The Interplay between Green Product Production and Advertising Investment under Green Reputation

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***Abstract*—Establishing a green reputation reduces environmental impacts, gains recognition and increases production efficiency. We develop a stochastic optimal control model for monopolies and stochastic differential game models for duopoly and two-tier supply chains. These models involve the sales of green products, the green reputation of products, and comprehensive capital stock. The model is transformed into the Hamilton–Jacobi–Bellman (HJB) equation (system). By solving the HJB equation (system), we obtain the analytical results and draw the following meaningful management conclusions: (i) Under a green reputation, the advertising and production investments of green products mutually promote each other when the firm's emissions decrease and the sales margin increases. (ii) Improving the green supply chain structure will increase the capital stock, green reputation, and sales of firms. However, as we modify our assumptions, the model becomes more complex, and analytical results cannot be obtained. We hence develop a machine learning algorithm based on the deep Galerkin method (DGM) to help solve it. We find that high sales profit margins can offset the investment constraints imposed by competitors' market presence and emission levels, thereby stimulating advertising investment. Finally, we use data from two electric vehicle firms to estimate our model and compare it with two econometric models. We show that our model performs better or equivalently in fitting. Our research aims to help engineering managers and firm executives use the green reputation to guide production planning and investment management.**

***Index Terms*—Green reputation, Equilibrium, Machine learning, Case study.**

Managerial Relevance Statement

In industrial engineering management, our research shows how emission levels impact green reputation. This offers a new way to assess product sustainability. Lower emissions increase green reputation and profits, creating a virtuous cycle. Emission levels are key in engineering production and promote a complementary relationship between advertising and production investments. Collaboration between manufacturers and retailers amplifies green reputation and profits, attracting more investments. Enhancing green reputation and reducing emissions strengthens cooperation and increases profits for all stakeholders. Engineering managers should embrace a green reputation to guide sustainable manufacturing practices. An increase in environmental disclosures establishes a robust green reputation, which attracts investments. Integrating sustainability into operations appeals to conscientious investors. This lowers financing costs and secures more advertising and production investments. Managers should also lead in green supply chain management. They should collaborate with suppliers to use eco-friendly materials and reduce waste. This boosts a firm's and supply chain's competitiveness. Governments should enforce strict environmental disclosure regulations. They should encourage diverse green supply chain practices, maintain market vitality, and foster green industrial chains. Engineering managers should stay informed about regulations and advocate for sustainable practices.

I. Introduction

Currently, the green economy, green reputation, and green supply chain have become critical priorities in engineering management and sustainable operations. Environmental challenges, particularly carbon emissions and climate change, are escalating. Enterprises are increasingly recognizing the operational and strategic value of integrating green practices into production systems, supply chain design, and marketing engineering [1], [2]. In engineering management terms, green reputation represents a quantifiable metric of a firm's environmental performance, reflecting its competence in implementing low-carbon technologies (e.g., energy-efficient production lines, circular material flows) and stakeholder engagement in sustainability initiatives [3]–[5]. Improving a green reputation not only helps reduce the ecological footprint through technical innovations (e.g., carbon capture in manufacturing) but also enhances market competitiveness by engineering branding strategies that align with consumer preferences [6], [7]. The 2023 China Consumer Trends Report highlighted that 73.8% of consumers prioritize products with certified low-carbon engineering attributes (e.g., ISO 14067 carbon footprint standards), with post-1990s consumers willing to pay a premium for brands demonstrating engineering transparency in sustainability practices [8]. The entry of green products into the market necessitates investment, which can be categorized into advertising investment and production investments [9], [10]. For example, electric vehicle (EV) manufacturer BYD leveraged digital engineering tools (e.g., media targeting driven by

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Artificial Intelligence (AI)) to collaborate with industry influencers, promoting their low-carbon powertrain technologies and achieving a top-10 global ranking in automotive brand value[1]. Similarly, Tesla increased its 2023 advertising spending by 37% to highlight battery engineering innovations (e.g., 4680 cell technology) and recycling capabilities, reinforcing its green brand image[2]. Moreover, IKEA allocated $100 million to multichannel engineering campaigns (e.g., VR-based sustainability showcases) to promote circular product design, driving 5.7% sales growth in 2023[3]. We refer to advertising investments as any "green activity" that contributes to carbon protection, as it helps firms establish or enhance their environmental image [3]. In terms of production investments, BYD's new energy vehicles have cumulatively reduced CO_2 emissions by more than 5.48 million tons[4], and Tesla has achieved a 92% recycling rate for raw battery materials[5]. Additionally, IKEA invests in renewable energy, resource-efficient processes, and the implementation of circular economy principles in product design and business operations[6]. In 2023, IKEA's sales reached 41.7 billion euros, an increase of 5.7% compared with the previous year[7]. We refer to production investments as efforts required to increase the low-carbon intensity of product manufacturing, and to expand the capital stock [11]. These companies' green efforts clearly integrate advertising investments and production investments, both of which are essential. Greenwashing risk may come from the dependence of advertising, and focusing exclusively on production may obscure the green attributes of products in consumers' opinion. Therefore, how to manage a firm's "green reputation" well under these investments has become crucial for participants.

In engineering management, sustainability has been validated as a key driver of innovation [12], [13]. The quality of a product is directly related to consumer experience and market competitiveness, whereas green reputation is related to the firm's image and sustainable development. The inherent connection between green reputation and investments in advertising and production lies in the fact that these investments are crucial strategies for businesses. Additionally, these investments can help build and maintain a competitive green reputation in the market. Particularly in engineering management, firms enhance the environmental performance of their products through production investment to meet market demand for green goods. They use advertising investments to promote their green practices, thereby increasing market competitiveness. These strategies not only help reduce pollution but also attract more consumers and investors, which leads to tax benefits and financing advantages. Therefore, firms must balance investments in product development and green reputation, primarily through advertising and production investments, to achieve both economic profits and environmental responsibility, which is essential for sustainable manufacturing and engineering management. For example, BYD must focus not only on advertising investments to attract consumers concerned with green travel but also on production investments to enhance the low-carbon performance of its products. This dual investment strategy effectively elevates the firm's green reputation and economic benefits. Moreover, it would be meaningful to quantify the carbon emissions of products in a standardized manner so that this practice affects the production process and green reputation. This can maximize the role of a green reputation to achieve the maximization of the firm's profits [14], [15]. Consumers not only purchase green products but also value their green reputation, which is one of the contributions of this study.

Analyzing various market structures is essential because of their significant differences. We consider a firm that focuses on producing green products. The firm believes that strategic investments in advertising and production can build a strong green reputation. This mitigates the environmental impact while helping the firm stand out in a competitive market to gain consumer recognition and higher profits. Initially, the firm, leveraging its first-mover advantage, becomes the sole supplier in the green product market, establishing a monopoly. Given the high costs of green technology R&D and low market awareness, the firm heavily invests in advertising to promote its products' environmental benefits. The firm also optimizes production processes through increased investment to reduce carbon emissions and enhance its green reputation. As demand grows and technology spreads, another firm, attracted by the potential of green products, enters the market with advanced green technologies and innovative business models, breaking the monopoly and creating a duopoly. The two firms recognize that cooperation could increase the entire industry's green reputation, thereby creating greater market opportunities and profits for both firms. Thus, they begin to explore collaborative possibilities for a win–win situation. Furthermore, driven by market expansion needs, the desire to optimize supply chain management, and the benefits of the division of labor, the firm realizes that it cannot achieve further breakthroughs alone. Thus, it introduces downstream retailers to form a two-tier supply chain. In this structure, the manufacturer focuses on production investment to improve the green attributes and quality of the products, whereas the retailers take on advertising investment to promote the products' advantages and increase market awareness. Analyzing these three scenarios provides targeted insights into the strategic management of green reputation for firms operating in monopolistic, competitive, and supply chain environments, highlighting the unique challenges and opportunities that each market structure presents.

[1]BYD Ranks among Top 10 Kantar BrandZ Most Valuable Global Brands in Auto Category for 2023. https://www.byd.com/us/news-list/BYD-Ranks-among-Top-10-Kantar-BrandZ-Most-Valuable-Global-Brands-in-Auto-Category-2023

[2]Tesla Advertising Spend Estimated at $6.5M in 2023, Says Report. https://teslanorth.com/2024/03/29/tesla-advertising-spend-6-5-million-2023/

[3]$100 Million Ikea Begins Global Media Audit. https://www.adweek.com/brand-marketing/100-mil-ikea-begins-global-media-audit-68107

[4]BYD's november NEV sales reach 230K units. https://pandaily.com/byds-november-nev-sales-reach-230k-units/

[5]Tesla Recycles 92% Of Battery Raw Materials, According To Impact Report 2021. https://www.torquenews.com/15475/tesla-recycles-92-battery-raw-materials-according-impact-report-2021

[6]IKEA: Sustainability Case Study. https://greenheroglobal.com/en/news-interviews/news/ikea-sustainability-case-study

[7]IKEA continues to support people to live a better everyday life at home–delivering a growth of 5.7%. https://www.ikea.com/us/en/newsroom/corporate-news/ikea-continues-to-support-people-to-live-a-better-everyday-life-at-home-delivering-a-growth-of-5-7-pub1b0865f0

Firms face several key challenges in advertising and production investments for green products. First, due to consumers' skepticism about greenwashing, firms must not only invest in advertising but also support their claims with substantial production investments. Second, the high costs of both advertising and production investments can be prohibitive, particularly for smaller firms, as they require significant resources to achieve meaningful impact. Finally, measuring returns is complex, as the long-term benefits related to brand reputation and sustainable production are difficult to quantify directly. Balancing these investments is crucial to avoid accusations of insincerity while ensuring that green efforts are recognized by consumers.

Given the importance of the link between green production and green reputation, as well as between advertising and production investments, firms and managers face the following challenges:

1) How can firms use the relationship between green products and green reputation to control emissions and achieve steady profits?
2) How can firms optimize their advertising and production investments to maximize profits while maintaining a green reputation and meeting sustainable manufacturing standards?
3) How can firms' managers adjust their investments and relationships with other firms to maintain a strong green reputation and maximize profits amidst complex market conditions?

To answer the above questions, we develop a stochastic optimal control model for a monopoly and stochastic differential game models for a duopoly and two-tier supply chain. Static models offer only a static description of the system at a particular instant, failing to capture its dynamic nature. In contrast, our models capture the dynamic and uncertain aspects of the system by modeling the continuous temporal evolution of variables, accounting for real-world randomness and fluctuations. This is crucial when analyzing complex systems involving elements such as green reputation, market demand, and firm investments, which are influenced by changing internal and external factors. By leveraging stochastic differential equation (SDE) models, we can comprehensively study the impact of green reputation on different market structures and provide targeted strategic recommendations for firms. "Green reputation" in this paper is precisely defined as the cumulative outcome of a firm's green production practices and efforts toward achieving sustainable objectives. It is incorporated into our model to quantify the impact of environmental stewardship on a firm's performance and to assess how it influences investment strategies in advertising and production. Consequently, this study provides a comprehensive framework for understanding the dynamics of sustainable industrial production management. Our models consist of the sales of green products, the green reputation of products, comprehensive capital stock, and advertising and production investments. By establishing a controlled diffusion model to describe the evolution of product green reputation and sales, we transform the model into the Hamilton–Jacobi–Bellman (HJB) equation (system). By solving the HJB equation (system), we obtain analytical results and draw some meaningful management conclusions.

Some necessary assumptions are made to analytically solve the models. However, when the settings change, the derived partial differential equation (system) are high-dimensional, traditional numerical methods, such as finite difference method and finite element method, have failed. To address this issue, a machine learning algorithm based on the deep Galerkin method (DGM) is introduced. The specific structure and principle of the algorithm are detailed in Online Supplementary Appendix M. Here, we provide an overview of the fundamental principles, methodologies, and theoretical underpinnings of the algorithm. The DGM algorithm is a novel approach that leverages deep neural networks to approximate solutions to partial differential equations within the Galerkin framework. It operates by formulating a loss function that incorporates the residual of the partial differential equations (PDEs) and the boundary conditions. The neural network is then trained to minimize this loss function, effectively learning the solution to the PDEs. This method is particularly advantageous for high-dimensional problems, as it avoids the exponential increase in computational complexity associated with traditional numerical methods when the number of dimensions increases. After extensive neural network training with the DGM algorithm, it was verified to fit the PDEs with the previous analytical solutions, stabilizing after sufficient training. However, for nonlinear PDE systems, the convergence accuracy is lower than that for linear systems, and high-dimensional problem focus demands more computational resources and time. Despite these challenges, we have demonstrated the robustness of the algorithm through numerical analysis under various assumptions, as detailed in Online Supplementary Appendix M. This robustness ensures that our model can withstand a variety of operational challenges and offer reliable support in practical applications, particularly in the context of engineering production and supply chain management.

