pace

LINEAR ALGEBRA:

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Assignment – 3

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① On R^n, define two properties: $\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$ and $c\bar{\alpha} = -c\bar{\alpha}$, which of the axioms for the vector space are satisfied by $(R, \oplus, \cdot)$?

Answer

Given, R^n, 2 properties are defined

① $\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$

② $c.\bar{\alpha} = -c.\bar{\alpha}$ ($c \in$ scalar)

Find: check for vector space properties (let V)

① Addition is NOT commutative ($\oplus$)

$\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$

$\bar{\beta} \oplus \bar{\alpha} = \bar{\beta} - \bar{\alpha}$

$\Rightarrow \boxed{\bar{\alpha} \oplus \bar{\beta} \neq \bar{\beta} \oplus \bar{\alpha}}$

② $\oplus$ is NOT Associative $(\bar{\alpha} \oplus \bar{\beta}) \oplus \bar{c} \neq \bar{\alpha} \oplus (\bar{\beta} \oplus \bar{c})$

L.H.S: $(\bar{\alpha} \oplus \bar{\beta}) \oplus \bar{c} = \bar{\alpha} - \bar{\beta} - \bar{c}$

R.H.S $= \bar{\alpha} \oplus (\bar{\beta} \oplus \bar{c})$ ($\because \bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$)

$= \bar{\alpha} \oplus (\bar{\beta} - \bar{c})$

$= \bar{\alpha} - (\bar{\beta} - \bar{c})$

R.H.S $= \bar{\alpha} - \bar{\beta} + \bar{c}$

$\therefore$ L.H.S $\neq$ R.H.S

③ $\exists$ unique vector $\bar{0}$ s/t

$\bar{\alpha} + \bar{0} = \bar{\alpha}$

Hence holds,

④ Additive Inverse under $\oplus$ does NOT holds

$\forall\ \bar{\alpha} \in V$

$\exists\ -\bar{\alpha} \in V$

s/t $\bar{\alpha} \oplus (-\bar{\alpha}) = \bar{\alpha} \oplus (-\bar{\alpha})$

$= \bar{\alpha} - (-\bar{\alpha})$

$= \bar{\alpha} + \bar{\alpha}$

$= 2\bar{\alpha}$

$\neq \bar{0}$ for $\bar{\alpha} \neq 0$ and $\bar{\alpha} \in V$

Multiplication:

Existence of multiplicative Identity:

① $1.\bar{\alpha} = -1.\bar{\alpha} = -\bar{\alpha}$ $(\because c\bar{\alpha} = -c\bar{\alpha})$ - Given

$\neq \bar{\alpha}$

$\therefore$ DOES NOT HOLD

② $(c_1.c_2)\bar{\alpha} \neq c_1.(c_2\bar{\alpha})$ - NOT Holds.

L.H.S $= (c_1.c_2)\bar{\alpha}$

$= -(c_1.c_2)\bar{\alpha}$

$= -c_1.c_2.\bar{\alpha}$

[let $c_1.c_2 = k$ the $k.\bar{\alpha} = -k\bar{\alpha}$]

R.H.S $= c_1.(c_2\bar{\alpha})$

$= c_1.(-c_2.\bar{\alpha})$

$= (+c_1.\bar{\beta})$ let $(-c_2.\bar{\alpha} = \bar{\beta})$

$= c_1.\beta$

$= -c_1.\beta$

$= -c_1 \ldots$

$\therefore$

$= +.c$

$\therefore$ L.H.

