Fuzzy Logic & Neural Networks (CS-514)

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Impact of Parameter on the Output

Derivatives

The derivative of a simple constant function:

$$f(x) = 1 \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}1 = 0$$

The derivative of a linear function:

$$f(x) = x \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}x = \frac{d}{dx}x^1 = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$$

$$f(x) = 3x^2 \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}3x^2 = 3 \cdot \frac{d}{dx}x^2 = 3 \cdot 2x^{2-1} = 3 \cdot 2x^1 = 6x$$

Impact of Parameter on the Output

Derivatives

$$f(x) = 3x^2 + 5x \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}[3x^2 + 5x] =$$

$$= \frac{d}{dx}3x^2 + \frac{d}{dx}5x^1 =$$

$$= 3 \cdot \frac{d}{dx}x^2 + 5 \cdot \frac{d}{dx}x^1 =$$

$$= 3 \cdot 2x^{2-1} + 5 \cdot 1x^{1-1} =$$

$$= 3 \cdot 2x^1 + 5 \cdot x^0 =$$

$$= 6x + 5$$

Impact of Parameter on the Output

Derivatives

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$$

$$f(x) = 5x^5 + 4x^3 - 5 \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}[5x^5 + 4x^3 - 5] =$$

$$= \frac{d}{dx}5x^5 + \frac{d}{dx}4x^3 - \frac{d}{dx}5 =$$

$$= 5 \cdot \frac{d}{dx}x^5 + 4 \cdot \frac{d}{dx}x^3 - \frac{d}{dx}5 =$$

$$= 5 \cdot 5x^{5-1} + 4 \cdot 3x^{3-1} - 0 =$$

$$= 5 \cdot 5x^4 + 4 \cdot 3x^2 =$$

$$= 25x^4 + 12x^2$$

Impact of Parameter on the Output

Partial Derivatives

- The **partial derivative** measures how much impact a single parameter has on the function's output.
- The ∂ operator means explicitly the partial derivative.

$$f(x, y, z) \quad \rightarrow \quad \frac{\partial}{\partial x} f(x, y, z), \frac{\partial}{\partial y} f(x, y, z), \frac{\partial}{\partial z} f(x, y, z)$$

Impact of Parameter on the Output

Partial Derivatives

- The Partial Derivative of a Sum

$$f(x,y) = x + y \quad \rightarrow \quad \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}[x+y] = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y = 1 + 0 = 1$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}[x+y] = \frac{\partial}{\partial y}x + \frac{\partial}{\partial y}y = 0 + 1 = 1$$

$$f(x,y) = 2x + 3y^2 \quad \rightarrow \quad \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}[2x + 3y^2] = \frac{\partial}{\partial x}2x + \frac{\partial}{\partial x}3y^2 =$$

$$= 2 \cdot \frac{\partial}{\partial x}x + 3 \cdot \frac{\partial}{\partial x}y^2 = 2 \cdot 1 + 3 \cdot 0 = 2$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}[2x + 3y^2] = \frac{\partial}{\partial y}2x + \frac{\partial}{\partial y}3y^2 =$$

$$= 2 \cdot \frac{\partial}{\partial y}x + 3 \cdot \frac{\partial}{\partial y}y^2 = 2 \cdot 0 + 3 \cdot 2y^1 = 6y$$

Impact of Parameter on the Output

Partial Derivatives

- The Partial Derivative of a Sum

$$f(x,y) = 3x^3 - y^2 + 5x + 2 \quad \rightarrow$$

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}[3x^3 - y^2 + 5x + 2] = \frac{\partial}{\partial x}3x^3 - \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial x}5x + \frac{\partial}{\partial x}2 =$$

$$= 3\cdot\frac{\partial}{\partial x}x^3 - \frac{\partial}{\partial x}y^2 + 5\cdot\frac{\partial}{\partial x}x + \frac{\partial}{\partial x}2 = 3\cdot 3x^2 - 0 + 5\cdot 1 + 0 = 9x^2 + 5$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}[3x^3 - y^2 + 5x + 2] = \frac{\partial}{\partial y}3x^3 - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial y}5x + \frac{\partial}{\partial y}2 =$$

$$= 3\cdot\frac{\partial}{\partial y}x^3 - \frac{\partial}{\partial y}y^2 + 5\cdot\frac{\partial}{\partial y}x + \frac{\partial}{\partial y}2 = 3\cdot 0 - 2y^1 + 5\cdot 0 + 0 = -2y$$

