DISCRETE STRUCTURES

Assignment-1

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Sec: B

① Let us consider the structure $\langle P(x), - \rangle$, where the operation $-$ (set difference) is defined by $A-B = \{x \mid x \in A \text{ and } x \notin B\} = A \cap B'$, where $P(X)$ is the power set of X.

(a) Show that the operation is neither commutative nor associative

(b) Verify whether $A \cup (B \Delta C) = (A \cup B) \Delta (A \cup C)$ holds (or) not, where Δ is the symmetric difference operator defined over sets.

(c) Show that the following properties hold for all $A, B, C \in P(X)$

(i) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

(ii) $A \Delta B = B \Delta A$

(iii) $A \Delta \phi = A$

(iv) $A \Delta X = A'$

(v) $A \Delta A' = X$, where $A \Delta B$ is the usual symmetric difference between A and B and A' the complement of A.

① Answer:-

(a)

Part-1:- '$-$' (set difference) is not commutative

Let A, B be 2 sets.

$A - B = A \cap B' = \{x \mid x \in A \text{ and } x \notin B\}$ - ①

$\Rightarrow B - A = B \cap A' = \{x \mid x \notin A \text{ and } x \in B\}$ - ②

① $\Rightarrow$ $x \in A$ only

② $\Rightarrow$ $x \in B$ only

Therefore, $A - B \neq B - A$

we can also prove this using a Counter Example:

Let, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

So, $A - B = \{1, 2, 3, 4\} - \{3, 4, 5, 6\}$

$\Rightarrow A - B = \{1, 2\}$ - ③

And $B - A = \{3, 4, 5, 6\} - \{1, 2, 3, 4\}$

$\Rightarrow B - A = \{5, 6\}$ - ④

From ③ & ④

$\{1, 2\} \neq \{5, 6\}$

$\Rightarrow A - B \neq B - A$

Part-2:- (RTP):- set difference ($-$) is not Associative i.e.,

for A, B, C $(A - B) - C \neq A - (B - C)$.

Let us Assume that the statement is True

that is

$(A - B) - C = A - (B - C)$

Now let $A = \{1, 2, 3\}$ and $C = \{5, 6, 3\}$

$B = \{3, 4, 5\}$

$\Rightarrow$ L.H.S $= (A-B)-C$
$= (\{1,2,3\} - \{3,4,5\}) - \{5,6,3\}$
$= \{1,2\} - \{5,6,3\}$
$= \{1,2\}$

And

R.H.S $= A-(B-C)$
$= \{1,2,3\} - (\{3,4,5\} - \{5,6,3\})$
$= \{1,2,3\} - (\{4\})$
$= \{1,2,3\}$

$\therefore$ L.H.S $\neq$ R.H.S

Thus, our Assumption is false and we have $(A-B)-C \neq A-(B-C)$

$\therefore$ set difference $(-)$ is NOT Associate and NOT commutative.

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b)

we can verify that, $A \cup (B \Delta C) \neq (A \cup B) \Delta (A \cup C)$ using a counter example

$[A \Delta B = (A-B) \cup (B-A)]$

Let, $A = \{1,2,3\}$
$B = \{3,4,5\}$
$C = \{5,6,3,7\}$

$\therefore A \cup (B \Delta C) = \{1,2,3\} \cup (\{3,4,5\} \Delta \{5,6,3,7\})$
$= \{1,2,3\} \cup (\{3,4,5\} - \{5,6,3,7\})$
$\cup (\{5,6,3,7\} - \{3,4,5\})$

