DISCRETE STRUCTURES (MA5.101)

Assignment - Set - 2

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Functions

1(a) Let an injective mapping f from a finite set A to itself, that is $f: A \to A$. Prove (or) disprove f is a bijective with proper justification.

(b) Let a function f is defined by $f: A \to B$. The function f is invertible iff f is bijective

1(a)

Answer:- Given that, $f: A \to A$ is an injection where 'A' is a finite set.

As 'A' is a finite set $\Rightarrow$ 'f' is surjection also

Proof:- proof by Contradiction:-

Let us assume that the conclusion is wrong, that is 'f' is not a surjection.

$\Rightarrow$ there exists an element $x_0 \in A$ such that it does'not have a pre-image in the Domain set.

Now say $B = A - \{x_0\}$ and a mapping $g : A \to B$, here 'g' is same as 'f' but it has an element less in codomain (Restriction of codomain)

$\therefore$ 'g' is also an injection

but for 'g' $|A| = m \Rightarrow |B| = m-1$

So $\boxed{|A| > |B|}$ This contradicts ~~one~~ our hypothesis that 'g' is an injection and this also implies that 'f' is also not an injection (As for a one-one func $|domain| <= |codomain|$)

So, as we reached at a contradiction.

We can tell that $f: A \to A$ is bijection, if if 'f' is injection and 'A' is a finite set)

5(b)
Answer:

Given, $f: A \to B$

To prove: f is invertible iff 'f' is a bijection

To prove this statement we have to prove 2 things.

(i) if 'f' is a bijection then 'f' is invertible
(ii) if 'f' is invertible then f is a bijective

(i) If 'f' is a bijection, we have 2 cases

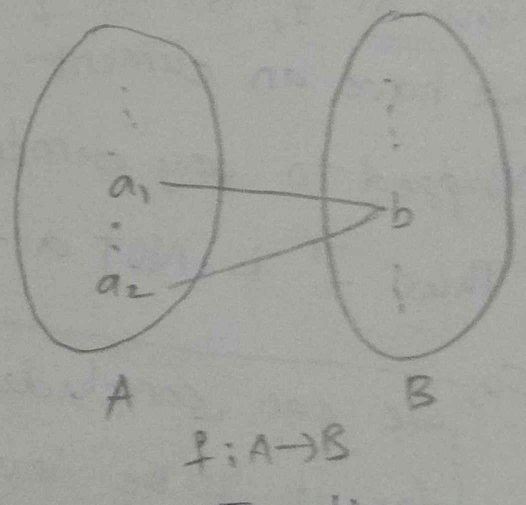
(a) 'f' is one-one
(b) 'f' is onto

we prove this by Contradiction

(a) Assume that 'f' is not a one-one function

So, $f: A \to B$, we have $a_1 \neq a_2 \Rightarrow f(a_1) = f(a_2)$ where $(a_1, a_2) \in A$

let $f(a_1) = f(a_2) = b \in B$



$f: A \to B$

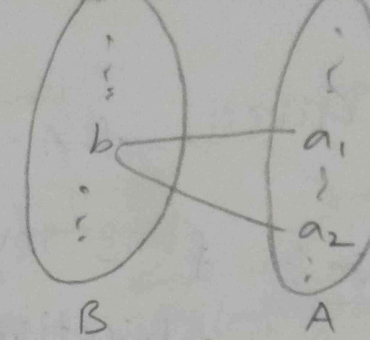
Now $f^{-1}: B \to A$

and we have for one $b \in B$, two Images $f(a_1), f(a_2)$.

This violates the definition of a Function that is "every element in domain has a unique image in codomain."

This contradicts the

$f^{-1}: B \to A$.

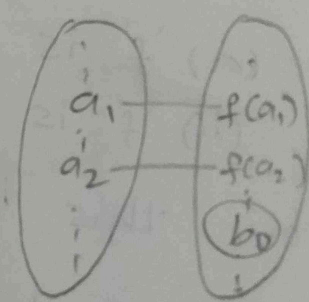


$f^{-1}: B \to A$

Thus, we can say that 'f' Must be one-one for it to be an inverse to exist

$\Rightarrow$ $\boxed{\text{'f' is one-one for it to be Invertible}}$

(b) Assume that 'f' is not a surjection, that is
$\exists b_0 \in B$ such that 'b_0' has "no Pre-image in A"
for $f: A \to B$



