Tempering with the Ambrosio Tortelli Approximation of Mumford-Shah Regularizer

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INTRODUCTION

Image regularization is used for the purpose of reducing the computational resources to represent, process and classify an image. For regularization of images, energy functionals are commonly used. In this study, I modify the energy functional for Ambrosio Tortelli (AT) Approximation of Mumford-Shah Regularizer, to incorporate *temperature* (affecting diffusivity speed for the edge set). Then I investigate two possible modifications. The modifications result in good texture removal, edge strengthening and noise reduction.

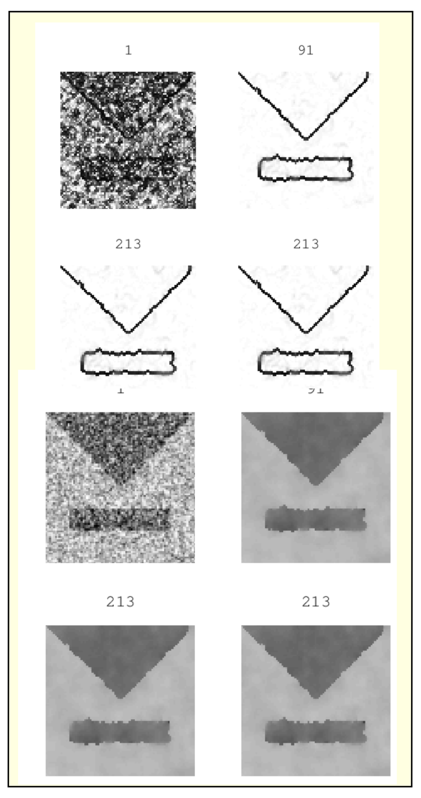


Fig. 1. Ambrosio Tortelli Approximation *with α = 5000, β = 200, delta = 0.25, ρ = 0.05 (edge set v)*

Fig. 2. Ambrosio Tortelli Approximation *with α = 5000, β = 200, delta = 0.25, ρ = 0.05 (reconstructed image u)*

THEORY & MODIFICATIONS

By minimizing the texture like details in the image and reducing edge count/length, a cartoon version of an image can be acquired via image regularization.

Energy functional proposed by Mumford-Shah:

$$E_{MS}(u,\S)=\beta\int_{\Omega}(u-g)^2\,dx+\alpha\int_{\Omega-\S}|\nabla u|^2\,dx+length(\S)$$

Ambrosio Tortelli Approximation to the Mumford-Shah:

$$E_{AT}(u,v)=\int_{\Omega}[\beta(u-g)^2+\alpha(v^2|\nabla u|^2)+\frac{1}{2}(\rho|\nabla v|^2+\frac{(1-v)^2}{\rho})]\,dx$$

My first modification to the Ambrosio Tortelli:

$$E_{AT}(u,v)=\int_{\Omega}[\beta|\nabla h|^2(u-g)^2+\alpha(v^2|\nabla u|^2)+\frac{1}{2}(\rho(h^2|\nabla v|^2)+\frac{(1-v)^2}{\rho})]\,dx$$

The model described in terms of PDEs:

$$\frac{\partial u}{\partial t}=\nabla.(v^2\nabla u)-\frac{\beta}{\alpha}|\nabla h|^2(u-g)$$

$$\frac{\partial v}{\partial t}=\nabla.(h^2\nabla v)-\frac{v-1}{\rho^2}-\frac{2\alpha v|\nabla u|^2}{\rho}$$

$$\frac{\partial h}{\partial t}=\nabla.(\beta(u-g)^2\nabla h)-\frac{h\rho|\nabla v|^2}{2}$$

Discretization of the above model:

$$u^{k+1}=(1+\Delta t\frac{\beta}{\alpha}|\nabla h|^2)^{-1}[u^k+\Delta t\frac{\beta}{\alpha}|\nabla h|^2 g+\frac{\Delta t}{2}(v^2.(Lu^k)-u^k.(Lv^2)+L(v^2.u^k))]$$

$$v^{k+1}=(1+\frac{\Delta t}{\rho^2}+\frac{2\alpha\Delta t U_s}{\rho})^{-1}[v^k+\frac{\Delta t}{2}[h^2.(Lv^k)-v^k.(Lh^2)+L(h^2.v^k)]+\frac{\Delta t}{\rho^2}]$$

$$h^{k+1}=(1+\frac{\rho\Delta t|\nabla v|^2}{2})^{-1}[h^k+\Delta t\nabla.(\beta(u-g)^2\nabla h^k)]$$

