# UNIT-I

## Matrices and Linear System of Equations.

(Mathematics – I)

### Definitions :→

Matrix :→ A System of "mn" numbers (real or complex) arranged in the form of an ordered set of "m" rows, each row consisting of an ordered set of "n" numbers between [ ] or ( ) or ‖ ‖ is called a "matrix" of order m×n

Ex:
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = [a_{ij}]_{m\times n} \quad [\text{where } 1\le i\le m,\ 1\le j\le n]$$

### Types of matrices :→

1. Square Matrix :→ $[a_{ij}]_{m\times n}$ is square matrix if $m=n$.
2. Rectangular Matrix :→ $[a_{ij}]_{m\times n}$ is Rectangular matrix if $m\neq n$
3. Row Matrix :→ $[a_{ij}]_{m\times n}$ is Row Matrix if $1\times n$ is order
4. Column Matrix :→ $[a_{ij}]_{m\times n}$ is Column matrix if $m\times 1$ is order
5. Unit Matrix :→ $[a_{ij}]_{n\times n}$ such that $a_{ij}=1$ for $i=j$ & $a_{ij}=0$ for $i\neq j$
6. Zero Matrix :→ $[a_{ij}]_{m\times n}$ such that $a_{ij}=0\ \forall\ i$ and $j$

7. Diagonal Matrix :-> $[a_{ij}]_{m \times n}$ where $i = j$

$$[a_{11} \ a_{22} \ a_{33} \ a_{44} \ a_{55} \ \cdots]$$

The line along which the diagonal elements lie is called the "Principl. diagonal" of that matrix.

8. The leading diagonal element may equal or may not.

8. Scalar Matrix :->

In diagonal matrix the leading diagonal elements must be equal.

9. Equal Matrices :->

Two Matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only iff (i) A and B are of Same Order.

(ii) $a_{ij} = b_{ij}$ for every i and j.

10. Transpose of a Matrix :->

inter Changing row's and column's of respective matrix called "Transpose of a matrix"

$$\text{Ex:-} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 4 \\ 0 & 7 & 2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 7 \\ 2 & 4 & 2 \end{bmatrix}$$

11. The Conjugate of a matrix :→

If $A$ is a given matrix, On replacing its elements by the corresponding conjugate complex numbers is called the conjugate of $A$.

$$A = [a_{ij}]_{m\times n}, \Rightarrow \bar{A} = [b_{ij}]_{n\times m} \text{ where } b_{ij} = \bar{a}_{ij}$$

12. The Complex Transpose of a Matrix.

$$A = [a_{ij}]_{m\times n} \Rightarrow A^{\theta} = [b_{ij}]_{n\times m}, \text{ where } b_{ij} = \bar{a}_{ji}$$

13. Triangular Matrix :→

A Square matrix all of whose elements below the leading diagonal are zero, is called an "Upper triangular matrix". A Square matrix all of whose elements above the leading diagonal are zero is called "lower triangular matrix".

Ex: upper triangular. lower triangular.

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} \qquad \begin{bmatrix} 7 & 0 & 0 \\ 5 & 3 & 0 \\ -4 & 6 & 5 \end{bmatrix}$$

14) Symmetric Matrix :→

$A = [a_{ij}]$. is Symmetric if $a_{ij} = a_{ji}$ $\forall$ i&j

thus $A = A^T$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 5 \end{bmatrix}$

15) Skew Symmetric Matrix :→

$A = [a_{ij}]$ is Skew Symmetric if $a_{ij} = -a_{ji}$ $\forall$ i&j

Thus $A = -A^T$

Ex: $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} \Rightarrow -A^T = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$

16). Idempotent :→ if $A^2 = A$.

17) Nill potent :→ $A^m = 0$ if m is any +ive integer. then A is called Nill potent matrix.

18) Involutary matrix :→ If A is Square matrix if $A^2 = I$ then A is called Involutary matrix.

19) Orthogonal matrix :→ If A is Square matrix if $A^T . A = A . A^T = I$ then A is called "Orthogonal matrix"

# UNIT-I

## MATRICES AND LINEAR SYSTEM OF EQUATIONS

Elementary transformations (Or operations) on a matrix:

i) Interchange of two rows: If ith row and jth row are interchanged, it is denoted by $R_1 \longleftrightarrow R_2$

ii> Multiplication of each element of a row with a non-zero scalar. If ith row is multiplied with k then it is denoted by $R_1 \leftrightarrow KR_1$

iii> Multiplying every element of a row with a non-zero scalar and adding to the corres-ponding elements of another row.

If all the elements of ith row are multiplied with k and added to the corresponding elements of jth row then it is denoted by $R_j \rightarrow R_j + KR_i$

The corresponding column transformations will be denoted by writing c, instead of R, i.e., by $c_i \leftrightarrow c_j$, $c_i \rightarrow c_i + Kc_j$, $c_j \rightarrow c_j + Kc_i$ resp.,

An elementary transformation is called 'row transformation' or a 'column transformation' according as it applies to rows or columns.

Important result:

We can prove that elementary operations on a matrix do not change its rank.

# Elementary Transformations.

**Rank :→** The order of a non-vanishing (or) non-zero minor is called the Rank of matrix.

(or)

A +ive no "r" is said to be the rank of matrix "A". if.

i) there exist at least one r-rowed minor whose value is not equal. zero.

ii) Every (r+1)-rowed. minor of A is zero It's denoted by $\rho(A)$.

## Echelon form of a matrix:

A matrix is said to be in echelon form if it has the following properties

1) zero rows, if any, are below any non-zero row.
2) The first non-zero entry in each non-zero row is equal to 1.
3) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note: The condition (ii) is optional.

Important result: The number of non-zero rows in the row echelon form of $A$ is the rank of $A$.

Ex. 1: Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into echelon form and hence find it's rank.

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_3$, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

Now Applying $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Finally Applying $R_4 \to R_4 - R_3$, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form and the number of non-zero is 3.

$\therefore$ Rank$(A) = \rho(A) = 3$.

Reduction to normal form:

There is another important method of finding rank of a matrix. we will discuss it and state some theorems with proof.

Theorem:

Every $m \times n$ matrix of rank $r$ can be reduced to the form $I_r$, $[I_r, 0]$ (or) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary row or column operations, where $I_r$ is the $r$-rowed unit matrix.

The above form is called 'normal form' or "1st canonical form" of a matrix.

Cor. 1: The rank of a $m \times n$ matrix A is 'r' if and only if it can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary row and coloumn operati-ons.

Cor. 2: If A is an $m \times n$ matrix of rank r, there exists non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Ex: Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to canonical form and hence find it's rank.

Given $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - C_1$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 8 & 5 & 0 \\ 1 & -2 & 1 & -8 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - R_1$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$

$C_3 \to \frac{C_3}{8}$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & \frac{-1}{4} & 1 & -8 \end{bmatrix}$

$C_3 \to C_3 - 5C_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{4} & \frac{9}{4} & -8 \end{bmatrix}$

$R_3 \to 4R_3$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 9 & 32 \end{bmatrix}$

$R_3 \rightarrow R_3 + R_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$

$C_3 \rightarrow \frac{C_3}{9}$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -32 \end{bmatrix}$

$C_4 \rightarrow C_4 + 32C_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

The above matrix is in the form $[I_3 \; 0]$

$\therefore$ Rank of A is 3

Elementary matrix:

Definition: It is a matrix obtained from a unit matrix by a single elementary transformation.

For example $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the elementary matrices obtained

from $I_3$ by applying the elementary operations $C_1 \leftrightarrow C_2$, $R_3 \rightarrow 2R_3$ and $R_1 \rightarrow R_1 + 2R_2$ respectively.

An Important note:

consider $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 7 \\ 8 & 1 & 1 \end{bmatrix}$

let us interchange 1st and 3rd rows $B = \begin{bmatrix} 8 & 1 & 1 \\ 1 & 2 & 7 \\ 2 & 3 & 4 \end{bmatrix}$

This B is same as the matrix obtained by pre-multi-plying A with the matrix $E_{13}$ obtained from unit matrix by interchanging 1st and 3rd rows in it

verification: $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$E_{13} \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 7 \\ 8 & 1 & 1 \end{bmatrix} \begin{bmatrix} 8 & 1 & 1 \\ 1 & 2 & 7 \\ 2 & 3 & 4 \end{bmatrix}$$

similarly interchange of two coloumns of a matrix is the result of the post-multiplication by an elementary matrix obtained from unit matrix by the interchange of the corresponding columns.

similarly observations can be made regarding row/column operations.

These ideas are used in the following examples

In this connection we have the following theorem.

Theorem: Every elementary row (column) transformation of a matrix can be obtained by pre-multiplication with corresponding elementary matrix.

Ex: Obtain non-singular matrices P and Q such that PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ when $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ and hence obtain its rank.

we know by cor. 2 of theorem in 1.9 we can find two non-singular matrices P and Q such that

$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. For this we adopt the following procedure

we write $A = I_3 A I_3$ i.e., $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we go on applying elementary row operations and column operations on the matrix. A until it is reduced to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Every row operation will also be applied to the pre-factor $I_3$ of the product on R.H.S and every column operation will be applied to the post factor $I_3$ of the product on the R.H.S

$R_2 \to R_2 - R_1$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 + R_2$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C_3 \to C_3 - C_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

This is of the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. we can verify that $\det P = 1 \neq 0$ and

$\det Q = 1 \neq 0$

$\therefore$ $P$ and $Q$ are non-singular such that $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$. Hence the rank of $A = 2$

## Echelon form (or) Triangular form:→

Reduce the given matrices to echelon form and find Rank

1) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

Solu: Given that $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

$R_1 \longleftrightarrow R_3$

$$A \approx \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5R_1$

$$A \approx \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$$

$R_3 \rightarrow \frac{R_3}{-12}$

$$\Rightarrow A \approx \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

Now, the above matrix is in echelon form

$\therefore \rho(A) = 3 =$ no of non-zero rows.

2. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$R_4 \longrightarrow R_4 - 6R_1$

$$A \approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$.

$$A \approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \longrightarrow \frac{R_2}{-4}$

$$A \approx \begin{bmatrix} 1 & 2 & 3 & 0. \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \longrightarrow R_4 + 4R_2$

$$A \approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{-3}$$

$$A \approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & 1 & -2/3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + 3R_3$$

$$A \approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the above matrix is in echelon form

$\therefore$ $\rho(A) = 3$ = no of non-zero rows.

3) $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Sol:

$$R_1 \to \frac{R_1}{2}$$

$$A \approx \begin{bmatrix} 1 & -1/2 & 3/2 & 2 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1$$

$$R_4 \to R_4 - 2R_1$$

$$A \approx \begin{bmatrix} 1 & -1/2 & 3/2 & 2 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{3}$$

$$A \approx \begin{bmatrix} 1 & -1/2 & 3/2 & 2 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - 4R_2$$

$$R_4 \to R_4 - 6R_2$$

$$A \approx \begin{bmatrix} 1 & -1/2 & 3/2 & 2 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 0 & -4/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{\frac{-4}{3}}$$

$$A \approx \begin{bmatrix} 1 & -1/2 & 3/2 & 2 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the above matrix is in echelon form.

$\therefore\ \rho(A) = 3$

4) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Sol $R_1 \to \dfrac{R_1}{-1}$

$$A \approx \begin{bmatrix} 1 & 3 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 + R_1$

$$A \approx \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

$R_3 \to \dfrac{R_3}{-2}$

$$A \approx \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

$R_3 \to R_3 + 11R_2$

$R_4 \to R_4 - 4R_1$

$$A \approx \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & -3 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 \to \frac{R_3}{-3}$

$$A \approx \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & -1 & -1/6 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$A \approx \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 0 & -1/6 \end{bmatrix}$$

$R_4 \to \frac{R_4}{-1/6}$

$$A \approx \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & -1 & +1/2 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, the above matrix is in echelon form.

$\therefore \rho(A) = 4$

Normal form (or) Canonical form :-

If $a_{n \times n}$ matrix is reduced to form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by using elementary transformation. where $I_r$ is unit matrix of order "r".

5 Find Rank of matrix $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ By using elementary & Column transformation reducing it into Normal form.

Sol: Given that

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$R_1 \longleftrightarrow R_3$

$$A \approx \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 4R_1$

$R_3 \longrightarrow R_3 - 2R_1$

$R_4 \longrightarrow R_4 - R_1$

$$A \approx \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$C_2 \to C_2 + C_1$

$C_4 \to C_4 - 3C_1$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \to \frac{R_2}{6}$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -8/3 \end{bmatrix}$$

$C_4 \to C_4 + \frac{5}{3} C_2$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{8}{3} \end{bmatrix}$$

$R_3 \leftrightarrow R_4$.

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_4 \longrightarrow C_4 + \frac{8}{3} C_3$$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \approx \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, the above matrix is in normal form, hence

$\rho(A)$ = order of unit matrix = 3.

6 Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to Canonical form & find Rank.

Sol: given that $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$A \approx \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - C_1$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$R_2 \to \frac{R_2}{8}$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 18/8 & -8 \end{bmatrix}$$

$C_3 \to C_3 - \frac{5}{8} C_2$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & +\frac{18}{8} & -8 \end{bmatrix}$$

$C_3 \to \frac{8}{18} C_3$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

$C_4 \to C_4 + 8C_3$

$$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A \approx \left[ I_3 \; 0 \right]$$

Now, the above matrix is in Normal form.

$\therefore \rho(A) =$ order of unit matrix $= 3$.

PAQ form of a matrix :-

1.) Obtain the non-singular matrices P and Q S/T PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ also find the rank of matrix A.

Sol:- Given that, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}_{3\times3}$

$$\text{Now } A = I_3 A I_3 \longrightarrow (1)$$

we reduce A on LHS to the normal form by applying elementary transformations. Each E' row transformation will be applied to pre-factor $I_3$ and each E' column transformation will be applied to post-factor $I_3$ of RHS of eq (1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \longrightarrow R_2 - R_1$$
$$R_3 \longrightarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \longrightarrow \frac{R_2}{-2} \quad , \quad R_3 \longrightarrow \frac{R_3}{-2}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 3/2 & 0 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \longrightarrow C_2 - C_1$$
$$C_3 \longrightarrow C_3 - C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 3/2 & 0 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \longrightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \longrightarrow C_3 - C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow PAQ \text{ form}$$

Here $P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ are non-singular matrices.

Rank of $A = \rho(A) = 2$ //

2: $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ obtain non-singular matrix P and Q show that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by using elementary transformations and determine the rank of A.

Sol:-

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}_{3\times 4}$$

$$A = I_3 A I_4 \longrightarrow (1)$$

We reduce A on LHS to the normal form by applying elementary transformations. Each E' row transformation with be applied to pre-factor $I_3$ and each E' column transformation will be applied to post-factor $I_4$ in RHS of eq (1)

$$\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 5 & 1 & 4 & -2 \\ 3 & 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \longrightarrow R_2 - 5R_1 \;,\; R_3 \longrightarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 0 & 21 & -51 & 93 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \to C_2 + 4C_1 \;;\; C_3 \to C_3 - 11C_1 \;;\; C_4 \to C_4 + 19C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 21 & -51 & 93 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 4 & -11 & 19 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \to \frac{C_2}{21}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -51 & 93 \\ 0 & 2/3 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 4/21 & -11 & 19 \\ 0 & 1/21 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{2}{3}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -51 & 93 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -2/3 & 1/3 \end{bmatrix} A \begin{bmatrix} 1 & 4/21 & -11 & 19 \\ 0 & 1/21 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 + 51C_2$$
$$C_4 \to C_4 - 93C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -2/3 & 1/3 \end{bmatrix} A \begin{bmatrix} 1 & 4/21 & 27/21 & 9/7 \\ 0 & 1/21 & 51/21 & -93/21 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -2/3 & 1/3 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 4/21 & 27/21 & 9/7 \\ 0 & 1/21 & 51/21 & -31/7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

are non-singular matrix and rank $\rho(A) = 2$

Ex: Find non-singular matrices P and Q so that PAQ is of the normal form, where $A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & 3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

$A = I_3 A I_4$ i.e., $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & 3 \\ -1 & 2 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Applying $R_2 + 2R_1$ and $R_3 + R_1$

$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 2 & 10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_3 - 2R_2$ $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\frac{1}{5} R_2$ $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_4 - C_3$ $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_1 - 3R_2$ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_2 + 2C_1$, $C_4 - C_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_2 \leftrightarrow C_3$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Inverse of a non-singular matrix by Elementary transformation :-

1) Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Sol:- Given that $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

we write $A = I_3 A$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \longrightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \longrightarrow R_2 - 2R_1 \; ; \; R_3 \longrightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} A$$

$R_2 \longrightarrow \frac{R_2}{-3}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & -1 & 0 \end{bmatrix} A$$

$R_3 \longrightarrow R_3 + 2R_1 \; ; \; R_3 \longrightarrow -3/2 \, R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$R_2 \longrightarrow R_2 + \frac{1}{3} R_3$

$R_1 \longrightarrow R_1 - R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3/2 & 3/2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$$R_1 \longrightarrow R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & 1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$$I_3 = BA$$

where $B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & 1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} = A^{-1}$

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2) Find the inverse of following matrices by using elementary transformations. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$$A = I_3 A$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 \longrightarrow R_2 - 2R_1$$

$$R_3 \longrightarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

$$R_2 \longleftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} A$$

$$R_2 \longrightarrow R_2/-1 \; ; \; R_3 \longrightarrow \frac{R_3}{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix} A$$

$$R_2 \longrightarrow R_2 - 3R_3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} A$$

$$R_1 \longrightarrow R_1 - 3R_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 0 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} A$$

$$R_1 \longrightarrow R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} A$$

$$I_3 = BA$$

where $B = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} = A^{-1}$

Thus we have $P = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ & $PAQ = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ Rank of given matrix is 2.

The inverse of a matrix by elementary transformations: (Gauss-Jordan method)

We can find the inverse of a non-singular square matrix using elementary row operations only. This method is known as gauss-Jordan method.

Working rule for finding the inverse of a matrix:

Suppose $A$ is a non-singular square matrix of order $n$. We write $A = I_n A$.

Now, we apply elementary row operations only to the matrix $A$ and the prefactor $I_n$ of the R.H.S. We will do this till we get an equation of the form.

$$I_n = BA$$

Then obviously $B$ is the inverse of $A$.

Ex: Find the inverse of the matrix $A$ using elementary operations.

We write $A = I_4 A$ i.e., $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_4$ and $R_3 \to R_3 + 6R_4$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -1 & -3 & 0 & -2 \\ -1 & -1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_3$ and $R_2 \to R_2 + R_3$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -2 & -2 & 2 & 4 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$\frac{R_1}{-2}$, $\frac{R_2}{-2}$ and $R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

This is of form $I_4 = BA$

$$\therefore A^{-1} = B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

System of linear stimultaneous equations:

Definition: An equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \quad \to (1)$$

Where $x_1, x_2 \ldots x_n$ are unknowns and $a_1, a_2 \ldots a_n, b$ are constants is called a linear equation with 'n' unknowns.

Definition: consider the system of m linear equations in n unknowns $x_1, x_2, x_3 \ldots x_n$ as given below:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

where $a_{ij}$ is and $b_1, b_2, \ldots b_m$ are constants. An ordered n-tuple $(x_1, x_2, \ldots x_n)$ satisfying all the equations in (2) simultaneously is called a solution of the system (2).

Given a system (2), we do not known in general whether it has a solution or not, If there is atleast one solution for (2), the system is said to be consistent. If (2) does not have any solution, the system is said to be inconsistent.

The system $AX=0$ is always consistent since $X=0$ (i.e., $x_1=0, x_2=0 \ldots x_n=0$) always a solution of $AX=0$

This solution is called a trivial solution of the system.

Given $AX=0$, we try to decide whether it has a solution $X \neq 0$. such a solution, if exists, is called a non-trivial solution

solution procedures to solve $AX=B$

let us first consider n equations in n unknowns (i.e.) $m=n$

The system will be of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

The above system can be written as $AX=B$ where A is an $n \times n$ matrix.

For non-homogeneous system:

The system $AX=B$ is consistent i.e it has a solution if and only if rank of $A$ = rank of $[A|B]$

i> If rank $A$ = rank of $[A|B] = r <$ no. of unknowns, the system is consistent, but there exists infinite no. of solns. giving arbitary values to $n-r$ of the unknowns we may express the other $r$ unknowns in terms of these.

ii> If rank of $A$ = rank of $[A|B] = r = n$ the system has unique solution.

iii> If rank of $A \neq$ rank of $[A|B]$ the system is inconsistent. It has no solution.

To obtain solutions, set $(n-r)$ variables any arbitary value and solve for the remaining unknowns.

Thus irrespective of $m<n$ or $m=n$, or $m>n$, we have decided about the existence and uniqueness of solutions of a system of linear simultaneous equations $AX=B$

Ex: Discuss for what values of $\lambda, \mu$ the simultaneous equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have i) no solution ii) a unique solution iii) an infinite no. of solutions.

The matrix form of given system is

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

we have the argumented matrix is $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$ $[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$

case 1): let $\lambda \neq 3$ then rank of $A = 3$ and rank of $[A,B] = 3$, so that they have same rank. Then the system of equations is consistent. Here the no. of unknowns is 3 which is same as the rank of A. The system of equations will have a unique solution. This is true for any value of $\mu$.

Thus if $\lambda \neq 3$ and $\mu$ has any value, the given system of equations will have a unique solution.

case 2): suppose $\lambda = 3$ and $\mu \neq 10$, then we can see that rank of $A = 2$ and rank of $[A,B] = 3$, since the ranks of A and $[A\ B]$ are not equal, we say that the system of equations has no solution.

case 3): let $\lambda = 3$ and $\mu = 10$. Then we can have rank of $A =$ rank of $[A|B] = 2$

$\therefore$ The given system of equations will be consistent.

But here the no. of unknowns $= 3 >$ rank of A.

Hence the system has infinitely many solutions.

linear dependence and linear independence of vectors:

linearly dependent set of vectors:

Definition: A set $\{x_1, x_2, x_3, x_n\}$ of $r$ vectors is said to be a linearly dependent set, if there exists $r$ scalars $K_1, K_2$ -- $K_r$, not all zero, such that $K_1 x_1 + K_2 x_2 + - + K_r x_r = 0$ where 0, ~~dented~~ denotes the $n$ vector with components all zero.

linearly dependent set of vectors:

Definition: A set $\{x_1, x_2, \text{--} x_n\}$ of $r$ vectors is said to be linearly independent set, if the set, is not linearly dependent i.e., if $K_1 x_1 + K_2 x_2 + \text{--} + K_r x_r = 0$ where '0' is denotes $n$ vector with components all zero

$\Rightarrow K_1 = 0, \ K_2 = 0, \text{---}, \ K_r = 0$

Important results:

1) If a set of vectors is linearly dependent, then atleast one no. of the set can be expressed as a linear combination of the rest of the members.

2) If a set of members is linearly independent then no member of the set can be expressed as a linear combination of the rest of the members.

Consistency of system of homogeneous linear equations:

consider a system of m homogeneous linear equations in n unknowns, namely

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + \cdots a_{3n}x_n = 0 \longrightarrow (1)$$
$$\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = 0$$

we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

Then (1) can be written as $AX=O$ which is the matrix equation. Here $A$ is called the coefficient matrix, It is clear that

$x_1 = 0 = x_2 = x_3 = x_4 = \cdots = x_n$ is a solution of (1) i.e.,

$X = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution of (2)

This is called trivial solu$^{n}$ of $AX=O$

This is $AX=O$ is always consistent (ie) it has a solution

The trivial solution is called the zero solution.

A zero solution is not linearly dependent

working rule for finding the solutions of $AX=O$:

let rank of $A = r$ and rank of $[A|B] = r_1$.

since all b's are zero, $r = r_1$, then

I. 1) If $r = n$ (no. of variables) $\Rightarrow$ the system of equations have only trivial solution (i.e., zero solution)

2) If $r < n \Rightarrow$ the system of equations have an infinite number of non-trivial solutions, we shall have $n-r$ linearly dependent solutions.

To obtain infinite solutions, set $(n-r)$ variables any arbitary value and solve for the remaining unknowns.

If the no. of equations is less than the no. of unknowns, it has a non-trivial solution. The number of solutions of the equation $AX=0$ will be infinite.

II. If the no. of equations is less than no. of variables, the solution is always other than trivial solution.

III. If the no. of equations = no. of variables, the necessary and sufficient condition for solutions other than a trivial solution is that the coefficient matrix is zero

Ex: show that the only real number $\lambda$ for which the system $x+2y+3z = \lambda x$ ; $3x+y+2z = \lambda y$ ; $2x+3y+z = \lambda z$ has non-zero solution is 6 and solve them, when $\lambda = 6$

Given system can be expressed as $AX=0$ where

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here no. of variables $= n = 3$

The given system of equations possess a non-zero (non-trivial) solution, if

Rank of A < number of unknowns i.e., Rank of A < 3

For this we must have $\det A = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1+R_2+R_3$
$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ $(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)\left[(-2-\lambda)(-1-\lambda)+1\right] = 0$

$\Rightarrow (6-\lambda)(\lambda^2+3\lambda+3) = 0 \Rightarrow \lambda = 6$ is the only real value

when $\lambda = 6$ the system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2 + 3R_1$, $R_3 \to 5R_3 + 2R_1 \Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & 19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x+2y+3z=0$ and $-19y+19z=0 \Rightarrow y=z$

Since Rank of $|A|$ < no. of unknowns, the given system has infinite no. of solution (non-trivial)

let $z=k \Rightarrow y=k$ and $-5x+2k+3k=0 \Rightarrow x=k$

$\therefore x=k, y=k, z=k$ is the solution.

Ex: Find $\lambda$ value for which the equations $(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$, $(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$, $2x+(3\lambda+1)y+3(\lambda-1)z=0$. are consistent and find the ratio of $x:y:z$ when $\lambda$ has the smallest of these values. what happens when $\lambda$ has the greater of these values.

$$AX=\begin{bmatrix}\lambda-1 & 3\lambda+1 & 2\lambda\\ \lambda-1 & 4\lambda-2 & \lambda+3\\ 2 & 3\lambda+1 & 3(\lambda-1)\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$$

The given equations will be consistent, if $|A|=0$

i.e., if $\begin{vmatrix}\lambda-1 & 3\lambda+1 & 2\lambda\\ \lambda-1 & 4\lambda-2 & \lambda+3\\ 2 & 3\lambda+1 & 3(\lambda-1)\end{vmatrix}=0$

$R_2-R_1$ $\begin{vmatrix}\lambda-1 & 3\lambda+1 & 2\lambda\\ 0 & \lambda-3 & 3-\lambda\\ 2 & 3\lambda+1 & 3\lambda-3\end{vmatrix}=0$ (expand by $R_2$)

(or) if $C_3+C_2$ $\begin{vmatrix}\lambda-1 & 3\lambda+1 & 5\lambda+1\\ 0 & \lambda-3 & 0\\ 2 & 3\lambda+1 & 6\lambda-2\end{vmatrix}=0$

(or) if $(\lambda-3)\begin{vmatrix}\lambda-1 & 5\lambda+1\\ 2 & 2(3\lambda-1)\end{vmatrix}=0$ or if $6\lambda(\lambda-3)^2=0$

or if $\lambda=0$ or $3$

i) when $\lambda = 0$, the equations become

$-x+y=0 \rightarrow (1) \quad -x-2y+3z=0 \rightarrow (2) \quad 2x+y-3z=0 \rightarrow (3)$

solving 2 & 3 we get

$x = y = z$

(ii) when $\lambda = 3$, equations become identical.

solution of linear systems- direct methods:

The solution of a linear system of equations can be found out by numerical method known as direct method (or) iterative methods.

1. Gaussian elimination method:

This method of solving a system of n linear equations in n unknowns consists of eleminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution. we discuss the method here n=3 The method is analagous for n>3

consider the system.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}\right\} \rightarrow (1)$$

The argumented matrix of system is

$$[A,B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow (2)$$

$R_2 \rightarrow R_2 - \frac{a_{21}}{a_{11}} R_1$ and $R_3 \rightarrow R_3 - \frac{a_{31}}{a_{11}} R_3$, we get

$$[A,B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & \alpha_{33} & \alpha_{33} & \beta_3 \end{bmatrix} \rightarrow (3)$$

where $\alpha_{22} = a_{22} - a_{12}\left[\frac{a_{21}}{a_{11}}\right]$ ; $\alpha_{23} = a_{23} - a_{13}\left[\frac{a_{21}}{a_{11}}\right]$

$$a_{32} = a_{32} - \left[\frac{a_{31}}{a_{11}}\right]a_{12} \; ; \; \alpha_{33} = a_{33} - \left[\frac{a_{31}}{a_{11}}\right]a_{13}$$

$$\beta_2 = b_2 - \left[\frac{a_{21}}{a_{11}}\right]b_1 \; ; \; \beta_3 = b_3 - \left[\frac{a_{31}}{a_{11}}\right]b_1$$

Here we assume $a_{11} \neq 0$

we call $\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}$ as multiplicrs for the first stage.

$a_{11}$ is called first pivot.

Now applying $R_3 \to R_3 - \frac{\alpha_{32}}{\alpha_{22}}(R_1)$, we get

$$[A, B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & 0 & \gamma_{33} & \Delta_3 \end{bmatrix}$$

where $\gamma_{33} = \alpha_{33} - \left[\frac{\alpha_{32}}{\alpha_{22}}\right]\alpha_{23}$ ; $\Delta_3 = \beta_3 - \left[\frac{\alpha_{32}}{\alpha_{22}}\right]\beta_2$

we have assumed $\alpha_{22} \neq 0$

Here the multiplier is $-\frac{\alpha_{32}}{\alpha_{22}}$ and new pivot is $\alpha_{22}$

The argumented matrix (4) corresponds to an upper triangular system with which can be solved by back ward substitution. The solution obtained is exact.

Ex: solve the system of equations: $3x+y-z=3$ ; $2x-8y+z=-5$
$x-2y+9z=8$ using gauss elimination method.

The argumented matrix is $[A, B] = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$

$R_2 \to R_2 - \frac{2}{3}R_1$ ; $R_3 \to R_3 - \frac{1}{3}R_1$

$$[A,B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & -7 \\ 0 & \frac{-7}{3} & \frac{28}{8} & 7 \end{bmatrix}$$

$R_3 \to R_3 - \frac{7}{26} R_2$, $[A,B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & -7 \\ 0 & 0 & \frac{693}{78} & \frac{231}{26} \end{bmatrix}$

From this, $3x+y-z=3$; $\frac{-26}{3}y+\frac{5}{3}z=-7$

$\frac{693}{78}z=\frac{231}{26} \Rightarrow z=1, y=1$ and $x=1$

Gauss- Jordan method:

This is a modification of the gauss's elimination method. In this method the unknowns are eliminated. so that the system is diagonal form. This can be done without using pivoting. The method is illustrated in the example given below.

Ex: solve the equations $10x_1+x_2+x_3=12$; $x_1+10x_2-x_3=10$ and $x_1-2x_2+10x_3=9$ by gauss- Jordan method.

The matrix form of the given system $AX=B$

$A = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$; $[A,B] = \begin{bmatrix} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{bmatrix}$

$R_1 \to R_1 - 9R_2 \sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{bmatrix}$; $\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -89 & 10 & -78 \\ 0 & 99 & -11 & 88 \\ 0 & 87 & 0 & 87 \end{bmatrix}$

$\begin{matrix} R_2 \to R_2/99 \\ R_3 \to R_3/87 \end{matrix} \sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 & 8 \\ 0 & 1 & 0 & 1 \end{bmatrix}$; $R_2 \to R_2 - 8R_3 \sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$R_1 \to R_1+89R_2$, $R_3 \to R_3-R_1$ $\sim \begin{bmatrix} 1 & 0 & -79 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ ; $R_1 \to R_1+79R_3$, $R_2 \to R_2+R_3$ $\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\therefore$ The system $AX=B$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

which gives $x_1 = x_2 = x_3 = 1$

Method of factorization (Triangularisation):

This method is based on the fact that a square matrix A can be factorized into the form LU where L is the unit lower triangular matrix and U is the upper triangular matrix. Here all the principle minors of A must be non-singular. This factorization, if it exists, is unique. consider the linear system

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$;

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$;

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$;

which can be written in the form $AX=B \to (1)$

let $A = LU \to (2)$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \to (3)$

and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \to (4)$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

then (1) becomes $LUX = B$

If we put $UX = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then (5) can be written as $LY = B$

The equivalent system is $y_1 = b_1$;

$l_{21}y_1 + y_2 = b_2$;

$l_{31}y_1 + l_{33}y_2 + y_3 = b$

This can be solved for $y_1, y_2, y_3$ by forward substitution

Using (6) and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ we get,

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$;

$u_{22}x_2 + u_{23}x_3 = y_2$;

$u_{33}x_3 = y_3$

which can be solved for $x_1, x_2, x_3$ by backward substitution. Thus when L and U are known we can calculate $y_1, y_2, y_3$ and $x_1, x_2, x_3$. From (2) we get

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{23} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating the corresponding elements from L.H.S and R.H.S

$u_{11} = a_{11},\ u_{12} = a_{12},\ u_{13} = a_{13}$;

$l_{21}u_{11} = a_{21} \Rightarrow l_{21} = \frac{a_{21}}{a_{11}}$ and $l_{31}u_{11} = a_{31} \Rightarrow l_{31} = \frac{a_{31}}{a_{11}}$

$l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}$; $l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - \frac{a_{2}}{a_{1}}$

$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = a_{32} - \frac{a_{31}}{a_{11}}a_{12}$

and $l_{31}u_{13} + l_{32}u_{23} = a_{33}$ from which $u_{33}$ can be calculated. These relations enables us to determine the elements of the matrices L and U.

Thus we have a systematic procedure to find the elements of L and U

1. we determine first row of U and the first column of L.
2. Next we determine the 2nd row of U and 2nd column of L.
3. Finally we compute the third row of U.

Ex: solve the equations $2x+3y+z=9$; $x+2y+3z=6$; $3x+y+2z=8$ by factorization method.

let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

The given system is of the form $AX=B$

let $LU=A$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

comparing we get $u_{11}=2$, $u_{12}=3$, $u_{13}=1$, $l_{21}u_{11}=1 \Rightarrow l_{21}=\frac{1}{2}$

$l_{31}u_{11}=3 \Rightarrow l_{31}=3/2$, $l_{21}u_{12}+u_{22}=2 \Rightarrow u_{22}=\frac{1}{2}$

$l_{21}u_{13}+u_{23}=3$, we $\Rightarrow u_{23}=\frac{5}{2}$, $l_{31}u_{12}+l_{32}u_{22}=1 \Rightarrow l_{32}=-7$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=2 \Rightarrow u_{33}=18$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$

and hence the given system of equations can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

(or) $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

Solving the system by forward substitution, weget $y_1 = 9$, $\frac{1}{2}y_1 + y_2 = 6 \Rightarrow y_2 = 3/2$ and $\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \Rightarrow y_3 = 5$

From $Ux = Y$, weget

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

we solve by backward substitution and get $x = \frac{35}{18}$, $y = \frac{29}{18}$, $z = 5/18$.

Solution of tridiagonal system:

Consider the system of equations defined by

$$\left.\begin{aligned} b_1u_1 + c_1u_2 &= d_1 \\ a_2u_1 + b_2u_2 + c_2u_3 &= d_2 \\ a_3u_2 + b_3u_3 + c_3u_4 &= d_3 \\ &\cdots \\ a_nu_{n-1} + b_nu_n &= d_n \end{aligned}\right\} \quad --- (1)$$

The coefficient matrix is

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & a_n & b_n \end{bmatrix} \quad --- (2)$$

Matrices of type given in (2) are called tridiagonal matrices. These equations occur frequently in the solution of differential equations by finite difference methods. The method of factorisation discussed previously can be applied to solve equations.

1) Solve completely the system of equations.

$$x - 2y + z - w = 0$$

$$x + y - 2z - 3w = 0$$

$$4x + y - 5z + 8w = 0$$

$$5x - 7y + 2z - w = 0$$

The given system can be written as $AX = 0$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$R_2 \to R_2 - R_1$$

$$R_3 \to R_3 - 4R_1 \; ; \; R_4 \to R_4 - 5R_1$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{3} \; ; \; R_4 \to \frac{R_4}{4} \; ; \; R_3 \to \frac{R_3}{-9}$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 4/3 \\ 0 & -1 & 1 & -4/3 \\ 0 & 3/4 & -3/4 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \; ; \; R_4 \to R_4 - \frac{3}{4} R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echlon form and rank of this matrix is 2 ($\because$ number of non-zero rows = 2)

$$\rho(A) = 2 = r$$

number of variables $(n) = 4$

here $r < n$ (Here $r=2, n=4$)

$\therefore$ The given solution has infinite no. of solutions.

