(CS1.301) Advanced Algorithm Design (Monsoon 2022) Due: 12.11.2022

Problem Set 4

Instructions:

- Discussions amongst the students are not discouraged, but all writeups must be done individually and must include names of all collaborators.
- Referring sources other than the lecture notes is discouraged as solutions to some of the problems can be found easily via a web search. But if you do use an outside source (eg., text books, other lecture notes, any material available online), do mention the same in your writeup. This will not affect your grades. However dishonesty of any sort when caught shall be heavily penalized.
- Be clear in your arguments. Vague arguments shall not be given full credit.
- Total marks will be normalized.

Question 1 [4 marks]

Prove or disprove the following. Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e; and let (A, B) be a mimimum s-t cut with respect to these capacities $c_e : e \in E$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum s-t cut with respect to these new capacities $\{1 + c_e : e \in E\}$

Question 2 [6 marks]

Network flow issues come up in dealing with natural disasters and other crises, since major unexpected events often require the movement and evacuation of large numbers of people in a short amount of time.

Consider the following scenario. Due to large-scale flooding in a region, paramedics have identified a set of n injured people distributed across the region who need to be rushed to hospitals. There are k hospitals in the region, and each of the n people needs to be brought to a hospital that is within a half-hour's driving time of their current location (so different people will have different options for hospitals, depending on where they are right now).

At the same time, one does not want to overload any one of the hospitals by sending too many patients its way. The paramedics are in touch by cell phone, and they want to collectively work out whether they can choose a hospital for each of the injured people in such a way that the load on the hospitals is balanced: Each hospital receives at most $\lceil \frac{n}{k} \rceil$ people. Give a polynomial-time algorithm that takes the given information about the people's locations and determines whether this is possible.

Question 3 [6 marks]

Suppose you are given a directed graph $G = (V, E)$, with a positive integer capacity c_e on each edge e, a source $s \in V$, and a sink $t \in V$.You are also given a maximum s-t flow in G, defined by a flow value f_e on each edge e. The flow f is acyclic: There is no cycle in G on which all edges carry positive flow. The flow f is also integer-valued.

Now suppose we pick a specific edge $e' \in E$ and reduce its capacity by 1 unit. Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of nodes.

Question 4 [6 marks]

Consider the following problem. You are given a flow network with unit capacity edges: It consists of a directed graph $G = (V, E)$, a source $s \in V$, and a sink $t \in V$; and $c_e = 1$ for every $e \in E$. You are also given a parameter k.

The goal is to delete k edges so as to reduce the maximum s-t flow in G by as much as possible. In other words, you should find a set of edges $F \subseteq E$ so that $|F| = k$ and the maximum s-t flow in $G' = (V, E - F)$ is as small as possible subject to this. Give a polynomial-time algorithm to solve this problem.

Question 5 [5 marks]

Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and nonnegative edge capacities c_e. Give a polynomial-time algorithm to decide whether G has a unique minimum s-t cut (i.e., an s-t of capacity strictly less than that of all other s-t cuts).

Question 6 [6 marks]

Given a bipartite matching $G = (V_1, V_2, E)$ ($|V_1| = |V_2| = n$) with a promise that G has a perfect matching, we would like to obtain it using network flow algorithm. Towards it, we do the following.

- Add a capacity of 1 to all edges in E.
- Add nodes s and t such that s is connected to all vertices in V_1 with outward edges with capacity 1, and t is connected from all vertices in V_2 with inward edges with capacity 1.

Show that the maximum flow in this new flow network gives us a perfect matching.