Impact of Parameter on the Output

Partial Derivatives

- The Partial Derivative of Multiplication

$$f(x, y) = x \cdot y \quad \rightarrow$$

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x}[x \cdot y] = y \frac{\partial}{\partial x} x = y \cdot 1 = y$$

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y}[x \cdot y] = x \frac{\partial}{\partial y} y = x \cdot 1 = x$$

Impact of Parameter on the Output

Partial Derivatives

- The Partial Derivative of Multiplication

$$f(x, y, z) = 3x^3z - y^2 + 5z + 2yz \quad \rightarrow$$

$$\frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial x}[3x^3z - y^2 + 5z + 2yz] =$$

$$= \frac{\partial}{\partial x} 3x^3z - \frac{\partial}{\partial x} y^2 + \frac{\partial}{\partial x} 5z + \frac{\partial}{\partial x} 2yz =$$

$$= 3z \cdot \frac{\partial}{\partial x} x^3 - \frac{\partial}{\partial x} y^2 + 5 \cdot \frac{\partial}{\partial x} z + 2 \cdot \frac{\partial}{\partial x} yz =$$

$$= 3z \cdot 3x^2 - 0 + 5 \cdot 0 + 2 \cdot 0 = 9x^2z$$

Impact of Parameter on the Output

Partial Derivatives

- The Partial Derivative of Multiplication

$$f(x, y, z) = 3x^3z - y^2 + 5z + 2yz \quad \rightarrow$$

$$\frac{\partial}{\partial y} f(x, y, z) = \frac{\partial}{\partial y}[3x^3z - y^2 + 5z + 2yz] =$$

$$= \frac{\partial}{\partial y}3x^3z - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial y}5z + \frac{\partial}{\partial y}2yz =$$

$$= 3 \cdot \frac{\partial}{\partial y}x^3z - \frac{\partial}{\partial y}y^2 + 5 \cdot \frac{\partial}{\partial y}z + 2z \cdot \frac{\partial}{\partial y}y =$$

$$= 3 \cdot 0 - 2y + 5 \cdot 0 + 2z \cdot 1 = -2y + 2z$$

Impact of Parameter on the Output

Partial Derivatives

- The Partial Derivative of Multiplication

$$f(x,y,z) = 3x^3z - y^2 + 5z + 2yz \quad \rightarrow$$

$$\frac{\partial}{\partial z} f(x,y,z) = \frac{\partial}{\partial z}[3x^3z - y^2 + 5z + 2yz] =$$

$$= \frac{\partial}{\partial z}3x^3z - \frac{\partial}{\partial z}y^2 + \frac{\partial}{\partial z}5z + \frac{\partial}{\partial z}2yz =$$

$$= 3x^3 \cdot \frac{\partial}{\partial z}z - \frac{\partial}{\partial z}y^2 + 5 \cdot \frac{\partial}{\partial z}z + 2y \cdot \frac{\partial}{\partial z}z =$$

$$= 3x^3 \cdot 1 - 0 + 5 \cdot 1 + 2y \cdot 1 = 3x^3 + 5 + 2y$$

Impact of Parameter on the Output

Partial Derivatives

➢ The Partial Derivative of Max

$$f(x) = max(x,0) \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}max(x,0) = 1(x > 0)$$

$$= \frac{d}{dx}\max(x,0) = 0(x \leqslant 0)$$

$$f(x,y) = max(x,y) \quad \rightarrow \quad \frac{\partial}{\partial x}f(x,y) = \frac{\partial}{\partial x}\max(x,y) = 1(x \geqslant y)$$

$$= \frac{\partial}{\partial x}\max(x,y) = 0(x < y)$$

Impact of Parameter on the Output

The Gradient

- The gradient is a vector composed of all the partial derivatives of a function:

$$f(x, y, z) = 3x^3 z - y^2 + 5z + 2yz \quad \rightarrow$$

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} f(x, y, z) \\ \frac{\partial}{\partial y} f(x, y, z) \\ \frac{\partial}{\partial z} f(x, y, z) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f(x, y, z) = \begin{bmatrix} 9x^2 z \\ -2y + 2z \\ 3x^3 + 5 + 2y \end{bmatrix}$$

