LINEAR ALGEBRA :-

Assignment-6

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① For a finite dimensional vector space V and a linear transformation $T: V \to W$ (V, W are over the same field F), prove:

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Answer :-

Given,

a linear Transformation $T: V \to W$, where V, W are vectorspaces over field F.

RTP : $\boxed{\text{rank}(T) + \text{nullity}(T) = \dim(V)}$

Let $\text{rank}(T) = m$

W.k.T

when a matrix A is invertible then $AX = 0$ has trivial solution only

hence $TX = 0$ has trivial solution $X = 0$

Hence

No 'α' in V map to '$\vec{0}$' in W

other than $(\alpha = 0)$

Hence nullspace of $(T) = \{0\}$

$\Rightarrow$ nullity $(T) = 0$

Now consider

rank$(T) = r < n$

$\therefore$ There will be $n - r > 0$ free variables

in solution, that can have any

arbitrary value

Now, $x_1, x_2, \ldots x_{n-r}$ denote the

soln's obtained sequentially setting each

free variable to '0',

and

now $\{x_1, x_2 - \ldots x_{n-r}\}$ is

linearly dependent

Moreover, every solution to

$AX = 0$ is a linear combination of $\{x_1, \ldots x_{n-r}\}$

thus $X = c_1x_1 + c_2x_2 + \ldots + c_{n-r}x_{n-r}$

$\Rightarrow \{x_1, x_2, \ldots, x_{n-r}\}$ spans nullspace(A)

Thus $\{x_1, x_2, \ldots, x_{n-r}\}$ is a basis for nullspace and $(\text{nullity}(A) = n-r)$

Hence $(n-r)+r = n$

$\Rightarrow \boxed{\text{rank}(T) + \text{nullity}(T) = \dim V}$

Hence, proved.

(2) Let V, W, Z be vector spaces over F and let $T: V \to W$, $U: W \to Z$ be linear tranformations. Then show that composition

$f: U.T$ is a linear tranformation

$(U.T): V \to Z$

Answer: Given, V, W, Z are vector spaces over F

and

(i) $T: V \to W$

(ii) $U: W \to Z$

RTP: $U.T$ is a linear Tranformation

i.e. we have to prove that

$U.T(c\alpha+\beta) = c(U.T)(\alpha) + (UT)(\beta)$

$\forall \alpha, \beta \in V$

and

scalar $c \in F$

$\Rightarrow$ L.H.S $= U.T(c\alpha+\beta)$

$= U[T(c\alpha+\beta)]$

$= U[c(T\alpha) + (T\beta)]$ ($\because T$ is a linear Tranformation)

$= c(UT$

$= U[c(T\alpha$

$= c(U.$

$= c(U.$

$= c(U$

$=$ R.H

$\therefore$ L.H.S

Hence a

the co

Hence

$\cancel{= c(UT)}$

$= U[c(T\alpha) + (T\beta)]$ ($\because$ U is a linear transformation)

$= c(U.T\alpha) + (UT\beta)$

$= c(U.T)(\alpha) + (U.T)(\beta)$

$= c.(U.T)(\alpha) + (U.T)(\beta).$

$= R.H.S$

$\therefore L.H.S = R.H.S$

Hence as $T: V \to W$ and $U: W \to Z$

the composition

$U.T : V \to Z$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "U.T"'] & W \arrow[r, "U"] & Z \end{tikzcd}$$

Hence $(U.T): V \to Z$ is a Linear Transformation

(3) Given the linear transformation $T: V \to W$. Prove that T is non-singular iff T takes each linearly independent subset of V onto a linearly independent subset of W

<u>Answer:</u>

Given,

$T: V \to W$ is linear Tranformation

<u>RTP</u>: T is non-singular iff T takes each linearly independent subset of V onto a linearly independent subset of W.

Linearly Independent subset

($\Rightarrow$) Let us suppose T be non-singular and let S be linearly independent subset of V.

If $\alpha_1, \ldots \alpha_n$ are in S, then the vectors $T\alpha_1, \ldots T\alpha_n$ are independent; for if

$$c_1(T\alpha_1) + \ldots + c_k(T\alpha_k) = 0$$

$$\Rightarrow T(c_1\alpha_1 + \ldots + c_k\alpha_k) = 0$$

($\because T$ is non-singular)

$\Rightarrow c_1\alpha_1 + \dots + c_k\alpha_k = 0$

$\Rightarrow$ each $c_i = 0$ as S is an independent set.

$\Rightarrow$ image of S under T is independent.

($\Leftarrow$) Suppose T takes independent subsets of V onto W.

Let α be a non-zero vector in V.

Then the set S containing 1 vector α is independent.

$\Rightarrow$ The image of S has $T\alpha$ and the image set is also independent

$\therefore T\alpha \neq 0$ as the set containing only zero vector is dependent.

Thus nullspace(T) is zero subspace

i.e., T is non-singular

Hence, proved.

④ Prove that every n-dimensional vector space over the field F is isomorphic to space of F^n?

