BT6270 Assignment 3: Coupled Hopf Oscillators

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I. HOPF MODEL

The Hopf oscillator represents the simplest canonical system exhibiting limit-cycle oscillations near a supercritical Hopf bifurcation. In Cartesian coordinates, its dynamics are described as

$$\dot{x} = \mu x - \omega y - (x^2 + y^2)x, \tag{1}$$
$$\dot{y} = \mu y + \omega x - (x^2 + y^2)y. \tag{2}$$

In complex form, with $z = x + iy$, this reduces to

$$\dot{z} = (\mu + i\omega - |z|^2)z. \tag{3}$$

For $\mu > 0$, trajectories converge to a stable limit cycle of radius $\sqrt{\mu}$, oscillating at frequency ω.

II. COUPLING MODELS

A. Complex Coupling

Two identical oscillators may be coupled as

$$\dot{z}_1 = (\mu + i\omega - |z_1|^2)z_1 + w_{12}z_2, \tag{1}$$
$$\dot{z}_2 = (\mu + i\omega - |z_2|^2)z_2 + w_{21}z_1, \tag{2}$$

where $w_{12} = Ke^{i\phi} = w_{21}^*$ is the complex coupling coefficient with magnitude K and phase ϕ. For identical natural frequencies ($\omega_1 = \omega_2$), the oscillators synchronize with a constant phase difference equal to ϕ. Real coupling enforces only 0 or π phase locking, whereas a complex coupling phase allows arbitrary steady-state phase offsets.

B. Power Coupling

For oscillators with different natural frequencies ($\omega_1 \neq \omega_2$), direct phase locking is not feasible. A generalized form termed *power coupling* is used:

$$\dot{z}_1 = (\mu + i\omega_1 - |z_1|^2)z_1 + w_{12}z_2^{\alpha_{12}}, \tag{3}$$
$$\dot{z}_2 = (\mu + i\omega_2 - |z_2|^2)z_2 + w_{21}z_1^{\alpha_{21}}, \tag{4}$$

where $\alpha_{12} = \omega_1/\omega_2$ and $\alpha_{21} = \omega_2/\omega_1$. The normalized phase difference

$$\psi = \frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} \tag{5}$$

converges to a steady value determined by the coupling phase and the initial conditions. The weights are $w_{12} = Ke^{i\phi/\omega_2}$, $w_{21} = Ke^{i\phi/\omega_1}$.

III. SIMULATION

Parameters used were $\mu = 1$, $K = 0.2$. Four coupling cases were simulated, as listed below:

Case	Coupling Type	(ω_1, ω_2)	Required Phase diff. (deg)
1	Complex	(5, 5)	-47
2	Complex	(5, 5)	98
3	Power	(5,15)	-47
4	Power	(5,15)	98

IV. RESULTS

Case 1: Complex Coupling ($\phi = -47°$)

Computed coupling coefficients and steady phase lag:

$$W_{12} = Ke^{i\phi} \quad = 0.1363996720 - 0.1462707403\,i,$$
$$W_{21} = Ke^{-i\phi} = 0.1363996720 + 0.1462707403\,i,$$
$$\Delta\theta_{\text{steady}} = -0.820305 \text{ rad} \quad (\approx -47.00°).$$

Figure 1 shows the time traces and phase-locked trajectories.

Case 2: Complex Coupling ($\phi = 98°$)

Computed coupling coefficients and steady phase lead:

$$W_{12} = Ke^{i\phi} \quad = -0.0278346202 + 0.1980536137\,i,$$
$$W_{21} = Ke^{-i\phi} = -0.0278346202 - 0.1980536137\,i,$$
$$\Delta\theta_{\text{steady}} = 1.710423 \text{ rad} \quad (\approx 98.00°).$$

Corresponding time traces and phase-locked trajectories are shown in Figure 2.