My second modification to the Ambrosio Tortelli:

$$E_{AT}(u,v)=\int_{\Omega}[\frac{\beta}{\alpha\omega}|\nabla h|^2(u-g)^2+(\frac{(1-\omega)}{\omega}v^2|\nabla u|^2+h^2|\nabla u|^2)+\frac{1}{2\alpha\omega}(\rho(h^2|\nabla v|^2)+\frac{(1-v)^2}{\rho})]\,dx$$

The model described in terms of PDEs:

$$\frac{\partial u}{\partial t}=\nabla.([(1-\omega)v^2+\omega h^2]\nabla u)-\frac{\beta}{\alpha}|\nabla h|^2(u-g)$$

$$\frac{\partial v}{\partial t}=\nabla.(h^2\nabla v)-\frac{(v-1)}{\rho^2}-\frac{2\alpha v(1-\omega)|\nabla u|^2}{\rho}$$

$$\frac{\partial h}{\partial t}=\nabla.((u-g)^2\nabla h)-\frac{h(\rho|\nabla v|^2+2\alpha\omega|\nabla u|^2)}{2\beta}$$

Discretization of the above model:

$$u^{k+1}=(1+\frac{\Delta t\beta|\nabla h|^2}{\alpha})^{-1}[u^k+\frac{\Delta t}{2}[b^2.(Lu^k)-u^k.(Lb^2)+L(b^2.u^k)]+\frac{\Delta t\beta|\nabla h|^2 g}{\alpha}]$$

