LINEAR ALGEBRA

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Assignment-4

(Q1)

Given,

'V' is a vector space spanned by the finite set of vectors, $\beta_1, \beta_2, \ldots \beta_m$

RTP:- Any independent set of vectors in V is finite & contains no more than m elements.

i.e., we have to prove if $\exists$ a set S in 'V' containing elements $\boxed{n>m}$ then 'S' is linearly dependent.

$\Rightarrow$ let 'S' be a non-empty subset of 'V' containing vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ s/t $n > m$

as $\beta_1, \beta_2 \ldots \beta_m$ span 'V'

$\Rightarrow$ ~~$\alpha_j = \Sigma$~~ $\Rightarrow \alpha_i = \sum_{i=1}^{n} A_{ij} \beta_i$

where A_{ij} are distinct scalars in Field F.

Now

consider

n scalars $x_1, x_2, \dots, x_n$ in F

~~set~~ then

$\Rightarrow x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$

$= \sum_{j=1}^{n} x_j\alpha_j \qquad (\because \alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i)$

$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i$

$= \sum_{j=1}^{n}\sum_{i=1}^{m} x_j A_{ij}\beta_i$

$= \sum_{i=1}^{m}\left[\sum_{j=1}^{n} A_{ij}x_j\right]\beta_i$

then,

W.K.T

For an $A_{m\times n}$ matrices if $n>m$ then

$AX=0$ has non-trivial solution

$\Rightarrow AX=0$ "for not-all x_j being equal to zero"

$\Rightarrow \sum_{j=1}^{n} A_{ij}x_j = 0$ (for some non-zero x_j)

$\Rightarrow x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} A_{ij}x_j\right]\beta_i$

$\boxed{\sum x_i\alpha_i = 0}$

~~for some~~ and $\exists$ some x_i's $\neq 0$

$\Rightarrow x_1\alpha_1 + x_2\alpha_2 + \dots$

and $\exists$ some

$\Rightarrow \alpha_1, \alpha_2, \dots$

i.e., the set S

Hence, any indepe

V is finite and

Hence, proved.

$\Rightarrow x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = 0$

and $\exists$ some $x_i \neq 0$

$\Rightarrow \alpha_1, \alpha_2, \cdots \alpha_n$ are linearly dependent

i.e., the set S is linearly dependent

Hence, any independent set of vectors in V is finite and contains $\leq m$ elements only,

Hence, proved

(Q2) ;

Answer :-

Given,

W_1, W_2 are finite dimensional sub-spaces of vector space V.

RTP :-

① $W_1 + W_2$ is finite dimensional

② $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

①

let

W_1 has the basis $\{\alpha_1, \dots \alpha_k, \beta_1 \dots \beta_m\}$

W_2 has the basis $\{\alpha_1, \dots \alpha_k, \gamma_1 \dots \gamma_n\}$

then W.K.T

Intersection of 2 subspaces is a subspace

Intersection of 2 subspaces is a subspace of V

$\Rightarrow W_1 \cap W_2$ is a subspace of V

and also

~~for~~ every linearly independent subset of $\boxed{W_1 \cap W_2}$ is finite and is part of a (finite) basis

Since W_1, W_2 are finite dimensional

$\Rightarrow W_1 \cap W_2$ has a finite basis which

is part of basis of W_1 and W_2

$= \{\alpha_1, \ldots \alpha_k\} \Rightarrow \boxed{\dim(W_1 \cap W_2) = k}$

Now $W_1 + W_2$ is spanned by the vectors

$\alpha_1 \ldots \alpha_k \quad \beta_1 \ldots \beta_m \quad \gamma_1 \ldots \gamma_n$

~~and~~

~~let $\alpha_1, \ldots \alpha_k, \beta_1 \ldots \beta_m$~~ and $\gamma_1 \ldots \gamma_n$ ~~form~~

~~an independent sets~~

let $\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$ for some scalars x_i, y_j, z_r

$\Rightarrow -\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_j \beta_j$

Hence,

as $\sum -z_r \gamma_r$ is a linear combination of

α_i, β_j $i, j = 1$ to k in W_1
$j = 1$ to m

$\Rightarrow \sum z_r \gamma_r \in W_1$

but as $\sum z_r \gamma_r \in W_2$

and $\sum x_i \alpha_i \in W_1$ and $\sum x_i \alpha_i \in W_2$

$\Rightarrow \sum z_r \gamma_r =$ linear combination of α_i in W_2

$= \sum c_i \alpha_i$

an
dependent
set
nd

also

Now,
W_1, W_2 are finite-dimensional subspaces
$\Rightarrow$ the basis is independent set

$$\sum z_r \gamma_r = \sum c_i \alpha_i$$

and $$-\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_j \beta_j \quad \text{— (a)}$$

Now as the set
$\{\alpha_1 \text{ -- } \alpha_k, \gamma_1 \text{ -- } \gamma_n\}$ is
independent set (W_2)
$\Rightarrow$ each $x_i = 0$ and $z_r = 0$ then only

$$\sum x_i \alpha_i + \sum z_r \gamma_r = 0$$

but as $\boxed{\sum z_r \gamma_r = \sum c_i \alpha_i}$

$\Rightarrow$ each $z_r = 0$ is a MUST! - for independent set. — (b)

(a) $\Rightarrow$ $\sum x_i \alpha_i + \sum y_j \beta_j = 0$

Now the set
$\{\alpha_1 \text{ -- } \alpha_k, \beta_1 \text{ -- } \beta_m\}$ is
~~is~~ independent (W_1)
$\Rightarrow$ each $x_i = 0$ and each $y_j = 0$ — (c)

From (b) & (c) the set

$\{\alpha_1, \ldots \alpha_k, \beta_1, \ldots \beta_m, \gamma_1 \ldots \gamma_n\}$

is an independent set and [it spans $W_1 + W_2$ (we know)]

thus $W_1 + W_2$ has the basis

$\{\alpha_1, \ldots \alpha_k, \beta_1, \ldots \beta_m, \gamma_1, \ldots \gamma_n\}$

which is finite-dimensional. ($\dim(W_1+W_2) = k+m+n$)

Now

(2) $\dim W_1 + \dim W_2 = (k+m) + (k+n)$

$= k + (m+k+n)$

$= \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

L.H.S = R.H.S

Hence, proved.