} \, dt$$