Impact of Parameter on the Output

The Chain Rule

- The chain of functions:

$$z = f(x)$$

$$y = g(z)$$

$$y = g(f(x))$$

Impact of Parameter on the Output

The Chain Rule

- Consider a chain of functions:

$$\frac{d}{dx}f(g(x)) = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} = f'(g(x)) \cdot g'(x)$$

$$\frac{\partial}{\partial x}f(g(y,h(x,z))) = \frac{\partial f(g(y,h(x,z)))}{\partial g(y,h(x,z))} \cdot \frac{\partial g(y,h(x,z))}{\partial h(x,z)} \cdot \frac{\partial h(x,z)}{\partial x}$$

Activation Function Derivatives

The Linear Activation Function Derivative

$$f(z) = y = z$$

$$\frac{\partial f(z)}{\partial z} = \frac{\partial z}{\partial z} = 1$$

Activation Function Derivatives

The ReLU Activation Function Derivative

$$f(z) = \mathrm{Re}LU(z) = \max(0, z)$$

$$\frac{\partial f(z)}{\partial z} = \frac{\partial \mathrm{Re}LU(z)}{\partial z} = \begin{cases} 1 & z > 0 \\ 0 & z \leqslant 0 \end{cases}$$

Activation Function Derivatives

The Leaky ReLU Activation Function Derivative

$$f(z) \;=\; \text{Leaky ReLU}(z) = \begin{cases} z & \text{if } z > 0 \\ \alpha z & \text{if } z \leq 0 \end{cases} ; 0 < \alpha < 0.1$$

$$\frac{\partial}{\partial z} \text{Leaky ReLU}(z) = \begin{cases} 1 & \text{if } z > 0 \\ \alpha & \text{if } z \leq 0 \end{cases}$$

Activation Function Derivatives

The Sigmoid Activation Function Derivative (logsig)

$$f(z) = y = \frac{1}{1+e^{-z}}$$

$$\frac{\partial f(z)}{\partial z} = \frac{\partial y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{1+e^{-z}}\right)$$

$$f(z) = \frac{g(z)}{h(z)}$$

$$\frac{\partial f(z)}{\partial z} = \frac{g'(z)h(z) - g(z)h'(z)}{[h(z)]^2}$$

$$g(z) = 1,\ h(z) = (1+e^{-z})$$

$$\frac{\partial g(z)}{\partial z} = g'(z) = 0,\ \frac{\partial h(z)}{\partial z} = h'(z) = -e^{-z}$$

Activation Function Derivatives

The Sigmoid Activation Function Derivative (logsig)

$$f(z) = y = \frac{1}{1+e^{-z}}$$

$$\frac{\partial f(z)}{\partial z} = \frac{\partial y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{1+e^{-z}}\right)$$

$$\frac{\partial f(z)}{\partial z} = \frac{e^{-z}}{(1+e^{-z})^{2}}$$

$$\frac{\partial f(z)}{\partial z} = \frac{1}{(1+e^{-z})}\frac{(1+e^{-z}-1)}{(1+e^{-z})}$$

$$\frac{\partial f(z)}{\partial z} = f(z)\,(1-f(z))$$

Activation Function Derivatives

The Sigmoid Activation Function Derivative (tansig)

$$f(z) = y = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$\frac{\partial f(z)}{\partial z} = \frac{\partial y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right)$$

$$f(z) = \frac{g(z)}{h(z)}$$

$$\frac{\partial f(z)}{\partial z} = \frac{g'(z)h(z) - g(z)h'(z)}{[h(z)]^{2}}$$

$$g(z) = (e^{z} - e^{-z}),\ h(z) = (e^{z} + e^{-z})$$

$$\frac{\partial g(z)}{\partial z} = g'(z) = (e^{z} + e^{-z}),\ \frac{\partial h(z)}{\partial z} = h'(z) = (e^{z} - e^{-z})$$

