# Minor-2 Solutions

Ans ① **a** $\|x-y\|_2^2 = (x-y)^T(x-y) = x^Tx - 2x^Ty + y^Ty = \|x\|^2 - 2x^Ty + \|y\|_2^2$

$= 1 - 2\cos 45^\circ + 1 = 2 - 2\frac{1}{\sqrt{2}} = 2-\sqrt{2}$

$\Rightarrow \boxed{\|x-y\|_2 = \sqrt{2-\sqrt{2}}}$

**b** $\because a_{ij} = ij \Rightarrow A$ is a symmetric matrix $\Rightarrow$ all its eigenvectors are orthonormal to each other

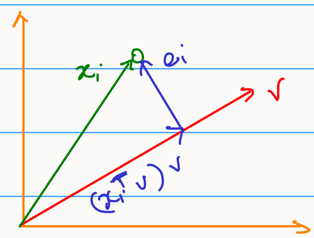
$\Rightarrow v_1^Tv_3 = 0 \Rightarrow \boxed{16\, v_1^Tv_2 = 0}$

② Likelihood function $\ell(\theta) = \prod_{i=1}^{10} P(x_i) = \theta^6(1-\theta)^4$

$\log \ell(\theta) = 6\log\theta + 4\log(1-\theta)$

$\frac{\partial}{\partial\theta}\log\ell(\theta) = \frac{6}{\theta} - \frac{4}{1-\theta} = 0 \Rightarrow \boxed{\theta = 0.6}$

③




ⓐ $\Rightarrow x_i = e_i +$ Projection of $x_i$ on $v$

$= e_i + (x_i^Tv)v = e_i + v(v^Tx_i)$

$\Rightarrow e_i = x_i - vv^Tx_i$

$\boxed{e_i = (I - vv^T)x_i}$

ⓑ $\|e_i\|_2^2 = e_i^Te_i = ((I-vv^T)x_i)^T(I-vv^T)x_i = x_i^T(I-vv^T)(I-vv^T)x_i$

Now, $(I-vv^T)(I-vv^T) = I - vv^T - vv^T + vv^Tvv^T = I - 2vv^T + vv^T = I - vv^T$

$\Rightarrow \|e_i\|_2^2 = x_i^T(I-vv^T)x_i = \underset{\uparrow \text{ constant w.r.t. } v.}{x_i^Tx_i} - x_i^Tvv^Tx_i = x_i^Tx_i - v^Tx_ix_i^Tv$ (since $x_i^Tv = v^Tx_i$)

$\Rightarrow \|e_i\|_2^2 = x_i^Tx_i - v^Tx_ix_i^Tv \Rightarrow \sum_{i=1}^{n}\|e_i\|_2^2 = \sum_i \|x_i\|_2^2 - v^T\left(\sum_i x_ix_i^T\right)v$

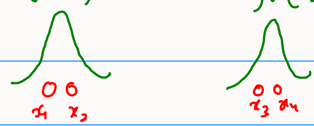
$\Rightarrow \underset{v,\, v^Tv=1}{\operatorname{argmin}} \frac{1}{n}\sum_{i=1}^{n}\|e_i\|_2^2 = \underset{v,\, v^Tv=1}{\operatorname{argmin}} -\frac{1}{n}v^T\left(\sum x_ix_i^T\right)v$

$\equiv \underset{v,\, v^Tv=1}{\operatorname{argmax}} \frac{1}{n}v^TXX^Tv$ since $\sum_{i=1}^{n} x_ix_i^T = XX^T$

ⓒ Derived in class.

(4)

$\mathcal{N}(\mu_1, \sigma_1^2)$ $\quad$ $\mathcal{N}(\mu_2, \sigma_2^2)$




$S = \{1, 2, 3, 4\}$

$(\mu_1, \sigma_1^2, \pi_1)$ $\quad (\mu_2, \sigma_2^2, \pi_2) \leftarrow$ unknowns

Initialize : $\mu_1 = 0, \ \sigma_1^2 = 1, \quad \mu_2 = 7, \ \sigma_2^2 = 1, \quad \pi_1 = \pi_2 = 1/2$

$$\mathcal{N}(x_j, \mu_1, \sigma_1^2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x_j - \mu_1)^2}{2\sigma_1^2}}$$

$$r(j1) = \frac{\pi_1 \mathcal{N}(x_j, \mu_1, \sigma_1^2)}{\pi_1 \mathcal{N}(x_j, \mu_1, \sigma_1^2) + \pi_2 \mathcal{N}(x_j, \mu_2, \sigma_2^2)}$$

$$\Rightarrow \quad r(11) = \frac{\pi_1 \mathcal{N}(1, 0, 1)}{\pi_1 \mathcal{N}(1, 0, 1) + \pi_2 \mathcal{N}(1, 7, 1)} = \frac{0.5 \times 0.242}{0.5 \times 0.242 + 0.5 \times 6.08 \times 10^{-9}} \approx 1$$

$$r(21) = \frac{\pi_1 \mathcal{N}(2, 0, 1)}{\pi_1 \mathcal{N}(2, 0, 1) + \pi_2 \mathcal{N}(2, 7, 1)} \approx 1$$

$$r(31) = \frac{\pi_1 \mathcal{N}(9, 0, 1)}{\pi_1 \mathcal{N}(9, 0, 1) + \pi_2 \mathcal{N}(9, 7, 1)} = \frac{1.03 \times 10^{-18}}{1.03 \times 10^{-18} + 0.5041} \approx 0$$

$$r(41) = \frac{\pi_1 \mathcal{N}(10, 0, 1)}{\pi_1 \mathcal{N}(10, 0, 1) + \pi_2 \mathcal{N}(10, 7, 1)} \approx 0$$

updated parameters.

$$n_1 = \sum_{j=1}^{4} r(j1) = 2 \qquad \Rightarrow \pi_1 = \frac{n_1}{n} = \frac{2}{4} = 1/2$$

$$\mu_1 = \frac{1}{2} \sum_{j=1}^{4} r(j1) x_j = \frac{1}{2}(1 \times 1 + 1 \times 2 + 0 \times 9 + 0 \times 10) = \frac{1}{2}(1 + 2) = 1.5$$

$$\sigma_1^2 = \frac{1}{2} \sum_{j=1}^{4} r(j1)(x_j - \mu_1)^2 = \frac{1}{2}\left(1 \times (1 - 1.5)^2 + 1 \times (2 - 1.5)^2 + 0 \times (9 - 1.5)^2 + 0 \times (10 - 1.5)^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}$$