Linear Algebra

NAGA MANOHAR Assignment-1
2021101128

①

(i) Given a vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

we have find the orthogonal basis of $\mathbb{R}^3$ that contains the vector $\underline{v}$

let $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3$ be s/t $u.v = 0$

$\Rightarrow \quad x_1 + 2x_2 + 3x_3 = 0$

$x_1 = -2x_2 + 3x_3$

Now

$$u = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

So, $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ span the $\underline{u's}$ that satisfy $u.v = 0$

Let $x_2 = 1$ and $x_3 = 0$

$\Rightarrow u = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is a valid u s/t $u.v = 0$

Now

consider

$$w = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$ ~~s.t~~

s/t $\{u, v, w\}$ forms an orthogonal set

thus forming an orthogonal basis of $\mathbb{R}^3$

~~s.t~~

$\Rightarrow$ $v.w = 0$ &

$u.w = 0$

$$\Rightarrow \left.\begin{aligned} y_1 + 2y_2 + 3y_3 &= 0 \\ -2y_1 + y_2 &= 0 \end{aligned}\right\} \Rightarrow 3y_1 + y_2 + 3y_3 = 0$$

$$y_1 = -\frac{1}{3}(y_2 + y_3)$$

$$\begin{bmatrix} -\frac{1}{3}(y_2 + y_3) \end{bmatrix}$$

$\Rightarrow \quad 3y_1 + y_2 + 3y_3 = 0$

$y_1 + 2y_2 + 3y_3 = 0 \quad -①$

$-2y_1 + y_2 = 0$

$y_2 = 2y_1 \quad -②$

① & ② $\Rightarrow y_1 + 2(2y_1) + 3y_3 = 0$

$\Rightarrow y_3 = 5y_1$

$y_2 = 2y_1$

$y_1 = y_1$

$$\therefore w = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ 2y_1 \\ 5y_1 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Hence,

v, u, w form an orthogonal set

$\Rightarrow$ They are linearly independent

they span $\mathbb{R}^3$

Thus $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

form an orthogonal Basis for $\mathbb{R}^3$.

(ii)

Given a vector $v = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$

we have to find an orthogonal basis containing $\underline{v}$ for $\mathbb{R}^3$

let $u = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} \in \mathbb{R}^3$ be s/t $u.v = 0$

$\Rightarrow 3x_1 + x_2 + 5x_3 = 0$

$x_2 = -3x_1 - 5x_3$

$\therefore u = \begin{bmatrix} x_1 \\ -3x_1 - 5x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$

$\therefore \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix} \right\}$ span's all $\underline{u}$ that satisfy $u.v = 0$

so let $x_1 = 1, x_3 = 0$

$\Rightarrow u = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ is valid u

Now we find a vector $\underline{w}$ s/t $\{u, v, w\}$ form an orthogonal set

$\Rightarrow$ let

$$w = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\Rightarrow$ $u \cdot w = 0$

$v \cdot w = 0$

$3y_1 + y_2 + 5y_3 = 0$

$y_1 - 3y_2 = 0$

$\Rightarrow$ $y_1 = 3y_2$

$y_2 = y_2$

$y_3 = -2y_2$

$$\therefore w = y_2 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

let $y_2 = 1$

$$\Rightarrow w = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\therefore \{v, u, w\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$ is an

orthogonal set

$\Rightarrow$ It is Linear Indipendent

$\therefore$ It forms an orthogonal Basis for $\mathbb{R}^3$

(2)

Given, to find the orthogonal basis of $\mathbb{R}^4$ that contains the

vector $v = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

Now we find 3 vectors v_1, v_2, v_3

s/t $\{v, v_1, v_2, v_3\}$ is an orthogonal set.

let $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

let

$v_1 \cdot v = 0$

$\Rightarrow x_1 + 2x_2 - x_3 = 0$

$x_3 = x_1 + 2x_2$

$$\therefore v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + 2x_2 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

