Lecture 6: Bayes Classification

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Slides are prepared from several information sources including Duda, Hart, Stork

Recap: Bayes' Classification

- Posterior, likelihood, prior, evidence

$$\boxed{P(\omega_j|x) = \frac{p(x|\omega_j)P(\omega_j)}{p(x)},}$$

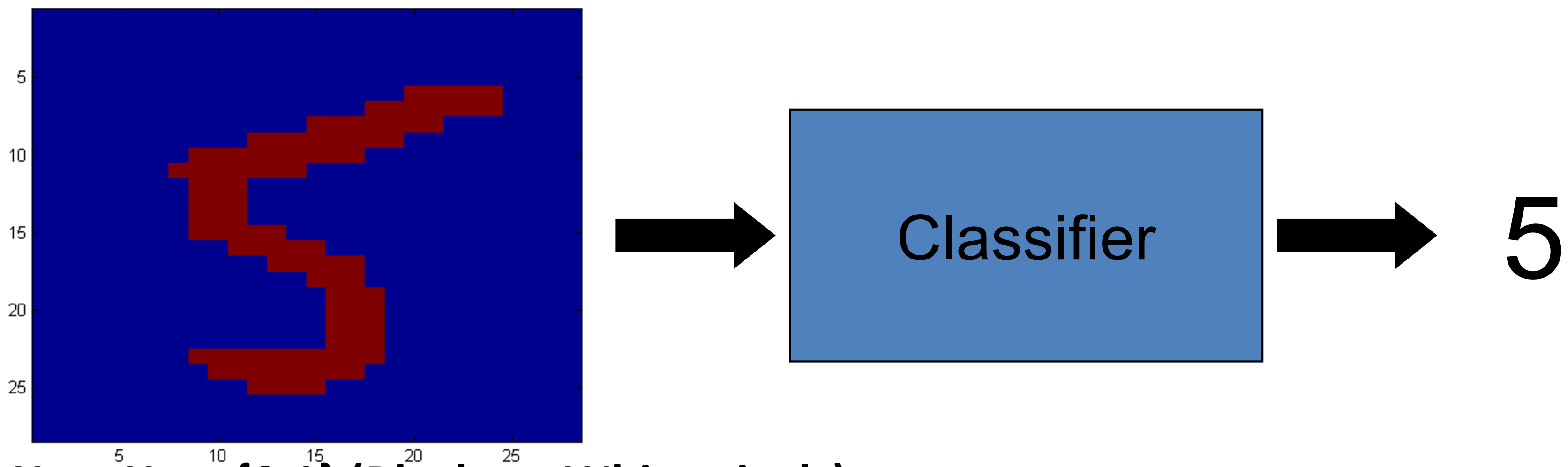
  – Evidence: In case of two categories

$$\boxed{p(x) = \sum_{j=1}^{2} p(x|\omega_j)P(\omega_j)}$$

$$\boxed{posterior = \frac{likelihood \times prior}{evidence}}$$

Another Application

- **Digit Recognition**



- **$X_1,\ldots,X_n \in \{0,1\}$ (Black vs. White pixels)**
- **$Y \in \{5,6\}$ (predict whether a digit is a 5 or a 6)**

The Bayes Classifier

- A good strategy is to predict:

$$\arg\max_{Y} P(Y|X_1, \ldots, X_n)$$

  – (for example: what is the probability that the image represents a 5 given its pixels?)

- So … How do we compute that?

The Bayes Classifier

- Use Bayes Rule!

$$P(Y|X_1,\ldots,X_n) = \frac{\overbrace{P(X_1,\ldots,X_n|Y)}^{\text{Likelihood}}\overbrace{P(Y)}^{\text{Prior}}}{\underbrace{P(X_1,\ldots,X_n)}_{\text{Normalization Constant}}}$$

- Why did this help? Well, we think that we might be able to specify how features are “generated” by the class label

The Bayes Classifier

- Let's expand this for our digit recognition task:

$$
\begin{aligned}
P(Y=5|X_1,\ldots,X_n) &= \frac{P(X_1,\ldots,X_n|Y=5)P(Y=5)}{P(X_1,\ldots,X_n|Y=5)P(Y=5)+P(X_1,\ldots,X_n|Y=6)P(Y=6)} \\
P(Y=6|X_1,\ldots,X_n) &= \frac{P(X_1,\ldots,X_n|Y=6)P(Y=6)}{P(X_1,\ldots,X_n|Y=5)P(Y=5)+P(X_1,\ldots,X_n|Y=6)P(Y=6)}
\end{aligned}
$$

- To classify, we'll simply compute these two probabilities and predict based on which one is greater

Model Parameters

- For the Bayes classifier, we need to “learn” two functions, the likelihood and the prior

- How many parameters are required to specify the prior for our digit recognition example?