$f: A \to B$

So Now we construct
'f^{-1}' from B to A
$f^{-1}: B \to A$

Now in f^{-1}, in the domain 'B'
we have an element $b_0 \in B$ which is NOT
Mapped to any element in codomain.
Thus f^{-1} is NOT a function! Again a Contradiction

So, $\boxed{\text{we can conclude that 'f' Must be onto for it to be invertible}}$

So from (a) & (b)

we can say that, if 'f' is a bijection, then

f in invertible, that is True

(ii) if 'f' is invertible, then f is a bijection

$f: A \to B$

let $g = f^{-1} \Rightarrow g: B \to A$

we have $g o f = 1_A$ and $f o g = 1_B$

As the Identity functions 1_A and 1_B are bijections, we can tell that 'f' is also a bijection

∴ From (i) & (ii) we have

if 'f' is bijection ⇒ 'f' is Invertible

if 'f' is invertible ⇒ f is bijection

$\boxed{\therefore \text{ 'f' is Invertible} \Leftrightarrow \text{'f' is bijection}}$

2(a) prove (or) disprove that for a non-empty set A, there is no surjection $g: A \to P(A)$, where $P(A)$ is a power set of A.

Answer:- Given a non-empty set A such that

$g: A \to P(A)$, $P(A)$ = power set of A

RTP:- g is not a surjection

Suppose that $g: A \to P(A)$ is a function,

we define

$D = \{a \in A \mid a \notin g(a)\}$. This is a good definition, since $g(a)$ is a subset of A, and a is an element of A, we can ask whether or not $a \in g(a)$.

So D is the set of those elements of A which do not have this property.

Of course that $D \in P(A)$ since it is clearly a subset of A. We will show that

$g(a) \neq D$ for all $a \in A$.

1. If $a \in D$ then, by definition of D, $a \notin g(a)$. So $g(a) \neq D$, since that the element $a \in D$ but $a \notin g(a)$.

2. If $a \notin D$ then, by definition of D, $a \in g(a)$

By the same argument again $g(a) \neq D$.

Either way, $D \neq g(a)$ for all $a \in A$.

Therefore, $\boxed{g \text{ is not surjective}}$

Hence, proved

[Note that for different functions we have different D's. It is possible that $D = A$ (e.g. if $f(a) = \phi \ \forall a$), or it could be ϕ (e.g if $f(a) = \{a\} \ \forall a$).

However regardless to its value it will not be in the range of f.]

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al?

2(b) Are the functions f and g equal?
Give Reasons.

(i) $f, g: D \to R$ defined $f(x) = \sin x - \cos x$, $x \in D$

$g(x) = \sqrt{1 - \sin 2x}$, $x \in D$ and $D = \{x \in R : 0 \le x \le \frac{\pi}{2}\}$

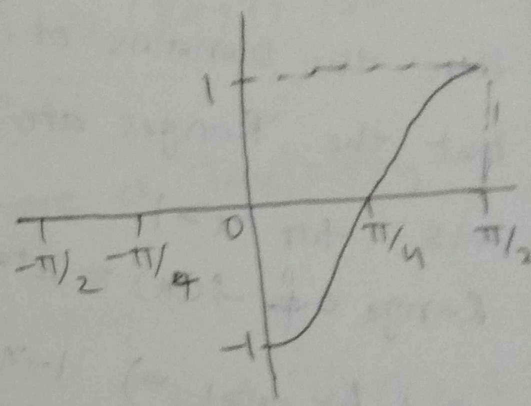
Sol: No, f and g are NOT equal.

As $f: D \to R$ $f(x) = \sin x - \cos x$, $D = \{x \in R : 0 \le x \le \frac{\pi}{2}\}$

From the graph

Range of $f(x)$

$= [-1, 1]$



$f(x) = \sin x - \cos x$

but for

$g: D \to R$

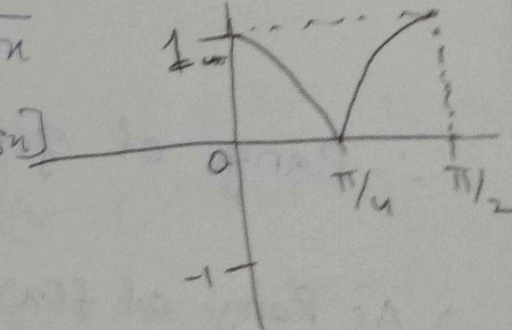
$g(x) = \sqrt{1 - \sin 2x}$

$= \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x}$

$[\sin^2 x + \cos^2 x = 1;\ \sin 2x = 2\sin x \cos x]$

$= \sqrt{(\sin x - \cos x)^2}$

$g(x) = |\sin x - \cos x|$



$g(x) = |\sin x - \cos x|$

$\therefore$ Again from the graph

Range of $g(x) = [0, 1]$

As, the $\boxed{\text{Ranges of } f(x) \neq \text{Range of } g(x)}$

$\boxed{\therefore \text{f and g are not equal}}$

2 (b)

