Chapter 1

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1.1 Introduction

Our aim is to abstract and generalize concepts such as **space**, **distance** and **continuity**.

As en example of the character of this theory, we intend not to distinguish subspaces A (circle) and B (square) of $\mathbb{R}^2$, but to distinguish these from C (ring), as in figure 1.1.

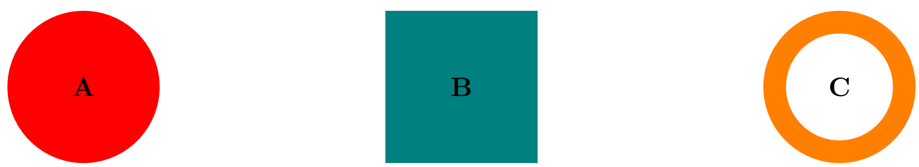

Figure 1.1: Examples of topological subspaces of $\mathbb{R}^2$.

1.2 Topological Spaces

Definition 1. Given a set X, we denote the the set of all subsets of X as $\mathcal{P}(X)$, called the **power set** of X.

In topology, the union of potentially infinite (and uncountable) subsets. We write the union of a family of subsets $\{A_i : i \in I\}$ as $\cup_{i\in I} A_i$.

Definition 2. Given a non-empty set X, we say $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** on X if:

1. $X, \emptyset \in \mathcal{T}$
2. if $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$
3. if $A_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i\in I} A_i \in \mathcal{T}$.

Then $(X, \mathcal{T})$ is a **topological space**, and the elements of X sometimes denotes by points.

Remark. For a finite family of subsets N, the reunion $\bigcap_{i\in N} A_i \in \mathcal{T}$.

Proof. The case for $n = 2$ is taken care by the second axiom in 2. Assume $\bigcap_{i=1}^{n} A_i = B \in \mathcal{T}$. Then

$$\bigcap_{i=1}^{n+1} A_i = \left(\bigcap_{i=1}^{n} A_i\right) \cap A_{n+1} = B \cap A \in \mathcal{T}$$

so the statement holds by induction. □

Example 1. Let $X_1 = \{a, b, c, d, e, f\}$ and $\mathcal{T}_1 = \{X_1, \emptyset, \{a\}, \{c, b\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then $\mathcal{T}_1$ is a topology on X_1.

Remark. For a finite set X, one does enough to test the union of each two subsets of $\mathcal{T}$, and the finite union follows by induction.

Example 2. Let $X = \{a, b, c, d\}$ and $\mathcal{T}_2 = \{X_2, \emptyset, \{a\}, \{c, d\}, \{a, c\}, \{a, c, d\}\}$. Then $\mathcal{T}_2$ is not a topology on X_2, for $\{a\} \cap \{a, c\} = \{c\} \notin \mathcal{T}$.

Example 3. Let $\mathcal{T}_3 \subseteq \mathcal{P}(\mathbb{N})$ be such that

$$\mathcal{T}_3 = \{\mathbb{N}\} \cup \{A \subseteq \mathbb{N} : A \text{ is finite}\}.$$

$\mathcal{T}_3$ is not a topology on $\mathbb{N}$. Let $B_n = \{2n + 1\}$ for $n \in \mathbb{N}$. Then $B_n \in \mathcal{T}_3$ but

$$\bigcup_{n=1}^{\infty} B_n \notin \mathcal{T}_3.$$

Definition 3. Let X be a non-empty set. Then $\mathcal{T} = \mathcal{P}(X)$ is a topology on X, called the **discrete topology**.

Remark. Let $(X, \mathcal{T})$ be a topological space. Then $\mathcal{T}$ is the discrete topology if and only if $\{x\} \in \mathcal{T}$ for all $x \in X$. These are called the **singular sets** of X.

Proof. Take $A \in \mathcal{P}(X)$ and let x_i for $i \in I$ be elements of X. If A is empty then it is simply the **empty union** of singleton sets. Otherwise, it implies that there does not exist a union such that

$$\bigcup_{i \in I} \{x_i\} = A.$$

Then there exists an element $x_k \in A$ that is not contained in the union. As every element of X is contained in the union of singletons sets of X, A contains an element that is not contained in X. Hence $A \notin \mathcal{P}(X)$, which is contradiction. □

Definition 4. Let X be a non empty set. Then $\mathcal{T} = \{X, \emptyset\}$ is a topology of X, called the **indiscrete topology**.

1.3 Open and Closed subsets

Definition 5. Given a topological space $(X, \mathcal{T})$, the elements of $\mathcal{T}$ are called **open** subsets of X.

Definition 6. Given a topological space $(X, \mathcal{T})$, a subset $A \in X$ is called **closed** if $X \setminus A \in \mathcal{T}$.

1.3.1 Properties of closed subsets

In a topological space $X, \mathcal{T}$:

- X and $\emptyset$ are closed
- the intersection of finitely many closed subsets is closed
- the union of an arbitrary family of closed subsets is closed.

Remark. The aforementioned properties of closed subsets are a direct consequence of the axioms in 2.

