Upper and Lower Error Bounds for a Compact Fourth-Order Finite-Difference Scheme for the Wave Equation with Nonsmooth Data

A. Zlotnik

Higher School of Economics University, Moscow, 109028 Pokrovskii bd. 11 Russia e-mail: azlotnik@hse.ru

Abstract

A compact three-level fourth-order finite-difference scheme for solving the 1d wave equation is studied. New error bounds of the fractional order $\mathcal{O}(h^{4(\lambda-1)/5})$ are proved in the mesh energy norm in terms of data, for two initial functions from the Sobolev and Nikolskii spaces with the smoothness orders λ and $\lambda-1$ and the free term with a dominated mixed smoothness of order $\lambda-1$, for $1 \leq \lambda \leq 6$. The corresponding lower error bounds are proved as well to ensure the sharpness in order of the above error bounds with respect to each of the initial functions and the free term for any λ . Moreover, they demonstrate that the upper error bounds cannot be improved if the Lebesgue summability indices in the error norm are weakened down to 1 both in x and t and simultaneously the summability indices in the norms of data are strengthened up to ∞ both in t and t. Numerical experiments confirming the sharpness of the mentioned orders for half-integer t and piecewise polynomial data have already been carried out previously.

Keywords: wave equation, nonsmooth data, compact fourth-order scheme, error bounds, lower error bounds.

1 Introduction

Compact finite-difference schemes of the fourth approximation order $\mathcal{O}(h^4 + \tau^4)$ combine high accuracy of results with simplicity of implementation and therefore serve as an important tool to solve numerically various partial differential equations. Extensive literature is devoted to the construction, stability analysis, fourth-order error bounds for smooth data and solutions and computer application of such schemes. In the case of the wave equation and its linear generalizations, in particular, see [1]- [13] (this list is far from being complete) and a lot of references therein. However, theoretical analysis of the error of these schemes in the case of non-smooth initial data and the free term of the wave-type equations is complicated and almost absent, although in this case, the fourth order schemes also have confirmed advantages in the practical error behavior over the second order schemes.

In this paper, we study in detail the error of the known implicit three-level in time compact fourth-order finite-difference scheme for the initial-boundary value problem for the 1d wave equation. The essential point is the application of the two-level implicit approximation of the second initial condition $\partial_t u|_{t=0} = u_1$, without the derivatives of the data of the problem, in contrast to known explicit 5th order approximation of the solution at the first time level involving such derivatives. New error bounds of the fractional order $\mathcal{O}(h^{4(\lambda-1)/5})$, with h being the mesh step in x, are proved in the uniform in time mesh energy norm in terms of the data, rather than under assumptions on the exact solution, for the two initial functions u_0 and u_1 from the Sobolev spaces (for integer λ) and Nikolskii spaces (for noninteger λ) with the smoothness orders λ and $\lambda - 1$ and the free term f with a dominated mixed smoothness of order $\lambda - 1$, for $1 \leq \lambda \leq 6$. In particular, the difficult case of the piecewise smooth u_0 and discontinuous u_1 is covered for $\lambda = \frac{3}{2}$, and the case of the piecewise smooth $\partial_x u_0$ and piecewise smooth u_1

is covered for $\lambda = \frac{5}{2}$. Importantly, for any $\lambda > 1$, the above error bounds are better in order than the corresponding error bounds $\mathcal{O}((h+\tau)^{2(\lambda-1)/3})$, $1 \leq \lambda \leq 4$, for the second order finite element method (FEM) with a weight in [14]. Such an effect is absent for the elliptic and parabolic equations, in particular, see [15], but is known for the time-dependent Schrödinger equation [16]. Recall that the cases of discontinuous or nonsmooth data are of interest, in particular, for the corresponding optimal control problems, for example, see [17,18].

The proof is based on the interpolation of the error operator using the uniform in time energy bounds for the exact and approximate solutions using the weakest data norms and the error bound $\mathcal{O}(h^4)$ under the minimal Sobolev regularity of the exact solution followed by its expression in terms of the data. In both the results for the compact scheme, its close connection to the bilinear FEM with a special weight plays an important role. The proof defers essentially from the corresponding studies of the error of finite-difference schemes in [19] and lead to the better results, although in the special case $u_1 = 0$ and f = 0, the similar error bound in a weaker mesh norm and under stronger assumptions on u_0 was proved in [20].

The corresponding lower error bounds for the difference quotient in x of the error and the error itself are derived as well ensuring the sharpness in order of the above error bounds with respect to each of the initial functions and the free term for each λ . Moreover, they demonstrate that the upper error bounds cannot be improved if the Lebesgue summability indices in the error norm are weakened down to 1 both in x and t and simultaneously the summability indices in the norms of the data are strengthened up to ∞ both in x and t.

The proof of the lower error bounds exploits the explicit Fourier-type formulas for the exact and approximate solutions for the harmonic initial data u_0 and u_1 and the harmonic both in x and t free term f accompanied by a careful choice of the harmonic frequencies in dependence with h. These lower bounds go back to the lower error bounds for the second order FEM in [21, Theorem 2] whose proof was given in [22]; also the modified versions of these bounds have recently been presented in [23]. Later the lower error bounds for the error of the semi-discrete (continuous in time) second and higher order FEMs were given in [24, 25].

Notice that numerical experiments confirming the sharpness of the proved error bound for half-integer λ and piecewise polynomial data have already been carried out previously in [7].

The paper is organized as follows. In Section 2, we state the initial-boundary value problem for the 1d wave equation and the compact fourth-order scheme to solve it. We also prove for the scheme the stability theorem in the mesh energy norm in the form required in this paper. In Section 3, we derive the 4th order error bound under the minimal 6th order Sobolev regularity assumptions on the exact solution. In Section 4, we obtain fractional-order error bounds in terms of data using the results of two previous sections and the interpolation of the error operator. The last Section 5 is devoted to the derivation of the corresponding lower error bounds based on the Fourier representations of the exact and approximate solutions.

2 An initial-boundary value problem for the 1d wave equation and the compact fourth-order scheme to solve it

We deal with the initial-boundary value problem (IBVP) for the 1d wave equation

$$\partial_t^2 u - a^2 \partial_x^2 u = f(x, t), \quad (x, t) \in Q = Q_T = I_X \times I_T,$$
 (2.1)

$$u|_{x=0,X} = 0$$
, $u|_{t=0} = u_0(x)$, $\partial_t u|_{t=0} = u_1(x)$, $x \in I_X$ (2.2)

under the homogeneous Dirichlet boundary condition. Here a = const > 0, $I = I_X = (0, X)$ and $I_T = (0, T)$. Define the collection of data $\mathbf{d} = (u_0, u_1, f)$.

For $u_0 \in H_0^1(I)$, $u_1 \in L^2(I)$ and $f \in L^{2,1}(Q)$, this problem has a unique weak solution from the energy class (space) having the properties $u \in C(\bar{I}_T; H_0^1(I))$ and $\partial_t u \in C(\bar{I}_T; L^2(I))$, and the energy bound holds

$$\max_{0 \le t \le T} \left(\|\partial_t u(\cdot, t)\|_{L^2(I)}^2 + a^2 \|u(\cdot, t)\|_{H_0^1(I)}^2 \right)^{1/2} \le \left(a^2 \|u_0\|_{H_0^1(I)}^2 + \|u_1\|_{L^2(I)}^2 \right)^{1/2} + 2 \|f\|_{L^{2,1}(Q)}, \quad (2.3)$$

see, for example, [26, 27] (with an addition concerning f in [14]). Hereafter $H_0^1(I) = \{w \in W^{1,2}(I); w|_{x=0,X} = 0\}$ is a Sobolev subspace, $L^{2,q}(Q)$ is the anisotropic Lebesgue space and $C(\bar{I}_T; H)$ is the space of continuous functions on \bar{I}_T with the values in a Hilbert space H, endowed with the norms

$$\|w\|_{H^1_0(I)} = \|\partial_x w\|_{L^2(I)}, \ \|f\|_{L^{2,q}(Q)} = \|\|f\|_{L^2(I)}\|_{L^q(I_T)}, \ 1 \leqslant q \leqslant \infty, \ \|u\|_{C(\bar{I}_T;H)} = \max_{0 \leqslant t \leqslant T} \|u(\cdot,t)\|_{H^1}.$$

Since $C(\bar{I}_T; H_0^1(I)) \subset C(\bar{Q})$, such solutions u are continuous in \bar{Q} .

Let $\bar{\omega}_h$ and $\bar{\omega}^{\tau}$ be the uniform meshes with the nodes $x_i = ih$, $0 \leq i \leq N$, in \bar{I}_X and $t_m = m\tau$, $0 \leq m \leq M$, in \bar{I}_T , with the steps h = X/N and $\tau = T/M$. Let $\omega_h = \bar{\omega}_h \setminus \{0, X\}$, $\omega^{\tau} = \bar{\omega}^{\tau} \setminus \{0, T\}$ and $\bar{\omega}_h = \bar{\omega}_h \times \bar{\omega}^{\tau}$ with $\mathbf{h} := (h, \tau)$. Let $w_i = w(x_i)$, $y^m = y(t_m)$ and $v_i^m = v(x_i, t_m)$.

We introduce the mesh averages and finite-difference operators in x and t:

$$s_N w_i = \frac{1}{12} (w_{i-1} + 10w_i + w_{i+1}), \quad Bw_i = \frac{1}{6} (w_{i-1} + 4w_i + w_{i+1}), \quad \Lambda_x w_i = \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2},$$
$$\bar{s}_t y = \frac{y + \check{y}}{2}, \quad s_{Nt} y = \frac{1}{12} (\hat{y} + 10y + \check{y}), \quad \delta_t y = \frac{\hat{y} - y}{\tau}, \quad \bar{\delta}_t y = \frac{y - \check{y}}{\tau}, \quad \Lambda_t y = \frac{\hat{y} - 2y + \check{y}}{\tau^2},$$

where $\check{y}^m = y^{m-1}$, $\hat{y}^m = y^{m+1}$. The formulas hold

$$s_N = \mathbf{I} + \frac{1}{12}h^2\Lambda_x$$
, $B = \mathbf{I} + \frac{1}{6}h^2\Lambda_x$, $s_{Nt} = \mathbf{I} + \frac{1}{12}\tau^2\Lambda_t$, $\Lambda_t = \delta_t\bar{\delta}_t$,

where **I** is the identity operator. Here s_N and s_{Nt} are the Numerov averaging operators in x and t, and B is the averaging operator (the scaled mass matrix) corresponding to the linear finite elements.

