LA - Assign 3 :-

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(a)
Given,
Hermitian matrice

W.K.T
let U = unitary matrix for $\underline{A}$ (n×n matrix)

then $U.U^* = I$ —① ⇒ $\boxed{U^* = U^{-1}}$

let D = diagonal matrix s/t $D = U^* A.U$

⇒ let $D^k = U^* A^k U$ (n = k) —②

Now we prove the statement for n = k+1

$\Rightarrow D^{k+1} = D^k . D = U^* A^k U . U^* A U$ (∵ ①,②)

$= U^* A^k (I) . A . U$ (∵ ①)

$\boxed{D^{k+1} = U^* A^{k+1} U}$

Hence $\boxed{D^n = U^* A^n U}$ ⇒ $U D^n U^* = \underbrace{U.U^*}_{I} A^n \underbrace{U.U^*}_{I}$

⇓

$\boxed{A^n = U . D^n . U^*}$ —③

(a) Given $A = \begin{bmatrix} 0 & 3+i \\ 3-i & -3 \end{bmatrix}$

$\Rightarrow A^* = (\bar{A})^T = \begin{bmatrix} 0 & 3-i \\ 3+i & -3 \end{bmatrix}^T = \begin{bmatrix} 0 & 3+i \\ 3-i & -3 \end{bmatrix} = A$

Hence $\boxed{A^* = A}$ A, is Hermitian

So

① Characteristic eqn $\Rightarrow |A-\lambda I| = 0$.

$$\begin{vmatrix} -\lambda & 3+i \\ 3-i & -3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda+3) - 10 = 0 \Rightarrow \lambda^2+3\lambda-10=0$

$\boxed{\lambda = 2, -5}$ are the eigen values

Now we compute the eigen vectors and
② find basis for eigen space

$\boxed{\lambda = 2} \Rightarrow (A-\lambda I)(x) = 0$

$$\Rightarrow \begin{bmatrix} -2 & 3+i \\ 3-i & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

~~let $x_2 = t$~~

$\Rightarrow$ $-2x_1 + (3+i)x_2 = 0$
$(3-i)x_1 - 5x_2 = 0 \Rightarrow x_2 = \left(\frac{3-i}{5}\right)x_1$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \left(\frac{3-i}{5}\right)x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ \left(\frac{3-i}{5}\right) \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ \frac{3-i}{5} \end{bmatrix} \right\}$$

$\downarrow$
v_1

$\boxed{\lambda = -5} \Rightarrow (A-\lambda I)(x) = 0$

$$\Rightarrow \begin{bmatrix} 5 & 3+i \\ 3-i & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$5x_1 + (3+i)x_2 = 0$

$(3-i)x_1 + 2x_2 = 0 \Rightarrow x_2 = \frac{-(3-i)}{2}x_1$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\ -\frac{(3-i)}{2}x_1\end{bmatrix} = x_1\begin{bmatrix}1\\ -\frac{(3-i)}{2}\end{bmatrix} = \text{span}\left\{\begin{bmatrix}1\\ -\frac{(3-i)}{2}\end{bmatrix}\right\}$$

$\Downarrow$

X_2

~~Now by Gram-schmidt process:-~~

$$\cancel{v_1 = X_1 = \begin{bmatrix}1\\ \frac{3-i}{3}\end{bmatrix}}$$

$$\cancel{v_2 = X_2 - \frac{\langle v_1, x_2\rangle}{\langle v_1, v_1\rangle}\cdot v_1 = \begin{bmatrix}1\\ -\frac{(3-i)}{2}\end{bmatrix} - \frac{\langle v_1, x_2\rangle}{\langle v_1, v_1\rangle}\begin{bmatrix}1\\ \frac{3-i}{5}\end{bmatrix}}$$

~~Hence after normalising the vectors are~~

Since the eigen vectors are orthogonal as A is Hermitian

~~After~~ normalising

$$\frac{v_1}{\|v_1\|} = \begin{bmatrix}\sqrt{5}/\sqrt{7}\\ \frac{\sqrt{5}}{\sqrt{7}}\cdot\left(\frac{3-i}{3}\right)\end{bmatrix}; \quad \frac{v_2}{\|v_2\|} = \begin{bmatrix}\sqrt{2}/\sqrt{7}\\ \sqrt{2}/\sqrt{7}\cdot\left(\frac{i-3}{2}\right)\end{bmatrix}$$