$-c_1c$

③ $c(\bar{\alpha} \oplus \bar{\beta})$

L.H.S $= c($

$= c.$

$= -$

$= -$

$= -$

R.H.S $= c$

$= (-c$

$= -$

$= -$

property h

④ $(c_1+c_2)\bar{\alpha}$

L.H.S $= (c_1$

$= -($

R.H.S $=$

$= -$

$= -$

$= c_1 \cdot \bar{\beta}$ (but $\bar{\beta} = -c_2 \cdot \bar{\alpha}$)

$= -c_1 \cdot \bar{\beta}$

$= -c_1 \cdot (-c_2 \cdot \bar{\alpha})$

$= c_1 \cdot c_2 \cdot \bar{\alpha}$

$\therefore -c_1 \cdot c_2 \cdot \bar{\alpha} \neq +c_1 \cdot c_2 \cdot \bar{\alpha}$

L.H.S $\neq$ R.H.S

The property Does NOT hold True

③ $c \cdot (\bar{\alpha} \oplus \bar{\beta}) = c\bar{\alpha} \oplus c\bar{\beta}$

L.H.S $= c(\bar{\alpha} \oplus \bar{\beta})$

$= c(\bar{\alpha} - \bar{\beta})$

$= -c(\bar{\alpha} - \bar{\beta})$

$= -c\bar{\alpha} + c\bar{\beta}$

R.H.S $= c\bar{\alpha} - c\bar{\beta}$ $(c\bar{\alpha} = -c\bar{\alpha})$

$= -c\bar{\alpha} - (-c\bar{\beta})$

$= -c\bar{\alpha} + c\bar{\beta}$

$\therefore$ L.H.S = R.H.S

The property Holds TRUE

④ $(c_1 \oplus c_2) \cdot \bar{\alpha} = c_1 \cdot \bar{\alpha} \oplus c_2 \bar{\alpha}$

L.H.S $= (c_1 \oplus c_2) \cdot \bar{\alpha}$

$= -(c_1 + c_2) \cdot \bar{\alpha} = -c_1 \cdot \bar{\alpha} - c_2 \bar{\alpha}$

R.H.S $= c_1 \cdot \bar{\alpha} - c_2 \cdot \bar{\alpha}$ $(\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta})$

$= -c_1 \cdot \bar{\alpha} - (-c_2 \cdot \bar{\alpha})$

$= -c_1 \cdot \bar{\alpha} + c_2 \cdot \bar{\alpha}$

$\therefore -c_1 \cdot \bar{\alpha} - c_2 \ldots$

The property ...

$\therefore -c_1 \cdot \bar{\alpha} - c_2 \bar{\alpha} \neq -c_1 \cdot \bar{\alpha} + c_2 \cdot \bar{\alpha}$

The property HOLDS NOT TRUE!

(2) Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers.

$$(x,y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x,y) = (cx, 0)$$

Is V a vector space?

Answer) NO, 'V' is not a Vector Space as all the properties of vector space doesn't HOLD TRUE.

Contradiction condition:

Additive identity Doesn't HOLD

let $\bar{\alpha} = (x, y)$

then $\bar{0} = (0, 0)$

$$(x,y) + (0,0) = (x, 0)$$
$$\neq (x,y) \quad \text{where } \boxed{y \neq 0}$$
$$\neq (\bar{\alpha})$$
$$\neq \bar{\alpha}$$

③ Let V be the set of all complex valued functions f on real line such $\forall t \in R$,

$$f(-t) = f((t))^* = f^*(t)$$

$f^*(t)$ = complex conjugation of $f(t)$

show that V with operations $(f+g)(t) = f(t) + g(t)$

and $(c.f)(t) = c.f(t)$ is a vector space over the field R

Give an example of a function f^n in V which is not real valued

Answer:

let $x, y \in R$

then $\overline{(x+y)} = \bar{x} + \bar{y}$

$\overline{xy} = \bar{x}\bar{y}$

RTP: $\forall$ f, g in V $(f+g) \in V$

and $f(-t) = \overline{f(t)}$

$g(-t) = \overline{g(t)}$

which implies $\Rightarrow (f+g)(t) = (f+g)(t)$.

① Addition on functions in V is defined by adding c to the values of functions in c

$\Rightarrow$ 'c' is commutative

② Associative: as C is associative. Addition in V ⟹ functions are also associative.

③ The zero function = Additive Identity $\in V$

as $g(t) = 0 \Rightarrow -0 = \bar{0} \Rightarrow f + g = f \quad \forall f \in V$

Hence Additive Identity exists!