We introduce the Euclidean space H_h of functions given on $\bar{\omega}_h$ and equal 0 at $x_i = 0, X$, endowed with the inner product

$$(v, w)_h = \sum_{i=1}^{N-1} v_i w_i h.$$

For a linear operator $A = A^* > 0$ acting in H_h , we also use the norm $||w||_A = (Aw, w)_h^{1/2}$ in H_h . Let ||A|| be its norm. The operators s_N , B and Λ_x can be considered as acting in H_h after setting $(s_N w)|_{i=0,N} = (Bw)|_{i=0,N} = (\Lambda_x w)|_{i=0,N} = 0$. It is well-known that then

$$\frac{2}{3}\mathbf{I} \leqslant s_N = s_N^* \leqslant \mathbf{I}, \quad \frac{1}{3}\mathbf{I} \leqslant B = B^* \leqslant \mathbf{I}, \quad 0 < -\Lambda_x = -\Lambda_x^* \leqslant \frac{4}{h^2}\mathbf{I}, \tag{2.4}$$

$$||w||_{-\Lambda_x}^2 = (-\Lambda_x w, w) = ||\bar{\delta}_x w||_{h^*}^2 := \sum_{i=1}^N (\bar{\delta}_x w_i)^2 h \text{ for } w \in H_h, \text{ with } \bar{\delta}_x w_i = \frac{w_i - w_{i-1}}{h}.$$

To solve the IBVP (2.1)–(2.2), we consider the known three-level fourth-order compact scheme

$$\left(s_N - \frac{1}{12}\tau^2 a^2 \Lambda_x\right) \Lambda_t v - a^2 \Lambda_x v = f_{\mathbf{h}} \quad \text{on} \quad \omega_h \times \omega^\tau, \tag{2.5}$$

$$\left(s_N - \frac{1}{12}\tau^2 a^2 \Lambda_x\right) \delta_t v^0 - \frac{\tau}{2} a^2 \Lambda_x v^0 = u_{1\mathbf{h}} + \frac{\tau}{2} f_{\mathbf{h}}^0 \quad \text{on} \quad \omega_h,$$
 (2.6)

$$v^0 = u_0 \text{ on } \bar{\omega}_h, \ v_i^m|_{i=0,N} = 0, \ 1 \le m \le M,$$
 (2.7)

in particular, see [1,7]. The usage of the two-level implicit initial condition (2.6) for equation (2.5) was suggested in [7], and this is essential for non-smooth data since it does not involve derivatives of the data in contrast to an alternative explicit formula for $v^m|_{m=1}$ having the 5th truncation order. The scheme has the 4th truncation order for smooth solutions u under the proper choice of $f_{\mathbf{h}}$, $f_{\mathbf{h}}^0$ and $u_{1\mathbf{h}}$. The standard choice of $f_{\mathbf{h}}$ is $f_{\mathbf{h}} = f + \frac{h^2}{12}\Lambda_x f + \frac{\tau^2}{12}\Lambda_t f$. It follows from [7] that, for example, one can also set

$$f_{\mathbf{h}} = q_h q_{\tau} f$$
 on $\omega_h \times \omega^{\tau}$, $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(0)} := s_N u_1 + \frac{\tau^2}{12} a^2 \Lambda_x u_1$ or $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(1)} := q_h u_1 + \frac{\tau^2}{12} a^2 \Lambda_x u_1$;

the form of $f_{\mathbf{h}}^0$ from [7] is omitted. The choice $f_{\mathbf{h}}^m = (q_h q_\tau f)^m$, $0 \le m \le M - 1$, is suitable for $f \in L^{2,1}(Q)$, and we use namely it below. But both the formulas for $u_{1\mathbf{h}}$ are not suitable for $u_1 \in L^2(I)$, and therefore another formula is suggested and applied below.

Hereafter, for $w \in L^1(I)$ and $z \in L^1(I_T)$ the following averages are applied

$$q_{0h}w_i = \frac{1}{2h} \int_{x_{i-1}}^{x_{i+1}} w(x) dx, \quad q_{0\tau}z^0 = \frac{1}{\tau} \int_0^{\tau} z(t) dt, \quad q_{0\tau}z^m = \frac{1}{2\tau} \int_{t_{m-1}}^{t_{m+1}} z(t) dt,$$

$$q_h w_i = \frac{1}{h} \int_{x_{i-1}}^{x_{i+1}} w(x) e_i^h(x) dx, \quad q_{\tau}z^0 = \frac{2}{\tau} \int_0^{\tau} z(t) e^{\tau,0}(t) dt, \quad q_{\tau}z^m = \frac{1}{\tau} \int_{t_{m-1}}^{t_{m+1}} z(t) e^{\tau,m}(t) dt,$$

where $1 \leq i \leq N-1$, $1 \leq m \leq M-1$, and the well-known "hat" functions

$$e_i^h(x) = \max\left\{1 - \left|\frac{x}{h} - i\right|, 0\right\}, \quad e^{\tau, m}(t) = \max\left\{1 - \left|\frac{t}{\tau} - m\right|, 0\right\}$$

from bases in the spaces of piecewise linear finite elements in x and t are applied.

We pass to a theorem on conditional stability of the introduced compact scheme. The known stability results, including that from [7], are not fully suitable for our purposes, and we need another one. Define the quantity

$$\|\{\check{v},v\}\|_{E_{\mathbf{h}}} = \left(\|\bar{\delta}_t v\|_B^2 + (\sigma_N - \frac{1}{4})a^2\|\bar{\delta}_t v\|_{-\Lambda_x}^2 + a^2\|\bar{s}_t v\|_{-\Lambda_x}^2\right)^{1/2}, \text{ with } \sigma_N := \frac{1}{12}\left(1 + \frac{h^2}{a^2\tau^2}\right).$$

It serves as the level energy norm for this scheme and is well defined under the stability condition

$$a^2 \tau^2 \leqslant \left(1 - \frac{\varepsilon_0^2}{2}\right) h^2 \tag{2.8}$$

with some $0 < \varepsilon_0 \le 1$. As it is shown below, the condition guarantees the lower bound

$$\varepsilon_0^2 \|\bar{\delta}_t v\|_B^2 + a^2 \|\bar{s}_t v\|_{-\Lambda_x}^2 \leqslant \|\{\check{v}, v\}\|_{E_{\mathbf{h}}}^2 \quad \text{for any} \quad \check{v}, v \in H_h.$$
 (2.9)

We also introduce the norm $||F||_{L_{\mathbf{h}}^{2,1}} = \tau \sum_{m=1}^{M-1} ||F^m||_h$ for functions F given on $\omega_h \times \omega^{\tau}$.

Theorem 2.1. Under the stability condition (2.8), for the compact scheme (2.5)–(2.7), the energy bound holds

$$\max_{1 \leq m \leq M} \|\{\check{v}, v\}^m\|_{E_{\mathbf{h}}} \\
\leq \left(a^2 \|(-\Lambda_x)^{1/2} v^0\|_h^2 + \varepsilon_0^{-2} \|B^{-1/2} u_{1\mathbf{h}}\|_h^2\right)^{1/2} + \varepsilon_0^{-1} \left(\|B^{-1/2} f_{\mathbf{h}}^0\|_h \tau + 2\|B^{-1/2} f_{\mathbf{h}}\|_{L^{2,1}}\right). \tag{2.10}$$

Here the initial data v^0 and u_{1h} and the free term f_h are arbitrary.

Proof. We rewrite equations (2.5)–(2.6) in virtue of the formula $s_N = B - \frac{h^2}{12} \Lambda_x$ in the form

$$(B - \sigma_N \tau^2 a^2 \Lambda_x) \Lambda_t v - a^2 \Lambda_x v = f_{\mathbf{h}} \quad \text{on} \quad \omega_h \times \omega^{\tau}, \tag{2.11}$$

$$\left(B - \sigma_N \tau^2 a^2 \Lambda_x\right) \delta_t v^0 - \frac{\tau}{2} a^2 \Lambda_x v^0 = u_{1\mathbf{h}} + \frac{\tau}{2} f_{\mathbf{h}}^0 \quad \text{on} \quad \omega_h.$$
 (2.12)

Importantly, these equations for $f_{\mathbf{h}} = q_h q_{\tau} f$ and $u_{1\mathbf{h}} = q_h u_1$ coincide with the equations of the bilinear FEM with the special weight $\sigma = \sigma_N$, see [14,21], which was indicated in [7].

Let $a^2(-\Lambda_x) \leq \alpha_h^2 B$ in H_h . According to [14, Section 2.2], under the condition

$$\sigma_N \geqslant \frac{1}{4} - \frac{1 - \varepsilon_0^2}{\tau^2 \alpha_h^2} \text{ with } 0 < \varepsilon_0 \leqslant 1,$$
 (2.13)

the following inequality holds

$$\varepsilon_0^2 \|w\|_B^2 \le \|w\|_B^2 + (\sigma_N - \frac{1}{4})\tau^2 a^2 \|w\|_{-\Lambda_x}^2 \text{ for } w \in H_h.$$

In virtue of the operator inequalities (2.4), one can set $\alpha_h^2 = \frac{12a^2}{h^2}$, then condition (2.13) becomes equivalent to (2.8), and the last inequality implies (2.9).

Now bound (2.10) follows from the stability result for an abstract three-level method with a weight [7, Theorem 1].

Recall that bounds (2.9) and (2.10) ensure existence and uniqueness of the approximate solution v.