(ii) $f, g : D \to R$ defined by $f(x) = 2\tan^{-1} x, x \in D$

$g(x) = \tan^{-1} \frac{2x}{1-x^2}, x \in D$, and $D = \{x \in R : x > 1\}$

Sol: NO, f and g are NOT equal

$f : D \to R$, $f(x) = 2\tan^{-1} x$, $x \in D$, $D = \{x \in R : x > 1\}$

$g : D \to R$, $g(x) = \tan^{-1} \frac{2x}{1-x^2}$, $x \in D$, $D = \{x \in R : x > 1\}$

Here the Domains of 'f' and 'g' are equal, but the Ranges are NOT equal

As for $x > 1$

Range of $f(x) = 2\tan^{-1} x \in R^+$ (+ve Real numbers)

But for $x > 1 \Rightarrow 1 - x^2 < 0 \Rightarrow \frac{2x}{1-x^2} < 0$

$\Rightarrow \tan^{-1} \frac{2x}{1-x^2} < 0$

$\Rightarrow g(x) < 0$

$\therefore$ Range of $g(x) = \tan^{-1} \frac{2x}{1-x^2} \in \overline{R}$ (-ve Real numbers)

$\therefore$ As Range of $f(x) \neq$ Range of $g(x)$

$\therefore$ So, 'f' and 'g' are not equal

9)

Countable Sets :-

3. Prove (or) disprove a subset of a countable set is countable or finite

Answer:-

Proof :- Let A be a countable set.

then 'A' can be written as a sequence

Let $A = \{a_1, a_2 \cdots\}$

we prove this by cases :-

Case 1 :- If 'A' is finite

$\Rightarrow$ subset of 'A' is finite

$\Rightarrow$ subset of 'A' is countable

Case 2 :-

Let 'A' is Denumerable set.

Let $B \subseteq A$

Sub case 1 :- If 'B' is finite / Empty

If 'B' is finite it is

$\Rightarrow$ countable

If 'B' is 'Empty' then the Result is "Trivial"

$\Rightarrow B \neq \phi$

Sub case 2 :- If 'B' is NOT finite

③ Answer continue

If is NOT finite

Let 'k_1' be the ~~subset~~ least positive integer

Such that $a_{k_1} \in B$

again consider

let $k_2 > k_1$

Be next least positive integer

Such that $a_{k_2} \in B$.

Similarly ... $a_{k_3}, a_{k_4} \ldots \in B$

Thus, B can be written as

$$B = \{a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4} \ldots\}.$$

which is a sequence

W.K.T any sequence is countable

$\therefore$ B is countable

Hence, a subset of a countable set is countable or finite

Hence, proved

4. Determine whether or not the following set is countable : the set $A = \{ a^2 \mid a \in N \}$ where N is the set of Natural numbers.

Answer:-

Yes, the set $A = \{ a^2 \mid a \in N \}$ is a countable set.

PROOF :- Let

A = set of consecutive squares of natural numbers.

N = set of Natural Numbers.

Claim:- There exists bijection between N and A that is

Now construct the mapping $f : N \to A$ defined by

$$f(n) = n^2 \; ; \; n \in N$$

$A = 1^2, 2^2, 3^2, \dots$

* To prove that 'f' is one-one, let $n_1, n_2 \in N$ such that $f(n_1) = f(n_2)$

then, $n_1^2 = n_2^2$

$\Rightarrow n_1^2 - n_2^2 = 0$

$\Rightarrow (n_1 - n_2)(n_1 + n_2) = 0$

As $n_1, n_2 \in N \Rightarrow n_1 + n_2 > 0 \Rightarrow n_1 - n_2 = 0$

$\Rightarrow n_1 = n_2$

Hence, $\boxed{f(n_1) = f(n_2) \Rightarrow n_1 = n_2 \;\; \therefore \text{'f' is one-one}}$ — 1