~~for~~ let $x_1 = 1,\ x_2 = 0,\ x_4 = 0$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a valid v_1 s/t $\boxed{v_1 \cdot v = 0}$ —(1)

also

$$\left\{ \underset{\downarrow \atop u_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\downarrow \atop u_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}}, \underset{\downarrow \atop u_3}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

is an orthogonal set that span's all v_1 s/t $v_1 \cdot v = 0$

Now we can

by Gram

as $v_1 =$

$\Rightarrow v_2 = u$

$\Rightarrow v_2 =$

$\Rightarrow v_3 =$

$\Rightarrow v_3 = u$

$v_3 =$

$v_3 =$

Hence

and

$\Rightarrow \{$

$\Rightarrow$

basis of

Now we can find the orthogonal vectors by Gram - Schimdt process

as $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = u_1(:,1) \Rightarrow \boxed{v_2 = u_2 - \text{Proj}_{v_1}(u_2)}$

$$\Rightarrow v_2 = u_2 - \frac{\langle u_1 \cdot u_2 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot \vec{v_1}$$

set

$$\Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_3 = u_3 - \text{Proj}_{v_1}(u_3) - \text{Proj}_{v_2}(u_3)$$

$$\Rightarrow v_3 = u_3 - \left[\frac{\langle v_1, u_3 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1 - \frac{\langle v_2, u_3 \rangle}{\langle v_2 \cdot v_2 \rangle} \cdot v_2 \right.$$

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - [0] - [0]$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence $\{v_1, v_2, v_3\}$ is an orthogonal set

and $v \cdot v_1 = 0$

$\Rightarrow \{v, v_1, v_2, v_3\}$ is an orthogonal set $\Rightarrow$ Linearly Independent

$\Rightarrow$ It forms an orthogonal Basis for $\mathbb{R}^4$

(3)

Given,

W is a subspace spanned by

$\{x_1, x_2, x_3\}$

where $x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

Now we see that (x_1, x_2, x_3) is a basis for W as.

x_1, x_2, x_3 are linearly independent

i.e., $c_1x_1 + c_2x_2 + c_3x_3 = 0$ iff $c_1 = c_2 = c_3 = 0$

Now let $\{v_1, v_2, v_3\}$ is an orthogonal set

By Gram-Schimdt process

$$v_1 = x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \text{proj}_{v_1} x_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\langle v_1 \cdot x_2 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$v_3 = x_3 - \text{proj}_{v_1} x_3 - \text{proj}_{v_2} x_3$$

$$= x_3 - \frac{\langle v_1 \cdot x_3 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1 - \frac{\langle v_2 \cdot x_3 \rangle}{\langle v_2 \cdot v_2 \rangle} \cdot v_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{3}{2} \cdot \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

$\therefore$ The orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} \right\}$

$\therefore$ The orthonormal basis is $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

$\|v_1\| = \sqrt{4} = 2$

$\|v_2\| = \sqrt{5} = \sqrt{5}$

$\|v_3\| = \sqrt{\frac{3}{2}}$

$\therefore$ orthonormal basis is

$$\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ \sqrt{2/3} \end{bmatrix} \right\}$$

(4)

Given,

vectors in $\mathbb{R}^2$, $\mathbb{R}^3$

Find orthogonal & orthonormal Basis for the vectors.

(a) $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Let orthogonal Basis be $\{v_1, v_2\}$

$\Rightarrow v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v_2 = x_2 - \text{proj}_{v_1} x_2$

$= x_2 - \frac{\langle v_1 \cdot x_2 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$

$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

The orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ +1/2 \end{bmatrix} \right\}$

The orthonormal basis $= \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \right\}$

$\|v_1\| = \sqrt{2}$ $\|v_2\| = \frac{1}{\sqrt{2}}$

$= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

(b) $x_1 = [$

Let $\{v_1, v_2$

By Gram

$v_1 = x_1 =$

$v_2 = x_2$

$=$

$v_2 =$

$\therefore$ ortho

$\|v_1\| =$

$\|v_2\| =$

$\therefore$ orth

(b) $x_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Let $\{v_1, v_2\}$ be the orthogonal basis for x_1, x_2

By Gram - Schmidt Process

$v_1 = x_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$v_2 = x_2 - \text{Proj}_{v_1} x_2$

$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{\langle v_1 . x_2 \rangle}{\langle v_1 . v_1 \rangle} . v_1$

$v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{\cancel{6}^{1}}{\cancel{18}_{3}} . \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\therefore$ orthogonal Basis $= \{ \underbrace{\begin{bmatrix} 3 \\ -3 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{v_2} \}$

$\| v_1 \| = \sqrt{18} = 3\sqrt{2}$

$\| v_2 \| = \sqrt{8} = 2\sqrt{2}$

$\therefore$ orthonormal Basis $= \left\{ \frac{v_1}{\| v_1 \|}, \frac{v_2}{\| v_2 \|} \right\}$

$= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

(x)

(c) $x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$

Let the orthogonal basis be $\{v_1, v_2, v_3\}$

then by Gram–Schimdt Process

$v_1 = x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$v_2 = x_2 - \text{Proj}_{v_1} x_2 = x_2 - \frac{\langle v_1 \cdot x_2 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$