Corollary 2.1. Let $u_0 \in H_0^1(I)$, $u_1 \in L^2(I)$ and $f \in L^{2,1}(Q)$. Under the stability condition (2.8) with some $0 < \varepsilon_0^2 \le 1$, bound (2.10) for $v^0 = u_0$, $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(2)} := (\mathbf{I} + \frac{\tau^2}{12}a^2\Lambda_x)q_hu_1$ with $q_hu_1|_{i=0,N} := 0$ and $f_{\mathbf{h}} = q_hq_\tau f$ implies the bound

$$\varepsilon_{0} \max \left\{ \max_{1 \leqslant m \leqslant M} \|\bar{\delta}_{t} v^{m}\|_{B}, \max_{0 \leqslant m \leqslant M} \frac{1}{\sqrt{6}} a \|\bar{\delta}_{x} v^{m}\|_{h*} \right\}
\leqslant \left(a^{2} \|u_{0}\|_{H_{0}^{1}(I)}^{2} + \varepsilon_{0}^{-2} \|u_{1}\|_{L^{2}(I)}^{2} \right)^{1/2} + 2\varepsilon_{0}^{-1} \|f\|_{L^{2,1}(Q)}. \tag{2.14}$$

Proof. According to [14, Lemma 2.1], under the condition

$$\sigma_N \geqslant \frac{1+\varepsilon_0^2}{4} - \frac{1}{\tau^2 \alpha_h^2}$$
 with $0 < \varepsilon_1 \leqslant 1$,

the following inequality holds

$$\varepsilon_1^2 a^2 \bar{s}_t(\|v\|_{-\Lambda_x}^2) \le \|\{\check{v}, v\}\|_{E_{\mathbf{h}}}^2 \text{ for any } \check{v}, v \in H_h.$$
 (2.15)

This condition on σ_N for $\alpha_h^2 = \frac{12a^2}{h^2}$ follows from the above condition (2.8) provided that $\varepsilon_1^2 \leqslant \varepsilon_0^2 \frac{h^2}{3a^2\tau^2}$. Thus, it is enough to assume that $\varepsilon_1^2 \leqslant \varepsilon_0^2/[3(1-\frac{\varepsilon_0^2}{2})]$ or simply to choose $\varepsilon_1^2 = \varepsilon_0^2/3$. Therefore, the left-hand side of the energy bound (2.10) is estimated from below by the left-hand side of bound (2.14).

Further, first, we have

$$\|(-\Lambda_x)^{1/2}u_0\|_h = \|\bar{\delta}_x u_0\|_{h^*} = \|q_{0h}^{(1)}\partial_x u_0\|_{h^*} \leqslant \|\partial_x u_0\|_{L^2(I)},\tag{2.16}$$

with $q_{0h}^{(1)}w_i := \frac{1}{h} \int_{x_{i-1}}^{x_i} w(x) dx$, $1 \leqslant i \leqslant N$. Second, under the stability condition (2.8), we have $0 < \mathbf{I} + \frac{\tau^2}{12} a^2 \Lambda_x < \mathbf{I}$ in H_h and thus $\|\mathbf{I} + \frac{\tau^2}{12} a^2 \Lambda_x\| \leqslant 1$. Therefore

$$||B^{-1/2}u_{1\mathbf{h}}^{(2)}||_h \leqslant ||B^{-1/2}q_hu_1||_h \leqslant ||u_1||_{L^2(I)},$$

where the latter inequality follows from [14]. Similarly, the following inequalities hold

$$||B^{-1/2}q_hq_{\tau}f^0||_h\tau + 2||B^{-1/2}q_hq_{\tau}f||_{L_{\mathbf{h}}^{2,1}} \leqslant \tau||q_{\tau}f^0||_{L^2(I)} + 2\tau \sum_{m=1}^{M-1} ||q_{\tau}f^m||_{L^2(I)}$$

$$\leqslant \tau(q_{\tau}||f(\cdot,t)||_{L^2(I)})^0 + 2\tau \sum_{m=1}^{M-1} (q_{\tau}||f(\cdot,t)||_{L^2(I)})^m \leqslant 2||f||_{L^{2,1}(Q)},$$

where the Minkowski generalized integral inequality is also applied. Consequently, for the chosen v^0 , $u_{1\mathbf{h}}$ and $f_{\mathbf{h}}$, the right-hand side of the energy bound (2.10) is majorized by the right-hand side of bound (2.14).

3 The 4th order error bound

In this Section, we consider solutions to the IBVP (2.1)–(2.2) having the additional regularity properties $\partial_x^2 u$, $\partial_t^2 u$, $\partial_x^4 u$, $\partial_t^4 u$, $\partial_x^4 \partial_t^2 u$, $\partial_x^2 \partial_t^4 u \in L^{2,1}(Q)$. Consequently, $\partial_x^4 u_1 = \partial_x^4 (\partial_t u)_0 \in L^2(I)$; hereafter $g_0 = g|_{t=0}$ for functions g depending on t.

Theorem 3.1. Under the stability condition (2.8), for the compact scheme (2.5)–(2.7) with $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(1)} = q_h u_1 + \frac{\tau^2}{12} a^2 \Lambda_x u_1$ and $f_{\mathbf{h}} = q_h q_{\tau} f$, the 4th order error bound holds

$$\max_{1 \leq m \leq M} \|\{\check{u} - \check{v}, u - v\}^m\|_{E_{\mathbf{h}}} \\
\leq ch^4 (\|\partial_x^4 \partial_t^2 u\|_{L^{2,1}(Q)} + \|\partial_x^2 \partial_t^4 u\|_{L^{2,1}(Q)} + \|\partial_x^2 \partial_t^3 u\|_{L^{2,1}(Q)} + \|\partial_x^4 u_1\|_{L^2(I)}). \tag{3.1}$$

Hereafter constants c, c_1 , c_2 , etc. can depend on X, T, ε_0 but are independent of \mathbf{h} .

Proof. Applying to the wave equation (2.1) the operator $q_h q_\tau$ leads to the equalities

$$\Lambda_t q_h u - a^2 \Lambda_x q_\tau u = q_h q_\tau f \text{ on } \omega_h \times \omega^\tau,$$

$$\delta_t q_h u^0 - \frac{\tau}{2} a^2 \Lambda_x q_\tau u^0 = q_h u_1 + \frac{\tau}{2} q_h q_\tau f^0 \text{ on } \omega_h.$$

They are mentioned in the proofs of Lemmas 1 and 2 in [7] and follow from the known formulas

$$\Lambda_x w = q_h(\partial_x^2 w), \quad \Lambda_t z = q_\tau(\partial_t^2 z), \quad \frac{\tau}{2} (q_\tau \partial_t^2 z)^0 = \delta_t z^0 - \partial_t z|_{t=0}, \tag{3.2}$$

for $w \in W^{2,1}(I)$ and $z \in W^{2,1}(I_T)$.

Therefore, in virtue of equations of the compact scheme (2.5)–(2.7), for the error r := u - v, we derive the following equation

$$(s_N - \frac{1}{12}\tau^2 a^2 \Lambda_x) \Lambda_t r - a^2 \Lambda_x r = (s_N - \frac{1}{12}\tau^2 a^2 \Lambda_x) \Lambda_t u - a^2 \Lambda_x u - (\Lambda_t q_h u - a^2 \Lambda_x q_\tau u)$$

$$= (s_N - q_h) \Lambda_t u - a^2 (s_{Nt} - q_\tau) \Lambda_\tau u \text{ on } \omega_h \times \omega^\tau$$

together with

$$(s_N - \frac{1}{12}\tau^2 a^2 \Lambda_x) \delta_t r^0 - \frac{\tau}{2} a^2 \Lambda_x r^0$$

$$= (s_N - \frac{1}{12}\tau^2 a^2 \Lambda_x) \delta_t u^0 - \frac{\tau}{2} a^2 \Lambda_x u^0 - (\delta_t q_h u^0 - \frac{\tau}{2} a^2 \Lambda_x q_\tau u^0) + q_h u_1 - u_{1\mathbf{h}}$$

$$= (s_N - q_h) \delta_t u^0 - a^2 \{ \left[\frac{\tau}{2} \mathbf{I} + \frac{1}{12} \tau^2 (\delta_t + \partial_t) - \frac{\tau}{2} q_\tau \right] \Lambda_x u \}^0 \text{ on } \omega_h,$$

$$r^0 = 0 \text{ on } \bar{\omega}_h, \quad r_i^m|_{i=0,N} = 0, \quad 1 \leqslant m \leqslant M.$$

Here we have used the formula $q_h u_1 - u_{1\mathbf{h}} = -a^2 \frac{\tau^2}{12} (\partial_t \Lambda_x u)^0$ for $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(1)}$. Therefore, due to the stability Theorem 2.1 and the estimate $||B^{-1/2}|| \leq \sqrt{3}$, the error bound holds

$$\max_{1 \leq m \leq M} \|\{\check{r}, r\}^m\|_{E_{\mathbf{h}}} \leq \sqrt{3}\varepsilon_0^{-1} \left(2\|(s_N - q_h)\Lambda_t u\|_{L_{\mathbf{h}}^{2,1}} + 2a^2\|(s_{Nt} - q_\tau)\Lambda_x u\|_{L_{\mathbf{h}}^{2,1}} + \|(s_N - q_h)\delta_t u^0\|_h + a^2\|\{\left[\frac{1}{2}\tau \mathbf{I} + \frac{1}{12}\tau^2(\delta_t + \partial_t) - \frac{1}{2}\tau q_\tau\right]\Lambda_x u\}^0\|_h\right).$$
(3.3)

Using the Taylor formula with the center at the points x_i , t_m and $t_0 = 0$ and the remainder in the integral form, one can verify the 4th order bounds

$$|(q_h w - w)_i| \leqslant ch^2(q_{0h}|\partial_x^2 w|)_i, \tag{3.4}$$

$$|(q_h w - s_N w)_i| \leqslant ch^4 (q_{0h} |\partial_x^4 w|)_i, \quad |(q_\tau z - s_{Nt} z)^m| \leqslant c\tau^4 (q_{0\tau} |\partial_t^4 z|)^m,$$

$$|\{\left[\frac{1}{2}\tau q_\tau - \frac{1}{2}\tau \mathbf{I} - \frac{1}{12}\tau^2 (\delta_t + \partial_t)\right] z\}^0| \leqslant c\tau^4 (q_{0\tau} |\partial_t^3 z|)^0,$$
(3.5)

where $1 \leq i \leq N-1$ and $1 \leq m \leq M-1$, for functions $w \in L^1(I)$ and $z \in L^1(I_T)$ having the derivatives $\partial_x^2 w$, $\partial_x^4 w \in L^1(I)$, $\partial_t^4 z \in L^1(I_T)$ and $\partial_t^3 z \in L^1(I_T)$, respectively. Clearly, the second and third of these bounds are quite similar. Bounds similar to all the presented ones are contained in the proofs of Lemmas 1 and 2 in [7].