$\exists$ $n-r = 4-2 = 2$ linearly independent solutions. The given system can be written as

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2y + z - \omega = 0 \longrightarrow \text{(i.)}$$

$$0 \cdot x + y - z - \frac{4}{3}\omega = 0 \longrightarrow \text{(ii.)}$$

To obtain $(n-r)$ L.I solution we have to assign $n-r = 2$ variables as Arbitary constants (A.C)

i.e let $z = K_1$, $\omega = K_2$

from (ii.) $y - K_1 + \frac{4}{3}K_2 = 0$

$$y = K_1 - \frac{4}{3}K_2$$

sub $y, z, \omega$ in (i.)

$$x - 2\left(K_1 - \frac{4}{3}K_2\right) + K_1 - K_2 = 0$$

$$x = 2K_1 - \frac{8}{3}K_2 - K_1 + K_2$$

$$x = K_1 - \frac{5}{3}K_2$$

Now the solution of A are

$$X = \begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix} = \begin{bmatrix} K_1 - 5/3\,K_2 \\ K_1 - 4/3\,K_2 \\ K_1 \\ K_2 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} -5/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix}$$

The linearly independent solutions of A are

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -5/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix}$$

The given system of given solution is.

$$X = c_1 x_1 + c_2 x_2$$

where $c_1, c_2$ are arbitary constants.

---

2) Solve the completely the following equations.

$$4x + 2y + z + 3u = 0$$
$$6x + 3y + 4z + 7u = 0$$
$$2x + y + u = 0$$

given system can be written in matrix form

$AX = 0 \longrightarrow (I)$ where $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ ; $X = \begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix}$

Reduce "A" into echlon form :-

consider $A \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

$$R_1 \longrightarrow \frac{R_1}{4}$$

$$A \approx \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 - 6R_1 \\ R_3 \longrightarrow R_3 - 2R_1 \end{matrix}$$

$$A \approx \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{5/2} \; ; \; R_3 \to \frac{R_3}{-1/2}$$

$$A \approx \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A \approx \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echlon form $\rho(A) = 2 = r$

$n =$ no. of unknown $= 4$

$m =$ no. of equations $= 3$

Here $r < m < n$

The given system has $\infty$ solution of those $(n-r) = 4-2 = 2$ are linear independent solution.

The given system can be written as

$$\begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + \frac{1}{2}y + \frac{1}{4}z + \frac{3}{4}u = 0 \longrightarrow \text{(i.)}$$

$$0 \cdot x + 0 \cdot y + 1 \cdot z + 1 \cdot u = 0 \longrightarrow \text{(ii.)}$$

To obtain L.I solution we take $x = k_1$, $z = k_2$

from (ii.) we have $u = -k_2$

(i.) $\frac{1}{2}y = -x - \frac{1}{4}z - \frac{3}{4}u$

$$= -k_1 - \frac{1}{4}k_2 + \frac{3}{4}k_2$$

$y = -2k_1 + k_2$

the solution of ① is $x = k_1$

$y = -2k_1 + k_2$

$z = k_2$

$u = -k_2$

let $x \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} k_1 + 0 \cdot k_2 \\ -2k_1 + k_2 \\ 0k_1 + k_2 \\ 0k_1 - k_2 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

The L.I solutions of A are

$$x_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

1) show that equations $x+y+z=-3$, $3x+y-2z=-2$, $2x+4y+7z=7$ are inconsistant.

The matrix eqⁿ of the given system is $AX=B$

where $A=\begin{bmatrix}1 & 1 & 1\\ 3 & 1 & -2\\ 2 & 4 & 7\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $B=\begin{bmatrix}-3\\ -2\\ 7\end{bmatrix}$

Reducing $[A \mid B]$ to Echlon form:-

consider $[A \mid B]=\left[\begin{array}{ccc|c}1 & 1 & 1 & -3\\ 3 & 1 & -2 & -2\\ 2 & 4 & 7 & 7\end{array}\right]$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$[A \mid B] \approx \left[\begin{array}{ccc|c}1 & 1 & 1 & -3\\ 0 & -2 & -5 & 7\\ 0 & 2 & 5 & 13\end{array}\right]$

$R_3 \to R_3 + R_2$

$[A \mid B] = \left[\begin{array}{ccc|c}1 & 1 & 1 & -3\\ 0 & -2 & -5 & 7\\ 0 & 0 & 0 & 20\end{array}\right]$

which is an echlon form $\rho(A)=2$ = no. of non zero's rows in A

$\rho([A \mid B]) = 3$ = no. of non-zero rows in $\approx [A \mid B]$

$\rho(A) \neq \rho[A \mid B]$

$\therefore$ The given system is inconsistant.

$\therefore$ It has no solution.

2) Show that the equations $3x+y+z=8$, $-x+y-2z=-5$, $2x+2y+2z=12$, $-2x+2y-3y=-7$ are consistant and solve them.

The matrix equation of given system is $AX=B$

where $A=\begin{bmatrix}3 & 1 & 1\\ -1 & 1 & -2\\ 2 & 2 & 2\\ -2 & 2 & -3\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $B=\begin{bmatrix}8\\ -5\\ 12\\ -7\end{bmatrix}$

Reduce $[A \mid B]$ to echlon form:-

consider $[A \mid B]=\begin{bmatrix}3 & 1 & 1 & 8\\ -1 & 1 & -2 & -5\\ 2 & 2 & 2 & 12\\ -2 & 2 & -3 & -7\end{bmatrix}$

$R_3 \rightarrow R_1$

$$\begin{bmatrix}1 & 1 & 1 & 6\\ -1 & 1 & -2 & -5\\ 3 & 1 & 1 & 8\\ -2 & 2 & -3 & -7\end{bmatrix}$$

$R_2 \rightarrow R_2+R_1$ ; $R_3 \rightarrow R_3-3R_1$ ; $R_4 \Rightarrow R_4+2R_1$

$$\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 2 & -1 & 1\\ 0 & -2 & -2 & -10\\ 0 & 4 & -1 & 5\end{bmatrix}$$

$R_3 \rightarrow R_3+R_2$ ; $R_4 \rightarrow R_4-2R_2$

$$\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 2 & -1 & 1\\ 0 & 0 & -3 & -9\\ 0 & 0 & 1 & 3\end{bmatrix}$$

$R_3 \rightarrow R_3/-3$

$$\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 2 & -1 & 1\\ 0 & 0 & -3 & -9\\ 0 & 0 & 1 & 3\end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad R_4 \longrightarrow R_4 - R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = \rho(A \mid B) = 3$$

$\therefore$ The given system of equation is consistant It has unique solution.

$\therefore$ The given system can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

$$x + y + z = 6 \longrightarrow ①$$

$$2y - z = 1 \Rightarrow 2y - 3 = 1 \Rightarrow \boxed{y = 2}$$

$$\boxed{z = 3}$$

Sub $y, z$ in ①

$$x + 2 + 3 = 6$$

$$\boxed{x = 1}$$

$\therefore$ The solution of ① is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\therefore$ The given system equations are consistant.

Guass - Elimination method (or) Partial pivoting method:-

1) Solve the following system $2x+y+z=10$

$3x+2y+3z=18$

$x+4y+9z=16$

The above system can be represented in matrix form $AX=B$

where $A=\begin{bmatrix}2&1&1\\3&2&3\\1&4&9\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}10\\18\\16\end{bmatrix}$

Consider the augmented matrix,

$$[A \mid B]=\left[\begin{array}{ccc|c}2&1&1&10\\3&2&3&18\\1&4&9&16\end{array}\right]$$

$R_2 \to 2R_2-3R_1$

$R_3 \to 2R_3-R_1$

$$[A \mid B]=\left[\begin{array}{ccc|c}2&1&1&10\\0&1&3&6\\0&7&17&22\end{array}\right]$$

$R_3 \to R_3-7R_2$

$$[A \mid B]=\left[\begin{array}{ccc|c}2&1&1&10\\0&1&3&6\\0&0&-4&-20\end{array}\right]$$ by Forward elimination.

The equivalent matrix equation is

$$\begin{bmatrix}2&1&1\\0&1&3\\0&0&-4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}10\\6\\-20\end{bmatrix}$$

$2x+y+z=10 \longrightarrow$ (i.)

$y+3z=6 \longrightarrow$ (ii.)

$-4z=-20 \longrightarrow$ (iii.)

from (iii.) $z = \frac{-20}{-4} = 5$

$$\boxed{z=5}$$

from (ii.) $y = 6-3(5) = -9 \Rightarrow \boxed{y=-9}$

from (i.) $x = \frac{10+9-5}{2} = 7 \Rightarrow \boxed{x=7}$

By backward substitution $x=7,\ y=-9,\ z=5.$

---

2) Solve the following system of equation by using the guass elimination $x_1 + 2x_2 + 3x_3 = 16$

$3x_1 + 5x_2 + 8x_3 = 43$

$4x_1 + 9x_2 + 10x_3 = 57$

The above system can be represent in matrix

$AX = B$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \\ 4 & 9 & 10 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 16 \\ 43 \\ 57 \end{bmatrix}$

consider the augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 16 \\ 3 & 5 & 8 & 43 \\ 4 & 9 & 10 & 57 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 16 \\ 0 & -1 & -1 & -5 \\ 0 & 1 & -2 & -7 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 16 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & -3 & -12 \end{array}\right]$$

The equivalent matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ -5 \\ -12 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 16 \longrightarrow \text{(i.)}$$

$$0 \cdot x_1 - 1 \cdot x_2 - 1 \cdot x_3 = -5 \longrightarrow \text{(ii.)}$$

$$0 \cdot x_1 + 0 \cdot x_2 - 3x_3 = -12 \longrightarrow \text{(iii.)}$$

from (iii.) $\boxed{x_3 = 4}$

From (ii.) $-x_2 - x_3 = -5$

$$-x_2 = -5 + 4$$

$$\cancel{-}x_2 = \cancel{-}1$$

$$\boxed{x_2 = 1}$$

from (i.) $x_1 + 2(1) + 3(4) = 16$

$$x_1 + 2 + 12 = 16$$

$$x_1 = 16 - 14$$

$$\boxed{x_1 = 2}$$

By backward substitution we get,

$$x_1 = 2 \quad , \quad x_1 = 1 \quad , \quad x_3 = 4$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

Factorization Method (or) LU decomposition method:-

1) Solve the system of equations $2x+3y+z=9$
$x+2y+3z=6$
$3x+y+2z=8$ by the factorization method.

The given matrix equation is $AX=B \longrightarrow ①$

where $A=\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B=\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

let $LU=A \longrightarrow ②$

① becomes $LUX=B \longrightarrow ③$

let $UX=Y \longrightarrow ④$ where $Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

③ becomes $LY=B \longrightarrow ⑤$

where $L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

from ② $LU=A$

i.e $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}=\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}=\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

comparing the above matrices we get

$\boxed{u_{11}=2}$ , $\boxed{u_{12}=3}$ , $\boxed{u_{13}=1}$

$l_{21}u_{11} = 1 \Rightarrow l_{21} = 1/u_{11} \Rightarrow \boxed{l_{21} = \frac{1}{2}}$

$l_{31}u_{11} = 3 \Rightarrow l_{31} = 3/u_{11} \Rightarrow \boxed{l_{31} = 3/2}$

$l_{21}u_{12} + u_{22} = 2 \Rightarrow \frac{1}{2}\times 3 + u_{22} = 2 \Rightarrow \boxed{u_{22} = \frac{1}{2}}$

$l_{21}u_{13} + u_{23} = 3 \Rightarrow \frac{1}{2}\times 1 + u_{23} = 3 \Rightarrow \boxed{u_{23} = \frac{5}{2}}$

$l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow \frac{3}{2}\times 3 + l_{32}\times\frac{1}{2} = 1$

$$\frac{l_{32}}{2} = 1 - \frac{9}{2}$$

$$\frac{l_{32}}{2} = \frac{-7}{2}$$

$$\boxed{l_{32} = -7}$$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$

$\frac{3}{2}\times 1 + (-7)\times\frac{5}{2} + u_{33} = 2 \Rightarrow \boxed{u_{33} = 18}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$

from (5) $LY = B$ i.e $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

$y_1 = 9 \rightarrow$ (i.)

$\frac{1}{2}y_1 + y_2 = 6 \rightarrow$ (ii.)

$\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \rightarrow$ (iii.)

from (i.) $\boxed{y_1 = 9}$

from (ii.) $\frac{9}{2} + y_2 = 6 \Rightarrow \boxed{y_2 = \frac{3}{2}}$

from (iii.) $\frac{3}{2}\times 9 - 7\times\frac{3}{2} + y_3 = 8 \Rightarrow \boxed{y_3 = 5}$

By forward substitution $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$

from ④ $UX = Y$

i.e $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$

$$2x + 3y + z = 9 \longrightarrow \text{(iv.)}$$

$$\tfrac{1}{2}y + \frac{5}{2}z = 3/2 \longrightarrow \text{(v.)}$$

$$18z = 5 \longrightarrow \text{(vi.)}$$

from eq (vi.) $\boxed{z = \frac{5}{18}}$

from (v.) $\frac{1}{2} \times y + \frac{5}{2} \times \frac{5}{18} = \frac{3}{2} \Rightarrow \boxed{y = \frac{29}{18}}$

from (iv.) $2 \times x + 3 \times \frac{29}{18} + \frac{5}{18} = 9 \Rightarrow \boxed{x = \frac{35}{18}}$

∴ By backward substitution, we get

$x = \frac{35}{18}$, $y = \frac{29}{18}$, $z = \frac{5}{18}$ is solution of equations

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35/18 \\ 29/18 \\ 5/18 \end{bmatrix}$$

LU decomposition by using Guass Elimination:-

1) Solve the system $2x+3y+z=9$

$$x+2y+3z=6$$

$$3x+y+2z=8$$

The matrix equation is $AX=B \longrightarrow (1)$

$$A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\ B=\begin{bmatrix}9\\ 6\\ 8\end{bmatrix}$$

let $LU=A \longrightarrow (2)$

from (1) $\Rightarrow LUX=B \longrightarrow (3)$

let $UX=Y \longrightarrow (4) \implies Y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

from (3) $\Rightarrow LY=B \longrightarrow (5)$

consider $A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$

$$R_2 \longrightarrow R_2-\tfrac{1}{2}R_1$$

$$R_3 \longrightarrow R_3-3/2\,R_1$$

$$A \approx \begin{bmatrix}2 & 3 & 1\\ 0 & 1/2 & 5/2\\ 0 & -7/2 & 1/2\end{bmatrix}$$

$$R_3 \longrightarrow R_3+7R_2$$

$$A \approx \begin{bmatrix}2 & 3 & 1\\ 0 & 1/2 & 5/2\\ 0 & 0 & 1/8\end{bmatrix} = U \text{ say}$$

and $L=\begin{bmatrix}1 & 0 & 0\\ 1/2 & 1 & 0\\ 3/2 & -7 & 1\end{bmatrix}$

from (5) $LY=B$

i.e $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

$y_1 = 9 \longrightarrow$ (i.)

$\frac{1}{2} y_1 + y_2 = 6 \longrightarrow$ (ii.)

$\frac{3}{2} y_1 - 7y_2 + y_3 = 8 \longrightarrow$ (iii.)

from (i.) $\boxed{y_1 = 9}$

from (ii.) $y_2 = 6 - \frac{1}{2} y_1$

$y_2 = 6 - \frac{1}{2} \times 9$

$y_2 = \frac{6-9}{2}$

$\boxed{y_2 = 3/2}$

from (iii.) $\frac{3}{2} y_1 - 7y_2 + y_3 = 8$

$\frac{3}{2} \times 9 - 7 \times \frac{3}{2} + y_3 = 8$

$\boxed{y_3 = 5}$

By forward substitution we get $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$

from (4) $UX = Y$

i.e $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$

$2x + 3y + z = 9 \longrightarrow$ (iv.)

$\frac{1}{2} y + \frac{5}{2} z = 3/2 \longrightarrow$ (v.)

$18z = 5 \longrightarrow$ (vi.)

from (vi.) $\boxed{z = \frac{5}{18}}$

from (v.) $\frac{1}{2} \times y + \frac{5}{2} \times \frac{5}{18} = \frac{3}{2} \Longrightarrow \boxed{y = \frac{29}{18}}$

from (iv.) $2 \times x + 3 \times \frac{29}{18} + \frac{5}{18} = 9 \Longrightarrow \boxed{x = \frac{35}{18}}$

By backward substitution, we get

$x = \frac{35}{18}$, $y = \frac{29}{18}$, $z = \frac{5}{18}$ is the solution of equations.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35/18 \\ 29/18 \\ 5/18 \end{bmatrix}$$

## Tridiagonal System:-

1) Solve the system of equations $2x - y = 0$

$$-x + 2y - z = 0$$
$$-y + 2z - u = 0$$
$$-z + 2u = 1$$

The given matrix is written as $AX = B \rightarrow (1)$

where $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

The given system is a tridiagonal system

let $LU = A \longrightarrow (2)$

$(1) \Longrightarrow LUX = B \longrightarrow (3)$

let $UX = Y \longrightarrow (4)$ , $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

$(3) \Longrightarrow LY = B \longrightarrow (5)$

where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$

from (2) $LU = A$

i.e $\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$$\begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & u_{23} & 0 \\ 0 & l_{32}u_{22} & l_{32}u_{23} + u_{33} & u_{34} \\ 0 & 0 & l_{43}u_{33} & l_{43}u_{34} + u_{44} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

comparing above matrices we get,

$\boxed{u_{11} = 2}$ , $\boxed{u_{12} = -1}$ , $\boxed{l_{21} = -1/2}$ , $\boxed{u_{22} = 3/2}$

$\boxed{u_{23} = -1}$ , $\boxed{l_{32} = -2/3}$ , $\boxed{u_{33} = 4/3}$ , $\boxed{u_{34} = -1}$

$\boxed{l_{43} = \frac{-3}{4}}$ $\boxed{u_{44} = 5/4}$

where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$

from ⑤ $\Rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{y_1 = 0} \longrightarrow$ (i.)

$-\frac{1}{2} y_1 + y_2 = 0 \longrightarrow$ (ii.)

$-\frac{2}{3} y_2 + y_3 = 0 \longrightarrow$ (iii.)

from (ii.) $-\frac{1}{2}(0) + y_2 = 0$

$\boxed{y_2 = 0}$

from (iii.) $-\frac{2}{3} y_2 + y_3 = 0$

$-\frac{2}{3}(0) + y_3 = 0$

$\boxed{y_3 = 0}$

from (iv.) $-\frac{3}{4} y_3 + y_4 = 1$

$-\frac{3}{4}(0) + y_4 = 1$

$\boxed{y_4 = 1}$

from ⑤ $UX = Y$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$2x - y = 0 \longrightarrow (v.)$$

$$\frac{3}{2}y - z = 0 \longrightarrow (vi.)$$

$$\frac{4}{3}z - u = 0 \longrightarrow (vii.)$$

$$\frac{5}{4}u = 1 \longrightarrow (viii.)$$

$$u = \frac{4}{5}$$

$$+\frac{4}{3}z - \frac{4}{5} = 0$$

$$\frac{4}{3}z = \frac{4}{5}$$

$$z = \frac{3}{5}$$

$$\frac{3}{2}y - \frac{3}{5} = 0$$

$$\frac{3}{2}y = 3/5$$

$$y = \frac{2}{5}$$

$$2x - y = 0$$

$$2x - \frac{2}{5} = 0$$

$$x = \frac{\cancel{2}}{5} \times \frac{1}{\cancel{2}}$$

$$x = \frac{1}{5}$$

$$X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 3/5 \\ 4/5 \end{bmatrix}$$

# 2. EIGEN VALUES AND EIGEN VECTORS.

Defination :-

Let $A = [a_{ij}]$ be an $n \times n$ matrix. A non-Zero vector $X$ is said to be character vector of $A$ if there exists a scalar $\lambda$ such that $AX = \lambda X$

If $AX = \lambda X$, $(X \neq 0)$ we say that $X$ is the Eigen vector or characteristic vector of $A$ corresponding to the Eigen value or characteristic value $\lambda$ of $A$.

TO FIND THE EIGEN VECTORS OF A MATRIX A :-

Let $A = [a_{ij}]$ be a $n \times n$ matrix. Let $X$ be an integer which is Eigen vector of $A$ corresponding to the Eigen value $\lambda$.

Then by defination $AX = \lambda X$.

i.e $AX = \lambda I X$.

$\Rightarrow AX - \lambda I X = 0 \Rightarrow (A - \lambda I) X = 0$

Note that this is a homogenous system of $n$ Equation in $n$ unknowns.

→ This will have a non-Zero solution $X$, if & only if $|A - \lambda I| = 0$.

$(A-\lambda I)$ is called characteristic matrix of A. Also $|A-\lambda I|$ is a polynomial in $\lambda$ of degree n & is called the characteristic polynomial of A. Also $|A-\lambda I|=0$ is called the characteristic equation of A. This will be a polynomial equation in $\lambda$ of degree n.

Solving this equation, we get the roots $\lambda=\lambda_1, \lambda_2 \dots \lambda_n$ of the characteristic equation. These are the characteristic roots or Eigen values of the matrix.

Corresponding to each one of these n Eigen values, we can find the characteristic vector x. Consider the homogenous system.

$$(A-\lambda_i I)\, x_i = 0 \quad \text{for } i=1,2 \dots n.$$

The non-zero solution $x_i$ of this system is the Eigen vector of A corresponding to the Eigen value $\lambda_i$

1) Find the Eigen values & the corresponding Eigen vectors of $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Its characteristic matrix $= A-\lambda I = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$.

Characteristic equation of A is

$$|A-\lambda I| = 0$$

i.e $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ — ①.

i.e $(5-\lambda)(2-\lambda) - 4 = 0$.

on simplification, we get $\lambda^2 - 7\lambda + 6 = 0$.

which gives $(\lambda-1)(\lambda-6) = 0$ — ②

Hence $\lambda = 1, 6$ are the roots of the Equation (These are Values of A)

Consider the system $\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — ③.

To get the Eigen Vector x Corresponding to each Eigen Value $\lambda$ we have to solve the above system.

Eigen Vector Corresponding to $\lambda = 1$

Put $\lambda = 1$ in the system ③ we get $\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The system of Equation is $4x_1 + 4x_2 = 0$

$x_1 + x_2 = 0$.

This implies that $x_1 = -x_2$. Taking $x_1 = \alpha$ we get & $x_2 = -\alpha$.

$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $\alpha \neq 0$ is a scalar.

Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is Eigen vector of A Corresponding to the Eigen Value $\lambda = 1$.

Eigen Vector Corresponding to the Eigen Value $\lambda = 6$.

Put $\lambda = 6$ in ③ we get $\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\therefore -x_1 + 4x_2 0$ and $x_1 - 4x_2 = 0$.

This implies that $x_1 = 4x_2$. Taking $x_2 = \alpha$, we get $x_1 = 4\alpha$.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Hence $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is Eigen vector of A corresponding to the Eigen Value $\lambda = 6$.

Find the characteristic roots of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and the corresponding Eigen Vectors?

The characteristic Equation of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2\left[-2(3-\lambda)+2\right] + 2\left[2 - 2(3-\lambda)\right] = 0$.

$\Rightarrow (6-\lambda)\left[9 + \lambda^2 - 6\lambda - 1\right] + 2\left[-6 + 2\lambda + 2\right] + 2\left[2 - 6 + 2\lambda\right] = 0$.

$\Rightarrow (6-\lambda)\left[\lambda^2 - 6\lambda + 8\right] + 2\left[2\lambda - 4\right] + 2\left[2\lambda - 4\right] = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda = 2, 2, 8$.

The Eigen Values of A are $2, 2, 8$.

The Eigen vector of A corresponding to $\lambda = 2$.

$(A-\lambda I)X = 0 \Rightarrow (A-2I)X = 0$.

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Applying $R_2 \to 2R_2 + R_1$ ; $R_3 \to 2R_3 - R_1$ we get.

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0 \Rightarrow 2x_1 - x_2 + x_3 = 0$.

Let $x_2 = k_1$, $x_3 = k_2$ then

$2x_1 - k_1 + k_2 = 0 \Rightarrow 2x_1 = k_1 - k_2 \Rightarrow x_1 = \frac{k_1}{2} - \frac{k_2}{2}$.

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_1}{2} - \frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to $\lambda = 2$ where $k_1$ and $k_2$ are arbitary constants.

The Eigen vector of A corresponding to $\lambda = 8$.

$$(A-8I)X = 0 \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Applying $R_2 \to R_2 - R_1$ ; $R_3 \to R_3 + R_1 \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Applying $R_3 \to R_3 - R_2$.

$$\begin{bmatrix} -2 & -2 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow \quad -2x_1 - 2x_2 + 2x_3 = 0 \Rightarrow x_1 + x_2 - x_3 = 0.$

and $\quad -3x_2 - 3x_3 = 0 \Rightarrow x_2 + x_3 = 0.$

Put $x_3 = k$ then $x_2 + k = 0 \Rightarrow x_2 = -k$.

and $\quad x_1 - k - k = 0 \Rightarrow x_1 - 2k = 0 \Rightarrow x_1 = 2k$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} k$ is the Eigen of vector of A corresponding to $\lambda = 8$ where $k$ is any non-zero arbitary constant.

## PROPERTIES OF EIGEN VALUES :-

### THEOREM 1 :-

The sum of the Eigen Values of a square matrix is Equal to its trace and Product of the Eigen Values is Equal to its determinant.

i.e if $A$ is an $n \times n$ matrix & $\lambda_1, \lambda_2 \cdots \lambda_n$ are its $n$ Eigen values, then $\lambda_1 + \lambda_2 + \cdots + \lambda_n = Tr(A)$ and $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = \det(A)$.

Proof :- characteristic Equation of A is $|A-\lambda I| = 0$.

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$.

Expanding this, we get.

$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) - a_{12}$ (a Polynomial of degree $n-2$) $+ a_{13}$ (a Polynomial of degree $n-2) + \cdots + = 0$.

i.e. $(-1)^n(\lambda-a_{11})(\lambda-a_{22})\cdots(\lambda-a_{nn})$ + a Polynomial of degree $(n-2) = 0$

i.e $(-1)^n\left[\lambda^n - (a_{11}+a_{22}\cdots+a_{nn})\right]\lambda^{n-1}$ + a Polynomial of degree $(n-2)]$ + a Polynomial of degree $(n-2)$ in $\lambda = 0$.

$\therefore (-1)^n\lambda^n + (-1)^{n+1}(\text{Trace } A)\lambda^{n-1}$ + a Polynomial of degree $(n-2)$ in $\lambda = 0$.

If $\lambda_1, \lambda_2 \cdots \lambda_n$ are the roots of this Equation

Sum of the roots $= -\dfrac{(-1)^{n+1}\,Tr(A)}{(-1)^n} = Tr(A)$.

Further $|A-\lambda I| = (-1)^n\lambda^n + \cdots + a_0$.

Put $\lambda = 0$. Then $|A| = a_0$.

$$(-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} - \dots + a_0 = 0 .$$

$\Rightarrow$ Product of the roots $= \dfrac{(-1)^n a_0}{(-1)^n} = a_0 = |A| = \det A$.

Hence the result.

THEOREM 2 :-

If $\lambda$ is an Eigen Value of $A$ Corresponding to the Eigen Vector $x$, then $\lambda^n$ is Eigen value of $A^n$ corresponding to the Eigen vector $x$.

PROOF :- Since $\lambda$ is an Eigen Value of $A$ Corresponding to the Eigen Vector $x$, we have $Ax = \lambda x$ —— (1).

Premultiply (1) by $A$, $A(Ax) = A(\lambda x)$.

(i.e) $(AA)x = \lambda(Ax)$ (i.e) $A^2 x = \lambda \cdot \lambda x = \lambda^2 x$ [using (1)]

Hence $\lambda^2$ is Eigen Value of $A^2$ with $x$ itself as the Corresponding Eigen Vector. Thus the theorem is true to $n=2$. Let the result be true for $n=k$.

Then $A^k x = \lambda^k x$.

Premultiplying this by $A$ and using $Ax = \lambda x$, we get

$$A^{k+1} x = \lambda^{k+1} x .$$

which implies that $\lambda^{k+1}$ is Eigen Value of $A^{k+1}$ with $x$ itself as the Corresponding Eigen vector.

THEOREM 3 :-

Show that if $\lambda_1, \lambda_2 \cdots \lambda_n$ are the latent roots of A then $A^3$ has latent roots $\lambda_1^3, \lambda_2^3 \cdots \lambda_n^3$.

PROOF :- Substitute $n=3$ in Theorem (2).

THEOREM 4 :-

A square matrix A & its transpose $A^T$ have the same Eigen values.

PROOF : we have $(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$\therefore |(A-\lambda I)^T| = |A^T - \lambda I|$ or $|A-\lambda I| = |A^T - \lambda I|$ $(\because |A^T| = |A|)$

$\therefore |A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$.

i.e. $\lambda$ is an Eigen value of A if & only if $\lambda$ is an Eigen value of $A^T$.

Thus the Eigen values of A and $A^T$ are same.

THEOREM 6 :-

If $\lambda_1, \lambda_2 \cdots \lambda_n$ are the Eigen values of a matrix A, then $k\lambda_1, k\lambda_2, \cdots k\lambda_n$ are the Eigen values of the matrix $kA$, where k is non-zero scalar.

PROOF :- Let A be a square matrix of order n.

Then $|KA - \lambda KI| = |K(A - \lambda I)| = K^n |A - \lambda I|$. $[\because |KA| = K^n |A|]$

Since $K \neq 0$, therefore $|KA - \lambda KI| = 0$ if and only if $|A - \lambda I| = 0$

i.e., $K\lambda$ is an Eigen value of $KA$ if and only if $\lambda$ is an Eigen value of $A$.

Thus $K\lambda_1, K\lambda_2, \ldots K\lambda_n$ are the Eigen values of the matrix $KA$ if $\lambda_1, \lambda_2 \ldots \lambda_n$ are the Eigen values of the matrix $A$.

THEOREM :-

If $\lambda$ is an Eigen value of a non-singular matrix $A$ corresponding to the Eigen vector $X$, then $\lambda^{-1}$ is an Eigen value of $A^{-1}$ and corresponding Eigen vector $X$ itself.

(OR)

Prove that the Eigen values of $A^{-1}$ are the reciprocals of the Eigen values of $A$

PROOF :- Since $A$ is non-singular & Product of the Eigen values is Equal to $|A|$, it follows that none of the Eigen values of $A$ is $0$.

$\therefore$ If $\lambda$ is an Eigen value of the non-singular matrix $A$ and $X$ is the corresponding Eigen vector, $\lambda \neq 0$ &

$AX = \lambda X$. Premultiplying this with $A^{-1}$, we get

$$A^{-1}(AX) = A^{-1}(\lambda X) \Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X.$$

$$\therefore X = \lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X \quad (\because \lambda \neq 0).$$

Hence by defination it follows that $\lambda^{-1}$ is an Eigen Value of $A^{-1}$ & $X$ is the corresponding Eigen vector.

THEOREM :

If $\lambda$ is an Eigen Value of a non-singular matrix $A$, then $\frac{|A|}{\lambda}$ is an Eigen value of the matrix adj $A$.

PROOF :- Since $\lambda$ is an Eigen value of a non-singular matrix therefore $\lambda \neq 0$.

Also $\lambda$ is an Eigen value of $A$ implies that there exists a non-zero vector $X$ such that

$$AX = \lambda X \quad \text{—(1)}.$$

$\Rightarrow (\text{adj } A)AX = (\text{adj } A)(\lambda X) \Rightarrow [(\text{adj}A)A]X = \lambda(\text{adj}A)X.$

$\rightarrow \quad |A|\lambda X = \lambda(\text{adj}A)X \quad [\because (\text{adj}A)A = |A|I]$

$\Rightarrow \quad \frac{|A|}{\lambda}X = (\text{adj } A)X \quad (\text{or}) \quad (\text{adj } A)X = \frac{|A|}{\lambda}X.$

Since $X$ is a non-zero vector, therefore from the relation (1) it is clear that $\frac{|A|}{\lambda}$ is an Eigen value of the matrix adj $A$.

THEOREM :-

If $\lambda$ is an Eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an Eigen value.

PROOF :- we know that if $\lambda$ is an Eigen value of a matrix A, then $\frac{1}{\lambda}$ is an Eigen value of $A^{-1}$.
Since A is an orthogonal matrix, therefore

$$A^{-1} = A'$$

$\therefore \frac{1}{\lambda}$ is an Eigen value of $A'$.

But the matrices A & $A'$ have the same Eigen values, since the determinants $|A - \lambda I|$ & $|A' - \lambda I|$ are same

Hence $\frac{1}{\lambda}$ is also an Eigen value of A.

COROLLARY :-

If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and $BA^{-1}$ have the same Eigen values

PROOF :- In the Previous theorem take $BA^{-1}$ in the Place of A and A in the Place of P.

we deduce that $A^{-1}(BA^{-1})A$ and $(BA^{-1})$ have the same Eigen values.

(i.e) $(A^{-1}B)(A^{-1}A)$ and $BA^{-1}$ have the same Eigen Values

(i.e) $(A^{-1}B)I$ and $BA^{-1}$ have the same Eigen Values.

(i.e) $A^{-1}B$ and $BA^{-1}$ have the same Eigen Values.

Find the Eigen Vectors & Eigen Values of the matrix A & its inverse where $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

The characteristic Equation of 'A' is given by $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)\right] = 0 \Rightarrow \lambda = 1, 2, 3.$$

$\therefore$ characteristic roots are 1, 2, 3.

To find characteristic vector of '1'.

For $\lambda = 1$, the Eigen vector of A is given by

$$(A - I)X = 0 \Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow 3x_2 + 4x_3 = 0$, $x_2 + 5x_3 = 0$ and $2x_3 = 0$.

$x_2 = 0, x_3 = 0$.

$\therefore X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the Eigen Vector Corresponding to $\lambda = 1$

To find characteristic Vector of '2'.

For $\lambda = 2$, the Eigen Vector of A is given by

$$(A - 2I)X = 0 \Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 + 3x_2 + 4x_3 = 0 \;,\; 5x_2 = 0$, and $x_3 = 0$

we take $x_1 = k$.

$\therefore X = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\therefore$ The characteristic vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

To find characteristic Vector of '3'

$$(A - 3I)X = 0 \Rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \; -2x_1 + 3x_2 + 4x_3 = 0 \;;\; -x_2 + 5x_3 = 0$.

Let $x_3 = k$ then $x_2 = 5k$ & $-2x_1 + 15k + 4k = 0$.

$\Rightarrow 2x_1 = 19k \Rightarrow x_1 = \frac{19}{2}k$ $\quad \therefore X = \begin{bmatrix} \frac{19}{2}k \\ 5k \\ k \end{bmatrix} = \frac{k}{2} \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$.

Eigen values of $A^{-1}$ are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ i.e $1, \frac{1}{2}, \frac{1}{3}$

& Eigen vector of $A^{-1}$ are same as Eigen vectors of the matrix A.

For the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the Eigen Values of $3A^3+5A^2-6A+2I$.

The characteristic Equation of A is $|A-\lambda I| = 0$.

i.e $\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$ [Expand by $C_1$]

$\Rightarrow (1-\lambda)\left[(3-\lambda)(-2-\lambda)-0\right] = 0$.

i.e., $(1-\lambda)\left[(3-\lambda)(2+\lambda)\right] = 0$ (or) $\lambda = 1, 3, -2$.

$\therefore$ Eigen Values of A are $1, 3, -2$.

we know that if $\lambda$ is an Eigen value of A & $f(A)$ is a Polynomial in A. then the Eigen Values of $f(A)$ is $f(\lambda)$.

Let $f(A) = 3A^3+5A^2-6A+2I$

Then, Eigen Values of $f(A)$ are $f(1)$, $f(3)$ and $f(-2)$

$\therefore f(1) = 3(1)^3+5(1)^2-6(1)+2(1) = 3+5-6-2 = 4$.

