LINEAR ALGEBRA:

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Class Assignment - (1a)

① Why study Linear Algebra? Mention your motivation(or) potential applications etc,

Answer:

Linear Algebra has many real world Applications:

① For solving any Real world Science/Math problem, we end up getting sum "unknowns" and some relation between them.
Thus they form a set of linear equations with
Number of unknowns = Number of variables.
Linear Algebra can be used in solving them more efficiently.

② In computers and programming say we define a new Abstract Data Type "complex" denoting complex numbers of the form (a+ib) s/t a,b∈Real numbers. Then to use it we need to check whether it is a group, Field(or) a ring so that all operations on them can be defined and the New Data Type can be used without errors. Here this Group/Field structures will be linked to vector spaces thus linking to Linear Algebra

③ In game Development as we need the "shadow, speed, angle, location" etc., for a player we must use vector spaces to see the view in Multi-dimensions and Solving all the sets of linear equations to calculate the exact value of each parameter.

Thus Linear Algebra is widely used everywhere in Real life for solving etc, ② Prove that the set of all complex numbers of the form $x+y\sqrt{2}$, where x and y are rational, is a sub-Field of $(C,+,\cdot)$.

Answer:- Given a Field $F = \langle C,+,\cdot \rangle$

C = Set of Complex Numbers

let $S = \{ k \mid k = x+y\sqrt{2},\ \forall x,y \in \text{Rational Numbers } (Q) \}$

RTP:- S is a sub-Field of F

i.e., ① $\langle S,+,\cdot \rangle$ is Field $(\langle S,+,\cdot \rangle)$

② $S \subset F$

① $\langle S,+,\cdot \rangle$ is a Field

① Addition (+)

(i) Closure => let $k_1 = x_1+y_1\sqrt{2} \in S$

$k_2 = x_2+y_2\sqrt{2}$ where $x_1,y_2 \in Q$

$\Rightarrow k_1+k_2 = x_1+y_1\sqrt{2}+x_2+y_2\sqrt{2}$

$= (x_1+x_2)+(y_1+y_2)\sqrt{2}$

$\Rightarrow (x_1+x_2), (y_1+y_2) \in Q$

let

$\Rightarrow k_3 = (x_1+x_2)+(y_1+y_2)\sqrt{2} \in S$

(ii) Commutative:-

$k_1+k_2 = x_1+y_1\sqrt{2}+x_2+y_2\sqrt{2}$

$= x_2+y_2\sqrt{2}+x_1+y_1\sqrt{2}$

$= k_2+k_1$

$\therefore k_1+k_2 = k_2+k_1$, Hence commutative under (+)

(iii) Associativity :-

let another $k_3 = x_3 + y_3\sqrt{2} \in S$ where $x_3, y_3 \in Q$

$$(k_1 + k_2) + k_3 =$$

$$= ((x_1 + y_1\sqrt{2}) + (x_2 + y_2\sqrt{2})) + (x_3 + y_3\sqrt{2})$$

$$= (x_1 + y_1\sqrt{2} + x_2 + y_2\sqrt{2}) + (x_3 + y_3\sqrt{2})$$

$$= x_1 + y_1\sqrt{2} + x_2 + y_2\sqrt{2} + x_3 + y_3\sqrt{2}$$

$$= x_1 + y_1\sqrt{2} + (x_2 + y_2\sqrt{2} + x_3 + y_3\sqrt{2})$$

$$= k_1 + (k_2 + k_3)$$

$$\therefore (k_1 + k_2) + k_3 = k_1 + (k_2 + k_3).$$

(iv) Existence of Identity under $(+)$

$\exists$ unique element $(0) \in S$ s/t

(let $k = x + y\sqrt{2}$)

$\Rightarrow \boxed{k + 0 = 0 + k = k}$

$$k + 0 = x + y\sqrt{2} + 0$$
$$= 0 + x + y.\sqrt{2}$$
$$= x + y.\sqrt{2}$$
$$= 0 + k$$
$$= k$$

(v) Existence of Inverse under $(+)$

$\forall k \in S$ $\exists$ a unique element $-k \in S$

s/t $k + (-k) = 0$

$\Rightarrow$ for some $k = x + y\sqrt{2}$

$-k = -(x + y\sqrt{2})$ is the Inverse

$\in \mathbb{Q}$

② **Multiplication: (·)**

(i) Closure: $\forall k_1, k_2 \in S$
$k_1 . k_2 \in S$

let
$k_1 = x_1 + y_1\sqrt{2}$
$k_2 = x_2 + y_2\sqrt{2}$
$k_3 = x_3 + y_3\sqrt{2}$