Example 4. Let X be a non-empty set. The family of subsets

$$\mathcal{T} = \{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$$

are a topology on X, called the **cofinite topology**. The closed sets of $\mathcal{T}$ are X and its finite subsets.

Proof. We have that $\emptyset$ and $X \in \mathcal{T}$.

Let $A, B \in X$. If A or B are empty the intersection is trivial. Otherwise

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

is finite because $X \setminus A$ and $X \setminus B$ are finite, therefore $A \cap B \in \mathcal{T}$.

Let $A_i \in \mathcal{T}$ for $i \in I$. Then

$$X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i)$$

is contained in $\mathcal{T}$, as the intersection of finite sets is finite. □

1.4 Functions

Definition 7. Let $f : X \to Y$ be a function. Given $A \subseteq Y$, we define **reciprocal imagine** of A

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Remark. The reciprocal image allows for the introduction of a topology on X from a given topology on Y. Let $f : X \to Y$ be a function with $X \neq \emptyset$. If $\mathcal{T}$ is a topology on Y, then

$$\mathcal{T}' = \{f^{-1}(A) : A \in \mathcal{T}\}$$

is a topology on X.

Proof. Because $\mathcal{T}$ is a topology on Y, we know that $\emptyset, Y \in \mathcal{T}$. Then $f^{-1}(\emptyset) \in \mathcal{T}'$ and $f^{-1}(Y) \in \mathcal{T}'$.

Let $A, B \in \mathcal{T}$. Hence $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \in \mathcal{T}'$.

Let $A_i \in \mathcal{T}$ for $i \in I$ be an arbitrary family of subsets of Y. We shall now prove that

$$\bigcup_{i \in I} f^{-1}(A_i) = f^{-1} \left(\bigcup_{i \in I} A_i \right) \in \mathcal{T}'$$

to conclude the proof.

Let $y \in \bigcup_{i \in I} A_i$, then $y \in A_k$ for some $k \in I$ and

$$f^{-1}(y) \in f^{-1}(A_k) \subseteq \bigcup_{i \in I} f^{-1}(A_i).$$

Now take $x \in \bigcup_{i \in I} f^{-1}(A_i)$ so, in particular, $x \in f^{-1}(A_k)$ for some $k \in I$. It follows that there exists $y = f(x) \in A_k$ and

$$y \in A_k \subseteq \bigcup_{i \in I} A_i \Rightarrow x \in f^{-1} \left(\bigcup_{i \in I} A_i \right)$$

as we wanted to show. □

Remark. Inheriting a topology in the reverse manner, that is, $\mathcal{T}' = \{f(A) : A \in \mathcal{T}\}$ does not work, even if f is surjective.

Example 5. Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$ and $f : X \to Y$ defined by

$$a \mapsto 1 \qquad b \mapsto 2 \qquad c \mapsto 3 \qquad d \mapsto 1$$

and $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{c, d\}\}$ on X. We notice that $\{1, 2\}, \{1, 3\} \in \mathcal{T}$ and

$$\{1, 2\} \cap \{1, 3\} = \{1\} \notin \mathcal{T}'.$$

1.5 Axioms of Separation

Definition 8. A topological space $(X, \mathcal{T})$ is T_0 if, given two distinct points $a, b \in X$, there exists an open subset that contains only one of them.

Notice that:

- Every discrete space is T_0.
- A indescrete space is T_0 if and only if $|X| \leq 1$.

Definition 9. A topological space $(X, \mathcal{T})$ is T_1 if, given two distinct points $a, b \in X$, there exists an open subset that contains a but not b.

Remark. Every T_1 topological space is T_0, but not the reverse.

Example 6. Let $X = \{a, b\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}\}$. Then $(X, \mathcal{T})$ is T_0 but not T_1, because there does not exist an open subset of X that contains b but not a.

Theorem 1. *A topological space $(X, \mathcal{T})$ is T_1 if and only if every singular subset of X is closed.*

Proof. Assume that every singular subset of X is closed. Take $a, b \in X$. We have that $\{b\}$ is closed, therefore $X \setminus \{b\}$ is open and contains a, hence the $(X, \mathcal{T})$ is T_1.

Assume now that $(X, \mathcal{T})$ is T_1. Then for every point y in $X \setminus \{x\}$ there exists an open subset A_y that does not contain x. Thus the reunion

$$\bigcup_{y \in X \setminus \{x\}} A_y = X \setminus \{x\}$$

is open and $\{x\}$ is a closed subset of X. $\square$

1.6 Cardinality

Definition 10. Two set X, Y are said to have the same cardinality ($|X| = |Y|$) if there exists a bijection between them.

Definition 11. A set is said to be **countable** if $|X| \leq |\mathbb{N}|$.

An equivalent definition is that if X is countable, its elements may be written as a succession $x_1, x_2, \ldots$

Definition 12. A set X is said to have **cardinality of the continuum** if $|X| = |\mathbb{R}|$.

1.6.1 Properties of Cardinality

- If I is countable and X_i is countable for all $i \in I$, then $\bigcup_{i \in I} X_i$ is countable
- $\mathbb{Q}$ is countable
- If $X_1, \ldots, X_n$ are countable, then $X_1 \times \cdots \times X_n$ is countable.