[$\because$ Eigen Values of I are 1,1,1]

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1) = 81+45-18+2 = 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2(1) = -24+20+12+2 = 10$

Thus, Eigen values of $3A^3+5A^2-6A+2I$ are $4, 110, 10$

(1) find the eigen Values and the Corresponding eigen Vector of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

The characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right] - 0\left[1(3-\lambda)-2\right] - 1\left[2-2(2-\lambda)\right] = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+4\right] + 2-2\lambda = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+4\right] + 2(1-\lambda) = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+6\right] = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2-2\lambda-3\lambda+6\right] = 0$

$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3) = 0$

$\therefore \lambda = 1, 2, 3.$

The Eigen Values are $\lambda = 1, 2, 3.$

The corresponding eigen vectors for this eigen Values are

The Eigen Vector for $\lambda = 1$:-

$$|A - \lambda I| X_1 = 0.$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

put $\lambda = 1$

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$-x_3 = 0 \longrightarrow (1)$$

$$x_1 + x_2 + x_3 = 0 \longrightarrow (2)$$

$$2x_1 + 2x_2 + 2x_3 = 0 \longrightarrow (3).$$

from (3) eq$^n$ $x_1 + x_2 + x_3 = 0$

from (2) eq$^n$. $x_1 + x_2 + x_3 = 0$

where $x_3 = 0 \Rightarrow x_1 + x_2 = 0$

$x_1 = -x_2$

Let $x_1 = k$

$x_2 = -k \quad x_3 = 0.$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$ $\Rightarrow$ The corresponding Eigen vectors for $\lambda = 1$

The Eigen Vector for $\lambda_2 = 2$:-

$$[A - \lambda I]\, X_2 = 0.$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 - x_3 = 0 \longrightarrow (1)$

$\Rightarrow \quad x_1 + x_3 = 0 \longrightarrow (2)$

$\Rightarrow \quad 2x_1 + 2x_2 + x_3 = 0 \longrightarrow (3)$

from eq$^n$ (2) $\quad x_1 = -x_3$. $\quad$ Let $x_1 = k$

$x_3 = -k$

Sub these values in eq$^n$ (3)

$$2(k) + 2x_2 - k = 0$$

$$\Rightarrow 2x_2 + k = 0$$

$$\Rightarrow x_2 = -k/2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k/2 \\ -k \end{bmatrix} = 2k \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}.$$

The corresponding eigen vectors for $\lambda = 2$ is $\begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}$

The Eigen vector for $\lambda_3 = 3$ :-

$$[A - \lambda I] X_3 = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow \quad -2x_1 - x_3 = 0 \longrightarrow (1)$

$\Rightarrow \quad x_1 - x_2 + x_3 = 0 \longrightarrow (2)$

$\Rightarrow \quad 2x_1 + 2x_2 = 0 \longrightarrow (3)$

from eqn (3) $\quad x_1 = -x_2 \quad$ Let $x_1 = k$

$x_2 = -k$

Sub $x_1$ value in eqn (1).

$\Rightarrow \quad -2(k) - x_3 = 0$

$\Rightarrow \quad -2k = x_3.$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k \\ -2k \end{bmatrix} = -k \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

$\therefore$ These are the corresponding Eigen vectors for $\lambda = 3$.

(2) find the eigen Values and the Corresponding eigen Vectors of $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$.

Given $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$.

The characteristic eqn $|A - \lambda I| = 0$

$$\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{bmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2 - 2\lambda + 2\right] + 2\lambda + 4 + 2(-\lambda + 1) = 0$$

$$\Rightarrow 3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda + 2\lambda - 4 - 2\lambda + 2 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\begin{array}{c|cccc} \lambda=1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline \lambda=2 & 1 & -4 & 4 & 0 \\ & 0 & 2 & -4 & \\ \hline & 1 & -2 & 0 & \end{array}$$

$$(\lambda-1)(\lambda-2)(\lambda-2) = 0$$

$$\lambda = 1, 2, 2$$

These are the eigen values of a given matrix

The Eigen Vector for $\lambda_1 = 1$:-

$[A - \lambda I] X_1 = 0$

$$\begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3-1 & 2 & 2 \\ 1 & 2-1 & 2 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 2x_1 + 2x_2 + 2x_3 = 0 \longrightarrow (1)$

$\Rightarrow x_1 + x_2 + 2x_3 = 0 \longrightarrow (2)$

$\Rightarrow -x_1 - x_2 - x_3 = 0 \longrightarrow (3)$

from eq^n (3) $\cancel{x_1} + \cancel{x_2} + x_3 = 0$

eq^n (2) $\cancel{x_1} + \cancel{x_2} + 2x_3 = 0$

$\quad - \quad - \quad -$

$-x_3 = 0$

Sub $x_3$ in eq^n (3)

$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2$

let $x_1 = k \quad x_2 = -k$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The Eigen vector for $\lambda = 2$:-

$[A - \lambda I] X_2 = 0.$

$$\begin{bmatrix} 3-2 & 2 & 2 \\ 1 & 2-2 & 2 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \quad x_1 + 2x_2 + 2x_3 = 0 \longrightarrow (1)$

$\Rightarrow \quad x_1 + 2x_3 = 0 \longrightarrow (2)$

$\Rightarrow \quad -x_1 - x_2 - 2x_3 = 0 \Rightarrow x_1 + x_2 + 2x_3 = 0 \longrightarrow (3)$

from eq$^n$ (2) $\quad x_1 = -2x_3 \quad$ let $x_1 = k$

$x_3 = -k/2$

Sub $x_1$ & $x_3$ in eq$^n$ (3)

$$k + x_2 + \cancel{2}\left(\frac{-k}{\cancel{2}}\right) = 0$$

$$x_2 = 0$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ -k/2 \end{bmatrix} = \frac{-k}{2}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

The corresponding eigen vectors for eigen values 1, 2, 2

are $\quad X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}; \quad X_3 = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}.$

# PROPERTIES:-

(i) If $\lambda$ is an eigen Value of a non-Singular matrix $A$, then $\frac{|A|}{\lambda}$ is an eigen Value of the matrix adj $A$

(ii) $A$ and $A^T$ have same eigen Values.

(iii) If $\lambda$ is an eigen Value of $A$ then the eigen value of $B = a_0A^2 + a_1A + a_2I$ is $a_0\lambda + a_1\lambda + a_1\lambda$

(iv) If $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_k$ are the eigen values of $A$, Then $\lambda_1 - k, \lambda_2 - k, \lambda_3 - k \dots \lambda_n - k$ are the eigen values of the matrix $|A - kI|$ (where $k$ is a non-zero Scalar)

(v) The two eigen vectors corresponding to the two diff eigen values are linearly Independent

(vi) If $\lambda$ is a given value of a non-Singular matrix $A$ then S/T $\frac{|A|}{\lambda}$ is a given value of Adj $A$

(vii) The product of the eigen Values is equal to the determinant of the matrix

(viii) The Sum of eigen values is equal to its trace

(ix) Any one of the eigen Value of $A$ is zero then $A$ is Singular matrix

(x) The eigen values of triangular matrix are just the diagonal elements of the matrix

# DIAGONALIZATION OF A MATRIX :-

## THEOREM :

If a square matrix A of order n has n linearly independent Eigen vectors $(x_1, x_2 \ldots x_n)$ corresponding to the n Eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

## PROOF :-

Given that $x_1, x_2 \ldots x_n$ be Eigen Vectors of A corresponding to the Eigen Values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively & these Eigen Vectors are linearly independent.

Define $P = (x_1, x_2 \ldots x_n)$

Since the n columns of P are linearly independent, $|P| \neq 0$.

Hence $P^{-1}$ exists.

Consider $AP = A[x_1 \; x_2 \ldots x_n] = [Ax_1 \; Ax_2 \ldots Ax_n]$.

$$= [\lambda_1 x_1 \; \lambda_2 x_2 \ldots \lambda_n x_n].$$

$$= [x_1 \; x_2 \ldots x_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

$$= PD.$$

where $D = \text{diag}(\lambda_1, \lambda_2 \ldots \lambda_n)$. $\therefore AP = PD$.

$\Rightarrow P^{-1}(AP) = P^{-1}(PD)$.

$\Rightarrow P^{-1}AP = (P^{-1}P)D = D = \text{diag}(\lambda_1, \lambda_2 \ldots \lambda_n)$

## MODAL AND SPECTRAL MATRICES :-

Def : The Matrix P in the above result which diagnolise the square matrix A is called the modal matrix of A & the resulting diagonal matrix D is known as Spectral matrix.

NOTE : 1.

If $x_1, x_2 \dots x_n$ are not linearly independent this result is not true.

2) An interesting special case can be considered here. Suppose A is a real symmetric matrix with n Pair wise distinct Eigen values $\lambda_1, \lambda_2 \dots \lambda_n$. Then the Corresponding Eigen vectors $x_1, x_2 \dots x_n$ are Pairwise orthogonal.

Hence if $P = (e_1, e_2 \dots e_n)$,

where $e_1 = (x_1 / \| x_1 \|)$, $e_2 = (x_2 / \| x_2 \|) \dots e_n = (x_n / \| x_n \|)$

then P will be an orthogonal matrix.

i.e $P^T P = P P^T = I$

Hence $P^{-1} = P^T$

$P^{-1} A D = D$

$\Rightarrow P^T A P = D.$

## CALCULATION OF POWERS OF A MATRIX :

We can obtain the Powers of a matrix by using diagonalisation.

Let A be the square matrix. Then a non-singular matrix P can be found such that

$$D = P^{-1}AP.$$

$\therefore D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P \quad (\because PP^{-1} = I)$

Similarly $D^3 = P^{-1}A^3P$.

In general $D^n = P^{-1}A^nP$. — (1)

To obtain $A^n$, Pre-multiply (1) by P & Post-multiply by $P^{-1}$

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = (PP^{-1})A^n(PP^{-1})A^n$.

Hence $A^n = P\begin{bmatrix} \lambda_1^n & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^n & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^n & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n^n \end{bmatrix} P^{-1}$.

Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate $A^4$.

Characteristic Equation of A is given by $|A - \lambda I| = 0$.

i.e., $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$ [Expand to $R_1$].

$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right] - 0 - 1\left[2-2(2-\lambda)\right] = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ (or) $(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\therefore \lambda = 1, 2, 3$.

Thus the Eigen values of A are 1, 2 and 3

If $x_1, x_2, x_3$ be the components of an Eigen vector corresponding to the Eigen value $\lambda$, we have.

$$[A-\lambda I]X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

For $\lambda = 1$, Eigen vectors are given by $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e $\quad 0\cdot x_1 + 0\cdot x_2 - x_3 = 0$ and $x_1 + x_2 + x_3 = 0$.

$\Rightarrow x_3 = 0$ and $x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = 0,\ x_1 = -x_2$

$\therefore x_1 = 1,\ x_2 = -1,\ x_3 = 0$

Eigen vector is $[1, -1, 0]^T$. Also every non-zero multiple of this vector is an Eigen vector corresponding to $\lambda = 1$.

For $\lambda = 2$, Eigen vectors are given by

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(or)} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 0\cdot x_2 + x_3 = 0$ and $2x_1 + 2x_2 + 2x_3 = 0$.

Solving $\frac{x_1}{0-2} = \frac{-x_2}{1-2} = \frac{x_3}{2-0}$ (or) $\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$

(or) $x_1 = -2,\ x_2 = 1,\ x_3 = 2$.

Eigen vector is $[-2, 1, 2]^T$.

Also every non-zero multiple of this vector is an Eigen vector corresponding to $\lambda = 2$.

For $\lambda = 3$, Eigen vectors are given by

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\text{Putting } \lambda = 3 \text{ in } (1)]$$

i.e $-2x_1 + 0.x_2 - x_3 = 0$ and $x_1 - x_2 + x_3 = 0$

Solving $\frac{x_1}{0-1} = \frac{-x_2}{-2+1} = \frac{x_3}{2-0}$ (or) $\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$.

Eigen vector is $[-1, 1, 2]^T$.

Writing the three Eigen vectors of the matrix A as the three columns, the required transformation matrix is

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

The matrix P is called Modal matrix of A.

$$P^{-1} = -\frac{1}{2} \begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}.$$

Now $P^{-1}AP$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D \text{(say)}$$

Hence $A^4 = PD^4P^{-1}$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & 1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

## THE CAYLEY - HAMILTON THEOREM

THEOREM :

Every square matrix satisfies its own characteristic Equation.

PROOF : Let $A$ be n-rowned square matrix. Then $|A-\lambda I| = 0$ is the characteristic Equation of $A$

Let $|A-\lambda I| = (-1)^n \left[\lambda^n + a_1 \lambda^{n-1} + a_2 I^{n-1} + a_3 I^{n-2} + \cdots + a_n\right]$.

Since all the elements of $A-\lambda I$ are atmost of $I^{st}$ degree in $\lambda$, all the elements of adj $(A-\lambda I)$ are Polynomials in $\lambda$ of degree $(n-1)$ or less and hence adj $(A-\lambda$ Can be written as a matrix Polynomials in $\lambda$.

Let $adj \cdot (A-\lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-2}\lambda^1 + B_{n-1}$

where $B_0, B_1 \dots B_{n-1}$ are n-rowed matrices.

$\Rightarrow$ Now $(A-\lambda I)\ adj\ (A-\lambda I)$

$$= |A-\lambda I| I_n \ (A-\lambda I)\left(B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}\right)$$

$$= (-1)^n \left[\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n\right] I .$$

Comparing coefficients of like powers of $\lambda$, we obtain

$$-B_0 = (-1)^n I .$$

$$AB_0 - B_1 = (-1)^n a_1 I ,$$

$$AB_1 - B_2 = (-1)^n a_2 I .$$

---

$$AB_{n-1} = (-1)^n a_n I .$$

Premultiplying the above equations successively by $A^n, A^{n-1}$ .... $I$ and adding we obtain

$$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + \cdots + (-1)^n a_n I .$$

$$(-1)^n \left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I\right] = 0 .$$

which implies that A satisfies its characteristic equation.

Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by using Cayley - Hamilton theorem.

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

The characteristic Equation of A is $\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0.$

i.e $\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)\left[2-\lambda-2\lambda+\lambda^2-1\right] + 2(-1) = 0$

$\Rightarrow \quad (1-\lambda)\left[1-3\lambda+\lambda^2\right] - 2 = 0$

$\Rightarrow \quad \lambda^3 - 4\lambda^2 + 4\lambda + I = 0$

By Cayley - Hamilton theorem, A satisfies its characteristic Equation.

$\therefore \quad A^3 - 4A^2 + 4A + I = 0$

$\Rightarrow \quad A^{-1}(A^3 - 4A^2 + 4A + I) = 0 \quad [\because |A| = -1 \neq 0)$

$\Rightarrow \quad A^2 - 4A + 4I + A^{-1} = 0$

$A^{-1} = -A^2 + 4A - 4I$

$= \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ -2 & -2 & -3 \\ -6 & -1 & -5 \end{bmatrix}$

$= \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}.$

$AA^{-1} = I.$

Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley - Hamilton theorem.

we have $|A-\lambda| = \begin{vmatrix} 0-\lambda & c & -b \\ -c & 0-\lambda & a \\ b & -a & 0-\lambda \end{vmatrix}$

$$\Rightarrow \quad -\lambda(\lambda^2+a^2) - c(c\lambda - ab) - b(ac+b\lambda)$$

$$= \quad -\lambda^3 - \lambda(a^2+b^2+c^2)$$

The characteristic Equation of matrix A is given as

$$\lambda^3 + \lambda(a^2+b^2+c^2) = 0$$

To Verify Cayley - Hamilton theorem, we have to Prove that

$$A^3 + (a^2+b^2+c^2)A = 0$$

we have $A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$$= \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$A^3 = A^2 \cdot A = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & a & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -c^3-b^2c-a^2c & bc^2+b^3+a^2b \\ c^3+a^2c+b^2c & 0 & ab^2+ac^2+a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+c^3 & 0 \end{bmatrix}$$

$$= (a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)A.$$

$\therefore A^3 + (a^2+b^2+c^2)A = 0$

$\therefore A$ satisfies Cayley-Hamilton theorem.

Using Cayley-Hamilton theorem find $A^8$ if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Characteristic Eq$^n$ of $A$ is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5 = 0$

By Cayley-Hamilton theorem, $A$ satisfies its characteristic Eq$^n$ so we must have $A^2 = 5I$.

$$A^8 = 5A^6 = 5(A^2)(A^2)(A^2)$$

$$= 5(5I)(5I)(5I)$$

$$= 625I.$$

# DIAGONALIZATION OF A MATRIX:-

(1) Diagonalize the matrix $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$.

Given $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$.

The characteristic eqn of $A$ is $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (8-\lambda)\left[(-3-\lambda)(1-\lambda) - 8\right] + 8\left[4(1-\lambda)+6\right] - 2\left[-16-3(-3-\lambda)\right] = 0$$

$$\Rightarrow (8-\lambda)\left[\lambda^2+2\lambda-11\right] + 8\left[10-4\lambda\right] - 2\left[\lambda-7\right] = 0$$

$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$$

$$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$$

$$\begin{array}{c|cccc}
\lambda=1 & 1 & -6 & 11 & -6 \\
 & 0 & 1 & -5 & 6 \\
\hline
\lambda=2 & 1 & -5 & 6 & 0 \\
 & 0 & 2 & -6 & \\
\hline
 & 1 & -3 & 0 &
\end{array}$$

$$(\lambda-1)(\lambda-2)(\lambda+(3)) = 0$$

$\lambda = 1, 2, 3.$ are the eigen values.

To find the eigen vectors to corresponding eigen value 's'

Eigen Vector for $\lambda = 1$

$$(A - \lambda I)\, X_1 = 0$$

$$\begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -3-1 & -2 \\ 3 & -4 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$7x_1 - 8x_2 - 2x_3 = 0 \longrightarrow (1)$$

$$4x_1 - 4x_2 - 2x_3 = 0 \longrightarrow (2)$$

$$3x_1 - 4x_2 = 0 \longrightarrow (3)$$

$$3x_1 = 4x_2 \Rightarrow x_1 = \frac{4x_2}{3} \quad \text{Let } x_1 = k$$

$$\Rightarrow x_2 = \frac{3k}{4}$$

from eqn (2)

$$2x_1 - 2x_2 - x_3 = 0$$

$$2k - \cancel{2}\left(\frac{3k}{\cancel{4}_{2}}\right) - x_3 = 0$$

$$\Rightarrow 2k - \frac{3k}{2} = x_3$$

$$\Rightarrow \frac{k}{2} = x_3.$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 3k/4 \\ k/2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 3/4 \\ 1/2 \end{bmatrix}.$$

$$X_1 = 4k \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

Eigen Vector for $\lambda = 2$ :-

$$[A - \lambda I]x_2 = 0$$

$$\begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 8-2 & -8 & -2 \\ 4 & -3-2 & -2 \\ 3 & -4 & 1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \quad 6x_1 - 8x_2 - 2x_3 = 0 \longrightarrow (1)$

$\Rightarrow \quad 4x_1 - 5x_2 - 2x_3 = 0 \longrightarrow (2)$

$\Rightarrow \quad 3x_1 - 4x_2 - x_3 = 0 \longrightarrow (3)$.

from eqn (2) & (3)

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -5 & & -2 & & 4 & & -5 \\ & \times & & \times & & \times & \\ -4 & & -1 & & 3 & & -4 \end{array}$$

$$\frac{x_1}{5-8} = \frac{x_2}{-6+4} = \frac{x_3}{-16+15}$$

$$\frac{x_1}{-3} = \frac{x_2}{-2} = \frac{x_3}{-1}$$

$$x_2 = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

$\therefore$ The corresponding eigen vectors for $\lambda = 2$ is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Eigen Vector for $\lambda = 3$:-

$$[A - \lambda I]\, X_3 = 0.$$

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$5x_1 - 8x_2 - 2x_3 = 0. \longrightarrow (1)$$

$$4x_1 - 6x_2 - 2x_3 = 0 \longrightarrow (2)$$

$$3x_1 - 4x_2 - 2x_3 = 0 \longrightarrow (3)$$

from eqn (2) & (3)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ -6 & \times & -2 & \times & 4 & \times & -6 \\ -4 & & -2 & & 3 & & -4 \end{array}$$

$$\frac{x_1}{12-8} = \frac{x_2}{-6+8} = \frac{x_3}{-16+18}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

$$|P| = -1 \neq 0.$$

$$P^{-1} = \frac{\operatorname{adj} P}{|P|} = \frac{1}{-1} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

$$P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & -2 \\ -1 & 2 & -1 \end{bmatrix}.$$

Now

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & -2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

After solving this we get

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$P^{-1}AP = D.$

(2) Diagonalize the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic eqn is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3 \end{vmatrix} = 0.$$

$\Rightarrow (8-\lambda)\left[(7-\lambda)(3) - 16\right] + 6\left[-6(3-\lambda) + 8\right] + 2\left[24 - 2(7-\lambda)\right] = 0$

$\Rightarrow 8\lambda^2 - 80\lambda + 40 - \lambda^2 + 10\lambda^2 - 5\lambda + 36\lambda - 60 + 4\lambda + 20 = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$

$$\lambda(\lambda-3)(\lambda-15)=0$$

$$\therefore \lambda = 0, 3, 15$$

Eigen vector for $\lambda = 0$ :- $(A - \lambda I)X_1 = 0$

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$8x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \longrightarrow (2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \longrightarrow (3)$$

from eq^n (1) & (3)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ -6 & & 2 & & 8 & & -6 \\ & \times & & \times & & \times & & \\ -4 & & 3 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{-18+8} = \frac{x_2}{4-24} = \frac{x_3}{-32+12}$$

$$\Rightarrow \frac{x_1}{-10} = \frac{x_2}{-20} = \frac{x_3}{-20}$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Eigen vector for $\lambda = 3$

$$(A - \lambda I)X_2 = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$5x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (1)$

$-6x_1 + 4x_2 - 4x_3 = 0 \longrightarrow (2)$

$2x_1 - 4x_2 = 0 \Rightarrow 2x_1 = 4x_2$

$x_1 = 2x_2$ put $x_2 = k$

$x_1 = 2k$

from eqn (2) $-6(2k) + 4k - 4x_3 = 0$

$\Rightarrow -12k + 4k = 4x_3$

$\Rightarrow -8k = 4x_3 \Rightarrow x_3 = -2k$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ -2k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Eigen vector for $\lambda = 15$:

$(A - \lambda I)X_3 = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-7x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (1)$

$-6x_1 - 8x_2 - 4x_3 = 0 \longrightarrow (2) \Rightarrow 3x_1 + 4x_2 + 2x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0 \longrightarrow (3) \Rightarrow x_1 - 2x_2 - 6x_3 = 0$

from eqn (2) & (3)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 4 & & 2 & & 3 & & 4 \\ & \times & & \times & & \times & \\ -2 & & -6 & & 1 & & -2 \end{array}$$

$$\Rightarrow \frac{x_1}{-24+4} = \frac{x_2}{2+18} = \frac{x_3}{-6-4}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{20} = \frac{x_3}{-10}$$

$$\Rightarrow \frac{x_1}{-2\cancel{0}} = \frac{x_2}{2\cancel{0}} = \frac{x_3}{-1\cancel{0}}$$

$$X_3 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Model matrix $P = [X_1 \quad X_2 \quad X_3]$.

$$P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

Verificatn of Orthogonality :-

$$X_1 X_2^T = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} [2 \quad 1 \quad -2] = 2+2-4 = 0$$

$$X_2 X_3^T = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} [2 \quad -2 \quad 1] = 4-2-2 = 0$$

$$X_3 X_1^T = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} [1 \quad 2 \quad 2] = 2-4+2 = 0$$

$X_1, X_2, X_3$ are mutually orthogonal vectors

$$e_1 = \frac{X_1}{\|X_1\|}, \quad e_2 = \frac{|X_2|}{\|X_2\|}, \quad e_3 = \frac{X_3}{\|X_3\|}$$

$$\|X_1\| = \sqrt{1+4+4} = 3 \qquad \|X_2\| = \sqrt{4+1+4} = 3 \qquad \|X_3\| = \sqrt{4+4+1} = 3$$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}.$$

Here $P$ is Orthogonal matrix

$$PP^T = I \Rightarrow P^{-1} = P^T$$

Diagonalizatn. $P^{-1}AP = P^TAP$.

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

$$P^TAP = D = diag(0, 3, 15)$$

POWERS OF A MATRIX:-

(1) Diagonalize the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ and calculate $A^4$

Given $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}.$

characteristic eqn of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

After solving we get $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

then $(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda = 1, 2, 3$

After Diagonalising we get $D^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^4 = D^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}.$

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

$$A^4 = \begin{bmatrix} 1146 & -1648 & 1226 \\ 454 & -639 & 476 \\ 359 & -528 & 407 \end{bmatrix}.$$

(2) Diagonalise the same matrix A and find $A^8$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}.$$

$$A^8 = P D^8 P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}.$$

$$A^8 = \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -1320 & 13120 & 6501 \end{bmatrix}.$$

## Cayley - Hamilton theorem :-

(1) Verify cayley - Hamilton and find the Inverse of A where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}.$$

The characteristic eqn is $|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{bmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(3-\lambda)(-4-\lambda) - 12\right] - 1\left[(-4-\lambda) - 6\right] + 3\left[-4 + 2(3-\lambda)\right] = 0$

$\Rightarrow (1-\lambda)\left[-12 + \lambda + \lambda^2 - 12\right] - \left[-\lambda - 10\right] + 3\left[-2\lambda + 2\right] = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2 + \lambda - 24\right] + (\lambda + 10) + 3(2 - 2\lambda) = 0$

$\Rightarrow \lambda^2 + \lambda - 24 - \lambda^3 - \lambda^2 + 24\lambda + \lambda + 10 + 6 - 6\lambda = 0$

$\Rightarrow -\lambda^3 + 20\lambda - 8 = 0$

$\Rightarrow \lambda^3 - 20\lambda + 8 = 0$

By cayley - Hamilton theorem every square matrix satisfyes its own - characteristic eqn. $A^3 - 20A + 8I = 0$

we have $A^2 = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$ $A^3 = \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix}$

Sub in $A^3 - 20A + 8I = 0$

$$\begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - 20\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ The characteristic eqn Satisfies the ~~eq~~ theorem

Finding $A^{-1}$:-

we got the characteristic eqn

$$A^{-1}[A^3-20A+8I]=0$$

$$\Rightarrow A^2-20+8A^{-1}=0$$

$$\Rightarrow A^{-1}=\frac{1}{8}\left\{20\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}-4&-8&-12\\10&22&6\\2&2&22\end{bmatrix}\right\}.$$

$$A^{-1}=\frac{1}{8}\begin{bmatrix}24&8&12\\-10&-2&-6\\-2&-2&-2\end{bmatrix}$$

(2) Verify the cayley Hamilton theorem for $A=\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}$. and also find $A^{-1}$

Given $A=\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}$.

The characteristic eqn of matrix $A$ is $|A-\lambda I|=0$.

$$\begin{vmatrix}1-\lambda&3&3\\1&4-\lambda&3\\1&3&4-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)[(4-\lambda)^2-9]-3[(4-\lambda)-3]+3[3-(4-\lambda)]=0$$

$$\Rightarrow (1-\lambda)[\lambda^2-8\lambda+7]+3\lambda-3+3\lambda-1=0$$

$$\Rightarrow \lambda^2-8\lambda+7-\lambda^3+8\lambda^2-7\lambda+6\lambda-6=0$$

$$\Rightarrow -\lambda^3+9\lambda^2-9\lambda+1=0$$

$$\Rightarrow \lambda^3-9\lambda^2+9\lambda-1=0.$$

By cayley - Hamilton theorem $A^3-9A^2+9A-1=0$ — eqn 1

$$A^3=\begin{bmatrix}55&189&189\\63&217&216\\63&216&217\end{bmatrix}\quad A^2=\begin{bmatrix}7&24&24\\8&28&27\\8&27&28\end{bmatrix}.$$

$$A^3-9A^2+9A-I = \begin{bmatrix} 55 & 189 & 189 \\ 63 & 217 & 216 \\ 63 & 216 & 217 \end{bmatrix} - 9\begin{bmatrix} 7 & 24 & 24 \\ 8 & 28 & 27 \\ 8 & 27 & 28 \end{bmatrix} + 9\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$\therefore$ The characteristic eqn satisfies the Cayley-Hamilton theorem

Finding $A^{-1}$:-

we have $A^3-9A^2+9A-I=0$

multiply $A^{-1}$ on b/s

$$A^2-9A+9-A^{-1}=0$$

$$\Rightarrow A^{-1} = A^2-9A+9I$$

$$= \begin{bmatrix} 7 & 24 & 24 \\ 8 & 28 & 27 \\ 8 & 207 & 28 \end{bmatrix} - 9\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Hermitian matrices

Show that $A=\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ is Hermitian matrix

Given, $A=\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$

$$A^{\theta}=(\bar{A})^{T}=\begin{bmatrix} 2 & 3\mp4i \\ 3-4i & 2 \end{bmatrix}^{T}$$

$$A^{\theta}=\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}=A$$

$\therefore A^{\theta}=A$

$\therefore$ A is a Hermitian matrix

Show that $A=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is Hermitian matrix

Given, $A=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$$\bar{A}=\begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$$

$$A^{\theta}=(\bar{A})^{T}=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

$\therefore A^{\theta} = A$

$\therefore A$ is hermitian matrix

Skewhermitian matrices:-

Showthat $A = \begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$ is a skew hermitian matrix

Given, $A = \begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} -2i & -3i \\ -3i & 0 \end{bmatrix}^T$$

$$A^{\theta} = \begin{bmatrix} -2i & -3i \\ -3i & 0 \end{bmatrix}$$

$$A^{\theta} = -\begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$$

$\therefore A = +A^{\theta}$

$\therefore A^{\theta} = -A$

$\therefore A$ is a skewhermitian matrix.

Find skew hermitian matrix of $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$

Given, $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$C = \frac{1}{2}(A - A^{\theta})$$

$$C = \frac{1}{2}\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -1 & 2-i & 2+i \end{bmatrix} - \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$C = \frac{1}{2}\begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & -6i+2 \\ 4i-4 & -6i-2 & 2i \end{bmatrix}$$

$$C = \frac{2}{2}\begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & -3i+1 \\ 2i-2 & -3i-1 & i \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} -i & -2+i & 2-2i \\ 2+i & 0 & +3i+1 \\ 2i-2 & +3i-1 & -i \end{bmatrix}$$

$$C^{\theta} = (\bar{C})^T = \begin{bmatrix} -i & 2+i & -2i-2 \\ -2+i & 0 & 3i-1 \\ 2-2i & 3i+1 & -i \end{bmatrix}$$

$$C^{\theta} = -\begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & -3i+1 \\ 2i-2 & -3i-1 & i \end{bmatrix}$$

$$C^{\theta} = -C$$

$\therefore$ Skewhermitian matrix of A is $\frac{1}{2}(A - A^{\theta})$

Unitary matrices:-

Show that $\frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$ is unitary matrix

Given, $A = \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$

$\bar{A} = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$

$A^{\theta} = (\bar{A})^T = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$

$AA^{\theta} = \frac{1}{4}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} -i^2+3 & i\sqrt{3}-\sqrt{3}i \\ -\sqrt{3}i+\sqrt{3}i & 3-i^2 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$= \frac{4}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AA^{\theta} = I$

lly $A^{\theta}A = I$

$\therefore A$ is unitary matrix.

Showthat $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary matrix

Given $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\bar{A} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^{T} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{3} \begin{bmatrix} 1+(1+i)(1-i) & (1+i)-(1+i) \\ (1-i)-1(1-i) & (1-i)(1+i)+1 \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{3} \begin{bmatrix} 1+1-i+i-i^2 & 1+i-1-i \\ 1-i-1+i & 1-i+i-i^2+1 \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$AA^{\theta} = \frac{3}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore AA^{\theta} = I$

$\therefore A$ is unitary matrix

Showthat $A=\begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$ is unitary matrix if $a^2+b^2+c^2+d^2=1$

Given, $A=\begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$

$\bar{A}=\begin{pmatrix} a-ic & -b-id \\ b-id & a+ic \end{pmatrix}$

$(\bar{A})^T=\begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

Now consider $AA^\Theta=\begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}\begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

$=\begin{pmatrix} (a+ic)(a-ic)+(-b+id)(-b-id) & (a+ic)(b-id)+(-b+id)(a+ic) \\ (b+id)(a-ic)+(a-ic)(-b-id) & (b+id)(b-id)+(a-ic)(a+ic) \end{pmatrix}$

$=\begin{pmatrix} a^2+c^2+b^2+d^2 & a+ic(b-id-b+id) \\ (a-ic)(b+id-b-id) & b^2+d^2+a^2+c^2 \end{pmatrix}$

$=\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$

$=a^2+b^2+c^2+d^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$AA^\Theta=(a^2+b^2+c^2+d^2)I$ —— (i)

If $A$ is unitary matrix $\Rightarrow AA^\Theta=(I)$ —— (ii)

from (i) & (ii) $\Rightarrow (a^2+b^2+c^2+d^2)I=1\cdot I$

$a^2+b^2+c^2+d^2=1$

$A$ is unitary ~~(functi)~~ matrix if $a^2+b^2+c^2+d^2=1$

Show that $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ is Hermitian matrix and find Eigen Value and Eigen vectors of A.

Given $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$

$A^{\theta} = (\bar{A})^T = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}^T$

$A^{\theta} = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix} = A$

$A^{\theta} = A$

$\therefore A$ is a hermitian matrix

The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$

$((2-\lambda)(2-\lambda) - (3+4i)(3-4i)) = 0$

$4 - 2\lambda - 2\lambda + \lambda^2 - (3^2 + 4^2) = 0$

$\lambda^2 - 4\lambda + 4 - (9+16) = 0$

$\lambda^2 - 4\lambda + 4 - (25) = 0$

$\lambda^2 - 4\lambda - 21 = 0$

$$\lambda^2 - 7\lambda + 3\lambda - 21 = 0$$

$$\lambda(\lambda-7) + 3(\lambda-7) = 0$$

$$(\lambda+3)(\lambda-7) = 0$$

$$\lambda = -3, \lambda = 7$$

$$\lambda = -3, 7$$

$\therefore$ The eigen ~~vector~~ Values of A are $-3, 7$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be any eigen vector of A then $AX = \lambda X$

$$AX = \lambda X$$

$$(AX - \lambda X) = 0$$

$$(A - \lambda I)X = 0 \quad \text{---} (1)$$

Finding eigen vector of A corresponding to eigen value $\lambda = -3$:-

Substitute $\lambda = -3$ in equation (1)

$$\left[\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$5x_1 + (3+4i)x_2 = 0 \quad \text{---} (ii)$$

$$(3-4i)x_1 + 5x_2 = 0 \quad \text{---} (iii)$$

from (ii)

$$-5x_1 = (3-4i)x_2$$

$$\frac{x_1}{3+4i} = \frac{x_2}{-5} = k \text{ say}$$

$$x_1 = (3+4i)k$$

$$x_2 = -5k$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (3+4i)k \\ -5k \end{pmatrix}$$

$$X = k\begin{pmatrix} 3+4i \\ -5 \end{pmatrix}$$

$\therefore$ The eigen vector of A is $X_1 = \begin{pmatrix} 3+4i \\ -5 \end{pmatrix}$ for $\lambda = -3$.

Finding eigen vector of A corresponding to eigen (vector) Value $\lambda = 7$:-

Substitute $\lambda = 7$ in eq$^n$ 1

$$\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-5(x_1) + (3+4i)(x_2) = 0 \quad \text{——— (iv)}$$

$$(3-4i)x_1 - 5x_2 = 0 \quad \text{——— (v)}$$

from (iv) $\Rightarrow -5x_1 = -(3+4i)x_2$

$$5x_1 = (3+4i)x_2$$

$$\frac{x_1}{3+4i} = \frac{x_2}{5} = k \text{ say}$$

$$x_1 = (3+4i)k$$

$$x_2 = 5k$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k\begin{pmatrix} 3+4i \\ 5 \end{pmatrix}$$

$\therefore$ The eigen vector of A is $X_2 = \begin{pmatrix} 3+4i \\ 5 \end{pmatrix}$ for $\lambda = 7$

# QUADRATIC FORMS :- [Q.F]

## MATRIX OF A QUADRATIC FORM :-

1) write the matrix of the following Quadratic form and verify that they can be written as matrix product $X'.A.X$ -?