In addition, we use the maximum likelihood etimation (MLE) method and data from two electric vehicle firms, BYD and Tesla, to estimate our model. The data include carbon credit data under the Dual Credit Policy (Passenger Cars Corporate Average Fuel Consumption and New Energy Vehicle Credit Regulation)[8] for passenger cars formulated by the Chinese government. Both BYD and Tesla fall under the supervision of this policy and have gained tangible benefits from carbon credit trading and enhanced market recognition. Carbon credit trading under the Dual Credit Policy is essentially in line with carbon emissions trading. Like carbon tax and carbon quota policies, this policy plays a crucial role in quantifying a firm's green reputation. By imposing constraints and incentives on carbon emissions, the Dual Credit Policy enables more accurate measurement of a firm's environmental performance. Given the positive impact of the Dual Credit Policy on BYD and Tesla, incorporating carbon quota policies into the model is essential. This is because such policies not

[8]The New Energy Vehicle Credits (NEVC) data are based on the requirements of the "Passenger Cars Corporate Average Fuel Consumption and New Energy Vehicle Credit Regulation." https://wap.miit.gov.cn/zwgk/zcwj/wjfb/gg/art/2020/art_c62e6bad9d05490db9cc73d5167e0c1a.htm.

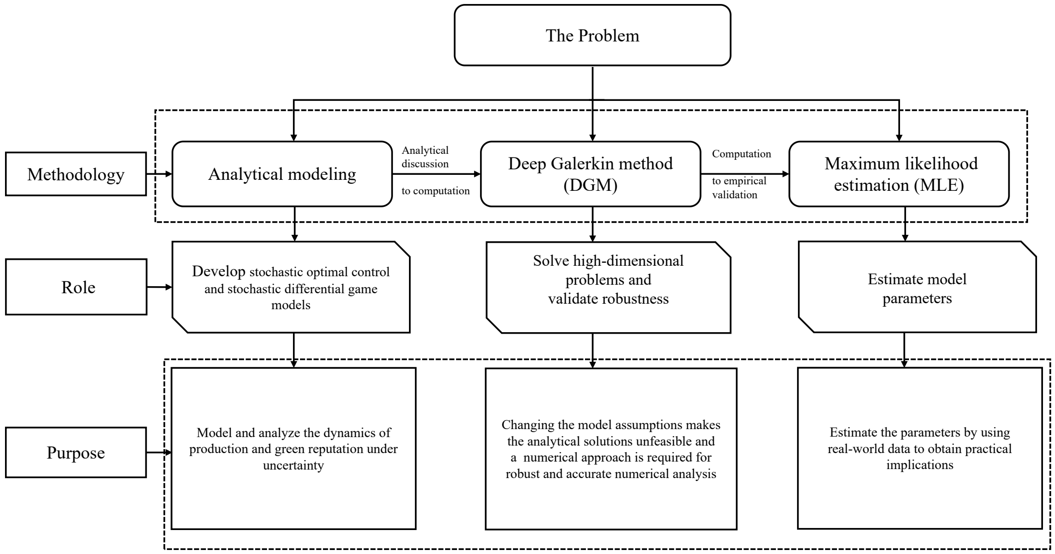

Fig. 1. Methodology illustration for our research methods.

only contribute to sustainable production but also help firms better demonstrate their green image. We compare the model estimated via MLE with econometric models including the autoregressive moving average (ARMA) [16] and the generalized autoregressive conditional heteroskedasticity (GARCH) [17] models, and show that our SDE models achieve better or equivalent fitting performance. The conclusions drawn from the empirical analysis confirm the results of our theoretical models.

Figure 1 outlines the structure of this paper. Stochastic optimal control and stochastic differential game models, with analytical solutions, are used to examine production and green reputation dynamics under uncertainty. Changing the model assumptions makes the analytical solutions unfeasible. A deep Galerkin method is developed to numerically solve the model and to show the robustness. By using real-world data and maximum likelihood estimation (MLE), we accurately estimate model parameters to enhance the practical significance. Through the obtained analytical results and the proposed machine learning algorithm, we gain some valuable management insights:

1) Green reputation, as a novel perspective in production practice, aids engineering managers and helps foster sustainable manufacturing. High (low) emission levels lead to reduced (increased) production and advertising investments, thereby decreasing (increasing) green reputation and sales, which differs from previous studies that focused solely on reputation models in single-scenario settings [3], [18], [19].
2) There is a complementary relationship between advertising and production investment when enjoying positive emission benefits. With the improvement of green reputation, complementarity becomes more intensive, increasing the value of investment for firms. This dynamic insight, which is unavailable in static models [18], [20], helps managers adjust investment levels in dynamic environments.
3) Improving the green supply chain structure, such as increasing the number of retailers and promoting cooperation between manufacturers and retailers, will increase the green reputation and profits of firms and promote their advertising and production investment. Furthermore, improving green reputation and reducing product emission levels will strengthen cooperation and increase profits for all members. These findings are absent in earlier studies that focused solely on individual green production structures [5], [21], [22].

Our research makes several contributions. First, we quantify carbon emissions in a standardized manner to examine production processes and green reputation. We find that a green reputation can bring positive benefits to firm profits at low emission levels but negative benefits at high emission levels. Engineering managers can use green reputation as a new perspective to guide production planning and investment management. Second, we establish stochastic models to analyze different scenarios: a stochastic optimal control model for a monopoly and stochastic differential game models for a duopoly and a two-tier supply chain. Importantly, we find that a high green reputation can enhance the complementarity between advertising and production investments, offering practical guidance for organizations in adjusting their investment levels. Third, when we modify our assumptions, the model becomes a complex high-dimensional problem. Traditional numerical methods fail because of the curse of dimensionality. To address this, we propose a machine learning method that eliminates the need for mesh construction in finite element analysis. This method is also designed to validate the robustness of the model under varying assumptions. By systematically altering the model's parameters and conditions, we analyze how changes impact the model's performance and stability. Finally, we use data from BYD and Tesla to estimate our model and compare it with econometric models, including ARMA and GARCH. Our stochastic differential equation (SDE) model achieves better or equivalent fitting performance, making it more practical.

The remaining parts of the paper are arranged as follows: Section II provides a summary of the pertinent literature. Our proposed models are presented in Sections III, IV and V under different situations, where we also analyze the equilibrium. Section VI compares the results. The case study is shown in Section VII. Finally, Section VIII concludes the paper. All the proofs are included in the Online Supplementary Appendix for conciseness and clarity.

II. Literature Review

This study is closely related to three main streams of literature: (A) green reputation, (B) impact overview of green products, and (C) differential games in green industries.

A. Green Reputation

The concept of green reputation has evolved in conjunction with corporate social responsibility (CSR) and has progressively matured into a quantitative environmental, social, and governance (ESG) system [23]. As a result, minimizing pollution-related behaviors and enhancing green reputation have emerged as prominent topics in recent engineering management discourses, particularly in optimizing sustainable production systems and supply chain operations. In the realm of industrial engineering production, Calzolari et al. [18] conclude that sustainable operations management can achieve a greater social and environmental reputation, highlighting the critical role of operations management in implementing sustainable development strategies. Sun et al. [24] explore the impact of leadership reputation on green behavior in cross-regional manufacturing networks and note that cultural factors and regional differences in leadership styles can affect the integration of green reputation into production planning and quality management systems for companies and products. In terms of engineering policy and decision-making, Wang et al. [25] examine the impact of fees on producers vs. consumers of electronics, revealing significant policy differences and stakeholder impacts on the basis of the product type and fee structure. Martin and Rubio [26] design a dynamic environmental tax on the basis of a monopolistic firm's emission intensity and find that regulatory efforts in innovation can promote cleaner technologies and sustainable firm governance.

An improvement in green reputation benefits firms. Technology involving green skills not only mitigates environmental stress and public fear but also improves a firm's reputation [27]. The research of Saha et al. [20] shows that green management enables companies to obtain more financing, and green innovation significantly enhances a firm's reputation by reducing pollution. Moreover, green reputation increases investors' confidence in a firm's green bonds, thereby attracting more investment and reducing financing costs [28]. Shahzad et al. [29] report that stakeholder pressure significantly impacts a company's environmental performance, whereas social reputation and green credit ratings positively moderate this influence and enhance managerial efficiency. Hou et al. [30] indicate that social performance (green reputation) plays a mediating role between green manufacturing and economic performance.

In the realm of engineering management, there is a dearth of research specifically examining the influence of green reputation on firms' investment strategies. In our study, we categorize firm investments in advertising and production and quantify the impact of product emissions on the production process and green reputation. Engineering managers and firm executives can leverage green reputation as a new perspective to guide production planning and investment management. We aim to investigate how green reputation influences firms' investment and sustainable manufacturing practices.

B. Impact Overview of Green Products

For a long time, nongreen products have been extensively studied in the field of engineering management. Chenavaz [31] demonstrates that product innovation and production optimization can enhance the cumulative product quality and a firm's expected profits. Ahn et al. [3] analyze the relationship between green product strategies and corporate reputation, revealing how investments in green innovation can enhance the firm's competitiveness. Ye et al. [32] explore how dynamic consumer attention drives iterative innovation via green products, highlighting the impact of consumer behavior and optimal strategies under different technology conditions. The concept of green products and their impact on environmental performance has garnered significant attention in the field of engineering management. Research has shown that the choices of eco-conscious consumers are influenced by a multitude of dynamics and have broader implications for sustainable manufacturing in firms [1]. Sustainable transformation in firms and organizations not only optimizes production processes but also enhances societal recognition [33]. Green products, often synonymous with eco-friendly or sustainable products, represent a response to increasing concerns about environmental degradation and climate change. Scholars have also researched groups related to green products. Recent research indicates that refurbished products, as part of a green production strategy, increase production enthusiasm and are favored by consumers for their superior cost-effectiveness. Tian and Bai [34] examine how low-carbon supply chain (LCSC) companies mitigate their carbon footprint through direct emission reduction (DER) efforts.

Products with a green reputation will also have an impact on manufacturers and supply chains. Zhou et al. [6] indicate that a green reputation may have a positive effect on firm profits and production. Word of mouth is an important means to promote green products and consumer awareness [35]. Liu et al. [36] reveal the value of green products under carbon-neutral activities and offer useful insights for operations managers in developing countries. To explore strategic evolution mechanisms, He et al. [5] study stakeholders (enterprises, investors, rating agencies) in ESG disclosure. Bian and Xiao [37] study a stylized game model of a dual-channel supply chain to explore strategies such as complementary return freight insurance (CRFI) under green acknowledgment.

Although capital allocation for green product manufacturing is pivotal for future market expansion in engineering management, existing studies incorporating green reputation into such investment decisions remain scarce, particularly when juxtaposed with research on conventional product portfolios. This study bridges this void by investigating the high-dimensional product investment game through the lens of green reputation—a nascent theme in the engineering economics literature. Unlike prior works that isolated green product attributes, our research integrates green reputation with carbon emission control metrics, delivering a holistic analysis aligned with engineering system optimization principles. We formulate a multifirm differential game model and convert it into a high-dimensional Hamilton–Jacobi–Bellman (HJB) equation system to deduce feedback equilibrium solutions. Empirical results demonstrate that reducing product carbon footprints not only elevates corporate green reputation but also fosters synergy between advertising strategies and production investment allocations, providing actionable insights for engineering managers in sustainable product development planning.

C. Differential Game In Green Industries

In the engineering management literature on green industry supply chain production, the extant research has predominantly employed static modeling frameworks [38], [39]. While these models offer critical insights, they characterize only the green product production system at discrete temporal nodes. The differential game formulation for green product systems encompasses multistakeholder dynamics, integrating consumers, original equipment manufacturers (OEMs), retail distributors, and regulatory authorities as active participants. Niu et al. [40] report that manufacturers should avoid outsourcing to contract manufacturers focused on production, especially if they have a green advantage. Collaborative relationships can enhance profitability and drive innovation. Chen et al. [22] study value cocreation in low-carbon tech innovation ecosystems via a triple helix framework and a three-party differential game model. Sun et al. [21] describe two pollution games where the government offers green funds and technologies and where firms select optimal production plans in response to government incentives, with the shared goal of emission reduction. Diebel et al. [41] study the impact of green manufacturing on firm performance via a game theoretic approach. The results show that retailer leadership aids customer satisfaction and that cost-sharing contracts work only under supplier leadership. The above three different game scenarios indicate that the recognition of green products can bring advantages to the industry.

With respect to green supply chain engineering management, Wu et al. [42] report that green innovation positively impacts logistics service providers (LSPs)' supply base stability, but radical changes may harm it. Executive attention and expertise influence these effects. Kang et al. [43] study low-carbon production in a two-stage supply chain via an uncertain differential game and report that cooperation yields greater emission reduction and profit. Wang et al. [44] examine green collaborative innovation drivers, finding that niche overlap and breadth increase it. Digital transformation and female executives moderate these relationships. We observe limited studies [13], [40], [44] on green reputation—a critical metric gaining prominence amid evolving consumer sustainability preferences and corporate ESG commitments. This study investigates the correlation between supply chain green reputation indices and stakeholder investment behaviors, with the objectives of optimizing interenterprise collaboration mechanisms, enhancing multiechelon profit margins, and deriving actionable management heuristics for sustainable supply chain engineering.