④ Additive Inverse:

let g be a function s/t $g(t) = -f(t)$

The $g(-t) = -f(-t) = -\overline{f(t)} = +\overline{g(t)}$

$\therefore (f+g)(t) = f(t) + g(t) = f(t) - f(t) = \bar{0}$

Hence, $\forall f \in V \ \exists g \in V$ = additive inverse of f

⑤ Multiplicative identity

$\Rightarrow 1.f = f$ (1 = multiplicative identity in $\mathbb{R}$)

⑥ Multiplicative Associativity in 'C'

$\Rightarrow (c_1 \cdot c_2).f = c_1.(c_2.f)$

⑦ As similarly, the distributive Property in $C \Rightarrow c.(f+g) = c.f + c.g$

⑧ Again by Distributive property in C implies

$(c_1+c_2).f = c_1.f + c_2.f$

Hence It forms vector space. Hence, proved

④ Prove the gi

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V is a subspa

each pair vect

$c \in F$, the vec

Answer:

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To prove:

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① let $W \subseteq V$

then let

the

$\Rightarrow c$

Hence, pro

② Given,

and

we have to

let

(4) Prove the given theorems:

(3) A non-empty subset W of vector space V is a subspace of V if and only if, for each pair vectors $\alpha, \beta \in W$ and each scalar $c \in F$, the vector $c\bar{\alpha} + \bar{\beta} \in W$

Answer:-

Given,

a non-empty subset W of vector space V

To prove:

(1)

(2)

(1) let $W \subseteq V$ be the subspace of V

then let $\bar{\alpha}, \bar{\beta} \in W$

the $c.\bar{\alpha} \in W$ (scalar multiplication)

$\Rightarrow c.\bar{\alpha} + \bar{\beta} \in W$ (vector addition of $c.\bar{\alpha}, \bar{\beta}$)

Hence, proved.

(2) Given, $c\bar{\alpha} + \bar{\beta} \in W \quad \forall \bar{\alpha}, \bar{\beta} \in W$

and $W \subseteq V$

we have to prove that W is a sub-space

let

① Put $c=1$ (vector Addition) holds

$\bar{\alpha}+\bar{\beta} \in W$

vector Addition holds True

② Scalar multiplication holds

$\bar{\beta}=0$

$\Rightarrow c\bar{\alpha} \in W$

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① Existence of Additive identity

$\bar{\alpha}=\bar{0}, \bar{\beta}=\bar{0}$

$\Rightarrow \bar{0}+\bar{0} \in W$

$\bar{0} \in W$

$\Rightarrow \bar{\alpha}+\bar{0} = \bar{0}+\bar{\alpha} = \bar{\alpha}$ – holds

② Existence of Additive inverse

let $\bar{\beta} = \bar{\alpha}$ and $c=-1$

then $-1.\bar{\alpha}+\bar{\beta} \in W$

$\bar{0} = (-\bar{\alpha})+\bar{\alpha} \in W$

$-\bar{\alpha} \in W$

so $\forall \bar{\alpha} \in W \ \exists -\bar{\alpha} \in W$ s/t $(\bar{\alpha}+(-\bar{\alpha}) = \bar{0})$

③ $1.\bar{\alpha} = \bar{\alpha}$ holds

put $c=1$ and $\bar{\beta}=\bar{0}$

$\Rightarrow c.\bar{\alpha}+\bar{\beta} \in W$

$1.\bar{\alpha}+\bar{0} \in W$

$\boxed{1.\bar{\alpha} = \bar{\alpha}} \in W$

④ $(c_1$

$c\bar{\alpha} \in$

let $c = c_2$

$\Rightarrow c_2\bar{\alpha}$

Now again

$\Rightarrow c_1(c_2\bar{\alpha}$

and simil

then $(c_1$

Hence $[c_1($

⑤ $c(\bar{\alpha}+\bar{\beta})$

vector ad

Scalar m

hence

$c(c\bar{\alpha}$

⑥ $(c_1+c_2).\bar{\alpha}$

if c_1+c_2

$c_1\bar{\alpha}$

Hence

④ $(c_1 . c_2).\bar{\alpha} = c_1.(c_2.\bar{\alpha})$

$c\bar{\alpha} \in W$ as scalar multiplication holds true

let $c = c_2$

$\Rightarrow c_2 . \bar{\alpha} \in W$

Now again as $c\bar{\alpha} \in W$ let $c = c_1$

$\Rightarrow c_1(c_2\bar{\alpha}) \in W$

and similarly let $k = c_1 . c_2 =$ scalar

then $(c_1 . c_2) . \bar{\alpha} \in W$

Hence $\boxed{c_1.(c_2 . \bar{\alpha}) = (c_1 . c_2) . \bar{\alpha}} \in W$ holds True