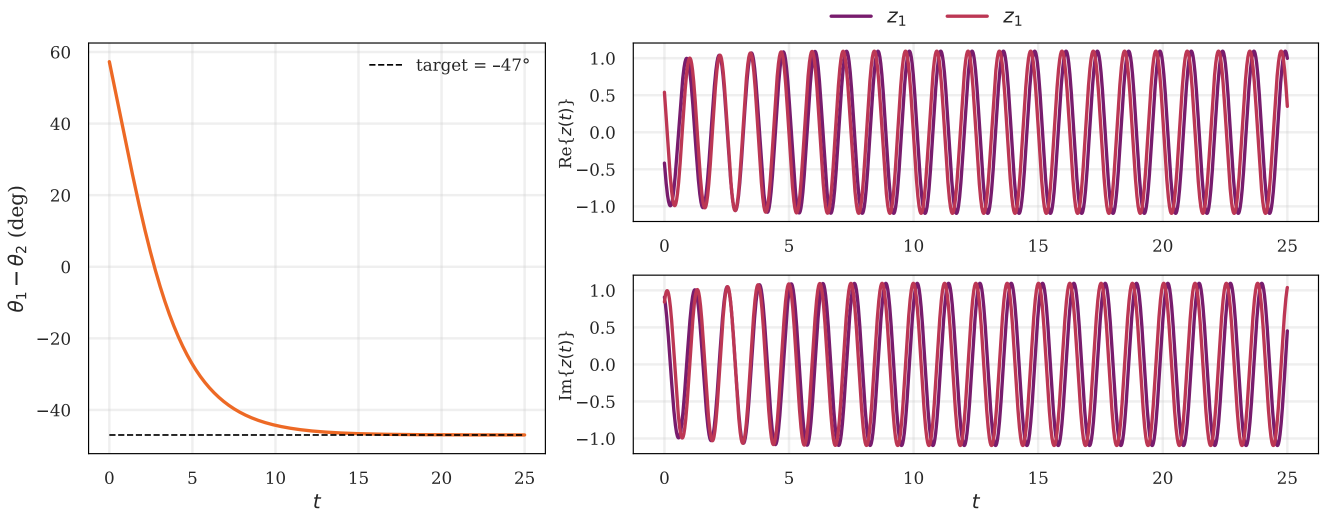

FIG. 1: **Case 1 – Complex Coupling ($\phi = -47°$):** The oscillators synchronize with a steady lag of -0.8203 rad ($-47.00°$). Amplitude remains constant at unity.

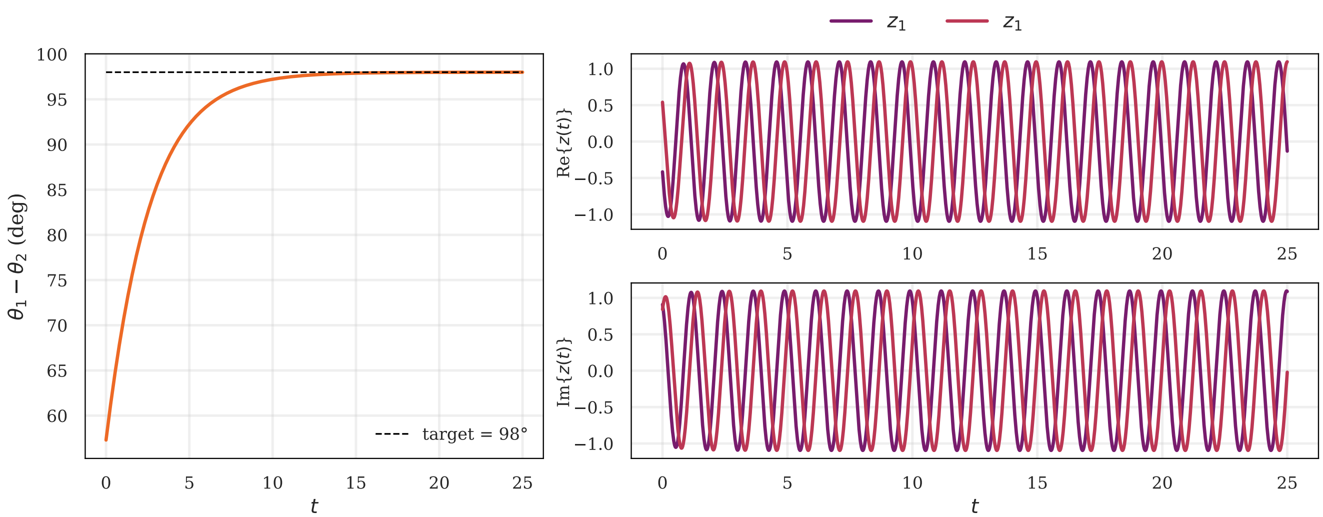

FIG. 2: **Case 2 – Complex Coupling ($\phi = 98°$):** Phase-locked state with $\Delta\theta \approx 1.7104$ rad ($98.00°$).