In engineering management circles, the existing corpus on supply chain production and operations has focused predominantly on static modeling frameworks, which typically offer a restricted and instantaneous view of system dynamics. Against this backdrop, few studies have applied differential game theory to supply chain production and management while considering green reputation. Few studies have focused on high-dimensional differential games from the vantage point of green product reputation. This study makes a substantial contribution to dynamic modeling by dissecting the intricate interactions among green reputation indices, investment synergy effects, and supply chain architecture. The proposed approach captures the continuous evolution of these variables, factoring in the inherent stochasticity of real-world supply chain environments. Empirical findings reveal that optimizing supply chain architecture can increase the scale of advertising and production investment portfolios. Moreover, enhancing green reputation metrics can foster complementarity between these investment streams, thereby strengthening cross-echelon collaboration and holistic supply chain performance. These insights offer actionable guidance for engineering managers to optimize green production investment strategies, strengthen interenterprise partnerships, and advance sustainable high-efficiency operations. Table I clearly shows the relevance and research areas of our article.

III. Monopoly

We first examine the monopoly scenario. In the early days of the green product market, the firm was a pioneer. Owing to the high costs of green technology development and application, coupled with low consumer awareness of green products, the firm remained the sole supplier for an extended period, thereby establishing a monopoly. During this period, the firm heavily invested in advertising to promote the environmental benefits of its products and in production to optimize processes, reduce carbon emissions, and build a strong green reputation. In the monopoly scenario, we focus on how government environmental policies and the firm's reputation directly influence its decision-making. We assume that the demand of the firm (denoted as $D(t)$) will be influenced by green reputation (denoted as $G(t)$). This is because, currently, consumers pay attention to the reputation of products when purchasing [45], especially in terms of environmental protection [4]. A good reputation will directly invreasing more demand, whereas a bad reputation will cause losses to the firm. Thus, the demand is governed by the stochastic differential equation [19], [46]:

TABLE I
THE RELEVANCE AND POSITION OF THIS PAPER

Paper	Green Production and Operations	Green Reputation in Engineering Management	Differential Game	Optimal Advertising and Product Investments	Complementarity /Substitutability of Investments
Ahn et al.(2024) [3]				√	√
He et al. (2024) [5]	√	√	√		
Calzolari et al. (2024) [18]	√	√			√
Saha et al. (2024) [20]		√		√	
Sun et al. (2025) [21]			√	√	
Chen et al. (2025) [22]	√			√	
Niu et al. (2024) [40]	√		√		
This paper	√	√	√	√	√

$$dD(t) = (\alpha(G(t) - \mu) + \gamma - \delta_D D(t))dt + \sigma_1(D(t))dW_1(t). \tag{1}$$

In Equation (1), the parameter α represents the impact of reputation generated by a firm in the production of green products on demand and can also indicate consumer sensitivity. This implies that a higher α indicates a stronger consumer preference for green products with superior reputation, or greater aversion to firms with severely poor reputations. The term $(G(t) - \mu)$ is a component representing reputation to capture the impact of product reputation on product demand, where μ represents the consumers' benchmark of product reputation. We have also revised the form of this item in Online Supplementary Appendix M to verify the robustness of our model. The trend parameter γ accounts for the impact of factors unrelated to green reputation on demand (e.g., the quality, experience, and after-sales service of products). The third term $\delta_D D(t)$ in Equation (1) represents the decay of demand, with the parameter δ_D indicating the degree of decay. This term reflects the natural decline in demand and indicates that demand will continuously decrease over time because of product obsolescence and forgetting [47]. The increment of the Wiener process $dW_1(t)$ represents the differential of Wiener motion, which satisfies a normal distribution with a zero mean and variance of dt. All randomness is captured by the stochastic term, with $\sigma_1(D(t))$ being a state-dependent function that determines the magnitude.

Considering environmental pollution during production, we use $e(t)$ to represent the pollutant emission (flow) as an inevitable byproduct of production. We assume $e(t) = \beta D(t)$, where $\beta > 0$ is the emission level per unit product. Parameter $\bar{e}$ can be interpreted as an emission standard, which is an objective level that regulatory agencies accept. For tools based on market regulation, assuming that regulators use a given ratio $\tau \geq 0$, taxation (subsidy) will be imposed on emissions above (below) the standard of firms. The quantity $\tau(\bar{e} - e(t))$ then represents the revenue of a firm if the pollution is below the emission standard (i.e., $e(t) < \bar{e}$); otherwise, it is a cost. For example, the European Union implements a carbon emission trading system that allows firms to trade carbon quotas between themselves[9]. Firms that exceed their quotas need to purchase more, whereas those with remaining quotas can earn profits by selling.

We assume that firms disclose their emissions to the public [48] to build a green reputation [49]. Many listed firms of green products, such as Tesla[10] and BYD[11], release their own environmental responsibility reports, which affects the public's perception of their reputation. Therefore, it is necessary to consider green reputation as a variable in our research. Following the approach of Nerlove and Arrow [50], we assume that a firm can increase its green reputation by advertising, denoted as $a(t)$, which can be explained not only as the firm's advertising investment but also more broadly as any "green activity" that contributes to carbon conservation, as it helps the firm establish or improve its environmental image. For example, BYD is committed to reducing carbon emissions throughout the entire production and supply chain process, which promotes its carbon neutrality planning and green production. Therefore, the green reputation of products $G(t)$ can be modeled via following stochastic differential equation:

$$\begin{aligned} dG(t) =& (a(t) f(K(t)) + \varphi(\bar{e} - e(t)) - \delta_G G(t))dt \\ &+ \sigma_2(G(t))dW_2(t). \end{aligned} \tag{2}$$

The introduction of green reputation links demand and production more closely. In Equation (2), the implicit function $f(K(t))$ is an increasing function, which is defined as the effect function of capital stock on reputation. To obtain the analytical results, we use $f(K(t)) = \sqrt{lK(t)}$ to represent it here, and we will also use numerical methods to calculate other forms in Online Supplementary Appendix M. Parameter l captures the impact of comprehensive capital stock on the effectiveness of green product advertising. Function $\sqrt{lK(t)}$ represents the intensity of advertising and environmental activities. The term $(a(t)\sqrt{lK(t)})$ aims to depict how a firm expands in scale and enhances advertising as capital stock grows during investment. The square root structure captures the diminishing marginal cost of capital stock on advertising, as documented in prior literature [51], [52]. Parameter δ_G is

[9] European Union (EU) Emissions Trading System (EU ETS). https://climate.ec.europa.eu/eu-action/eu-emissions-trading-system-eu-ets_en

[10] Tesla Environmental Responsibility Report. https://www.tesla.cn/impact

[11] BYD Environmental Responsibility Report. https://www.bydglobal.com/cn

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the decay rate, which characterizes the natural decay rate of green reputation over time. A firm's green reputation increases with $\varphi(\bar{e} - e(t))$, where $\varphi \geq 0$ is a sensitivity parameter capturing how reputation responds to the firm's environmental protection efforts. The environmental image of a firm can vary depending on the symbol of the term $(\bar{e} - e(t))$. We quantify the product's emissions in terms of its impact on the product's reputation [53], [54]. Here, $\bar{e}$ represents the emission standard accepted by regulatory agencies, and $e(t)$ denotes the emissions at time t. If $e(t) \leq \bar{e}$, it indicates that the firm is environmentally responsible, making it a well-received individual among the public. Conversely, if $\bar{e} < e(t)$, the firm will be regarded as a polluter by the public and society. In this case, the firm not only has to bear the corresponding pollution-related losses but also suffers damage to its business reputation. Similarly, $dW_2(t)$ represents the random component, the differential of the Wiener motion, which satisfies a normal distribution with zero mean and variance of dt, and $\sigma_2(x(t))$ is the magnitude of the stochastic component.

Prior research has examined a firm's dynamic model of production processes with energy as input. The input of energy can be generated from both traditional fossil fuels and renewable sources. We use the comprehensive capital stock $K(t)$ to represent the total energy input, whose evolution is characterized by the following stochastic differential equation [52]:

$$dK(t) = (I(t) - \delta_K K(t))dt + \sigma_3(K(t))dW_3(t), \quad (3)$$

where $I(t)$ denotes the production investment in the capital stock, and δ_K represents the rate of equipment depreciation. We use the stochastic (diffusion) term $dW_3(t)$ to represent the random component, which satisfies a normal distribution with zero mean and variance of dt, and $\sigma_3(x(t))$ is the magnitude of the stochastic component. We assume that $dW_1(t)$, $dW_2(t)$ and $dW_3(t)$ are independent, a simplification that does not affect the paper's main managerial insights. However, we recognize that these variables may be correlated in reality, particularly between demand and green reputation. To address this, we introduce correlation coefficients between these variables and conduct a detailed analysis in the Online Supplementary Appendix M. This analysis shows that the key management conclusions remain robust. This ensures that the core insights provided by our model are still valid and applicable in a more realistic setting where correlations exist. Notably, although comprehensive capital stock does not directly influence demand, it significantly impacts green reputation. There is a positive correlation between capital stock and green reputation, as well as between green reputation and demand. Thus, capital stock indirectly has a positive effect on demand.

The conceptual model of a firm's production-to-profitability pathway is depicted in Figure 2. In our model, managers must focus on the complementary or substitutable dynamics between advertising and production investments, and the effects of these investments on green reputation $(G(t))$ and capital stock $(K(t))$ to drive sales and profits. Advertising investment enhances green reputation by promoting eco-friendly initiatives, whereas increased production investment enables firms to adopt advanced green technologies, thereby improving product green credentials. Industrial managers should integrate these investments into decision-making and operations, ensuring that advertising aligns with actual eco-practices and that production investments streamline processes, enhance efficiency, and reduce energy consumption and costs. To build a competitive edge, firms should leverage synergies between advertising and production investments to strengthen product competitiveness and capture a larger market share.

We consider that a firm aims to maximize total discounted profit within the planned scope and specified emissions. The costs of two types of control efforts, advertising investment effort and production investment effort, are represented as $C(a)$ and $C(I)$, respectively. These costs are modeled as quadratic functions, i.e., $\frac{1}{2}a^2(t)$ and $\frac{1}{2}I^2(t)$, to characterize the increase in marginal costs, which has been applied in many studies [45], [49], [55]. Consequently, the profit of the firm at time t is modeled as follows:

$$\max_{I(t),a(t)} \int_0^\infty e^{-rt} \left\{ \eta D(t) + \tau(\bar{e} - e(t)) - \frac{1}{2}a^2(t) - \frac{1}{2}I^2(t) \right\} dt, \quad (4)$$

where η represents the sales margin and r represents the discount rate. Parameter τ represents the taxation (subsidy) imposed on emissions that are above (below) the regulatory standard. We define $\eta - \tau\beta$ as the emission-benefit indicator per unit product, which is the sales margin minus the emission costs and will become an important parameter in future discussions. The term $e(t) = \beta D(t)$ represents the total product emissions. The term $[(\eta - \tau\beta)D(t) + \tau\bar{e}]$ captures the firm's revenue, including carbon-tax subsidies, over the interval dt. The total profit is calculated by integrating the profit over the time interval from $t = 0$ to $t = \infty$. To clearly illustrate the meanings of the variables and parameters, Table II summarizes the definitions of all the parameters.