Hence

$$U = \begin{bmatrix}\sqrt{5}/\sqrt{7} & \sqrt{2}/\sqrt{7}\\ \frac{\sqrt{5}}{\sqrt{7}}\cdot\left(\frac{3-i}{5}\right) & \frac{\sqrt{2}}{\sqrt{7}}\cdot\left(\frac{i-3}{2}\right)\end{bmatrix}$$

and

$$D = \begin{bmatrix}2 & 0\\ 0 & -5\end{bmatrix}$$

(-: 3)

so $A^{100} = U D^{100} U^*$

$$A^{100} = \begin{bmatrix} \sqrt{5}/\sqrt{7} & \sqrt{2}/\sqrt{7} \\ \frac{\sqrt{5}}{\sqrt{7}}\cdot\left(\frac{3-i}{5}\right) & \frac{\sqrt{2}}{\sqrt{7}}\cdot\left(\frac{i-3}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5}/\sqrt{7} & \frac{\sqrt{5}}{\sqrt{7}}\cdot\left(\frac{3+i}{5}\right) \\ \sqrt{2}/\sqrt{7} & \frac{\sqrt{2}}{\sqrt{7}}\left(\frac{-i-3}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{7}\cdot 2^{100} + \frac{2}{7}\cdot 5^{100} & \frac{3+i}{7}\cdot(2^{100} - 5^{100}) \\ \frac{3-i}{7}\cdot(2^{100} - 5^{100}) & \frac{1}{7}\cdot(2^{101} + 5^{101}) \end{bmatrix}$$

1(b)

$$A = \begin{bmatrix} 6 & 2+2i \\ 2-2i & 4 \end{bmatrix}$$

$$\Rightarrow A^* = (\bar{A})^T = \begin{bmatrix} 6 & 2-2i \\ 2+2i & 4 \end{bmatrix}^T = \begin{bmatrix} 6 & 2+2i \\ 2-2i & 4 \end{bmatrix} = A$$

Hence A is Hermitian

Now

① Finding Eigen values (λ)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & 2+2i \\ 2-2i & 4-\lambda \end{vmatrix} = 0 \Rightarrow (6-\lambda)(4-\lambda) - (2+2i)(2-2i) = 0$$

$$\Rightarrow (\lambda-4)(\lambda-6) - 8 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$\boxed{\lambda = 2, 8}$$

Now

(2) Find eigenvectors for basis of eigenspace

$(A-\lambda I)(x)=0$

$\boxed{\lambda=2}$ $(A-2I)(x)=0$

$$\Rightarrow \begin{bmatrix} 4 & 2+2i \\ 2-2i & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1+(2+2i)x_2=0$$

$$(2-2i)x_1+2x_2=0 \Rightarrow x_2=(i-1)x_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_1 \\ (i-1)x_1 \end{bmatrix}=x_1\begin{bmatrix} 1 \\ i-1 \end{bmatrix}=\text{span}\left\{\begin{bmatrix} 1 \\ i-1 \end{bmatrix}\right\}$$

v_1 = basis vector 1

$\boxed{\lambda=8}$ $(A-8I)(x)=0$

$$\Rightarrow \begin{bmatrix} -2 & 2+2i \\ 2-2i & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (2-2i)x_1 - 4x_2=0 \Rightarrow x_2=\frac{1-i}{2}x_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_1 \\ \frac{(1-i)}{2}x_1 \end{bmatrix}=x_1\begin{bmatrix} 1 \\ \frac{(1-i)}{2} \end{bmatrix}=\text{span}\left\{\begin{bmatrix} 1 \\ \left(\frac{1-i}{2}\right) \end{bmatrix}\right\}$$

v_2 = basis vector 2

(3) As $\underline{A}$ is Hermitian v_1, v_2 are orthogonal

Now normalise the vectors to get $\underline{U}$

$\Rightarrow \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ \frac{1}{\sqrt{3}}(i-1) \end{bmatrix}, \quad \frac{v_2}{\|v_2\|} = \begin{bmatrix} \sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \cdot \left(\frac{1-i}{2}\right) \end{bmatrix}$

Hence

$$U = \begin{bmatrix} 1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ \frac{1}{\sqrt{3}} \cdot (i-1) & \sqrt{2}/\sqrt{3} \cdot \left(\frac{1-i}{2}\right) \end{bmatrix}$$

W.K.T $\quad A^n = U \cdot D^n U^*$

and

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

Hence $\quad A^{100} = \begin{bmatrix} 1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ \frac{1}{\sqrt{3}} \cdot (i-1) & \sqrt{2}/\sqrt{3} \cdot \left(\frac{1-i}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} 2^{100} & 0 \\ 0 & 8^{100} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{3}}(i+1) \\ \sqrt{2}/\sqrt{3} & \frac{\sqrt{2}}{\sqrt{3}} \cdot \left(\frac{1+i}{2}\right) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{3} \cdot 2^{100} + \frac{2}{3} \cdot 8^{100} & \frac{(1+i)}{3}(8^{100} - 2^{100}) \\ \frac{(1-i)}{3} \cdot (8^{100} - 2^{100}) & \frac{1}{3} \cdot (8^{100} - 2^{101}) \end{bmatrix}$$