(i) $x_1^2 - 22x_1x_2 + 12x_2^2$.

let $\phi = x_1^2 - 22x_1x_2 + 12x_2^2$

The Q.F can be written as

$\phi = x_1x_1 - 11x_1x_2 - 11x_2x_1 + 12x_2x_2$

Let A be the matrix eqn of Q.F

$\phi = X'.A.X$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $A = \begin{bmatrix} 1 & -11 \\ -11 & 12 \end{bmatrix}$

verification :- $X'AX = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -11 \\ -11 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 - 11x_2 \\ -11x_1 + 12x_2 \end{bmatrix}$

$= \left[x_1^2 - 11x_1x_2 + x_2(-11x_1 + 12x_2)\right]$

$= x_1^2 - 22x_1x_2 + 12x_2^2$

$= \phi$

(ii) $x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1$

$\phi = x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1$

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ x_1 & 1 & -1/2 & -3/2 \\ x_2 & -1/2 & 2 & 2 \\ x_3 & -3/2 & 2 & -5 \end{array}$$

$$\phi = x_1x_1 - \frac{1}{2}x_1x_2 - \frac{3}{2}x_1x_3 - \frac{1}{2}x_2x_1 + 2x_2x_2 + 2x_2x_3 - \frac{3}{2}x_3x_1 + 2x_3x_2 - 5x_3 \cdot x_3$$

Let A be the matrix of Q.F '$\phi$'

$\phi = x' \cdot A \cdot X$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $A = \begin{bmatrix} 1 & -1/2 & -3/2 \\ -1/2 & 2 & 2 \\ -3/2 & 2 & -5 \end{bmatrix}$

## Quadratic form of a Matrix :-

① Write the corresponding Quadratic forms of given Matrices

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -2 & -3 \\ 3 & -2 & 5 & 6 \\ 4 & -3 & 6 & -7 \end{bmatrix}$

Let 'A' be the matrix of the Q.F '$\phi$'

$\phi = x' \cdot A \cdot X$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -2 & -3 \\ 3 & -2 & 5 & 6 \\ 4 & -3 & 6 & -7 \end{bmatrix}$

$$x' \cdot A \cdot X = \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}}_{1\times 4} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -2 & -3 \\ 3 & -2 & 5 & 6 \\ 4 & -3 & 6 & -7 \end{bmatrix}}_{4\times 4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{4\times 1}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ 2x_1 - x_2 - 2x_3 - 3x_4 \\ 3x_1 - 2x_2 + 5x_3 + 6x_4 \\ 4x_1 - 3x_2 + 6x_3 - 7x_4 \end{bmatrix}}_{4\times 1}$$

$= x_1(x_1+2x_2+3x_3+4x_4) + x_2(2x_1-x_2-2x_3-3x_4) + x_3(3x_1-2x_2+$

$5x_3+6x_4) + x_4(4x_1-3x_2+6x_3-7x_4)$

$= x_1^2+2x_1x_2+3x_1x_3+4x_1x_4+2x_1x_2-x_2^2-2x_2x_3-3x_2x_4+3x_1x_3$

$-2x_2x_3+5x_3^2+6x_3x_4+4x_1x_2-3x_2x_4+6x_3x_4-7x_4^2$

$\phi = x_1^2-x_2^2+5x_3^2-7x_4^2+4x_1x_2+6x_1x_3+8x_1x_4-4x_2x_3-6x_2x_4$

$+12x_3x_4$

(ii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A$

$\phi = x'.A.x$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times 1}$ $x' = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}_{1\times 5}$

$\phi = x'.A.x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

$\phi = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

$\phi = x_1^2+x_2^2+x_3^2+x_4^2+x_5^2$

Linear transformation of a matrix to its Canonical form and Index, Rank, Signature, value clause of a Quadratic form?

1) Reduce the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ to a diagnol form and interpret the result in terms of Quadratic form and find signature, index, rank and value clause of Q.F -?

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

let $\phi$ be the Q.F of matrix A

$$\phi = X'.A.X \quad \text{where} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$X'.A.X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6x_1 - 2x_2 + 2x_3 \\ -2x_1 + 3x_2 - x_3 \\ 2x_1 - x_2 + 3x_3 \end{bmatrix}$$

$$= x_1(6x_1 - 2x_2 + 2x_3) + x_2(-2x_1 + 3x_2 - x_3) + x_3(2x_1 - x_2 + 3x_3)$$

$$\phi = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3 \longrightarrow (1)$$

$$A = I_3 \cdot A \cdot I_3$$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \longrightarrow 3R_2 + R_1$

$R_3 \longrightarrow 3R_3 - R_1$

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & 7 & -1 \\ 0 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \longrightarrow 3C_2 + C_1$

$C_3 \longrightarrow 3C_3 - C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & -3 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_3 \longrightarrow 7R_3 + R_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & 19 & 144 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$C_3 \longrightarrow 7C_3 + C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

$\text{Diag}(6, 21, 1008) = P^T \cdot A \cdot P$ where $P = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$

The linear transformation $X = PY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 &= y_1 + y_2 - 6y_3 \\ x_2 &= 3y_2 + 3y_3 \\ x_3 &= 21y_3 \end{aligned}\right\} \text{---} (2)$$

$X = PY$ transforms the Q.F $X^1 \cdot A \cdot X$ to $6y_1^2 + 21y_2^2 + 1008y_3^2$

sub eqns ② in eqn ① we get

$$X^1 \cdot A \cdot X = 6y_1^2 + 21y_2^2 + 1008y_3^2$$

canonical form of $\phi$ is $y_1^2 + y_2^2 + y_3^2$

rank of Q.F = no: of square terms $= r = 3$

index of Q.F = no: of positive terms $= p = 3$

Signature of Q.F $= 2p - r$

$$s = 6 - 3 = 3$$

$n =$ no: of variables $= 3$

since $s = r = n$

The value clause of Quadratic form is positive definite.

② Find the value clause of Quadratic form using Eigen values -?

$$\phi = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$$

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $x_1$ | 1 | -2 | 1 |
| $x_2$ | -2 | 4 | -2 |
| $x_3$ | 1 | -2 | 1 |

Let A be the matrix of Q.F $= \phi$

$\phi = X^1 \cdot A \cdot X$ where $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

is a symmetric matrix of order 3, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The charachteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$1-\lambda\left[(4-\lambda)(1-\lambda)-4\right]+2\left[-2(1-\lambda)+2\right]+1\left[4-(4-\lambda)\right]=0$$

$$1-\lambda\left[4-4\lambda-\lambda+\lambda^2-4\right]+2\left[-2+2\lambda+2\right]+\left[4-4+\lambda\right]=0$$

$$1-\lambda\left[\lambda^2-5\lambda\right]+2\left[2\lambda\right]+\lambda=0$$

$$\lambda^2-5\lambda-\lambda^3+5\lambda^2+4\lambda+\lambda=0$$

$$-\lambda^3+6\lambda^2=0$$

$$\lambda^3-6\lambda^2=0$$

$$\lambda^2(\lambda-6)=0$$

$$\lambda^2=0 \qquad \lambda-6=0$$

$$\lambda=0, \qquad \lambda=6,$$

$$\lambda=0,$$

$$\therefore \lambda=0, \lambda=0, \lambda=6$$

The eigen value of 'A' is +ve and atleast one eigen value is zero

$\therefore$ The given Q.f is positive Semidefinite.

Reduction of Quadratic form to Canonical form by orthogonal transformation

① Reduce the Q·F to canonical form by an orthogonal reduction

$3x^2+2y^2+3z^2-2xy-2yz$

Let $\phi = 3x^2+2y^2+3z^2-2xy-2yz$

The given Q·F can be written as

$\phi = 3x\cdot x + 2yy + 3z\cdot z - 2xy - 2yz$

Let A be the matrix of Q·F $\phi$ then $\phi = X'\cdot A\cdot X$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ of order 3

Let $\lambda$ be the eigen value of A then $|A-\lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$3-\lambda\left[(2-\lambda)(3-\lambda)-1\right]+1\left[-1(3-\lambda)\right]=0$$

$$(3-\lambda)(6-2\lambda-3\lambda+\lambda^2-1)-3+\lambda=0$$

$$(3-\lambda)(\lambda^2-5\lambda+5)-3+\lambda=0$$

$$3\lambda^2-15\lambda+15-\lambda^3+5\lambda^2-5\lambda-3+\lambda=0$$

$$\lambda^3-8\lambda^2+19\lambda+12=0$$

$$
\begin{array}{r|rrrr}
 & 1 & -8 & 19 & -12 \\
\lambda=3 & 0 & 3 & -15 & 12 \\
\hline
 & 1 & -5 & 4 & \boxed{0} \\
\lambda=4 & 0 & 4 & -4 & \\
\hline
 & 1 & -1 & \boxed{0} & \\
\lambda=1 & 0 & 1 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}
$$

$\therefore \lambda = 1, 3, 4$ let $x$ be the eigen vector of $A$ then $AX = \lambda X$

$$(A - \lambda I)X = 0 \longrightarrow (1)$$

<u>Finding the eigenvector of A corresponding to eigenvalue $\lambda = 1$ :-</u>

$$(A - \lambda I)X = 0$$

$$(A - I)X = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 2x - y &= 0 \\ -x + y - z &= 0 \\ -y + 2z &= 0 \end{aligned}\right\} (2)$$

Solving eqns (2) $\dfrac{x}{1+0} = \dfrac{y}{0+2} = \dfrac{z}{2-1}$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1} = k \text{ say}$$

$$x = k, \; y = 2k, \; z = k$$

The solution is $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$||X_1|| = \sqrt{1^2+2^2+1^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$\frac{X_1}{||X_1||} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Finding eigen vector of A corresponding to $\lambda = 3$ :-

$\lambda = 3$

$$(A-3I)X = 0$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-y = 0$$
$$-x-y-z = 0$$
$$-y = 0$$

Let $z = k$, then $x = -k$

Solution of $\lambda = 3$ is $X_2 = \begin{pmatrix} -k \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$||X_2|| = \sqrt{1+0+1} = \sqrt{2}$$

$$\frac{X_2}{||X_2||} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Finding the eigen vector of A for $\lambda = 4$

$$(A-4I)X = 0$$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-x-y=0 \longrightarrow ①$$

$$-x-2y-z=0 \longrightarrow ②$$

$$-y-z=0 \longrightarrow ③$$

Solving ① and ②

$$\begin{array}{cccc} & x & y & z \\ -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -2 \end{array}$$

$$\frac{x}{1-0} = \frac{y}{0-1} = \frac{z}{2-1}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1} = k \text{ say}$$

$$X_3 = \begin{bmatrix} 1k \\ -k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} k \qquad \|X_3\| = \sqrt{1+1+1} = \sqrt{3}$$

$$\frac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

The eigen values and eigen vectors of A are

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \; X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \; X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{for } \lambda = 1, 3, 4$$

$$\text{orthogonal matrix} = P = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

There exist a orthogonal linear transformation $x = Py$

where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$X = PY$

$$\left.\begin{aligned} x &= \frac{1}{\sqrt{6}}y_1 - \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{3}}y_3 \\ y &= \frac{2}{\sqrt{6}}y_1 - \frac{1}{\sqrt{3}}y_3 \\ z &= \frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{3}}y_3 \end{aligned}\right\} \text{eqn ②}$$

Substituting eqn ② in $X^1 \cdot A \cdot X$ we get $y_1^2 + 3y_2^2 + 4y_3^2$

The canonical form of $\phi$ is $Z_1^2 + Z_2^2 + Z_3^2$

The rank of $\phi = r = 3$

index of $\phi = p = 3$

signature of $\phi$, $s = 2p - r = 3$

$s = 3$

no: of variables $= n = 3$

$\therefore s = r = n$

The Q.F $\phi$ is positive definite.

Reduction of Quadratic form by using lagranges method :-

① reduce the Q.F $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

let the given Q.F be $X^1 \cdot AX = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

$$= x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1(x_2 - 2x_3)$$

$$= x_1^2 - 2x_1 2(x_2 - 2x_3) + [2(x_2 - 2x_3)]^2 - [2(x_2 - 2x_3)]^2 + 2x_2^2 - 7x_3^2$$

$$= [x_1 - 2(x_2 - 2x_3)]^2 - 4(x_2^2 + 4x_3^2 - 4x_2x_3) + 2x_2^2 - 7x_3^2$$

$$= [x_1 - 2x_2 + 4x_3]^2 - 4x_2^2 - 16x_3^2 + 16x_2x_3 + 2x_2^2 - 7x_3^2$$

$$= (x_1 - 2x_2 + 4x_3)^2 - 2x_2^2 + 16x_2x_3 - 23x_3^2$$

$$= (x_1 - 2x_2 + 4x_3)^2 - 2(x_2^2 + 2x_2 4x_3 + 16x_3^2) + 32x_3^2 - 23x_3^2$$

$$= (x_1 - 2x_2 + 4x_3)^2 - 2(x_2 - 4x_3)^2 + (3x_3)^2 \longrightarrow ①$$

let
$$\left.\begin{aligned} y_1 &= x_1 - 2x_2 + 4x_3 \\ y_2 &= x_2 - 4x_3 \\ y_3 &= 3x_3 \end{aligned}\right\} ②$$

From ① and ②

$$x' \cdot A \cdot x = y_1^2 - 2y_2^2 + y_3^2$$

The canonical form of $x' \cdot A \cdot x = z_1^2 - z_2^2 + z_3^2$

The rank of Q.F $= r = 3$

index of Q.F $= p = 2$

Signature $= 2p - r = 4 - 3 = 1 = s$

$|s| \neq r$ The Q.F is indefinite.

MATHEMATICS-I

# MEAN VALUE THEOREMS FUNCTIONS OF SINGLE & SEVERAL VARIABLES

I YEAR B.TECH



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## SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

| Name of the Unit | Name of the Topic |
|---|---|
| Unit-I<br>Sequences and Series | • Basic definition of sequences and series<br>• Convergence and divergence.<br>• Ratio test<br>• Comparison test<br>• Integral test<br>• Cauchy's root test<br>• Raabe's test<br>• Absolute and conditional convergence |
| Unit-II<br>Functions of single variable | • Rolle's theorem<br>• Lagrange's Mean value theorem<br>• Cauchy's Mean value theorem<br>• Generalized mean value theorems<br>• Functions of several variables<br>• Functional dependence, Jacobian<br>• Maxima and minima of function of two variables |
| Unit-III<br>Application of single variables | • Radius , centre and Circle of curvature<br>• Evolutes and Envelopes<br>• Curve Tracing-Cartesian Co-ordinates<br>• Curve Tracing-Polar Co-ordinates<br>• Curve Tracing-Parametric Curves |
| Unit-IV<br>Integration and its applications | • Riemann Sum<br>• Integral representation for lengths<br>• Integral representation for Areas<br>• Integral representation for Volumes<br>• Surface areas in Cartesian and Polar co-ordinates<br>• Multiple integrals-double and triple<br>• Change of order of integration<br>• Change of variable |
| Unit-V<br>Differential equations of first order and their applications | • Overview of differential equations<br>• Exact and non exact differential equations<br>• Linear differential equations<br>• Bernoulli D.E<br>• Newton's Law of cooling<br>• Law of Natural growth and decay<br>• Orthogonal trajectories and applications |
| Unit-VI<br>Higher order Linear D.E and their applications | • Linear D.E of second and higher order with constant coefficients<br>• R.H.S term of the form *exp(ax)*<br>• R.H.S term of the form *sin ax and cos ax*<br>• R.H.S term of the form exp(ax) v(x)<br>• R.H.S term of the form exp(ax) v(x)<br>• Method of variation of parameters<br>• Applications on bending of beams, Electrical circuits and simple harmonic motion |
| Unit-VII<br>Laplace Transformations | • LT of standard functions<br>• Inverse LT –first shifting property<br>• Transformations of derivatives and integrals<br>• Unit step function, Second shifting theorem<br>• Convolution theorem-periodic function<br>• Differentiation and integration of transforms<br>• Application of laplace transforms to ODE |
| Unit-VIII<br>Vector Calculus | • Gradient, Divergence, curl<br>• Laplacian and second order operators<br>• Line, surface , volume integrals<br>• Green's Theorem and applications<br>• Gauss Divergence Theorem and applications<br>• Stoke's Theorem and applications |

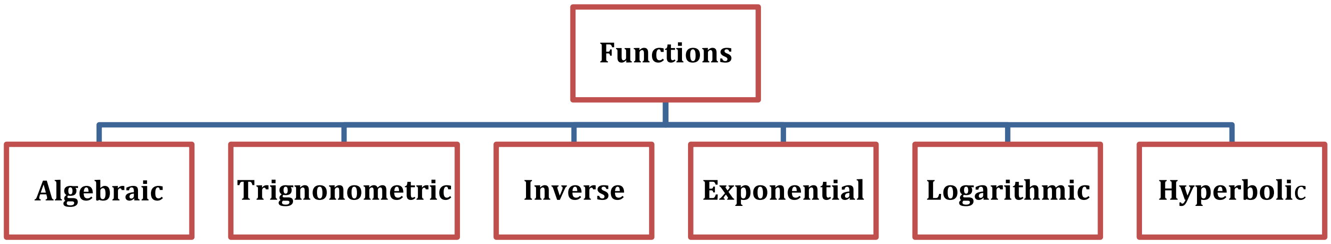
# CONTENTS

UNIT-2

## Functions of Single & Several Variables

## Introduction

**Real valued function**: Any function $\boldsymbol{f}: \boldsymbol{S} \longrightarrow \mathbb{R}$ $(\boldsymbol{S} \subseteq \mathbb{R})$ is called a Real valued function.




**Limit of a function:** Let $f: S \longrightarrow \mathbb{R}$ is a real valued function and $a \in S$. Then, a real number $l \in \mathbb{R}$ is said to be limit of $f$ at $x = a$ if for each $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$ $\forall$ $x \in S$. It is denoted by $\lim_{x \longrightarrow a} f(x) = l$.

**Continuity:** Let $f: S \longrightarrow \mathbb{R}$ is a real valued function and $a \in S$. Then, $f$ is said to be continuous at $x = a$ if for each $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$ $\forall$ $x \in S$. It is denoted by $\lim_{x \longrightarrow a} f(x) = f(a)$.

**Note:** 1) The function $y = \sin x$ is continuous every where

2) Every $\cos x$ function is continuous every where.

3) Every $\tan x$ function is not continuous, but in $\left(0, \frac{\pi}{2}\right)$ the function $\tan x$ is continuous.

4) Every polynomial is continuous every where.

5) Every exponential function is continuous every where.

6) Every log function is continuous every where.

**Differentiability:** Let $y = f(x)$ be a function, then $f$ is said to be differentiable or derivable at a point $x = a$, if $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

# MEAN VALUE THEOREMS

Let $y = f(x)$ be a given function.

## ROLLE'S THEOREM

Let $f: [a, b] \longrightarrow \mathbb{R}$ such that

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable (or) derivable in $(a, b)$

(iii) $f(a) = f(b)$

then $\exists$ atleast one point $c$ in $(a, b)$ such that $f'(c) = 0$

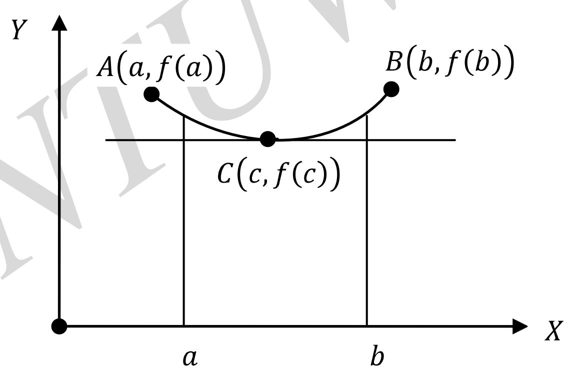
**Geometrical Interpretation of Rolle's Theorem:**

From the diagram, it can be observed that

(i) there is no gap for the curve $y = f(x)$ from $A(a, f(a))$ *to* $B(b, f(b))$. Therefore, the function is continuous

(ii) There exists unique tangent for every intermediate point between $A$ and $B$

(ii) Also the ordinates of $a$ and $b$ are same, then by Rolle's theorem, there exists atleast one point $C(c, f(c))$ in between $A$and $B$ such that tangent at $C$ is parallel to $X$ −axis.




## LAGRANGE'S MEAN VALUE THEOREM

Let $f: [a, b] \longrightarrow \mathbb{R}$ such that

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable (or) derivable in $(a, b)$

then $\exists$ atleast one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

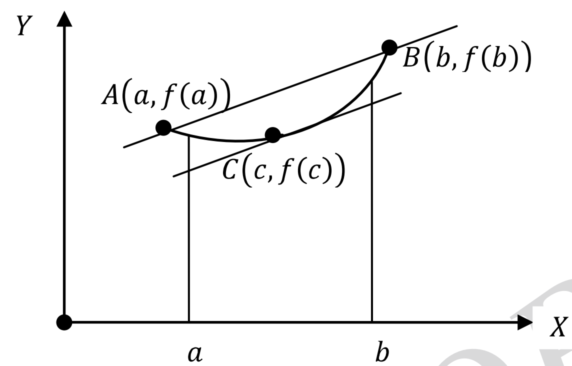
**Geometrical Interpretation of Rolle's Theorem:**

From the diagram, it can be observed that

(i) there is no gap for the curve $y = f(x)$ from $A(a, f(a))$ $to$ $B(b, f(b))$. Therefore, the function is continuous

(ii) There exists unique tangent for every intermediate point between $A$ and $B$

Then by Lagrange's mean value theorem, there exists atleast one point $C(c, f(c))$ in between $A$and $B$ such that tangent at $C$ is parallel to a straight line joining the points $A$ and $B$




## CAUCHY'S MEAN VALUE THEOREM

Let $f: [a, b] \longrightarrow \mathbb{R}, g: [a, b] \longrightarrow \mathbb{R}$ are such that

(i) $f, g$ are continuous in $[a, b]$

(ii) $f, g$ are differentiable (or) derivable in $(a, b)$

(iii) $g'(x) \neq 0 \ \forall \, x \in (a, b)$

then $\exists$ atleast one point $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-g(a)}{g(b)-g(a)}$



## Generalised Mean Value Theorems

**Taylor's Theorem:** If $f: [a, b] \longrightarrow \mathbb{R}$ such that

(i) $f, f, f'', \dots, f^{(n-1)}$ are continuous on $[a, b]$

(ii) $f, f, f'', \dots, f^{(n-1)}$ are derivable or differentiable on $(a, b)$

(iii) $p \in \mathbb{Z}^+$ , then $\exists \, c \ \in (a, b)$ such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f(a) + \frac{(b-a)^2}{2!} f(a) + \frac{(b-a)^3}{3!} f(a) + \ \dots + R_n$$

where, $R_n$ is called as Schlomilch – Roche's form of remainder and is given by

$$R_n = \frac{(b-a)^n (b-c)^{n-p}}{(n-1)!\,p} f^{(n)}(c)$$

- **Lagrange's form of Remainder:**

  Substituting $\boldsymbol{p = n}$ in $R_n$ we get Lagrange's form of Remainder

  *i.e.* $R_n = \frac{(b-a)^n}{(n)!} f^{(n)}(c)$

- **Cauchy's form of Remainder:**

  Substituting $\boldsymbol{p = 1}$ in $R_n$ we get Cauchy's form of Remainder

  *i.e.* $R_n = \frac{(b-a)^n (b-c)^{n-1}}{(n-1)!} f^{(n)}(c)$

**Maclaurin's Theorem:** If $f: [0, x] \longrightarrow \mathbb{R}$ such that

(i) $f, f, f'', \dots, f^{(n-1)}$ are continuous on $[0, x]$

(ii) $f, f, f'', \dots, f^{(n-1)}$ are derivable or differentiable on $(0, x)$

(iii) $p \in \mathbb{Z}^+$, then $\exists\, \theta \in (0, x)$ such that

$$f(x) = f(0) + \frac{x}{1!} f(0) + \frac{x^2}{2!} f(0) + \frac{x^3}{3!} f(0) + \dots + R_n$$

where, $R_n$ is called as Schlomilch – Roche's form of remainder and is given by

$$R_n = \frac{x^n (1-\theta)^{n-p}}{(n-1)!\,p} f^{(n)}(\theta x)$$

- **Lagrange's form of Remainder:**

  Substituting $p = n$ in $R_n$ we get Lagrange's form of Remainder

  *i.e.* $R_n = \frac{x^n}{(n)!} f^{(n)}(\theta x)$

- **Cauchy's form of Remainder:**

  Substituting $p = 1$ in $R_n$ we get Cauchy's form of Remainder

  *i.e.* $R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x)$

# Jacobian

Let $u = u(x, y), v = v(x, y)$ are two functions , then the Jacobian of $u$ and $v$ w.r.t $x$ and $y$ is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ $or\ J\left(\frac{u,v}{x,y}\right)$ and is defined as

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \ or\ \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

## Properties:

- $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$
- If $u, v$ are functions of $r, s$ and $r, s$ are functions of $x, y$ then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

**Functional Dependence:**

Two functions $u(x, y), v(x, y)$ are said to be functional dependent on one another if the Jacobian of $(u, v)\ w.r.t\ (x, y)$ is zero.

If they are functionally dependent on one another, then it is possible to find the relation between these two functions.

## MAXIMA AND MINIMA

Maxima and Minima for the function of one Variable:

Let us consider a function $y = f(x)$

To find the Maxima and Minima, the following procedure must be followed:

**Step 1:** First find the first derivative and equate to zero. i.e. $\frac{dy}{dx} = 0$

**Step 2:** Since $y = f(x)$ is a polynomial $\Longrightarrow \frac{dy}{dx} = 0$ is a polynomial equation. By solving this equation we get roots.

**Step 3:** Find second derivative i.e. $\frac{d^2y}{dx^2}$

**Step 4:** Now substitute the obtained roots in $\frac{d^2y}{dx^2}$

**Step 5:** Depending on the Nature of $\frac{d^2y}{dx^2}$ at that point we will solve further. The following cases will be there.

**Case (i):** If $\frac{d^2y}{dx^2} < 0$ at a point say $= a$ , then $f$ has maximum at $x = a$ and the maximum value is given by $[f(x)]_{x=a} = f(a)$

**Case (ii):** If $\frac{d^2y}{dx^2} > 0$ at a point say $= a$ , then $f$ has minimum at $x = a$ and the minimum value is given by $[f(x)]_{x=a} = f(a)$

**Case (iii):** If $\frac{d^2y}{dx^2} = 0$ at a point say $= a$ , then $f$ has neither minimum nor maximum. i.e. stationary.

## Maxima and Minima for the function of Two Variable

Let us consider a function $z = f(x, y)$

To find the Maxima and Minima for the given function, the following procedure must be followed:

**Step 1:** First find the first derivatives and equate to zero. i.e. $\frac{\partial z}{\partial x} = 0$ , $\frac{\partial z}{\partial y} = 0$

(Here, since we have two variables, we go for partial derivatives, but not ordinary derivatives)

**Step 2:** By solving $\frac{\partial z}{\partial x} = 0$ , $\frac{\partial z}{\partial y} = 0$ , we get the different values of $x$ and $y$.

Write these values as set of ordered pairs. i.e. $(x, y)$

**Step 3:** Now, find second order partial derivatives.

*i.e.,* $\frac{\partial^2 z}{\partial x^2}$ , $\frac{\partial^2 z}{\partial x\, \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$

**Step 4:** Let us consider $l = \frac{\partial^2 z}{\partial x^2}$ , $m = \frac{\partial^2 z}{\partial x\, \partial y}$ , $n = \frac{\partial^2 z}{\partial y^2}$

**Step 5:** Now, we have to see for what values of $x$ & $y$, the given function is maximum/minimum/ does not have extreme values/ fails to have maximum or minimum.

- If at a point, say $(a, b)$ : $\boldsymbol{ln - m^2} > 0$ *and* $l < 0$, then $f$ has **maximum** at this point and the maximum value will be obtained by substituting $(a, b)$ in the given function.
- If at a point, say $(a, b)$ : $\boldsymbol{ln - m^2} > 0$ *and* $l > 0$, then $f$ has **minimum** at this point and the minimum value will be obtained by substituting $(a, b)$ in the given function.
- If at a point, say $(a, b)$ : $\boldsymbol{ln - m^2} < 0$ , then $f$ has **neither maximum nor minimum** and such points are called as saddle points.
- If at a point, say $(a, b)$ : $\boldsymbol{ln - m^2 = 0}$ , then $f$ fails to have maximum or minimum and case needs further investigation to decide maxima/minima. *i.e.* **No conclusion**

## Problem

**1) Examine the function for extreme values $f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ $(x > 0, y > 0)$**

**Sol:** Given $f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

The first order partial derivatives of $f$ are given by

$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x$ and

$\frac{\partial f}{\partial y} = 6xy - 6y$

Now, equating first order partial derivatives to zero, we get

$\frac{\partial f}{\partial x} = 0 \Longrightarrow 3x^2 + 3y^2 - 6x = 0 \quad \dots(1)$

$\frac{\partial f}{\partial y} = 0 \Longrightarrow 6xy - 6y = 0 \quad \dots(2)$

Solving $(1)$ & $(2)$ we get

$(2) \Longrightarrow 6y(x-1) = 0$

$\Longrightarrow y = 0, x = 1$

Substituting $y = 0$ in $(1)$ $\Longrightarrow 3x^2 - 6x = 0$

$\Longrightarrow 3x(x-2) = 0$

$\Longrightarrow x = 0, x = 2$

Substituting $x = 1$ in $(1)$ $\Longrightarrow 3y^2 - 3 = 0$

$\Longrightarrow y = 1, -1$

$\therefore$ All possible set of values are $(0,0), (2,0), (1,1), (1,-1)$

Now, the second order partial derivatives are given by

$l = \frac{\partial^2 f}{\partial x^2} = 6x - 6$

$m = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(6xy - 6y) = 6y$

$n = \frac{\partial^2 f}{\partial y^2} = 6x - 6$

Now, $ln - m^2 = (6x-6)(6x-6) - (6y)^2 = (6x-6)^2 - 36y^2$

At a point $(0,0) \Longrightarrow ln - m^2 = 36 > 0$ & $l = -6 < 0$

$\therefore f$ has maximum at $(0,0)$ and the maximum value will be obtained by substituting $(0,0)$ in the function

i.e. $[f(x,y)]_{(0,0)} = 4$ is the maximum value.

Also, at a point $(2,0) \Longrightarrow \boldsymbol{ln - m^2} = 36 > \mathbf{0}$ & $\boldsymbol{l} = 6 > 0$

$\therefore f$ has minimum at $(2,0)$ and the minimum value is $[f(x,y)]_{(2,0)} = 0$.

Also, at a point $(1,1) \Longrightarrow \boldsymbol{ln - m^2} < \mathbf{0}$

$\therefore f$ has neither minimum nor maximum at this point.

Again, at a point $(1,-1) \Longrightarrow \boldsymbol{ln - m^2} < \mathbf{0}$

$\therefore f$ has neither minimum nor maximum at this point.

## Lagrange's Method of Undetermined Multipliers

This method is useful to find the extreme values (i.e., maximum and minimum) for the given function, whenever some condition is given involving the variables.

To find the Maxima and Minima for the given function using Lagrange's Method , the following procedure must be followed:

**Step 1:** Let us consider given function to be $f(x,y,z)$ subject to the condition $\phi(x,y,z) = 0$

**Step 2:** Let us define a Lagrangean function $F = f + \lambda\,\phi$ , where $\lambda$ is called the Lagrange multiplier.

**Step 3:** Find first order partial derivatives and equate to zero

$$i.e.\ \frac{\partial F}{\partial x} = 0 \Longrightarrow \frac{\partial f}{\partial x} + \lambda\,\frac{\partial \phi}{\partial x} = 0 \quad \dots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow \frac{\partial f}{\partial y} + \lambda\,\frac{\partial \phi}{\partial y} = 0 \quad \dots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow \frac{\partial f}{\partial z} + \lambda\,\frac{\partial \phi}{\partial z} = 0 \quad \dots (3)$$

Let the given condition be $\phi(x,y,z) = 0 \quad \dots (4)$

**Step 4**: Solve $(1),(2),(3)$ & $(4)$ , eliminate $\lambda$ to get the values of $x,y,z$

**Step 5:** The values so obtained will give the stationary point of $f(x,y,z)$

**Step 6:** The minimum/maximum value will be obtained by substituting the values of $x,y,z$ in the given function.

## Problem

**1) Find the minimum value of $\boldsymbol{x^2 + y^2 + z^2}$ subject to the condition $\boldsymbol{xyz = a^3}$**

**Sol:** Let us consider given function to be $f = x^2 + y^2 + z^2$ and $\phi = xyz - a^3$

Let us define Lagrangean function $F = f + \lambda\,\phi$ , where $\lambda$ is called the Lagrange multiplier.

$$\Longrightarrow F = (x^2 + y^2 + z^2) + \lambda\,(xyz - a^3)$$

Now, $\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x + \lambda yz = 0 \quad \Longrightarrow \frac{\lambda}{2} = -\frac{x}{yz} \quad \dots (1)$

$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y + \lambda xz = 0 \quad \Longrightarrow \frac{\lambda}{2} = -\frac{y}{xz} \dots (2)$

$\frac{\partial F}{\partial z} = 0 \Longrightarrow 2z + \lambda xy = 0 \quad \Longrightarrow \frac{\lambda}{2} = -\frac{z}{xy} \quad \dots (3)$

Solving $(1), (2)\ \&\ (3) \Longrightarrow \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xz}$

Now, consider $\frac{x}{yz} = \frac{y}{xz} \Longrightarrow x^2 = y^2 \quad \dots (4)$

Again, consider $\frac{y}{xz} = \frac{z}{xz} \Longrightarrow y^2 = z^2 \dots (5)$

Again solving $(4)\ \&\ (5) \Longrightarrow x^2 = y^2 = z^2$

$$\Longrightarrow x = y = z$$

Given $\phi = xyz - a^3 = 0$

$At\ x = y = z \Longrightarrow x^3 = a^3$

$$\Longrightarrow x = a$$

Similarly, we get $y = a, z = a$

Hence, the minimum value of the function is given by $(f)_{(a,a,a)} = a^2 + a^2 + a^2 = 3a^2$

$\underline{\overline{* * *}}$

**MATHEMATICS-I**

# Curvature, Evolutes & Envelopes Curve Tracing

I YEAR B.Tech



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UNIT-III

# CURVATURE

The shape of a plane Curve $C$ is characterized by the degree of Bentness or Curvedness.

## RADIUS OF CURVATURE

The reciprocal of the curvature of a curve is called the radius of curvature of curve.

## FORMULAE FOR THE EVALUATION OF RADIUS OF CURVATURE

In this we have three types of problems

- Problems to find Radius of Curvature in Cartesian Co-ordinates
- Problems to find Radius of Curvature in Polar Co-ordinates
- Problems to find Radius of Curvature in Parametric Form.

### In Cartesian Co-ordinates

Let us consider $y = f(x)$ be the given curve, then radius of curvature is given by

$$(\rho)_{at\ P} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)}$$

If the given equation of the curve is given as $x = g(y)$ , then the radius of curvature is given by

$$(\rho)_{at\ P} = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2x}{dy^2}\right)}$$

## In Parametric Form
(i.e. $x$ in terms of other variable and $y$ in terms of other variable say $t, \theta$ $etc$)

If the equation of the curve is given in parametric form $x = x(t), y = y(t)$ then

$$(\rho)_{at\ P} = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right)}$$

## In Polar co-ordinates

If the equation of the curve is given in polar form i.e. $r = f(\theta)$ then

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}$$

where, $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$.

## Radius of Curvature at origin (Newton's Theorem)

Suppose a curve is passing through the origin and $x$-axis or $y$-axis is tangent to the curve at the origin. Then the following results will be useful to find $\rho$ at (0,0).

If $x$-axis is tangent at $(0,0)$ , then

$$(\rho)_{at\ (0,0)} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y} = \lim_{x\to 0} \frac{x^2}{2y}$$

If $y$-axis is tangent at $(0,0)$ , then

$$(\rho)_{at\ (0,0)} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{y^2}{2x} = \lim_{y\to 0} \frac{y^2}{2x}$$

***Note:*** If the given curve $y = f(x)$ passes through $(0,0)$ and neither $x$-axis nor $y$-axis is tangent to the curve, then

Using Maclaurin's series expansion of $f(x)$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ \dots$$

$$= 0 + px + qx^2 + \ \dots$$

then, $p = f'(0), q = \frac{f''(0)}{2}$

By this we can calculate $\rho$ at $(0,0)$ using the formula for finding Radius of Curvature in Cartesian Co-ordinates.

## Problems on Radius of Curvature

**1) Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the curve $x^3 + y^3 = 3axy$.**

**Sol:** Clearly, the given equation of curve belongs to Cartesian coordinates.

We know that, the radius of curvature $(\rho)$ at any point on the curve $y = f(x)$ is given by

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Now, consider $x^3 + y^3 = 3axy$

Differentiate w.r.t $x$ , we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

$$\Longrightarrow\ x^2 + y^2\frac{dy}{dx} - ax\frac{dy}{dx} - ay = 0$$

$$\Longrightarrow\ (y^2 - ax)\frac{dy}{dx} = (ay - x^2)$$

$$\Longrightarrow\ \boxed{\frac{dy}{dx} = \frac{(ay-x^2)}{(y^2-ax)}} \quad \dots \quad (1)$$

Here, $\boxed{\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1}$

Again, differentiating $(1)$ , w.r.t $x$ , we get

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left(a\frac{dy}{dx}-2x\right)-(ay-x^2)\left(2y\frac{dy}{dx}-a\right)}{(y^2-ax)^2}$$

$$\Longrightarrow\ \boxed{\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{32}{3a}}$$

Now, the Radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is given by $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$$\Rightarrow (\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{[1+(-1)^2]^{\frac{3}{2}}}{\left(-\frac{32}{3a}\right)} = -\frac{2^{\frac{3}{2}}}{32}\ .3a = -\frac{2\sqrt{2}}{32}3a$$

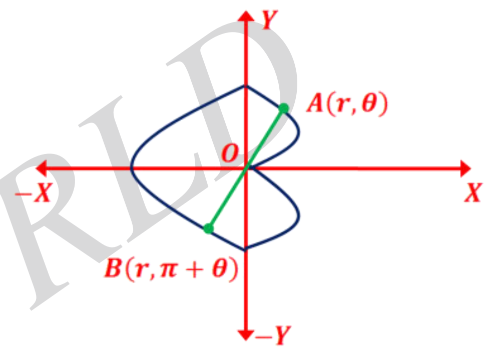
$$\Rightarrow (\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{3\sqrt{2}}{16}a \text{ (numerically... since radius cannot be negative)}$$

**2) If $\rho_1, \rho_2$ be the radius of curvature at the extremities of an chord of the cardioid $r = a(1-\cos\theta)$ which passes through the pole, show that $\rho_1{}^2 + \rho_2{}^2 = \frac{16a^2}{9}$**

**Sol:** Let us consider the equation of the cardioids to be
$r = a(1-\cos\theta)$

Let us consider $A$ and $B$ to be the extremities of the chord, whose coordinates are given by $A(r,\theta)$ and $B(r, \pi+\theta)$

Let $\rho_1, \rho_2$ be the radius of curvature at the point $A$ and $B$ respectively.