$v_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$v_3 = x_3 - \text{proj}_{v_1} x_3 - \text{proj}_{v_2} x_3$

$= x_3 - \frac{\langle v_1 \cdot x_3 \rangle}{\langle v_1 \cdot v_2 \rangle} \cdot v_1 - \frac{\langle v_2 \cdot x_3 \rangle}{\langle v_2 \cdot v_2 \rangle} \cdot v_2$

$= \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} - \frac{(-\cancel{3})}{\cancel{3}} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \frac{(12)}{(6)} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$v_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ orthogonal basis $= \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\|v_1\| = \sqrt{3}$

$\|v_2\| = \sqrt{6}$

$\|v_3\| = \sqrt{2}$

$\therefore$ orthonormal basis $= \left\{ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

(d) $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Let the orthogona basis be $\{v_1, v_2, v_3\}$

then by Gram-Schimdt process,

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \text{proj}_{v_1} x_2 = x_2 - \frac{\langle v_1 . x_2 \rangle}{\langle v_1 . v_1 \rangle} . v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow v_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$v_3 = x_3 - \text{proj}_{\{v_1 v_2\}} x_3 = x_3 - \frac{\langle v_1 . x_3 \rangle}{\langle v_1 . v_1 \rangle} . v_1 - \frac{\langle v_2 . x_3 \rangle}{\langle v_2 . v_2 \rangle} . v_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} . \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} . \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$\therefore$ orthogonal Basis $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} \right\}$

$\|v_1\| = \sqrt{3}, \ \|v_2\| = \sqrt{\frac{2}{3}}, \ \|v_3\| = \frac{1}{\sqrt{2}}$

$\therefore$ orthonormal Basis $= \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \end{bmatrix}, \begin{bmatrix} +1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$

(5)

let u, v be 2 unit vectors

RTP: $\langle u, v \rangle \geq -1$ (or) $\langle u, v \rangle$ cannot be less than -1

From Cauchy-Schwarz inequality:

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

$$|\langle u, v \rangle| \leq 1.1$$

$$|\langle u, v \rangle| \leq 1$$

$$\Rightarrow -1 \leq (\langle u, v \rangle) \leq 1$$

i.e., $\boxed{\langle u, v \rangle \geq -1}$

$\therefore$ There cannot be unit vectors u, v

s/t $\boxed{\langle u, v \rangle < -1}$

Hence, proved

(6)

Let
u, v are 2 vectors in the inner product

space V.

<u>RTP:</u> $||u+v|| \leq ||u|| + ||v||$

<u>Triangle Inequality:</u>

$||u+v||^2 = \langle u+v, v+u \rangle$

$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$

$= ||u||^2 + 2 Re \langle u,v \rangle + ||v||^2$

$\leq ||u||^2 + 2 ||u|| ||v|| + ||v||^2$

$\leq (||u|| + ||v||)^2$

$\Rightarrow \boxed{||u+v|| \leq ||u|| + ||v||}$

(∵ $2 Re \langle u,v \rangle \leq 2|\langle u,v \rangle|$

and

$|\langle u,v \rangle| \leq ||u|| ||v||$

Cauchy's Inequality)

Hence, proved

(7)

roduct

$\langle u, v \rangle \leq$

$2|\langle u, v \rangle|$

$|\leq \|u\| \|v\|$

y's

quality)

(7)

Given, In P_2 let $p(x) = a_0 + a_1 x + a_2 x^2$

& $q(x) = b_0 + b_1 x + b_2 x^2$

RTP'r $\langle p(x), q(x) \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$ is

an Inner Product

$\Rightarrow$ (i) $\langle p(x), q(x) \rangle = \langle q(x), p(x) \rangle$

(ii) $\langle a p(x) + b.q(x), r(x) \rangle = a.\langle p(x), r(x) \rangle + b \langle q(x), r(x) \rangle$

(iii) $\langle p(x), p(x) \rangle \geq 0$ and $\langle p(x), p(x) \rangle = 0$ iff $p(x) =$

(i) $\underset{\text{L.H.S}}{\langle p(x), q(x) \rangle} = a_0 b_0 + a_1 b_1 + a_2 b_2$

$= b_0 a_0 + b_1 a_1 + b_2 a_2$

$= \langle q(x), p(x) \rangle$

$= R.H.S$

$\therefore L.H.S = R.H.S$

(ii) $\langle a.p(x) + b.q(x), r(x) \rangle$

$= \langle (aa_0 + bb_0 + (aa_1 + bb_1)x + (aa_2 + bb_2)x^2, c_0 + c_1 x + c_2 x$