1(c)

$$A = \begin{bmatrix} 2 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 2 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}.$$

Now $A^{*} = (\bar{A})^T = \begin{bmatrix} 2 & \frac{-i}{\sqrt{2}} & \frac{+i}{\sqrt{2}} \\ \frac{+i}{\sqrt{2}} & 2 & 0 \\ \frac{-i}{\sqrt{2}} & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 2 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}.$

$\boxed{A^{*} = A}$

A is Hermitian

(1) Finding eigen values $\vdash$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & \frac{i}{\sqrt{2}} & -i/\sqrt{2} \\ -i/\sqrt{2} & 2-\lambda & 0 \\ i/\sqrt{2} & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(2-\lambda)(2-\lambda) - \frac{i}{\sqrt{2}}\left((2-\lambda)\cdot\left(\frac{-i}{\sqrt{2}}\right)\right) - \frac{i}{\sqrt{2}}\left(0 - \frac{i}{\sqrt{2}}(2-\lambda)\right) = 0$

$\Rightarrow (2-\lambda)^3 - \left(\frac{2-\lambda}{2}\right) + \frac{2-\lambda}{2} = 0$

$\Rightarrow (2-\lambda)^3 - 2 - \lambda = 0$

$(2-\lambda)\left((2-\lambda)^2 - 1\right) = 0$

$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$

$(2-\lambda)(\lambda-3)(\lambda-1) = 0$

$\Rightarrow \boxed{\lambda = 1, 2, 3}$

② Finding basis eigenvectors

$\boxed{\lambda = 1}$ $(A - I)(x) = 0$

$$\Rightarrow \begin{bmatrix} 1 & i/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1 & 0 \\ +i/\sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + \frac{i}{\sqrt{2}} x_2 + \frac{i}{\sqrt{2}} x_3 = 0$ $\Rightarrow x_2 = \frac{i}{\sqrt{2}} x_1$

$-\frac{i}{\sqrt{2}} x_1 + x_2 = 0$ $\Rightarrow x_2 + x_3 = 0$ $x_3 = \frac{-i}{\sqrt{2}} x_1$

$\frac{i}{\sqrt{2}} x_1 + x_3 = 0$ $x_3 = -x_2$

$\cancel{x_2 = i}$

Eigen space of $\lambda = 1$ is $\left\{ x_1 \begin{bmatrix} 1 \\ \frac{i}{\sqrt{2}} \\ -i/\sqrt{2} \end{bmatrix} \right\}$

$$v_2 = \begin{bmatrix} 1 \\ i/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}$$

For $\lambda = 2$

$$\begin{bmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 0 & 0 \\ +i/\sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{i}{\sqrt{2}}(x_2 - x_3) = 0$

$-\frac{i}{\sqrt{2}} x_1 = 0 \Rightarrow x_1 = 0, \; x_2 = x_3$

$\therefore$ Eigen Space of $\underline{\lambda = 2}$ is $\left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\|v_1\| = \sqrt{2}$

$\underline{\lambda = 3}$:

$(A - 3I)(x) = 0$

$$\begin{bmatrix} -1 & i/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & -1 & 0 \\ i/\sqrt{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + i/\sqrt{2}\,(x_2 - x_3) = 0$

$-i/\sqrt{2}\, x_1 - x_2 = 0 \quad \Rightarrow x_2 = -i/\sqrt{2}\, x_1$

$\Rightarrow x_3 = i/\sqrt{2}\, x_1$

Hence Eigen space for $\lambda = 3$ is span $\left\{ x_1 \begin{bmatrix} 1 \\ -i/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} \right\}$

Basis vector $v_3 = \begin{bmatrix} 1 \\ -i/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$

$\Rightarrow \|v_3\| = \sqrt{1 + \frac{1}{2} + \frac{1}{2}} = \sqrt{2}$

$\therefore$ The orthonormal set of vector is $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

forms the unitary matrix,

$$U = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & i/2 & -i/2 \\ 1/\sqrt{2} & -i/2 & i/2 \end{bmatrix}$$