**Let us find $\rho_1$ :**

We know that the radius of curvature for the curve $r = f(\theta)$ is given by

$$\boldsymbol{\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}}$$

Now, consider $r = a(1-\cos\theta)$

$\Rightarrow r_1 = a\sin\theta$ and $r_2 = a\cos\theta$

Hence, $\boldsymbol{\rho_1} = \frac{\left((a(1-\cos\theta))^2+(a\sin\theta)^2\right)^{\frac{3}{2}}}{(a(1-\cos\theta))^2+2(a\sin\theta)^2-a(1-\cos\theta)\,a\cos\theta}$

$$\Rightarrow \boldsymbol{\rho_1} = \frac{\left(a^2[(1-\cos\theta)^2+\sin^2\theta]\right)^{\frac{3}{2}}}{(a^2)\left[((1-\cos\theta))^2+2(\sin\theta)^2-(1-\cos\theta)\cos\theta\right]}$$

$$= \frac{a^3(2-2\cos\theta)^{\frac{3}{2}}}{a^2[1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{3}{2}}}{(3-3\cos\theta)} = \frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{3}{2}}}{3(1-\cos\theta)}$$

$$\Longrightarrow \boxed{\boldsymbol{\rho_1} = \frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{1}{2}}}{3}}$$

Similarly, $\boldsymbol{\rho_2} = \frac{a\, 2^{\frac{3}{2}}(1-\cos(\pi+\theta))^{\frac{1}{2}}}{3} = \frac{a\, 2^{\frac{3}{2}}(1+\cos\theta)^{\frac{1}{2}}}{3}$

$$\Longrightarrow \boxed{\boldsymbol{\rho_2} = \frac{a\, 2^{\frac{3}{2}}(1+\cos\theta)^{\frac{1}{2}}}{3}}$$

Now, consider L.H.S:

i.e. $\rho_1{}^2 + \rho_2{}^2 = \left(\frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{1}{2}}}{3}\right)^2 + \left(\frac{a\, 2^{\frac{3}{2}}(1+\cos\theta)^{\frac{1}{2}}}{3}\right)^2$

$$= \frac{8a^2}{9}([1-\cos\theta] + [1+\cos\theta])$$

$$= \frac{8a^2}{9}\,(2) = \frac{16a^2}{9}$$

$$\therefore \boxed{\boldsymbol{\rho_1}^2 + \boldsymbol{\rho_2}^2 = \frac{\mathbf{16}a^2}{\mathbf{9}}}$$

Hence the result.

**3) Show that the radius of curvature at each point of the curve $x = a\Big($cos $t$ +**

**logtan$t2$,**

**$y = a\sin t$ is inversely proportional to the length of the normal intercepted between the point on the curve and the $x$ −axis.**

**Sol:** Clearly, the equation of the curve is in parametric form

Let us change the problem of solving radius of curvature in parametric form to problem of solving radius of curvature in Cartesian form.

$$i.e.\quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

Here, given $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$

$$\Rightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \,.\sec^2\frac{t}{2} \,.\frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2}\,\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \,.\frac{1}{\cos^2\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2\,\sin\frac{t}{2}\,\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right) = a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = a\left(\frac{\cos^2 t}{\sin t}\right)}$$

Similarly, $\boxed{\frac{dy}{dt} = a\ \cos t}$

Now, $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \tan t$

$$\Rightarrow \boxed{\frac{dy}{dx} = \tan t}$$

Again, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \sec^2 t \,.\frac{dt}{dx}$

$$= \sec^2 t \,.\frac{1}{\left(\frac{dx}{dt}\right)} = \sec^2 t \,.\frac{1}{a\left(\frac{\cos^2 t}{\sin t}\right)}$$

$$= \frac{\sec^4 t\sin t}{a}$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{\sec^4 t\sin t}{a}}$$

Hence, the radius of curvature for the equation of the curve $y = f(x)$ at any point is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\Rightarrow \rho = \frac{\left[1 + (\tan t)^2\right]^{\frac{3}{2}}}{\frac{\sec^4 t\sin t}{a}}$$

$$= \frac{a\,(\sec^2 t)^{\frac{3}{2}}}{\sec^4 t \sin t} = a\frac{\sec^3 t}{\sec^4 t \sin t}$$

$$= a\frac{\cos t}{\sin t} = a\cot t$$

$$\Rightarrow \boxed{\boldsymbol{\rho = a\cot t}} \quad \dots \quad (\mathbf{1})$$

Also, we know that the length of the normal is given by $L = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow L = (a\sin t)\sqrt{1 + \tan^2 t}$$

$$= a\sin t \sec t$$

$$= a\tan t = \frac{a}{\cot t}$$

$$\Rightarrow \boxed{\boldsymbol{L = \frac{a}{\cot t}}} \quad \dots \quad (\mathbf{2})$$

Hence, from $(\mathbf{1})$ & $(\mathbf{2})$ we can say that, the radius of curvature at each point of the curve $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$ is inversely proportional to the length of the normal intercepted between the point on the curve and the $x$ –axis.

Hence the result.

## CENTRE OF CURVATURE

***Definition :*** The Centre of Curvature at a point 'P' of a curvature is the point "C" which lies on the Positive direction of the normal at 'P' and is at a distance $'\rho'$ (in magnitude) from it.

## CIRCLE OF CURVATURE

***Definition:*** The circle of curvature at a point 'P' of a curve is the circle whose centre is at the centre of Curvature 'C' and whose radius is $'\rho'$ in magnitude

## Problems on Centre of Curvature and circle of curvature

**1) Find the centre of curvature at the point $\left(\frac{a}{4},\frac{a}{4}\right)$ of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$. Find also the equation of the circle of curvature at that point.**

**Sol:** We know that, if $(X,Y)$ are the coordinates of the centre of curvature at any point $P(x,y)$ on the curve $y=f(x)$, then

$$\boxed{(X,Y)=\left(x-\frac{y_1(1+y_1^2)}{y_2}\ ,y+\frac{(1+y_1^2)}{y_2}\right)}$$

Now, given $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiate w.r.t $x$ , we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}} \qquad \ldots \quad (1)$$

Now, $\left(\frac{dy}{dx}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=-1$

Again, Differentiate (1) w.r.t $x$ , we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\left(\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}\frac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}\right)$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{4}{a}$$

Hence, the coordinates of the centre of curvature are

$$(X,Y)=\left(x-\frac{y_1(1+y_1^2)}{y_2}\ ,y+\frac{(1+y_1^2)}{y_2}\right)$$

$$=\left(\frac{a}{4}-\frac{(-1)\,(1+(-1)^2)}{\left({}^4/_a\right)}\ ,\frac{a}{4}+\frac{(1+(-1)^2)}{\left({}^4/_a\right)}\right)=\left(\frac{3a}{4},\frac{3a}{4}\right)$$

Hence, $\boxed{(\boldsymbol{X},\boldsymbol{Y})=\left(\frac{3a}{4},\frac{3a}{4}\right)}$

Now, radius of curvature at the point $\left(\frac{a}{4},\frac{a}{4}\right)$ is given by

$$\rho_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$=\frac{[1+(-1)^2]^{\frac{3}{2}}}{\frac{4}{a}}=\frac{a}{\sqrt{2}}$$

$\Longrightarrow \boxed{\boldsymbol{\rho}_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{\boldsymbol{a}}{\sqrt{2}}}$

Hence, the equation of circle of curvature at the given point $\left(\frac{a}{4},\frac{a}{4}\right)$ is given by

$$(x-X)^2+(y-Y)^2=\rho^2$$

$\Longrightarrow \left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$

*Hence the result.*

## EVOLUTE

Corresponding to each point on a curve we can find the curvature of the curve at that point. Drawing the normal at these points, we can find Centre of Curvature corresponding to each of these points. Since the curvature varies from point to point, centres of curvature also differ. The totality of all such centres of curvature of a given curve will define another curve and this curve is called the evolute of the curve.

The Locus of centres of curvature of a given curve is called the evolute of that curve.

The locus of the centre of curvature $C$ of a variable point $P$ on a curve is called the evolute of the curve. The curve itself is called involute of the evolute.

Here, for different points on the curve, we get different centre of curvatures. The locus of all these centres of curvature is called as Evolute.

The external curve which satisfies all these centres of curvature is called as Evolute. Here Evolute is nothing but an curve equation.

To find Evolute, the following models exist.

1) If an equation of the curve is given and If we are asked to show / prove L.H.S=R.H.S, Then do as follows.

   First find Centre of Curvature $C(X, Y)$, where $X = x - \frac{y_1[1+(y_1)^2]}{y_2}$

   $$Y = y + \frac{[1+(y_1)^2]}{y_2}$$

   And then consider L.H.S: In that directly substitute $X$ in place of $x$ and $Y$ in place of $y$. Similarly for R.H.S. and then show that L.H.S=R.H.S

2) If a curve is given and if we are asked to find the evolute of the given curve, then do as follows:

   First find Centre of curvature $C(X, Y)$ and then re-write as

   $x$ in terms of $X$ and $y$ in terms of $Y$. and then substitute in the given curve, which gives us the required evolute.

3) If a curve is given, which is in parametric form, then first find Centre of curvature, which will be in terms of parameter. then using these values of $X$ and $Y$ eliminate the parameter, which gives us evolute.

## Problem

**1) Find the coordinates of centre of curvature at any point of the parabola $y^2 = 4ax$ and also show its evolute is given by $27ay^2 = 4(x-2a)^2$**

**Sol:** Given curve is $y^2 = 4ax$

We know that, if $(X, Y)$ are the coordinates of the centre of curvature at any point $P(x, y)$ on the curve $y = f(x)$, then

$$\boxed{(X,Y) = \left(x - \frac{y_1(1+y_1^2)}{y_2}\ , y + \frac{(1+y_1^2)}{y_2}\right)}$$

Now, $y^2 = 4ax \Longrightarrow 2y\frac{dy}{dx} = 4a$

$$\Longrightarrow \boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

Also, $\frac{d^2y}{dx^2} = -\frac{2a}{y^2} \cdot \frac{dy}{dx}$

$$= -\frac{2a}{y^2} \cdot \frac{2a}{y} = -\frac{4a^2}{y^3}$$

$$\Longrightarrow \boxed{\frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}}$$

$$\therefore (X,Y) = \left(x - \frac{\left(\frac{2a}{y}\right)\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}\ , y + \frac{\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}\right)$$

Consider, $\boldsymbol{X} = \boldsymbol{x} - \dfrac{\left(\frac{2a}{y}\right)\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}$

$$= x + \frac{y^2}{2a}\left(1 + \frac{a}{x}\right)$$

$$= x + \frac{4ax}{2a}\left(\frac{x+a}{x}\right)$$

$$= x + 2(x+a)$$

$$= 3x + 2a$$

$\Longrightarrow \boxed{\boldsymbol{X = 3x + 2a}}$

Consider, $\boldsymbol{Y = y + \frac{\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}}$

$$= y - \left(1 + \left(\frac{2a}{y}\right)^2\right)\frac{y^3}{4a^2}$$

$$= y - \left(1 + \frac{4a^2}{y^2}\right)\frac{y^3}{4a^2}$$

$$= y - \left(\frac{y^2+4a^2}{y^2}\right)\frac{y^3}{4a^2}$$

$$= y - (x + a)\frac{y}{a} = -\frac{xy}{a}$$

$\Longrightarrow \boxed{\boldsymbol{Y = -\frac{xy}{a}}}$

Now, required to prove is $27aY^2 = 4(X - 2a)^3$

**L.H.S** $\Longrightarrow 27aY^2 = 27a\left(-\frac{xy}{a}\right)^2 = 108x^3$

**R.H.S** $\Longrightarrow 4(X - 2a)^3 = 4(3x + 2a - 2a)^3 = 4(3x)^3 = 108x^3$

Hence, $\mathbf{L.H.S} = \mathbf{R.H.S}$

Hence the Result.



## ENVELOPE

A curve which touches each member of a given family of curves is called envelope of that family.

**<u>Procedure to find envelope for the given family of curves:</u>**

**Case 1:** Envelope of one parameter

Let us consider $y = f(x)$ to be the given family of curves.

**Step 1:** Differentiate w.r.t to the parameter partially, and find the value of the parameter

**Step 2:** By Substituting the value of parameter in the given family of curves, we get required envelope.

**Special Case:** If the given equation of curve is quadratic in terms of parameter, then envelope is given by $discriminant = 0$

**Case 2:** Envelope of two parameter

Let us consider $y = f(x)$ to be the given family of curves, and a relation connecting these two parameters

**Step 1:** Obtain one parameter in terms of other parameter from the given relation

**Step 2:** Substitute in the given equation of curve, so that the problem of two parameter converts to problem of one parameter.

**Step 3:** Use one parameter technique to obtain envelope for the given family of curve

## Problem

**1) Find the envelope of the family of straight line $y = mx + \sqrt{a^2m^2 + b^2}$, $m$ is the parameter.**

**Sol:** Given equation of family of curves is $y = mx + \sqrt{a^2m^2 + b^2}$

$\Rightarrow (y - mx) = \sqrt{a^2m^2 + b^2}$

$\Rightarrow (y - mx)^2 = (a^2m^2 + b^2)$

$\Rightarrow y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$

$\Rightarrow \boxed{\boldsymbol{m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0}}$

**Step 1:** Differentiate partially w.r.t the parameter $(i.e. m)$

$\Rightarrow 2m(x^2 - a^2) - 2xy = 0$

$\Rightarrow \boxed{\boldsymbol{m = \frac{xy}{(x^2 - a^2)}}}$

**Step 2:** Substitute the value of $m$ in the given family of curves

$$\Longrightarrow m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$$

$$\Longrightarrow \left(\frac{xy}{(x^2-a^2)}\right)^2 (x^2 - a^2) - 2\frac{xy}{(x^2-a^2)}\, xy + (y^2 - b^2) = 0$$

$$\Longrightarrow \frac{x^2y^2}{x^2-a^2} - \frac{2x^2y^2}{x^2-a^2} + (y^2 - b^2) = 0$$

$$\Longrightarrow -\frac{x^2y^2}{x^2-a^2} + (y^2 - b^2) = 0$$

$$\Longrightarrow \frac{x^2y^2}{x^2-a^2} = (y^2 - b^2)$$

$$\Longrightarrow x^2y^2 = (x^2 - a^2)(y^2 - b^2)$$

$$\Longrightarrow x^2y^2 = x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2$$

$$\Longrightarrow x^2b^2 + a^2y^2 = a^2b^2$$

$$\Longrightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

∴ The envelope of the given family of straight lines is an ellipse.

**2) Find the envelope of family of straight line $\frac{x}{a} + \frac{y}{b} = 1$, where $a, b$ are two parameters which are connected by the relation $a + b = c$.**

**Sol:** Given equation of family of straight lines is $\boxed{\frac{x}{a} + \frac{y}{b} = 1}$ ... $(1)$

Also given, $a + b = c$

$$\Longrightarrow \boxed{b = c - a} \quad \ldots(2)$$

Substituting $(2)$ in $(1)$, we get $\frac{x}{a} + \frac{y}{c-a} = 1$

**Step 1:** Differentiate w.r.t $a$ partially, we get $-\frac{x}{a^2} + \frac{y}{(c-a)^2} = 0$

$$\Longrightarrow \frac{x}{a^2} = \frac{y}{(c-a)^2}$$

$$\Longrightarrow \frac{(c-a)^2}{a^2} = \frac{y}{x}$$

$$\Longrightarrow \left(\frac{c-a}{a}\right)^2 = \frac{y}{x}$$

$$\Longrightarrow \frac{c-a}{a} = \sqrt{\frac{y}{x}}$$

$$\Longrightarrow \frac{c}{a} - 1 = \sqrt{\frac{y}{x}}$$

$\Rightarrow \frac{c}{a} = 1 + \sqrt{\frac{y}{x}}$

$\Rightarrow \frac{c}{a} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$

$\Rightarrow \boxed{\boldsymbol{a} = \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}}$

Now, substitute the value of $a$ in $b = c - a$

$\Rightarrow b = c - \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$

$= \frac{c\sqrt{x}+c\sqrt{y}-c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$

$\Rightarrow \boxed{\boldsymbol{b} = \frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}}$

**Step 2:** Substitute the values of $a$ & $b$ in the given family of curves $\frac{x}{a} + \frac{y}{b} = 1$, we get

$\frac{x}{\left(\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}\right)} + \frac{y}{\left(\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)} = 1$

$\Rightarrow \frac{x(\sqrt{x}+\sqrt{y})}{c\sqrt{x}} + \frac{y(\sqrt{x}+\sqrt{y})}{c\sqrt{y}} = 1$

$\Rightarrow \frac{x(\sqrt{x}+\sqrt{y})}{\sqrt{x}} + \frac{y(\sqrt{x}+\sqrt{y})}{\sqrt{y}} = c$

$\Rightarrow (\sqrt{x} + \sqrt{y})\left(\frac{x}{\sqrt{x}} + \frac{y}{\sqrt{y}}\right) = c$

$\Rightarrow (\sqrt{x} + \sqrt{y})(\sqrt{x} + \sqrt{y}) = c$

$\Rightarrow (\sqrt{x} + \sqrt{y})^2 = c$

$\Rightarrow (\sqrt{x} + \sqrt{y}) = \sqrt{c}$ is the required envelope

# CURVE TRACING

Drawing a rough sketch of the curve is called as Curve Tracing.

**Aim:** To find the appropriate shape of a curve for the given equation.

## METHOD OF TRACING CURVES

***Cartesian Co-ordinates:*** In order to obtain general shape of the curve from the given equations, we have to examine the following properties.

### 1) SYMMETRY

a) If the equation contains even powers of $y$ only, the curve is symmetrical about $x - axis$.

**Example:** $y^2 = 4ax, \; y^4 = x, \; xy^2 = 4(2-x)$

b) If the equation contains even powers of $x$ only, the curve is symmetrical about $y - axis$.

**Example:** $x^2 = 4ay, x^4 = y, \; y = x^{100} + x^2 + 7$

c) If all the powers of $x$ and $y$ in the given equation are even, the curve is symmetrical about the both the axes. i.e. about the origin.

**Example:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x^2 + y^2 = a^2, x^2y^2 = a^2(y^2 - x^2)$

d) If the equations of the curve is not changed by interchanging $x$ and $y$, then the curve is symmetrical about the line $y = x$.

**Example**: $xy = c^2, x^3 + y^3 = 3axy$.

e) If the equation of a curve remains unchanged when both $x$ and $y$ are interchanged by $-x$ and $-y$ respectively, then the curve is symmetrical in opposite-quadrants.

**Examples**: $xy = c^2, x^3 + y^3 = 3ax, y = x^3$.

## 2) ORIGIN

If the equation of a curve is satisfied by $x = 0, y = 0$ then the curve passes through the origin.

**Example:** $y^2 = 4ax, x^3 + y^3 = 3axy$.

## 3) INTERSECTION WITH CO-ORDINATE AXIS

Put $x = 0$ in the given equation to get points of intersection with $y - axis$.

Put $y = 0$ in the given equation to get points of intersection with $x - axis$.

## 4) REGION

If possible write the given equation in the form of $y = f(x)$. Give values of $x$ to make $y$ imaginary. Let $y$ be imaginary for the values of $x$ lying between $x = a$ and $x = b$. Then, no part of the curve lies between $x = a$ and $x = b$. Similarly, the curve does not lies between those values of $y$ for which $x$ is imaginary.



## 5) TANGENTS

a) If the curve passes through the origin, the tangents at the origin are given by equating the lowest degree terms to zero.

**Example**: For the curve $x^3 + y^3 = 3axy$, the tangents are given by equating the lowest degree terms to zero. i.e. $3axy = 0 \Rightarrow x = 0, y = 0$ are tangents at origin.

b) If the curve is not passing through the origin, the tangents at any point are given by finding $\frac{dy}{dx}$ at that point and this indicates the direction of the tangent at that point.

**Note:** If there are two tangents at the origin, the origin is a double point.

***A)*** If the two tangents are real and coincident, the origin is a **cusp**.

***B)*** If the two tangents are real and different, the origin is a **node**.

***C)*** If the two tangents are Imaginary, the origin is a conjugate point or Isolated point.

## 6) EXTENSION OF THE CURVE TO INFINITY

Give values to $x$ for which the value of $y$ is Infinity and also give values to $y$ for which $x$ is Infinity. These values indicate the direction in which the curve extends to Infinity.

## 7) ASYMPTOTES

An Asymptote is a straight line which cuts a curve in two points, at an infinite distance from the origin and yet is not itself wholly at Infinity.

***To find Asymptotes***

***a) Asymptotes parallel to axis***

$\boldsymbol{x-axis}$: Asymptotes parallel to $x$-axis are obtained by equating the coefficient of the highest power of $x$ to zero.

$\boldsymbol{y-axis}$: Asymptotes parallel to $y$-axis are obtained by equating the coefficient of the highest power of $y$ to zero.

**b) Oblique Asymptotes:** ( Asymptotes which are not parallel to axis)

Let $y = mx + c$ be an asymptote. Put $y = mx + c$ in the given equation of the curve. Equate the coefficients of highest powers of $x$ to zero and solve for $m$ and $c$.

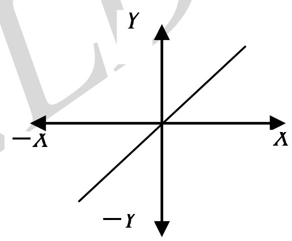
## Problems on Curve Tracing - Cartesian Coordinates

### 1) Trace the curve $x^3 + y^3 = 3axy$

**Sol:** In order to trace a curve, we need to check the following properties

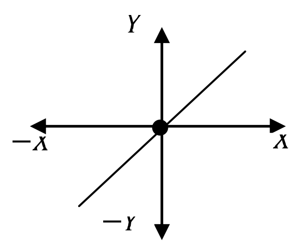
- ✓ Symmetry
- ✓ Origin
- ✓ Intersection with coordinate axis
- ✓ Region
- ✓ Tangents
- ✓ Intersection of curve to ∞
- ✓ Asymptotes

Let us consider given equation of curve to be $\boldsymbol{F(x,y) = x^3 + y^3 = 3axy}$



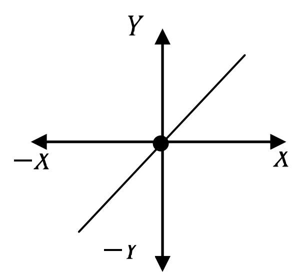

**Symmetry:** If we interchange $x$ and $y$, the equation of curve is not changing.

Hence, the curve is symmetric about the line $y = x$




**Origin:** If we substitute $x = 0$ and $y = 0$, the equation of curve is satisfied.

Hence, we can say that the curve passes through the origin. $i.e. (0,0)$




**Intersection with coordinate axis:**

Put $y = 0 \Longrightarrow x = 0$

$\therefore$ The curve meets the $X$ −axis only at the Origin

Again $x = 0 \Longrightarrow y = 0$

$\therefore$ The curve meets the $Y$ −axis only at the Origin

**Region:** Consider $x^3 + y^3 = 3axy$

- $x$ is positive $\Longrightarrow$ $y$ is also positive

  $\therefore$ The curve lies in 1st Quadrant
- $x$ is negative $\Longrightarrow$ $y$ is positive

  $\therefore$ The curve lies in 2nd Quadrant
- $x$ is negative & $y$ is negative

Then the equation of curve is not satisfied

∴ The curve does not lie in 3rd Quadrant

- $x$ is positive $\Longrightarrow y$ is negative

∴ The curve lies in 4th Quadrant

**Tangents:** The given equation of curve $\boldsymbol{F(x, y) = x^3 + y^3 - 3axy = 0}$ passes through Origin.

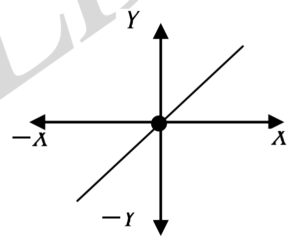
Hence, the tangents at origin are given by equating lowest degree terms to zero.

i.e., $\boldsymbol{3axy = 0 \Longrightarrow xy = 0}$

$\Longrightarrow \boldsymbol{x = 0, y = 0}$

∴ There exists two tangents namely, $\boldsymbol{x = 0, y = 0}$

Here, the two tangents at origin are real and distinct. Hence, we get a node




Now, let us check, at what points the curve meets the line $\boldsymbol{y = x}$

$\Longrightarrow x^3 + x^3 - 3ax^2 = 0$

$\Longrightarrow x^2(2x - 3a) = 0$

$\therefore x = 0, x = \frac{3a}{2}$

$\Longrightarrow y = 0, y = \frac{3a}{2}$

Hence, the curve meets at $\boldsymbol{(0, 0)}$ , $\boldsymbol{\left(\frac{3a}{2}, \frac{3a}{2}\right)}$

**Extension of the curve to Infinity:**

As $y \longrightarrow \infty \Longrightarrow \boldsymbol{x} \longrightarrow -\infty$

∴ The curve extends to infinity in the second quadrant.

Also, as $x \longrightarrow \infty \Longrightarrow \boldsymbol{y} \longrightarrow -\infty$

The curve extends to infinity in the fourth quadrant.

**Asymptotes:**

**Asymptotes parallel to axis:**

No Asymptote is parallel to $X - axis$

No Asymptote is parallel to $Y - axis$

**Asymptotes not parallel to axis (Oblique Asymptotes)**

Let us consider $y = mx + c$ to be the required asymptote.

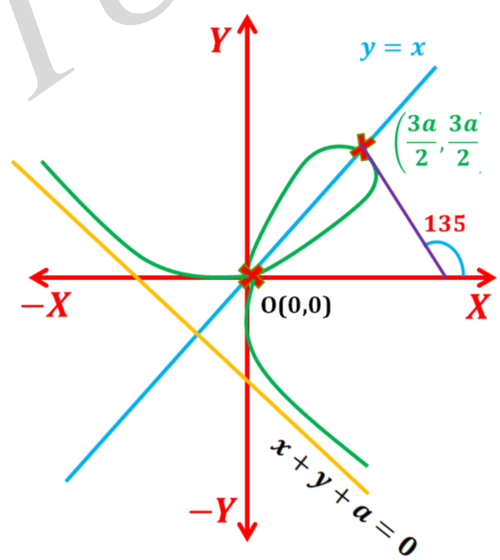
Now, substitute in the given equation of the curve and solving for $m$ and $c$, we get

$$m = -1, c = -a$$

Hence, the required asymptote is $x + y + a = 0$

Hence the shape of the curve is as follows





## PROCEDURE FOR TRACING CURVES IN POLAR CO-ORDINATES

### 1) SYMMETRY

i) If the equation does not alter by changing $\theta$ to $-\theta$, the curve is symmetrical about the initial line or $X$-axis.

**Example:** $r = a(1 + \cos\theta)$

ii) If the equation does not alter by changing $r$ to $-r$, the curve is symmetrical about the pole.

**Example:** $r^2 = a^2 \cos 2\theta$

iii) If the equation of the curve remains unaltered when $\theta$ is changed to $\pi - \theta$ or by changing $\theta$ to $-\theta$, $r$ to $-r$, the curve is symmetric about the line $\theta = \frac{\pi}{2}$ or $Y$-axis.

**Example:** $r = a \sin 3\theta$

iv) If the equation of the curve remains unaltered when $\theta$ is changed to $\frac{\pi}{2} - \theta$, then the curve is symmetrical about the line $\theta = \frac{\pi}{4}$ $or\ y = x$.

**Example:** $r = a \sin 2\theta$

v) If the equation of the curve does not alter by changing $\theta$ to $\frac{3\pi}{2} - \theta$, then the curve is symmetrical about the line $\theta = \frac{3\pi}{4}$ $or\ y = -x$.

**Example:** $r = a \sin 2\theta$.

### 2) Discussion for r and θ

Give certain values to $\theta$ and find the corresponding values of $r$ and then plot the points. Sometimes it is inconvenient to find the corresponding values of $r$ for certain values of $\theta$. In such cases, a particular region for $\theta$ may be considered and find out whether $r$ increases or decreases in that region.

***For example:*** It is inconvenient to find the value of $r$ for $\theta = 32^o$ but it is equal to know whether $r$ increases or decreases in the region from 0 to $45^0$ in which $\theta = 32^o$ value is also included.

### *3) Region*

No part of the curve exists for those values of $\theta$ which make corresponding value of $r$ imaginary.

### ***4) Tangents***

Find $tan\,\emptyset = r\,\frac{d\theta}{dr}$ , where $\emptyset$ is the angle between the radius vector $OP$ and the tangent at $P\,(r,\theta)$. It will indicate the direction of the tangents at any point $P\,(r,\theta)$.

### **5) Asymptotes**

Find the value of $\theta$ which makes $r$ infinity. The curve has an asymptote in that direction.

* * *

**MATHEMATICS-I**

# APPLICATIONS OF INTEGRATION

I YEAR B.Tech



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## UNIT-4
## APPLICATIONS OF INTEGRATION

### Riemann Integrals:

Let us consider an interval $[a, b]$ with $a < b$

If $= x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, then a finite set $\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ is called as a partition of $[a, b]$ and it is denoted by $P$.

The sub intervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ are called $n-$ segments (or) $n-$ sub intervals.

The $r^{th}$ sub interval in this process is $I_r = [x_{r-1}, x_r]$ and its length is given by $\delta_r = x_r - x_{r-1}$

**Note:** For every interval $[a, b]$, it is possible to define infinitely many partitions.

**Norm (or) Mesh of the partition:** The maximum of the lengths of the sub intervals w.r.t the partition $P$ is called as Norm of the partition $P$ (or) Mesh of the partition $P$ and it is denoted by $\|P\|$ $or\ \mu_P$

**Refinement:** If $P$ and $P$ are two partitions of $[a, b]$ and if $\subset P$, then $P$ is called as Refinement of $P$.

### Lower and Upper Reimann Sum's

Let $f: [a, b] \longrightarrow \mathbb{R}$ is bounded and $P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ be a partition on $[a, b]$, then $r^{th}$ sub interval is given by $I_r = [x_{r-1}, x_r]$ and its length is given by $\delta_r = x_r - x_{r-1}$

If $f$ is bounded on $[a, b]$, then $f$ is bounded on $I_r$

let $m_r$ and $M_r$ be Infimum and supremum of $f$ on $I_r$, then

- The sum $\sum_{r=1}^{n} m_r\, \delta_r$ is called as lower Reimann sum and it is denoted by $L(P, f)$
- The sum $\sum_{r=1}^{n} M_r\, \delta_r$ is called as upper Reimann sum and it is denoted by $U(P, f)$

**Note:** Always, $U(P, f) \geq L(P, f)$

## Problem

**1) If $f(x) = x \ \forall\ x \in [0,1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of $[0,1]$ then compute $U(P,f), L(P,f)$**

**Sol:** Given $f(x) = x$ defined on $[0,1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of $[0,1]$

Here, let $I_1 = \left[0, \frac{1}{3}\right]$ , $I_2 = \left[\frac{1}{3}, \frac{2}{3}\right]$ , $I_3 = \left[\frac{2}{3}, 1\right]$

And, $\delta_1 = \frac{1}{3} - 0 = \frac{1}{3}$ , $\delta_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ , $\delta_3 = 1 - \frac{2}{3} = \frac{1}{3}$

Let $m_r$ and $M_r$ be Infimum and supremum of $f$ on $I_r$ , then

$$m_1 = 0 \, , M_1 = \frac{1}{3}$$

$$m_2 = \frac{1}{3} \, , M_2 = \frac{2}{3}$$

$$m_3 = \frac{2}{3} \, , M_2 = 1$$

Hence, $U(P,f) = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 = \frac{2}{3}$

Also, $L(P,f) = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 = \frac{1}{3}$

**Lower Reimann Integral:** Let $f: [a,b] \longrightarrow \mathbb{R}$ be a bounded function and $P$ is a partition of $[a,b]$, then supremum of $\{L(P,f) | P \in \phi[a,b]\}$ is called as Lower Reimann integral on $[a,b]$ and it is denoted by $\underline{\int_a^b} f(x)\,dx$

**Upper Reimann Integral:** Let $f: [a,b] \longrightarrow \mathbb{R}$ be a bounded function and $P$ is a partition of $[a,b]$, then Infimum of $\{U(P,f) | P \in \phi[a,b]\}$ is called as Upper Reimann integral on $[a,b]$ and it is denoted by $\overline{\int_a^b} f(x)\,dx$

## Riemann Integral

If $f: [a,b] \longrightarrow \mathbb{R}$ be a bounded function and $P$ is a partition of $[a,b]$ and if $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ , then $f$ is said to be Riemann integrable on $[a,b]$ and it is denoted by $\int_a^b f(x)\ dx$

**Rectification:** The process of finding the length of the arc of the curve is called as Rectification

## Length of the arc of the curve

| Equation of the curve | Arc Length |
|---|---|
| **Cartesian Form**<br>(i) $y = f(x)$ and $x = a$ and $x = b$<br>(ii) $x = f(y)$ and $y = a$ and $y = b$ | $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$<br>$s = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ |
| **Parametric Form**<br>$x = x(\theta), y = y(\theta)$ and $\theta = \theta_1$ and $\theta = \theta_2$ | $s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ |
| **Polar Form**<br>(i) $r = f(\theta)$ and $\theta = \alpha$ and $\theta = \beta$<br>(ii) $\theta = f(r)$ and $r = r_1$ and $r = r_2$ | $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$<br>$s = \int_{r_1}^{r_2} \sqrt{1 + \left(\frac{d\theta}{dr}\right)^2}\, dr$ |



## Problems on length of the arc of the curve

**1) Find the length of the arc of the curve $y = \log\left(\frac{e^x-1}{e^x+1}\right)$ from $x = 1$ to $x = 2$.**

**Solution:** We know that, the equation of the length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad .... \ (1)$$

Given $y = \log\left(\frac{e^x-1}{e^x+1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{e^x-1}{e^x+1}\right)} \frac{d}{dx}\left(\frac{e^x-1}{e^x+1}\right)$$

$$= \frac{2e^x}{e^{2x}-1}$$

$\therefore$ The required length of the arc of the curve is given by $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$\Rightarrow S = \int_1^2 \sqrt{1+\left(\frac{2e^x}{e^{2x}-1}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx$$

$$= \int_1^2 \frac{e^{2x}+1}{e^{2x}-1} dx$$

$$= \int_1^2 \frac{e^x(e^x+e^{-x})}{e^x(e^x-e^{-x})} dx$$

$$= \int_1^2 \frac{(e^x+e^{-x})}{(e^x-e^{-x})} dx \quad = [\log(e^x-e^{-x})]_1^2$$

$$= \left[\log\left(\frac{e^{2x}-1}{e^x}\right)\right]_1^2$$

$$= [\log(e^{2x}-1)-\log e^x]_1^2$$

$$= \log(e^4-1)-\log e^2-\log(e^2-1)+\log e^1$$

$$= \log\left(\frac{e^4-1}{e^2-1}\right)-\log\left(\frac{e^2}{e}\right) = \log(e^2+1)-\log e$$

$$= \log\left(\frac{e^2+1}{e}\right) = \log\left(e+\frac{1}{e}\right)$$

**2) Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$**

**Solution:** We know that, the equation of the length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad .... \quad (1)$$

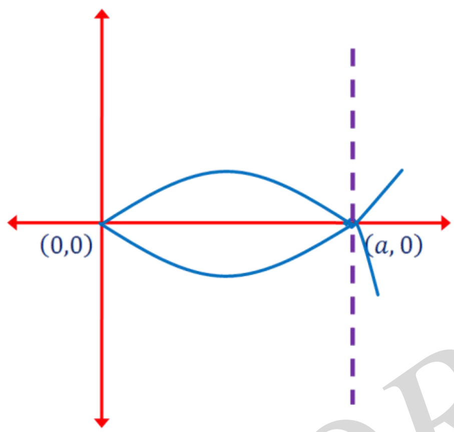
Given $3ay^2 = x(x-a)^2$

$$\Rightarrow y = \frac{1}{\sqrt{3a}}\sqrt{x}\,(x-a)$$

$$= \frac{1}{\sqrt{3a}}\left(x^{\frac{3}{2}} - ax^{\frac{1}{2}}\right)$$

Now, $\frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left(\frac{3}{2}x^{\frac{1}{2}} - a\frac{1}{2}x^{-\frac{1}{2}}\right)$

$$= \frac{1}{\sqrt{3a}}\left(\frac{3}{2}\sqrt{x} - \frac{a}{2\sqrt{x}}\right) = \frac{1}{\sqrt{3a}}\left(\frac{3x-a}{2\sqrt{x}}\right)$$




Here the curve is symmetrical about the $X-axis$. Hence the length of the arc will be double that of the arc of the loop about the $X-axis$.