Model Parameters

- How many parameters are required to specify the likelihood?
  - (Supposing that each image is 30x30 pixels)

?

Model Parameters

- The problem with explicitly modeling $P(X_1,\ldots,X_n|Y)$ is that there are usually way too many parameters:
  - We'll run out of space
  - We'll run out of time
  - And we'll need tons of training data (which is usually not available)

The Naïve Bayes Model

- The *Naïve Bayes Assumption*: Assume that all features are independent **given the class label Y**
- Equationally speaking:

$$P(X_1, \ldots, X_n|Y) = \prod_{i=1}^{n} P(X_i|Y)$$

- (We will discuss the validity of this assumption later)

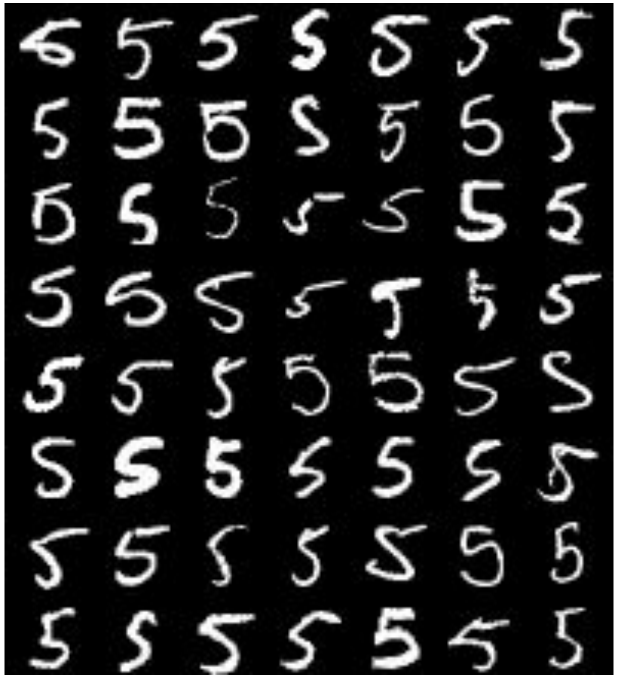
Why is this useful?

- # of parameters for modeling $P(X_1,\ldots,X_n|Y)$:
- - Given each x_i is a binary attribute and y is boolean
    - $2(2^n-1)$

- # of parameters for modeling $P(X_1|Y),\ldots,P(X_n|Y)$
    - $2n$

Naïve Bayes Training

- Now that we've decided to use a Naïve Bayes classifier, we need to train it with some data:



MNIST Training Data

Naïve Bayes Training

- Training in Naïve Bayes is **easy**:
  - Estimate P(Y=v) as the fraction of records with Y=v

$$P(Y = v) = \frac{Count(Y = v)}{\#\ records}$$

  - Estimate P(X_i=u|Y=v) as the fraction of records with Y=v for which X_i=u

$$P(X_i = u|Y = v) = \frac{Count(X_i = u \wedge Y = v)}{Count(Y = v)}$$

- (This corresponds to Maximum Likelihood estimation of model parameters)

