- Moments and Moment Generating Functions

- The *moments* (or *raw moments)* of a random variable or of a distribution are the expectations of the powers of the random variable which has the given distribution.

Moments

- If *X* is a random variable, the *rth moment* of *X,* usually denoted by μ'_r, is defined as

$$\mu'_r = \mathscr{E}[X^r]$$

if the expectation exists.

- Note that $\mu'_1 = \mathscr{E}[X] = \mu_X$, the mean of X.

Central moments

- If *X* is a random variable, the *rth central moment* of *X* about *a* is defined as $\mathscr{E}[(X-a)^r]$.
- If $a = \mu_X$, we have the rth central moment of X about μ_X, denoted by μ_r . which is

$$\mu_r = \mathscr{E}[(X-\mu_X)^r].$$

Note that

- $\mu_1 = \mathscr{E}[(X - \mu_X)] = 0$ and $\mu_2 = \mathscr{E}[(X - \mu_X)^2]$, the variance of X.

- Also note that all odd moments of X about μ_X are 0 if the density function of X is symmetrical about μ_X, provided such moments exist.

Moment Generating Functions (mgf)

- The moment generating function *φ(t)* of the random variable *X* is defined for all values *t* by

$$\phi(t) = E[e^{tX}]$$
$$= \begin{cases} \sum_{x} e^{tx} p(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & \text{if } X \text{ is continuous} \end{cases}$$

We call *φ(t)* the moment generating function

- because all of the moments of *X* can be obtained by successively differentiating *φ(t)*.
- For example,

$$\begin{aligned}\phi'(t) &= \frac{d}{dt}E[e^{tX}] \\ &= E\left[\frac{d}{dt}(e^{tX})\right] \\ &= E[Xe^{tX}]\end{aligned}$$

- Hence, $\phi'(0) = E[X]$

Similarly,

$$
\begin{aligned}
\phi''(t) &= \frac{d}{dt}\phi'(t) \\
&= \frac{d}{dt}E[Xe^{tX}] \\
&= E\left[\frac{d}{dt}(Xe^{tX})\right] \\
&= E[X^2 e^{tX}]
\end{aligned}
$$

- And so $\phi''(0) = E[X^2]$

- In general,

- the nth derivative of $\varphi(t)$ evaluated at $t = 0$ equals $E[X^n]$,

- that is,

$$\phi^n(0) = E[X^n], \qquad n \geq 1 \cdot$$

An important property

- An important property of moment generating functions is that the *moment generating function of the sum of independent random variables is just the product of the individual moment generating functions*.

To see this,

- suppose that X and Y are independent and have moment generating functions $\varphi_X(t)$ and $\varphi_Y(t)$, respectively.
- Then $\varphi_{X+Y}(t)$, the moment generating function of $X + Y$, is given by

$$\begin{aligned}
\phi_{X+Y}(t) &= E[e^{t(X+Y)}] \\
&= E[e^{tX}e^{tY}] \\
&= E[e^{tX}]E[e^{tY}] \text{, since } X \text{ and } Y \text{ are independent.} \\
&= \phi_X(t)\phi_Y(t)
\end{aligned}$$

- mgf of the Binomial Distribution with Parameters *n* and *p*

$$
\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\
&= (pe^t + 1 - p)^n
\end{aligned}
$$

Hence, $\phi'(t) = n(pe^t + 1 - p)^{n-1} pe^t$

and so $E[X] = \phi'(0) = np$

Differentiating a second time yields

$$\phi''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

and so

$$E[X^2] = \phi''(0) = n(n-1)p^2 + np$$

Thus, the variance of X is given

$$\begin{aligned}\mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= np(1-p) \quad \blacksquare\end{aligned}$$

mgf of the Poisson Distribution with Mean λ

$$
\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\
&= e^{-\lambda} e^{\lambda e^t} \\
&= \exp\{\lambda(e^t - 1)\}
\end{aligned}
$$

Differentiation yields

$$\phi'(t) = \lambda e^t \exp\{\lambda(e^t - 1)\},$$

$$\phi''(t) = (\lambda e^t)^2 \exp\{\lambda(e^t - 1)\} + \lambda e^t \exp\{\lambda(e^t - 1)\}$$

and so

$$\begin{aligned} E[X] &= \phi'(0) = \lambda, \\ E[X^2] &= \phi''(0) = \lambda^2 + \lambda, \\ \mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \lambda \end{aligned}$$

Thus, both the mean and the variance of the Poisson equal λ.