∴ The required length of the loop is $=2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\int_0^a \sqrt{1+\frac{(3x-a)^2}{12ax}}\,dx$$

$$= \frac{2}{\sqrt{12a}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx$$

$$= \frac{2}{\sqrt{12a}}\int_0^a \left(\frac{3x}{\sqrt{x}}+\frac{a}{\sqrt{x}}\right)dx$$

$$= \frac{2}{\sqrt{12a}}\int_0^a \left(3\sqrt{x}+\frac{a}{\sqrt{x}}\right)dx$$

$$= \frac{2}{\sqrt{12a}}\left[3.\frac{2}{3}x^{\frac{3}{2}} + a.2\sqrt{x}\right]_0^a$$

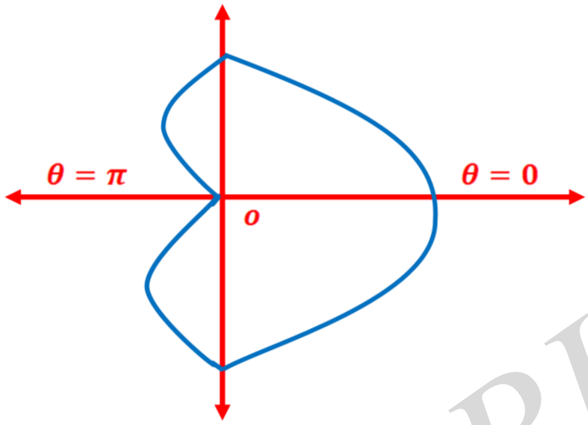
$$= \frac{2}{\sqrt{3a}}\left[x^{\frac{3}{2}} + a\sqrt{x}\right]_0^a$$

$$= \frac{4a}{\sqrt{3}} \text{ units}$$

**3) Find the perimeter of the cardioids $r = a(1 + \cos\theta)$.**

**Solution:** We know that the length of the arc of the curve $r = f(\theta)$ and $\theta = \alpha, \theta = \beta$ is given by

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$




Given $r = a(1 + \cos\theta)$

$$\Longrightarrow \frac{dr}{d\theta} = -a\sin\theta$$

The cardioids is symmetrical about the initial line and passes through the pole.

Hence the length of the arc will be double that of the arc of the loop about the *pole*.

$\therefore$ The required length of the loop is $=2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2 sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2a^2(1+\cos\theta)}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{4a^2 cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 8a$$

**4) Find the perimeter of the curve $x^2 + y^2 = r^2$.**

**Solution**: Given equation of the curve is $x^2 + y^2 = r^2$ ... (1)

The given curve is an equation of circle with radius $'a'$.

We know that the length of the arc of the curve $y = f(x)$ between the absissae $x = a$ and $x = b$ is given by

$$S = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now, let us find the length of the arc AB of the given curve.

Differentiating (1) w.r.t $'x'$, we get

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Now, the length of the curve $AB$ is

$$S = \int_{x=0}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Rightarrow S = \int_{x=0}^{a} \sqrt{1 + \left(-\frac{x}{y}\right)^2}\, dx$$

$$\Rightarrow S = \int_{x=0}^{a} \sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= \int_{x=0}^{a} \sqrt{\frac{y^2 + x^2}{y^2}}\, dx$$

$$= \int_{x=0}^{a} \sqrt{\frac{a^2}{y^2}}\, dx \quad = a \int_{x=0}^{a} \frac{1}{y}\, dx$$

$$= a \int_{x=0}^{a} \frac{1}{\sqrt{a^2 - x^2}}\, dx$$

$$= a \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a}$$

$$= \frac{\pi a}{2}$$

$\therefore$ Required Perimeter of the sphere $= 4 \times S = 4 \times \frac{\pi a}{2} = 2\pi a$

## Volume of solid of Revolution

| Region (R) | Axis | Volume of the solid generated |
|---|---|---|
| **Cartesian form** | | |
| (i) $y = f(x)$, the $x - axis$ and the lines $x = a$and $x = b$ | $x - axis$ | $V = \pi \int_a^b y^2\, dx$ |
| (ii) $x = g(y)$, the $y - axis$ and the lines $y = c$ and $y = d$ | $y - axis$ | $V = \pi \int_c^d x^2\, dy$ |
| (iii) $y = y_1(x), y = y_2(x), the\ x - axis\ and\ the\ ordinates\ x = a, x = b$ | $x - axis$ | $V = \pi \int_a^b (y_2^2 - y_1^2)\, dx$ |
| (iv) $x = x_1(y), x = x_2(y), the\ y - axis\ and\ the\ ordinates\ y = a, y = b$ | $y - axis$ | $V = \pi \int_a^b (x_2^2 - x_1^2)\, dy$ |
| **Parametric form** | | |
| (i) $x = \phi(t), y = \psi(t),$ $the\ ordinates\ t = t_1, t = t_2$ | $x - axis$ | $V = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt}\, dt$ |
| (ii) $x = \phi(t), y = \psi(t),$ $the\ abscissae\ t = t_1, t = t_2$ | $y - axis$ | $V = \pi \int_{t_1}^{t_2} x^2 \frac{dy}{dt}\, dt$ |
| **Polar form** | | |
| (i) $r = f(\theta), the\ initial\ line\ \theta = 0\ and\ the\ radii\ vectors\ \theta = \alpha, \theta = \beta$ | The initial line $\theta = 0$ | $V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \sin\theta\, d\theta$ |
| (ii) $r = f(\theta), the\ line\ \theta = \frac{\pi}{2}\ \perp^r\ to\ the\ initial\ line\ and\ the\ ra$ $\alpha, \theta = \beta$ | The line $\theta = \frac{\pi}{2}$ | $V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \cos\theta\, d\theta$ |
| (iii) $r = f(\theta), the\ initial\ line\ \theta = r\ and\ the\ radii\ vectors\ \theta = \alpha, \theta = \beta$ | The line $\theta = r$ | $V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \sin(\theta - r)\, d\theta$ |



## Problems on Volume of solid of Revolution

**1) Find the volume of the solid that result when the region enclosed by the curve $y = x^3$, $y = 0$, $x = 1$ is revolved about the $y - axis$.**

**Sol:** We know that the volume of the solid generated by the revolution of the area bounded by the curve $x = f(y)$ , the $y - axis$ and the lines $y = a, y = b$ is given by $V = \pi \int_a^b x^2\, dy$

Now, given curve $y = x^3$

∴ Required volume is given by $V = \pi \int_0^1 x^{\frac{2}{3}}\, dy$

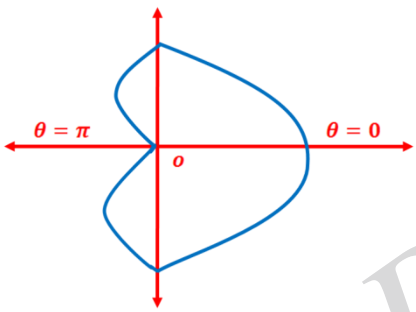
$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^1 = \frac{3\pi}{5}$$

**2) Find the volume of the solid generated by the revolution of the cardioids $r = a(1 + \cos\theta)$ about the initial line.**

**Sol:** We know that the volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$, the initial line and $\theta = \alpha, \theta = \beta$ is given by

$$V = \frac{2}{3}\pi \int_0^{\pi} r^3 \sin\theta \; d\theta \quad \dots(1)$$

Here, the given cardioids is symmetrical about the initial line. The upper half of the curve formed when $\theta$ varies from $0$ to $\pi$.




$$V = \frac{2}{3}\pi \int_0^{\pi} r^3 \sin\theta \; d\theta$$

$$= \frac{2}{3}\pi \int_0^{\pi} a^3(1 + \cos\theta)^3 \sin\theta \; d\theta$$

$$= -\frac{2}{3}\pi \int_0^{\pi} a^3(1 + \cos\theta)^3 (-\sin\theta) \; d\theta$$

$$= -\frac{2}{3}\pi a^3 \left[\frac{(1 + \cos\theta)^4}{4}\right]_0^{\pi}$$

$$= \frac{8\pi a^3}{3} \text{ cu. units}$$

## Surface area of solid of Revolution

| Equation of the curve | Arc Length |
|---|---|
| **Cartesian Form** (i) $y = f(x)$ and $x = a$ and $x = b$, $X - axis$ | $s = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ |
| (ii) $x = f(y)$ and $y = a$ and $y = b$, $Y - axis$ | $s = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ |
| **Parametric Form** (i) $x = x(\theta), y = y(\theta)$ and $\theta = \theta_1$ and $\theta = \theta_2$ $X - axis$ | $s = 2\pi \int_{\theta_1}^{\theta_2} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ |
| (ii) $x = x(\theta), y = y(\theta)$ and $\theta = \theta_1$ and $\theta = \theta_2$ $Y - axis$ | $s = 2\pi \int_{\theta_1}^{\theta_2} x \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ |
| **Polar Form** (i) $r = f(\theta)$ and $\theta = \alpha$ and $\theta = \beta$ | $s = 2\pi \int_\alpha^\beta r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ |
| (ii) $\theta = f(r)$ and $r = r_1$ and $r = r_2$ | $s = 2\pi \int_\alpha^\beta r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ |



## Problem

**1) Find the surface area generated by the revolution of an arc of the catenary $y = c\cosh\frac{x}{c}$ about the $x - axis$.**

**Sol:** We know that the surface area of the solid generated by the revolution of an arc $y = f(x)$ about the $-axis$ , $x = a, x = b$ is given by

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^c c\cosh\frac{x}{c} \sqrt{1 + \sinh^2\frac{x}{c}}\, dx$$

$$= 2\pi c \int_0^c \cosh^2 \frac{x}{c}\, dx$$

$$= 2\pi c \int_0^c \frac{1 + \cosh \frac{2x}{c}}{2}\, dx$$

$$= \pi c \int_0^c \left(1 + \cosh \frac{2x}{c}\right) dx$$

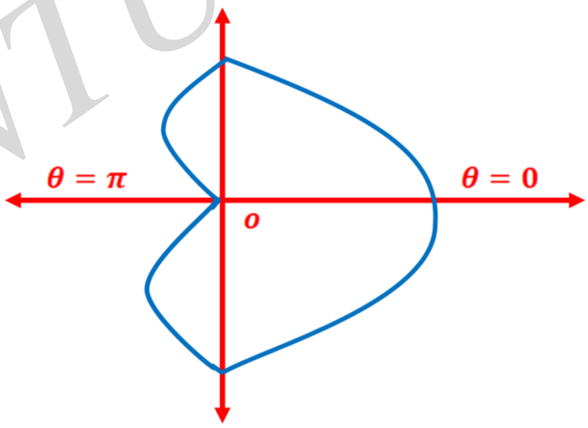
$$= \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]_0^c$$

$$= \pi c \left[c + \frac{c}{2} \sinh 2\right] = \pi c^2 \left(1 + \frac{1}{2} \sinh 2\right)$$

**2) Find the surface area of the solid formed by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.**

**Sol:** We know that the surface area of the solid formed by revolving the cardioid $r = f(\theta)$, the initial line and $\theta = \alpha, \theta = \beta$ is given by

$$S = 2\pi \int_\alpha^\beta r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$




Given $r = a(1 + \cos\theta)$

$$\Rightarrow \frac{dr}{d\theta} = -a \sin\theta$$

The cardioid is symmetrical about the initial line and passes through the pole.

Hence, required surface area is given by

$$= 2\pi \int_0^\pi r \sin\theta \sqrt{a^2(1 + \cos\theta)^2 + a^2 sin^2\theta}\, d\theta$$

$$= 2\pi \int_0^{\pi} a(1+\cos\theta)\sin\theta\,\sqrt{2a^2(1+\cos\theta)}\;d\theta$$

$$= 2\sqrt{2}\pi a^2 \int_0^{\pi} (1+\cos\theta)^{\frac{3}{2}}\,\sin\theta\;d\theta$$

Let $(1+\cos\theta) = t$

$\Longrightarrow -\sin\theta\;d\theta = dt$

Lower Limit: $\theta = 0 \Longrightarrow t = 2$

Upper Limit: $\theta = \pi \Longrightarrow t = 0$

$\therefore$ Surface area $= 2\sqrt{2}\pi a^2 \int_{t=2}^{0} t^{\frac{3}{2}}\,(-dt)$

$$= 2\sqrt{2}\pi a^2 \int_0^{2} t^{\frac{3}{2}}\;dt$$

$$= \frac{32}{5}\pi a^2$$

## Change of variables in Double Integral

**Problem:Evaluate the following integral by transforming into polar coordinates.**

$$\boldsymbol{I = \int_0^{a}\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\;dx\,dy}$$

**Solution: Clearly, given coordinates are in Cartesian.**

Now, let us consider given integral to be $I = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\;dx\,dy$

$$\Longrightarrow I = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\;dy\,dx$$

$\Longrightarrow y = 0$ to $y = \sqrt{a^2-x^2}$ and $x = 0$ to $x = a$

$\Longrightarrow y = 0$ to $x^2+y^2 = a^2$

The given region is a circle with centre $O(0,0)$ and with radius $a$

Now, in order to change to polar coordinates, let us substitute

$x = r\,cos\theta$

$y = r\,sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$The\ limits\ for\ r: 0\ to\ a\ and\ for\ \theta: 0\ to\ \dfrac{\pi}{2}$

$$\therefore\ \ I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} (r\ sin\theta)\,.\,r.\,(r\ dr\ d\theta)$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^3\ sin\theta\ dr\ d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left\{ \int_{r=0}^{a} r^3\ dr \right\} sin\theta\ \ d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{r=0}^{a}\ sin\theta\ \ d\theta$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\frac{\pi}{2}} sin\theta\ \ d\theta$$

$$= \frac{a^4}{4}\ [-cos\theta]_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{a^4}{4}$$

$$\therefore\ \ I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\ dx\ dy = \frac{a^4}{4}$$

**2) By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$.**

**Solution:** In order to change coordinates Cartesian to Polar, let us substitute

$$\begin{aligned} x &= r\ cos\theta \\ y &= r\ sin\theta \end{aligned} \quad dx\ dy = r\ dr\ d\theta$$

The limits are $r \longrightarrow a$ to $b$ $\ and\ \ \theta \longrightarrow 0$ to $2\pi$

$$\therefore \text{ Let } I = \int\int \frac{x^2y^2}{x^2+y^2}\ dx\ dy$$

$$\Longrightarrow I = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{(r\ cos\ \theta)^2 (r\ sin\theta)^2}{(r\ cos\ \theta)^2 + (r\ sin\theta)^2}\ \ r\ dr\ d\theta$$

$$\Rightarrow I = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{r^4 \cos^2\theta \ \sin^2\theta}{r^2} \ r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=a}^{b} r^3 \cos^2\theta \ \sin^2\theta \ dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[ \int_{r=a}^{b} r^3 dr \right] \cos^2\theta \ \sin^2\theta \ d\theta$$

$$= \frac{b^4 - a^4}{4} \int_{0}^{2\pi} \cos^2\theta \ \sin^2\theta \ d\theta$$

$$= \frac{b^4 - a^4}{4.4} \int_{0}^{2\pi} (2\sin\theta\cos\theta)^2 \ d\theta$$

$$= \frac{b^4 - a^4}{16} \int_{0}^{2\pi} (\sin 2\theta)^2 \ d\theta$$

$$= \frac{b^4 - a^4}{16} \int_{0}^{2\pi} \left( \frac{1 - \cos 4\theta}{2} \right) d\theta$$

$$\Rightarrow I = \frac{b^4 - a^4}{32} \int_{0}^{2\pi} (1 - \cos 4\theta) \ d\theta$$

$$= \frac{b^4 - a^4}{32} \left[ \theta - \frac{\sin 4\theta}{4} \right]_{\theta=0}^{2\pi}$$

$$= \frac{\pi}{32}(b^4 - a^4)$$

$$\therefore \ I = \int\int \frac{x^2y^2}{x^2 + y^2} \ dx\, dy = \frac{\pi}{32}(b^4 - a^4)$$

## Change of Order of Integration

**Problem 1: Change the order of integration and evaluate**

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\ dx$$

**Solution**: Let us consider

$$I=\int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\ dx$$

Hence, given limits are $x=0$ to $x=4a$ and $y=\dfrac{x^2}{4a}$ to $y=2\sqrt{ax}$

Now, we have to convert $y$ limits interms of

$constants\ and\ x\ limits\ in\ terms\ of\ y.$

Given $\displaystyle\int_{x\to const}\int_{y\to(x)} dy\,dx\,.$ To find $\displaystyle\int_{y\to const}\int_{x\to(y)} dx\,dy$

$$x=0\ to\ x=4a\ \ and\ y=\frac{x^2}{4a}\ to\ y=2\sqrt{ax}$$

$$\Longrightarrow\ x^2=4ay\ \text{ to }\ y^2=4ax$$

$$\Longrightarrow x=2\sqrt{ay}\ \text{ to }\ x=\frac{y^2}{4a}$$

Also, if $x=0\Longrightarrow y=0$

if $x=4a\Longrightarrow y=4a$

$$\therefore I=\int_{y=0}^{4a}\int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dy\ dx$$

$$=\int_{y=0}^{4a}\left[\int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dx\right]dy$$

$$=\int_{y=0}^{4a}[x]_{x=2\sqrt{ay}}^{y^2/4a}\ dy=\frac{16a^2}{3}$$

**2) Change the order of integration and evaluate**

$$\int_{0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)dx\ dy$$

**Solution**: Let us consider

$$I=\int_{x=0}^{a}\int_{y=\frac{x}{a}}^{\sqrt{x/a}} (x^2+y^2)dy\ dx$$

Hence, given limits are $x=0$ to $x=a$ and $y=\frac{x}{a}$ to $y=\sqrt{x/a}$

$\Longrightarrow\ x=ay$ to $y^2=\frac{x}{a}$

$\Longrightarrow x=ay$ to $x=ay^2$

Also, if $x=0 \Longrightarrow y=0$

if $x=a \Longrightarrow y=1$

$$\therefore I=\int_{y=0}^{1}\int_{x=ay^2}^{ay} (x^2+y^2)dy\ dx$$

$$=\int_{y=0}^{4a}\left[\int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dx\right]dy$$

$$=\frac{a^3}{28}+\frac{a}{20}$$

**3) Change the order of integration and evaluate $\int_0^1\int_{x^2}^{2-x} xy\ dx\ dy$ and hence evaluate the integral.**

**Solution**: Let us consider

$$I=\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\ dx$$

Hence, given limits are $x=0$ to $x=1$ *and* $y=x^2$ to $y=2-x$

$\Longrightarrow\ x=ay$ to $y^2=\frac{x}{a}$

$\Longrightarrow x=ay$ to $x=ay^2$

Also, if $x = 0 \Longrightarrow y = 0$

if $x = a \Longrightarrow y = 1$

$$\therefore I = \int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2 + y^2) dy\ dx = \int_{y=0}^{4a} \left[ \int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dx \right] O(0,1)$$

$$\iint_R xy\, dx\ dy = \iint_{R_1} xy\, dx\ dy + \iint_{R_2} xy\, dx\, dy$$

Now, $x^2 = 2 - x \Longrightarrow x^2 + x - 2 = 0$

$\Longrightarrow x = -2, x = 1$

$$\therefore I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\, dx\ dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\, dx\ dy$$

$$= \frac{3}{8}$$

**4) Change the order of integration and evaluate**

$$\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$$

**Sol**: Let us consider $I = \int_{y=0}^{b} \int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$

Given limits are $x = 0, x = \frac{a}{b}\sqrt{b^2 - y^2}$ and $y = 0, y = b$

$\Longrightarrow x = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Now, by changing the order of integration, we get

Here, $x$ limits are in terms of $y$ / $y$ limits are constants $\Longrightarrow$ $y$ limits should be in terms of $x$ / $x$ limits should be constants

$Now, y = 0, y = \frac{b}{a}\sqrt{a^2 - x^2}$ and $x = 0, x = a$

$$\text{Now, } I = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dx\, dy = \int_{x=0}^{a} \left[ \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dy \right] dx$$

$$= \int_{x=0}^{a} x \left[\frac{y^2}{2}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{a} x \frac{b^2}{a^2}(a^2 - x^2) dx$$

$$= \frac{b^2}{2a^2} \int_{x=0}^{a} (xa^2 - x^3)\, dx = \frac{b^2a^2}{8}$$

* * *

# MATHEMATICS-I

# DIFFERENTIAL EQUATIONS-I

I YEAR B.TECH



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# SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

| Name of the Unit | Name of the Topic |
|---|---|
| Unit-I<br>Sequences and Series | 1.1 Basic definition of sequences and series<br>1.2 Convergence and divergence.<br>1.3 Ratio test<br>1.4 Comparison test<br>1.5 Integral test<br>1.6 Cauchy's root test<br>1.7 Raabe's test<br>1.8 Absolute and conditional convergence |
| Unit-II<br>Functions of single variable | 2.1 Rolle's theorem<br>2.2 Lagrange's Mean value theorem<br>2.3 Cauchy's Mean value theorem<br>2.4 Generalized mean value theorems<br>2.5 Functions of several variables<br>2.6 Functional dependence, Jacobian<br>2.7 Maxima and minima of function of two variables |
| Unit-III<br>Application of single variables | 3.1 Radius , centre and Circle of curvature<br>3.2 Evolutes and Envelopes<br>3.3 Curve Tracing-Cartesian Co-ordinates<br>3.4 Curve Tracing-Polar Co-ordinates<br>3.5 Curve Tracing-Parametric Curves |
| Unit-IV<br>Integration and its applications | 4.1 Riemann Sum<br>4.3 Integral representation for lengths<br>4.4 Integral representation for Areas<br>4.5 Integral representation for Volumes<br>4.6 Surface areas in Cartesian and Polar co-ordinates<br>4.7 Multiple integrals-double and triple<br>4.8 Change of order of integration<br>4.9 Change of variable |
| Unit-V<br>Differential equations of first order and their applications | 5.1 Overview of differential equations<br>5.2 Exact and non exact differential equations<br>5.3 Linear differential equations<br>5.4 Bernoulli D.E<br>5.5 Newton's Law of cooling<br>5.6 Law of Natural growth and decay<br>5.7 Orthogonal trajectories and applications |
| Unit-VI<br>Higher order Linear D.E and their applications | 6.1 Linear D.E of second and higher order with constant coefficients<br>6.2 R.H.S term of the form $exp(ax)$<br>6.3 R.H.S term of the form $sin\ ax$ and $cos\ ax$<br>6.4 R.H.S term of the form exp(ax) v(x)<br>6.5 R.H.S term of the form exp(ax) v(x)<br>6.6 Method of variation of parameters<br>6.7 Applications on bending of beams, Electrical circuits and simple harmonic motion |
| Unit-VII<br>Laplace Transformations | 7.1 LT of standard functions<br>7.2 Inverse LT –first shifting property<br>7.3 Transformations of derivatives and integrals<br>7.4 Unit step function, Second shifting theorem<br>7.5 Convolution theorem-periodic function<br>7.6 Differentiation and integration of transforms<br>7.7 Application of laplace transforms to ODE |
| Unit-VIII<br>Vector Calculus | 8.1 Gradient, Divergence, curl<br>8.2 Laplacian and second order operators<br>8.3 Line, surface , volume integrals<br>8.4 Green's Theorem and applications<br>8.5 Gauss Divergence Theorem and applications<br>8.6 Stoke's Theorem and applications |

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UNIT-5

## Differential Equations-I

- **Overview of differential equations**
- **Exact and non exact differential equations**
- **Linear differential equations**
- **Bernoulli D.E**
- **Orthogonal Trajectories and applications**
- **Newton's Law of cooling**
- **Law of Natural growth and decay**

## DIFFERENTIAL EQUATIONS

Differentiation: The rate of change of a variable w.r.t the other variable is called as a Differentiation.

In this case, changing variable is called Dependent variable and other variable is called as an Independent variable.

***Example:*** $\frac{dy}{dx}$ is a Differentiation, Here $y$ is dependent variable and $x$ is Independent variable.

DIFFERENTIAL EQUATION: An equation which contains differential coefficients is called as a D.E.

***Examples:*** 1) $\frac{dy}{dx} + 1 = 0$ 2) $\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial z} + 1 = 0$.

Differential Equations are separated into two types

Differential Equations → Ordinary D.E, Partial D.E

Ordinary D.E: In a D.E if there exists single Independent variable, it is called as Ordinary D.E

***Example:*** 1) $\frac{dy}{dx} + 2y = 0$ is a Ordinary D.E 2) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = 0$ is a Ordinary D.E

Partial D.E: In a D.E if there exists more than one Independent variables then it is called as Partial D.E

***Example***: 1) $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} + 1 = 0$ is a Partial D.E, since $x, t$ are two Independent variables.

2) $\frac{\partial^2 y}{\partial x\, \partial z} + 1 = 0$ is a Partial D.E, since $x, z$ are two Independent variables

ORDER OF D.E: The highest derivative in the D.E is called as Order of the D.E

***Example:*** 1) Order of $\frac{d^2y}{dx^2} + 2y = 0$ is one.

2) Order of $\frac{d^5y}{dx^5} + \left[\frac{d^3y}{dx^3}\right]^8 + 3y = 0$ is Five.

DEGREE OF D.E: The Integral power of highest derivative in the D.E is called as degree of the D.E

***Example:*** 1) The degree of $\left[\frac{d^2y}{dx^2}\right]^1 + 2\frac{dy}{dx} + 1 = 0$ is One.

2) The degree of $x\left[\frac{d^2y}{dx^2}\right]^8 + \left[\frac{dy}{dx}\right]^{11} + \left[\frac{d^3y}{dx^3}\right]^2 = 0$ is Two.

**NOTE:** Degree of the D.E does not exist when the Differential Co-efficient Involving with exponential functions, logarithmic functions, and Trigonometric functions.

***Example:*** 1) There is no degree for the D.E $e^{\frac{dy}{dx}} + 1 = 0$

2) There is no degree for the D.E $log\left(\frac{d^2y}{dx^2}\right) + 1 = 0$

3) There is no degree for the D.E $sin\left(\frac{d^3y}{dx^3}\right) + 1 = 0$ .

**NOTE:** 1) The degree of the D.E is always a +ve Integer, but it never be a negative (or) zero (or) fraction.

2) Dependent variable should not include fraction powers. It should be perfectly Linear.

Ex: For the D.E $\frac{d^2y}{dx^2} + \sqrt{y} = 0$ Degree does not exist.

## FORMATION OF DIFFERNTIAL EQUATION

A D.E can be formed by eliminating arbitrary constants from the given D.E by using Differentiation Concept. If the given equation contains 'n' arbitrary constants then differentiating it 'n' times successively and eliminating 'n' arbitrary constants we get the corresponding D.E.

**NOTE:** If the given D.E contains 'n' arbitrary constants , then the order of its corresponding D.E is 'n' .

**NOTE:** For $y = c_1e^{\alpha_1 x} + c_2e^{-\alpha_2 x}$ then the corresponding D.E is given by $(D - \alpha_1)(D - \alpha_2)y = 0$

In general, if $y = c_1e^{\alpha_1 x} + c_2e^{\alpha_2 x} + --- + c_ne^{\alpha_n x}$ then its D.E is given by

$$(D - \alpha_1)(D - \alpha_2) --- (D - \alpha_n)y = 0, \text{ where } D = \frac{d}{dx}$$

**Special Cases:** If $y = (c_1 + c_2x)e^{\alpha x}$ then D.E is given by $(D - \alpha)^2y = 0$ , where $D = \frac{d}{dx}$.

In general, if $y = (c_1 + c_2x + c_3x^2 + --- + c_kx^{k-1})e^{\alpha x}$ then the corresponding D.E is given by

$$(D - \alpha)^k y = 0.$$

**NOTE:** For $y = e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x)$ then D.E is $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]y = 0.$

### WRANSKIAN METHOD

Let $y = Ax + Bx^2$ be the given equation then its corresponding D.E is given by

$$\begin{vmatrix} y & x & x^2 \\ y^1 & 1 & 2x \\ y^2 & 0 & 2 \end{vmatrix} = 0$$

This method is applicable when there are two arbitrary constants only.

## DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A D.E of the form $\frac{dy}{dx} = f(x, y)$ is called as a First Order and First Degree D.E in terms of dependent variable $y$ and independent variable $x$.

In order to solve above type of Equation's, following methods exists.

1) Variable Separable Method.
2) Homogeneous D.E and Equations reducible to Homogeneous.
3) Exact D.E and Equations made to exact.
4) Linear D.E and Bernoulli's Equations.

## Method-I: VARIABLE SEPERABLE METHOD

***First Form***: Let us consider given D.E $\frac{dy}{dx} = f(x, y)$

If $f(x, y) = \frac{f(x,y)}{g(x,y)}$ then proceed as follows

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \Longrightarrow g(y)dy = f(x)dx$

$\Longrightarrow \int g(y)dy = \int f(x)dx + C$ is the required general solution.

***Second Form:*** If $\frac{dy}{dx} = f(ax + by + c)$ then proceed as follows

Let $ax + by = c = z \Longrightarrow \frac{dy}{dx} = \frac{1}{b}\left[\frac{dz}{dx} - a\right]$

$\Longrightarrow \frac{dz}{dx} = bf(z) + a$

By using variable separable method we can find its general solution.

Let it be $\emptyset(z, x, c_1) = 0$ . But $z = ax + by + c$

$\Longrightarrow \emptyset(ax + by + c, x, c_1) = 0$ .

## Method-2: HOMOGENEOUS DIFFERENTIAL EQUATION METHOD

Homogeneous Function: A Function $f(x, y)$ is said to be homogeneous function of degree 'n' if

$$f(kx, ky) = k^n f(x, y)$$

***Example:*** 1) If $f(x, y) = \frac{x^3+y^3}{x^3+y^3}$ is a homogeneous function of degree '0'

2) If $f(x, y) = \frac{xy+y}{x^2+y^2}$ is not a homogeneous function.

Homogeneous D.E: A D.E of the form $\frac{dy}{dx} = f(x, y)$ is said to be Homogeneous D.E of first order and first degree in terms of dependent variable 'y' and independent variable 'x' if $f(x, y)$ is a homogeneous function of degree '0'.

**Ex:** 1) $\frac{dy}{dx} = \frac{x+y}{x-y}$ is a homogeneous D.E

2) $\frac{dy}{dx} = \frac{x^2+y^2}{x^2-y^2}$ is a homogeneous D.E

3) $\frac{dy}{dx} = \frac{xy+y^2}{x^2+y^2}$ is a homogeneous D.E

4) $\frac{dy}{dx} = \frac{xy+y}{x^2+y^2}$ is not a homogeneous D.E

5) $\frac{dy}{dx} = \frac{x^3+y^3}{x+y}$ is not a homogeneous D.E

**Working Rule:** Let us consider given homogeneous D.E $\frac{dy}{dx} = f(x, y)$

Substituting $y = vx \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ we get

$\Longrightarrow v + x\frac{dv}{dx} = f(x, vx)$

$\Longrightarrow x\frac{dv}{dx} = f(x, vx) - v$

By using variable separable method we can find the General solution of it

Let it be $\emptyset(v, x, c) = 0$ . But $v = \frac{y}{x}$

$\emptyset\left(\frac{y}{x}, x, c\right) = 0$ be the required general solution.

## NON-HOMOGENEOUS DIFFERENTIAL EQUATION

A D.E of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is called as a Non-Homogeneous D.E in terms of independent variable $x$ and dependent variable $y$ , where $a_1, b_1, c_1, and\ \ a_2, b_2, c_2$ are real constants.

Case (I): If $a_1b_2 - a_2b_1 \neq 0$ then procedure is as follows

Let us choose constants h & k in such a way that $\left.\begin{array}{l} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{array}\right\} \Longrightarrow \mathbf{I}$

Let $x = X + h$ , $y = Y + k$ and also $\frac{dy}{dx} = \frac{dY}{dX}$ then above relation becomes

$$\frac{dY}{dX} = \frac{(a_1X+b_1Y)+(a_1h+b_1k+c_1)}{(a_2X+b_2Y)+(a_2h+b_2k+c_2)}$$

$$\Longrightarrow \frac{dY}{dX} = \frac{a_1X+b_2Y}{a_2X+b_2Y} \qquad \text{(From I)}$$

Which is a Homogeneous D.E of first order and of first degree in terms of $X$ and $Y$.

By using Homogeneous method, we can find the General solution of it. Let it be $\emptyset(Y, X, C) = 0$ .

But $x = X + h , y = Y + k$

$\Longrightarrow \emptyset(y - k, x - h, c) = 0$ is the required General Solution of the given equation.

Case (II): If $a_1b_2 - a_2b_1 = 0$ , then By Using Second form of Variable Seperable method we can find the General Solution of the given equation.

## Method-3: EXACT DIFFERENTIAL EQUATION

A D.E of the form $Mdx + Ndy = 0$ is said to be exact D.E if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Its general solution is given by $\int^x Mdx + \int (terms\ independent\ of\ \ in\ N)dy = C$

$$(OR)$$

$$\int^x Mdx + \int (free\ from\ x\ terms\ in\ N)dy = C$$

$$(OR)$$

$$\int^x M\,dx + \int (terms\ not\ containing\ x\ in\ N)\,dy = C$$

## NON EXACT DIFFERENTIAL EQUATION

A D.E of the form $Mdx + Ndy = 0$ is said to be Non-Exact D.E if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

In order to make above D.E to be Exact we have to multiply with $\mu(x, y) \neq 0$ which is known as an

Integrating Factor.

To Solve such a type of problems, we have following methods.

1) Inspection Method
2) Method to find Integrating Factor I.F $\frac{1}{Mx+Ny}$
3) Method to find Integrating Factor I.F $\frac{1}{Mx-Ny}$
4) Method to find Integrating Factor I.F $e^{\int f(x)\,dx}$
5) Method to find Integrating Factor I.F $e^{\int g(y)dy}$

## Method 1: INSPECTION METHOD

This Method is used for both exact and non-exact D.E

Some Formulae:

- $d\left(\frac{x^2+y^2}{2}\right) = xdx + ydy$
- $d(xy) = xdy + ydx$
- $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$
- $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy-e^ydx}{x^2}$
- $d(xe^y) = xe^ydy + e^ydx$
- $d\left(log\frac{y}{x}\right) = \frac{xdy-ydx}{xy}$
- $d\left(log\frac{x}{y}\right) = \frac{ydx-xdy}{xy}$
- $d\left(Tan^{-1}\frac{y}{x}\right) = \frac{xdy-ydx}{x^2+y^2}$
- $d\left(Tan^{-1}\frac{x}{y}\right) = \frac{ydx-xdy}{x^2+y^2}$

**Hints while solving the problems using Inspection Method**

- If in a problem $e^{\blacksquare}$ term is there then select another $e^{\blacksquare}$ term.
- Always take $\boldsymbol{ydx}$ combination with $\boldsymbol{xdy}$ .

99% of the problems can be solved using Inspection method

## Method-2: Method to find Integrating Factor $\frac{1}{Mx+Ny}$

If given D.E is $Mdx + Ndy = 0$ is Non-Exact and it is Homogeneous and also $Mx + Ny \neq 0$

Then $\frac{1}{Mx+Ny}$ is the Integrating Factor (I.F).

## Method-3: Method to find Integrating Factor $\frac{1}{Mx-Ny}$

Let the given D.E is $Mdx + Ndy = 0$ is Non-Exact, and if given D.E can be expressed as

$yf(x,y)dx + xg(x,y)dy = 0$ And also $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an I.F

**Note:** Here in $f(x,y), g(x,y)$, there should be only $xy$ combination (with constants also)

i.e. With same powers $x^3y^3$, $x^ny^n$ , $(1 + xy + x^2y^2)$ $etc.$

## Method-4: Method to find the I.F $e^{\int f(x)dx}$

Let $Mdx + Ndy = 0$ be the Non-Exact D.E. If $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = f(x)$

Where $f(x)$= Function of x-alone or constant then I.F is $e^{\int f(x)dx}$

**NOTE:** In this case number of terms in M is greater than or equal to number of terms in N i.e. M≥N

## Method-5: Method to find the I.F $e^{\int g(y)dy}$

Let $Mdx + Ndy = 0$ be the Non-Exact D.E. If $\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M} = g(y)$

Where $g(y)$=Function of y-alone or constant then I.F is $e^{\int g(y)dy}$

**NOTE:** In this case number of terms in N is greater than or equal to number of items in M i.e. N≥M

## LINEAR DIFFERENTIAL EQUATION

A D.E of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is called as a First order and First degree D.E in terms of dependent variable $y$ and independent variable $x$ where $P(x), Q(x)$ functions of x-alone (or) constant.

