Math 323 - Assignment 1

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Problem 1

Suppose a family contains two children of different ages, and we are interested in the gender of these children. Let F denote a female child, and M denote male child, such that the ordered pair FM denotes an older female child and a younger male child without loss of generality. In the set S, we thus have:

$$S = \{FF, FM, MF, MM\}$$

Let A denote the subset of possibilities containing no males, B denote the subset containing two males, and C the subset containing at least one male. List the elements of $A, B, C, A \cap B, A \cup B, A \cap C, A \cup C, B \cap C, B \cup C$, and $C \cap \bar{B}$.

Solution.

$$\begin{aligned}
A &= \{FF\} \\
B &= \{MM\} \\
\bar{B} &= \{FF, FM, MF\} \\
C &= \{FM, MF, MM\} \\
A \cap B &= \{\} = \varnothing \\
A \cup B &= \{FF, MM\} \\
A \cap C &= \{\} = \varnothing \\
A \cup C &= \{FF, FM, MF, MM\} = S \\
B \cap C &= \{MM\} = B \\
B \cup C &= \{FM, MF, MM\} = C \\
C \cap \bar{B} &= \{FM, MF\}
\end{aligned}$$

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Problem 2

Suppose two dice are tossed, and the numbers on he upper faces are observed. Let S denote the set of all possible pairs that can be observed. [These pairs can be listed, for example, by letting $(2,3)$ denote that a 2 was observed on the first die, and a 3 on the second.]

a) *Define the following subsets of S:*

$$\begin{aligned} A &: \textit{The number on the second die is even.} \\ B &: \textit{The sum of the two numbers is even.} \\ C &: \textit{At least one number in the pair is odd.} \end{aligned}$$

b) *List the points in $A, \bar{C}, A \cap B, A \cap \bar{B}, \bar{A} \cup B$, and $\bar{A} \cap C$*

Solution.

a) We begin by defining the a particular subset of the natural numbers $\mathbb{N}_k$. Let k be a positive integer such that:

$$\mathbb{N}_k = \{1, 2, \ldots, k\}$$

The particular subset N_k has the following multiplicative property for $\alpha \in \mathbb{N}$:

$$\mathbb{N}_{\alpha(k)} = \{\alpha(1), \alpha(2), \ldots, \alpha(k)\}$$

We can thus define the set $S = \{(a, b) : a, b \in \mathbb{N}_6\}$ where a denotes the result of the first die, and b denotes the result of the second die. Now, we can use the same notation to define the required subsets:

$$\begin{aligned} A &= \{(a, b) : b \in \mathbb{N}_{2(3)}\} \\ B &= \{(a, b) : a + b \in \mathbb{N}_{2(6)}\} \\ C &= \{(a, b) : a \notin \mathbb{N}_{2(3)} \quad \nabla \quad b \notin \mathbb{N}_{2(3)}\} \end{aligned}$$

Note the use of ∇ (XOR), which is the exclusive $\vee$ (OR).

b)

$$
\begin{aligned}
A &= \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6),(3,2),(3,4),(3,6),(4,2),(4,4),(4,6),\\
&\qquad (5,2),(5,4),(5,6),(6,2),(6,4),(6,6)\}\\
\bar{C} &= \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}\\
A \cap B &= \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6),(3,2),(3,4),(3,6),(4,2),(4,4),(4,6),\\
&\qquad (5,2),(5,4),(5,6),(6,2),(6,4),(6,6)\} \cap \{(1,1),(1,3),(1,5),(2,2),\\
&\qquad (2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),(5,3),(5,5),\\
&\qquad (6,2),(6,4),(6,6)\}\\
&= \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\} = \bar{C}\\
A \cap \bar{B} &= \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6),(3,2),(3,4),(3,6),(4,2),(4,4),(4,6),\\
&\qquad (5,2),(5,4),(5,6),(6,2),(6,4),(6,6)\} \cap \{(1,2),(1,4),(1,6),(2,1),\\
&\qquad (2,3),(2,5),(3,2),(3,4),(3,6),(4,1),(4,3),(4,5),(5,2),(5,4),(5,6),\\
&\qquad (6,1),(6,3),(6,5)\}\\
&= \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}\\
\bar{A} \cup B &= \{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5),(3,1),(3,3),(3,5),(4,1),(4,3),(4,5),\\
&\qquad (5,1),(5,3),(5,5),(6,1),(6,3),(6,5)\} \cup \{(1,1),(1,3),(1,5),(2,2),\\
&\qquad (2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),(5,3),(5,5),\\
&\qquad (6,2),(6,4),(6,6)\}\\
&= \{(1,1),(1,3),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,3),(3,5),\\
&\qquad (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,3),(5,5),(6,1),(6,2),\\
&\qquad (6,3),(6,4),(6,5),(6,6)\}\\
\bar{A} \cap C &= \{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5),(3,1),(3,3),(3,5),(4,1),(4,3),(4,5),\\
&\qquad (5,1),(5,3),(5,5),(6,1),(6,3),(6,5)\} \cap \{(1,1),(1,2),(1,3),(1,4),\\
&\qquad (1,5),(1,6),(2,1),(2,3),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\
&\qquad (4,1),(4,3),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,3),\\
&\qquad (6,5)\}\\
&= \{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5),(3,1),(3,3),(3,5),(4,1),(4,3),(4,5),\\
&\qquad (5,1),(5,3),(5,5),(6,1),(6,3),(6,5)\} = \bar{A}
\end{aligned}
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Problem 3