and
$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

thus $A^{100} = U D^{100} U^{*}$

$$= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 1^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -i/2 & i/2 \\ \frac{1}{\sqrt{2}} & i/2 & -i/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/\sqrt{2} & \frac{1}{\sqrt{2}}\cdot(3^{100}) \\ \frac{1}{\sqrt{2}}\cdot(2^{100}) & i/2 & -\frac{i}{2}\cdot(3^{100}) \\ \frac{1}{\sqrt{2}}\cdot 2^{100} & \left(-\frac{i}{2}\right) & \frac{i}{2}(3^{100}) \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -i/2 & i/2 \\ 1/\sqrt{2} & i/2 & -i/2 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{0} & \cancel{1/\sqrt{2}} & \cancel{1/\sqrt{2}} \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} \frac{1}{2}(3^{100}+1) & \frac{i}{2\sqrt{2}}(3^{100}-1) & \frac{-i}{2\sqrt{2}}\cdot(3^{100}-1) \\ \frac{-i}{2\sqrt{2}}\cdot(3^{100}-1) & \frac{1}{4}\cdot(2^{101}+3^{100}+1) & \frac{1}{4}(2^{101}-1-3^{100}) \\ \frac{i}{2\sqrt{2}}\cdot(3^{100}-1) & \frac{1}{4}(2^{101}-1-3^{100}) & \frac{1}{4}\cdot(2^{100}+1+3^{100}) \end{bmatrix}$$

2

(a) Given matrix

$$A = \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 4 & 1+i \\ 1-i & 5 \end{bmatrix}^T = \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix} = A$$

Hence A is Hermitian.

① Finding eigen values for $\underline{A}$: $|A-\lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 1-i \\ 1+i & 5-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(5-\lambda) - (1+1) = 0 \Rightarrow \lambda^2 - 9\lambda + 20 - 2 = 0$

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda-3)(\lambda-6) = 0$

$\boxed{\lambda = 3, 6}$

Now

② Finding eigen vectors in $\underline{\text{basis}}$: $(A-\lambda I)(x) = 0$

For $\boxed{\lambda = 3}$

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + (1-i)x_2 = 0$

$(1+i)x_1 + 2x_2 = 0 \Rightarrow x_2 = -\frac{(1+i)}{2}x_1$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{(1+i)}{2}x_1 \end{bmatrix}$

Eigen space for $\lambda = 3 \Rightarrow \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -\frac{(1+i)}{2} \end{bmatrix}}_{V_1} \right\}$

$\Rightarrow \|V_1\| = \sqrt{1 + \frac{1+1}{4}} = \sqrt{3/2}$

For $\boxed{\lambda = 6}$

$$\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + (1-i)x_2 = 0$

$(1+i)x_1 - x_2 = 0 \Rightarrow x_2 = (1+i)x_1$

$\therefore$ Eigen space for $\lambda = 6 \Rightarrow \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1+i \end{bmatrix}}_{v_2} \right\}$

$$\|v_2\| = \sqrt{1 + 1 + 1} = \sqrt{3}$$

$$\therefore \frac{v_1}{\|v_1\|} = \begin{bmatrix} \sqrt{2}/\sqrt{3} \\ -\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{2} \cdot (1+i) \end{bmatrix}, \quad \frac{v_2}{\|v_2\|} = \begin{bmatrix} 1/\sqrt{3} \\ \frac{1}{\sqrt{3}} \cdot (1+i) \end{bmatrix}$$

$$\therefore \text{unitary matrix } (U) = \begin{bmatrix} \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \\ -\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{2} \cdot (1+i) & \frac{1}{\sqrt{3}} \cdot (1+i) \end{bmatrix}$$

s.t. $\boxed{D = U^* A U}$ holds true

where $D = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$

2(b)

Given

$$A = \begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix}$$

$$\therefore A^{\theta} = (\bar{A})^T = \begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix} = A$$

Hence $\underline{A}$ is hermitian

① Finding eigen values : $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -i \\ i & 3-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(3-\lambda) - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 2)(\lambda - 4) = 0 \quad \boxed{\lambda = 2, 4}$$

② Finding eigen vectors : $(A - \lambda I)(x) = 0$
in $\underline{\text{basis}}$

For $\boxed{\lambda = 2}$

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - i x_2 = 0$$
$$i x_1 + x_2 = 0 \Rightarrow \boxed{x_2 = -i x_1}$$

Eigen space $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -i x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\}$

$\downarrow$
v_1

$$\|v_1\| = \sqrt{1+1} = \sqrt{2}$$

$$\therefore \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}$$

For $\boxed{\lambda = 4}$

$$\begin{bmatrix} -1 & -i \\ i & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 - i x_2 = 0$

$i x_1 - x_2 = 0 \Rightarrow x_2 = i x_1$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ i x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ i \end{bmatrix}$

Eigen space $\Rightarrow$ span$\left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\}$ $\downarrow$ V_2

$\|V_2\| = \sqrt{1+1} = \sqrt{2}$

$$\frac{V_2}{\|V_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$$

$\therefore$ Unitary matrix$(U) = \begin{bmatrix} \frac{V_1}{\|V_1\|} & \frac{V_2}{\|V_2\|} \end{bmatrix}$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$$

s/f ~~A = 0~~ $\boxed{D = U^{\theta} A U}$ holds true.