Case 3: Power Coupling ($\psi_{\text{target}} = -47°$)

For $(\omega_1, \omega_2) = (5, 15)$, the required coupling phase is

$$\psi_{\text{target}} = \frac{\phi}{\omega_1 \omega_2},$$

$$\phi = \psi_{\text{target}}\, \omega_1 \omega_2 = (-47°) \times 75 = -3525°.$$

Modulo 360°, this corresponds to an effective $\phi \equiv 75°$ in the coupling term. Using this ϕ, the coupling coefficients become:

$$w_{12} = K e^{i\phi/\omega_2} = 0.1997 + 0.0664i,$$

$$w_{21} = K e^{-i\phi/\omega_1} = 0.1888 - 0.1203i.$$

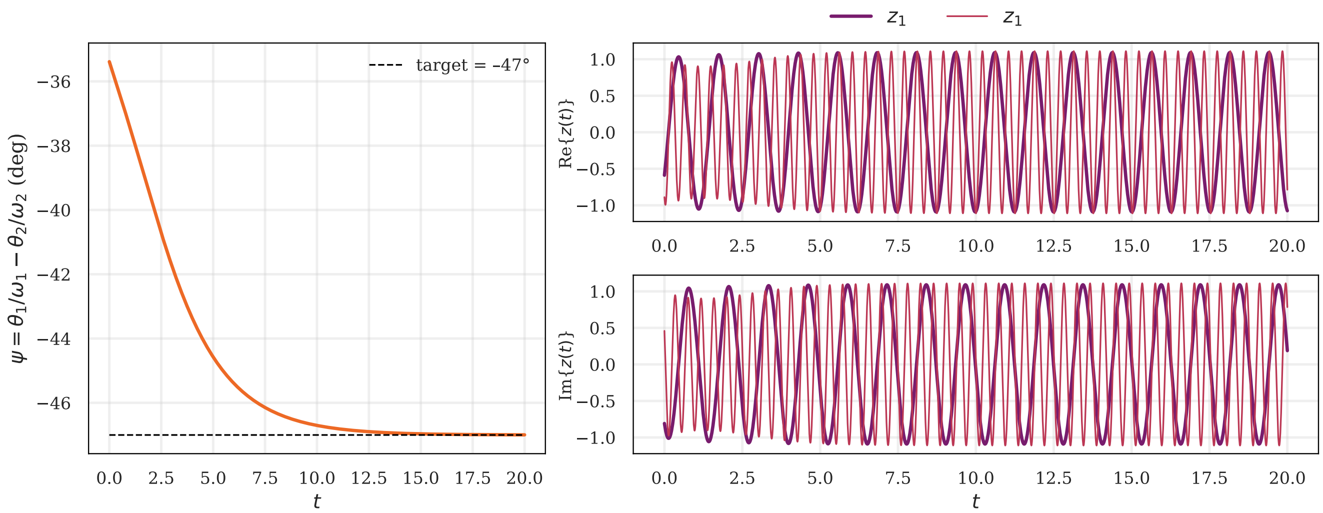

FIG. 3: **Case 3 – Power Coupling ($\psi_{\mathbf{target}} = -47°$):** Normalized phase difference ψ converges to the specified target value of $-47°$ despite the frequency ratio $\omega_2/\omega_1 = 3$.