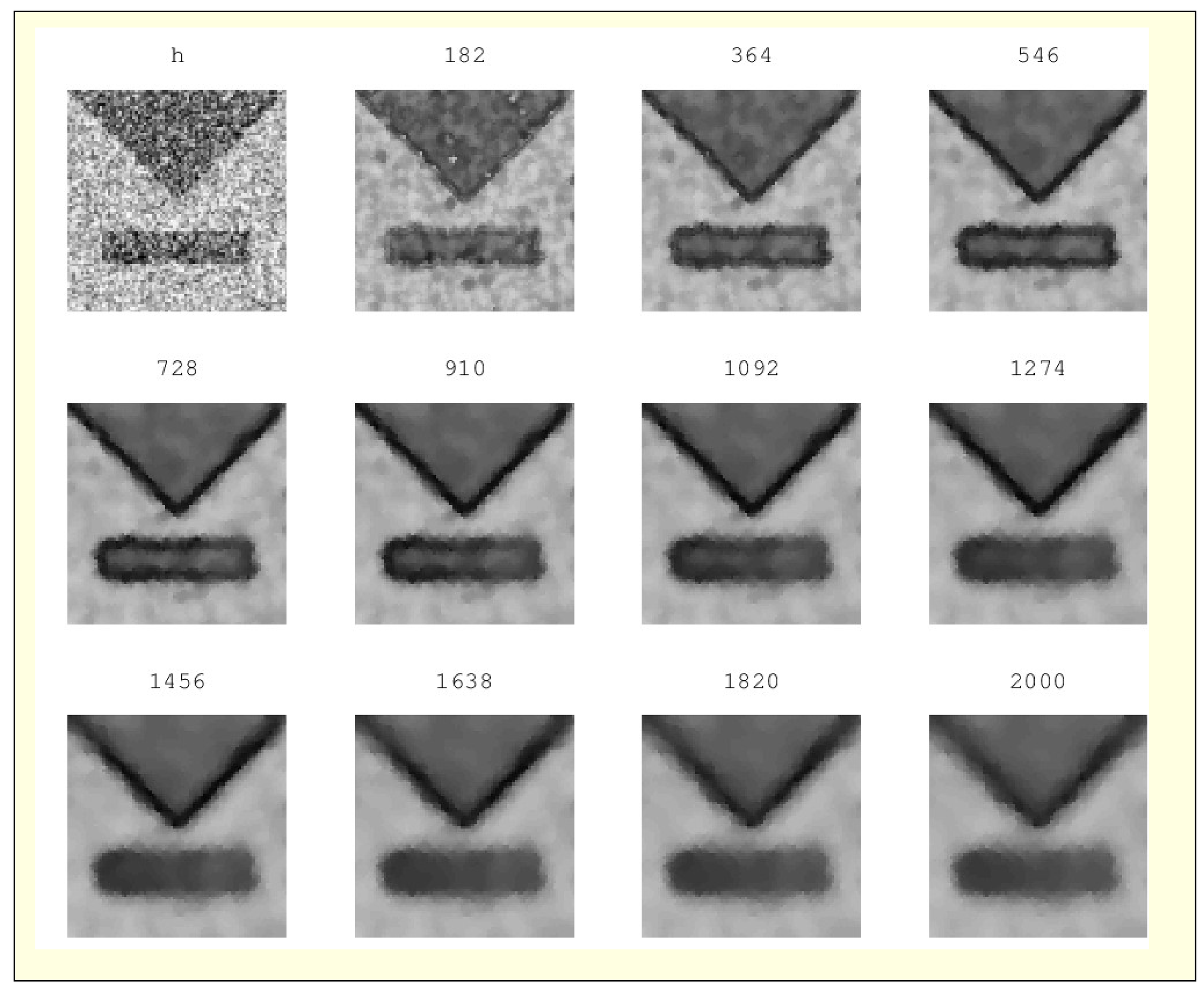
$$(b=(1-\omega)v^2+\omega h^2)$$

$$v^{k+1}=(1+\frac{\Delta t}{\rho^2}+\frac{2\Delta t\alpha(1-\omega)|\nabla u|^2}{\rho})^{-1}[v^k+\Delta t\nabla.(h^2\nabla v^k)+\frac{\Delta t}{\rho^2}]$$

$$h^{k+1}=(1+\frac{\Delta t(\rho|\nabla v|^2+2\alpha\omega|\nabla u|^2)}{2\beta})^{-1}[h^k+\Delta t\nabla.((u-g)^2\nabla h^k)]$$