$= \{1,2,3\} \cup (\{4\} \cup \{6,7\})$

$= \{1,2,3,4,6,7\}$ — ①

$(A \cup B) \Delta (A \cup C) = (\{1,2,3\} \cup \{3,4,5\}) \Delta$

$(\{1,2,3\} \cup \{5,6,3,7\})$

$= \{1,2,3,4,5\} \Delta \{1,2,3,5,6,7\}$

$[\because A \Delta B = (A-B) \cup (B-A)]$

$= \{4\} \cup \{6,7\}$

$= \{4,6,7\}$ — ②

From ① & ②

$\{1,2,3,4,6,7\} \neq \{4,6,7\}$

$\Rightarrow A \cup (B \Delta C) \neq (A \cup B) \Delta (A \cup C)$

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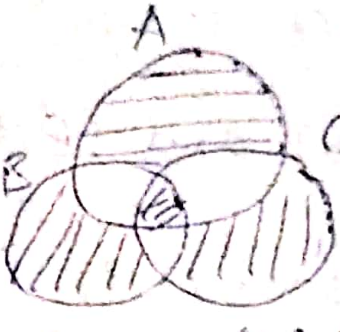
(c)

(i) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$



$(A \Delta B) \Delta C = A \Delta (B \Delta C)$

Given that,

$A, B, C \in P(X)$

Now consider L.H.S

$A \Delta (B \Delta C) = \{x \mid x \in A$ only (or) $x \in B \Delta C$ only but not both$\}$ $[\because A \Delta B = A \cup B - A \cap B]$

$= \{x \mid x \in A$ and $x \notin B \Delta C$ (or)
$x \notin A$ and $x \in B \Delta C\}$

$= \{x \mid x \in A$ and $x \notin B$ and $x \notin C$ (or)
$x \in A$ and $x \in B$ and $x \in C$ (or)
$x \notin A$ and $x \in B$ and $x \notin C$ (or)
$x \notin A$ and $x \notin B$ and $x \in C\}$

$[A \Delta B = (A \cap B') \cup (B \cap A')]$

$= \{x \mid x$ is exactly in one of A, B, C (or)
x is in all the three $(A, B, C)\}$

So for R.H.S also consider

$(A \Delta B) \Delta C = \{x \mid x \in A \Delta B$ only (or) $x \in C$ only but not both$\}$

$= \{x \mid x \in A \Delta B$ and $x \notin C$ (or) $x \notin A \Delta B$ and $x \in C\}$

$= \{x \mid x \in A$ and $x \notin B$ and $x \notin C$ (or)
$x \notin A$ and $x \in B$ and $x \notin C$ (or)
$x \in A$ and $x \in B$ and $x \in C$ (or)
$x \notin A$ and $x \notin B$ and $x \in C\}$

$= \{x \mid x$ is exactly in one of A, B, C (or) x is in all the three $(A, B, C)\}$

$\therefore$ L.H.S = R.H.S

So, $\boxed{A \Delta (B \Delta C) = (A \Delta B) \Delta C}$

ii) $A \Delta B = B \Delta A$

(RTP): 'Δ' is commutative (or) $A \Delta B = B \Delta A$

Now consider L.H.S

L.H.S $= A \Delta B$
$= (A - B) \cup (B - A)$
$= (B - A) \cup (A - B)$ $[\because A \cup B = B \cup A]$
$= B \Delta A$
$=$ R.H.S

$\therefore$ L.H.S = R.H.S

$\Rightarrow \boxed{A \Delta B = B \Delta A}$

So, 'Δ' is commutative.

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(iii) $A \Delta \phi = A$ — (RTP)

We prove the above statement by proving

$L.H.S = R.H.S$

So, $L.H.S = A \Delta \phi$ [since, $A \Delta B = (A-B) \cup (B-A)$]

$= (A-\phi) \cup (\phi - A)$

~~Here~~, as there is no common element between an empty set(ϕ) and a non-empty set (A)

we have $A - \phi = A$ and $\phi - A = \phi$ itself

~~Therefore~~, $\Rightarrow L.H.S = (A-\phi) \cup (\phi - A)$

$= A \cup \phi$

$= A$

$= R.H.S$

$\therefore L.H.S = R.H.S$

That is, $A \Delta \phi = A$

(iv) (RTP) :- $A \Delta X = A'$

It is given that $A, B, C \in P(X)$ that is

$\Rightarrow A, B, C$ are subsets of X

and we consider that 'X' is a universal set for this instance.