* To prove that 'f' is onto, let $m \in A$

Then, $\exists n \in N$ such that $m = f(n) = n^2$

as "m is perfect square"

for every $m \in A$

$$m = f(n) = n^2$$

$$\Rightarrow n = \sqrt{m} = \sqrt{n^2} = |n|$$

[as 'm' is perfect square]

$$\Rightarrow |n| \Rightarrow n \geq 1 \quad [m \text{ starts from } 1^2]$$

$$\Rightarrow n \in N.$$

$\therefore$ for every $m \in A$, $\exists n \in N$ such that $\boxed{f(n) = m}$

Hence, 'f' is an onto function from $N \to A$ —②

From ① & ②

'f' is a bijection, $f: N \to A$

Now W.K.T when $f: N \to S$ is a bijection then

the set 'S' is countable

$\therefore$ 'A' is countable, where $A = \{a^2 \mid a \in N\}$

Hence proved

Propositional Logic :-

5 (a) Show that if P, q and r are compound propositions such that p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent.

Answer: Given 3 compound propositions P, q and r

~~$P \leftrightarrow q$ is a Tautology~~

To Prove: $(P \leftrightarrow q) \wedge (q \leftrightarrow r) \rightarrow (P \leftrightarrow r)$ is a Tautology. $[P \equiv r]$

So if $(P \leftrightarrow q)$ & $(q \leftrightarrow r)$ are a Tautology

that is $P \equiv q$

$q \equiv r$

we have to show that $P \equiv r$ that is $(P \leftrightarrow r)$ is a Tautology.

So consider $(P \leftrightarrow q) \wedge (q \leftrightarrow r) \equiv T \wedge T \equiv T$

$$\equiv (P \rightarrow q) \wedge (q \rightarrow P) \wedge (q \rightarrow r) \wedge (r \rightarrow q)$$

$$\equiv (P \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow q) \wedge (q \rightarrow P) \quad -(1)$$

[By commutativity of $\wedge$]

Now consider

$$(P \rightarrow q) \wedge (q \rightarrow r) \equiv (\sim P \vee q) \wedge (\sim q \vee r) \quad [P \rightarrow q = \sim P \vee q]$$

$$\equiv (\sim P \wedge \sim q) \vee (\sim P \wedge r) \vee (q \wedge \sim q) \vee (q \wedge r)$$

[By distributivity of $\wedge$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$]

$\equiv \sim p \vee \sim p \vee (q \wedge \sim q) \vee r$

[weakening]

$\equiv \sim p \vee F \vee r$ $\left[\begin{array}{l} q \wedge \sim q \equiv F \\ r \vee r \equiv r \end{array}\right]$

$\equiv \sim p \vee r$ $[q \vee F \equiv q]$

$\equiv p \to r$

So from $(p \to q) \wedge (q \to r) \equiv (p \to r)$

① $\Rightarrow$ $(r \to q) \wedge (q \to p) \equiv (r \to p)$

$\equiv (p \leftrightarrow q) \wedge (q \leftrightarrow r) \equiv T$

$\equiv (p \to r) \wedge (r \to p) \equiv T$

$\equiv (p \leftrightarrow r) \equiv T$

$\therefore (p \leftrightarrow r)$ is a Tautology. ~~Hence~~ 'p' and 'r' are logically equivallent.

We can also say that, given to say that p and q are logically equivalent is to say that the truth tables for p and q are identical, similarly, to say that q and r are logically equivalent is to say that the truth tables of q and r are identical.

Clearly if the truth Tables for p & q are identical, and the Truth Tables for q and r are identical, then the

truth t

Therefore

(b) Show

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Answer

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truth tables for 'p' and 'r' are identical.
Therefore 'p' and 'r' are logically equivalent.

(b) Show that $\sim p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent by a series of logical equivalences.

<u>Answer</u>:- Given, 2 propositions $\sim p \to (q \to r)$ & $q \to (p \vee r)$

<u>RTP</u>:- $\sim p \to (q \to r) \equiv q \to (p \vee r)$

Consider

$L.H.S \equiv \sim p \to (q \to r)$

$\equiv \sim(\sim p) \vee (q \to r)$ $[\because p \to q \equiv \sim p \vee q]$

$\equiv p \vee (\sim q \vee r)$ $[\sim(\sim p) \equiv p]$ $[p \to q \equiv \sim p \vee q]$

$\equiv (p \vee \sim q) \vee r$ By Associativity

$\equiv (\sim q \vee p) \vee r$ By commutativity

$\equiv \sim q \vee (p \vee r)$ By Associativity

$\equiv q \to (p \vee r)$ Here a is q & b is $p \vee r$ then $[a \to b \equiv \sim a \vee b$