** Discretization of div(grad(xy)) skipped for v and h, since they can be calculated as described in u.*

RESULTS



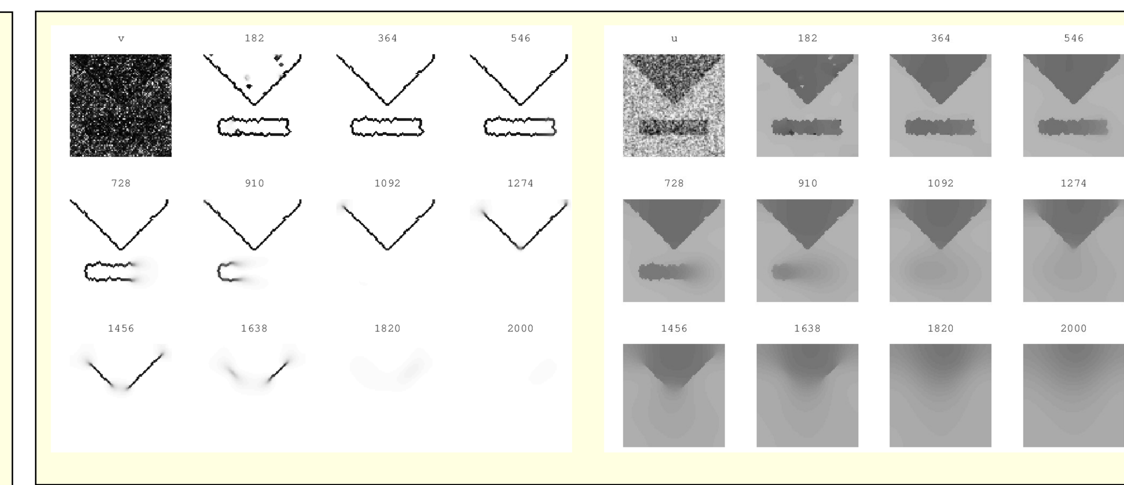
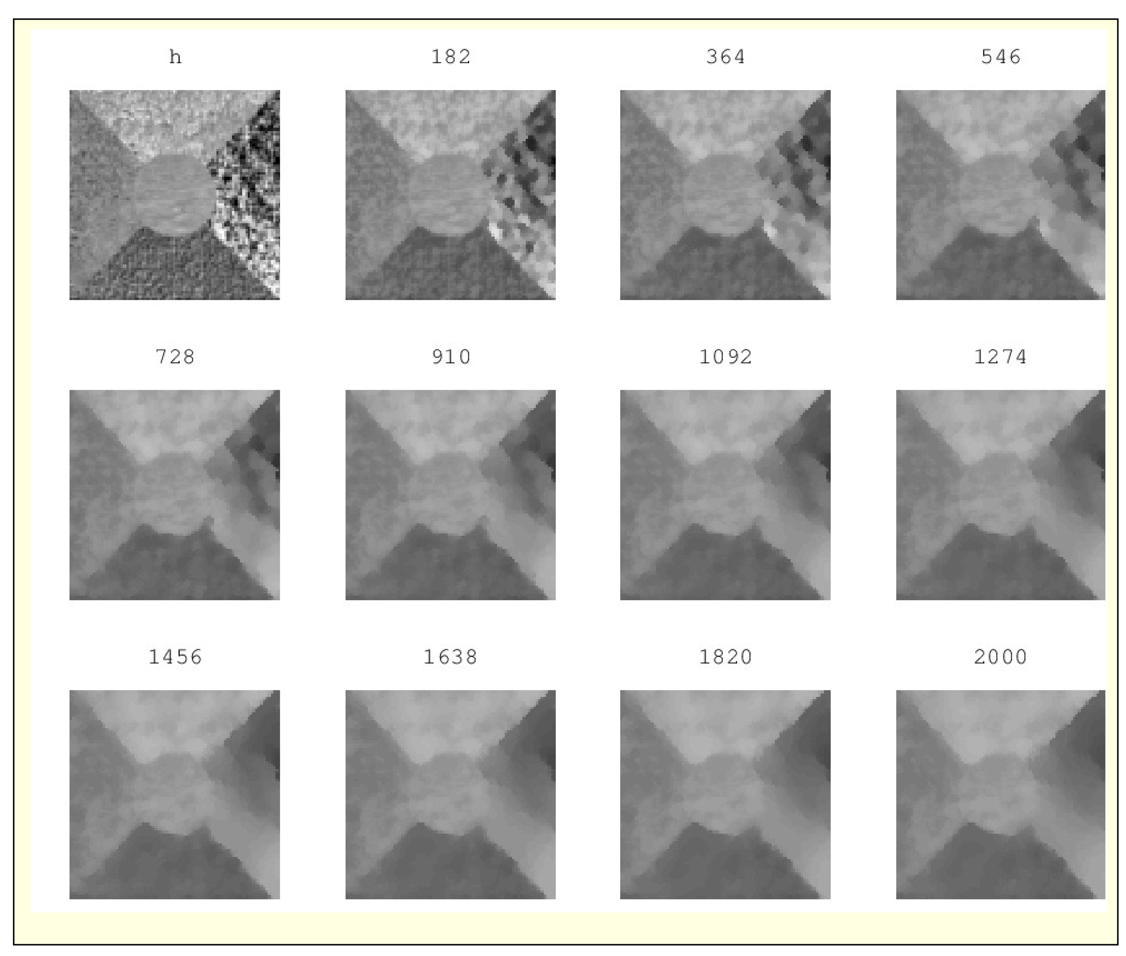


Fig. 3-4. *Modification 1.* Strengthening of edges according to the edge set, (left:h, right:u,v, *α = 4000, β =1, ρ=0.08*)

Increasing *α* dramatically,results in *cold* regions in the temperature field that correspond to the edges in the edge set. This is due to the fact, if *α is relatively much higher than β,* the main problem becomes decreasing edge(v)-gradient(u) inconsistency in the system, *step by step*. The solution can be either allowing the system to quickly dissolve into uniform state (where temporary allowing edge-shape inconsistency) or settling in a local minimum where the effects of interactions are minimized, as illustrated in left.