Now, $L.H.S = A \Delta X$ [$\because A \Delta B = (A-B) \cup (B-A)$]

$= (A-X) \cup (X-A)$

as $A \subseteq X$, there is no element in 'A' which is not in X

(iv) Answer continued...

So, $A - X = \emptyset$ and as X is universal set we have $X - A = A'$ (complement of A)

So we have

$$L.H.S = (A - X) \cup (X - A)$$
$$= \emptyset \cup A'$$
$$= A'$$
$$= R.H.S$$

$$\therefore L.H.S = R.H.S$$
$$A \Delta X = A'.$$

Hence proved.

(v) (RTP) :- $A \Delta A' = X$

$L.H.S = A \Delta A'$
$= (A - A') \cup (A' - A)$ [$\because A \Delta B = (A - B) \cup (B - A)$]

Here, $A - B = \{x \mid x \in A \text{ and } x \notin B\}$ and there are no common elements in A and A'

we have $A - A' = A$
$A' - A = A'$

Therefore, $L.H.S = (A - A') \cup (A' - A)$
$= A \cup A'$
$= X$ (universal set)
$= R.H.S$

$$\therefore L.H.S = R.H.S$$
$$A \Delta A' = X$$

Hence proved.

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2. Given two sets S and T and $S-T = S\cap T'$, prove that $S\Delta T = S\cup T - S\cap T$

Answer:-

Given two sets S and T,

$$S-T = S\cap T'$$

(RTP):- $S\Delta T = S\cup T - S\cap T$

$$\begin{aligned}
L.H.S &= S\Delta T \\
&= (S-T)\cup(T-S) \\
&= (S\cap T')\cup(T\cap S') \\
&\qquad [\because A\cup(B\cap C) = (A\cup B)\cap(A\cup C)] \\
&= ((S\cap T')\cup T)\cap((S\cap T')\cup S') \\
&\qquad \text{By distributive laws} \\
&= (S\cup T)\cap(T'\cup T)\cap(S\cup S')\cap(T'\cup S') \\
&= (S\cup T)\cap U\cap U\cap(T'\cup S') \qquad [\because A\cup A' = U] \\
&= (S\cup T)\cap(T'\cup S') \\
&\qquad \text{By DeMorgan's laws} \\
&\qquad [(A\cap B)' = A'\cup B'] \\
&= (S\cup T)\cap(T\cap S)' \\
&= (S\cup T) - (S\cap T) \qquad [A-B = A\cap B'] \\
&= R.H.S
\end{aligned}$$

$\therefore$ L.H.S = R.H.S

$$S\Delta T = S\cup T - S\cap T$$

Hence proved.

3. A Binary relation on a set that is reflexive and symmetric is called a compatible relation. Let A be a set of English words and R be a binary relation on A such that w_1 and w_2 (2 words) in A are related if they have one (or) more letters in common. Show that R is a compatible Relation.

Answer:-

Given that,

A is a set of English words

and a Relation R on A is such that

w_1 and w_2 are words related on R

if they have a common one (or) more letters

$\Rightarrow w_1 R w_2$ iff w_1, w_2 have 1(or) more letters in common

~~Case 1~~ (RTP):- 'R' is compatible Relation

To prove this we have to prove 2 cases:-

Case 1:- 'R' is Reflexive

Case 2:- 'R' is symmetric

Case 1:- 'R' is Reflexive

Here $w_1 R w_1$ holds $\forall w_1 \in A$, because a given word has all the letters common with itself. Even if it is a single letter word there is one letter common.

Therefore, 'R' is Reflexive

Case 2:- 'R' is symmetric

Let $w_1 R w_2$ holds where $\forall w_1, w_2 \in A$

$\Rightarrow w_1, w_2$ - have one or more letters in common

$\Rightarrow w_2, w_1$ - have one or more letters in common

$\Rightarrow w_2 R w_1$ holds

Therefore, $w_1 R w_2 \Rightarrow w_2 R w_1$

'R' is symmetric.