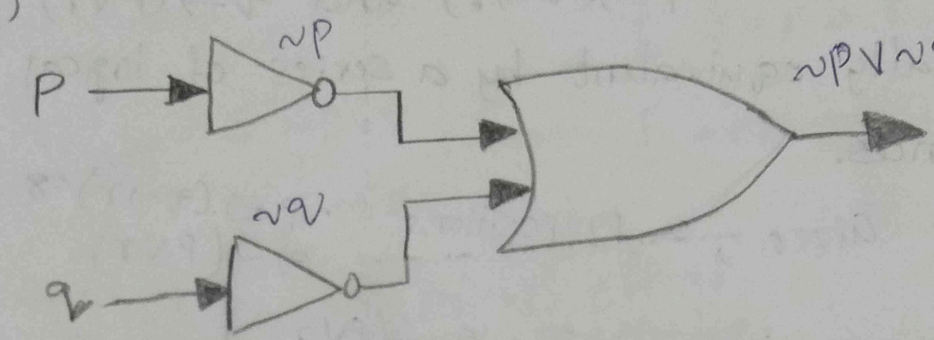
$\equiv R.H.S$

$\therefore L.H.S \equiv R.H.S$

Hence, proved

6 (i) Find the output of each of these combinational circuits shown in figure below.

(a)



p	q	~p	~q	~p V ~q OUTPUT
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The Equation for the given Combinational circuit is = ~p V ~q and column (~p V ~q) represents all the possible outputs that we get.

6(ii)

(a) sho

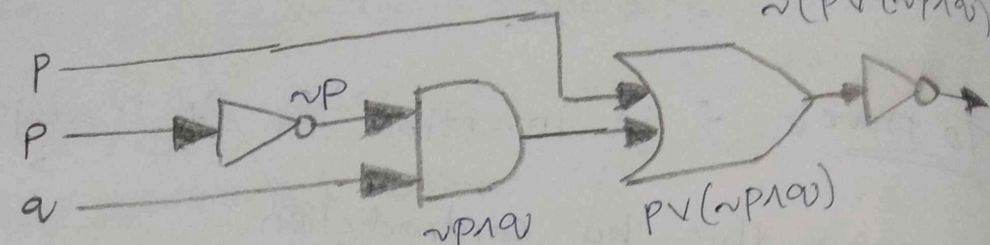
Tautolo

p	q	r	p→q	q→r	p→r	(p→q) ∧ (q→r)	(p→q) ∧ (q→r) → (p→r)
			T	T	T	T	T

6(i)

(b)

Sol:



The Final Expression for the circuit is:

$\equiv \sim(p \vee (\sim p \wedge q))$ $\quad [\sim(a \vee b) = \sim a \wedge \sim b]$

$\equiv \sim p \wedge \sim(\sim p \wedge q)$ $\quad [\sim(a \wedge b) = \sim a \vee \sim b]$

$\equiv \boxed{\sim p \wedge (p \vee \sim q)}$

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \wedge (p \vee \sim q)$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	T	T

From the above Truth Table:

$\boxed{\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge (p \vee \sim q)}$

6(ii)

(a) show that $(p \to q) \land (q \to r) \to (p \to r)$ is a Tautology.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \land (q \to r)$	$(p \to q) \land (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the above truth Table it is clear that $(p \to q) \land (q \to r) \to (p \to r)$ is a Tautology.

Hence, proved

(b) Show that $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$ is a Tautology.

p	q	r	$\sim p$	$p \vee q$	$\sim p \vee r$	$q \vee r$	$(p\vee q) \wedge (\sim p \vee r)$	$(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	T	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	T	T	F	T
F	F	F	T	F	T	F	F	T

From the above Truth Table, we have proved that $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$ is a Tautology.

Permutations:-

7. Let f, g be permutations on the set $\{1,2,3,4,5,6\}$. Find fg, gf, f^{-1} and g^{-1}, when $f = (12456)$, $g = (26345)$.

Sol Given that,

$Set = S = \{1,2,3,4,5,6\}$ there are
(let)

2 permutations on $S \Rightarrow f, g$

and $f = (12456)$
$g = (26345)$.

Find? (i) fg
(ii) gf
(iii) f^{-1}
(iv) g^{-1}.