**Working Rule:** Given that $\frac{dy}{dx} + P(x)y = Q(x)$ -------- (1)

I.F is given by $e^{\int Pdx}$

Multiplying with this I.F to (1), it becomes $e^{\int Pdx}\left[\frac{dy}{dx} + P(x)y\right] = Qe^{\int Pdx}$

$\Rightarrow d\left[ye^{\int Pdx}\right] = Qe^{\int Pdx}$ Now Integrating both sides we get

$\Rightarrow ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$ is the required General Solution.

**ANOTHER FORM**

A D.E of the form $\frac{dx}{dy} + P(y)x = Q(y)$ is also called as a Linear D.E where $P(y)$, $Q(y)$ functions of y-alone. Now I.F in this case is given by I.F=$e^{\int Pdy}$ and General Solution is given by

$$xe^{\int Pdy} = \int Qe^{\int Pdy}dy + C$$

## Equations Reducible to Linear Form

An Equation of the form $f^{I}(y)\frac{dy}{dx} + P(x)f(y) = Q(x)$ is called as an Equation Reducible to Linear Form

**Working Rule:**

> **Hint:** First make $\frac{dy}{dx}$ coefficient as 1, and then make R.H.S term purely function of $x$ alone

Given that $f^I(y)\frac{dy}{dx} + P(x)f(y) = Q(x) \longrightarrow$ ( I )

Let $f(y) = z \Rightarrow f^I(y)\frac{dy}{dx} = \frac{dz}{dx}$

( I ) $\Rightarrow \frac{dz}{dx} + P(x)z = Q(x)$ which is Linear D.E in terms of $z, x$

By using Linear Method we can find its General Solution.

Let it be $\emptyset(z, x, c) = 0$ But $f(y) = z$

$\Rightarrow \emptyset(f(y), x, c) = 0$ is the required solution

## BERNOULLIS EQUATION

A D.E of the form $\frac{dy}{dx} + P(x)y = Qy^n$ is called as Bernoulli's equation in terms of dependent variable y and independent variable x. where P and Q are functions of x-alone (or) constant.

**Working Rule:**

Given that $\frac{dy}{dx} + P(x)y = Q(x)y^n \longrightarrow 1$

$$\Rightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \longrightarrow 2$$

Let $y^{1-n} = z$

Differentiating with respect to x, we get

$$\Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$\Rightarrow \frac{1}{1-n}\frac{dz}{dx} + P(x)z = (1-n)Q$ (From 2), which is a Linear D.E in terms of $z, x$

By using Linear Method we can find general solution of it.

Let it be $\emptyset(z, x, c) = 0 \Rightarrow \emptyset(y^{-n}, x, c) = 0$ which is general solution of the given equation.

## Orthogonal Trajectories

Trajectory: A Curve which cuts given family of curves according to some special law is called as a Trajectory.

Orthogonal Trajectory: A Curve which cuts every member of given family of curves at $90^o$ is called as an Orthogonal Trajectory.

## Orthogonal Trajectory in Cartesian Co-ordinates

Let $f(x,y,c)=0$ be given family of curves in Cartesian Co-ordinates.

Differentiating it w.r.t $x$, we get $f\left(x,y,\frac{dy}{dx}\right)=0$.

Substituting $\frac{dy}{dx}=-\frac{dx}{dy}$, we get $f\left(x,y,-\frac{dx}{dy}\right)=0$ . By using previous methods we can find general solution of it. Let it be $g(x,y,c)=0$, which is Orthogonal Trajectory of the given family of curves.

For O.T, $\frac{dy}{dx}=-\frac{dx}{dy}$ because, two lines are $\perp^r$ if product of $slopes=-1$

$$\Rightarrow \frac{dy}{dx}\cdot\frac{dy}{dx}=-1$$

$$\Rightarrow \frac{dy}{dx}=-\frac{dx}{dy}$$

## Orthogonal Trajectory in Polar Co-ordinates

Let $f(r,\theta\,,c)=0$ be given family of curves in Polar Co-ordinates.

Differentiating it w.r.t $\theta$, we get $f\left(r,\theta,\frac{dr}{d\theta}\right)=0$.

Substituting $\frac{dr}{d\theta}=-r^2\frac{d\theta}{dr}$, we get $f\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0$. By using previous methods we can find general solution of it. Let it be $g(\theta,r,c)=0$, which is Orthogonal Trajectory of the given family of curves.

$$\left(\frac{1}{r}\frac{dr}{d\theta}\right)\cdot\left(\frac{1}{r}\frac{dr}{d\theta}\right)=-1$$

$$\Rightarrow \frac{dr}{d\theta}=-r^2\frac{d\theta}{dr}$$

Self Orthogonal: If the Orthogonal Trajectory of given family of Curves is family of curves itself then it is called as Self Orthogonal.

Mutual Orthogonal: Given family of curves $f(x,y,c)=0$ & $g(x,y,c)=0$ are said to be Mutually Orthogonal if Orthogonal Trajectory of one given family of curves is other given family of curves.

## NEWTON'S LAW OF COOLING

**Statement:** The rate of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let θ be the temperature of the body at the time $t$ and θ be the temperature of its surrounding medium (air). By the Newton's Law of cooling, we have

$$\frac{d\theta}{dt}\propto(\theta-\theta_o)$$

$$\Rightarrow \frac{d\theta}{dt}=-k(\theta-\theta_o)$$ , where $k$ is a positive constant

$$\therefore \frac{d\theta}{(\theta-\theta_o)}=-k\,dt$$

Integrating, we get $\int \frac{d\theta}{(\theta-\theta_o)} = -k \int dt$

$$\Rightarrow \boxed{\theta = \theta_o + ce^{-kt}}$$

## Problem

A body is originally at 80°C and cools down to 60°C in 20 minutes. If the temperature of the air is 40°C, find the temperature of the body after 40 minutes.

***Sol:*** Let θ be the temperature of the body at a time $t$

| $t$ | $\theta$ |
|---|---|
| 0 | 80 |
| 20 | 60 |
| 40 | ? |

We know that from Newton's Law of cooling

$$\frac{d\theta}{dt} \propto (\theta - \theta_o)$$

$\Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_o)$ , where $k$ is a positive constant

Given temperature of the air $\theta_o = 40^oC$

$$\therefore \frac{d\theta}{(\theta-40)} = -k\, dt$$

Integrating, we get $\int \frac{d\theta}{(\theta-40)} = -k \int dt$

$$\Rightarrow \theta = 40 + ce^{-kt} \longrightarrow \text{I}$$

Now, given at $t = 0$, $\theta = 80^oC$

I $\Rightarrow 80 = 40 + c$

$\Rightarrow c = 40$

Substituting this value of $c$ in I , we get

$$\Rightarrow \theta = 40 + 40e^{-kt} \longrightarrow \text{II}$$

Again, given at $t = 20$, $\theta = 60^oC$

II $\Rightarrow 60 = 40 + 40\, e^{-20k}$

$$\Rightarrow k = \frac{1}{20}\log 2$$

Substituting this value of $k$ in II we get

$$\Rightarrow \theta = 40 + 40e^{-\left(\frac{1}{20}\log 2\right)t}$$

$$\Rightarrow \theta = 40 + 40e^{-\left(\frac{t}{20}\log 2\right)} \longrightarrow \text{III}$$

Again, when $t = 40, \theta = ?$

III $\Rightarrow \theta = 40 + 40e^{-\left(\frac{40}{20}\log 2\right)}$

$\Rightarrow$ At $t = 40, \theta = 50^{o}C$

## LAW OF NATURAL GROWTH (Or) DECAY

If $x(t)$ be the amount of substance at time $t$ , then the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of the substance available at that time.

$$\frac{dx}{dt} \propto x \Longrightarrow \frac{dx}{dt} = -kx$$

where $k$ is a proportionality constant.

Note: If as $t$ increases, $x$ increases we can take $\frac{dx}{dt} = kx \ \ (k > 0)$, and if as $t$ increases, $x$ decreases we can take $\frac{dx}{dt} = -kx \ \ (k > 0)$

## RATE OF DECAY OF RADIOACTIVE MATERIALS

If $u$ is the amount of the material at any time $t$, then $\frac{du}{dt} = -ku$, where $k$ is any constant.

* * *

# MATHEMATICS-I

# DIFFERENTIAL EQUATIONS-II

I YEAR B.TECH

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# SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

| Name of the Unit | Name of the Topic |
|---|---|
| Unit-I<br>Sequences and Series | 1.1 Basic definition of sequences and series<br>1.2 Convergence and divergence.<br>1.3 Ratio test<br>1.4 Comparison test<br>1.5 Integral test<br>1.6 Cauchy's root test<br>1.7 Raabe's test<br>1.8 Absolute and conditional convergence |
| Unit-II<br>Functions of single variable | 2.1 Rolle's theorem<br>2.2 Lagrange's Mean value theorem<br>2.3 Cauchy's Mean value theorem<br>2.4 Generalized mean value theorems<br>2.5 Functions of several variables<br>2.6 Functional dependence, Jacobian<br>2.7 Maxima and minima of function of two variables |
| Unit-III<br>Application of single variables | 3.1 Radius , centre and Circle of curvature<br>3.2 Evolutes and Envelopes<br>3.3 Curve Tracing-Cartesian Co-ordinates<br>3.4 Curve Tracing-Polar Co-ordinates<br>3.5 Curve Tracing-Parametric Curves |
| Unit-IV<br>Integration and its applications | 4.1 Riemann Sum<br>4.3 Integral representation for lengths<br>4.4 Integral representation for Areas<br>4.5 Integral representation for Volumes<br>4.6 Surface areas in Cartesian and Polar co-ordinates<br>4.7 Multiple integrals-double and triple<br>4.8 Change of order of integration<br>4.9 Change of variable |
| Unit-V<br>Differential equations of first order and their applications | 5.1 Overview of differential equations<br>5.2 Exact and non exact differential equations<br>5.3 Linear differential equations<br>5.4 Bernoulli D.E<br>5.5 Newton's Law of cooling<br>5.6 Law of Natural growth and decay<br>5.7 Orthogonal trajectories and applications |
| Unit-VI<br>Higher order Linear D.E and their applications | 6.1 Linear D.E of second and higher order with constant coefficients<br>6.2 R.H.S term of the form *exp(ax)*<br>6.3 R.H.S term of the form *sin ax and cos ax*<br>6.4 R.H.S term of the form exp(ax) v(x)<br>6.5 R.H.S term of the form exp(ax) v(x)<br>6.6 Method of variation of parameters<br>6.7 Applications on bending of beams, Electrical circuits and simple harmonic motion |
| Unit-VII<br>Laplace Transformations | 7.1 LT of standard functions<br>7.2 Inverse LT –first shifting property<br>7.3 Transformations of derivatives and integrals<br>7.4 Unit step function, Second shifting theorem<br>7.5 Convolution theorem-periodic function<br>7.6 Differentiation and integration of transforms<br>7.7 Application of laplace transforms to ODE |
| Unit-VIII<br>Vector Calculus | 8.1 Gradient, Divergence, curl<br>8.2 Laplacian and second order operators<br>8.3 Line, surface , volume integrals<br>8.4 Green's Theorem and applications<br>8.5 Gauss Divergence Theorem and applications<br>8.6 Stoke's Theorem and applications |

# CONTENTS

UNIT-6

## Differential Equations-II

- **Linear D.E of second and higher order with constant coefficients**
- **R.H.S term of the form *exp(ax)***
- **R.H.S term of the form *sin ax and cos ax***
- **R.H.S term of the form exp(ax) v(x)**
- **Method of variation of parameters**

## LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

A D.E of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ \ldots + \ P_{n-1}\frac{dy}{dx} + P_n = Q(x)$ is called as a Linear Differential Equation of order $n$ with constant coefficients, where $P_1, P_2, \ldots, P_n$ are Real constants.

Let us denote $\frac{d}{dx} \equiv D, \frac{d^2}{dx^2} \equiv D^2, \frac{d^3}{dx^3} \equiv D^3$ $etc$, then above equation becomes

$(D^n + P_1 D^{n-1} + \ \ldots \ + P_{n-1}D + P_n)y = Q(x)$ which is in the form of $f(D)y = Q(x)$, where

$f(D) = (D^n + P_1 D^{n-1} + \ \ldots \ + P_{n-1}D + P_n)$.

The **General Solution** of the above equation is $y = C.F + P.I$ C.F= Complementary Function

(or) $\boldsymbol{y = y_c + y_p}$ P.I= Particular Function

Now, to find Complementary Function $y_c$ , we have to find Auxillary Equation

**Auxillary Equation:** An equation of the form $f(m) = 0$ is called as an Auxillary Equation.

Since $f(m) = 0$ is a polynomial equation, by solving this we get roots. Depending upon these roots we will solve further.

**Complimentary Function**: The General Solution of $f(D)y = 0$ is called as Complimentary Function and it is denoted by $y_c$

Depending upon the Nature of roots of an Auxillary equation we can define $y_c$

Case I: If the Roots of the A.E are real and distinct, then proceed as follows

If $\alpha_1, \alpha_2$ are two roots which are real and distinct (different) then complementary function is given by $y_c = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$

**Generalized condition:** If $\alpha_1, \alpha_2, \alpha_3, \ \ldots, \alpha_n$ are real and distinct roots of an A.E then

$$y_c = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + c_3 e^{\alpha_3 x} + \ \ldots + c_n e^{\alpha_n x}$$

Case II: If the roots of A.E are real and equal then proceed as follows

If $\alpha_1 = \alpha_2 = \alpha$ then $y_c = e^{\alpha x}(c_1 + c_2 x)$

**Generalized condition:** If $\alpha_1 = \alpha_2, = \alpha_3, = \ \ldots, = \alpha_n = \alpha$ then

$$y_c = e^{\alpha x}(c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1})$$

Case III: If roots of A.E are Complex conjugate i.e. $m = \alpha \pm i\beta$ then

$$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

(Or) $y_c = c_1 e^{\alpha x} \cos(\beta x + c_2)$

(Or) $y_c = c_1 e^{\alpha x} \sin(\beta x + c_2)$

**Note:** For repeated Complex roots say, $m = \alpha \pm i\beta,\ \alpha \pm i\beta$

$$y_c = e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x]$$

Case IV: If roots of A.E are in the form of Surds i.e. $m = \alpha \pm \sqrt{\beta}$, where $\beta$ is not a perfect square then,

$$y_c = e^{\alpha x}\left(c_1\cosh\sqrt{\beta}x + c_2\sin\sqrt{\beta}x\right)$$

$$\text{(Or)}\quad y_c = c_1e^{\alpha x}\cos\left(\sqrt{\beta}x + c_2\right)$$

$$\text{(Or)}\quad y_c = c_1e^{\alpha x}\sin\left(\sqrt{\beta}x + c_2\right)$$

**Note:** For repeated roots of surds say, $m = \alpha \pm \sqrt{\beta},\ \alpha \pm \sqrt{\beta}$

$$y_c = e^{\alpha x}\left[(c_1 + c_2x)\cosh\sqrt{\beta}x + (c_3 + c_4x)\sinh\sqrt{\beta}x\right]$$

## Particular Integral

The evaluation of $\frac{1}{f(D)}Q(x)$ is called as Particular Integral and it is denoted by $y_p$

i.e. $y_p = \frac{1}{f(D)}Q(x)$

**Note:** The General Solution of $f(D)y = Q(x)$ is called as Particular Integral and it is denoted by $y_p$

## Methods to find Particular Integral

### Method 1: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = e^{ax}$, where $a$ is a constant.

We know that $y_p = \frac{1}{f(D)}Q(x)$

$$= \frac{1}{f(D)}e^{ax}$$

$$\therefore\ y_p = \frac{1}{f(a)}e^{ax} \quad \text{if } f(a) \neq 0 \qquad \boxed{\text{Directly substitute } a \text{ in place of } D}$$

$$= e^{ax}\frac{1}{f(D+a)} \quad \text{if } f(a) = 0 \qquad \boxed{\text{Taking } e^{ax} \text{ outside the operator by replacing } D \text{ with } D + a}$$

Depending upon the nature of $f(D + a)$ we can proceed further.

**Note:** while solving the problems of the type $\frac{1}{f(D)}Q(x)$, where Denominator =0, Rewrite the Denominator quantity as product of factors, and then keep aside the factor which troubles us. I.e the term which makes the denominator quantity zero, and then solve the remaining quantity. finally substitute $D + a$ in place of $D$.

**Method 2: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = \sin ax \ (or) \cos ax$, a is constant**

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} \sin ax \ \ (or) \ \frac{1}{f(D)} \cos ax$$

Let us consider $f(D) = \emptyset(D^2)$, then the above equation becomes

$$\therefore \ y_p = \frac{1}{\emptyset(D^2)} \sin ax \ \ (or) \ \frac{1}{\emptyset(D^2)} \cos ax$$

Now Substitute $D^2 = -a^2$ if $\emptyset(D^2) \neq 0$

If $\emptyset(D^2) = 0$ then i.e. $y_p = \frac{1}{D^2+a^2} \sin ax \ \ (or) \ \frac{1}{D^2+a^2} \cos ax$

Then $y_p = \frac{x}{2} \int \sin ax \ dx \ \ (or) \ \frac{x}{2} \int \cos ax \ dx$ respectively.

**Method 3: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = x^k, k \in \mathbb{Z}^+$**

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} x^k$$

Now taking Lowest degree term as common in $f(D)$, above relation becomes $y_p = \frac{1}{[1+\emptyset(D)]} x^k$

$\Rightarrow \ y_p = [1 + \emptyset(D)]^{-1} x^k$

Expanding this relation upto $k^{th}$ derivative by using Binomial expansion and hence get $y_p$

**Important Formulae:**

1) $(1 - D)^{-1} = 1 + D + D^2 + \ ...$
2) $(1 + D)^{-1} = 1 - D + D^2 - \ ...$
3) $(1 - D)^{-2} = 1 + 2D + 3D^2 + \ ...$
4) $(1 + D)^{-2} = 1 - 2D + 3D^2 - \ ...$
5) $(1 - D)^{-3} = 1 + 3D + 6D^2 + \ ...$
6) $(1 + D)^{-3} = 1 - 3D + 6D^2 - \ ...$

**Method 4: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = e^{ax} V$, where $V$ is a function of $x$ and $a$ is constant**

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} e^{ax} V$$

In such cases, first take $e^{ax}$ term outside the operator, by substituting $D + a$ in place of $D$.

$$\Rightarrow \ y_p = e^{ax} \frac{1}{f(D + a)} V$$

Depending upon the nature of $V$ we will solve further.

### Method 5: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = x^k.v$ , where $k \in \mathbb{Z}^+$, $v$ is any function of $x$ ( i.e. $v = \sin ax$ $(or) \cos ax$ )

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\cos\theta$ → R.P, $i\sin\theta$ → I.P

$\cos\theta = R.P(e^{i\theta})$

$\sin\theta = I.P(e^{i\theta})$

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} x^k.v$$

Case I: Let $k = 1$, then $y_p = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} v$

Case II: Let $k \neq 1$ and $v = \sin ax$

$$y_p = \frac{1}{f(D)} x^k \sin ax$$

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$$y_p = \frac{1}{f(D)} x^k\, I.P(e^{iax})$$

$$= I.P \frac{1}{f(D)} x^k e^{iax}$$

$$= I.P\, e^{iax} \frac{1}{f(D+ia)} x^k$$

By using previous methods we will solve further

Finally substitute $e^{iax} = \cos ax + i \sin ax$

Let $k \neq 1$ and $v = \sin ax$

$$y_p = \frac{1}{f(D)} x^k \cos ax$$

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$$y_p = \frac{1}{f(D)} x^k\, R.P(e^{iax})$$

$$= R.P \frac{1}{f(D)} x^k e^{iax}$$

$$= R.P\, e^{iax} \frac{1}{f(D+ia)} x^k$$

By using previous methods we will solve further

Finally substitute $e^{iax} = \cos ax + i \sin ax$

## General Method

To find P.I of $f(D)y = Q(x)$ where $Q(x)$ is a function of $x$

We know that $y_p = \frac{1}{f(D)} Q(x)$

Let $f(D) = (D - \alpha)$ then $y_p = \frac{1}{(D-\alpha)} Q(x)$

$$= e^{\alpha x} \int e^{-\alpha x} Q(x)\, dx$$

Similarly, $f(D) = (D + \alpha)$ then $y_p = \frac{1}{(D+\alpha)} Q(x)$

$$= e^{-\alpha x} \int e^{\alpha x} Q(x)\, dx$$

Note: The above method is used for the problems of the following type

- $(D^2 - 3D + 2)y = \sin(e^{-x})$
- $(D^2 + a^2)y = \sec ax$
- $(D^2 + a^2)y = \tan ax$
- $(D^2 + a^2)y = \operatorname{cosec} ax$

## Cauchy's Linear Equations (or) Homogeneous Linear Equations

A Differential Equation of the form $[x^n D_n + A_1 x^{n-1} D_{n-1} + \ ... + A_{n-1} x D + A_n]y = Q(x)$ where $D \equiv \frac{d}{dx}$ is called as $n^{th}$ order Cauchy's Linear Equation in terms of dependent variable $y$ and independent variable $x$, where $A_1, A_2, A_3, \dots, A_n$ are Real constants and $D \equiv \frac{d}{dx}$.

Substitute $x = e^z \implies \log x = z$ and

$xD = \theta, x^2D^2 = \theta(\theta - 1), x^3D^3 = \theta(\theta - 1)(\theta - 2), \dots \quad \theta \equiv \frac{d}{dz}$

Then above relation becomes $(\theta)y = Q(z)$ , which is a Linear D.E with constant coefficients. By using previous methods, we can find Complementary Function and Particular Integral of it, and hence by replacing $z$ with $\log x$ we get the required General Solution of Cauchy's Linear Equation.

## Legendre's Linear Equation

An D.E of the form $[(ax + b)^n D_n + A_1(ax + b)^{n-1} D_{n-1} + \ ... + A_{n-1}(ax + b)D + A_n]y = Q(x)$ is called as Legendre's Linear Equation of order , where $a, b, A_1, A_2, A_3, \dots, A_n$ are Real constants.

Now substituting, $(ax + b) = e^z \implies z = \log(ax + b)$

$(ax + b)D = a\theta, (ax + b)^2D^2 = a^2\theta(\theta - 1), (ax + b)^3D^3 = a^3\theta(\theta - 1)(\theta - 2), \dots \quad \theta \equiv \frac{d}{dz}$

Then, above relation becomes $f(\theta)y = Q(z)$ which is a Linear D.E with constant coefficients. By using previous methods we can find general solution of it and hence substituting $z = \log(ax + b)$ we get the general solution of Legendre's Linear Equation.



## Method of Variation of Parameters

To find the general solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x)$

Let us consider given D.E $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x) \longrightarrow$ **( I )**

Let the Complementary Function of above equation is $y_c = c_1u + c_2v$

Let the Particular Integral of it is given by $y_p = Au + Bv$ , where

$$A = \int \frac{-vR}{uv' - vu'} dx \qquad B = \int \frac{uR}{uv' - vu'}$$

* * *

# MATHEMATICS-I

# LAPLACE TRANSFORMS

## I YEAR B.Tech



**By**

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# SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

| Name of the Unit | Name of the Topic |
|---|---|
| Unit-I<br>Sequences and Series | 1.1 Basic definition of sequences and series<br>1.2 Convergence and divergence.<br>1.3 Ratio test<br>1.4 Comparison test<br>1.5 Integral test<br>1.6 Cauchy's root test<br>1.7 Raabe's test<br>1.8 Absolute and conditional convergence |
| Unit-II<br>Functions of single variable | 2.1 Rolle's theorem<br>2.2 Lagrange's Mean value theorem<br>2.3 Cauchy's Mean value theorem<br>2.4 Generalized mean value theorems<br>2.5 Functions of several variables<br>2.6 Functional dependence, Jacobian<br>2.7 Maxima and minima of function of two variables |
| Unit-III<br>Application of single variables | 3.1 Radius , centre and Circle of curvature<br>3.2 Evolutes and Envelopes<br>3.3 Curve Tracing-Cartesian Co-ordinates<br>3.4 Curve Tracing-Polar Co-ordinates<br>3.5 Curve Tracing-Parametric Curves |
| Unit-IV<br>Integration and its applications | 4.1 Riemann Sum<br>4.3 Integral representation for lengths<br>4.4 Integral representation for Areas<br>4.5 Integral representation for Volumes<br>4.6 Surface areas in Cartesian and Polar co-ordinates<br>4.7 Multiple integrals-double and triple<br>4.8 Change of order of integration<br>4.9 Change of variable |
| Unit-V<br>Differential equations of first order and their applications | 5.1 Overview of differential equations<br>5.2 Exact and non exact differential equations<br>5.3 Linear differential equations<br>5.4 Bernoulli D.E<br>5.5 Newton's Law of cooling<br>5.6 Law of Natural growth and decay<br>5.7 Orthogonal trajectories and applications |
| Unit-VI<br>Higher order Linear D.E and their applications | 6.1 Linear D.E of second and higher order with constant coefficients<br>6.2 R.H.S term of the form *exp(ax)*<br>6.3 R.H.S term of the form *sin ax and cos ax*<br>6.4 R.H.S term of the form exp(ax) v(x)<br>6.5 R.H.S term of the form exp(ax) v(x)<br>6.6 Method of variation of parameters<br>6.7 Applications on bending of beams, Electrical circuits and simple harmonic motion |
| Unit-VII<br>Laplace Transformations | 7.1 LT of standard functions<br>7.2 Inverse LT –first shifting property<br>7.3 Transformations of derivatives and integrals<br>7.4 Unit step function, Second shifting theorem<br>7.5 Convolution theorem-periodic function<br>7.6 Differentiation and integration of transforms<br>7.7 Application of laplace transforms to ODE |
| Unit-VIII<br>Vector Calculus | 8.1 Gradient, Divergence, curl<br>8.2 Laplacian and second order operators<br>8.3 Line, surface , volume integrals<br>8.4 Green's Theorem and applications<br>8.5 Gauss Divergence Theorem and applications<br>8.6 Stoke's Theorem and applications |

# CONTENTS

UNIT-7

## LAPLACE TRANSFORMS

- **Laplace Transforms of standard functions**
- **Inverse LT- First shifting Property**
- **Transformations of derivatives and integrals**
- **Unit step function, second shifting theorem**
- **Convolution theorem – Periodic function**
- **Differentiation and Integration of transforms**
- **Application of Laplace Transforms to ODE**

# LAPLACE TRANSFORMATION

## INTRODUCTION

Laplace Transformations were introduced by Pierre Simmon Marquis De Laplace (1749-1827), a French Mathematician known as a Newton of French.

Laplace Transformations is a powerful Technique; it replaces operations of calculus by operations of Algebra.

Suppose an Ordinary (or) Partial Differential Equation together with Initial conditions is reduced to a problem of solving an Algebraic Equation.

**Definition of Laplace Transformation:** Let $f(t)$ be a given function defined for all $t \geq 0$, then the Laplace Transformation of $f(t)$ is defined as $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{dt} = F(s)$

Here, $L$ is called Laplace Transform Operator. The function $f(t)$ is known as determining function, depends on $t$. The new function which is to be determined (i.e. F($s$) ) is called generating function, depends on $s$.

Here $F(s) = \overline{f}(s)$

**NOTE:** Here Question will be in $t$ and Answer will be in $s$.

**Laplace Transformation** is useful since

- Particular Solution is obtained without first determining the general solution
- Non-Homogeneous Equations are solved without obtaining the complementary Integral
- Solutions of Mechanical (or) Electrical problems involving discontinuous force functions (R.H.S function $F(x)$ ) (or) Periodic functions other than *cos* and *sin* are obtained easily.
- The Laplace Transformation is a very powerful technique, that it replaces operations of calculus by operations of algebra. For e.g. With the application of L.T to an Initial value problem, consisting of an Ordinary( or Partial ) differential equation (O.D.E) together with Initial conditions is reduced to a problem of solving an algebraic equation ( with any given Initial conditions automatically taken care )

## APPLICATIONS

Laplace Transformation is very useful in obtaining solution of Linear D.E's, both Ordinary and Partial, Solution of system of simultaneous D.E's, Solutions of Integral equations, solutions of Linear Difference equations and in the evaluation of definite Integral.

Original Equation (In terms of t)

↓

Laplace Transform

↓

Final Equation (In terms of s)

Thus, Laplace Transformation transforms one class of complicated functions $f(t)$ to produce another class of simpler functions $F(s)$.

## ADVANTAGES

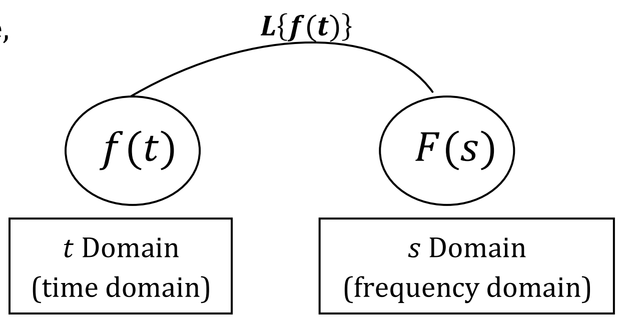
- With the application of Linear Transformation, Particular solution of D.E is obtained directly without the necessity of first determining general solution and then obtaining the particular solution (by substitution of Initial Conditions).
- L.T solves non-homogeneous D.E without the necessity of first solving the corresponding homogeneous D.E.
- L.T is applicable not only to continuous functions but also to piece-wise continuous functions, complicated periodic functions, step functions, Impulse functions.
- L.T of various functions are readily available.

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\ dt = F(s)$$

The symbol '$L$' denotes the L.T operator, when it operated on a function $f(t)$, it transforms into a function $F(s)$ of complex variable $s$. We say the operator transforms the function $f(t)$ in the $'t'$ domain (usually called time domain) into the function $F(s)$ in the $'s'$ domain (usually called complex frequency domain or simply the frequency domain)

Because the Upper limit in the Integral is Infinite, the domain of Integration is Infinite. Thus the Integral is an example of an Improper Integral.

$$\int_0^{\infty} e^{-st} f(t)\ dt = \lim_{T \to \infty} \int_0^{T} e^{-st} f(t)\ dt$$




The Laplace Transformation of $f(t)$ is said to exist if the Integral $\int_0^{\infty} e^{-st} f(t)\ dt$ Converges for some values of $s$, Otherwise it does not exist.

Definition: A function $f(t)$ is said to be piece wise Continuous in any Interval $[a, b]$, if it is defined on that Interval and is such that the Interval can be broken up into a finite number of sub-Intervals in each of which $f(t)$ is Continuous.

In Mathematics, a transform is usually a device that converts one type of problem into another type.

The main application of D.E using Laplace Transformation and Inverse Laplace Transformation is that, By solving D.E directly by using Variation of Parameters, etc methods, we first find the general solution and then we substitute the Initial or Boundary values. In some cases it will be more critical to find General solution.

By suing Laplace and Inverse Laplace Transformation, we will not going to find General solution and in the middle we substitute the Boundary conditions, so the problem may becomes simple.

Note: Some Problems will be solved more easier in Laplace than by doing using Methods (variation of Parameter etc) and vice-versa.

## PROPERTIES OF LAPLACE TRANSFORMATION

- Laplace Transformation of constant $K$ is $\frac{K}{s}$

*Sol*: *We know that*

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\{k\} = \int_0^{\infty} e^{-st}\, k\, dt$$

$$\Rightarrow L\{k\} = k \int_0^{\infty} e^{-st}\, dt$$

$$\Rightarrow L\{k\} = k \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} = \frac{k}{s}$$

- The Laplace Transformation of $e^{at}$ is $\frac{1}{s-a}$

*Sol: We know that*

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\{f(t)\} = \int_0^{\infty} e^{-st} e^{at}\, dt$$

$$= \int_0^{\infty} e^{(-s+a)t}\, dt$$

$$= \frac{1}{(-s+a)} \left[e^{-(s-a)}\right]_{t=0}^{\infty} = \frac{1}{s-a}$$

- Definition of Gama Function

$$\Gamma(n) = \int_0^{\infty} e^{-t}\, t^{n-1}\, dt, n \geq 0$$

*(OR)*

$$\Gamma(n) = \int_0^{\infty} e^{-x}\, x^{n-1}\, dt, n \geq 0.$$

Note: i) $\Gamma(n+1) = n\Gamma(n) = n!$

ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

▶ The Laplace Transformation of $t^n$ , where $n$ is a non-negative Real number.

*Sol:* We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$\Rightarrow L\{f(t)\} = \int_0^\infty e^{-st} t^n \, dt$$

$Put\ st = x \ \Rightarrow t = \frac{x}{s}$

$dt = \frac{1}{s}\, dx$

$As\ t \to 0\ to\ \infty \quad \Rightarrow\ x \to 0\ to\ \infty$

$$\Rightarrow L\{t^n\} = \int_{x=0}^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \frac{1}{s} dx$$

$$\Rightarrow L\{t^n\} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \qquad \left[\because\ \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dt\, , n \geq 0\right]$$

$$\Rightarrow L\{t^n\} = \frac{n!}{s^{n+1}} \qquad [\because\ \Gamma(n+1) = n!]$$

## Problems

1) Find the Laplace Transformation of $t^{\frac{1}{2}}$

*Sol:* We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\left(t^{\frac{1}{2}}\right) = \frac{\left(\frac{1}{2}\right)!}{s^{n+1}}$$

$$= \frac{1}{s^{\frac{1}{2}+1}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{s^{\frac{3}{2}}} \frac{\sqrt{\pi}}{2} \qquad [\because\ n\Gamma(n) = n!\,]$$

2) Find the Laplace Transformation of $t^{-\frac{1}{2}}$

*Sol:* We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\left(t^{-\frac{1}{2}}\right) = \frac{\left(-\frac{1}{2}\right)!}{s^{-\frac{1}{2}+1}}$$

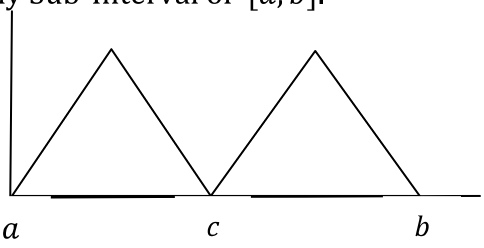
$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{\frac{1}{2}}} \qquad [\because\ \Gamma(n+1) = n!\,]$$

$$\Rightarrow L\left(t^{-\frac{1}{2}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

## SECTIONALLY CONTINUOUS CURVES (Or) PIECE-WISE CONTINUOUS

A function $f(x)$ is said to be Sectionally Continuous (or) Piece-wise Continuous in any Interval $[a, b]$, if it is continuous and has finite Left and Right limits in any Sub-Interval of $[a, b]$.

In the above case Laplace Transformation holds good.



## FUNCTIONS OF EXPONENTIAL ORDER

A function $f(x)$ is said to be exponential of order $'a'$ as $x \to \infty$ if $\lim_{x\to\infty} e^{-ax} f(x)$ is a finite value.

***Example:* Verify $x^n$ is an exponential order (or) not?**

**Sol:** $lim_{x\to\infty} e^{-ax} f(x) = lim_{x\to\infty} e^{-ax} x^n = lim_{x\to\infty} \frac{x^n}{e^{ax}} = \frac{\infty}{\infty}$

$$= lim_{x\to\infty} \frac{n!}{a^n e^{ax}} = \frac{n!}{\infty} = 0, \text{ a finite value.}$$

$\therefore$ $x^n$ is an exponential order.

## Sufficient conditions for the Existence of Laplace Transformation

The Laplace Transformation of $f(t)$ exists i.e. The Improper Integral $\int_0^\infty e^{-st} f(t)\, dt$ of $L\{f(t)\}$ Converges (finite value) when the following conditions are satisfied.

1) $f(t)$ is a piece-wise continuous
2) $f(t)$ is an exponential of order $'a'$.

### PROPERTIES OF LAPLACE TRANSFORMATION

## LINEAR PROPERTY

*Statement:* If $L\{f(t)\} = F(s), L\{g(t)\} = G(s)$, then $L\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s)$

*Proof:* Given that $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$

***L.H.S:*** $L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}$

$= c_1 F(s) + c_2 G(s)$ ***=R.H.S***

## FIRST SHIFTING PROPERTY (or) FIRST TRANSLATION PROPERTY

*Statement:* If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$

*Proof:* We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt$$

*(Here we have taken exponential quantity as negative, but not positive, because as $t \to \infty \Rightarrow e^t \to \infty \Rightarrow e^{-t} \to 0$ )*

*Put* $s - a = p,\ p > 0$

$$\Rightarrow L\{e^{at} f(t)\} = \int_0^\infty e^{-pt} f(t) dt$$

$$= F(p)$$

$$= F(s-a)$$

*Hence, If* $L\{f(t)\} = F(s)$ *then* $L\{e^{at} f(t)\} = F(s-a)$

- whenever we want to evaluate $\boldsymbol{L\{e^{at} f(t)\}}$, first evaluate $\boldsymbol{L\{f(t)\}}$ which is equal to $\boldsymbol{F(s)}$ and then evaluate $\boldsymbol{L\{e^{at} f(t)\}}$, which will be obtained simply, by substituting $\boldsymbol{s - a}$ in place of $\boldsymbol{a}$ in $\boldsymbol{F(s)}$.