Considering the stochastic differential equations satisfied by the three state variables and the value function, we express the stochastic optimal control problem as follows:

$$\begin{aligned}
&\max_{I(t),a(t)} \int_0^\infty e^{-rt} \left\{ \eta D(t) + \tau(\bar{e} - e(t)) - \frac{1}{2}a^2(t) - \frac{1}{2}I^2(t) \right\} dt \\
&\quad s,t \quad dD(t) = (\alpha(G(t) - \mu) + \gamma - \delta_D D(t))dt \\
&\qquad\qquad + \sigma_1(D(t))dW_1(t), \\
&\qquad\quad dG(t) = (a(t)\sqrt{lK(t)} + \varphi(\bar{e} - \beta D(t)) - \delta_G G(t))dt \\
&\qquad\qquad + \sigma_2(G(t))dW_2(t), \\
&\qquad\quad dK(t) = (I(t) - \delta_K K(t))dt + \sigma_3(K(t))dW_3(t).
\end{aligned}$$

To solve the above stochastic optimal control problem, we use the Itô lemma and derive the Hamilton–Jacobi–Bellman (HJB) equation as follows:

$$\begin{aligned}
rV = \max_{a(t),I(t)} \Big\{ & [\alpha(G(t) - \mu) + \gamma - \delta_D D(t)] V_D + \frac{1}{2}\sigma_1^2(D(t))V_{DD} \\
& + \left(a(t)\sqrt{lK(t)} + \varphi(\bar{e} - \beta D(t)) - \delta_G G(t) \right) V_G \\
& + \frac{1}{2}\sigma_2^2(G(t))V_{GG} + (I(t) - \delta_K K(t))V_K \\
& + \frac{1}{2}\sigma_3^2(K(t))V_{KK} \\
& + \left((\eta - \tau\beta)D(t) + \tau\bar{e} - \frac{1}{2}a^2(t) - \frac{1}{2}I(t)^2 \right) \Big\}.
\end{aligned} \quad (5)$$

We define V as the value function of the firm, thus V_D, V_G and V_K (V_{DD}, V_{GG} and V_{KK}) denote the first-order (second-

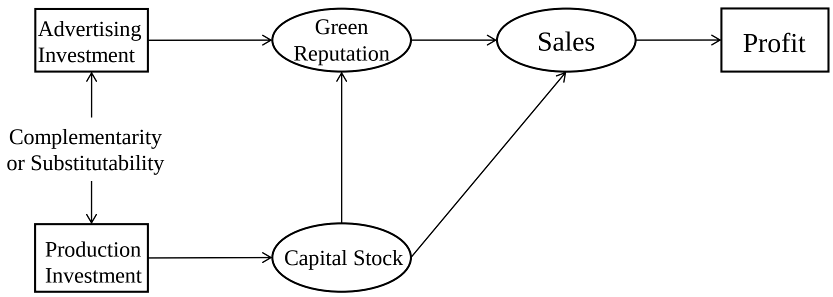

Fig. 2. Conceptual model.

TABLE II
NOTATIONS OF THE PARAMETERS AND VARIABLES

Notations	Definition
$i = 1, 2$	The number of players in a duopoly or two-tier supply chain
r	Discount factor
$\bar{e}$	Emission standard that regulatory agencies accept
τ	Taxation (subsidy) on emissions if above (below) the standard
l	Impact coefficient of capital stock on advertising investment
φ	Impact of firm's environmental protection on reputation
α, α_i	Consumer sensitivity of green reputation
μ, μ_i	Consumers' benchmark of different player's reputation
γ, γ_i	Impact of factors unrelated to green reputation on demand
$\delta_D, \delta_G, \delta_K$	Degree of decay on demand, reputation and comprehensive capital stock
β, β_i, β_M	Emission level per unit product of different players
η, η_i, M	Sales margin of different players
$e(t), e_i(t)$	pollutant emission (flow) of different players
$D(t), D_i(t)$	Demand at time t of firm i
$G(t)$	Reputation at time t of green products
$a(t), a_i(t)$	Advertising investment effort at time t
$I(t), I_i(t)$	Production investment effort at time t
$I_M(t)$	Production investment effort at time t of manufacture
$dW_i(t)$	Differential of Brownian motion for online reputation of different state variable
$\sigma_i(*)$	The magnitude of the fluctuation in Wiener motion

order) partial derivative operator with respect to variables D, G and K, respectively. Subsequently, taking the first-order conditions for $a(t)$ and $I(t)$, respectively, we obtain the optimal controls as follows:

$$a^*(t) = \sqrt{lK(t)}V_G, \quad I^*(t) = V_K.$$

By solving the above HJB equation, we obtain the results under a monopoly.

Lemma 3.1: Under a monopoly, the optimal advertising and production investments of the firm are as follows:

$$\begin{aligned} a^*(t) &= \frac{\sqrt{lK(t)}\alpha(\eta - \tau\beta)}{(r+\delta_G)(r+\delta_D)+\alpha\varphi\beta}, \\ I^* &= \frac{\alpha^2(\eta-\tau\beta)^2 l}{2(r+\delta_K)\left[(r+\delta_G)(r+\delta_D)+\alpha\varphi\beta\right]^2}. \end{aligned} \tag{6}$$

Proof of Lemma 3.1: See Online Supplementary Appendix A. □

The firm's value function is provided in Online Supplementary Appendix A. Lemma 3.1 shows that optimal advertising investment depends on capital stock, meaning that firm managers can increase product scale investments to increase both product output and green reputation. Notably, while firms pursue economic benefits, they also actively reduce pollutant emissions. However, the profit motive consistently outweighs emission reduction incentives [22], [56], reflecting fundamental firm behavior. Otherwise, a no-production strategy would be optimal. On the other hand, a nonnegative emission benefit ensures positive investment returns, leading us to make the following assumption:

Assumption 3.1: Under a monopoly, the emission benefit indicator is nonnegative, that is, $\eta - \tau\beta \geq 0$.

When the emission benefit indicator is nonnegative, the marginal effects on demand and reputation are positive, and the firm's profit increases with the increasing demand and reputation. Conversely, if $\eta - \tau\beta < 0$, the firm's profit will be negative, and shutdown is more reasonable than continuing production. The nonnegativity of the emission benefit indicator is vital in industrial engineering management, reflecting a firm's focus on environmental sustainability and its profitability. A practical example of a firm with a nonnegative emission benefit indicator is Tesla. Tesla's commitment to reducing emissions has led to significant market growth and profitability, demonstrating the business case for maintaining a nonnegative emission benefit indicator. This makes it hold a dominant position in the U.S. EV market, with its market share peaking at 79.4% in 2020[12]. In contrast, Hanma Technology faced a negative emission benefit indicator due to strict environmental regulations and decreasing demand for traditional fuel vehicles. As a result, Hanma Technology announced that it will stop producing traditional fuel vehicles by 2025[13]. In what follows, we study the impact of different parameters on advertising promotion and production investment under a monopoly in Online Supplementary Appendix B.

[12] 2020 U.S. Electric Car Sales: Tesla Captures 80% Of BEV Market. https://insideevs.com/news/487969/2020-us-electric-car-sales-tesla-share/

[13] Hanma Tech to Stop Production of Traditional Fuel Vehicles by 2025. https://www.chinatrucks.com/news/2022/0714/article_10149.html

Proposition 3.2: Under a monopoly, when there is a slight change in α in the neighborhood of $a^*(t)$ and I^*, the optimal control efforts $a^*(t)$ and I^* are complementary to each other.

Proof of Proposition 3.2: See Online Supplementary Appendix C. □

Proposition 3.2 shows that under a monopoly, the two investments vary in the same direction: efforts in one investment promote efforts in the other. The first insight is that increasing investment in green product promotion enhances green reputation, which in turn drives higher demand. To meet this demand, the firm increases production investment to produce more low-emission products. The second insight is that higher production investment improves product quality and environmental friendliness. To ensure sufficient product visibility, the firm increases advertising investment to stimulate consumer demand for high-quality, low-emission products. Thus, a stronger green reputation not only motivates higher production investment but also drives additional advertising investment.

Proposition 3.3: Under a monopoly, the steady states of comprehensive capital stock K_{ss}, reputation G_{ss}, demand D_{ss} and pollutant emission e_{ss} are as follows:

$$K_{ss} = \frac{I^*}{\delta_K}, \tag{7}$$

$$G_{ss} = \frac{\delta_D}{\delta_G\delta_D + \alpha\varphi\beta}\left(\frac{l\lambda I^*}{\delta_K} + \varphi\bar{e} - \frac{\varphi\beta(\gamma - \alpha\mu)}{\delta_D}\right), \tag{8}$$

$$\begin{aligned} D_{ss} =& \frac{\alpha}{\delta_D}\left(\frac{\delta_D}{\delta_G\delta_D + \alpha\varphi\beta}\left(\frac{l\lambda I^*}{\delta_K} + \varphi\bar{e} - \frac{\varphi\beta(\gamma - \alpha\mu)}{\delta_D}\right)\right) \\ &+ \frac{(\gamma - \alpha\mu)}{\delta_D}, \end{aligned} \tag{9}$$

$$e_{ss} = \beta D_{ss}, \tag{10}$$

where $\lambda = \frac{\alpha(\eta - \tau\beta)}{(r+\delta_G)(r+\delta_D)+\alpha\varphi\beta}$, $I^* = \frac{\alpha^2(\eta-\tau\beta)^2 l}{2(r+\delta_K)[(r+\delta_G)(r+\delta_D)+\alpha\varphi\beta]^2}$.

Proof of Proposition 3.3: See Online Supplementary Appendix D. □

From Equation (8), it is evident that when the firm enjoys a positive emission benefit $(\eta - \tau\beta) > 0$, the impact of comprehensive capital stock K_{ss} on product green reputation is positive in the steady state. This suggests that enhancing green reputation through increased production investment aligns with long-term sustainability goals by encouraging environmentally friendly production practices. Similarly, the positive effect of green reputation G_{ss} on demand D_{ss} indicates that building a strong green reputation can drive sustainable consumer behavior and market demand over time. Regarding the firm's environmental pollution, we apply the established function $e(t) = \beta D(t)$ to represent its steady state emission level as (10). This underscores the importance of achieving low emission levels in steady state operations, which is critical for meeting long-term environmental objectives. Decision-makers can leverage this insight to strategically adjust production processes and investment strategies, ensuring alignment with sustainable development principles. The effects of different parameters on steady state outcomes are discussed in Online Supplementary Appendix D.

Section III analyzes the production problem in a monopoly scenario. However, with the recent rise of green products and their recognition by governments worldwide, markets typically feature multiple firms producing similar products—a scenario of significant research relevance. In the following sections, we investigate the differential game problem in a two-firm duopoly setting.

IV. Duopoly

As the green product market matures and demand grows, another firm enters, investing in similar green products with advanced technologies and innovative business models. Both firms recognize that cooperation can increase industry-wide green reputation, creating greater market opportunities and mutual profits, thus initiating collaborative explorations. For example, the self-developed GL8 series jointly launched by SAIC Group and General Motors focuses on carbon reduction, where both firms' production and advertising investments collectively influence product green reputation. We analyze a differential game between two firms that produce identical green products, where each firm adjusts its investment strategy through control efforts. While the firms share comprehensive capital stock $K(t)$ and product reputation $G(t)$, they independently bear their own advertising investment $a_i(t)$ and production investment $I_i(t)$ $(i = 1, 2)$. For product reputation $G(t)$, we use an additive form to show that an increase (decrease) in emissions from one firm will reduce (increase) the reputation of the product [49], [50]. Therefore, the comprehensive capital stock $K(t)$ and product reputation $G(t)$ satisfy the following stochastic differential equations:

$$\begin{aligned} dK(t) &= (I_1(t) + I_2(t) - \delta_K K(t))dt + \sigma_1(K(t))dW_1(t), \\ dG(t) &= \Big((a_1(t) + a_2(t))\sqrt{lK(t)} + \varphi(2\bar{e} - e_1(t) - e_2(t)) \\ &\quad -\delta_G G(t)\Big)\, dt + \sigma_2(G(t))dW_2(t). \end{aligned}$$

The demand will be separated into $D_1(t)$ and $D_2(t)$, with each firm aiming to maximize its respective demand to reflect the competitive dynamics between them. The demands $D_1(t)$ and $D_2(t)$ are represented as changes in dt time that satisfy the following stochastic differential equations:

$$\begin{aligned} dD_1(t) =& (\alpha_1(G(t) - \mu_1) + \gamma_1 - \delta_D D_1(t))dt \\ &+ \sigma_3(D_1(t))dW_3(t), \\ dD_2(t) =& (\alpha_2(G(t) - \mu_2) + \gamma_2 - \delta_D D_2(t))dt \\ &+ \sigma_4(D_2(t))dW_4(t). \end{aligned}$$

The profits of the two firms at time t satisfy the following form:

$$\max_{I_i(t), a_i(t)} \int_0^\infty e^{-rt}\left\{\eta_i D_i(t) + \tau(\bar{e} - e_i(t)) - \frac{1}{2}a_i^2(t) - \frac{1}{2}I_i^2(t)\right\} dt, \quad i = 1, 2.$$

The value functions of two firms are denoted as V_1 and V_2. Next, we use the Itô lemma and derive a system of HJB equations (see Online Supplementary Appendix E). The optimal advertising and production investments satisfy the first-order condition as follows:

$$\begin{aligned} a_1^*(t) &= \sqrt{lK(t)}\frac{\partial V_1}{\partial G}, \quad I_1^*(t) = \frac{\partial V_1}{\partial K}, \\ a_2^*(t) &= \sqrt{lK(t)}\frac{\partial V_2}{\partial G}, \quad I_2^*(t) = \frac{\partial V_2}{\partial K}. \end{aligned}$$

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By solving the above equation, we show the results under duopoly as follows.

