BT6270 Assignment 2: FitzHugh-Nagumo Model

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I. THE FN MODEL

The FitzHugh–Nagumo (FN) model is a simplified, **dimensionless** two-variable reduction of the Hodgkin–Huxley (HH) equations. It captures neuronal spiking behavior through an activator variable $\boldsymbol{V}$ and a recovery variable $\boldsymbol{w}$ as:

$$\frac{dV}{dt} = f(V) - w + I_{\text{ext}}, \tag{1}$$

$$\frac{dw}{dt} = bV - rw, \tag{2}$$

$$\text{where} \quad f(V) = V(a - V)(V - 1) \tag{3}$$

here $\boldsymbol{a} \in (\mathbf{0}, \mathbf{1})$, $\boldsymbol{b}$, & $\boldsymbol{r}$ determine the type of response shown by the model for an input current $\boldsymbol{I}_{\text{ext}}$.

II. SIMULATION

For the first three cases, values $a = 0.5$, $b = 0.1$, $r = 0.1$ were used. An Euler integration scheme (time step $\Delta t = 0.01$, total duration $T = 200$) was used to perform the simulations.

The code used to generate the results and figures in this report is provided in the accompanying Jupyter notebook.

III. RESULTS

A. Case-1: $I_{\text{ext}} = 0$ (Excitability)

For initial conditions $w(0) = 0$, (i) $V(0) = 0.3 (< a)$ and (ii) $V(0) = 0.7 (> a)$, trajectories settle to the stable fixed point at $V = 0, w = 0$ without oscillations. The phase portrait, trajectories, and $V - t, w - t$ plots are shown in Fig. 1.

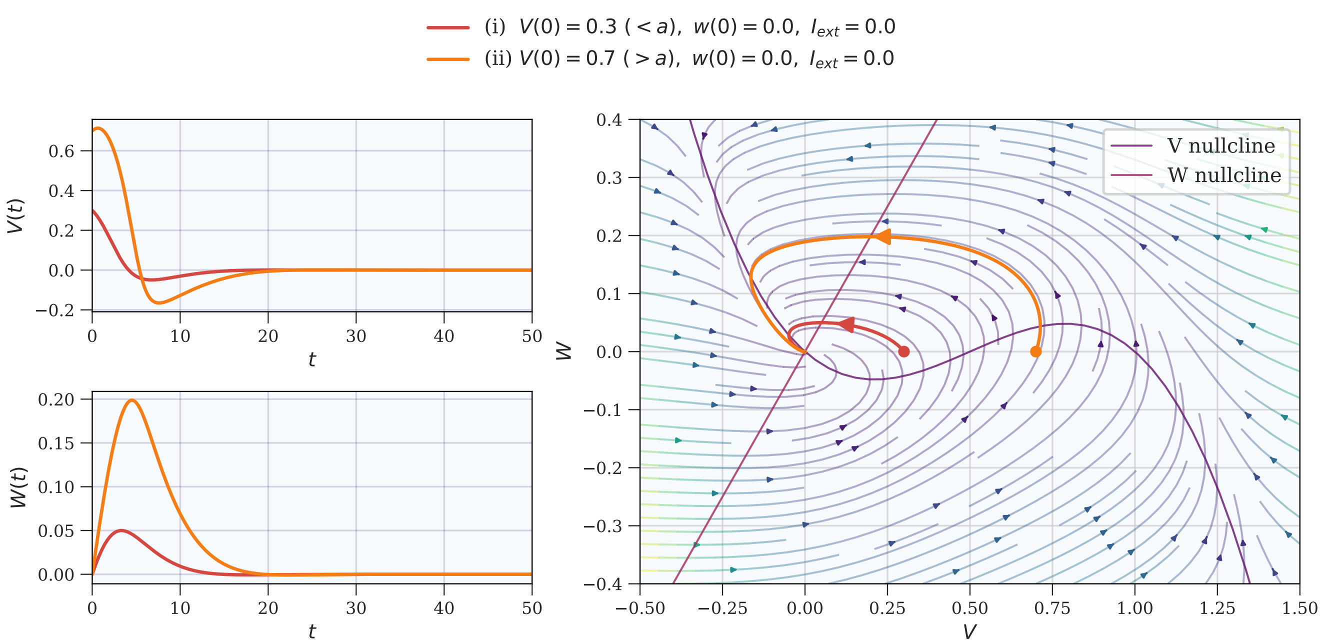

FIG. 1: Top-left: $V(t)$ for two initial conditions, (i) $V(0) < a$ (red) and (ii) $V(0) > a$ (orange), with $w(0) = 0$ and $I_{\text{ext}} = 0$. Bottom-left: corresponding $W(t)$ dynamics. Right: phase portrait in the (V, W) plane with the nullclines and trajectories for the two initial conditions superimposed.