## Problem

❖ Find the Laplace Transformation of

$f(t) = t^{\frac{7}{2}}.e^{3t}$

Sol: To find $L\left\{t^{\frac{7}{2}}.e^{3t}\right\}$, first we shall evaluate

$L\left\{t^{\frac{7}{2}}\right\}$

Now, $L\left\{t^{\frac{7}{2}}\right\} = \frac{(7/2)!}{s^{(7/2)+1}}\left[\because L\{t^n\} = \frac{n!}{s^{n+1}}\right]$

$\Longrightarrow L\left\{t^{\frac{7}{2}}\right\} = \frac{\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}\right)}{s^{\frac{9}{2}}}$ $[\because n! = n\,\Gamma(n)]$

$= \frac{\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{s^{\frac{9}{2}}}$ $[\because\ \Gamma(n+1) = n\Gamma(n)]$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{9}{2}}} = \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}{s^{\frac{9}{2}}}$ $\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

$\Longrightarrow L\left\{t^{\frac{7}{2}}\right\} = \frac{105\sqrt{\pi}}{16\,s^{\frac{9}{2}}} = F(s)$

$\therefore L\left\{t^{\frac{7}{2}}.e^{3t}\right\} = \frac{105\sqrt{\pi}}{16\,(s-a)^{\frac{9}{2}}}$

## CHANGE OF SCALE PROPERTY

Statement: If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

*Proof:* We know that $L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,\mathrm{dt} = F(s)$

$$\Longrightarrow L\{f(at)\} = \int_0^\infty e^{-st}f(at)\,\mathrm{dt}$$

Put $at = x \quad \Longrightarrow t = \frac{x}{a}$

$$\Longrightarrow dt = \frac{1}{a}\,dx$$

$$\Longrightarrow L\{f(at)\} = \int_0^\infty e^{-\frac{s}{a}x}f(x)\,\frac{1}{a}\,dx$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}x}f(x)\,dx$$

Put $\frac{s}{a} = p$ then, $L\{f(at)\} = \frac{1}{a}\int_0^\infty e^{-px}f(x)\,dx$

$$= \frac{1}{a}F(p) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

## Problems

**1)** Find the Laplace Transformation of $\cosh at$ $\quad i.e.\, L\{\cosh at\}$

*Sol:* We know that $\cosh at = \frac{e^{at}+e^{-at}}{2}$

and $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{dt} = F(s)$

$$\text{Now, } L\{\cosh at\} = \frac{1}{2}[L\{e^{at} + e^{-at}\}]$$
$$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$$
$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$
$$\therefore L[\cosh at] = \frac{s}{s^2 - a^2}$$

**2)** Find the Laplace Transformation of $\sinh at$ $\quad i.e.\, L\{\sinh at\}$

*Sol:* We know that $\sinh at = \frac{e^{at}-e^{-at}}{2}$

and $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{dt} = F(s)$

$$\text{Now, } L\{\sinh at\} = \frac{1}{2}[L\{e^{at} - e^{-at}\}]$$
$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$
$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$
$$\therefore L[\sinh at] = \frac{a}{s^2-a^2}$$

**3)** Find $L\{\sin at\}$ and $L\{\cos at\}$

*Sol:* We know that $e^{iat} = \cos at + i \sin at$

$$\Rightarrow L\{e^{iat}\} = L\{\cos at + i\sin at\} \longrightarrow \quad \text{I}$$

$$\text{But } L\{e^{iat}\} = \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} \longrightarrow \quad \text{II}$$

$$\therefore \text{From I \&II, } L\{\cos at + i \sin at\} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + \frac{a}{s^2+a^2}$$

Equating the corresponding coefficients, we get

$$L\{\sin at\} = \frac{a}{s^2+a^2}, \text{ and } L\{\cos at\} = \frac{s}{s^2+a^2}$$

## LAPLACE TRANSFORMATION OF DERIVATIVES

*Statement:* If $L\{f(t)\} = F(s)$, then $L\{f'(t)\} = sF(s) - f(0)$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$
$$\vdots$$
$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$$

*Proof:* We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{dt} = F(s)$$

$$\text{Let us consider } L\{f'(t)\} = \int_0^\infty e^{-st} \frac{df}{dt}\, \mathrm{dt}$$

$$= \left[e^{-st}\int \frac{df}{dt}\, dt - \int\left\{\frac{d}{dt}(e^{-st})\int \frac{df}{dt}\, dt\right\} dt\right]_{t=0}^{\infty}$$

$$= [e^{-st} f(t)]_{t=0}^{\infty} - [\int_{t=0}^{\infty}(-se^{-st} f(t)\, dt]$$
$$= -f(0) + s\, L\{f(t)\}$$
$$= -f(0) + s\, F(s)$$
$$\therefore L\{f'(t)\} = sF(s) - f(0)$$

In view of this, $L\{g'(t)\} = s\, G(s) - g(0)$

Now, put $g(t) = f'(t)$

$$\Rightarrow L\{g(t)\} = L\{f''(t)\} = G(s)$$

Since, $L\{g'(t)\} = s\, G(s) - g(0)$

$$\therefore L\{f''(t)\} = s\, L\{g(t)\} - g(0)$$
$$\Rightarrow L\{f''(t)\} = s\, L\{f'(t)\} - f'(0)$$
$$\Rightarrow L\{f''(t)\} = s\big(sF(s) - f(0)\big) - f'(0)$$
$$\Rightarrow L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Generalizing this, we get finally

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots . - f^{n-1}(0)$$

***Problem:*** *Find the $L\{t^2\}$ by using derivatives method.*

*Sol*: we know that $L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

Given that $f(t) = t^2 \Rightarrow f(0) = 0$

$\Rightarrow f'(t) = 2t \Rightarrow f'(0) = 0$

$\therefore L\{f''(t)\} = s^2 F(s)$

$\Rightarrow L(2) = s^2 F(s)$

$\Rightarrow \frac{2}{s} = s^2 F(s)$

$\Rightarrow F(s) = \frac{2}{s^3}$

## DIVISION BY 't' METHOD (or) Laplace Integrals

*Statement:* If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$

*Proof:* Let us consider $\boldsymbol{F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt}$

Now, Integrate on both sides w.r.t $'s'$ by taking the Limits from $s$ *to* $\infty$ then

$$\int_s^{\infty} F(s)\, ds = \int_{s=s}^{\infty} \left( \int_{t=0}^{\infty} e^{-st} f(t)\, dt \right) ds$$

Since $'s'$ and $'t'$ are Independent variables, by Interchanging the order of Integration,

$$\Rightarrow \int_s^{\infty} F(s)\, ds = \int_{t=0}^{\infty} \left( \int_{s=s}^{\infty} e^{-st}\, ds \right) f(t)\, dt$$

$$= \int_{t=0}^{\infty} \left[ \frac{e^{-st}}{-t} \right]_{s=s}^{\infty} f(t)\, dt$$

$$= \int_{t=0}^{\infty} e^{-st} \left( \frac{f(t)}{t} \right) dt$$

$$\Rightarrow \int_s^{\infty} F(s)\, ds = L\left\{\frac{f(t)}{t}\right\}$$

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$$

## Problems

- Find the $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

Sol: Here $f(t) = e^{-at} - e^{-bt}$

$$\Rightarrow F(s) = L\{f(t)\} = L\{e^{-at} - e^{-bt}\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

We know that $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$

$$\Rightarrow L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \left[\frac{1}{s+a} - \frac{1}{s+b}\right] dsv$$

$$= [\log|s+a| - \log|s+b|]_s^{\infty}$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_{s=s}^{\infty}$$

$$= \log 1 - \log\left(\frac{s+a}{s+b}\right) = log\left(\frac{s+a}{s+b}\right)$$

- If $\{f(t)\} = F(s)$ , then $L\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}\ F(s), s > 0$ .

**Sol:** Let us consider $g(t) = \int_0^t f(t)\ dt$ and $g(0) = 0$

$$\Rightarrow g'(t) = f(t)$$

Now, $L\{g'(t)\} = s\ L\{g(t)\} - g(0)$

$$L\{f(t)\} = s\ L\left\{\int_0^t f(t)\ dt\right\} - 0$$

$$\Rightarrow F(s) = s\ L\left\{\int_0^t f(t)\ dt\right\}$$

$$\Rightarrow L\left\{\int_0^t f(t)\ dt\right\} = \frac{F(s)}{s}$$

Generalization of above one:

$$L\left\{\underbrace{\int_0^t \int_0^t \int_0^t \dots \int_0^t f(t)}_{n\text{-}times}\ dt\right\} = \frac{F(s)}{s^n}$$

## Multiplication of $'t'$

*Statement: If* $L\{f(t)\} = F(s)$, *then* $L\{t.f(t)\} = -\frac{d}{ds}[F(s)]$

*Proof:* We know that $F(s) = \int_0^\infty e^{-st}\ f(t)dt$

$$\Rightarrow \frac{d}{ds}\ F(s) = \frac{d}{ds}\ \left[\int_0^\infty e^{-st}\ f(t)dt\right]$$

$$= \int_0^\infty\ f(t)\ \frac{\partial}{\partial s}(e^{-st})dt$$

$$= \int_0^\infty f(t)\ e^{-st}\ (-t)\ dt$$

$$= -\int_0^\infty e^{-st}\ t\ f(t)dt$$

$$= -L\{t\ f(t)\}$$

$$\therefore \boldsymbol{L\{t.f(t)\} = -\frac{dF}{ds}}$$

Generalization: $L\{t^n.f(t)\} = -\frac{d^n}{ds^n}[F(s)] = -\frac{d^nF}{ds^n}$

## Problem

- Find $L\{t\ \sin at\}$

*Sol:* we know that $L\{t.f(t)\} = -\frac{dF}{ds}$

*Here* $f(t) = \sin at$

$$\Rightarrow F(s) = \left[\frac{a}{s^2+a^2}\right]$$

$$\therefore\ L\{t\ \sin at\} = -\frac{\boldsymbol{d}}{\boldsymbol{ds}}\left\{\frac{\boldsymbol{a}}{\boldsymbol{s^2+a^2}}\right\} = \frac{\boldsymbol{2as}}{\boldsymbol{s^2+a^2}}$$

## INVERSE LAPLACE TRANSFORMATION

Definition: If $\{f(t)\} = F(s)$ , then $f(t)$ is known as Inverse Laplace Transformation of $F(s)$ and it is denoted by $L^{-1}[F(s)] = f(t)$ , where $L^{-1}$ is known as Inverse Laplace Transform operator and is such that $L\,L^{-1} = L^{-1}\,L = 1$.

### Inverse Elementary Transformations of Some Elementary Functions

- $L^{-1}\left(\frac{1}{s}\right) = 1$
- $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
- $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
- $L^{-1}\left(\frac{1}{(s-a)^n}\right) = e^{at}\,\frac{t^{n-1}}{(n-1)!}$
- $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$
- $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
- $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$
- $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
- $L^{-1}\left(\frac{s}{(s-a)^2+b^2}\right) = \frac{1}{b}\,e^{at}\sin bt$
- $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

### Problems based on Partial Fractions

A fraction of the form $\frac{a_0\,s^m + a_1\,s^{m-1} + \ldots + a_m}{b_0\,s^n + b_1\,s^{n-1} + \cdots + b_n}$ in which both powers $m$ and $n$ are positive numbers is called rational algebraic function.

When the degree of the Numerator is Lower than the degree of Denominator, then the fraction is called as Proper Fraction.

To Resolve Proper Fractions into Partial Fractions, we first factorize the denominator into real factors. These will be either Linear (or) Quadratic and some factors may be repeated.

From the definitions of Algebra, a Proper fraction can be resolved into sum of Partial fractions.

| S.No | Factor of the Denominator | Corresponding Partial Fractions |
|---|---|---|
| **1.** | Non-Repeated Linear Factor<br>Ex: $\frac{1}{s-a}$, [ $(s-a)$occurs only one time] | $\frac{A}{s-a}$ |
| **2.** | Repeated Linear Factor, repeated 'r' times<br>Ex: $(s-a)^r$ | $\frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \ldots + \frac{A_r}{(s-a)^r}$ |
| **3.** | Non-repeated Quadratic Expression<br>Ex: $s^2 + as + b$ | $\frac{As+B}{s^2+as+b}$ , Here atleast one of $A$ or $B \neq 0$ |
| **4.** | Repeated Quadratic Expression, repeated 'r' times<br>Ex: $(s^2 + as + b)^r$ | $\frac{A_1 s + B_1}{s^2 + as + b} + \frac{A_2 s + B_2}{(s^2 + as + b)^2} + \cdots \frac{A_r s + B_r}{(s^2 + as + b)^r}$ |

## SHIFTING PROPERTY OF INVERSE LAPLACE TRANSFORMATION

We know that $L\{e^{at} f(t)\} = F(s-a)$

$$\Rightarrow L^{-1}[F(s-a)] = e^{at} f(t)$$

## FORMULAS

- If $L^{-1}[F(s)] = f(t)$ then, $L^{-1}[F(s-a)] = e^{at} f(t)$
- If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$ then, $L^{-1}[s\,F(s)] = \frac{d}{dt}[f(t)]$

  In general, $L^{-1}[s^n F(s)] = \frac{d^n}{dt^n}[f(t)]$, provided $f(0) = f'(0) = f''(0) = \dots = f^{n-1} = 0$
- If $L^{-1}[F(s)] = f(t)$ then, $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$
- If $L^{-1}[F(s)] = f(t)$ then, $t.f(t) = L^{-1}\left[-\frac{d}{ds}\{F(s)\}\right]$
- If $L^{-1}[F(s)] = f(t)$ then, $\frac{f(t)}{t} = L^{-1}\left[\int_0^\infty F(s)\,ds\right]$

## CONVOLUTION THEOREM

(A Differential Equation can be converted into Inverse Laplace Transformation)

(In this the denominator should contain atleast two terms)

Convolution is used to find Inverse Laplace transforms in solving Differential Equations and Integral Equations.

**Statement:** *Suppose two Laplace Transformations $F(s)$ and $G(s)$ are given. Let $f(t)$ and $g(t)$ are their Inverse Laplace Transformations respectively i.e.* $L^{-1}[F(s)] = f(t)$

$$L^{-1}[G(s)] = g(t)$$

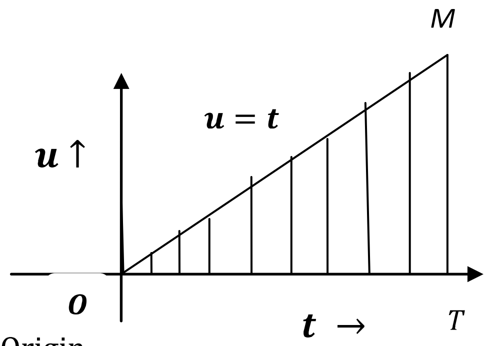
*Then,* $L^{-1}[F(s).G(s)] = \int_0^t f(u)\,g(t-u)du = F * G$

*Where $F * G$ is called Convolution. (Or) Falting of $F$ & $G$.*

*Proof:* Let $\phi(t) = \int_0^t f(u)\; g(t-u)\,du$

Now $L\{\phi(t)\} = \int_0^\infty e^{-st}\left[\int_{u=0}^t f(u)g(t-u)\,du\right] dt$

$\Rightarrow L\{\phi(t)\} = \int_0^\infty \left[\int_{u=0}^t e^{-st} f(u)g(t-u)\,du\right] dt$



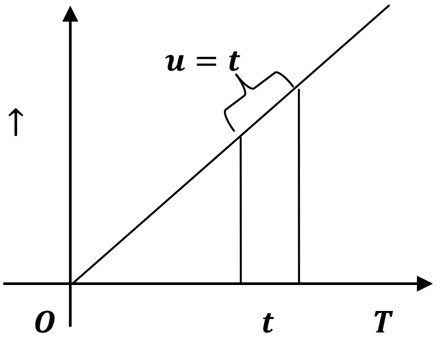

The above Integration is within the region lying below the line, and above $OT$.

(Here equation of $OM$ is $u = t$)

Let $'t'$ is taken on $OT$ line and $u$ is taken on $OM$ line, with $'O'$ as Origin.

The axes are perpendicular to each other.

If the order of Integration is changed, the strip will be taken parallel to $OT$. So that the limits of $t$ are from $u\ to\ \infty$ and $u$ is taken as $0\ to\ \infty$.




$$L\{\phi(t)\} = \int_{u=0}^{\infty} \left[\int_{t=u}^{t} e^{-st} f(u) g(t-u)\, du\right] dt$$

$$= \int_{u=0}^{\infty} \left[\int_{t=u}^{t} e^{-s(t-u)} e^{-su} f(u) g(t-u)\, du\right] dt$$

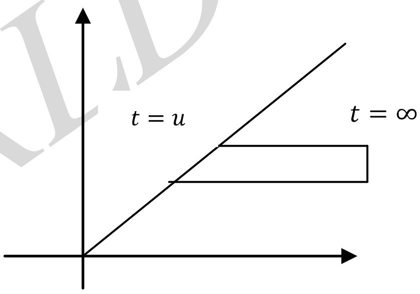
$$= \int_{u=0}^{\infty} (e^{-su} f(u) du) \left[\int_{t=u}^{t} e^{-s(t-u)} g(t-u)\, dt\right]$$

Put $t-u=v$, then Lower Limit: $t=u \implies v=0$

Upper Limit: $t=\infty \implies v=\infty$

Then, Consider $\int_{t=u}^{\infty} e^{-s(t-u)} g(t-u)\ dt \ = \int_{v=0}^{\infty} e^{-sv}\, g(v)\, dv$

$$= L\{g(v)\} = G(v)$$




$$L\{\phi(t)\} = \left[\int_{u=0}^{\infty} e^{-su} f(u) du\right] \{G(v)\}$$

$$= F(u)G(v)$$

Again, $\phi(t) = L^{-1}[F(u).G(v)]$

$$= L^{-1}[F(u).G(t-u)]$$

## Problem

- Apply Convolution Theorem to evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

*Sol:* Given $L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$

Let us choose one quantity as $F(s)$ and other quantity as $G(s)$

Now, Let $F(s) = \frac{1}{s^2+a^2}, G(s) = \frac{s}{s^2+a^2}$

Now $f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

$$\implies f(u) = \frac{1}{a}\sin au$$

Again $g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$\therefore$ By Convolution Theorem, $L^{-1}[F(s).G(s)] = \int_0^u f(u)\, g(t-u) du$

$$\implies L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_{u=0}^{t} \frac{1}{a}\sin au \cos a(t-u)\ du$$

$$= \frac{1}{2a} \int_{u=0}^{t} 2 \sin au \cos a(t-u)\ du$$

$$= \frac{1}{2a} \int_{u=0}^{t} [\{\sin(au + at - au\} + \sin\{au - at + au\}]\ du$$

$$= \frac{1}{2a} \int_{u=0}^{t} \sin at\ du + \frac{1}{2a} \int_{u=0}^{t} \sin(2au - at)\ du$$

$$= \frac{1}{2a} \sin at\ [u]_0^t + \frac{1}{2a} \left[\frac{-\cos(2au-at)}{2a}\right]_{u=0}^{t}$$

$$= \frac{1}{2a} \sin at\ t + \frac{(-1)}{4a^2} \{-\cos at + \cos at\}$$

$$= \frac{1}{2a}\ t \sin at$$

## APPLICATIONS OF D.E's BY USING LAPLACE AND INVERSE LAPLACE TRANSFORMATIONS

Laplace Transform Method of solving Differential Equations yields particular solutions without necessity of first finding General solution and elimination of arbitrary constants.

Suppose the given D.Eq is of the form $a\ \frac{d^2y}{dt^2} + b\ \frac{dy}{dt} + y = f(t) \longrightarrow$ I

is a Linear D.Eq of order 2 with constants a, b.

Case 1: Suppose in Equation I, we assume a,b are constants and the boundary conditions are $y(0) = y'(0) = 0$.

We Know that $L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0)$

and $L[f''(t)] = s^2 F(s) - sf(0) - f'(0)$

(Or) $L[y^2(t)] = s^2\bar{y}(s) - s\ y(0) - y'(0)$ Here $\bar{y}(s) = L[y(t)]$ and $y^2(t)$ is Second derivative

and $L[y'(t)] = s\bar{y}(s) - y(0)$

Procedure: Apply Laplace Transformation to equation ( I )

i.e. $a\ L\{y''\} + b\ L\{y'\} + L\{y\} = L\{f(t)\}$

$\Longrightarrow a\{s^2\bar{y}(s) - s\ y(0) - y'(0)\} + b\ \{s\bar{y}(s) - y(0)\} + \bar{y}(s) = F(s)$

$\Longrightarrow as^2\bar{y}(s) + bs\ \bar{y}(s) + \bar{y}(s) = F(s)$

($\because$ *we have taken Initial conditions as* $y(0) = y'(0) = 0$ )

$\Longrightarrow (as^2 + bs + 1)\ \bar{y}(s) = F(s)$

$$\Longrightarrow \bar{y}(s) = \frac{F(s)}{(as^2 + bs + 1)}$$

Now, apply Inverse Laplace Transformation i.e. $L^{-1}\{\bar{y}(s)\} = L^{-1}\left\{\frac{F(s)}{(as^2+bs+1)}\right\}$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{F(s)}{(as^2+bs+1)}\right\}$$

By solving this, we get the required answer.

Case 2: If a, b are not constants (i.e. D.E with Variable Co-efficient)

Let the D.E is of the form $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = f(t) \longrightarrow$ II

Here $a, b$ are some functions of $'t'$, with Initial conditions $y(0) = y'(0) = 0$

We know that $L\{t^m f^n(t)\} = (-1)^m \frac{d^m}{ds^m}[L\{f^n(t)\}]$

Now, $L\{t^2y''\} = (-1)^2 \frac{d^2}{ds^2}[L\{y''(t)\}]$

$$= \frac{d^2}{ds^2}[s^2\bar{y}(s) - s\,y(0) - y'(0)]$$

And, $L\{t\,y'\} = (-1)\frac{d}{ds}L\{y'(t)\} = -\frac{d}{ds}\{s\bar{y}(s) - y(0)\}$

Apply Laplace Transformation on both sides to ( II ), we get

$$\Rightarrow L\{t^2y''\} + L\{t\,y'\} + L\{y\} = L\{f(t)\}$$

$$\Rightarrow \frac{d^2}{ds^2}[s^2\bar{y}(s) - s\,y(0) - y'(0)] - \frac{d}{ds}\{s\bar{y}(s) - y(0)\} + \bar{y}(s) = F(s)$$

Substituting the boundary conditions in equation II and get the values of $\bar{y} \longrightarrow$ III

Required solution is obtained by taking Inverse Laplace Transformation for equation III.

## Problem

- Solve by the method of Transformations, the equation $y''' + 2y'' - y' - 2y = 0$ and $y(0) = y'(0) = 1$ and $y''(0) = -6$.

Sol: Given $y''' + 2y'' - y' - 2y = 0 \longrightarrow$ I

Apply Laplace Transformation on both sides, we get

$$\Rightarrow L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = L\{0\}$$

$$\Rightarrow \{s^3\bar{y} - s^2\bar{y}(0) - s\,y'(0) - y''(0)\} + 2\{s^2\bar{y}(s) - s\,y(0) - y'(0)\} - \{s\bar{y}(s) - y(0)\} - 2\bar{y} = 0 \longrightarrow \text{II}$$

Now substitute boundary conditions Immediately before solving in equation II, we get

$$\Rightarrow \{s^3\bar{y} + 6\} + 2\{s^2\bar{y}\} - \{s\bar{y}\} - 2\bar{y} = 0$$

$$\Rightarrow \bar{y}(s^3 + 2s^2 - s - 2) = -6$$

$$\Rightarrow \bar{y} = \frac{-6}{s^3 + 2s^2 - s - 2}$$

$$\Rightarrow \bar{y} = \frac{-6}{(s-1)(s+1)(s-2)}$$

$$= \frac{-1}{s-1} + \frac{3}{s+1} - \frac{2}{s+2} \qquad \text{(By resolving into partial fractions)}$$

$$\Rightarrow L^{-1}\{\bar{y}(s)\} = -L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{s+1}\right\} - 2L^{-1}\left\{\frac{1}{s+2}\right\}$$

$\Rightarrow y(t) = -e^t + 3e^{-t} - 2e^{-2t}$ is the required solution.

- Solve the D.E $ty'' + 2y' + ty = \cos t$ , $y(0) = 1, y'(0) = 1$

*Sol:* Taking Laplace Transform on both sides, we get

$$L\{ty''\} + 2L\{y'\} + 2L\{ty\} = L\{Cos\ t\}$$

$$\Rightarrow (-1)\frac{d}{ds}[s^2\,Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + (-1)\frac{d}{ds}[Y(s)] = \frac{s}{s^2+1}$$

Here, given Initial/Boundary conditions are $y(0) = 1, y'(0) = 1$

$$\Rightarrow (-1)\frac{d}{ds}[s^2\,Y(s) - s.1 - 1] + 2[sY(s) - 1)] + (-1)\frac{d}{ds}[Y(s)] = \frac{s}{s^2+1}$$

$$\Rightarrow -s^2\frac{d}{ds}Y(s) - 2sY(s) + 1 + 2sY(s) - 2 - \frac{d}{ds}Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow -\frac{dY}{ds}\{s^2+1\} = \frac{s}{s^2+1} + 1$$

$$\Rightarrow -\frac{dY}{ds} = \frac{s}{(s^2+1)^2} + \frac{1}{s^2+1}$$

Apply Inverse Laplace Transformation on both sides, we get

$$\Rightarrow -L^{-I}\left(\frac{dY}{ds}\right) = L^{-I}\left(\frac{s}{(s^2+1)^2}\right) + L^{-I}\left(\frac{1}{s^2+1}\right)$$

We know that $L\{t\,f(t)\} = -\frac{d}{ds}F(s)$

$$\Rightarrow t.y(t) = \frac{1}{2}t\sin t + \sin t$$

(Or) $y(t) = \frac{1}{2}\sin t + \frac{\sin t}{t}$

## Hints for solving problems in Inverse Laplace Transformation

- If it is possible to express denominator as product of factors then use partial fraction method (i.e. resolve into partial fractions and solve further)
- Sometimes it is not possible to express as product of partial fractions. In such case, express denominator quantity in the form of $[(s-a)^2 + b^2]$ *or* $[(s-a)^2 - b^2]$ etc
- Note that, the problems in which the denominator is possible to express as product of partial fractions can be solved in other methods also.

## Some Formulas

- If $L\{f(t)\} = F(s)$ then $L\{e^{at}\, f(t)\} = F(s-a)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{F(s-a)\} = e^{at} f(t)$

  $$\Rightarrow L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$$

- If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}\, F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\ dt$

  ***Generalization:*** If $L\{f(t)\} = F(s)$ then $L\left\{\underbrace{\int_0^t \int_0^t \ldots \int_0^t f(t)}_{n\ times}\ dt\right\} = \frac{1}{s^n}\, F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \underbrace{\int_0^t \int_0^t \ldots \int_0^t f(t)}_{n\ times}\ dt$

- If $L\{f(t)\} = F(s)$ then $L\{t\, f(t)\} = -\frac{d}{ds} F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t\, f(t)$

  ***Generalization***: If $L\{f(t)\} = F(s)$ then $L\{t^n\, f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{d^n}{ds^n} F(s)\right\} = (-1)^n\, t^n\, f(t)$

- If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_{s=s}^{\infty} F(s)\ ds$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{\int_{s=s}^{\infty} F(s)\ ds\} = \frac{f(t)}{t}$

- If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = [s\, F(s) - f(0)]$

  Now, If $L^{-1}\{F(s)\} = f(t)$, and $f(0) = 0$, then $L^{-1}[s\, F(s)] = f'(t)$

  ***Generalization:*** If $L\{f(t)\} = F(s)$ then $L\{f''(t)\} = [s^2\, F(s) - s\, f(0) - f'(0)]$

  Now, If $L^{-1}\{F(s)\} = f(t)$, and $f(0) = f'(0) = 0$, then $L^{-1}[s^2\, F(s)] = f''(t)$

  ***Similarly,*** If $L\{f(t)\} = F(s)$ then $L\{f^n(t)\} = [s^n\, F(s) - s^{n-1}\, f(0) - \ \ldots - f^{n-1}(0)]$

  Now, If $L^{-1}\{F(s)\} = f(t)$, and $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$, then $L^{-1}[s^n\, F(s)] = f^n(t)$

- If $L\{f(t)\} = F(s)$ and $g(t) = f(t-a)\, u(t-a) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t > a \end{cases}$ Then,

  $L\{g(t)\} = L\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{e^{-as} F(s)\} = g(t) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t > a \end{cases}$

## Problems

❖ Find $L^{-1}\left\{\frac{1}{s(s+2)}\right\}$

Sol: Here, if we observe, the denominator is in the product of factors form. So we can use partial fractions method. Or we can use the following method.

We know that, If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}\ F(s)$

Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\ dt \longrightarrow \mathrm{I}$

Let us consider $F(s) = \frac{1}{s+2}$

$$\Rightarrow L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow f(t) = e^{-2t}$$

$\therefore$ Substituting in I, we get $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\ dt$

$$\Rightarrow L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t e^{-2t}\ dt$$

$$= \left(\frac{1-e^{-2t}}{2}\right)$$

❖ Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

**Sol:** Here if we observe, it is possible to express denominator as product of partial fractions. So we can use partial fractions method also. But the method will be lengthy.

So, we go for another method, by which we can solve the problem easily.

Now, We know that If $L\{f(t)\} = F(s)$ then $L\{t\ f(t)\} = -\frac{d}{ds}F(s)$

Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\ f(t) \longrightarrow \mathrm{I}$

Let us consider $F(s) = \frac{1}{s^2+a^2}$

$$\Rightarrow L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$\Rightarrow \boxed{f(t) = \frac{1}{a}\sin at}$$

If Numerator is $'s'$ and denominator is $(\quad)^2$ term, then always use $t\ f(t)$ model.

$\therefore$ From I, we have $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\ f(t)$

$$\Rightarrow -tf(t) = L^{-1}\left\{\frac{d}{ds}(s^2+a^2)^{-1}\right\}$$

$$= L^{-1}\{-2s(s^2+a^2)^{-2}\}$$

$$\Rightarrow -tf(t) = -2\ L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow tf(t) = 2\ L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow \frac{t}{2}f(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow \frac{t}{2}\frac{1}{a}\sin at = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}\ \text{(Or)}$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

- If Numerator is $'s'$ and denominator is $(\quad)^2$ term, then always use $t\,f(t)$ model.
- If Numerator is 1, and denominator is $(\quad)^2$ term, then always use $t\,f(t)$ model.
- If we are asked to find $L^{-1}$ of $\log(\quad)$, $\tan(\quad)$, $\cot(\quad)$ etc. or any unknown quantity, where we don't have any direct formula, in such cases always use $t\,f(t)$ model.

## PERIODIC FUNCTIONS OF LAPLACE TRANSFORMATIONS

Periodic Function: A Function $f(t)$ is said to be periodic function of the period $T\ (>0)$ if

$$f(t) = f(t+T) = f(t+2T) = \ \dots = f(t+nT)$$

Example: $\sin t$, $\cos t$ are the periodic functions of period $2\pi$.

Theorem: The Laplace Transformation of Piece-wise periodic function $f(t)$ with period $'p'$ is

$$L\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st}\ f(t)\ dt$$

Proof: Let $f(t)$ be the given function then

$$L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)dt$$

$$= \int_0^P e^{-st}\ f(t)dt + \int_P^{2P} e^{-st}\ f(t)dt + \int_{2P}^{3P} e^{-st}\ f(t)dt + \ \dots$$

Put $t = u + p$ in the second Integral

$t = u + 2p$ in the third Integral

⋮

$t = u + np$ in the $n^{th}$ Integral etc

$$\therefore L\{f(t)\} = \int_0^p e^{-st}\ f(t)dt + \int_{u=0}^p e^{-s(u+p)}\ f(u+p)du + \ \dots + \int_0^p e^{-s[u+(n-1)p]}\ f(u+(n-1)p)du + \ \dots$$

Since by substituting $t = u \implies dt = du$ in 1st Integral and also by the definition of Periodic function $f(t) = f(t+T) = f(t+2T) = \ \dots = f(t+nT)$

$$\implies L\{f(t)\} = \int_0^p e^{-su}\ f(u)du + e^{-sp}\int_{u=0}^p e^{-su}\ f(u)du + \ e^{-2sp}\int_0^p e^{-su}\ f(u)du + \ \dots$$

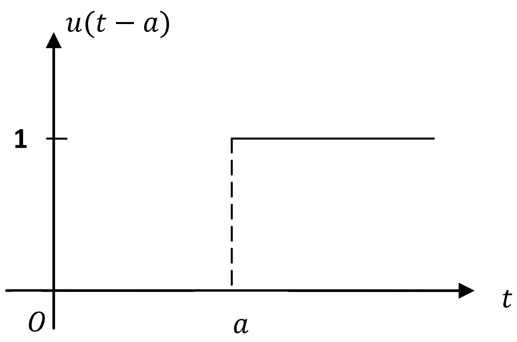
$$\implies L\{f(t)\} = [1 + e^{-sp} + e^{-2sp} + \ \dots] \int_0^p e^{-su}\ f(u)du$$

This is a Geometric Progression $S_\infty = \frac{a}{1-r}$

$$\implies \boxed{L\{f(t)\} = \left(\frac{1}{1-e^{-sp}}\right)\left[\int_0^p e^{-su}\ f(u)du\right]}$$

## Unit Step Function (Heaviside's Unit Function)

The Unit Step Function $u(t-a)$ or $H(t-a)$ is defined as $H(t-a) = \begin{cases} 0\,, & for\ t < a \\ 1\,, & for\ t \geq a \end{cases}$, where $'a'$ is positive number always.




## Second Shifting Property (or) Second Translation Property

*Statement:* If $L\{f(t)\} = F(s)$ and the shifted function

$g(t) = f(t-a)\,u(t-a) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t > a \end{cases}$ Then, $L\{g(t)\} = L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

Proof: We know that $L\{g(t)\} = \int_0^\infty e^{-st}\,g(t)\,dt$

$$= \int_0^a e^{-st}\,g(t)\,dt + \int_a^\infty e^{-st}\,g(t)\,dt$$

$$= 0 + \int_a^\infty e^{-st}\,f(t-a)\,dt$$

Put $t - a = u \implies dt = du$ then

$$\implies L\{g(t)\} = \int_0^\infty e^{-s(u+a)}\,f(u)du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)\,du$$

$$\implies L\{g(t)\} = e^{-sa}F(s)$$

*Note:* The Laplace Transform of Unit Step Function (put $f(t) = 1$) is $L\{H(t-a)\} = \frac{e^{-as}}{s}$

## Unit Impulse Function (or Diract delta Function)

Suppose a large force (like Earthquake, collision of two bodies) acts on a system, produces large effect when applied for a very short interval of time. To deal with such situations, we introduce a function called unit impulse function, which is a discontinuous function.

If a large force acts for a short time, the product of the force and time is called impulse. To deal with similar type of problems in applied mechanics, the unit impulse is to be introduced.

## Laplace Transform of Unit Step Function

$$L\{H(t-a)\} = \int_0^\infty e^{-st}\, H(t-a)\, dt$$

$$\Rightarrow\; L\{H(t-a)\} = \int_0^a e^{-st}\, H(t-a)\, dt + \int_a^\infty e^{-st}\, H(t-a)\, dt$$

$$= \int_0^a e^{-st}.0\, dt + \int_a^\infty e^{-st}.1\, dt$$

$$= \frac{e^{-as}}{s}$$

❖ Find Laplace Transformation of $J_0(x)$

Sol: We know that $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots$

$$\Rightarrow L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2.4^2}\frac{4!}{s^5} + \ldots$$

$$\Rightarrow L\{J_0(x)\} = \frac{1}{s}\left[1 - \frac{1}{2}\frac{1}{s^2} + \frac{1\,3}{2\,4}\frac{1}{s^4} - \frac{1\,3\,5}{2\,4\,6}\frac{1}{s^6} + \ldots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{\frac{-1}{2}} = \frac{1}{\sqrt{s^2+1}}$$

❖ Find Laplace Transformation of $J_1(x)$

Sol: We know that $J_1(x) = -J_0^I(x)$

$$= -\frac{2x}{2^2} + 4\frac{x^3}{2^2 4^2} - 6\frac{x^5}{2^2 4^2 6^2} + \ldots$$

$$\Rightarrow J_1(x) = -\frac{x}{2} + \frac{x^3}{2^2.4} - \ldots$$

$$\Rightarrow L^{-I}[J_1(x)] = -L^{-I}\left[\frac{x}{2}\right] + \frac{1}{2^2.4}L^{-I}[x^3] - \ldots$$

$$= -\frac{1}{2}\frac{1}{s^2} + \frac{1}{2^2.4}\frac{3!}{s^4} - \ldots$$

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