where $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

2(c)

Given

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}.$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1-i \\ 0 & -1+i & 0 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1+i & 0 \end{bmatrix} = A$$

$\therefore$ A is Hermitian

① Finding eigen values of $\underline{A}$: $|A-\lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & -1+i \\ 0 & -1-i & -\lambda \end{vmatrix} = 0$$

$\cancel{(\lambda-5)(\lambda+1)(\lambda) \quad (2)\lambda\lambda \quad =)}$

$\cancel{\lambda = 0, -1, 5}$

$(5-\lambda)(\lambda(\lambda+1) - 2) = 0$

$(5-\lambda)(\lambda+2)(\lambda-1) = 0$

$$\boxed{\lambda = 1, -2, 5}$$

② Finding eigen vectors: $(A-\lambda I)(x) = 0$
in $\underline{\text{basis}}$

~~For~~

$(A-I)(x)=0$

For $\boxed{\lambda=1}$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & -1+i \\ 0 & -1-i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1=0 \Rightarrow x_1=0$

$-2x_2+(-1+i)x_3=0 \Rightarrow x_3=-(1+i)x_2$

$(-1-i)x_2-x_3=0$

$\therefore$ Eigen space $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ x_2 \\ -(1+i)x_2 \end{bmatrix}=x_2\begin{bmatrix} 0 \\ 1 \\ -(1+i) \end{bmatrix}=\text{Span}\left\{\underbrace{\begin{bmatrix} 0 \\ 1 \\ -(1+i) \end{bmatrix}}_{v_1}\right\}$

$\|v_1\|=\sqrt{0+1+1+1}=\sqrt{3}$

$$\frac{v_1}{\|v_1\|}=\begin{bmatrix} 0 \\ 1/\sqrt{3} \\ -\frac{1}{\sqrt{3}}(1+i) \end{bmatrix}$$

For $\boxed{\lambda=-2}$ $(A+2I)(x)=0$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & -1+i \\ 0 & -1-i & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$7x_1=0$

$x_2+(-1+i)x_3=0 \Rightarrow x_3=\frac{(1+i)}{2}x_2$

$-(1+i)x_2+2x_3=0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_2\left(\frac{1+i}{2}\right) \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ (1+i)/2 \end{bmatrix}$$

Eigen space for $\underline{\lambda = -2} \Rightarrow$ span $\left\{ \begin{bmatrix} 0 \\ 1 \\ \frac{(1+i)}{2} \end{bmatrix} \right\}$ $\rightarrow v_2$

$$\|v_2\| = \sqrt{0+1+\frac{1+1}{4}} = \sqrt{\frac{3}{2}}$$

$$\therefore \frac{v_2}{\|v_2\|} = \begin{bmatrix} 0 \\ \sqrt{2}/\sqrt{3} \\ \frac{1}{\sqrt{6}} \cdot (1+i) \end{bmatrix}$$

$(A-5I)(x) = 0$

For $\boxed{\lambda = 5}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & -4i \\ 0 & -1-i & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-6x_2 - 4ix_3 = 0 \Rightarrow x_2 = x_3 = 0$$
$$(-1-i)x_2 - 5x_3 = 0$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{eigen space} \Rightarrow \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\downarrow$
v_3

$$\|v_3\| = \sqrt{1} = 1, \quad \frac{v_3}{\|v_3\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \text{Unitary mat}(U) = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{3} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}}(1+i) & \frac{1}{\sqrt{6}} \cdot (1+i) & 0 \end{bmatrix}$$

and $\boxed{D = U^* A U}$ holds true

(5)

Given,

A, B. are orthogonally diagonalisable

$AB = BA$

RTP: AB is orthogonally diagonalisable

W.k.T

from Spectral Theorem,

A is symmetric iff A is orthogonally diagonalisable

i.e.) If A is orthogonally diagonalisable then A is symmetric

If A is symmetric then it is orthogonally diagonalisable

Thus $A = A^T$, $B = B^T$ ($\because$ A, B are orthogonally diagonalisable)

Now

$$
\begin{aligned}
(AB)^T &= B^T A^T \\
&= BA \quad (\because AB = BA) \\
&= AB
\end{aligned}
$$

$$\boxed{\therefore AB = (AB)^T}$$

AB is symmetric (⇔) AB is orthogonally diagonalisable

$\therefore$ AB is orthogonally diagonalisable (from Spectral theorem)

Hence, proved

(4) orthogonally diagonalise.