In virtue of the listed bounds and the above formulas (3.2) for u, bound (3.3) implies

$$\max_{1 \leq m \leq M} \|\{\check{r}, r\}^m\|_{E_{\mathbf{h}}} \leq c \left(h^4 \|q_{0h}q_{\tau}|\partial_x^4 \partial_t^2 u|\|_{L_{\mathbf{h}}^{2,1}} + \tau^4 \|q_h q_{0\tau}|\partial_x^2 \partial_t^4 u|\|_{L_{\mathbf{h}}^{2,1}} + h^4 \|q_{0h}(q_{0\tau}|\partial_x^4 \partial_t u|)^0|\|_h + \tau^4 \|q_h(q_{0\tau}|\partial_x^2 \partial_t^3 u|)^0|\|_h\right).$$

The mesh norms of the averages of functions on the right-hand side of this bound are less or equal to the corresponding norms of the functions themselves, in general, with a multiplier c. Therefore, using also the stability condition (2.8), we obtain the further bound

$$\max_{1 \leqslant m \leqslant M} \| \{ \check{r}, r \}^m \|_{E_{\mathbf{h}}} \leqslant ch^4 (\| \partial_x^4 \partial_t^2 u \|_{L^{2,1}(Q)} + \| \partial_x^2 \partial_t^4 u \|_{L^{2,1}(Q)} + \| \partial_x^4 \partial_t u \|_{L^{2,\infty}(Q_\tau)} + \| \partial_x^2 \partial_t^3 u \|_{L^{2,\infty}(Q_\tau)}).$$

Applying the simple estimates for the 5th order derivatives of u of the form

$$\|\partial_x^4 \partial_t u\|_{L^{2,\infty}(Q_\tau)} \leq \|\partial_x^4 u_1\|_{L^2(I)} + \|\partial_x^2 \partial_t^4 u\|_{L^{2,1}(Q)},$$

$$\|\partial_x^2 \partial_t^3 u\|_{L^{2,\infty}(Q_\tau)} \leq \frac{1}{T} \|\partial_x^2 \partial_t^3 u\|_{L^{2,1}(Q)} + \|\partial_x^2 \partial_t^4 u\|_{L^{2,1}(Q)},$$
(3.6)

we complete the proof.

The following addition is important below.

Remark 3.1. The error bound (3.1) remains valid for $u_{1h} = u_{1h}^{(0)}$ and, in the case $u_1|_{x=0,X} = \partial_x^2 u_1|_{x=0,X} = 0$, also for $u_{1h} = u_{1h}^{(2)}$.

Proof. In virtue of the stability Theorem 2.1, the replacement of $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(1)}$ with $u_{1\mathbf{h}}^{(0)}$ or $u_{1\mathbf{h}}^{(2)}$ implies the appearance of the additional summands $\sqrt{3}\varepsilon_0^{-1}\|(s_N - q_h)u_1\|_h \leqslant ch^4\|\partial_x u_1\|_{L^2(I)}$ or $S := \sqrt{3}\varepsilon_0^{-1}a^2\frac{1}{12}\tau^2\|\Lambda_x(u_1 - q_hu_1)\|_h$, respectively, on the right-hand side of bound (3.3).

Let us bound S. To this end, we enlarge the above introduced notation for any $x \in I_X$:

$$(\tilde{\Lambda}_x w)(x) := \frac{1}{h^2} (w(x+h) - 2w(x) + w(x-h)), \quad (\tilde{q}_h w)(x) := \frac{1}{h} \int_{x-h}^{x+h} w(\xi) \left(1 - \frac{|\xi - x|}{h}\right) d\xi$$

for $w \in L^1(\tilde{I})$, where $\tilde{I} := (-X, 2X)$. Then the following formulas hold

$$(\Lambda_x q_h w)_i = \frac{1}{h^3} \int_{-h}^{X+h} w(x) (e_{i+1}^h - 2e_i^h + e_{i-1}^h)(x) dx = \frac{1}{h} \int_0^X (\tilde{\Lambda}_x w)(x) e_i^h(x) dx = (q_h(\tilde{\Lambda}_x w))_i,$$
$$(\tilde{\Lambda}_x w)(x) = \tilde{q}_h(\partial_x^2 w)(x), \quad x \in \bar{I}_X,$$

where $1 \leq i \leq N-1$ and, in the latter formula, it is assumed that $\partial_x^2 w \in L^1(\tilde{I})$.

If $w \in W^{4,2}(I)$, $w|_{x=0,X} = \partial_x^2 w|_{x=0,X} = 0$ and w is extended oddly with respect to x=0,X on \tilde{I} , then there exist $\partial_x^2 w$, $\partial_x^4 w \in L^2(\tilde{I})$ and $q_h w|_{x=0,X} = 0$. Therefore, the bound holds

$$\|\Lambda_x(u_1 - q_h u_1)\|_h = \|\tilde{\Lambda}_x u_1 - q_h \tilde{\Lambda}_x u_1\|_h \leqslant c_1 h^2 \|q_{0h}| \partial_x^2 \tilde{\Lambda}_x u_1\|_h \leqslant c_2 h^2 \|\partial_x^4 u_1\|_{L^2(I)}, \tag{3.7}$$

since bound (3.4) is valid and $\partial_x^2 \tilde{\Lambda}_x u_1 = \tilde{\Lambda}_x \partial_x^2 u_1 = \tilde{q}_h(\partial_x^4 u_1)$. Thereby $S \leqslant c\tau^2 h^2 \|\partial_x^4 u_1\|_{L^2(I)}$.

4 Fractional-order error bounds in terms of data

We first consider the Fourier series with respect to sines $w(x) = \sum_{k=1}^{\infty} \tilde{w}_k \sin \frac{\pi kx}{X}$ for functions $w \in L^2(I)$ and define the well-known Hilbert spaces of functions

$$\mathcal{H}^{\alpha}(I) = \left\{ w \in L^{2}(I); \ \|w\|_{\mathcal{H}^{\alpha}}^{2} = \sum_{k=1}^{\infty} \left(\frac{\pi k}{X}\right)^{2\alpha} \tilde{w}_{k}^{2} < \infty, \quad \tilde{w}_{k} = \sqrt{\frac{2}{X}} \int_{0}^{X} w(x) \sin \frac{\pi k x}{X} dx \right\}, \quad \alpha \geqslant 0.$$

Here $\mathcal{H}^0(I) \equiv L^2(I)$, and the space $\mathcal{H}^{\alpha}(I)$ for $\alpha = k \in \mathbb{N}$ coincides with the subspace in the Sobolev space

$$\mathcal{H}^k(I) = \left\{ w \in W^{k,2}(I); \ \partial_x^{2j} w|_{x=0,X} = 0, \ 0 \leqslant 2j < k \right\}, \quad \|w\|_{\mathcal{H}^k(I)} = \|\partial_x^k w\|_{L^2(I)}.$$

In particular, $\mathcal{H}^1(I) = H_0^1(I)$.

We also define the Banach spaces $S_{2,1}^{3,2}W(Q)$ and $S_{2,1}^{2,3}W(Q)$ of functions $f \in L^{2,1}(Q)$ having the dominating mixed smoothness, endowed with the norms

$$||f||_{S_{2,1}^{3,2}W(Q)} = ||\partial_t^2 f||_{L^1(I_T;\mathcal{H}^3(I))} + ||f_0||_{\mathcal{H}^4(I)} + ||(\partial_t f)_0||_{\mathcal{H}^3(I)},$$

$$||f||_{S_{2,1}^{2,3}W(Q)} = ||\partial_t^3 f||_{L^1(I_T;\mathcal{H}^2(I))} + ||f_0||_{\mathcal{H}^4(I)} + ||(\partial_t f)_0||_{\mathcal{H}^3(I)} + ||(\partial_t^2 f)_0||_{\mathcal{H}^2(I)};$$

in this respect see, in particular, [28–30]. Here it is assumed that respectively $\partial_t^{\ell} f \in L^1(I_T; \mathcal{H}^3(I))$ for $0 \leq \ell \leq 2$, $f_0 \in \mathcal{H}^4(I)$ or $\partial_t^{\ell} f \in L^1(I_T; \mathcal{H}^2(I))$ for $0 \leq \ell \leq 3$, $f_0 \in \mathcal{H}^4(I)$, $(\partial_t f)_0 \in \mathcal{H}^3(I)$. Note that the additional properties respectively $(\partial_t f)_0 \in \mathcal{H}^3(I)$ or $(\partial_t^2 f)_0 \in \mathcal{H}^2(I)$ follow from the listed ones. To simplify the notation of these spaces, we do not mark the arisen particular conditions of taking zero values at x = 0, X.

Let $H^{(k)} = \mathcal{H}^k(I)$ for integer $k \ge 0$, $F^{(0)} = L^{2,1}(Q)$ and $F^{(5)} = S_{2,1}^{3,2}W(Q) + S_{2,1}^{2,3}W(Q)$ be the sum of (compatible) Banach spaces, for example, see [31]. We define the Banach spaces of functions having an intermediate smoothness

$$H^{(\varkappa)} = (\mathcal{H}^k(I), \mathcal{H}^{k+1}(I))_{\varkappa - k, \infty}, \quad F^{(\lambda)} = (L^{2,1}(Q), S_{2,1}^{3,2}W(Q) + S_{2,1}^{2,3}W(Q))_{\lambda/5, \infty}, \quad 0 < \lambda < 5,$$

where $\varkappa > 0$ is noninteger and k is the integer part of \varkappa . Also $(B_0, B_1)_{\alpha,\infty}$ are the Banach spaces constructed by $K_{\alpha,\theta}$ -method of real interpolation between Banach spaces B_0 and B_1 , with $0 < \alpha < 1$ and $\theta = \infty$ [31]; hereafter we need only the case $B_1 \subset B_0$. The spaces $H^{(\varkappa)}$ for noninteger \varkappa are subspaces in the Nikolskii spaces $H_2^{\varkappa}(I)$, in more detail, see [31,32]. Let

 $V(\bar{I})$ be the space of functions of bounded variation on \bar{I} . Then $V(\bar{I}) \subset H^{(1/2)}$ that allows one to cover the practically important cases of discontinuous piecewise differentiable functions for $\varkappa = \frac{1}{2}$ and then functions having discontinuous kth order Sobolev derivative on \bar{I} for $\varkappa = k + \frac{1}{2}$, $k \ge 1$.