B. Case-2: $I_1 < I_{\mathbf{ext}} < I_2$ (Oscillations)

Oscillations typically occur between two critical values of the external current I_1 and I_2, where the w-nullcline and the V-nullcline both have a positive slope at their intersection. If $V_{\min}$ and $V_{\max}$ denote the locations of the local maximum and minimum of $f(V)$. Then:

$$V_{min} = \frac{1-\sqrt{1/3}}{2}, \quad V_{max} = \frac{1+\sqrt{1/3}}{2} \tag{4}$$

and

$$I_1 = \frac{b}{r}V_{\min} - f(V_{\min}), \quad I_2 = \frac{b}{r}V_{\max} - f(V_{\max}) \tag{5}$$

yielding $\boldsymbol{I_1} \approx \mathbf{0.259}$ and $\boldsymbol{I_2} \approx \mathbf{0.740}$. Choosing $\boldsymbol{I}_{\text{ext}} = \mathbf{0.5}$ places the system in the oscillatory regime with a fixed point at (V,w) = (0.5, 0.5). Two trajectories shown in Fig. 2. have initial conditions (i) (V(0), w(0)) = (0.51, 0.52) and (ii) (V(0), w(0)) = (-0.1, 0.3).

Trajectory-(i) represents a small perturbation from the fixed point. As seen in the figure, it does not return back to the fixed point but instead evolves into a limit cycle, indicating the fixed point is **not stable**. Both (i) and (ii) converge to the same limit cycle.

Numerical stability analysis of the fixed point is done in the accompanying code, confirming that the fixed point is unstable.

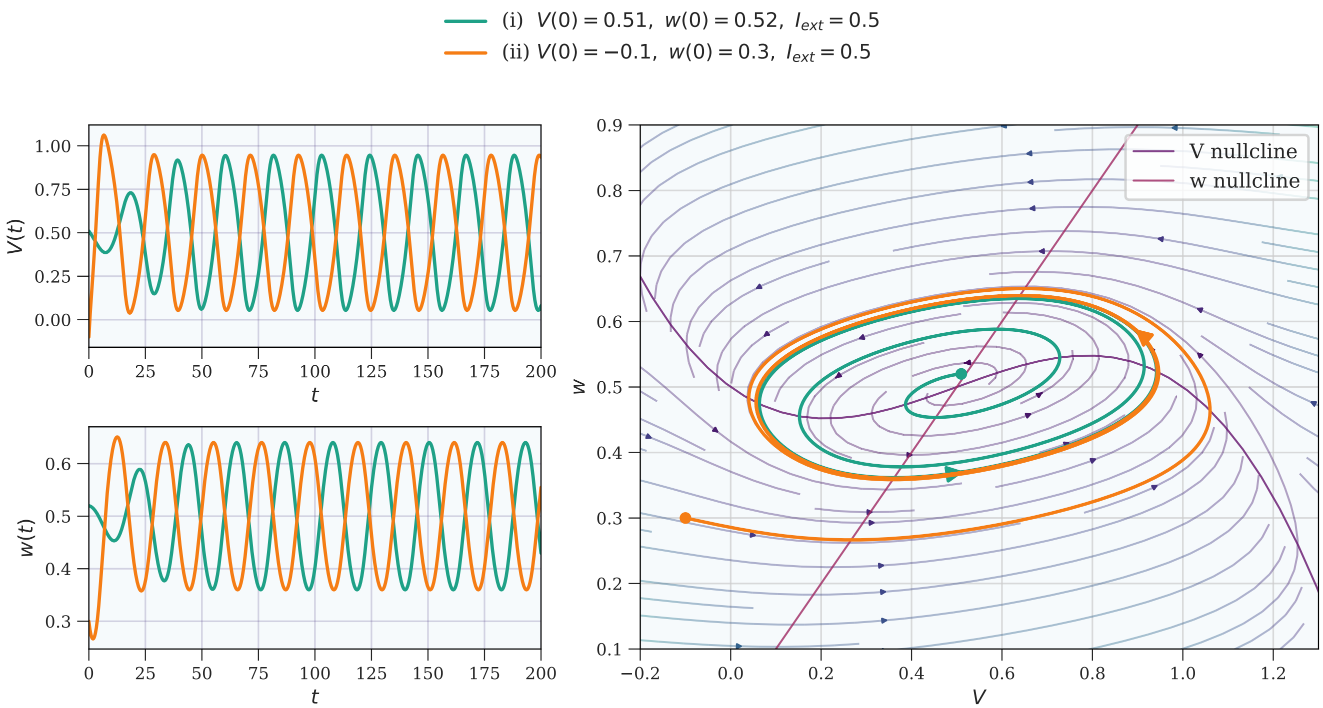

FIG. 2: Top-left: $V(t)$ for two initial conditions. Bottom-left: corresponding $w(t)$ dynamics. Right: phase portrait in the (V, W) plane with the nullclines and trajectories for the two initial conditions superimposed. Trajectory-(i) is a small perturbation from the fixed point at $(0.5, 0.5)$