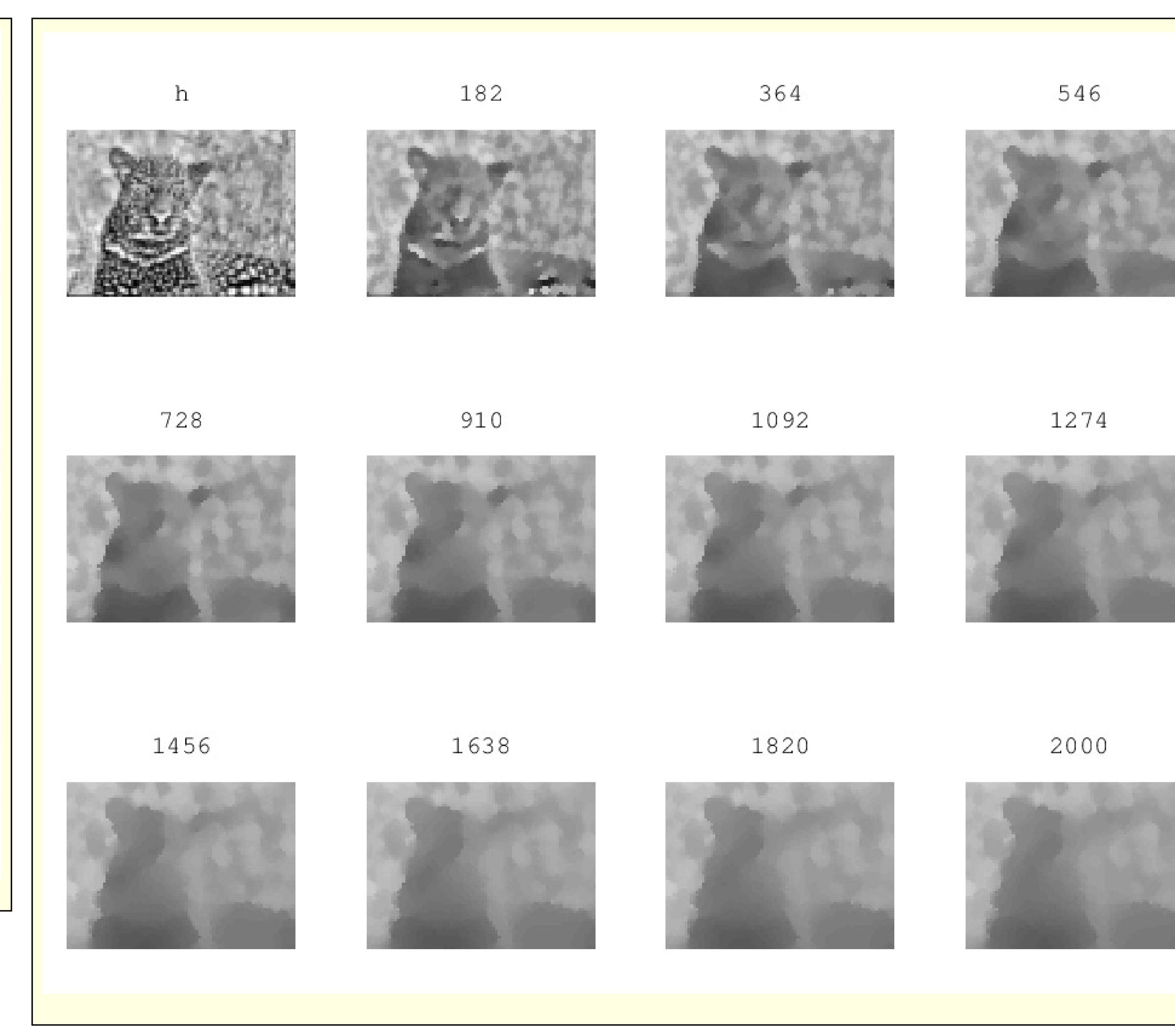