Naïve Bayes Training

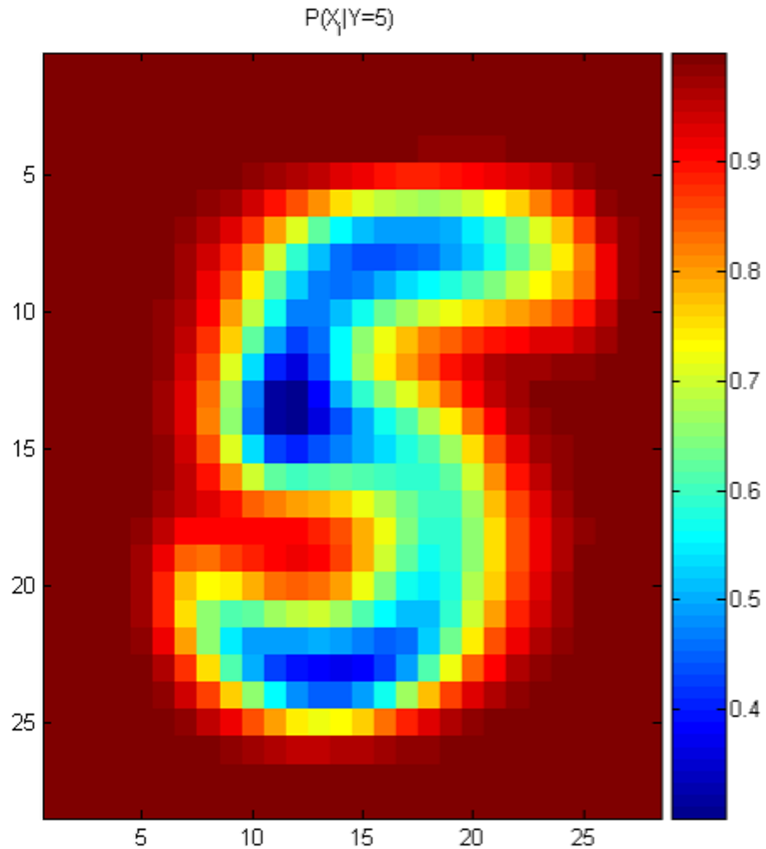
- In practice, some of these counts can be zero
- Fix this by adding "virtual" counts:

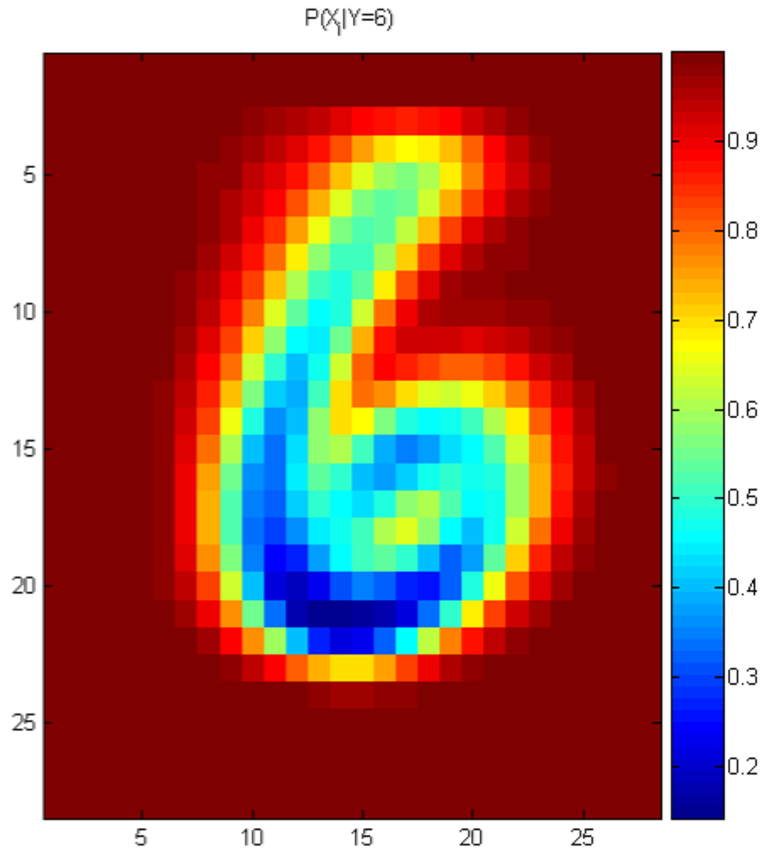
$$P(X_i = u|Y = v) = \frac{Count(X_i = u \wedge Y = v) + 1}{Count(Y = v) + 2}$$

  - (This is like putting a prior on parameters and doing MAP estimation instead of MLE)
  - This is called *Smoothing*