(i) $f.g = fog = f(g) = (12456).(26345)$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 2 & 6 & 3 & 4 & 5 \\ 1 & 6 & 3 & 4 & 5 & 2 \end{pmatrix}$$

$$\boxed{f.g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 4 & 3 \end{pmatrix}}$$

This can also be written as $= (12).(3546)$

(ii) $gf = gof$

$= g(f) = (2\ 6\ 3\ 4\ 5).(1\ 2\ 4\ 5\ 6)$

$$= \begin{pmatrix} 1 & 2 & 6 & 3 & 4 & 5 \\ 1 & 6 & 3 & 4 & 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix}$$

$$\boxed{g.f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 2 & 3 & 1 \end{pmatrix}}$$

$= (1\ 6).(3\ 4\ 2\ 5)$

(iii) f^{-1}

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix}$$

$$\Rightarrow f^{-1} = \begin{pmatrix} 2 & 4 & 3 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$\boxed{f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 2 & 4 & 5 \end{pmatrix}}$$

(iv) g^{-1}

$$g = \begin{pmatrix} 1 & 2 & 6 & 3 & 4 & 5 \\ 1 & 6 & 3 & 4 & 5 & 2 \end{pmatrix}$$

$$\Rightarrow g^{-1} = \begin{pmatrix} 1 & 6 & 3 & 4 & 5 & 2 \\ 1 & 2 & 6 & 3 & 4 & 5 \end{pmatrix}$$

$$\boxed{g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 3 & 4 & 2 \end{pmatrix}}$$

8. The order of a permutation 'f' is the least positive integer 'n' such that $f^n = I$, 'I' being the Identity ~~function~~ permutation

prove that :-

(i) The order of an r- cycle permutation is r.

Answer :-

Given, 'f' is a permutation of length - r

RTP :- The order of an r-cycle permutation is r.

W.K.T

The order of a permutation 'f' is the smallest positive integer 'n' such that

$f^n = I$ = Identity permutation

Now suppose we have an r-cycle permutation

$f = (a_0 \, a_1 \, a_2 \, a_3 \, \cdots \, a_{r-1})$.

Its order cannot be less than r, because if $0 < k < r$ then $f^k(a_0) = a_k$ and it's implicit in the concept of a cycle that $a_0 \neq a_k$

On the other hand, f^{γ} is certainly the identity permutation, because it acts on each element of the cycle by moving it around the entire cycle one time. (As it is an γ-cycle permutation)

And formally we can prove that (By Induction)

$$f^{k}(a_j) = a_{(j+k) \bmod \gamma}, \text{ and since}$$

$n \equiv 0 \pmod{n}$ we have

$(j+\gamma) \bmod \gamma = j$ whenever $0 \leq j < \gamma$.

$\therefore$ After moving the element 'γ' times we have

$$f^{\gamma}(a_j) = a_j$$ thus forming an

identity permutation with elements in the γ-cycle

And elements that are not in the cycle at all are left alone by f and for them any how $f^{\gamma}(a_m) = a_m$, for any 'γ', $a_m \notin \gamma$-cycle

and therefore they are left alone by f^{γ} also.

[$\because f^{\gamma} = I$, and $f^{k} \neq I$ when $0 \leq k < \gamma$]

Thus we can say that $\boxed{\text{"The order of an } \gamma\text{-cycle permutation is '}\gamma\text{'"}}$

Hence, proved

(ii) the order of a permutation f is the LCM of the lengths of its disjoint cycles.

Answer: Given a permutation 'f'

RTP: The order of 'f' is LCM of the lengths of its disjoint cycles.

let $\alpha_1, \alpha_2, \alpha_3 \dots, \alpha_k$ be the k-disjoint cycles of 'f'. then

$$f = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \dots \alpha_k$$

these cycles commute with each other.

Hence

$$f^r = \alpha_1^r \cdot \alpha_2^r \cdot \alpha_3^r \dots \alpha_k^r = I \Leftrightarrow$$

$$\alpha_1^r = \alpha_2^r = \alpha_3^r = \dots = \alpha_k^r = I \quad \text{— (1)}$$

W.K.T The order of an r-cycle permutation is r i.e., the length itself.

So for (1) to hold 'r' should be a common multiple of order of α_1,
order of α_2,
⋮
order of α_k

And the smallest of these r's is by definition the least common multiple of the orders of the cycles.

Hence, It is proved that,

the order of f is r = LCM of orders of / lengths of its disjoint cycles.