C. Case-3: $I_2 < I_{\mathbf{ext}}$ (Depolarization)

Setting $I_{\text{ext}} = 1.0$ ($> I_2$) leads to one fixed point at $(V, W) = (1.0, 1.0)$. Two trajectories are again picked as: (i) $(V(0), w(0)) = (1.0, 1.05)$ and (ii) $(V(0), w(0)) = (0.5, 1.4)$; (i) is a small perturbation from the fixed point and here it returns back to the fixed point, unlike Case-2, indicating the fixed point **is stable**. This is again confirmed by stability analysis performed in the code.

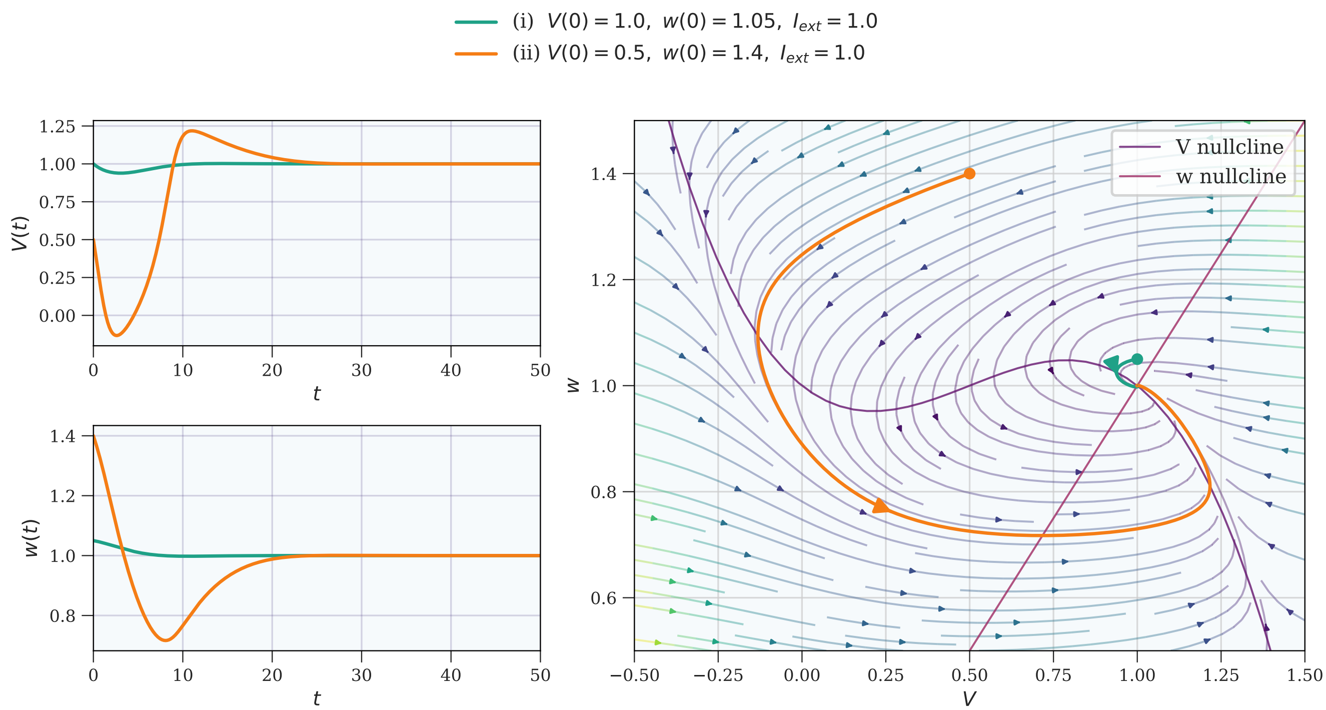

FIG. 3: Top-left: $V(t)$ for two initial conditions. Bottom-left: corresponding $w(t)$ dynamics. Right: phase portrait in the (V, W) plane with the nullclines and trajectories for the two initial conditions superimposed. Trajectory-(i) is a small perturbation from the fixed point at $(1.0, 1.0)$

D. Case-3: $I_2 < I_{\mathbf{ext}}$ (Bistability)

For the w-nullcline to intersect the V-nullcline at three distinct points, we need to set the slope of the w-nullcline such that:

$$\text{slope}\left(=\frac{b}{r}\right) < \frac{f(V_{max}) - f(V_{min})}{V_{max} - V_{min}}$$

Choosing $b = 0.05$, $r = 2.0$, $I_{\text{ext}} = 0.0$ now leads to three intersections, i.e., three fixed points, labelled $P_1, P_2, \&\ P_3$ in increasing order of their V-coordinate. Numerically, we find found that $P_1 = (0.0, 0.0), P_2 \approx (0.56, 0.01), P_3 \approx (0.94, 0.02)$. Numerical stability analysis shows that P_1 & P_3 are stable fixed points, whereas P_2 is a saddle point.