Activation Function Derivatives

The Sigmoid Activation Function Derivative (tansig)

$$f(z) = y = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$\frac{\partial f(z)}{\partial z} = \frac{\partial y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right)$$

$$\frac{\partial f(z)}{\partial z} = \frac{(e^{z} + e^{-z})^{2} - (e^{z} - e^{-z})^{2}}{(e^{z} + e^{-z})^{2}}$$

$$\frac{\partial f(z)}{\partial z} = (1 - f^{2}(z))$$

Activation Function Derivatives

The Softmax Activation Function Derivative

$$S_i = \frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}$$

$$\frac{\partial S_i}{\partial z_m} = \frac{\partial}{\partial z_m}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right); \; j = 1, 2, ..m...., K$$

Activation Function Derivatives

The SoftMax Activation Function Derivative

We will consider two cases: when i=m and when i≠m.

Case: 1 i=m

$$\frac{\partial S_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right)$$

$$\frac{\partial S_i}{\partial z_i} = \frac{\frac{\partial e^{z_i}}{\partial z_i}\sum_{j=1}^{K} e^{z_j} - e^{z_i}\frac{\partial}{\partial z_i}\sum_{j=1}^{K} e^{z_j}}{\left(\sum_{j=1}^{K} e^{z_j}\right)^2}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

We will consider two cases: when i=m and when i≠m.

Case: 1 i=m

$$\frac{\partial S_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right)$$

$$\frac{\partial S_i}{\partial z_i} = \frac{e^{z_i}\sum_{j=1}^{K} e^{z_j} - e^{z_i}e^{z_i}}{\left(\sum_{j=1}^{K} e^{z_j}\right)^2}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

We will consider two cases: when i=m and when i≠m.

Case: 1 i=m

$$\frac{\partial S_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right)$$

$$\frac{\partial S_i}{\partial z_i} = = \frac{e^{z_i}}{\left(\sum_{j=1}^{K} e^{z_j}\right)} \frac{\left(\sum_{j=1}^{K} e^{z_j} - e^{z_i}\right)}{\left(\sum_{j=1}^{K} e^{z_j}\right)}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

We will consider two cases: when i=m and when i≠m.

Case: 1 i=m

$$\frac{\partial S_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right)$$

$$\frac{\partial S_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right) = S_i(1 - S_i)$$

Activation Function Derivatives

The Softmax Activation Function Derivative

We will consider two cases: when i=m and when i≠m.

Case: 2 i≠m

$$\frac{\partial S_i}{\partial z_m} = \frac{\partial}{\partial z_m}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right)$$

$$\frac{\partial S_i}{\partial z_m} = \frac{\frac{\partial e^{z_i}}{\partial z_m}\sum_{j=1}^{K} e^{z_j} - e^{z_i}\frac{\partial}{\partial z_m}\sum_{j=1}^{K} e^{z_j}}{\left(\sum_{j=1}^{K} e^{z_j}\right)^2}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

We will consider two cases: when *i*=m and when *i*≠m.

Case: 2 *i*≠m

$$\frac{\partial S_i}{\partial z_m} = \frac{\partial}{\partial z_m}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right)$$

$$\frac{\partial S_i}{\partial z_m} = \frac{e^{z_i}}{\left(\sum_{j=1}^{K} e^{z_j}\right)} \frac{-e^{z_m}}{\left(\sum_{j=1}^{K} e^{z_j}\right)}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

We will consider two cases: when *i*=m and when *i*≠m.