<u>Answer</u>:

Let V be n-dimensional vectorspace in F

with Basis $B = \{\alpha_1, \dots, \alpha_n\}$

where $\alpha_i \in V$

Now define a map $\boxed{T: V \to F^n}$ by

sending each vector $\alpha \in V$ to its coordinate vector $[\alpha]_B$ with

$\Rightarrow \alpha = c_1\alpha_1 + \dots + c_n\alpha_n$ $c_1, \dots, c_n \in F$

then the coordinate matrix of α relative to B is

$$[\alpha]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Now we prove that T is injective, surjective.

① T is inj

for some α

⇒ T

So, the co-ordina

hence, $V = 0V_1 +$

Thus $N(T) = \{$

② T is surje

Let $k = [k$

Then consid

vector

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∴ T is

From ①

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∴

① T is injective

for some $\alpha \in$ Nullspace(T)

$\Rightarrow T(\alpha) = 0 = [\alpha]_B$

so, the co-ordinative vector of V is zero

hence,

$V = 0v_1 + \cdots + 0v_n = 0$

Thus $N(T) = \{0\}$ and T is injective

② T is surjective

let $k = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$ be an arbitrary vector in F^n

Then consider the

vector

$V = k_1v_1 + \cdots + k_nv_n$ in $\underline{V}$ (vector spac

Then it follows from definition of

linear tranformation T that

$T(V) = [V]_B = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = k$

$\therefore$ T is surjective

From ① & ② $\boxed{T : V \to F^n}$ is a linear

Transformation

$\therefore \underline{V}$ over field F is isomorphic to $\underline{F^n}$

Hence proved

(5) Let V, W and Z be finite-dimensional vector spaces over the field F.
Let T be a linear transformation from V into W and U a linear transformation from W into Z.
If β, β', β'' are ordered basis for the spaces V, W and Z respectively, if A is the matrix of T relative to the pair β, β' and B is the matrix of U relative to the pair β', β'', then prove that the matrix composition U.T relative to the pair β, β'' is the product matrix C = BA.

Answer

Given,
V, W, Z are finite-dimensional vector spaces over F.

and T : V → W are linear transformations
U : W → Z

Given, ordered Basis
β — for vector space V
β' — for vector space W
β'' — for vector space Z

let dimension of V → p
dimension of W → q
dimension of Z → r

A is matrix

$\Rightarrow T\beta_s = \sum_{t=1}^{q}$

$\Rightarrow \boxed{T\beta_s = \sum_{t=1}^{q} A}$

Again
B is mat

$\Rightarrow \boxed{U\beta'_t =}$

Apply U on

$\Rightarrow U(T\beta_s)$

$\Rightarrow U(T\beta_s) =$

$\Rightarrow U(T\beta_s) =$

$\Rightarrow (U.T)(\beta_s)$

$\Rightarrow (U.T)(\beta_s$

$\Rightarrow (U.T)(\beta$

A is matrix of T relative to β, β'

$$\Rightarrow T\beta_s = \sum_{t=1}^{q} A_{ts}\beta'_t$$

$$\Rightarrow \boxed{T\beta_s = \sum_{t=1}^{q} A_{ts}\beta'_t} \quad - (1)$$

Again

B is matrix of U relative to β', β''

$$\Rightarrow \boxed{U\beta'_t = \sum_{u=1}^{r} B_{ut}\beta''_u} \quad - (2)$$

Apply U on (1) on b/s

$$\Rightarrow U(T\beta_s) = U\left(\sum_{t=1}^{q} A_{ts}\beta'_t\right)$$

(∵ U is a linear transformation)

$$\Rightarrow U(T\beta_s) = \sum_{t=1}^{q} A_{ts}\, U(\beta'_t) \quad (\because (2))$$

$$\Rightarrow U(T\beta_s) = \sum_{t=1}^{q} A_{ts}\cdot\left(\sum_{u=1}^{r} B_{ut}\beta''_u\right)$$

$$\Rightarrow (U.T)(\beta_s) = \sum_{t=1}^{q}\sum_{u=1}^{r} A_{ts}\cdot B_{ut}\cdot \beta''_u$$

$$\Rightarrow (U.T)(\beta_s) = \sum_{t=1}^{q}\sum_{u=1}^{r} B_{ut}\cdot A_{ts}\cdot \beta''_u$$

$$\Rightarrow (U.T)(\beta_s) = \sum_{u=1}^{r}\left(\sum_{t=1}^{q} B_{ut}A_{ts}\right)\beta''_u$$

$(\because ①)$ M be a $p \times r$ matrix

$\Rightarrow$ let

$$M_{us} = \sum_{t=1}^{r} B_{ut} A_{ts}$$

Now let $C = BA$

$$\Rightarrow C_{us} = \sum_{t=1}^{r} B_{ut} A_{ts}$$

Hence Matrix $\boxed{M = C = BA}$

~~Thus matrix $C = BA$ of (U.T)~~

Thus matrix C of (U.T) relative to β, β' is $\boxed{C = BA}$

Hence, proved