Three trajectories with initial conditions $(V(0), w(0))$ = (i) $(-0.1, -0.05)$, (ii) $(0.5, 0.05)$, (iii) $(1.0, 0.05)$ are chosen close to the three fixed points. As expected (i) and (iii) evolve towards the nearest fixed point whereas (ii) first moves towards and then away from P_2, characteristic of saddle points.

Unlike the previous cases (were all trajectories either evolved towards a single stable point or into a common limit cycle), here, all trajectories evolve into one of two stable nodes- i.e, the system is **bistable**. The final state can be determined based on the initial conditions: all trajectories to the right of the separatrix corresponding to P_2 evolve towards P_3 and all trajectories to the left evolve towards P_1.

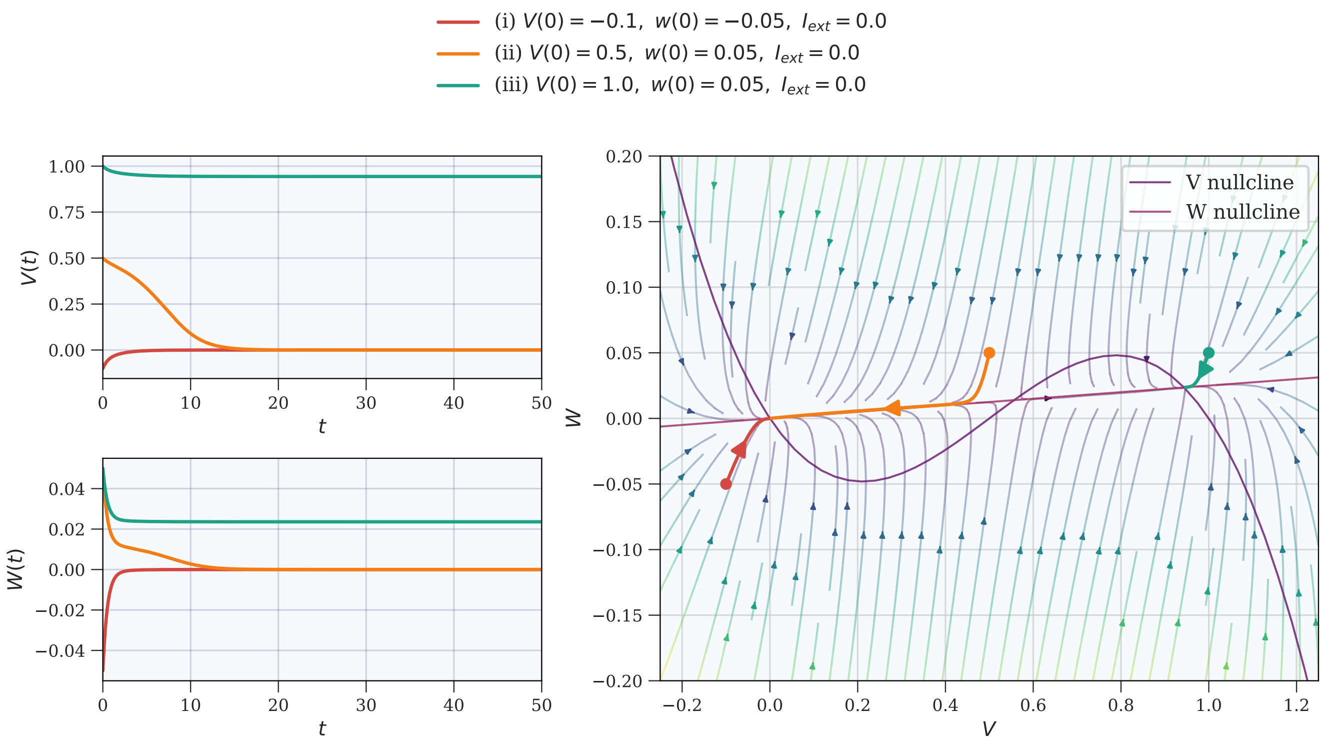

FIG. 4: Top-left: $V(t)$ for two initial conditions. Bottom-left: corresponding $w(t)$ dynamics. Right: phase portrait in the (V, W) plane with the nullclines and trajectories for the three initial conditions superimposed.