Problem Set 6

MA2.101: Linear Algebra (Spring 2022)

May 11, 2022

Due date: May 15, 2022

General Instructions: All symbols have the usual meanings. Remember to prove all your intermediate claims, starting from basic definitions and theorems used in class to show whatever is being asked. Try proving everything from definitions and keep arguments mathematically well formed and concise.

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1. For a finite dimensional vector space V and a linear transformation $T : V \mapsto W$ (V, W are over the same field $\mathbf{F}$), prove:
$$rank(T) + nullify(T) = dim(V)$$

2. Let V, W, Z be vector spaces over $\mathbf{F}$ and let $T : V \mapsto W, U : W \mapsto Z$ be linear transformations. Then show that the composition $f : U \cdot T$ is a linear transformation $(U \cdot T) : V \mapsto Z$.

   Hint: $U \cdot T$ is defined by
$$U \cdot T(\bar{\alpha}) = U(T(\bar{\alpha})) \qquad \forall \bar{\alpha} \in V$$

3. Given the linear transformation $T : V \mapsto W$, prove that T is non-singular *iff* T takes each linearly independent subset of V onto a linearly independent subset of W.

4. Prove that every n-dimensional vector space over the field F is isomorphic to the space of F^n.

5. Let V, W, and Z be finite-dimensional vector spaces over the filed F. Let T be a linear transformation from V into W and U a linear transformation from W into Z. If $\beta, \beta^{'}, \beta^{''}$ are ordered bases for the spaces V,W, and Z, respectively, if A is the matrix of T relative to the pair $\beta, \beta^{'}$, and B is the matrix of U relative to the pair $\beta^{'}, \beta^{''}$, then prove that the matrix of the composition UT relative to the pair β, $\beta^{''}$ is the product matrix C=BA.