Case: 2 *i*≠m

$$\frac{\partial S_i}{\partial z_m} = \frac{\partial}{\partial z_m}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right)$$

$$\frac{\partial S_i}{\partial z_m} = \frac{\partial}{\partial z_m}\left(\frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}\right) = S_i\left(0 - S_m\right)$$

Activation Function Derivatives

The Softmax Activation Function Derivative

Combining both the Cases:

$$\frac{\partial S_i}{\partial z_m} = \begin{cases} S_i(1 - S_m) & for\ i = m \\ S_i(0 - S_m) & for\ i \neq m \end{cases}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

For better understanding consider:

General Case:

$$S = \begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix}$$

Example Case:

$$S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

General Case:

$$\frac{\partial S}{\partial z_m} = \frac{\partial}{\partial z_m}\begin{bmatrix} S_1 \\ S_2 \\ \dots \\ S_K \end{bmatrix}; \; For\ all\ m\!:\!1\ to\ K$$

Example Case:

$$\frac{\partial S}{\partial z_m} = \frac{\partial}{\partial z_m}\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}; \; For\ all\ m\!:\!1\ to\ 3$$

Activation Function Derivatives

The Softmax Activation Function Derivative

General Case:

$$\frac{\partial S}{\partial z_1} = \frac{\partial}{\partial z_1}\begin{bmatrix} S_1 \\ S_2 \\ \dots \\ S_K \end{bmatrix} = \begin{bmatrix} \frac{\partial S_1}{\partial z_1} \\ \frac{\partial S_2}{\partial z_1} \\ \dots \\ \frac{\partial S_K}{\partial z_1} \end{bmatrix}$$

Example Case:

$$\frac{\partial S}{\partial z_1} = \frac{\partial}{\partial z_1}\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial S_1}{\partial z_1} \\ \frac{\partial S_2}{\partial z_1} \\ \frac{\partial S_3}{\partial z_1} \end{bmatrix}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

For overall Softmax outputs:

General Case:

$$\frac{\partial S}{\partial z} = \begin{bmatrix} \frac{\partial S_1}{\partial z_1} & \frac{\partial S_1}{\partial z_2} & \cdots & \frac{\partial S_1}{\partial z_K} \\ \frac{\partial S_2}{\partial z_1} & \frac{\partial S_2}{\partial z_2} & \cdots & \frac{\partial S_2}{\partial z_K} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial S_K}{\partial z_1} & \frac{\partial S_K}{\partial z_2} & \cdots & \frac{\partial S_K}{\partial z_K} \end{bmatrix}$$

The above expression is called Jacobian Matrix

Activation Function Derivatives

The Softmax Activation Function Derivative

For overall Softmax outputs:

Example Case:

$$\frac{\partial S}{\partial z} = \begin{bmatrix} \frac{\partial S_1}{\partial z_1} & \frac{\partial S_1}{\partial z_2} & \frac{\partial S_1}{\partial z_3} \\ \frac{\partial S_2}{\partial z_1} & \frac{\partial S_2}{\partial z_2} & \frac{\partial S_2}{\partial z_3} \\ \frac{\partial S_3}{\partial z_1} & \frac{\partial S_3}{\partial z_2} & \frac{\partial S_3}{\partial z_3} \end{bmatrix}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

$$\frac{\partial S_i}{\partial z_m} = \begin{cases} S_i(1 - S_m) & for\ i = m \\ S_i(0 - S_m) & for\ i \neq m \end{cases}$$

For overall Softmax outputs:

General Case:

$$\frac{\partial S}{\partial z} = \begin{bmatrix} S_1(1 - S_1) & -S_1 S_2 & \dots & -S_1 S_K \\ -S_1 S_2 & S_2(1 - S_2) & \dots & -S_2 S_K \\ \dots & \dots & \dots & \dots \\ -S_1 S_K & -S_2 S_K & \dots & S_K(1 - S_K) \end{bmatrix}$$

Activation Function Derivatives

The Softmax Activation Function Derivative

$$\frac{\partial S_i}{\partial z_m} = \begin{cases} S_i(1 - S_m) & for\ i = m \\ S_i(0 - S_m) & for\ i \neq m \end{cases}$$

For overall Softmax outputs:

Example Case:

$$\frac{\partial S}{\partial z} = \begin{bmatrix} S_1(1 - S_1) & -S_1 S_2 & -S_1 S_3 \\ -S_1 S_2 & S_2(1 - S_2) & -S_2 S_3 \\ -S_1 S_3 & -S_2 S_3 & S_3(1 - S_3) \end{bmatrix}$$