$= (aa_0 c_0 + bb_0 c_0) + (aa_1 c_1 + bb_1 c_1) + (aa_2 c_2 + bb_2 c_2)$ — (1)

$=$

Now

$a\langle p(x), r(x)\rangle + b\langle q(x), r(x)\rangle$

$= a.[a_0c_0 + a_1c_1 + a_2c_2] + b[b_0c_0 + b_1c_1 + b_2c_2]$

$= (aa_0c_0 + bb_0c_0) + (aa_1c_1 + bb_1c_1) + (aa_2c_2 + bb_2c_2)$ — (2)

From (1) & (2)

$\langle ap(x) + b.q(x), r(x)\rangle = a.\langle p(x), r(x)\rangle + b.\langle q(x), r(x)\rangle$

(iii) $\langle p(x), p(x)\rangle = a_0a_0 + a_1a_1 + a_2a_2 \geq 0$

$= a_0^2 + a_1^2 + a_2^2 \geq 0$

$\Rightarrow \langle p(x), p(x)\rangle \geq 0$

let $\langle p(x), p(x)\rangle = 0 \Rightarrow a_0^2 + a_1^2 + a_2^2 = 0$

$\Rightarrow a_0 = a_1 = a_2 = 0$

$\Rightarrow p(x) = 0$

if $\langle p(x), p(x)\rangle = 0$ then $p(x) = 0$

and if $p(x) = 0$ the $\langle p(x), p(x)\rangle = 0 + 0 + 0 = 0$

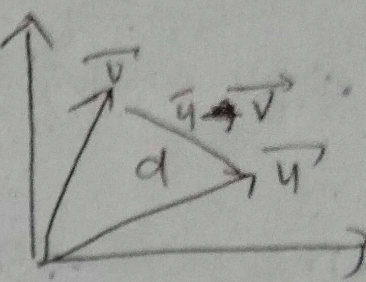
$\therefore \langle p(x), p(x)\rangle = 0$ iff $p(x) = 0$

⑧

Given

u, v vectors

RTP: $d(u,v) = \sqrt{\|u\|^2 + \|v\|^2}$ iff u, v are orthogonal



$(\Rightarrow)$

given

$d(u,v) = \sqrt{\|u\|^2 + \|v\|^2}$

w.k.T

$d(u,v) = \|u - v\|$

$\Rightarrow \|u - v\| = \sqrt{\|u\|^2 + \|v\|^2}$

$\Rightarrow (\|u - v\|)^2 = \|u\|^2 + \|v\|^2$

$\Rightarrow \cancel{\|u\|^2} + \cancel{\|v\|^2} + \langle -v, u\rangle + \langle u, -v\rangle = \cancel{\|u\|^2}$

$\Rightarrow \langle -v, u\rangle + \langle u, -v\rangle = 0$ $\quad$ ($\because$ $\langle v, u\rangle = \langle u, v\rangle$, $\langle cv, u\rangle = c\langle u,$

$\Rightarrow -2\langle v, u\rangle = 0$

$\boxed{\langle v, u\rangle = 0}$

$\therefore$ v and u are orthogonal

($\Leftarrow$) If $\underline{u}$ and $\underline{v}$ are orthogonal

then $\quad \langle u, v\rangle = \langle v, u\rangle = 0$

$\langle -v, u\rangle = \langle u, -v\rangle = 0$

$d(u,v) = \|u-v\|$

$\Rightarrow (\|u-v\|)^2 = \|u\|^2 + \langle -v, u\rangle + \langle u, -v\rangle + \|v\|^2$

($\because$ u, v are orthog

$\Rightarrow (\|u-v\|)^2 = \|u\|^2 + \|v\|^2$

$\Rightarrow \boxed{d(u,v) = \|u-v\| = \sqrt{\|u\|^2 + \|v\|^2}}$

Hence, ~~proved~~

$d(u,v) = \sqrt{\|u\|^2 + \|v\|^2}$ iff u, v are orthogonal

(9) Given,

2 vectors u, v

RTP: $\|u+v\| = \|u-v\|$ iff u & v are orthogonal

$(\Rightarrow)$ $\|u+v\| = \|u-v\|$ then $u.v = 0$ ($\langle u|v\rangle = 0$)