$= k_1 . k_2$
$= (x_1 + y_1\sqrt{2}).(x_2 + y_2\sqrt{2})$
$= x_1 . x_2 + x_1 y_2\sqrt{2} + x_2 y_1\sqrt{2} + 2 y_1 y_2$
$= \underbrace{(x_1 x_2 + 2 . y_1 . y_2)}_{X_1} + \underbrace{(x_1 y_2 + x_2 y_1)}_{Y_1}.\sqrt{2}$ $\quad X_1, Y_1 \in \mathbb{Q}$

Hence
$= X_1 + Y_1\sqrt{2} \in S$

(v) commutativity of (·) $\Rightarrow k_1 . k_2 = k_2 . k_1$

$k_1 . k_2 = (x_1 + y_1\sqrt{2}).(x_2 + y_2\sqrt{2})$
$= (x_2 + y_2\sqrt{2}).(x_1 + y_1\sqrt{2})$

$\boxed{k_1 . k_2 = k_2 . k_1}$

(vi) Associativity of (·) $(k_1 . k_2) k_3 = k_1 . (k_2 . k_3)$

$(k_1 . k_2) . k_3 = [(x_1 + y_1\sqrt{2}).(x_2 + y_2\sqrt{2})](x_3 + y_3\sqrt{2})$

$= (x_1 + y_1\sqrt{2}).(x_2 + y_2\sqrt{2}).(x_3 + y_3\sqrt{2})$
$= (x_1 + y_1\sqrt{2}).[(x_2 + y_2\sqrt{2}).(x_3 + y_3\sqrt{2})]$
$= k_1 (k_2 . k_3)$

(vii) Enistence of Identity under $(\cdot)$

$\exists$ a unique non-zero element '1' in S

s/t $\quad k.1 = 1.k = k$

let

$k = x + y\sqrt{2}$

$\Rightarrow (x+y\sqrt{2}).1 = k.1$

$= (x+y\sqrt{2}) = k$

$= 1.(x+y\sqrt{2}) = 1.k$

$= k$

(viii) Enistence of Inverse under $(\cdot)$

$\exists$ a unique non-zero element k^{-1} in S

s/t $\quad k.k^{-1} = k^{-1}.k = 1$

let

$k = x + y\sqrt{2}$

$\Rightarrow k^{-1} = \dfrac{1}{x+y\sqrt{2}}$

$\Rightarrow k.k^{-1} = \cancel{(x+y\sqrt{2})}.\dfrac{1}{\cancel{(x+y\sqrt{2})}} = 1$

$\Rightarrow k^{-1}.k = \dfrac{1}{\cancel{(x+y\sqrt{2})}}.\cancel{(x+y\sqrt{2})} = 1$

$\Rightarrow \boxed{k.k^{-1} = k^{-1}.k = 1}$

(ix) (·) is Distributive over (+)

i.e. for $k_1 = x_1 + y_1\sqrt{2}$, $k_2 = x_2 + y_2\sqrt{2}$, $k_3 = x_3 + y_3\sqrt{2}$ $\in S$

$$k_1.(k_2+k_3) = k_1.k_2 + k_1.k_3$$

$$= (x_1+y_1\sqrt{2}).((x_2+y_2\sqrt{2})+(x_3+y_3\sqrt{2}))$$

$$= x_1(x_2+y_2\sqrt{2}) + \underline{y_1\sqrt{2}}.(x_2+y_2\sqrt{2}) + x_1.(x_3+y_3\sqrt{2})$$
$$+ y_1\sqrt{2}.(x_3+y_3\sqrt{2})$$

$$= (x_1x_2 + 2y_1y_2 + x_1x_3 + 2y_1y_3)$$
$$+ \sqrt{2}(x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1)$$

$$= (x_1x_2 + x_1y_2\sqrt{2} + x_2y_1\sqrt{2} + 2y_1y_2)$$
$$+ (x_1x_3 + x_1y_3\sqrt{2} + x_3y_1\sqrt{2} + 2y_1y_3)$$

$$= (x_1+y_1\sqrt{2}).(x_2+y_2\sqrt{2}) + (x_1+y_1\sqrt{2}).(x_3+y_3\sqrt{2})$$

$$= k_1.k_2 + k_1.k_3$$

$$\boxed{\therefore k_1.(k_2+k_3) = k_1.k_2 + k_1.k_3}$$

Hence by (i) to (ix) (1-to-9) Properties 'S' is a Field and

as

(2) $S = \{k \mid k = x + y\sqrt{2}, \text{ where } x, y \in \mathbb{Q}\}$

W.K.T The complex number set is the biggest set of numbers so every other set of numbers is a subset of it.