Lemma 4.1: The optimal advertising investments $a_i^*(t)(i = 1, 2)$ are both influenced by the comprehensive capital stock $K(t)$, and the optimal production investments $I_i^*(i = 1, 2)$ are as follows:

$$\begin{aligned}
a_1^*(t) &= \frac{\sqrt{lK(t)}\alpha_1(\eta_1 - \tau\beta_1)}{(r+\delta_G)(r+\delta_D) + \alpha_1\varphi\beta_1 + \alpha_2\varphi\beta_2}, \\
I_1^* =& \frac{l\alpha_1(\eta_1 - \tau\beta_1)}{r+\delta_K}\left(\frac{\alpha_1(\eta_1-\tau\beta_1) + 2\alpha_2(\eta_2-\tau\beta_2)}{2[(r+\delta_D)(r+\delta_G) + \alpha_1\varphi\beta_1 + \alpha_2\varphi\beta_2]^2}\right), \\
a_2^*(t) &= \frac{\sqrt{lK(t)}\alpha_2(\eta_2 - \tau\beta_2)}{(r+\delta_G)(r+\delta_D) + \alpha_1\varphi\beta_1 + \alpha_2\varphi\beta_2}, \\
I_2^* =& \frac{l\alpha_2(\eta_2 - \tau\beta_2)}{r+\delta_K}\left(\frac{\alpha_2(\eta_2-\tau\beta_2) + 2\alpha_1(\eta_1-\tau\beta_1)}{2[(r+\delta_D)(r+\delta_G) + \alpha_1\varphi\beta_1 + \alpha_2\varphi\beta_2]^2}\right).
\end{aligned}$$

Proof of Lemma 4.1: See Online Supplementary Appendix E. □

We provide the value functions of the two firms in Online Supplementary Appendix E.Moreover, regarding the impact of self-parameters on self-control effort, we find that the conclusion is consistent with the monopoly model in Section III, so it will not be repeated here. Unlike the monopoly scenario, a firm's advertising investment is influenced by competitors' consumer sensitivity and emission levels, creating a negative competitive effect: competition makes investments in rival products more dependent on consumer evaluations and product emissions. The impact of competitors' parameters on optimal investment is detailed in Online Supplementary Appendix F. Similarly, to ensure long-term profit maximization and the rationality of positive advertising investment $a_i^*(t)(i = 1, 2)$, we propose the following assumptions:

Assumption 4.1: Under a duopoly, the emission benefit indicator of two firms is positive, that is, $\eta_i - \tau\beta_i > 0(i = 1, 2)$.

Under a duopoly, it is crucial to maintain a positive emission benefit indicator. This ensures that both firms remain competitive and profitable under environmental regulations and consumer sustainability demands. For example, in the EV market, Tesla and BYD compete in China. BYD's positive emission benefit indicator is evident in its continuous innovation and significant market share. Tesla, aiming to reduce emissions and meet consumer expectations, also invests in EV technology. This competition drives advancements in battery technology and more efficient production processes, benefiting both firms and the environment. Next, we discover the relationship between the two controls in a duopoly situation given by:

Proposition 4.2: Under a duopoly, when there is a slight change in β_i in the neighborhood of ${a_i}^*(t)$ and ${I_i}^*$, the optimal control efforts ${a_i}^*(t)$ and ${I_i}^*$ are complementary to each other if and only if the emission benefit indicator $\eta_i - \tau\beta_i \geq 0, (i = 1, 2)$.

Proof of Proposition 4.2: See Online Supplementary Appendix G. □

Proposition 4.2 demonstrates that each firm's two optimal controls are complementary under a duopoly. This result mirrors the conclusion of Proposition 3.2 under a monopoly, but the duopoly model highlights that product promotion and production investment remain equally critical when multiple firms are involved. As long as profit opportunities exist, firms prioritize product quality to enhance their green reputation and produce lower-emission products. The second implication is that increased production investment strengthens product quality. High-quality and high-reputation products not only stimulate consumer demand but also bring spillover benefits to peer firms, which can leverage this complementarity to increase profits. Our research shows that reducing product emissions not only enhances a firm's green reputation but also strengthens the complementarity between advertising and production investments.

V. Two-Tier Supply Chain

As the market continues to develop, firms recognize the advantages of market expansion, supply chain management optimization, and labor division. It is difficult to achieve greater market breakthroughs solely through their own efforts. To expand market share and enhance operational efficiency, they establish a two-tier supply chain with downstream retailers. In this structure, the manufacturer focuses on production investment to enhance product green attributes and quality, whereas retailers undertake advertising investment to promote product advantages and increase market awareness. Thus, we extend our analysis to examine the differential game problem between one manufacturer and two retailers in a two-tier supply chain. In this scenario, the manufacturer bears product emission costs and production investment costs, whereas retailers focus on adjusting their advertising investments. Building on our earlier analysis, the manufacturer and retailers make strategic investments to maximize their respective profits. The comprehensive capital stock and product reputation follow the previously defined stochastic differential equations as follows:

$$\begin{aligned}
dK(t) &= (I_M(t) - \delta_K K(t))\,dt + \sigma_1(K(t))dW_1(t), \\
dG(t) &= \Big((a_1(t) + a_2(t))\sqrt{lK(t)} + \varphi(2\bar{e} - e_1(t) - e_2(t)) \\
&\quad - \delta_G G(t)\Big)\,dt + \sigma_2(G(t))dW_2(t).
\end{aligned} \tag{11}$$

From Equation (11), as the upstream of the supply chain, the manufacturer first announces their production investment I_M in the absence of cooperation, and retailers then choose their advertising investments a_i $(i = 1, 2)$, The demand of the upstream manufacturer at time t in the supply chain is obtained by adding the demand of retailers $D_M(t) = D_1(t) + D_2(t)$ [57], and the demand of retailers satisfies the following stochastic differential equations:

$$\begin{aligned}
dD_1(t) =& (\alpha_1(G(t) - \mu_1) + \gamma_1 - \delta_D D_1(t))dt \\
&+ \sigma_3(D_1(t))dW_3(t), \\
dD_2(t) =& (\alpha_2(G(t) - \mu_2) + \gamma_2 - \delta_D D_2(t))dt \\
&+ \sigma_4(D_2(t))dW_4(t).
\end{aligned}$$

With the manufacturer's involvement, product manufacturing becomes more standardized, and production pollution decreases, enabling supply chain players to achieve higher profits [58]. Thus, we assume that the manufacturer's emission-benefit indicator $M - \tau\beta$ is greater than $\eta_i - \tau\beta_i$ in the previous duopoly model (Section IV). For example, Lenovo has implemented top-level green supply chain management through Full Material Declaration initiatives, promoting harmful substance substitution and emission reduction across the entire industry chain[14]. Therefore, we propose the following assumptions:

Assumption 5.1: The emissions of products produced by the manufacturer are lower, whereas the emissions benefit indicator is greater than that of the model of duopoly, that is, $\beta_M < \beta_i$, and $(M - \tau\beta_M) > (\eta_i - \tau\beta_i) > 0, (i = 1, 2)$.

Assumption 5.1 suggests that the manufacturer's introduction standardizes production and reduces pollution, both complying with environmental regulations and enhancing market position and profitability. This is particularly significant in engineering management, where the focus is on optimizing production processes to reduce emissions and improve efficiency within the supply chain framework. Building on the previous model, we formulate a differential game model for the two-tier supply chain, where the manufacturer acts as a Stackelberg leader and the retailers serve as followers. Concurrently, a Nash game exists between retailers. We will explore the optimal investments between the manufacturer and retailers under noncooperation and cooperation scenarios. For ease of discrimination, the trademarks "N" and "C" are used to represent these two scenarios.

A. Noncooperation

In the noncooperation scenario, the manufacturer and retailers bear their own investment efforts. Retailers may not be willing to engage in green marketing initiatives. They worry that extra expenses, such as promoting environmentally friendly products or renovating stores to be more sustainable, will not be compensated for by increased consumer demand. Since each party is focused on maximizing its self-interest, there is no coordination in green investment strategies. A recent example is that in 2023, approximately half of Ford's U.S. dealers opted not to sell electric vehicles in 2024, despite earlier plans to invest \$500,000 in Ford's "Model E" initiative. They feared that the costs of new equipment, tools, and staff training for EVs would exceed sales revenue and were uncertain about the EV market's future,choosing not to partner with Ford on EV sales[15]. Under this noncooperative framework, the manufacturer, as the leader, first announces its production investment I_M^N, and then, the retailers independently select their advertising investments $a_i^N (i = 1, 2)$. The profit of the manufacturer at time t satisfies the following:

$$\max_{I_M^N(t)} \int_0^\infty e^{-rt} \left\{ M(D_1(t) + D_2(t)) + \tau(2\bar{e} - e_1(t) - e_2(t)) - \frac{1}{2} I_M^2(t) \right\} dt,$$

and the retailers' profits at time t satisfies the following:

$$\max_{a_i^N(t)} \int_0^\infty e^{-rt} \left\{ \eta_i D_i(t) - \frac{1}{2} a_i^2(t) \right\} dt, i = 1, 2.$$

[14]Typical Cases of Green Supply Chain Management in Enterprises. https://wap.miit.gov.cn/jgsj/jns/zhlyh/art/2020/art_2852f388305648daac3d34544fb18819.html

[15]Ford confirms half of dealers will sell EVs in 2024 as some are opting out. https://electrek.co/2023/12/21/ford-confirms-half-dealers-sell-evs-2024-several-opt-out/

For simplicity, we use the superscript of $V^{S,P}(S = N, C$ and $P = M, 1, 2)$ to represent the value functions of the manufacturer and retailers under different scenarios. By applying the Itô lemma, the value functions $V^{N,M}$, $V^{N,1}$ and $V^{N,2}$ satisfy a system of HJB equations (we put it in Online Supplementary Appendix H). By taking the first-order conditions for $I_M^N(t)$ and $a_i^N(t)$ $(i = 1, 2)$ respectively, we obtain the optimal controls as follows:

$$I_M^{N^*} = V_K^{N,M},$$
$$a_1^{N^*}(t) = \sqrt{lK(t)} V_G^{N,1}, a_2^{N^*}(t) = \sqrt{lK(t)} V_G^{N,2}.$$

Then, we provide the specific forms of the system of HJB equations that satisfy the value functions $V^{N,M}$, $V^{N,1}$ and $V^{N,2}$. Using the previously adopted guess-and-verify method for value functions, we solve for the three optimal value functions and provide the corresponding optimal control expressions as follows:

Lemma 5.1: The optimal advertising investments $a_i^{N^*}(t)(i = 1, 2)$ of retailers are both influenced by the comprehensive capital stock $K(t)$ and the optimal production investments $I_M^{N^*}$ of the manufacturer are as follows:

$$a_1^{N^*}(t) = \frac{\sqrt{lK(t)}\alpha_1\eta_1}{(r + \delta_G)(r + \delta_D) + \alpha_1\varphi\beta_M + \alpha_2\varphi\beta_M},$$
$$a_2^{N^*}(t) = \frac{\sqrt{lK(t)}\alpha_2\eta_2}{(r + \delta_G)(r + \delta_D) + \alpha_1\varphi\beta_M + \alpha_2\varphi\beta_M},$$
$$I_M^{N^*} = \frac{l(M - \tau\beta_M)}{r + \delta_K} \left(\frac{\alpha_1^2\eta_1 + \alpha_1\alpha_2\eta_1 + \alpha_1\alpha_2\eta_2 + \alpha_2^2\eta_2}{[(r + \delta_D)(r + \delta_G) + \alpha_1\varphi\beta_M + \alpha_2\varphi\beta_M]^2} \right).$$

Proof of Lemma 5.1: See Online Supplementary Appendix H. □

Lemma 5.1 presents the optimal investments for the manufacturer and retailers in the noncooperative scenario, showing that the manufacturer's production investment remains stable over time. This stability ensures consistent production input, addressing long-term investment requirements. In contrast, retailers' optimal advertising investments fluctuate with the product's comprehensive capital stock $K(t)$, indicating that retailers' strategies are influenced by the manufacturer's decisions. Consequently, retailers must carefully align their advertising and marketing strategies with the manufacturer's production activities to effectively enhance green reputation. The optimal value functions for the manufacturer and retailers are provided in Online Supplementary Appendix H.

Proposition 5.2: Under noncooperation, when there is a slight change in β_M in the neighborhood of ${a_i}^{N^*}(t)$ and $I_M^{N^*}$, the optimal control efforts $a_i^{N^*}(t)$ and $I_M^{N^*}$ are complementary to each other if and only if the emission benefit indicator $\eta_i - \tau\beta_i \geq 0 \quad (i = 1, 2)$.

Proof of Proposition 5.2: See Online Supplementary Appendix I. □

Proposition 5.2 demonstrates that in a two-tier supply chain, complementarity also exists between the manufacturer's and retailers' optimal controls. Unlike prior conclusions, the investments of the manufacturer and two retailers can mutually reinforce each other under a two-tier structure. This implies that when profit opportunities arise, the manufacturer prioritizes product quality control to produce lower-emission products, whereas retailers enhance product promotion to build a stronger green reputation. Notably, mutual promotion between the manufacturer and one retailer does not influence the other retailer, meaning that a better product marketing environment can be created. Reducing product emissions not only enhances green reputation but also strengthens the complementarity between advertising and production investments among supply chain members. Next, we explore the scenario where the manufacturer and retailers collaborate to share emission costs, aiming to increase retailers' commitment to green products.