Hence, proved

Find the order of the following permutation using the result (ii)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}.$$

Solution:- Given,

say $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

Find:- order of f.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

$$= \underset{①}{\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}} \cdot \underset{②}{\begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \end{pmatrix}}$$

① - length of Transposition = 2

② - length of permutation = 3

$\boxed{\text{LCM of lengths} = 2 \times 3 = 6}$

$\boxed{\therefore \text{order of } f = 6}$

Mathematical Induction:-

9. prove by Induction that the sum of the cubes of 3 consecutive integers is divisible by 9

Answer:-

RTP:- Sum of cubes of 3 consecutive integers is divisible by 9.

Let $n, n+1, n+2$ be 3 consecutive Integers

$\Rightarrow n, n+1, n+2 \in \mathbb{Z}$

Proof By Induction:-

Let $p(n) = n^3 + (n+1)^3 + (n+2)^3$ be divisible by 9 be a statement.

[Basis step.] Here $n_0 = 1$. Then,

$$p(1) = (1)^3 + (1+1)^3 + (1+2)^3 = 36$$

which is divisible by 9 $(9 \times 4 = 36)$.

Thus, the statement $p(n)$ is True for $n = n_0 = 1$.

[Induction Hypothesis] Assume that the statement is true for $n = k$, that is $p(k)$ is True

[Induction Step] :-

Now consider

$$p(k+1) - p(k) = \left[(k+1)^3 + (k+1+1)^3 + (k+1+2)^3\right] - \left[k^3 + (k+1)^3 + (k+2)^3\right]$$

$$= \cancel{(k+1)^3} - \cancel{(k+1)^3} + \cancel{(k+2)^3} - \cancel{(k+2)^3} + (k+3)^3 - k^3$$

$$= (k+3)^3 - k^3$$

$$= \cancel{k^3} + 3^3 + 3\cdot 3k^2 + 3\cdot 3^2 k - \cancel{k^3}$$

$$= 3^2\left[3 + k^2 + 3k\right] \quad \left[\text{let } k^2 + 3k + 3 = m\right]$$

$$p(k+1) - p(k) = 9.m$$

Thus, $p(k+1) - p(k)$ is divisible 9

by Induction Hypothesis $p(k)$ is also divisible by 9.

Hence, $p(k+1)$ is divisible by 9

By the first principle of mathematical Induction it follows that $p(n)$ is true for all $n \in \mathbb{N}$

10. It is known that for any positive integer $n \geq 2$

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - A \geq 0.$$

Where A is a constant. How large can A be?

Sol: Given, that,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - A \geq 0 \quad \forall n \geq 2, \; n \in \mathbb{Z}$$

Find: 'A'

Here we use "Definite integral as a limit of a sum" to get the largest value of 'A'

So; $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} - A \geq 0$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{n+r} - A \geq 0 \quad - \circledast$$

consider

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{n+r} - A \geq 0$$

we apply limit $(n \to \infty)$ to get Maximum value of 'A'.

$$\Rightarrow \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n+r} - A \geq 0 \quad - (1)$$

Now consider

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{1}{1+r/n}\right)$$

$$= \lim_{n\to\infty} 1/n \cdot \sum_{r=1}^{n}\frac{1}{1+r/n}$$

Now take

$r/n = x$

$1/n = dx$

U.L of $r = n$

$n/n = x$

$x = 1$

L.L of $r = 1$

$x = 1/n = 1/\infty = 0$

$$= \int_0^1 \frac{1}{1+x}\,dx$$

$\left[\because \int \frac{1}{x}\,dx = \log_e|x|\right]$

$$= \left[\log_e|1+x|\right]_0^1$$

$$= \log_e|1+1| - \log_e|1+0|$$

$$= \ln 2 - \ln 1 \qquad [\log_e 1 = 0]$$

$$= \ln 2 - 0$$

$$\boxed{\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \ln 2}$$

Now

① $\Rightarrow$ $\ln 2 - A \geq 0$

$\Rightarrow$ $\boxed{A \leq \ln 2}$

$\boxed{\therefore \text{Largest value of } A = \ln 2}$

11. Let A and B be square matrices. If $AB = BA$, then prove that $(AB)^n = A^n . B^n$, for $n \geq 1$.

Answer:- Given that

A, B are 2 square matrices

S/t $AB = BA$

RTP:- $(AB)^n = A^n . B^n \quad \forall n \geq 1$

Proof By Induction:-

Step 1:- Let "$p(n)$: If $AB = BA$, then $(AB)^n = A^n . B^n$" be a statement

Step 2:- prove that the statement is True for $n = 1$

For $n = 1$

$L.H.S = (AB)^1 = AB$

$R.H.S = A^1 B^1 = AB$

$\therefore L.H.S = R.H.S$

$\therefore p(n)$ is True for $n = 1$

Step 3:- Assume that the statement is True for $n = k$, then prove that $p(k+1)$ is True