From case 1 and case 2, we can say that 'R' is a compatible relation

4. For a given set A, consider the relation

$R = \{(x, y) \mid x \in P(A), y \in P(B), \text{ and } x \subseteq y\}$

where $P(X)$ is the power set of X

show that R is a partial order relation.

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(4)

Answer :-

Given that, R is a Relation on Set A such that

$R = \{(x,y) \mid x \in P(A), y \in P(B), \text{ and } x \subseteq y\}$

(RTP) :- 'R' is a partial order Relation

To prove this we have to prove that R is Reflexive, Anti-Symmetric and Transitive

Thus, we have 3 cases.

Case 1 :- 'R' is Reflexive

For xRx $x \in P(A), x \in P(A)$ and $x \subseteq x$

The condition for 'R' is satisfied. [since, every set is subset to itself]

Therefore, 'R' is reflexive

Case 2 :- 'R' is Anti-symmetric

Let xRy and yRx hold ~~for all x, y ∈ A~~, $x \in A, y \in B$

$\Rightarrow x \subseteq y$ and $y \subseteq x$

$\Rightarrow x = y$ [By Equality of sets]

So, as xRy and $yRx \Rightarrow x = y$

'R' is Anti-symmetric

Case 3: 'R' is Transitive

Let $_xR_y$ and $_yR_z$ holds $\forall x, y, z \in A, B, C$

$\Rightarrow x \subseteq y$ and $y \subseteq z$

$\Rightarrow x \subseteq z$

$\Rightarrow {}_xR_z$ holds $\forall x \in A, z \in C$

Therefore, $_xR_y$ and $_yR_z \Rightarrow {}_xR_z$

'R' is Transitive.

From case 1, case 2, case 3:

'R' is Reflexive, Anti-symmetric and Transitive

So, 'R' is a partial-order Relation

5. Let R and R' be two equivalence relations on a set A. prove that

(i) $R \cap R'$ is an equivalence relation in A

(ii) $R \cup R'$ is not necessarily an equivalence relation in A.

Answer: Given that,

R and R' are two equivalence relations on a set A

(5) continued ...

(i) (RTP) :- $R \cap R'$ is an equivalence relation

Let (x,y) in $R \cap R'$

Now, (x,y) in $R \cap R' \Rightarrow (x,y)$ in R and (x,y) in R'

As, R and R' are symmetric we have

$\Rightarrow (y,x) \in R$ and $(y,x) \in R'$

$\Rightarrow (y,x) \in R \cap R'$

So $(x,y) \in R \cap R' \Rightarrow (y,x) \in R \cap R' \ \forall x,y \in A$

$R \cap R'$ is symmetric — (1)

As R and R' are reflexive

$(x,x) \in R$ and $(x,x) \in R'$

$\Rightarrow (x,x) \in R \cap R'$

Thus $(x,x) \in R \cap R' \ \forall x \in A$

$R \cap R'$ is Reflexive — (2)

Now let $(x,y) \in R \cap R'$ and $(y,z) \in R \cap R'$

Now, $(x,y) \in R \cap R' \Rightarrow (x,y) \in R$ and $(x,y) \in R'$

Again, $(y,z) \in R \cap R' \Rightarrow (y,z) \in R$ and $(y,z) \in R'$

As R and R' are Transitive

$\Rightarrow (x,z) \in R$ and $(x,z) \in R'$

$\Rightarrow (x,z) \in R \cap R'$

Therefore, for $(x,y) \in R \cap R'$ and $(y,z) \in R \cap R'$

$\Rightarrow (x,z) \in R \cap R'$

$R \cap R'$ is Transitive — (3)

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From (1), (2) & (3)

$R \cap R'$ is reflexive,
Symmetric and
Transitive

$\therefore R \cap R'$ is an equivalence Relation

(ii) $R \cup R'$

(RTP): $R \cup R'$ is not necessarily an equivalence relation

$R \cup R'$ is an equivalence relation (might be) when $R \cup R'$ is Transitive, otherwise $R \cup R'$ is not necessarily equivalence relation.