$\Rightarrow \|u+v\|^2 = \|u-v\|^2$ — ①

But $\|u+v\|^2 = \|u\|^2 + \langle u|v\rangle + \langle v|u\rangle + \|v\|^2$

and $\|u-v\|^2 = \|u\|^2 + \langle u|-v\rangle + \langle -v|+u\rangle + \|v\|^2$

~~But as u, v~~

Now ① $\Rightarrow$ $\|u\|^2 + \langle u|v\rangle + \langle v|u\rangle + \|v\|^2$

$= \|u\|^2 + \langle u|-v\rangle + \langle v|+u\rangle + \|v\|^2$

$\Rightarrow \langle u|v\rangle + \langle v|u\rangle + \langle u|-v\rangle + \langle -v|u\rangle = 0$ ($\because \langle u|v\rangle = \langle v|$ $\langle -v|u\rangle = -\langle$

$\Rightarrow 4.\langle u|v\rangle = 0$

$\langle u|v\rangle = 0$ (Inner product $= 0$)

Hence u, v are orthogonal

$(\Leftarrow)$ $\langle u|v\rangle = 0$ / $\underline{u.v = 0}$ then $\|u+v\| = \|u-v\|$

Now $\|u+v\|^2 = \|u\|^2 + \langle u|v\rangle + \langle v|u\rangle + \|v\|^2$

$\langle u|v\rangle = 0$ ($\because$ u, v are orthogonal)

$\langle v|u\rangle = 0$

$\Rightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2$ — ②

Now

$\|u-v\|^2 = \|u\|^2 + \langle u|-v\rangle + \langle -v|u\rangle + \|v\|^2$

again $\langle u|-v\rangle = 0$ ($\because$ u,v are orthogonal)

$\langle -v|u\rangle = 0$

$\Rightarrow \|u-v\|^2 = \|u\|^2 + \|v\|^2$ — ③

From ② & ③

$\|u+v\|^2 = \|u-v\|^2$

Since the norm is always positive

$\Rightarrow \|u+v\| = \|u-v\|$

---

⑩ Given,

W is a subspace of an inner product space V. and $v \in V$

then

RTP: $perp_W(v)$ is orthogonal to all $\underline{w}$ in W

~~To~~ let $\{u_1, u_2, \cdots u_n\}$ be the orthogonal basis of W

$\therefore proj_W(v)$

$\Rightarrow perp_W(v$

To prove

in W

we need

$\langle perp_W(v$

consider

$\Rightarrow \Big[\frac{\langle u_1}{\langle u}$

$\Rightarrow \Big[\frac{\langle u}{\langle}$

$\Rightarrow \Big[\frac{\langle u}{\langle}$

$= proj$

$$\therefore \text{proj}_W(v) = \frac{\langle u_1 v\rangle}{\langle u_1, u_1\rangle} \cdot u_1 + \frac{\langle u_2 v_2\rangle}{\langle u_2 \cdot u_2\rangle} \cdot u_2 + \cdots + \frac{\langle u_n v_n\rangle}{\langle u_n u_n\rangle} \cdot u_n$$

$\Rightarrow \text{perp}_W(v) = v - \text{proj}_W(v)$

To prove $\text{perp}_W(v)$ is orthogonal to all $\underline{w}$ in W

we need to prove $\langle \text{perp}_W(v), w\rangle = 0$

$\langle \text{perp}_W(v), w\rangle = \langle v, w\rangle - \langle \text{proj}_W(v), w\rangle$

consider $\langle \text{proj}_W(v), w\rangle$

$$\Rightarrow \left[\frac{\langle u_1, v\rangle}{\langle u_1, u_1\rangle} \cdot u_1 + \cdots + \frac{\langle u_n v\rangle}{\langle u_n u_n\rangle} \cdot u_n\right] \cdot w$$

$$\Rightarrow \left[\frac{\langle u_1 v\rangle}{\langle u_1 u_1\rangle} \cdot u_1 w + \cdots + \frac{\langle u_n v\rangle}{\langle u_n u_n\rangle} \cdot u_n w\right]$$

$$\Rightarrow \left[\frac{\langle u_1, w\rangle}{\langle u_1, u_1\rangle} \cdot u_1 + \cdots + \frac{\langle u_n w\rangle}{\langle u_n u_n\rangle} \cdot u_n\right] v$$

$$\Downarrow$$

$$= \text{proj}_W(w) = w$$

~~pr~~ $\Rightarrow \boxed{proj_W(w) = w}$

$\therefore \langle proj_W(v), w \rangle = v \cdot w$

$\therefore \langle perp_W(v), w \rangle = \langle v, w \rangle - \langle v, w \rangle = 0$

$\therefore$ $perp_W(v)$ and $w \in W$ are always orthogonal