The exact explicit description of the spaces $F^{(\lambda)}$ is a separate problem in theory of functions of a real variable, but obviously they are more broad than when using the standard Sobolev spaces. Indeed, consider the Sobolev subspace $W_{(0)}^{5;2,1}(Q)$ endowed with the norm

$$||f||_{W_{(0)}^{5;2,1}(Q)} = \sum_{k=0}^{5} \sum_{l=0}^{5-k} ||\partial_x^k \partial_t^l f||_{L^{2,1}(Q)},$$

where all the appearing derivatives belong to $L^{2,1}(Q)$ and additionally $f|_{x=0,X}=\partial_x^2 f|_{x=0,X}=\partial_x^2 \partial_t^2 f|_{x=0,X}=0$. Then $W_{(0)}^{5;2,1}(Q)\subset S_{2,1}^{3,2}W(Q)\cap S_{2,1}^{2,3}W(Q)$ and consequently

$$(L^{2,1}(Q), W_{(0)}^{5;2,1}(Q))_{\lambda/5,\infty} \subset F^{(\lambda)}, \quad 0 < \lambda < 5.$$

Here we take into account the simple imbeddings

$$S_{2,1}^{1,1}W(Q) \subset C(\bar{I}_T; W^{1,2}(I)) \subset C(\bar{Q}),$$

where $S_{2,1}^{1,1}W(Q)$ is the space of functions $f \in W^{1;2,1}(Q)$ having the dominating mixed derivative $\partial_x \partial_t f \in L^{2,1}(Q)$, endowed with the norm

$$||f||_{S_{2,1}^{1,1}W(Q)} = ||f||_{L^{2,1}(Q)} + ||\partial_x f||_{L^{2,1}(Q)} + ||\partial_t f||_{L^{2,1}(Q)} + ||\partial_x \partial_t f||_{L^{2,1}(Q)}.$$

Namely, these imbeddings guarantee that all the derivatives up to and including the 3rd order of the functions $f \in W^{5;2,1}_{(0)}(Q)$ are continuous in \bar{Q} . Therefore, for such functions, we have $f(x,t)|_{x=0,X} = \partial_x^2 f(x,t)|_{x=0,X} = 0$ for all $0 \le t \le T$ and, consequently, the conditions of vanishing at x=0,X, including the corner points $(0,0),(X,0) \in \bar{Q}$, which participate in the above definitions of the spaces $S^{3,2}_{2,1}W(Q)$ and $S^{2,3}_{2,1}W(Q)$, are valid.

Let $w \in L^1(I)$ and w(x) be extended oddly with respect to x = 0, X on \tilde{I} . We introduce the one more average

$$q_{2h}w_i = q_h w_i - \frac{h^2}{12} \Lambda_x q_h w_i = \frac{1}{12} (-q_h w_{i-1} + 14q_h w_i - q_h w_{i+1}), \quad 1 \leqslant i \leqslant N - 1.$$

Here $q_h w|_{i=0,N} = 0$. If $\partial_x^4 w \in L^2(I)$ and $w|_{x=0,X} = \partial_x^2 w|_{x=0,X} = 0$, the bound holds

$$||w - q_{2h}w||_h \leqslant ch^4 ||\partial_x^4 w||_{L^2(I)}. \tag{4.1}$$

It follows from the formula

$$q_{2h}w - w = (q_h - s_N)w - \frac{h^2}{12}\Lambda_x(q_hw - w)$$

in virtue of the first bound (3.5) and bound (3.7).

Now we are ready to state and prove the main result of this Section, i.e., the general error bound of the fractional order in terms of the data.

Theorem 4.1. Consider any $1 \le \lambda \le 6$. Let $u_0 \in H^{(\lambda)}$, $u_1 \in H^{(\lambda-1)}$ and $f \in F^{(\lambda-1)}$. Under the stability condition (2.8), for the compact scheme (2.5)–(2.7) with $v^0 = u^0$, $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(2)}$ and $f_{\mathbf{h}} = q_h q_\tau f$, the error bound holds

$$\max_{1 \leq m \leq M} \left(\|\bar{\delta}_t (q_{2h} u - v)^m\|_h + \|\bar{\delta}_x (u - v)^m\|_{h*} \right)$$

$$\leq ch^{4(\lambda - 1)/5} \left(\|u_0\|_{H^{(\lambda)}} + \|u_1\|_{H^{(\lambda - 1)}} + \|f\|_{F^{(\lambda - 1)}} \right).$$
(4.2)

Proof. First, for $\lambda = 1$, we have

$$\max_{1 \leq m \leq M} \left(\|\bar{\delta}_{t}(q_{2h}u - v)^{m}\|_{h} + \|\bar{\delta}_{x}(u - v)^{m}\|_{h*} \right)$$

$$\leq \max_{1 \leq m \leq M} \left(\|\bar{\delta}_{t}q_{2h}u^{m}\|_{h} + \|\bar{\delta}_{x}u^{m}\|_{h*} \right) + \max_{1 \leq m \leq M} \left(\|\bar{\delta}_{t}v^{m}\|_{h} + \|\bar{\delta}_{x}v^{m}\|_{h*} \right)$$

$$\leq \max_{0 \leq t \leq T} \left(c\|\partial_{t}u(\cdot, t)\|_{L^{2}(I)} + \|\partial_{x}u(\cdot, t)\|_{L^{2}(I)} \right) + 4 \max \left\{ \max_{1 \leq m \leq M} \|\bar{\delta}_{t}v^{m}\|_{B}, \max_{1 \leq m \leq M} \|\bar{\delta}_{x}v^{m}\|_{h*} \right\},$$

where the norms of $\bar{\delta}_t q_{2h} u$ and $\bar{\delta}_x u$ are bounded similarly as above using the inequalities $||q_{2h}w||_h \leqslant \frac{4}{3}||q_hw||_h$ and (2.16). Therefore, in virtue of the energy bound (2.3) and Corollary 2.1, we get

$$\max_{1 \le m \le M} \left(\|\bar{\delta}_t (q_{2h} u - v)^m\|_h + \|\bar{\delta}_x (u - v)^m\|_{h*} \right) \le c \left(\|u_0\|_{H_0^1(I)} + \|u_1\|_{L^2(I)} + \|f\|_{L^{2,1}(Q)} \right).$$

This is the error bound (4.2) for $\lambda = 1$.

Second, let $u_0 \in \mathcal{H}^6(I)$ and $u_1 \in \mathcal{H}^5(I)$. In virtue of [14, Proposition 1.3], for $f \in S_{2,1}^{3,2}W(Q)$, the following regularity property holds for the solution to the IBVP (2.1)–(2.2):

$$\|\partial_{t}u\|_{C(\bar{I}_{T};\mathcal{H}^{5}(I))} + \|\partial_{t}^{2}u\|_{C(\bar{I}_{T};\mathcal{H}^{4}(I))} + \|\partial_{t}^{3}u\|_{C(\bar{I}_{T};\mathcal{H}^{3}(I))} + \|\partial_{t}^{4}\partial_{x}^{2}u\|_{L^{2,1}(Q)}$$

$$\leq c(\|u_{0}\|_{\mathcal{H}^{6}(I)} + \|u_{1}\|_{\mathcal{H}^{5}(I)} + \|f\|_{S_{2,1}^{3,2}W(Q)}), \tag{4.3}$$

and, for $f \in S_{2,1}^{2,3}W(Q)$, another regularity property holds

$$\|\partial_t^2 u\|_{C(\bar{I}_T;\mathcal{H}^4(I))} + \|\partial_t^3 u\|_{C(\bar{I}_T;\mathcal{H}^3(I))} + \|\partial_t^4 u\|_{C(\bar{I}_T;\mathcal{H}^2(I))} + \|\partial_t^5 \partial_x u\|_{L^{2,1}(Q)}$$

$$\leq c (\|u_0\|_{\mathcal{H}^6(I)} + \|u_1\|_{\mathcal{H}^5(I)} + \|f\|_{S_{2,1}^{2,3}W(Q)}). \tag{4.4}$$

The derivatives of u appearing in their left-hand sides exist and belong to the spaces in whose norms they stand. Therefore, with the help of inequalities (2.9) and (2.15) for $\varepsilon_1^2 = \varepsilon_0^2/3$, from the 4th order error bound (3.1) and Remark 3.1, for $f \in S_{2,1}^{3,2}W(Q) + S_{2,1}^{2,3}W(Q)$, the bound in terms of the data holds

$$\max_{1 \leq m \leq M} \left(\|\bar{\delta}_t (u - v)^m\|_h + \|\bar{\delta}_x (u - v)^m\|_{h*} \right)$$

$$\leq ch^4 \left(\|u_0\|_{\mathcal{H}^6(I)} + \|u_1\|_{\mathcal{H}^5(I)} + \|f\|_{S_{2,1}^{3,2}W(Q) + S_{2,1}^{2,3}W(Q)} \right). \tag{4.5}$$

Further, using bound (4.1) for $w - q_{2h}w$, we obtain the bound

$$\max_{1 \le m \le M} \|\bar{\delta}_t(u - q_{2h}u)\|_h \leqslant ch^4 \|\partial_t \partial_x^4 u\|_{L^{2,\infty}(Q)}.$$

Here we use the properties $u(x,t)|_{x=0,X}=\partial_x^2 u(x,t)|_{x=0,X}=0$ for any $0\leqslant t\leqslant T$ (notice that $a^2\partial_x^2 u=\partial_t^2 u-f$, where $\partial_t^2 u,f\in S_{2,1}^{1,1}W(Q)\subset C(\bar{Q})$). Therefore, with the help of estimate (3.6) for u, bound (4.5) remains valid for $\bar{\delta}_t(u-v)$ replaced with $\bar{\delta}_t(q_{2h}u-v)$, i.e., the error bound (4.2) holds for $\lambda=6$.