B. Cooperation

Under the cooperation scenario, the manufacturer and retailers collaborate strategically. The manufacturer recognizes that joint marketing campaigns with retailers can highlight the supply chain's compliance with new regulations. By investing in green production, the manufacturer not only meets regulatory requirements but also gains a competitive edge for its products. In turn, retailers are more willing to support these efforts, as they benefit from the positive brand image created by green products. This collaboration enhances consumer loyalty and expands the market share for both parties. In 2021, the EU's stricter single-use plastic regulations prompted Unilever to collaborate with retailers such as Carrefour and Tesco. Unilever invested in developing sustainable packaging solutions, while retailers joined marketing campaigns to promote Unilever's compliant new products and emphasize the company's environmental efforts[16]. In our cooperative model, retailers support green product manufacturing by jointly sharing the emission costs with the manufacturer. In this two-tier supply chain, the manufacturer first determines its production investment I_M^C and the cost-sharing rates θ_i^C for retailers. Subsequently, retailers select their advertising investments a_i^C. Accordingly, the manufacturer's profit at time t is governed by:

$$\max_{I_M^C(t),\theta_1^C,\theta_2^C} \int_0^\infty e^{-rt}\left\{M\sum_i^2 D_i(t)+\tau(2\bar{e}-\sum_i^2 e_i(t)) -\frac{1}{2}(1-\sum_i^2\theta_i^C)I_M^C(t)^2\right\}dt,$$

and the retailers' profits at time t satisfies the following:

$$\max_{a_i^C(t),\theta_i^C} \int_0^\infty e^{-rt}\left\{\eta_i D_i(t)-\frac{1}{2}a_i^C(t)^2-\frac{1}{2}\theta_i^C I_M^C(t)^2\right\}dt, \quad i=1,2.$$

where $i=1,2$ and $(1-\sum_i^2\theta_i^C)\geq 0$, which represents the proportion of the manufacturer's emission costs dispersed by two retailers. We define the manufacturer and the retailers' value functions as $V^{C,M}$, $V^{C,1}$ and $V^{C,2}$ which satisfy a system of HJB equations (see Online Supplementary Appendix J). The optimal cost-sharing rates of retailers satisfy the first-order condition, as follows:

$$\frac{\partial rV^{C,1}}{\partial\theta_1^C}=\frac{\partial rV^{C,2}}{\partial\theta_2^C}=0.$$

Then, we can obtain the optimal cost-sharing rates of retailers as follows:

$$\theta_1^{C^*}=\frac{2V_K^{C,1}-V_K^{C,M}}{2(V_K^{C,1}+V_K^{C,2})},\quad \theta_2^{C^*}=\frac{2V_K^{C,2}-V_K^{C,M}}{2(V_K^{C,1}+V_K^{C,2})}.$$

By substituting the optimal control and cost-sharing rates into the system of HJB equations, which satisfy the value functions $V^{C,M}$, $V^{C,1}$ and $V^{C,2}$, we use the previous guessing value function method to solve three optimal value functions and provide corresponding optimal control expressions as follows:

Lemma 5.3: The optimal cost-sharing rates of the retailers $\theta_i^{C^*}$ have the following forms:

$$\theta_1^{C^*}=\frac{2\eta_1\alpha_1(\alpha_1\eta_1+2\alpha_2\eta_2)-(M-\tau\beta_M)(\alpha_1+\alpha_2)(\alpha_1\eta_1+\alpha_2\eta_2)}{2\left[\alpha_1\eta_1(\alpha_1\eta_1+2\alpha_2\eta_2)+\alpha_2\eta_2(2\alpha_1\eta_1+\alpha_2\eta_2)\right]},$$
$$\theta_2^{C^*}=\frac{2\eta_2\alpha_2(2\alpha_1\eta_1+\alpha_2\eta_2)-(M-\tau\beta_M)(\alpha_1+\alpha_2)(\alpha_1\eta_1+\alpha_2\eta_2)}{2\left[\alpha_1\eta_1(\alpha_1\eta_1+2\alpha_2\eta_2)+\alpha_2\eta_2(2\alpha_1\eta_1+\alpha_2\eta_2)\right]}.$$

The optimal advertising efforts $a_i^{C^*}(t)(i=1,2)$ of retailers are both influenced by the comprehensive capital stock $K(t)$, and the optimal production investment $I_M^{C^*}$ of the manufacturer is as follows:

$$a_1^{C^*}(t)=\frac{\sqrt{lK(t)}\alpha_1\eta_1}{(r+\delta_G)(r+\delta_D)+\alpha_1\varphi\beta_M+\alpha_2\varphi\beta_M},$$
$$a_2^{C^*}(t)=\frac{\sqrt{lK(t)}\alpha_2\eta_2}{(r+\delta_G)(r+\delta_D)+\alpha_1\varphi\beta_M+\alpha_2\varphi\beta_M},$$
$$I_M^{C^*}=\frac{l(M-\tau\beta_M)}{(1-\theta_1^{C^*}-\theta_2^{C^*})(r+\delta_K)}\left(\frac{\alpha_1^2\eta_1+\alpha_1\alpha_2\eta_1+\alpha_1\alpha_2\eta_2+\alpha_2^2\eta_2}{\left[(r+\delta_D)(r+\delta_G)+\alpha_1\varphi\beta_M+\alpha_2\varphi\beta_M\right]^2}\right).$$

Proof of Lemma 5.3: See Online Supplementary Appendix J. □

The optimal value functions of the manufacturer and retailers are provided in Online Supplementary Appendix J. Lemma 5.3 provides the optimal cost-sharing rates for retailers, which allows the manufacturer to reduce the pressure on product production emissions. Obviously, we have $\frac{\partial\theta_i^{C^*}}{\partial M}<0$ and $\frac{\partial\theta_i^{C^*}}{\partial\beta_M}>0$ $(i=1,2)$. When the manufacturer's marginal profit increases, this enhances its efficiency or market pricing power, allowing the manufacturer to invest more in emission reduction, thus reducing the retailers' need to compensate for emissions. Conversely, as product emission levels increase, retailers' optimal cost-sharing rates rise. This is because higher emissions necessitate greater environmental compliance efforts, and the retailers, being closer to consumers, play a leading role in maintaining product image and regulatory compliance, thus taking a greater share of emissions responsibility. The cost-sharing rates reflect the balance and interdependence between manufacturers and retailers in a two-tier supply chain, emphasizing their collaborative impact on engineering management and the environment. In the cooperation situation, we give the relationship between the two controls proposed as follows:

[16]Deal on new rules for more sustainable packaging in the EU. https://www.europarl.europa.eu/news/en/press-room/20240301IPR18595
Unilever's Global Strategy for Reducing Supply Chain Plastic. https://sustainabilitymag.com/sustainability/unilevers-global-strategy-for-reducing-supply-chain-plastic

Proposition 5.4: Under cooperation, when there is a slight change in β_M in the neighborhood of $a_i^{C^*}(t)$ $(i = 1, 2)$ and $I_M^{C^*}$, the optimal control efforts $a_i^{C^*}(t)$ and $I_M^{C^*}$ are complementary to each other, respectively.

Proof of Proposition 5.4: See Online Supplementary Appendix I. □

Proposition 5.4 demonstrates that in the cooperative two-tier supply chain, the manufacturer's and retailers' optimal controls exhibit a complementary relationship. Considering the complementary relationship between green reputation and product emissions is crucial, as this synergy strengthens collaboration and enables retailers to secure more emission subsidies for the manufacturer. Furthermore, when the manufacturer increases production investment, retailers can correspondingly increase their advertising investment, and vice versa. We present and compare the results of all preceding models in the next section.

VI. Comparison and Analysis

On the basis of the four preceding models, specific expressions for optimal investment efforts and steady state outcomes in each scenario are provided in Online Supplementary Appendix K. We first analyze the steady state values of comprehensive capital stock, product reputation, and demand, followed by cross-scenario comparisons of advertising and production investments.

We trace the evolution of a firm that initially establishes a monopoly by leveraging significant advertising and production investments to build a strong green reputation, then transitions to a duopoly and ultimately to a two-tier supply chain. Throughout this progression, the firm adjusts its strategies to maintain competitive advantage and enhance its green reputation. By examining these four distinct scenarios, we derive the following meaningful managerial propositions and conclusions:

Proposition 6.1: For $\alpha\beta > \alpha_1\beta_1 + \alpha_2\beta_2$, consider four scenarios: a monopoly, a duopoly, and a two-tier supply chain under noncooperation and cooperation,

(i) Steady state comprehensive capital stock, the common reputation of products, and the demand hold for

$$K_{ss}^C > K_{ss}^N > K_{ss}^D > K_{ss}, \quad G_{ss}^C > G_{ss}^N > G_{ss}^D > G_{ss},$$
$$D_{1ss}^C > D_{1ss}^N > D_{1ss}^D > D_{1ss}, \quad D_{2ss}^C > D_{2ss}^N > D_{2ss}^D > D_{2ss}.$$

(ii) The steady state production investments of the manufacturer, duopoly and monopoly hold for

$$I_M^{C^*} > I_M^{N^*} > \sum_{i=1}^{2} I_i^* > I^*.$$

(iii) The steady state advertising investments of the retailers, duopoly and monopoly hold for

$$a_{1ss}^{C^*} > a_{1ss}^{N^*} > a_{1ss}^*, \quad a_{2ss}^{C^*} > a_{2ss}^{N^*} > a_{2ss}^*,$$
$$\text{and } \sum_{i=1}^{2} a_{iss}^{C^*} > \sum_{i=1}^{2} a_{iss}^{N^*} > \sum_{i=1}^{2} a_{iss}^* > a_{ss}^*.$$

Proof of Proposition 6.1: See Online Supplementary Appendix L. □

Proposition 6.1 *(i)* indicates that as market demand and the imperative to build a green reputation increase, the comprehensive capital stock, product reputation, and market demand grow progressively across four scenarios: monopoly, duopoly, noncooperative two-tier supply chain, and cooperative two-tier supply chain. In the duopoly, the new entrant not only drives additional demand but also collaborates with the incumbent firm to enhance the product's green reputation and accumulate comprehensive capital stock. Furthermore, motivated by market expansion and supply chain optimization goals, the establishment of a two-tier supply chain more effectively boosts production efficiency and product reputation, leading to even higher demand. Under the cooperative scenario, the comprehensive capital stock, product reputation, and demand for green products exceed those under the other scenarios. This is because the manufacturer recognizes that collaborating with retailers accelerates market expansion and supply chain optimization. Through the cooperative mechanism, retailers share the manufacturer's emission costs, enabling higher production investment and improved product quality. This quality improvement directly enhances product reputation and stimulates demand. Thus, adopting a cooperative strategy in a two-tier supply chain significantly elevates green product reputation and demand, making it an optimal choice for investors.

Proposition 6.1 *(ii)* compares the green production investment under four scenarios and explores the impacts of monopoly, duopoly, noncooperation, and cooperation mechanisms on the green production investment. The first inequality indicates that in the two-tier supply chain, the production investment effort under cooperative strategies is the greatest because when retailers share the production expenses of the product, it increases the production investment I_M^C of the green product. In addition, the production investment under the noncooperative scenario is greater than that under a duopoly scenario; this finding indicates that the introduction of a two-tier supply chain can lead to more unified and standardized production, reduce more emissions, and invest excess funds in production efforts. Therefore, cooperation strategies can stimulate investment in the production of green products, and retailers should collaborate with the manufacturer to stimulate product investment and produce better green products. The third inequality indicates that when the product of the green reputation sensitivity and emission under a monopoly $(\alpha\beta)$ exceeds the combined product $(\alpha_1\beta_1 + \alpha_2\beta_2)$ of two firms under duopoly, the monopoly's production investment is lower than the total production investment of the duopoly firms. In a monopoly, the firm's strong market power allows it to control supply and pricing effectively. This dominance means that even if the product's emission level β increases, the firm is not forced to increase production investment to compete for market share. In contrast, in a duopoly, the two firms often need to make greater production investments to reduce their products' emission levels to capture more market share and jointly enhance the green reputation of products. This results in higher total production investments than those in the monopolistic scenario.

Proposition 6.1 *(iii)* indicates that, from a long-term investment perspective, advertising investment under the cooperative scenario of the two-tier supply chain is greater than that under other scenarios. This is because cooperative production investment I_M^C reaches its peak, ensuring an abundance of high-quality products and incentivizing retailers to promote them actively. The inequality $a_{iss}^{N^*} > a_{iss}^*(i = 1, 2)$ further indicates that the two-tier supply chain drives the production of high-quality green products. Combining Proposition 6.1 *(ii)* and *(iii)*, investors benefit most from participating in a two-tier supply chain and signing cooperation contracts with manufacturers (or retailers, if the investor is a manufacturer). This strategy stimulates green production investment, enhances green product promotion, and increases product visibility, thereby advancing the green product industry. The relationship $\sum_{i=1}^{2} a_{iss}^* > a_{ss}^*$ also shows that monopoly's advertising investment is lower than the total advertising investment of duopoly firms under comparable conditions. A monopoly's market power eliminates the need for aggressive advertising to compete for market share, even as emission levels (β) rise. In a duopoly, firms must increase advertising to enhance product green reputation and attract consumers, leading to higher total advertising investments than in a monopoly.