Thus $S \subset F = \mathbb{C}$

$S \subset F$

$\therefore$ S' is a sub-field of $F = \langle \mathbb{C}, +, \cdot \rangle$

Hence, the set of all numbers of the form $x + y\sqrt{2}$, where x & y are rational is a sub-field of $\langle \mathbb{C}, +, \cdot \rangle$

③ prove that each sub-field of the field of complex numbers contains every rational number?

Answer:

Given,

a Field F = set of complex numbers

$F = \langle C, +, \cdot \rangle$

RTP: let S be sub-field of F,

then, S contains all rational numbers in it.

{Rational numbers (Q) is in the form $\frac{m}{n}$, $n \neq 0$, $m, n \in Z$}

Now

W.K.T as S is a sub-field. It obeys all field properties

$\Rightarrow \exists$ Additive Identity $= 0 \in S$ (under $+$)

and $\exists$ Multiplicative Identity $= 1 \in S$ (under $\cdot$)

Now $\langle S, +, \cdot \rangle$

Every ~~characteristic~~ sub-field of F has a characteristic zero.

$\because S$ is a sub-field

then

$1 \in S$

$n.1 = 0$ in $S \Rightarrow n.1 = 0$ in F

But $n.1 = 0$ in F

$\Rightarrow n = 0$

Hence the set of Natural numbers $\in S$

$\Rightarrow \boxed{1, 2, 3 \text{------} \infty \in S}$ — ① $\quad \langle S, +, \cdot \rangle$

Now all these numbers have

Additive inverses (under +) $\Rightarrow \boxed{x + (-x) = 0}$

$\Rightarrow \boxed{-1, -2, -3, \text{------} \in S}$ as well — ②

Now they also have multiplicative inverses

From ① & ② and $0 \in S$

we have $\quad$ Set of Integers $\in S$

ie, $n \in S, \forall n \in \mathbb{Z}$

Now for every 'n' we have multiplicative inverse

$\Rightarrow \pm \frac{1}{n} \in S \quad \forall n \in \mathbb{Z} \ (n \neq 0)$

Now consider some other $m \in S, \forall m \in \mathbb{Z}$

Now

$m \in S$ and $\pm \frac{1}{n} \in S \quad (n \neq 0)$

By closure property of multiplication $(\cdot)$

$\Rightarrow \pm m \cdot \frac{1}{n} \in S$

$\Rightarrow \boxed{\pm \frac{m}{n} \in S}$

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Thus, $\left\{ \frac{p}{q} \in S \quad \forall\ p, q \in \mathbb{Z},\ q \neq 0 \right\}$

Hence, '$\hat{S}$ has every rational number it

Hence, proved

(4) Prove that the inverse operation (function) of an elementary row operation exists, and is an elementary row operation of the same type

Answer:

Given, an elementary row operation

RTP: Inverse of it exists and it is of same type

We have 3 elementary row operations. We prove that these 3 will have inverses of same type

Type 1: swap two Rows

let R_i, R_j be any two rows

then $R_i \to R_j$ and $R_j \to R_i$ $(i \neq j)$

Now the inverse of this is again of same type

$\Rightarrow$ $\boxed{R_j \to R_i \text{ and } R_i \to R_j}$

Thus we get back to original matrix and hence is an inverse of same-type

Type 2

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Type 2 : Multiplying a row by a scalar 'k'

$\Rightarrow e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = k \cdot A_{rj}$

Now $k \in$ Field F of Matrix A.

$\Rightarrow 1/k \in$ Field F

So applying elementary operation of same type on matrix we get

$\Rightarrow e(A)_{ij} = A_{ij}$ if $i \neq r$ $e(A)_{rj} = \frac{1}{k} \cdot (k \cdot A_{rj})$

$e(A)_{rj} = A_{rj}$

Thus we get back original matrix again

Type 3 : Multiplying a row by a scalar and adding it to another row

$R_i \rightarrow R_i + k \cdot R_j$ — (1)

Now the inverse of this operation is

$R_i \rightarrow \underline{R_i} - k \cdot R_j$

This is previous R_i that is changed by (1)

Now $k \in$ Field F $\Rightarrow ((R_i + k R_j) - k \cdot R_j)$

$\Rightarrow -k \in$ Field F $(= R_i)$

let $c = -k$

Inverse operation $\Rightarrow R_i \rightarrow R_i + c \cdot R_j$

Thus it is also of same type

Hence, proved