A survey classified a large number of adults according to whether they were diagnosed as needing eyeglasses to correct their reading vision, and whether they use eyeglasses when reading. The proportions falling into the four resulting categories are given in the following table:

	Uses Eyeglasses for Reading?	
Needs glasses?	Yes	No
Yes	0.44	0.14
No	0.02	0.40

If a single adult is selected from the large group, find the probabilities of the events defined below. The adult

a) needs glasses.

b) needs glasses but does not use them.

c) uses glasses whether the glasses are needed or not.

Solution.

a) An adult from the large group that needs glasses has a probability of $0.44 + 0.14 = 0.60$.

b) An adult from the large group that needs glasses, but does not use them, has a probability of 0.14.

c) An adult from the large group that uses glasses regardless of whether they are needed or not has a probability $0.44 + 0.02 = 0.46$.

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Problem 4

An oil prospecting firm hits oil or gas on 10% of its drillings. If the firm drills two wells, the four possible simple events and three of their associated probabilities are as given in the accompanying table:

Simple Event	Outcome of First Drilling	Outcome of Second Drilling	Probability
E_1	Hit	Hit	0.01
E_2	Hit	Miss	-
E_3	Miss	Hit	0.09
E_4	Miss	Miss	0.81

Find the probability that the company will hit oil or gas

a) On the first drilling and miss on the second.

b) On at least one of the two drilling.

Solution.

a) The company will hit oil or gas on the first drilling and miss on the second with a probability of $1 - 0.01 - 0.09 - 0.81 = 0.09$

b) The company will hit oil or gas on at least one of the two drillings with a probability of $1 - 0.81 = 0.19$. Also note that this probability can be found through $0.01 + 0.09 + 0.09 = 0.19$.

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Problem 5

Extend Theorem 5, proved in class, to three events, A, B, and C, by finding an expression for $P(A \cup B \cup C)$ in terms of the probabilities of $A, B,$ and C, of their pair-wise intersections, and the intersection of all three events. (Hint: Begin by considering $A \cup B$ as a single event).

Proof.

Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Thus, we may extend this to three sets by:

$$\begin{aligned}
P(A \cup B \cup C) &= P\big((A \cup B) \cup C\big) \\
&= P(A \cup B) + P(C) - P\big((A \cup B) \cap C\big) \\
&= P(A) + P(B) - P(A \cap B) + P(C) - P\big((A \cap C) \cup (B \cap C)\big) \\
&= P(A) + P(B) + P(C) - P(A \cap B) - \big(P(A \cap C) + P(B \cap C) - P\big((A \cap C) \cap (B \cap C)\big)\big) \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\end{aligned}$$

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Problem 6

Here is a subtle question. Criticize the reasoning of Example 2.3 p37 given by Wackerly, Mendenhall, and Scheaffer. They argue that just because the coin is balanced, each outcome (the result of three tosses) must have probability $1/8$. *Note that a coin is balanced if* $P(H) = P(T) = 1/2$ *at each toss. Is is enough to criticize the reasoning by considering just a two-toss experiment, for which WMS would (incorrectly) argue that the probability of each pair of outcome is automatically* $1/4$*, from the fairness of the coin alone. This mistake is made in many introductory statistics books. (This is not complicated, but you will need to look at your notes carefully before submitting.)*

Solution.