(a)

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

$A = A^T \Rightarrow A$ is orthogonally diagonalisable

① Find eigen values of A i.e. $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)^2 - 1 = 0$$

$$\boxed{\lambda = 3, 5}$$

② Finding eigen vectors:- $(A - \lambda I)(x) = 0$

For $\boxed{\lambda = 3}$. $(A - 3I)(x) = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$\Rightarrow \boxed{x_2 = -x_1}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$ - eigen space of $\boxed{\lambda = 3}$

$\downarrow v_1$

~~For $\lambda =$~~ i.e. $\frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\|v_1\| = \sqrt{1+1} = \sqrt{2}$

For $\boxed{\lambda = 5}$ $(A - 5I)(x) = 0$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$\Rightarrow \boxed{x_2 = x_1}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{eigen space} \Rightarrow \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v_2} \right\}$$

$$\|v_2\| = \sqrt{1+1} = \sqrt{2}$$

$$\frac{v_2}{\|v_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\therefore$ A can be orthogonally diagonalised as

$$A = \sum \lambda_i v_i v_i^T$$

where v_i = eigenvector (unit) corresponding to $\underline{\lambda_i}$

$\rightarrow$ $\lambda_1 = 3$, $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $\lambda_2 = 5$, $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore A = 3 \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} + 5 \cdot \begin{bmatrix} 1/\sqrt{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Hence, A is orthogonally diagonalised.

4(b) Given

$$A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} = A$$

$\therefore$ A is orthogonally diagonalisable

(1) Finding eigen value: $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)^2 - 9 = 0$$

$$\lambda^2 + 2\lambda - 8 = 0$$

$$(\lambda+4)(\lambda-2) = 0 \qquad \boxed{\lambda = 2, -4}$$

(2) Finding eigen vectors: $(A - \lambda I)(x) = 0$

For $\boxed{\lambda = 2}$:

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3x_1 + 3x_2 = 0 \Rightarrow \boxed{x_2 = x_1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ $\Rightarrow$ eigen space $\Rightarrow$ span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\downarrow$

v_1

$\|v_1\| = \sqrt{2}$ $\qquad \dfrac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

For $\boxed{\lambda = -4}$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 = 0$$

$$\Rightarrow \boxed{x_2 = -x_1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \text{eigen space} \Rightarrow \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{v_2} \right\}$$

$$\|v_2\| = \sqrt{2} \qquad \frac{v_2}{\|v_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\therefore A = \sum \lambda_i v_i v_i^T$ where λ_i — v_i

eigenvalue — eigen vector (corresponding)

So

$$\lambda_1 = 2, \quad v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \lambda_2 = -4, \quad v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\therefore A = 2 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + (-4) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Hence A is orthogonally diagonalised

4(c) Given

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$A = A^T \Rightarrow$ symmetric $\Rightarrow$ orthogonally diagonalizable

① Eigen values of A : $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)((1-\lambda)^2 - 9) = 0$$

$\lambda = 5, -2, 4$

② Eigen vectors : $(A - \lambda I)(x) = 0$

For $\boxed{\lambda = 5}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -4x_2 + 3x_3 = 0 \Rightarrow x_2 = 0$

$3x_2 - 4x_3 = 0 \qquad x_3 = 0$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ eigen space $\Rightarrow$ span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\Downarrow$ v_1

$\|v_1\| = 1 \qquad \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\boxed{\lambda = -2}$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$7x_1 = 0 \qquad x_1 = 0$

$3x_2 + 3x_3 = 0 \qquad x_3 = -x_2$

$\therefore$ eigen space $\Rightarrow$ span $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ $\Downarrow$ v_2 $\qquad \|v_2\| = \sqrt{2}$

$$\frac{v_2}{\|v_2\|} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

For $\boxed{\lambda = 4}$ $(A - 4I)(x) = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0 \Rightarrow \boxed{x_1 = 0}$

$-3x_2 + 3x_3 = 0 \Rightarrow \boxed{x_3 = x_2}$

$3x_2 - 3x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ eigen space $\Rightarrow$ span $\left\{\underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{v_3}\right\}$

$\|v_3\| = \sqrt{2}$ $\quad \frac{v_3}{\|v_3\|} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore A = \sum \lambda_i q_i q_i^T$ $\quad \lambda_i - q_i$

$\lambda_1 = 5, \ q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = -2, \ q_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\lambda_3 = 4, \ q_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Hence

$$A = 5 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\therefore$ A is orthogonally diagonalized

(5)