We can prove that $R \cup R'$ is sometimes not an equivalence relation using a counter example

Counter Example:

Let R, R' be 2 equivalence relations on a Non-Empty Set A such that, for $A = \{1,2,3\}$

$$R_t = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$$

and

$$R' = \{(1,1),(2,2),(3,3),(3,1),(1,3)\}$$

Now

$\Rightarrow R \cup R' = \{(1,1),(2,2),(3,3),(2,3),(3,2),(3,1),(1,3)\}$

Here, we see that RUR' is NOT Transitive

as for $(2,3), (3,1) \in RUR'$, we don't have $(2,1) \in RUR'$

i.e., $(2,1) \notin RUR'$

Therefore, RUR' is NOT an equivalence relation

So, we can conclude that

RUR' may (or) may not be an equivalence relation, when R, R' are equivalence relations.

6. On $R = R \times R$, the following relation ρ is defined, where R is the set of real numbers. Check whether it is an equivalence relation (or) not. If yes, find the equivalence classes.

$(a,b) \rho (c,d)$ iff both the points lie on the same curve : $9x^2 + 16y^2 = K^2$, for some $K \in R$.

<u>Answer</u>:

Given that a relation 'ρ' is on $R \times R$ such that $(a,b) \rho (c,d)$ iff both the points lie on the same curve : $9x^2 + 16y^2 = K^2$, $K \in R$.

<u>To prove</u> that 'ρ' is an equivalence relation we have to prove that 'ρ' is (i) Reflexive

(ii) Symmetric

(iii) Transitive

(i) 'ρ' is Reflexive

for $(a,b) \rho_{(a,b)}$ holds, because if (a,b) is on the given curve $9x^2+16y^2=k^2$ then vacously (a,b) itself is again on the same curve

so, 'ρ' is Reflexive. - ①

(ii) 'ρ' is Symmetric

let $(a,b), (c,d)$ be 2 points on the ellipse $9x^2+16y^2=k^2$ then we can say that $(a,b) \rho_{(c,d)}$ holds and again (c,d), (a,b) are also on the same curve $9x^2+16y^2=k^2$

so, $(c,d) \rho_{(a,b)}$ also holds $\forall a,b,c,d \in R$

so, 'ρ' is symmetric - ②

(iii) 'ρ' is Transitive

let $(a,b), (c,d)$, be 2 points on the ellipse,

then $(a,b) \rho_{(c,d)}$ holds

Similarly (e,f) is also a point on the ellipse

then as (c,d) and (e,f) both points are on the same ellipse $(c,d) \rho_{(e,f)}$ holds

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Now, once again

(a,b) and (e,f) are on the same ellipse

So $(a,b) \rho (e,f)$ holds

$\Rightarrow$ 'ρ' is Transitive — ③

From ①, ② & ③

'ρ' is Reflexive, Symmetric, Transitive

$\Rightarrow$ 'ρ' is an Equivalence Relation

Now we have to find the equivalence classes of 'ρ'.

If $(a,b), (c,d), (e,f) \ldots$ satisfies the given curves $9x^2 + 16y^2 = k^2$ then, It is clear that equivalence class for (a,b) is

$[(a,b)] = \{(a,b), (c,d), (e,f) \ldots\}$ $\forall k \in R$

all the points which satisfy the curve $9x^2 + 16y^2 = k^2$ &

Similarly

$[(c,d)] = \{(a,b), (c,d), (e,f) \ldots\}$

For any pair which satisfy the curve $9x^2 + 16y^2 = k$

$\Rightarrow$ $[(a,b)] = [(c,d)]$

So, For this relation, equivalence class of any pair always has all the points which the both points satisfy $9x^2 + 16y^2 = k^2$.