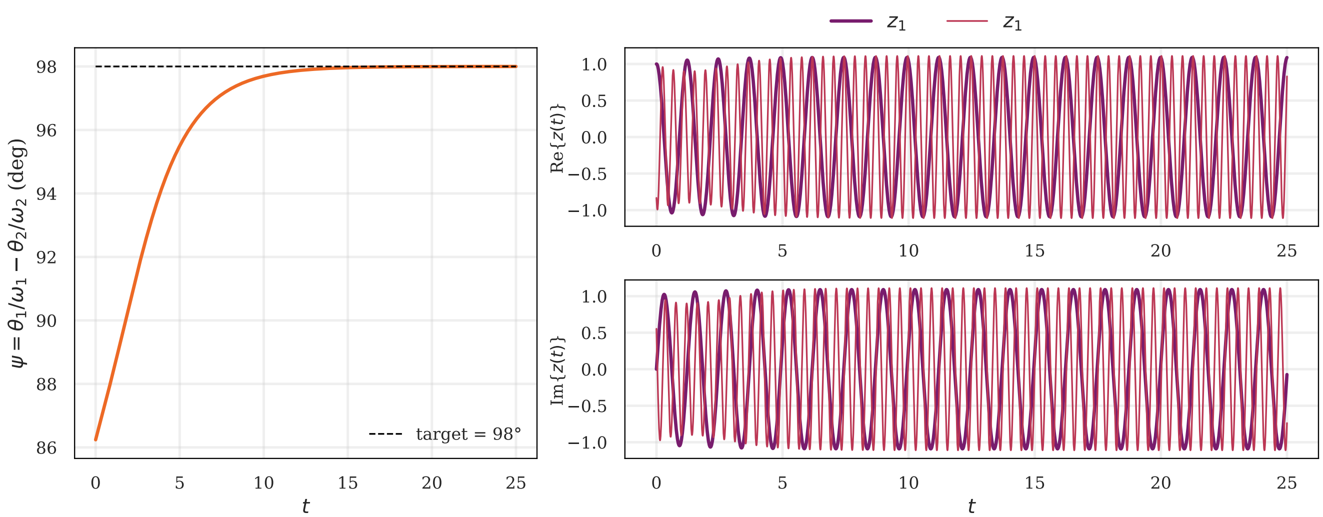

FIG. 4: **Case 4 – Power Coupling ($\psi_{\mathbf{target}} = 98°$):** Normalized phase difference ψ locks at 98° after selecting the correct integer branch ($m = -5$).

Case 4: Power Coupling ($\psi_{\text{target}} = 98°$)

Here the target normalized phase difference is $\psi_{\text{target}} = 98°$, implying a coupling phase

$$\phi = \psi_{\text{target}}\,\omega_1\omega_2 = 98° \times 75 = 7350°.$$

Since ϕ appears inside the exponentials $e^{i\phi/\omega_1}$ and $e^{i\phi/\omega_2}$, multiple equivalent coupling phases differing by integer multiples of 2π correspond to distinct dynamical branches. Not all branches produce stable synchronization.

In the attached notebook, ϕ was scanned over integer branch offsets

$$\phi_m = \psi_{\text{target}}\omega_1\omega_2 + 2\pi m,$$

to identify the branch that yielded a stable steady ψ. This search revealed that for $m = -20, -5, 10, 25, ...$ the system converged cleanly to the desired $\psi_{\text{steady}} \approx 98°$.

Coupling coefficients used were:

$$w_{12} = Ke^{i\phi_m/\omega_2} = 0.1987 + 0.0228i,$$

$$w_{21} = Ke^{-i\phi_m/\omega_1} = 0.1884 - 0.0671i.$$

The normalized phase difference then converged to

$$\psi_{\text{steady}} \approx 98.0°,$$

Multistability in Power Coupling

The normalized phase difference in power coupling can settle to multiple values in steady sate depending on initial conditions. These correspond to distinct fixed points of the phase relation that arise because ϕ enters the coupling terms modulo 2π. Figure 5 shows the multistability map obtained by simulating multiple initial conditions when the target normalised phase difference was $-47°$, demonstrating that only specific branches yield the intended phase alignment.

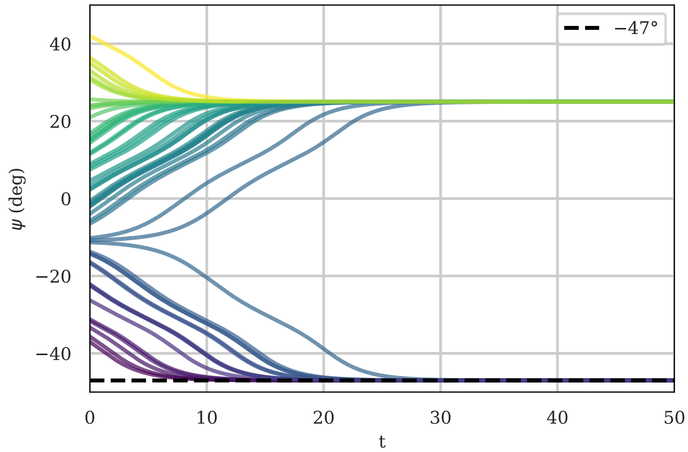

FIG. 5: **Multistability in Power Coupling:** Distinct initial condisions produce different steady normalized phases ψ, showing the system's inherent multistability.

V. CONCLUSION

- Real coupling can only allow phase locking with a difference of 0 or π.
- Complex coupling allows the steady state phase difference to be set to any value.
- If the natural frequencies of the oscillators are different, Power coupling allows the normalized phase difference to be locked, subject to initial conditions.
- in all cases, increasing the magnitude of weights (K) ensures faster convergence.