Given

To show that

(i) Every Hermitian matrix is normal

(ii) Every Unitary matrix is normal

a matrix is normal when $\boxed{A^*A = A^*A^*}$

(i) Let A be Hermitian matrix $(n \times n)$ then $A = A^*$

$\Rightarrow A^*A = A \cdot A = A^2$

$\Rightarrow A \cdot A^* = A \cdot A = A^2$

$\Rightarrow \boxed{A^* \cdot A = A \cdot A^*}$

Hence, every hermitian matrix is normal

(ii) Let Q be any Unitary matrix $\Rightarrow Q^* \cdot Q = I$

$\Rightarrow (Q^*)^{-1} \cdot Q^* \cdot Q = (Q^*)^{-1} I$

$\Rightarrow \boxed{Q = (Q^*)^{-1}}$ $\rightarrow$ Inverse of Q is Q^* — (1)

Now $(Q^* \cdot Q) Q^{-1} = I \cdot Q^{-1}$

$\Rightarrow Q^* \cdot \underbrace{(Q \cdot Q^{-1})}_{I} = Q^{-1}$

$\boxed{Q^* = Q^{-1}}$ $\Rightarrow$ Inverse of Q^* is Q — (2)

From (1) & (2)

From $Q \cdot Q^* = Q \cdot Q^{-1} = I$

$\boxed{Q^* \cdot Q = Q Q^* = I}$

Hence every Unitary matrix is normal

(6)

let

A be a matrix $(n \times n)$ which is unitarily diagonalisable i.e.

let D = diagonal matrix

$\Rightarrow D = P^* A P$

RTP: A matrix $\underline{A}$ is normal i.e. $\boxed{A \cdot A^* = A^* A}$

$D = P^* A P$ where P = unitary matrix

$\Rightarrow P \cdot D \cdot P^* = \underline{P \cdot P^*} A \cdot \underline{P \cdot P^*} \Rightarrow \boxed{P \cdot P^* = I}$

$\Rightarrow P \cdot D \cdot P^* = I \cdot A \cdot I$ $\quad (\because P^* P = I \Rightarrow P P^* = I)$

$\Rightarrow \boxed{A = P \cdot D \cdot P^*}$ $\quad$ $\underline{P}$ is inverse of $\underline{P^*}$ and P^* is inverse of P

$\Rightarrow A^* = (P \cdot D \cdot P^*)^*$

$= (P^*)^* \cdot D^* \cdot P^*$

$\boxed{A^* = P \cdot D^* \cdot P^*}$

Now

$A \cdot A^* = P \cdot D \cdot \underbrace{P^* \cdot P}_{I} \cdot D^* \cdot P^*$

$A \cdot A^* = P (D \cdot D^*) P^*$ — (1)

consider $A^* A = A^* \cdot A = P \cdot D^* \cdot \underbrace{P^* \cdot P}_{I} \cdot D \cdot P^*$

$A^* \cdot A = P (D^* \cdot D) P^*$ — (2)

Now as D is a diagonal matrix

and product a number and its

complex conjugate is real

$((a+ib)(a-ib) = a^2+b^2)$ $\Rightarrow$ entries in $D.D^*$ are real

entries in $D^*.D$ are real

$\Rightarrow$ $D.D^* = D^*.D$

from ① & ②

$\Rightarrow$ $A.A^* = P.(D.D^*).P^* = P.(D^*.D).P^* = A^*.A$

$$\boxed{A.A^* = A^*.A}$$

Hence A is normal

$\therefore$ If a square complex matrix is unitarily diagonalisable then it must be normal

Hence, proved.

7(a) Given,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find singular value decomposition of A

as $A = UWV^T$

$\Rightarrow A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times 3}$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

① Eigen values of $A^T A$: $|A^T A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda)^2 - (1-\lambda) = 0$$

$$(1-\lambda)[\lambda^2 - 2\lambda] = 0$$

$$\lambda = 1, 0, 2$$

$$\lambda_2, \lambda_3, \lambda_1$$

∴ The singular values are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{2}$

$\sigma_2 = \sqrt{\lambda_2} = 1$

$\sigma_3 = \sqrt{\lambda_3} = 0$

Vectors in eigenspace

For $\boxed{\lambda_1 = 2}$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$
$x_1 - x_2 = 0$ $\Rightarrow x_1 = x_2$
$-x_3 = 0$ $\Rightarrow x_3 = 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ $\rightarrow v_1$

$\|v_1\| = \sqrt{2}$

$$\therefore \boxed{\frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}$$

For $\boxed{\lambda = 1}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$
$x_1 = 0$
x_3

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Eigenspace is span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\rightarrow v_2$

for $\boxed{\lambda = 1}$

$\|v_2\| = 1$ $\quad \frac{v_2}{\|v_2\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\boxed{\lambda = 8}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0 \Rightarrow \boxed{x_2 = -x_1}$

$\boxed{x_3 = 0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \text{eigen space is span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{v_3} \right\}$$