Proposition 6.2: (i) In the monopoly, duopoly, and two-tier supply chains under noncooperation and cooperation, steady state advertising investments, production investments, comprehensive capital stocks, and green reputations increase with greater consumer sensitivity to green reputation and sales margins but decrease with higher emission levels per unit product. Moreover, steady state demands increase with higher sales margins but decrease with higher emission levels per unit of product.

(ii) For $\alpha\beta > \alpha_1\beta_1 + \alpha_2\beta_2$ and $2(\eta_{3-i} - \tau\beta_{3-i}) < M - \tau\beta_M$, the sales margin affects the steady state production investments and comprehensive capital stocks differently across the four scenarios, satisfying the following relationship:

$$\frac{\partial I_M^{C^*}}{\partial \eta_i} > \frac{\partial I_M^{N^*}}{\partial \eta_i} > \frac{\partial \sum_{i=1}^{2} I_i^*}{\partial \eta_i} > \frac{\partial I^*}{\partial \eta}$$
$$\text{and } \frac{\partial K_{ss}^C}{\partial \eta_i} > \frac{\partial K_{ss}^N}{\partial \eta_i} > \frac{\partial K_{ss}^D}{\partial \eta_i} > \frac{\partial K_{ss}}{\partial \eta} \ (i = 1, 2).$$

Proof of Proposition 6.2: See Online Supplementary Appendix L. □

To understand Proposition 6.2 *(i)*, we begin by analyzing the evolution of the firm. Initially, the firm uses its first-mover edge to monopolize. It invests heavily in advertising and production to build a strong green image. Higher sales margins increase investments, stock, and reputation, whereas higher emission levels per unit of product harm these factors. Moreover, demands increase with higher sales margins but decrease with higher emission levels per unit of product. When the market evolves and demand for a greener reputation grows, a new firm enters, forming a duopoly. They cooperate to enhance the industry's green reputation. Cooperative moves increase advertising, production, stock, and reputation. Finally, for better supply chain management and market expansion, the firm adds downstream retailers, creating a two-tier supply chain. This improves efficiency and targeted advertising, strengthening positive factors and weakening the negative impact of emissions. In summary, across the four scenarios, higher sales margins lead to increased investments and better performance, whereas higher emission levels per unit product detract from these outcomes. This serves as a clear guide for firms at different developmental stages to maximize market impact and sustainability goals across various market structures.

Proposition 6.2 *(ii)* examines how the product's sales margin affects steady state production investments and comprehensive capital stocks across four scenarios. The results show that the impact is greatest in the cooperative two-tier supply chain, followed by the noncooperative two-tier supply chain, the duopoly, and finally the monopoly. In the monopoly scenario, the impact of the sales margin on production investment and comprehensive capital stock is the smallest. This is because, in a monopoly, the firm faces no direct competition, reducing the incentive to invest heavily in production and capital. The lack of competitive pressure means that the firm can maintain its market position with lower investment levels. As the market evolves and the demand for enhancing green reputation increases, a duopoly is created, driving firms to invest more in production and capital to capture market share, resulting in higher values for these metrics. In the noncooperative two-tier supply chain, the division of labor between the manufacturer and retailers allows for some optimization. This optimization ensures that the manufacturer's emission benefit coefficient is sufficiently large, exceeding twice that of the duopoly scenario (i.e., $2(\eta_{3-i} - \tau\beta_{3-i}) < M - \tau\beta_M$). As a result, investors are more willing to invest, which leads to higher production investment and comprehensive capital stock. leading to even higher production investment and comprehensive capital stock. Finally, in the cooperative two-tier supply chain, collaboration between the manufacturer and retailers maximizes efficiency and resource allocation, resulting in the highest production investment and comprehensive capital stock.

VII. Case Study

In this section, we present an empirical analysis based on data collected from two leading electric vehicle (EV) firms, BYD and Tesla, spanning the period from 2018 to 2023. These firms have extensive influence and market share in China and even globally. We collect the specific data of two firms over the past six years, which include capital stock, new energy vehicle credits (NEVC), total greenhouse gas emissions (TGGE), emissions per vehicle (EPV), sales, net profit, and average profit per vehicle (APPV). Except for the new energy vehicle credit (NEVC) data from the Ministry of Industry and Information Technology of the Chinese government[17], all other data come from the annual financial reports of BYD[18] and

[17]The New Energy Vehicle Credits (NEVC) data are based on the requirements of the "Passenger Cars Corporate Average Fuel Consumption and New Energy Vehicle Credit Regulation" https://ythxxfb.miit.gov.cn/ythzxfwpt/hlwmh/tzgg/xzxk/clsczr/.

[18]BYD's annual report data source: BYD's official website financial report https://www.bydglobal.com/cn/Investor/InvestorAnnals.html, Sina Finance financial statistics summary. https://vip.stock.finance.sina.com.cn/corp/go.php/vFD_FinanceSummary/stockid/002594/displaytype/4.phtml?source=lrb

TABLE III
PARAMETER ESTIMATION RESULTS

	BYD		Tesla	
	Parameters	SE	Parameters	SE
Consumer Sensitivity (α)	1.242**	2.225	1.051***	1.553
Consumers' Benchmark of Reputation (μ)	5.201**	8.662	8.134***	9.237
Impact of Factors on Demand (γ)	7.173**	12.112	10.173***	10.278
Sales Margin (η)	13.684***	15.464	16.812***	18.61
Taxation (Subsidy) on Emissions (τ)	0.058**	0.163	0.049***	0.173
Emission Level Per Unit Product (β)	8.758***	8.588	14.532***	16.09
Impact of Capital Stock on Advertisement (l)	0.049*	0.011	0.078***	0.110
Impact of Environmental Protection on Reputation (φ)	0.572***	0.672	0.913*	2.195
Emission Standard of Firms ($\bar{e}$)	1.042**	1.850	0.827***	1.020
Degree of decay on Demand (δ_D)	0.052***	0.061	0.056*	0.167
Degree of decay on Green Reputation (δ_G)	0.048**	0.562	0.053*	0.127
Degree of decay on Capital Stock (δ_K)	0.005**	0.010	0.047***	0.059
Volatility of Capital Stock (σ_1)	0.493**	0.179	0.489**	1.117
Volatility of Demand (σ_2)	0.505***	0.655	0.303***	0.077
Volatility of Green Reputation (σ_3)	0.829***	0.979	0.523***	0.686

Note: SE represents the standard error. The asterisk " * " represents the significance level corresponding to different p-values: $^{***}p < 0.01$; $^{**}p < 0.05$; $^{*}p < 0.1$.

Tesla[19]. We introduce a simple idea of how to use MLE to estimate the parameters of the stochastic differential equation (SDE) in the Online Supplementary Appendix N [45]. We utilize time series data of capital stock on the comprehensive capital stock $K(t)$. The data also include carbon credit data under the Dual Credit Policy for passenger cars formulated by the Chinese government, in which BYD and Tesla participate. Carbon credit trading under the Dual Credit Policy is essentially consistent with carbon emissions trading and therefore has practical significance. We utilize these carbon credit data to estimate the value of the term $(e - e(t))$ in (2). We subsequently use each firm's total greenhouse gas emissions (TGGE) and emissions per vehicle (EPV) to estimate the parameter of emission level per unit product β. We then adopt their net profit and average profit per vehicle (APPV) data to estimate the parameter of their own sales margin η. Therefore, combined with the time series data of sales $D(t)$, we can continue to estimate the remaining parameters in three state equations.

The above parameters can be fully recognized. The final results of the parameter estimation are listed in Table III, and the standard error (SE) is given next to the estimated parameters. The asterisk " * " represents the different significance levels corresponding to the p-value. We used the Nelder-Mead simplex algorithm to numerically calculate the MLE estimates. The estimation procedure is subsequently implemented in MATLAB.

To assess the performance of our SDE model, we compare its predictive accuracy with that of two well-established econometric models: the autoregressive moving average (ARMA) [16] and generalized autoregressive conditional heteroscedasticity (GARCH) [17]. The detailed results are presented in Online Supplementary Appendix N. We employ a series of statistical indicators—root mean square error (RMSE), mean absolute error (MAE), and symmetric mean absolute percentage error (SMAPE)—to evaluate model performance. Compared with ARMA and GARCH, our SDE model exhibits superior or comparable fitting and predictive capabilities. As shown in Table III, all the parameters are identifiable, with p-values for different parameters being significant at $p < 0.1$. This significance reflects both inherent data volatility and the influence of nonlinear and stochastic terms in the SDEs. The estimated parameters for BYD and Tesla exhibit distinct characteristics, likely due to differences in their market dominance periods in China. Notably, our model demonstrates stability in estimating MLE parameters across firms. A key finding is the noncontradictory coexistence of low emissions and high subsidy rates. For instance, both BYD and Tesla hold positive carbon credits and receive regulatory subsidies. Although BYD has lower unit product emissions (β) than Tesla does, it has a higher subsidy rate (τ). This aligns with the hypothesis that greener production yields greater returns, as regulatory agencies incentivize such behavior through increased subsidies, as evident in Table III.

Using the estimated parameters in Table III, we calculate the derivatives of the steady states with respect to the parameters and their variation patterns, as shown in Table IV. The base-case parameters are: $\alpha = 2.7$, $\alpha_1 = 1.3$, $\alpha_2 = 1.1$, $l = 0.05$, $\bar{e} = 1.0$, $\eta = 11$, $\eta_1 = 13$, $\eta_2 = 12$, $M = 30$, $\beta = 9$, $\beta_1 = \beta_2 = \beta_M = 8$, $\theta_1^C = 0.1$, $\theta_2^C = 0.2$, $\mu = 5.2$, $\mu_1 = 5.2$, $\mu_2 = 5.2$, $\gamma = \gamma_1 = \gamma_2 = 7$, $\tau = r = 0.05, \phi = 0.6$, $\delta_K = 0.005$, and $\delta_D = \delta_G = 0.05$. Table IV presents the results when one parameter is varied while all others remain constant. Notably, these results are computed under the conditions $\alpha\beta > \alpha_1\beta_1 + \alpha_2\beta_2$ and $2(\eta_2 - \tau\beta_2) < M - \tau\beta_M$, which are the prerequisites for Proposition 6.2. First, the table shows that consumer sensitivity to green reputation positively influences steady state outcomes across all scenarios, with the cooperative two-tier supply chain

[19]Tesla's annual report data source: Tesla's official website financial report https://ir.tesla.com, EastMoney financial statistics summary https://ir.tesla.com/.

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TABLE IV
COMPARISONS: THE IMPACT OF THE PARAMETERS ON STEADY STATES

Parameter	$\frac{\partial I^*}{\partial \alpha}$	$\frac{\partial \left(\sum_{i=1}^{2} I_i^*\right)}{\partial \alpha_1}$	$\frac{\partial I_M^{N*}}{\partial \alpha_1}$	$\frac{\partial I_M^{C*}}{\partial \alpha_1}$	$\frac{\partial a_{ss}^*}{\partial \alpha}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^*\right)}{\partial \alpha_1}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^{N*}\right)}{\partial \alpha_1}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^{C*}\right)}{\partial \alpha_1}$
$\alpha = 2.7, \alpha_1 = 1.2$	$1.243 * 10^{-3}$	2.634	5.268	7.526	0.006	18.097	24.672	29.489
$\alpha = 2.8, \alpha_1 = 1.3$	$1.160 * 10^{-3}$	0.865	2.591	3.702	0.005	8.080	12.146	14.518
$\alpha = 2.9, \alpha_1 = 1.4$	$1.084 * 10^{-3}$	0.040	0.222	0.317	0.005	0.625	1.040	1.243

Parameter	$\frac{\partial K_{ss}}{\partial \alpha}$	$\frac{\partial K_{ss}^D}{\partial \alpha_1}$	$\frac{\partial K_{ss}^N}{\partial \alpha_1}$	$\frac{\partial K_{ss}^C}{\partial \alpha_1}$	$\frac{\partial G_{ss}}{\partial \alpha}$	$\frac{\partial G_{ss}^D}{\partial \alpha_1}$	$\frac{\partial G_{ss}^N}{\partial \alpha_1}$	$\frac{\partial G_{ss}^C}{\partial \alpha_1}$	$\frac{\partial D_{1ss}}{\partial \alpha}$	$\frac{\partial D_{1ss}^D}{\partial \alpha_1}$	$\frac{\partial D_{1ss}^N}{\partial \alpha_1}$	$\frac{\partial D_{1ss}^C}{\partial \alpha_1}$
$\alpha = 2.7, \alpha_1 = 1.2$	0.249	526.872	1053.698	1505.283	0.886	47.107	44.447	40.743	-0.003	-1029.055	-914.442	-805.551
$\alpha = 2.8, \alpha_1 = 1.3$	0.231	173.037	518.224	740.320	0.824	23.122	21.817	19.978	-0.002	-506.365	-448.736	-394.778
$\alpha = 2.7, \alpha_1 = 1.4$	0.215	7.933	44.329	63.327	0.768	1.975	1.863	1.704	-0.001	-43.343	-38.305	-33.657