Finally, we consider the error operator R: $\mathbf{d} = (u_0, u_1, f) \to R\mathbf{d} = \{\bar{\delta}_t(q_{2h}u - v), \bar{\delta}_x(u - v)\};$ note that $(u-v)^0 = 0$ on $\bar{\omega}_h$. The appeared pair of errors belongs to the Banach space B_0 of pairs whose components $\{y, z\}$ are given on the meshes $\omega_h \times (\omega^\tau \cup \{T\})$ and $(\omega_h \cup \{X\}) \times (\omega^\tau \cup \{T\})$, endowed with the norm $\|\{y, z\}\|_{B_0} = \max_{1 \le m \le M} (\|y^m\|_h + \|z^m\|_{h^*}).$

We apply the basic theorem on interpolation of linear operators [31] to the operator R using the proven error bound (4.2) for $\lambda = 1$ and 6 which expresses bounds for the corresponding norms of R. Since R is linear, it is convenient to apply the theorem separately in u_0 (i.e., for $u_1 = f = 0$), u_1 (i.e., for $u_0 = f = 0$) and f (i.e., for $u_0 = u_1 = 0$), then add the results and finally derive the error bound (4.2) for $1 < \lambda < 6$. Here we applied the well-known imbedding $\mathcal{H}^{k+j}(I) \subset (\mathcal{H}^j(I), \mathcal{H}^{5+j}(I))_{k/5,\infty}$ for k = 1, 2, 3, 4 and j = 0, 1.

Note that, in the error bound (4.2), one can use subspaces in the Nikolskii spaces $H_2^{\lambda}(I)$ and $H_2^{\lambda-1}(I)$ instead of the Sobolev subspaces $H^{(\lambda)}$ and $H^{(\lambda-1)}$ for integer $\lambda=2,3,4,5$ as well (the former subspaces are slightly broader than the latter ones), but this is less explicit and has few applications for the wave equation.

5 Lower error bounds

To simplify further formulas, we can assume that $X = \pi$ and a = 1 by scaling. Recall the well-known spectral formulas

$$-\Lambda_x \sin kx = \lambda_k \sin kx \quad \text{on} \quad \omega_h, \quad \lambda_k = \left(\frac{2}{h} \sin \frac{kh}{2}\right)^2, \quad 1 \leqslant k \leqslant N - 1. \tag{5.1}$$

For $\mathbf{d}(x,t) = (\alpha_0, \alpha_1, g(t)) \sin kx$, where α_0 and α_1 are constants and $g \in L^1(I_T)$, the classical Fourier formula represents the solution to the IBVP (2.1)–(2.2):

$$u(x,t) = \left(\alpha_0 \cos kt + \frac{\alpha_1}{k} \sin kt + \frac{1}{k} \int_0^t g(\theta) \sin k(t-\theta) d\theta\right) \sin kx \text{ on } \bar{Q}.$$
 (5.2)

We need its counterpart for the compact scheme. For $y \in C(\bar{I}_T)$, let $\hat{s}_t y$ be its piecewise linear interpolant, i.e., $\hat{s}_t y(t_m) = y(t_m)$ on $\bar{\omega}^{\tau}$ and $\hat{s}_t y$ is linear on the segments $[t_{m-1}, t_m]$, $1 \leq m \leq M$.

Lemma 5.1. Let the stability condition (2.8) be valid and $1 \le k \le N-1$. For $\mathbf{d}(x,t) = (\alpha_0, \alpha_1, g(t)) \sin kx$ with $g \in L^1(I_T)$, the solution to the compact scheme (2.5)–(2.7) with $v^0 = u_0$, some $u_{1\mathbf{h}}$ such that $u_{1\mathbf{h}} = a_{1k}\alpha_1 \sin kx$ on $\bar{\omega}_h$ and $f_{\mathbf{h}} = q_h q_t f$, is represented as

$$v(x,t) = \left(\alpha_0 \cos \mu_k t + \hat{\gamma}_{1k} \frac{\alpha_1}{k} \sin \mu_k t + \frac{\gamma_{1k}}{k} \int_0^t g(\theta) \hat{s}_{\theta} \sin \mu_k (t - \theta) d\theta\right) \sin kx \quad on \quad \bar{\omega}_{\mathbf{h}}, \quad (5.3)$$

where the coefficients μ_k , $\hat{\gamma}_{1k}$ and γ_{1k} are given by the formulas

$$\mu_k = \frac{2}{\tau} \arcsin \frac{\tau \varphi_k}{2}, \quad \varphi_k = \left(\frac{\lambda_k}{1 - (h^2/6)\lambda_k + \tau^2 \sigma_N \lambda_k}\right)^{\frac{1}{2}},$$
 (5.4)

$$\hat{\gamma}_{1k} = a_{1k} \frac{2k}{\lambda_k \tau} \tan \frac{\mu_k \tau}{2}, \quad \gamma_{1k} = \frac{2}{k\tau} \tan \frac{\mu_k \tau}{2}. \tag{5.5}$$

In particular, for $u_{1\mathbf{h}} = u_{1\mathbf{h}}^{(0)}$, $u_{1\mathbf{h}}^{(1)}$ and $u_{1\mathbf{h}}^{(2)}$, we respectively have

$$a_{1k} = 1 - \frac{h^2 + \tau^2}{12} \lambda_k, \quad \frac{\lambda_k}{k^2} \left(1 - \frac{\tau^2 k^2}{12} \right), \quad \frac{\lambda_k}{k^2} \left(1 - \frac{\tau^2 \lambda_k}{12} \right).$$
 (5.6)

Hereafter, for brevity, the dependence of the coefficients on \mathbf{h} is not indicated.

This lemma follows from the more general one [23, Lemma 1.1] since the compact scheme can be considered as the bilinear FEM with the special weight $\sigma = \sigma_N$ that has already been used above.

Notice that $1 - \frac{h^2}{6}\lambda_k + \tau^2\sigma_N\lambda_k = 1 + \frac{\tau^2 - h^2}{12}\lambda_k$. Due to the stability condition (2.8), it is not difficult to check that

$$\min\left\{\frac{2}{3}, (1+\varepsilon_1)\frac{\tau^2}{4}\lambda_k\right\} \leqslant 1 + \frac{\tau^2 - h^2}{12}\lambda_k \leqslant 1 \text{ with } \varepsilon_1 = \frac{2}{3}\frac{\varepsilon_0^2}{1 - \varepsilon_0^2/2}$$

and thus

$$\frac{\tau}{2}\sqrt{\lambda_k} \leqslant \frac{\tau\varphi_k}{2} \leqslant \min\left\{\frac{1}{\sqrt{1+\varepsilon_1}}, \sqrt{\frac{3}{2}}\frac{\tau}{2}\sqrt{\lambda_k}\right\}. \tag{5.7}$$

Let us expand the terms μ_k , $\hat{\gamma}_{1k}$ and γ_{1k} .

Lemma 5.2. Let the stability condition (2.8) be valid and $1 \le k \le N-1$. The asymptotic formulas hold: $\mu_k \approx k$, i.e., $\underline{c}k \le \mu_k \le \bar{c}k$ with some $0 < \underline{c} < \bar{c}$ independent of \mathbf{h} and k, and

$$\mu_k = k - k^5 \nu_{\mathbf{h}} + \mathcal{O}(k^7 h^6) \quad with \quad \nu_{\mathbf{h}} := \frac{1}{480} (h^4 - \tau^4), \quad \frac{\varepsilon_0^2}{2} h^4 \leqslant 480 \nu_{\mathbf{h}} \leqslant h^4,$$
 (5.8)

$$\hat{\gamma}_{1k} = 1 + \mathcal{O}((kh)^2), \quad \gamma_{1k} = 1 + \mathcal{O}((kh)^2)$$
 (5.9)

for any $a_{1k} = 1 + \mathcal{O}((kh)^2)$ including those given by formulas (5.6).

Proof. The relation $\mu_k \simeq k$ together with $\mu_k \frac{\tau}{2} \leqslant \frac{\pi}{2} \frac{\tau \varphi_k}{2} \leqslant \frac{\pi}{2\sqrt{1+\varepsilon_1}}$ follow from inequalities (5.7) and the similar elementary relation $\sqrt{\lambda_k} \simeq k$.

and the similar elementary relation $\sqrt{\lambda_k} \approx k$. We set $\hat{h} = \frac{h}{2}$ and $\hat{\tau} = \frac{\tau}{2}$. Clearly the expansion holds

$$\sqrt{\lambda_k} = k \left[1 - \frac{1}{6} (k\hat{h})^2 + \frac{1}{120} (k\hat{h})^4 + \mathcal{O}((kh)^6) \right]$$

and therefore $\frac{\lambda_k}{k^2} = 1 - \frac{1}{3}(k\hat{h})^2 + \mathcal{O}((kh)^4)$ and $\frac{\lambda_k^2}{k^4} = 1 + \mathcal{O}((kh)^2)$. We also set $a := \frac{1}{3}(\frac{\hat{\tau}}{\hat{h}} - 1)$. Then we can write

$$\frac{\varphi_k}{k} = \left(\frac{\lambda_k}{1 + a\hat{h}^2\lambda_k}\right)^{1/2} = \frac{\sqrt{\lambda_k}}{k} \left[1 - \frac{1}{2}a\frac{\lambda_k}{k^2}(k\hat{h})^2 + \frac{3}{8}a^2\frac{\lambda_k^2}{k^4}(k\hat{h})^4 + \mathcal{O}((kh)^6)\right]
= \left[1 - \frac{1}{6}(k\hat{h})^2 + \frac{1}{120}(k\hat{h})^4\right] \left[1 - \frac{a}{2}(k\hat{h})^2 + \left(\frac{a}{6} + \frac{3}{8}a^2\right)(k\hat{h})^4\right] + \mathcal{O}((kh)^6)
= 1 - \left(\frac{1}{6} + \frac{a}{2}\right)(k\hat{h})^2 + \left(\frac{1}{120} + \frac{a}{4} + \frac{3}{8}a^2\right)(k\hat{h})^4 + \mathcal{O}((kh)^6).$$