~~Hence~~ $\|v_3\| = \sqrt{1+1} = \sqrt{2}$

$$\frac{v_3}{\|v_3\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

Hence, orthonormal set of eigen vectors $\left(\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right)$

$$\Rightarrow V = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

Now w = diagonal matrix s/t $\sigma_1 > \sigma_2 > \sigma_3$

$$\Rightarrow W = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Now for matrix U

$$u_1 = \frac{1}{\sigma_1}(Av_1)$$

$\Rightarrow u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}_{3\times 1} = \frac{1}{\sigma_1}(Av_1)$

$\underline{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \underline{u_2} = \frac{1}{\sigma_2}(Av_2) = \frac{1}{1} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore U$ is a 2×2 matrix

$U = [u_1 \; u_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \cdot \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{W} \cdot \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}}_{V^T}$

$\Rightarrow \boxed{A = UWV^T}$

---

7 (h)

$A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix} \qquad A_{3\times 2} = U_{3\times 3} W_{3\times 2} V^T_{2\times 2}$

$A^T = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix}$

$\therefore A^T \cdot A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}_{2\times 2}$

(1) Eigen values $|A^TA - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(9-\lambda) = 0$

$\lambda = 4, 9$

$\lambda_2 = 4, \lambda_1 = 9$

singular values $\Rightarrow$ $\boxed{\sigma_2 = \sqrt{\lambda_1} = 2, \quad \sigma_1 = \sqrt{\lambda_2} = 3}$

(2) Eigen vectors

for $\lambda = 9$

$\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\quad -5x_1 = 0 \Rightarrow x_1 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \therefore$ eigen space $\Rightarrow$ span $\left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{v_1} \right\}$

$\|v_1\| = 1$ $\quad \boxed{\frac{v_1}{\|v_1\|} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

For $\lambda = 4$

$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 5x_2 = 0 \Rightarrow x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$ eigen space is span $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v_2} \right\}$

$\|v_2\| = 1$ $\quad \boxed{\frac{v_2}{\|v_2\|} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

Now

V matrix

$$\Rightarrow V = \begin{bmatrix} \frac{v_1}{\|v_1\|} & \frac{v_2}{\|v_2\|} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \sigma_1 > \sigma_2$$

For matrix $\underline{U}$

$$u_i = \frac{1}{\sigma_i} \cdot Av_i$$

$$u_1 = \frac{1}{\sigma_1} \cdot Av_1 = \frac{1}{3} \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}_{3\times 2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} \cdot Av_2 = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

U is a 3×3 matrix with column vectors as $u_1, u_2,$ and u_3

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\therefore A = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{U} \cdot \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}_{W} \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V^T}$$

$$\boxed{A = UWV^T}$$

(5) Given

To sh

(i) Eve

(ii) Ev

a matrix

Her

(i) Let matr

A be

$\Rightarrow A^*A$

$\Rightarrow A \cdot A$

$\Rightarrow$

Hence, e

(ii) Let (

$\Rightarrow (Q^*)$

I

$\Rightarrow$

Now (

From (

~~From~~

(8)

Let A be a matrix $(m \times n)$

RTP: A and A^T have same singular values.

Let A has r non-zero eigen values.

$(r < m$ and $r < n)$

now A can be decomposed as

$$\boxed{A = UWV^T}$$

where U, V are orthogonal matrices

(Here $U - m \times m$
$V - n \times n$)

Thus $A^T = (UWV^T)^T = VWU^T$

now since V, U are orthogonal matrices and

also V is $n \times n$

i.e, (rows of A^T) $\times$ (rows of A^T)

and

U is $m \times m$ $\Rightarrow$ (colums of A^T) $\times$ ((columns of A^T)

Thus $\boxed{A^T = VW^TU^T}$ is the singular

value decomposition of A^T

Now

$U' = V \quad V' = U \quad W' = W^T$

And from properties of W

$$W = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & 0 & \cdots & \sigma_r & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{m \times n}$$

and from properties of Transpose

we have

$$(W^T)_{ij} = (W)_{ji}$$

$$\Rightarrow (W^T)_{11} = (W)_{11}, \quad (W^T)_{22} = (W)_{22}$$

let

$$W^T = \begin{bmatrix} \sigma_1' & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2' & \cdots & 0 & \cdots & & & & 0 \\ 0 & \cdots & \sigma_3' & \cdots & & & & & \vdots \\ \vdots & & & \sigma_r' & \cdots & & & & \vdots \\ 0 & \cdots & & 0 & & & & & \vdots \\ 0 & \cdots & & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times m}$$

$$\boxed{W_{ij} = (W^T)_{ji}}$$

$\Rightarrow \sigma_1' = \sigma_1, \ \sigma_2' = \sigma_2, \ \cdots \ \sigma_r' = \sigma_r$

Hence, singular values of $\underline{A}$ and $\underline{A^T}$ are equal (same)