Recall that two events A and B are said to be independent if $P(B \mid A) = P(B)$ or $P(A \mid B) = P(A)$ or $P(A \cap B) = P(A)P(B)$. Thus, consider the case of a balanced (or fair) coin being tossed twice, where $P(H) = P(T) = 1/2$. We define the following events:

$$\begin{aligned} A &= \text{ First toss is Heads (HH, HT)} \\ B &= \text{ Second toss is Tails (HT, TT)} \\ C &= \text{ There is at least one Heads (HH, HT, TH)} \end{aligned}$$

Therefore, we have the following probabilities:

$$\begin{aligned} P(A) &= 1/2 \\ P(B) &= 1/2 \\ P(C) &= 1/2 \\ P(A \cap B) &= 1/4 \\ P(A \cap C) &= 1/2 \\ P(B \cap C) &= 1/4 \end{aligned} \qquad \begin{aligned} P(A \mid B) &= \frac{1/4}{1/2} = 1/2 \\ P(B \mid A) &= \frac{1/4}{1/2} = 1/2 \\ P(A \mid C) &= \frac{1/2}{1/2} = 1 \\ P(C \mid A) &= \frac{1/2}{1/2} = 1 \\ P(B \mid C) &= \frac{1/4}{1/2} = 1/2 \\ P(C \mid B) &= \frac{1/4}{1/2} = 1/2 \end{aligned}$$

We can therefore observe that A and B **are independent** as $P(A \mid B) = 1/2 = P(A)$, A and C **are not independent** as $P(C \mid A) = 1 \neq 1/2 = P(C)$ and $P(A \cap C) = 1/2 \neq 1/4 = P(A)P(C)$, and finally B and C **are independent** as $P(B \cap C) = 1/4 = P(B)P(C)$. The independence of the events in question must therefore be considered, as well as the fairness of the coin before assuming that the probability of each outcome is $1/4$ for a two-toss coin experiment. This can thus be expanded upon the three-toss experiment in the book. ⊡

Problem 7

We previously considered a situation where cars entering an intersection each could turn right, turn left, or go straight. An experiment consists of observing two vehicles moving through the intersection.

a) How many sample points are there in the sample space?

b) Assuming that all sample points are equally likely, what is the probability that at least one vehicle turns left?

c) Again assuming all sample points are equally likely, what is the probability that at most one vehicle turns?

Solution.

a) There are 2 vehicles, and each car has 3 possible directions in which to turn. Thus, the total sample points can be calculated by:

$$(3 \text{ directions})^{(2 \text{ vehicles})} = 9 \text{ sample points}$$

Let us list these 9 points through pairs (A, B) where A denotes the direction of the first car and B the direction of the second. We have:

$(Left,\ Left)$	$(Straight,\ Left)$	$(Right,\ Left)$
$(Left,\ Straight)$	$(Straight,\ Straight)$	$(Right,\ Straight)$
$(Left,\ Right)$	$(Straight,\ Right)$	$(Right,\ Right)$

b) We can see above that out of all 9 possibilities, only 5 of them have at least one vehicle turning left. Therefore, the probability that at least one vehicle turns left is $5/9$. We can also see that the amount of possibilities where no vehicle turns left at all is $4/9$, and so $1 - 4/9 = 5/9$.

c) We again have a total 9 possibilities, only that this time we consider the cases where both of the vehicles turn either left or right. Thus, we can see that there are only 4 cases where this occurs. As we are looking for the number of possibilities where at most one car turns, we calculate this again by subtracting the number of cases where both vehicles turn from all possibilities: $1 - 4/9 = 5/9$. If we count the number of possibilities that include a straight direction (of which at most one car turns), we see that there are 5 possibilities, confirming the result.

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Problem 8

A boxcar contains six complex electronic systems. Two of the six are to be randomly selected for thorough testing, and they can be classified as defective or not defective.

a) *If two of the six systems are actually defective, find the probability that at least one of the two systems tested will be defective.*

b) *If four of the six systems are actually defective find the probability that at least one of the two systems tested will be defective.*

Solution.

a) Let us consider the case where no defective systems are chosen. If two of the six systems are actually defective, then the probability of choosing two non-defective systems is:

$$\frac{4}{6} \cdot \frac{3}{5} = \frac{12}{30} = \frac{2}{5}$$

Therefore, the probability that at least one of the two systems is defective is:

$$1 - \frac{2}{5} = \frac{3}{5}$$

b) Similarly, if four of the six systems are actually defective, the probability of choosing two non-defective systems is:

$$\frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$$

Therefore, the probability that at least one of the two systems is defective is:

$$1 - \frac{1}{15} = \frac{14}{15}$$

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