Thus we obtain the expansion

$$\frac{\varphi_k}{k} = 1 - \frac{1}{6}(k\hat{\tau})^2 + b(k\hat{h})^4 + \mathcal{O}((kh)^6)$$
 with $b := \frac{1}{24}\frac{\tau^2}{h^2} - \frac{1}{30}$

Based on this expansion, further we derive

$$\frac{\mu_k}{k} = \frac{1}{k\hat{\tau}} \arcsin \hat{\tau} \varphi_k = \frac{\varphi_k}{k} + \frac{1}{6} \left(\frac{\varphi_k}{k}\right)^3 (k\hat{\tau})^2 + \frac{3}{40} \left(\frac{\varphi_k}{k}\right)^5 (k\hat{\tau})^4 + \mathcal{O}\left((k\hat{\tau})^6\right)$$

$$= 1 - \frac{1}{6} (k\hat{\tau})^2 + b(k\hat{h})^4 + \frac{1}{6} \left((k\hat{\tau})^2 - 3\frac{1}{6} (k\hat{\tau})^4\right) + \frac{3}{40} (k\hat{\tau})^4 + \mathcal{O}\left((kh)^6\right)$$

$$= 1 + k^4 \left(\frac{1}{24} \hat{\tau}^4 - \frac{1}{30} \hat{h}^4 - \frac{1}{120} \hat{\tau}^4\right) + \mathcal{O}\left((kh)^6\right) = 1 + k^4 \frac{1}{30} (\hat{\tau}^4 - \hat{h}^4) + \mathcal{O}\left((kh)^6\right),$$

and expansion (5.8) is proved.

Expansions (5.9) and $a_{1k} = 1 + \mathcal{O}((kh)^2)$ for a_{1k} given by formulas (5.6) follow from formulas $\lambda_k = k^2(1 + \mathcal{O}((kh)^2))$, $\mu_k = k(1 + \mathcal{O}((kh)^2))$ and the above inequality $\mu_k \frac{\tau}{2} \leqslant \frac{\pi}{2\sqrt{1+\varepsilon_1}}$.

Corollary 5.1. Let the stability condition (2.8) be valid. For any $\alpha > 0$ independent of \mathbf{h} , there exists an integer $1 \leq k_{\mathbf{h}} \leq N-1$ such that $k_{\mathbf{h}} \approx h^{-4/5}$ and

$$\mu_{k_{\mathbf{h}}} = k_{\mathbf{h}} - \alpha + \mathcal{O}(h^{2/5}).$$

Proof. We define $k_{\mathbf{h}} = [\rho_{\mathbf{h}}] + 1$, where $\rho_{\mathbf{h}} = \left(\frac{\alpha}{\nu_{\mathbf{h}}}\right)^{1/5}$ and $[\rho_{\mathbf{h}}]$ is its integer part. For h so small that $(\rho_{\mathbf{h}} + 2)h \leqslant \pi$ (without loss of generality), we get $1 \leqslant k_{\mathbf{h}} \leqslant N - 1$ and $0 \leqslant k_{\mathbf{h}} - \rho_{\mathbf{h}} \leqslant 1$. Using relations (5.8), we obtain $k_{\mathbf{h}} \approx \rho_{\mathbf{h}} \approx h^{-4/5}$ and

$$\mu_{k_{\mathbf{h}}} - (k_{\mathbf{h}} - \alpha) = (\rho_{\mathbf{h}}^5 - k_{\mathbf{h}}^5)\nu_{\mathbf{h}} + \mathcal{O}(k_{\mathbf{h}}^7 h^6) = \mathcal{O}(\rho_{\mathbf{h}}^4 \nu_{\mathbf{h}}) + \mathcal{O}(k_{\mathbf{h}}^7 h^6) = \mathcal{O}(h^{2/5})$$

that proves the result.

Let $x_{i-1/2} = (i - 0.5)h$ and $w_{i-1/2} = w(x_{i-1/2}), 1 \le i \le N$. We define the mesh L^1 norms

$$||w||_{L_h^1(I_\pi)} = \sum_{i=1}^N \frac{1}{2} (|w_{i-1}| + |w_i|) h, \quad ||w||_{L_{h*}^1(I_\pi)} = \sum_{i=1}^N |w_{i-1/2}| h, \quad ||\bar{\delta}_x w||_{L_h^1(I_\pi)} = \sum_{i=1}^N |\delta_x w_i| h,$$

$$||y||_{L_\tau^1(I_T)} = \sum_{i=1}^M \frac{1}{2} (|y_{j-1}| + |y_j|) \tau, \quad ||v||_{L_h^1(Q_T)} = |||v||_{L_h^1(I_\pi)} ||_{L_\tau^1(I_T)}.$$

For $w \in W^{1,1}(I_{\pi})$ and $y \in W^{1,1}(I_T)$, the simple bounds for differences between the continuous and mesh L^1 norms hold

$$\max\{\left|\|w\|_{L^{1}(I_{\pi})} - \|w\|_{L^{1}_{h}(I_{\pi})}\right|, \left|\|w\|_{L^{1}(I_{\pi})} - \|w\|_{L^{1}_{h,s}(I_{\pi})}\right| \le \|w'\|_{L^{1}(I_{\pi})}h, \tag{5.10}$$

$$\left| \|y\|_{L^{1}(I_{T})} - \|y\|_{L^{1}_{\tau}(I_{T})} \right| \leqslant \|y'\|_{L^{1}(I_{T})} \tau. \tag{5.11}$$

For any a > 0 and $y \in W^{1,1}(I_T)$, the estimate holds

$$\left| \int_0^T |y(t)\sin at| \, dt - \frac{2}{\pi} \int_0^T |y(t)| \, dt \right| \le \left(\|y'\|_{L^1(I_T)} + \frac{3}{2} \left(1 + \frac{\pi}{2} \right) \|y\|_{L^{\infty}(I_T)} \right) \frac{2}{a},\tag{5.12}$$

and it remains valid for $\sin at$ replaced with $\cos at$ (that is used below), see [23].

Below we use the collections of the harmonic data

$$\mathbf{d}_k^{(0)} = (\sin kx, 0, 0), \ \mathbf{d}_k^{(1)} = (0, \sin kx, 0), \ \mathbf{d}_k^{(2)} = (0, 0, (\sin kx)\sin(k-1)t).$$

Let $(u - v_h)[\mathbf{d}]$ be the compact scheme error for a given \mathbf{d} .

The proof of the lower error bounds is based on the following asymptotic behavior of the error norms for these specific data.

Theorem 5.1. Let l = 0, 1 and j = 0, 1, 2. For some $k = k_h \approx h^{-4/5}$, the following formula for the norms of the compact scheme error holds

$$\|\bar{\delta}_x^l(u - v_{\mathbf{h}})[\mathbf{d}_k^{(j)}]\|_{L_{\mathbf{h}}^1(Q_T)} = k^{-p_j + l} \left(\frac{4}{\pi} c_j(T) + \mathcal{O}(h^{1/5})\right), \tag{5.13}$$

with $p_0 = 0$ and $p_1 = p_2 = 1$. Also $c_0(T) = c_1(T) = 2(2K_T + 1 - \cos(T - K_T\pi))$, where K_T is the integer part of $\frac{T}{\pi}$, and $c_2(T) = T - \sin T$.

Proof. Let l=0,1 and $(x,t) \in \bar{\omega}_h$ excluding x=0 for l=1. We use $\alpha=2$ in Corollary 5.1.

1. First we consider the cases j = 0, 1. Due to the Fourier-type representations of solutions (5.2)–(5.3), we obtain the formulas for the error

$$\bar{\delta}_x^l(u - v_{\mathbf{h}})[\mathbf{d}_k^{(0)}](x, t) = \left[\cos kt - \cos(k - 2)t + r_k^{(0)}(t)\right]\bar{\delta}_x^l\sin kx,\tag{5.14}$$

$$\bar{\delta}_x^l(u - v_{\mathbf{h}})[\mathbf{d}_k^{(1)}](x, t) = \frac{1}{k} [\sin kt - \sin(k - 2)t + r_k^{(1)}(t)] \,\bar{\delta}_x^l \sin kx. \tag{5.15}$$

Here clearly we have

$$\cos kt - \cos(k-2)t = -2(\sin t)\sin(k-1)t, \quad \sin kt - \sin(k-2)t = 2(\sin t)\cos(k-1)t. \quad (5.16)$$

The remainders are

$$r_k^{(0)}(t) = \cos(k-2)t - \cos\mu_k t, \quad r_k^{(1)}(t) = \sin(k-2)t - \sin\mu_k t - (\hat{\gamma}_{1k} - 1)\sin\mu_k t,$$

and they satisfy the bounds

$$||r_k^{(0)}||_{C(\bar{I}_T)} \le |\mu_k - (k-2)|, ||r_k^{(1)}||_{C(\bar{I}_T)} \le |\mu_k - (k-2)| + |\hat{\gamma}_{1k} - 1|.$$

Moreover, due to estimates (5.10), (5.11) and (5.12), we obtain

$$\|\sin kx\|_{L^1_1(I_\pi)} = \|\sin kx\|_{L^1(I_\pi)} + \mathcal{O}(kh) = 2 + \mathcal{O}(kh), \tag{5.17}$$

$$\|\bar{\delta}_x \sin kx\|_{L_h^1(I_\pi)} = \sqrt{\lambda_k} \|\cos kx\|_{L_{h_*}^1(I_\pi)} = k(\|\cos kx\|_{L^1(I_\pi)} + \mathcal{O}(kh)) = k(2 + \mathcal{O}(kh)), \quad (5.18)$$

$$\|2(\sin t)\sin(k-1)t\|_{L_\pi^1(I_T)} = \|2(\sin t)\sin(k-1)t\|_{L^1(I_T)} + \mathcal{O}(k\tau)$$

$$= \frac{4}{\pi} \|\sin t\|_{L^1(I_T)} + \mathcal{O}(k\tau + k^{-1}) = \frac{2}{\pi} c_0(T) + \mathcal{O}(k\tau + k^{-1}) \text{ for } k \geqslant 2,$$
 (5.19)

and the last formula remains valid for $\sin(k-1)t$ replaced with $\cos(k-1)t$.

