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Class Assignment - 2a

① Prove that if 'A' is a matrix of size $m \times n$ and $m < n$, then the homogenous system of linear equations $AX = 0$ has a non-trivial solution.

Answer :-

Given, a matrix $A_{m \times n}$ s/t $m < n$

and there is homogenous system of linear equations $AX = 0$

RTP :- $AX = 0$ has a non-trivial solution.

Let 'R' be a row-Reduced Echelon matrix formed by Elementary Row operations on A.

i.e., 'A' is row-equivalent to 'R'

W.K.T

If 'B' is row-equivalent to 'A' then the system of linear equations $AX = 0$ and $BX = 0$ have exactly the same solutions.

Hence, $AX = 0$ and $RX = 0$ have exactly the same solution

Now let r = Number of non-zero rows in 'R'

$\Rightarrow r \leq m$ $m =$ No. of rows in R
$n =$ No. of cols in R

and $m < n$ ($\because$ Given)

$\Rightarrow r \leq m < n$

$\Rightarrow r < n$

$\Rightarrow \boxed{(n-r) > 0}$

$\therefore$ for n variables in m equations r variables are independent and $(n-r)$ variables (x_i) can have any arbitrary value other than zero.

Thus $A_{m \times n}$ & $m < n$ ~~has~~ $(Ax=0)$ has a non-trivial solution

Hence, proved.

② prove that if A is a square matrix of size ~~m~~ $n \times n$ then A is row equivalent to $I_{n \times n}$ (iff) the system of linear equation. $AX = 0$ only has the trivial solution.

Answer:-

Given,

$A_{n \times n}$ - square matrix

$I_{n \times n} = I_{n \times n}$ - Identity matrix

$AX = 0$

RTP:- A is row equivalent $I_{n \times n}$ (iff) $AX = 0$ has the trivial solution only,

i.e,

① If A is row equivalent $I_{n \times n}$, then $AX = 0$ has trivial solution only.

② If $AX = 0$ has trivial solution only, the A is row-equivalent to $I_{n \times n}$

① ⇒ $A\cancel{x} = 0$ ~~and~~ $IX = 0$ have same solution as 'A is row equivalent to 'I'

as I has trivial solution only

⇒ A also has trivial solution only.

⇒ $Ax = 0$ has trivial solution only.

② $Ax = 0$ has trivial solution only

then let R be an $n \times n$ ~~matrix~~ row-reduced echelon matrix which is row equivalent to A and

let

r be no. of non zero rows in R

as $\Rightarrow$ $RX=0$ has no non-trivial solution

$r \geq n$ - ⓐ

but as Total no. of rows in $R' = n$

$r \leq n$ - ⓑ

from ⓐ & ⓑ

$r == n$

Thus there is only one possibility for 'R' that is

'R' has a leading non-entry in each of its n-rows and one in each column

and as it is row-reduced echelon form

'R' MUST be an Identity Matrix ($I_{n\times n}$)

Thus $Ax=0$ has Trivial solution only

$\Rightarrow$ A is row equivalent to $I_{n\times n} = I$

From ① & ②

The condition is proved

③ If A, B, C are matrices over a field F such that the products BC and $A(BC)$ are well defined, and so are the products AB and $(AB)C$, then prove

$$A(BC) = (AB)C$$

Answer:

Given, A, B, C are matrices over a field F

s/t – BC and $A(BC)$ are well defined
– AB and $(AB)C$ are well defined

RTP: $A(BC) = (AB)C$

let B be an $n \times p$ matrices

$\Rightarrow BC$ has n-rows $(\because (BC)$ is defined$)$

$\because A(BC)$ is defined $\Rightarrow A$ has n columns and A is $m \times n$ matrix

$\because AB$ is defined $\Rightarrow AB$ is $m \times p$ matrix

$\Rightarrow (AB)C$ exists

Now we have to prove that:

$$[A(BC)]_{ij} = [(AB)C]_{ij}$$

W.K.T

Matrix multiplication of matrices

$A_{m \times n}, B_{n \times p} \Rightarrow C_{m \times p}$

$$C_{m \times p} = A_{m \times n} \cdot B_{n \times p}$$

$$\boxed{C_{ij} = \sum_{r=1} A_{ir} B_{rj}}$$

$$\Rightarrow [A(BC)]_{ij} = \sum_{r} A_{ir} (BC)_{rj}$$

Now applying same on $(BC)_{rj}$ we get

$$\boxed{(BC)_{rj} = \sum_{s} B_{rs} C_{sj}}$$

$$= \sum_{r} A_{ir} \sum_{s} B_{rs} C_{sj}$$

$$= \sum_{r} \sum_{s} A_{ir} \cdot B_{rs} C_{sj}$$

$$= \sum_{s} \left(\sum_{r} A_{ir} B_{rs} \right) C_{sj}$$

$$= \sum_{s} (AB)_{is} C_{sj} \qquad \left[\because \sum_{r} A_{ir} B_{rs} = (AB)_{is} \right]$$

again $= [(AB).C]_{ij}$

$$\therefore [A(BC)]_{ij} = [(AB)C]_{ij}$$

$$\boxed{\therefore A(BC) = (AB)C}$$

Hence, proved.

(4) Let e be an elementary row operation and let E be elementary matrix of size $m \times m$ such that $E = e(I_{m \times m})$ then prove that

$$e(A) = EA$$

holds $\forall$ matrices A of size $m \times m$.

Answer :

Given that,

e = elementary row operation

E = elementary matrix $(m \times m)$

S/t $E = e(I_{m \times m})$

$= (AB)_{is}$

To prove : $e(A) = EA$. $\forall A_{m \times m}$

we can prove this by proving that $(EA)_{ij}$ is obtained by ith row of E and jth row of A only.

=) we have to prove this for all 3 conditions (or) elementary row operations :

(1) ~~s~~ r = row on which operation is made

e = operation of multiplying a row by a scalar c

then

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ c\delta_{rk}, & i = r \end{cases}$$

$$\therefore (EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ij}, & i \neq r \\ cA_{rj}, & i = r \end{cases}$$

That is $EA = e(A)$

(2) $r \neq s$ $\quad r, s =$ rows of E/A

$e =$ operation of swapping 2 rows r, s

then
$$E_{ik} = \begin{cases} \delta_{ik} & i \neq r \\ \delta_{sk} & i = r \\ \delta_{rk} & i = s \end{cases} \Big\} \text{ swap}$$

$$\therefore (EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ik} & i \neq r \\ A_{sj} & i = r \\ A_{rj} & i = s \end{cases}$$

That is $EA = e(A)$

(3) $r \neq s$ $\quad r, s =$ rows of $A_{m \times m}$

$e =$ operation of 'replacement of row r by row r plus c times rows'

then
$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ \delta_{rk} + c\delta_{sk} & i = r \end{cases}$$

$$\therefore (EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ik}, & i \neq r \\ A_{rj} + c A_{sj}, & i = r \end{cases}$$

That is again $EA = e(A)$

Thus

of elementar

Hence it

Hence, proved

Thus the condition

$$EA = e(A)$$ holds $\forall$ 3 types

of elementary row operations

Hence it holds for all matrices $A_{m \times m}$

$$\boxed{e(A) = EA}$$

Hence, proved