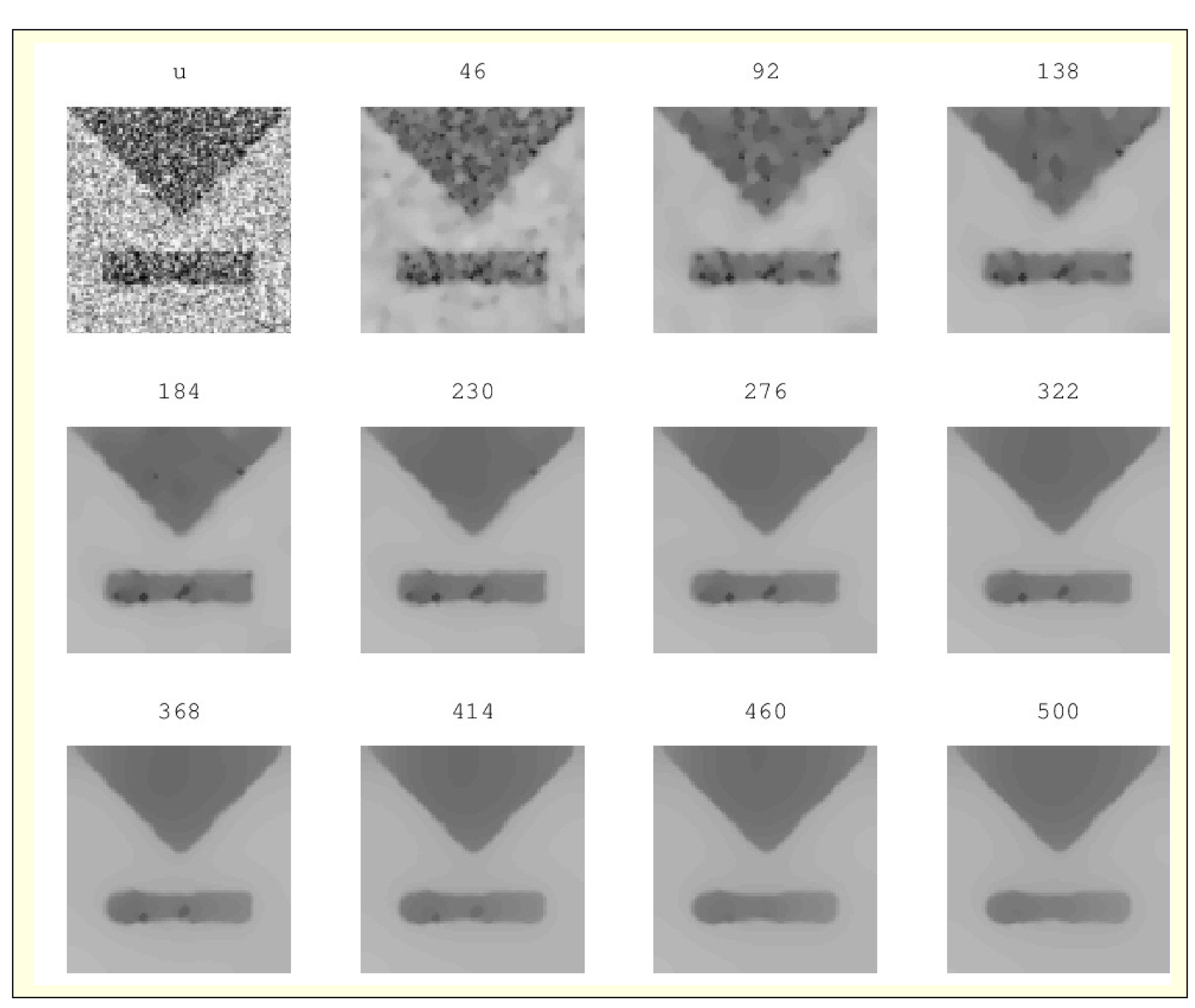