Parameter	$\frac{\partial I^*}{\partial \beta}$	$\frac{\partial \left(\sum_{i=1}^{2} I_i^*\right)}{\partial \beta_1}$	$\frac{\partial I_M^{N*}}{\partial \beta_M}$	$\frac{\partial I_M^{C*}}{\partial \beta_M}$	$\frac{\partial a_{ss}^*}{\partial \beta}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^*\right)}{\partial \beta_1}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^{N*}\right)}{\partial \beta_M}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^{C*}\right)}{\partial \beta_M}$
$\beta, \beta_1, \beta_M = 8.0$	-0.081	-0.082	-0.233	-0.333	-0.378	-0.391	-0.753	-0.900
$\beta, \beta_1, \beta_M = 9.0$	-0.056	-0.067	-0.163	-0.233	-0.264	-0.285	-0.529	-0.632
$\beta, \beta_1, \beta_M = 10.0$	-0.041	-0.057	-0.119	-0.170	-0.192	-0.217	-0.385	-0.460

Parameter	$\frac{\partial K_{ss}}{\partial \beta}$	$\frac{\partial K_{ss}^D}{\partial \beta_1}$	$\frac{\partial K_{ss}^N}{\partial \beta_M}$	$\frac{\partial K_{ss}^C}{\partial \beta_M}$	$\frac{\partial G_{ss}}{\partial \beta}$	$\frac{\partial G_{ss}^D}{\partial \beta_1}$	$\frac{\partial G_{ss}^N}{\partial \beta_M}$	$\frac{\partial G_{ss}^C}{\partial \beta_M}$	$\frac{\partial D_{1ss}}{\partial \beta}$	$\frac{\partial D_{1ss}^D}{\partial \beta_1}$	$\frac{\partial D_{1ss}^N}{\partial \beta_M}$	$\frac{\partial D_{1ss}^C}{\partial \beta_M}$
$\beta, \beta_1, \beta_M = 8.0$	-16.105	-16.156	-46.601	-66.573	-0.002	-0.002	-0.006	-0.009	-0.099	-0.133	-0.166	-0.231
$\beta, \beta_1, \beta_M = 9.0$	-11.271	-13.449	-32.690	-46.700	-0.001	-0.002	-0.004	-0.005	-0.058	-0.066	-0.096	-0.132
$\beta, \beta_1, \beta_M = 10.0$	-8.186	-11.312	-23.800	-34.000	-0.001	-0.001	-0.002	-0.003	-0.036	-0.040	-0.059	-0.080

Parameter	$\frac{\partial I^*}{\partial \eta}$	$\frac{\partial \left(\sum_{i=1}^{2} I_i^*\right)}{\partial \eta_1}$	$\frac{\partial I_M^{N*}}{\partial \eta_1}$	$\frac{\partial I_M^{C*}}{\partial \eta_1}$	$\frac{\partial a_{ss}^*}{\partial \eta}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^*\right)}{\partial \eta_1}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^{N*}\right)}{\partial \eta_1}$	$\frac{\partial \left(\sum_{i=1}^{2} a_{iss}^{C*}\right)}{\partial \eta_1}$
$\eta, \eta_1 = 13.0$	0.329	0.392	0.632	0.902	1.541	1.623	2.082	2.489
$\eta, \eta_1 = 14.0$	0.360	0.404	0.632	0.902	1.687	1.778	2.125	2.540
$\eta, \eta_1 = 15.0$	0.391	0.415	0.632	0.902	1.833	1.933	2.167	2.590

Parameter	$\frac{\partial K_{ss}}{\partial \eta}$	$\frac{\partial K_{ss}^D}{\partial \eta_1}$	$\frac{\partial K_{ss}^N}{\partial \eta_1}$	$\frac{\partial K_{ss}^C}{\partial \eta_1}$	$\frac{\partial G_{ss}}{\partial \eta}$	$\frac{\partial G_{ss}^D}{\partial \eta_1}$	$\frac{\partial G_{ss}^N}{\partial \eta_1}$	$\frac{\partial G_{ss}^C}{\partial \eta_1}$	$\frac{\partial D_{1ss}}{\partial \eta}$	$\frac{\partial D_{1ss}^D}{\partial \eta_1}$	$\frac{\partial D_{1ss}^N}{\partial \eta_1}$	$\frac{\partial D_{1ss}^C}{\partial \eta_1}$
$\eta, \eta_1 = 13.0$	65.707	78.414	126.306	180.438	0.033	0.042	0.148	0.212	1.782	2.580	3.856	5.509
$\eta, \eta_1 = 14.0$	71.933	80.725	126.306	180.438	0.040	0.049	0.155	0.221	2.136	2.623	4.017	5.739
$\eta, \eta_1 = 15.0$	78.160	83.037	126.306	180.438	0.047	0.056	0.161	0.230	2.521	2.667	4.178	5.968

exhibiting the highest impact, followed by the noncooperative two-tier supply chain, duopoly, and monopoly. This pattern reflects the firm's strategic evolution: initially, the monopolist's market dominance allowed it to overlook consumer sensitivity to green reputation, but as the market evolved, this sensitivity became increasingly significant. Heightened consumer sensitivity to green reputation reduces demand for low-green products, with this effect being most pronounced in a duopoly and least pronounced in a monopoly. In a duopoly, firms compete fiercely for market share, and increased green reputation sensitivity amplifies competition, leading to steeper demand declines for products with lower green attributes. In contrast, monopolists face minimal competition, enabling them to sustain demand even with higher sensitivity due to limited consumer alternatives. Second, unit product emission levels negatively affect steady state outcomes, with the impact intensity decreasing in the following order: cooperative two-tier supply chain, noncooperative two-tier supply chain, duopoly, and monopoly. Monopolistic markets initially limit consumer choices, weakening the market penalty for high emissions. However, as firms advance and green reputation gains importance, consumer sensitivity to emissions has increased, making emission reductions a critical focus in later stages. Finally, the sales margin continues to affect production investment and comprehensive capital stocks in the steady state in accordance with Proposition 6.2 *(ii)*. Additionally, we have extended this analysis to advertising investments, green reputations, and demands. The impact follows the same order in a duopoly supply chain: cooperation has the greatest effect, followed by noncooperation, duopoly, and monopoly. This is because continuous firm development and supply chain optimization increase the profit margins of products, thereby maximizing the impact on these variables. During the monopoly period, despite dominating the market, building a strong green reputation is the firm's goal. Therefore, the sales margin has little effect on the firm.

TABLE V
THE OPTIMAL INVESTMENT STRATEGIES

	BYD	Tesla
Consumer Sensitivity (α)	1.232	1.051
Emission Level Per Unit Product (β)	8.758	14.532
Optimal Production Investment (I^*)	17.389	4.476
Optimal Steady State Advertising Investment (a_{ss}^*)	4.621	1.883

Using the MLE-estimated parameters in Table III, we derive the optimal production investment and steady state advertising investment for both firms via Lemma 3.1 and Proposition 3.3, as presented in Table V. Analyzing the optimal strategies in Table V yields two key insights: higher consumer sensitivity necessitates increased advertising investments, whereas production investments increase as product emissions decrease. For practical implementation, BYD and Tesla should prioritize consumer needs by enhancing engagement—such as launching online product discussion forums—while intensifying energy conservation and emission reduction efforts. These strategies can attract additional investments and increase profits. The investment complementarity identified in our model suggests that Tesla should adopt BYD-like carbon reduction measures, such as developing green technologies and constructing zero-carbon parks, to increase its green reputation. These actions can simultaneously increase advertising and production investment levels, creating a positive feedback loop. We hope these recommendations provide actionable steps for firms advancing their green initiatives. Our case study enhances the practical relevance of our theoretical conclusions, demonstrating their applicability to real-world scenarios. However, the study has limitations: while NEVC data are suitable for electric vehicles, unified carbon emission indicators for other green products require further development. Additionally, we used annual data to mitigate seasonality effects, although incorporating seasonal factors into future models could refine the analysis and represent an interesting methodological extension.

VIII. Conclusion Remarks

A. Conclusions

Our research examines the interplay between production investment, advertising investment, and green reputation-building strategies. The firm begins by establishing a monopoly through advertising and production investments to build a strong green reputation. As competition intensifies, a new entrant forms a duopoly, after which the two firms collaborate to increase industry-wide green reputation. Finally, to optimize supply chain management and expand market presence, the firm integrates downstream retailers, establishing a two-tier structure where the manufacturer focuses on production investment and retailers drive advertising efforts. By integrating market competition dynamics into game-theoretic models, we analyze how green reputation influences firms' profit returns and drives the development of green products in the green economy. Through realistic mathematical modeling, our goal is to uncover the competitive advantages of green reputation in sustainable markets and provide decision-making support for firms designing green supply chain strategies.

First, green reputation generates positive (negative) profit effects at low (high) product emission levels, a pattern consistent across the monopoly, duopoly, and two-tier supply chain scenarios. Unlike prior studies that focused solely on reputation models in single-scenario settings [3], [19], we have conducted a detailed comparative analysis across the above three scenarios. This provides engineering managers and executives with a new perspective to guide production planning and investment management through green reputation strategies. Second, we identify a complementary relationship between advertising and production investments, which enhances the value of firms' capital allocation. Notably, enhancing green reputation strengthens this complementary relationship, as a product's green reputation is inversely linked to its emission level. This dynamic insight, which is unavailable in static models [3], [34], [39], offers practical guidance for adjusting investment levels in dynamic environments, enabling managers to visually assess how green reputation influences production and investment decisions. Firms must prioritize both product quality and sustainable development, as these dimensions are equally critical. Additionally, our analysis underscores the importance of monitoring competitors and reveals that competition can increase product green reputation. While competitors' market presence and emission levels may constrain firms' investment incentives, higher sales profit margins can offset this effect and stimulate production investment. This aligns with our model's narrative of firms adapting strategies to maximize competitive advantage by agilely responding to competitive and market dynamics. These insights are neglected in prior studies that focused solely on the interaction of low-carbon production between upstream and downstream firms [42], [44]. Finally, optimizing the green supply chain structure, such as expanding retailer networks and fostering manufacturer–retailer collaboration, enhances firms' green reputation and profitability while increasing their advertising and production investments. This finding mirrors the model's evolutionary trajectory from a monopoly to a duopoly to a two-tier supply chain, where enhanced green reputation and reduced emissions strengthen collaboration and shared profits. Such a finding is absent in earlier studies that focused solely on individual green production structures [21], [43]. The above insights offer feasible management strategies for businesses developing green products and supply chains, enabling them to enhance competitiveness and achieve sustainable growth.

B. Policy Implications

Firms should actively increase the quality of environmental information disclosure. This practice serves as a foundation for building a green reputation, as it reduces information asymmetry and strengthens investor trust. Second, firms should prioritize building a strong green reputation, particularly as markets saturate. This strategy not only reinforces the firm's social image but also attracts green-focused investors and financial institutions. Consequently, it lowers financing costs and drives higher advertising and production investments. Finally, firms should actively implement green supply chain management, collaborate with suppliers and partners to promote eco-friendly materials and reduce waste and emissions. This approach enhances not only the firm's own green reputation but also the competitiveness of the entire supply chain.

The government can promote environmental information disclosure and transparency by formulating stricter disclosure regulations (such as the Dual Credit Policy studied in this article). Thus, the government can ensure the truth and reliability of the information disclosed by firms. In addition, the government should encourage diversity in green supply chain management to maintain market vitality and promote the development of green industrial chains.

C. Limitations and Future Directions

There is still space for improvement in our research. First, our models presume an immediate impact of green reputation on sales, potentially ignoring delays in consumer behavior. Furthermore, although our study divides investment into advertising and production investment, this categorization remains insufficiently detailed. In reality, investment encompasses numerous complex dimensions. The model's applicability might be constrained by its generalizability across different industries and its reliance on quantifiable measures of capital stock's impact on green reputation. Second, we assume that there is competition among firms, but due to geographical conditions and time intervals, some firms may not have strong relationships, whereas others may have closer relationships. Therefore, building a more refined market network structure and supply chain structure is a direction for future research. Finally, for the case study, we collect data via the NEVC, which might not apply to other green products. Additionally, we overlooked seasonal effects by using annual data to represent the firm's performance over time, aiming to mitigate the influence of seasonality on our model analysis. The inclusion of seasonal factors and universal models in future work is of methodological and intellectual interest.

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