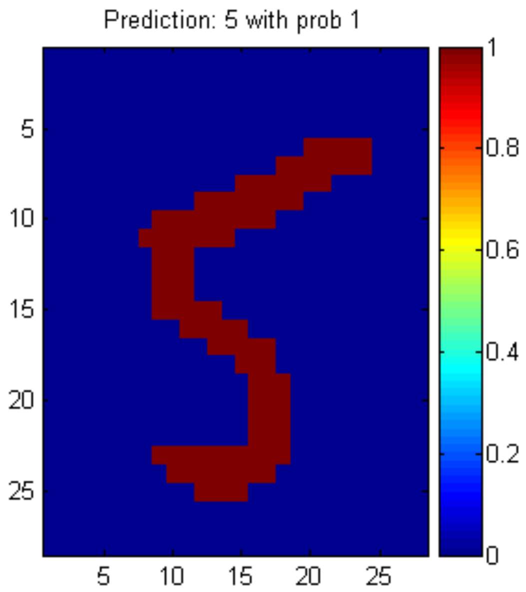
Naïve Bayes Training

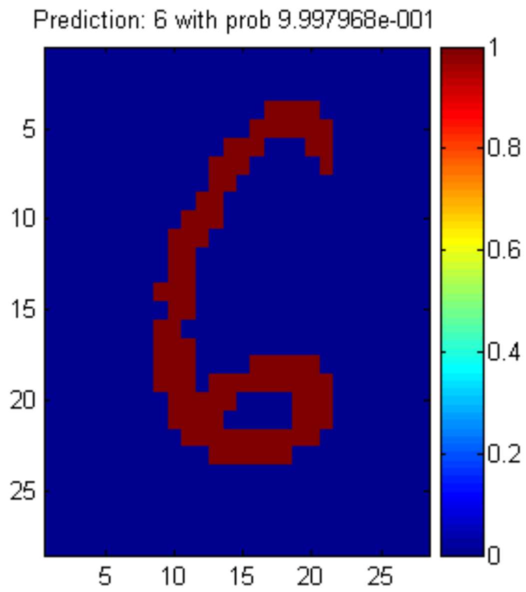
- For binary digits, training amounts to averaging all of the training fives together and all of the training sixes together.

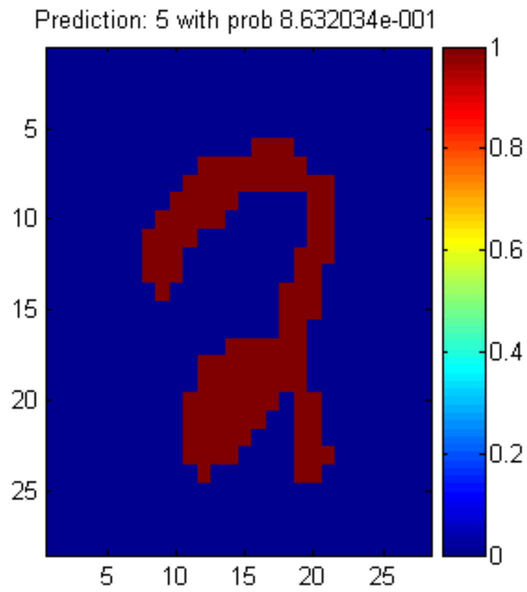




Naïve Bayes Classification







Another Example of the Naïve Bayes Classifier

The weather data, with counts and probabilities														
outlook			temperature			humidity			windy			play	
	yes	no		yes	no		yes	no		yes	no	yes	no
sunny	2	3	hot	2	2	high	3	4	false	6	2	9	5
overcast	4	0	mild	4	2	normal	6	1	true	3	3		
rainy	3	2	cool	3	1								
sunny	2/9	3/5	hot	2/9	2/5	high	3/9	4/5	false	6/9	2/5	9/14	5/14
overcast	4/9	0/5	mild	4/9	2/5	normal	6/9	1/5	true	3/9	3/5		
rainy	3/9	2/5	cool	3/9	1/5								

A new day				
outlook	temperature	humidity	windy	play
sunny	cool	high	true	?

- Likelihood of yes

$$=\frac{2}{9}\times\frac{3}{9}\times\frac{3}{9}\times\frac{3}{9}\times\frac{9}{14}=0.0053$$

- Likelihood of no

$$=\frac{3}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{3}{5}\times\frac{5}{14}=0.0206$$

- Therefore, the prediction is No

The Naive Bayes Classifier for Data Sets with Numerical Attribute Values

- One common practice to handle numerical attribute values is to assume normal distributions for numerical attributes.