Fig. 5-6. *Modification 1.* Texture removal (*α = 0.005, β=1.0 ρ=0.5*) Small *α/β* yields isolated regions to diffuse quickly in a scale dependent way (***almost hierarchically***)



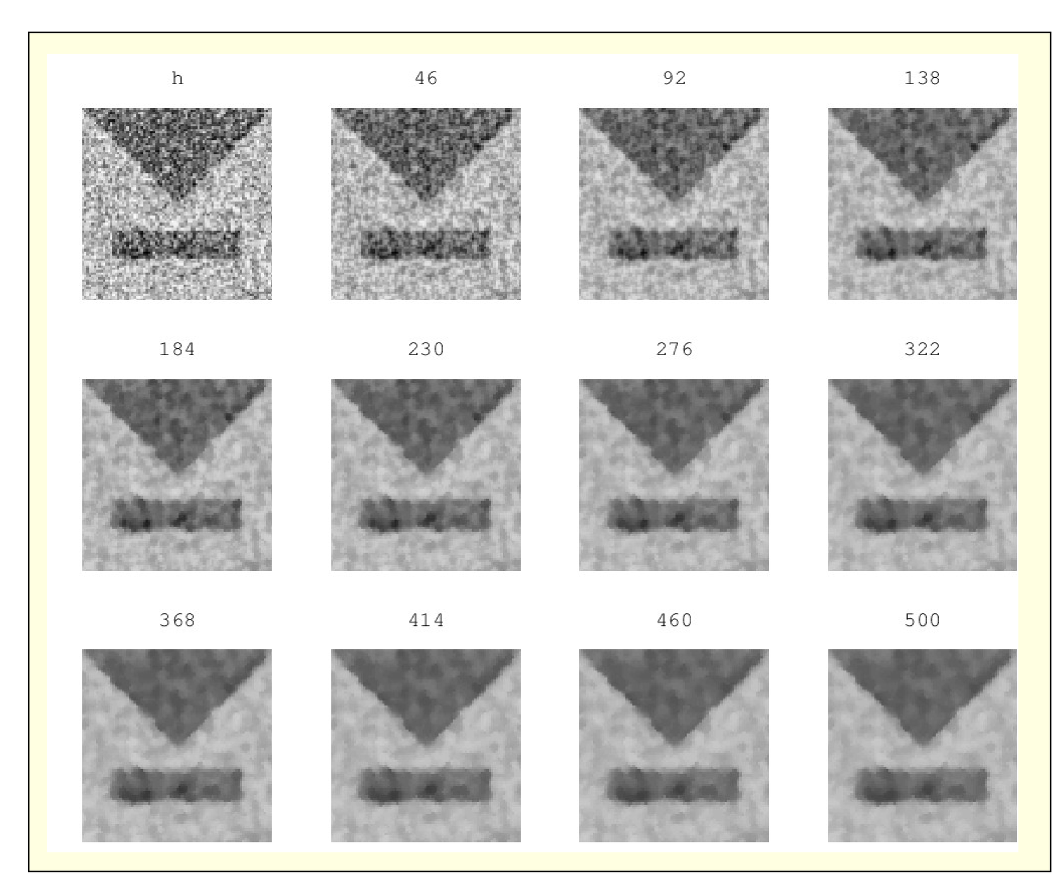


Fig. 7-8. *Modification 2.* Noise reduction (*α=10000, β=5000, rho=0.2, ω=0.2, h=g*) Thanks to ω, both temperature field (h) and edge set (v) contributes to u. This improves the results of original AT *(Fig 2)*

DISCUSSIONS & FUTURE WORK

Image regularization requires one to use a formal representation that can incorporate assumptions in a compact but expressive way. Mumford-Shah Model and its AT approximation is able to efficiently incorporate the assumption that edges and texture should be minimum for a cartoon like image. However the model does not express a bias on which edges are more important.

In my model, I assumed that we can ignore edges that are on small isolated regions and incorporated this assumption by modifying the AT model. I added diffusion speed to the edge set v and called it h (temperature field). I then assumed that for the regions where the gradient of the temperature field is zero, fidelity component for u is not important and also added this idea into the AT model (modification 1). Finally, I also reasoned that aside from edge set v, temperature field h can also be used when adjusting the edges of u and also incorporated this into the my modified model (the term with ω, modification 2). This allowed me to control the contribution of h and v to u (at each update).

The results showed that the temperature field h can be flexibly used for various purposes. Fig 3-6 illustrates how the temperature field can be used for removing texture and strengthening edges). Fig 7-8 illustrates a more controlled h using ω. Overall, the idea of incorporating assumptions in the form of new fields may deserve a more rigorous investigation. After a literature survey, this will be further investigated in a research paper.