⑤ $c(\bar{\alpha} + \bar{\beta}) = c\bar{\alpha} + c\bar{\beta}$

vector addition holds $\Rightarrow$ $\bar{\alpha} + \bar{\beta} \in W$

scalar multiplication holds $\Rightarrow$ $c.\bar{\alpha} \in W$ and $c.(\bar{\alpha} + \bar{\beta}) \in W$

$c.\bar{\beta} \in W$

again $c\bar{\alpha} + c.\bar{\beta} \in W$

hence $c.(\bar{\alpha} + \bar{\beta}) = c.\bar{\alpha} + c\bar{\beta} \in W$ holds True

$\bar{0}$)

⑥ $(c_1 + c_2).\bar{\alpha} = c_1.\bar{\alpha} + c_2.\bar{\alpha}$

if $c_1 + c_2 = k$ $\Rightarrow$ $k\bar{\alpha} \in W \Rightarrow (c_1 + c_2).\bar{\alpha} \in W$

~~$c_1.\bar{\alpha}$~~ $c_1.\bar{\alpha} \in W$, $c_2.\bar{\alpha} \in W$ $\Rightarrow (c_1.\bar{\alpha} + c_2.\bar{\alpha}) \in W$

Hence $(c_1 + c_2).\bar{\alpha} = c_1.\bar{\alpha} + c_2.\bar{\alpha} \in W$ holds true

Hence from

① & ②

$\therefore$ $W \subseteq V$ is a subspace of V

(iff) $c\bar{\alpha}+\bar{\beta} \in W$ $\forall\, \bar{\alpha}, \bar{\beta} \in W$ hold True

Hence, proved.

hold True

(4)

(2) Let V be a vector space over the field F. The intersection of any collection of subspaces of the vectorspace V is a subspace of V.

Answer:

Given, V is a vector space over Field F

RTP: Intersection of any collection of subspaces of V is a subspace of V.

Let the collection subspaces by denoted by $\{W_k\}_{k=1 \text{ to } n}$ of vector space V.

s/t $W = \bigcap_{k=1}^{n} W_k$

Now we prove that W is a sub-space of V by proving $\boxed{c\bar{\alpha}+\bar{\beta} \in W \ \forall \bar{\alpha},\bar{\beta} \in W}$

W.K.T $\bar{0}$ = additive identity is in every subspace = $\{W_k\}$

hence W is non-empty.

Now $\forall \bar{\alpha},\bar{\beta} \in W \Rightarrow \bar{\alpha},\bar{\beta}$ all $\{W_k\}$

($\because$ W = intersection of subspaces)

$\Rightarrow \bar{\alpha},\bar{\beta} \in \bigcap_{k=1}^{n} W_k \in W_k$

Now

$\bar{\alpha} \in$ all W_k

$\Rightarrow c\bar{\alpha} \in$ all W_k ($\because$ Scalar multiplication holds in all sub-spaces)

again

$c.\bar{\alpha}, \bar{\beta} \in$ all W_k

then

$c.\bar{\alpha} + \bar{\beta} \in$ all W_k

($\because$ vector Addition holds in all sub-spaces)

$\therefore \forall \bar{\alpha}, \bar{\beta} \in W = \bigcap_{k=1}^{n} W_k$

$$\boxed{c.\bar{\alpha} + \bar{\beta} \in W} = \bigcap_{k=1}^{n} W_k$$

$\therefore$ W is a sub-space of vectorspace V

So the intersection of subspaces of a vectorspace V is also a subspace

Hence, proved