So, we assume that $p(k)$ is True

$p(k)$: If $AB = BA$, then $(AB)^k = A^k . B^k$

Now we prove that $p(k+1)$ is True as follows

$p(k+1)$: If $AB = BA$, then $(AB)^{k+1} = A^{k+1} . B^{k+1}$

Taking LHS

$$L.H.S = (AB)^{k+1}$$

$$= (AB)^k (AB) \quad [\because (AB)^k = A^k . B^k \ (p(k))]$$

$$= A^k . B^k . (AB) \quad [\because AB = BA]$$

$$= A^k . B^k . (BA)$$

$$= A^k . (B^{k+1} . A)$$

$$\boxed{\begin{array}{l} \text{W.K.T} \\ \text{If } AB = BA, \text{ then} \\ \quad AB^n = B^n A \end{array}}$$

$$= A^k (AB^{k+1})$$

$$= A^{k+1} B^{k+1}$$

$$= R.H.S$$

$$\therefore L.H.S = R.H.S$$

Hence, $p(k+1)$ is True

So, By the First principle of Mathematical Induction it follows that '$p(n)$' is True $\forall n \in N. \ (n \geq 1)$

Pigeonhole Principle

12. Show that if any 30 people are selected, then one may choose a subset of five so that all five were born on the same day of the week

Answer:

If 30 people are selected, there are 7 days in a week

Here, consider 30 people – be pigeons and
7 days – 7 pigeonholes

Then by the "Generalised Pigeonhole Principle",

the minimum number students born on same day of the week $= \left\lfloor \frac{m-1}{n} \right\rfloor + 1$

$= \left\lceil \frac{30-1}{7} \right\rceil + 1$

$= 4 + 1$

$= 5$

∴ one can choose a subset of 5 people who were born on the same day of the week.

13. Show that if any eight positive integers are chosen, two of them will have the same remainder when divided by 7.

Answer:- let $x \in Z^{+}$

then eight consecutive +ve Integers are:-

$x, x+1, x+2, \text{------}, x+7$

And there are 7 remainders (possible) $= 0, 1, 2, \text{--} \to 6$

we get when we divide any number by 7.

So, Here consider

8 integers – to 8 pigeons (m)

7 remainders – 7 pigeonholes (n)

So, by "Generalised Pigeonhole Principle"

No. of Integers with same remainder $= \left\lfloor \frac{m-1}{n} \right\rfloor + 1$

$= \left\lfloor \frac{8-1}{7} \right\rfloor + 1$

$= \left\lfloor \frac{7}{7} \right\rfloor + 1$

$= 2$

$\therefore$ Two of 8 +ve Integers have same remainder when divided by 7.

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14. A chess player wants to prepare for a championship match by playing some practice games in 77 days. She wants to play atleast one game a day but no more than 132 games altogether. Show that no matter how she schedules the games there is a period of consecutive days within which she plays exactly 21 games.

Answer:-

Given that,

A chess player has 77 days to practice.

Each day

$1 \leq$ No. of games played ≤ 132

RTP:- There is period of consecutive days within which she plays exactly 21 games.

Let $a_1, a_2, a_3, \ldots a_{77}$ be the number of games played in the n^{th} day, such that

$a_i \leq 132$, $i \in [1, 77]$

Now consider the sequence

$a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$

Since $1 \leq a_1$ and $a_{77} + 21 \leq 132 + 21 = 153$, we have a collection of 154 positive integers that are among of the positive integers 1 through 153.

The 154 sequences $a_1, a_2, a_3, \cdots a_{77}$ and $a_1 + 21, a_2 + 21, \cdots, a_{77} + 21$ are all less than 153. Thus by "Pigeon hole Principle"

$$\left\lceil \frac{154}{153} \right\rceil = 2$$

there's atleast $2a_j$ and $a_k + 21$, such that $a_j = a_k + 21$. We know that since a_j cannot equal another a_l, $j \neq l$ since all the values in the sequence $a_1, a_2, a_3, \cdots, a_{77}$ are distinct (atleast one game a day), thus one of a_j have to equal $a_k + 21$, for some $j, k \in \mathbb{Z}$.

That means that there exist from k^{th} day to j^{th} day, there's 21 consecutive games.

Hence, There is a period of consecutive days within which she plays exactly 21 games.

Hence, proved