2. To consider the last case j=2, we define the functions

$$\begin{aligned}
y^{(\varkappa)}(t) \\
&:= \int_0^t (\sin(k-1)\theta) \sin \varkappa(t-\theta) \, d\theta = -\frac{1}{2} \frac{\sin(k-1)t - \sin \varkappa t}{(k-1) - \varkappa} + \frac{1}{2} \frac{\sin(k-1)t + \sin \varkappa t}{(k-1) + \varkappa}, \quad (5.20) \\
y_\varkappa(t) &:= \int_0^t (\sin(k-1)\theta) \hat{s}_\theta \sin \varkappa(t-\theta) \, d\theta,
\end{aligned}$$

with the parameter \varkappa such that $|\varkappa| \neq k-1$.

Once again due to the Fourier-type representations of solutions (5.2)–(5.3), we can write the formula for the error

$$\bar{\delta}_x^l(u - v_{\mathbf{h}})[\mathbf{d}_k^{(2)}](x, t) = \frac{1}{k} \left[(y^{(k)} - y^{(k-2)})(t) + r_k^{(2)}(t) \right] \bar{\delta}_x^l \sin kx.$$
 (5.21)

Here according to formula (5.20), we get

$$(y^{(k)} - y^{(k-2)})(t) = -\frac{1}{2}[\sin kt - 2\sin(k-1)t + \sin(k-2)t] + \frac{\theta_k}{k} = (1 - \cos t)\sin(k-1)t + \frac{\theta_k}{k}$$

with some $\theta_k \in [-1, 1]$. The reminder is

$$r_k^{(2)}(t) := (y^{(k-2)} - y^{(\mu_k)})(t) + (y^{(\mu_k)} - y_{\mu_k})(t) - (\gamma_{1k} - 1)y_{\mu_k}(t),$$

and it obeys the bound

$$||r_k^{(2)}||_{C(\bar{I}_T)} \leq T|\mu_k - (k-2)| + \max_{0 \leq t \leq T} \int_0^t |\sin \mu_k(t-\theta) - \hat{s}_\theta \sin \mu_k(t-\theta)| \, d\theta + T|\gamma_{1k} - 1|$$

$$\leq T(|\mu_k - (k-2)| + \mu_k^2 \tau^2 + |\gamma_{1k} - 1|),$$

where the elementary bound for the error of the linear interpolation has been applied.

3. Now we choose $k = k_h \approx h^{-4/5}$ according to Corollary 5.1 for $\alpha = 2$. Then, for j = 0, 2, using the above bounds for the reminders, expansions (5.9) and Corollary 5.1, we obtain

$$\left|\bar{\delta}_x^l(u-v_{\mathbf{h}})[\mathbf{d}_{k_{\mathbf{h}}}^{(j)}](x,t)\right| = k_{\mathbf{h}}^{-p_j} \left(\left|\zeta_j(t)\sin(k_{\mathbf{h}}-1)t\right|\left|\bar{\delta}_x^l\sin k_{\mathbf{h}}x\right| + \mathcal{O}(h^{2/5})\right)$$

where $\zeta_0(t) = 2\sin t$, $\zeta_2(t) = 1 - \cos t$ and \mathcal{O} -term is independent of (x, t). For j = 1, the same formula with $\sin(k_{\mathbf{h}} - 1)t$ replaced with $\cos(k_{\mathbf{h}} - 1)t$ is valid.

To derive formula (5.13), it remains to apply formulas (5.17)–(5.19) and the formula similar to (5.19) with $2 \sin t$ and $c_0(T)$ replaced with $1 - \cos t$ and $c_1(T)$.

Now we are ready to pass to the lower error bounds. For real $\lambda \geqslant 0$, let $C^{\lambda}(\bar{I}_{\pi})$ and $C^{\lambda}(\bar{Q}_{T})$ be the Hölder spaces of functions defined on \bar{I}_{π} and \bar{Q}_{T} , for example, see [33]. Recall that, for integer $\ell = \lambda$, they consist of functions continuous, for $\ell = 0$, and ℓ times continuously differentiable, for $\ell \geqslant 1$, in \bar{I}_{π} and \bar{Q}_{T} , respectively. Denote by $C^{\lambda}_{(0)}(\bar{I}_{\pi})$ and $C^{\lambda}_{(0)}(\bar{Q}_{T})$ their subspaces (equipped with the same norms) containing functions such that $\partial_{x}^{2k}w(x)|_{x=0,\pi} = 0$ and $\partial_{x}^{2k}f(x,t)|_{x=0,\pi} = 0$ together with $\partial_{t}^{2k}f(x,t)|_{t=0} = 0$, respectively, for $0 \leqslant 2k \leqslant [\lambda]$; note that here only the derivatives of the even order are involved.

Theorem 5.2. Let the stability condition (2.8) hold and l = 0, 1. There exist $h_0 > 0$ and $c_1 > 0$ such that, for $h \leq h_0$, the following lower error bounds with respect to u_0 , u_1 and f hold

$$\sup_{u_{j}(x)=\sin kx, k \in \mathbb{N}} \frac{\|\bar{\delta}_{x}^{l}(s_{x}u - v_{\mathbf{h}})[\mathbf{d}^{(j)}]\|_{L_{\mathbf{h}}^{1}(Q_{T})}}{\|u_{j}\|_{C_{(0)}^{\lambda-j}(\bar{I}_{\pi})}} \geqslant c_{1}h^{4(\lambda-l)/5}, \quad j = 0, 1,$$

$$(5.22)$$

$$\sup_{f(x,t)=(\sin kx)\sin(k-1)t, k\in\mathbb{N}, k\geqslant 2} \frac{\|\bar{\delta}_x^l(s_x u - v_{\mathbf{h}})[\mathbf{d}^{(2)}]\|_{L^1_{\mathbf{h}}(Q_T)}}{\|f\|_{C^{\lambda-1}_{(0)}(\bar{Q}_T)}} \geqslant c_1 h^{4(\lambda-l)/5}, \tag{5.23}$$

for any $l \leq \lambda \leq 5 + l$ such that, for l = 0, $\lambda \geq 1$ in (5.22) for j = 1 and in (5.23).

Proof. For natural k, it is not difficult to check (for example, see [23]) that the bounds hold

$$\|\sin kx\|_{C^{\lambda}_{(0)}(\bar{I}_{\pi})} \le ck^{\lambda}, \|(\sin kx)\sin(k-1)t\|_{C^{\lambda}_{(0)}(\bar{Q}_{T})} \le ck^{\lambda} \text{ for } \lambda \ge 0.$$
 (5.24)

Due to Theorem 5.1 and for $k = k_h$ chosen in it, the lower bound holds

$$\|\bar{\delta}_x^l(s_x u - v_{\mathbf{h}})[\mathbf{d}_{k_{\mathbf{h}}}^{(j)}]\|_{L^1(Q_T)} \geqslant c_1 k_{\mathbf{h}}^{-p_j+l}, \quad j = 0, 1, 2,$$

for $h \leq h_0$ and some $c_1 > 0$, with $h_0 > 0$ small enough. Combining it with bound (5.24), we get

$$\frac{\|\bar{\delta}_{x}^{l}(s_{x}u - v_{\mathbf{h}})[\mathbf{d}_{\mathbf{k_{h}}}^{(j)}]\|_{L_{\mathbf{h}}^{1}(Q_{T})}}{\|\sin k_{\mathbf{h}}x\|_{C_{(0)}^{\lambda-j}(\bar{I}_{\pi})}} \geqslant \frac{c_{1}}{k_{\mathbf{h}}^{\lambda-l}}, \quad j = 0, 1, \quad \frac{\|\bar{\delta}_{x}^{l}(s_{x}u - v_{\mathbf{h}})[\mathbf{d}_{\mathbf{k_{h}}}^{(2)}]\|_{L_{\mathbf{h}}^{1}(Q_{T})}}{\|(\sin k_{\mathbf{h}}x)\sin(k_{\mathbf{h}} - 1)t\|_{C_{(0)}^{\lambda-1}(\bar{Q}_{T})}} \geqslant \frac{c_{1}}{k_{\mathbf{h}}^{\lambda-l}}$$

for any $l \leqslant \lambda \leqslant 5 + l$ such that, for l = 0, we assume that $\lambda \geqslant 1$ in the former bound with j = 1 and in the latter one. Due to $k_{\mathbf{h}} \asymp h^{-4/5}$, these bounds lead to bounds (5.22)–(5.23). \square

Note that actually bounds (5.22)–(5.23) are valid for any $l \leqslant \lambda \leqslant 5 + l$ without exceptions, provided that the Hölder-type spaces of distributions $C_{(0)}^{\lambda}(\bar{I}_{\pi})$ and $C_{(0)}^{\lambda}(\bar{Q}_{T})$ for $-1 \leqslant \lambda < 0$ are suitably defined, see [23], but we do not dwell on that in the present paper.

Finally, the following imbeddings hold

$$||w||_{H^{(\lambda)}} \leqslant c||w||_{C^{\lambda}_{(0)}(\bar{I}_{\pi})}, \ 0 \leqslant \lambda \leqslant 6; \ ||f||_{F^{\lambda}} \leqslant c||f||_{C^{\lambda}_{(0)}(\bar{Q}_{T})}, \ 0 \leqslant \lambda \leqslant 5,$$

for any $w \in C_{(0)}^{\lambda}(\bar{I}_{\pi})$, $f \in C_{(0)}^{\lambda}(\bar{Q}_{T})$ and $X = \pi$. The latter one follows from the elementary property that if the four Banach spaces $B_{1} \subset B_{0}$ and $\hat{B}_{1} \subset \hat{B}_{0}$ are such that $\hat{B}_{i} \subset B_{i}$, i = 0, 1, then $(\hat{B}_{0}, \hat{B}_{1})_{\alpha,\infty} \subset (B_{0}, B_{1})_{\alpha,\infty}$ for $0 < \alpha < 1$. According to the lower bounds (5.22)–(5.23) for l = 1, the error bound (4.2) is sharp in order for each $0 \leq \lambda \leq 5$ with respect to each of the functions u_{0} , u_{1} and f. Moreover, the error bound cannot be improved if the summability index in the error norm is weakened down to 1 both with respect to x and x and

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