The numeric weather data with summary statistics

outlook			temperature			humidity			windy			play	
	yes	no		yes	no		yes	no		yes	no	yes	no
sunny	2	3		83	85		86	85	false	6	2	9	5
overcast	4	0		70	80		96	90	true	3	3		
rainy	3	2		68	65		80	70					
				64	72		65	95					
				69	71		70	91					
				75			80						
				75			70						
				72			90						
				81			75						
sunny	2/9	3/5	mean	73	74.6	mean	79.1	86.2	false	6/9	2/5	9/14	5/14
overcast	4/9	0/5	std dev	6.2	7.9	std dev	10.2	9.7	true	3/9	3/5		
rainy	3/9	2/5											

- Let $x_1, x_2, \ldots, x_n$ be the values of a numerical attribute in the training data set.

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\sigma = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \mu)^2$$

$$f(w) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\mu)^2}{\sigma^2}}$$

- For examples,

$$f(\text{temperature} = 66 \mid \text{Yes}) = \frac{1}{\sqrt{2\pi}(6.2)} e^{-\frac{(66-73)^2}{2(6.2)^2}} = 0.0340$$

- Likelihood of Yes = $\frac{2}{9} \times 0.0340 \times 0.0221 \times \frac{3}{9} \times \frac{9}{14} = 0.000036$

- Likelihood of No = $\frac{3}{5} \times 0.0291 \times 0.038 \times \frac{3}{5} \times \frac{5}{14} = 0.000136$

Bayesian Decision Theory

- Generalization of the preceding ideas
  - Use of more than one feature
  - Use more than two states of nature
  - Allowing actions other than decide on the state of nature
    - Allowing actions other than classification primarily allows the possibility of rejection
    - Refusing to make a decision in close or bad cases!
  - Introduce a loss function which is more general than the probability of error
    - The loss function states how costly each action taken is

Bayesian Decision Theory – Continuous Features…

- Let $\{\omega 1, \omega 2, \ldots, \omega c\}$ be the set of c states of nature (or "categories")
- Let $\{\alpha 1, \alpha 2, \ldots, \alpha a\}$ be the set of possible actions
- Let $\lambda(\alpha i \mid \omega j)$ be the loss incurred for taking action αi when the true state of nature is ωj

Two-category Classification

- $\alpha 1$: deciding $\omega 1$
- $\alpha 2$: deciding $\omega 2$
- $\lambda ij = \lambda(\alpha i \mid \omega j)$
- Loss incurred for deciding αi when the true state of nature is ωj

Two-category Classification

- $\alpha 1$: deciding $\omega 1$
- $\alpha 2$: deciding $\omega 2$
- $\lambda ij = \lambda(\alpha i \mid \omega j)$
- Loss incurred for deciding αi when the true state of nature is ωj
- Conditional risk:

$$\begin{aligned} R(\alpha_1|\mathbf{x}) &= \lambda_{11}P(\omega_1|\mathbf{x}) + \lambda_{12}P(\omega_2|\mathbf{x}) \\ R(\alpha_2|\mathbf{x}) &= \lambda_{21}P(\omega_1|\mathbf{x}) + \lambda_{22}P(\omega_2|\mathbf{x}). \end{aligned}$$

Two-category Classification

- Our rule is the following:

  if R(α1 | x) < R(α2 | x)
- Action α1: "decide ω1" is taken
- This results in the equivalent rule :
- Decide ω1 if:

$$(\lambda_{21} - \lambda_{11})p(\mathbf{x}|\omega_1)P(\omega_1) > (\lambda_{12} - \lambda_{22})p(\mathbf{x}|\omega_2)P(\omega_2)$$

- and decide ω2 otherwise

Bayesian Decision Theory – Continuous Features…

- Overall risk

  R = Sum of all $\underbrace{R(\alpha i \mid x)}_{\textbf{Conditional risk}}$ for i = 1,…,a

- Minimizing R $\Longleftrightarrow$ Minimizing R(αi | x) for i = 1,…, a

- $$R(\alpha_i \mid x) = \sum_{j=1}^{j=c} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid x)$$ for i = 1,…,a