# Notes on Topology

# Chapter 1

# 22/09/2020

## 1.1 Introduction

Our aim is to abstract and generalize concepts such as **space**, **distance** and **continuity**.

As en example of the character of this theory, we intend not to distinguish subspaces A (circle) and B (square) of $\mathbb{R}^2$, but to distinguish these from C (ring), as in figure 1.1.

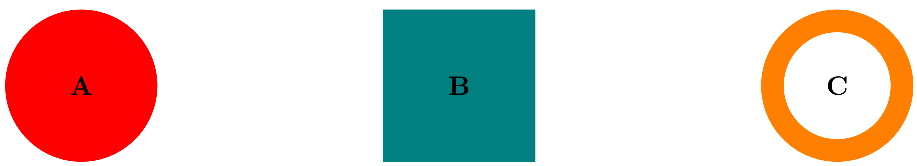


Figure 1.1: Examples of topological subspaces of $\mathbb{R}^2$.

## 1.2 Topological Spaces

**Definition 1.** Given a set $X$, we denote the the set of all subsets of $X$ as $\mathcal{P}(X)$, called the **power set** of $X$.

In topology, the union of potentially infinite (and uncountable) subsets. We write the union of a family of subsets $\{A_i : i \in I\}$ as $\cup_{i \in I} A_i$.

**Definition 2.** Given a non-empty set $X$, we say $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** on $X$ if:

1. $X, \emptyset \in \mathcal{T}$
2. if $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$
3. if $A_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}$.

Then $(X, \mathcal{T})$ is a **topological space**, and the elements of $X$ sometimes denotes by points.

**Remark.** For a finite family of subsets $N$, the reunion $\bigcap_{i \in N} A_i \in \mathcal{T}$.

*Proof.* The case for $n = 2$ is taken care by the second axiom in 2. Assume $\bigcap_{i=1}^{n} A_i = B \in \mathcal{T}$. Then

$$\bigcap_{i=1}^{n+1} A_i = \left( \bigcap_{i=1}^{n} A_i \right) \cap A_{n+1} = B \cap A \in \mathcal{T}$$

so the statement holds by induction. □

**Example 1.** Let $X_1 = \{a, b, c, d, e, f\}$ and $\mathcal{T}_1 = \{X_1, \emptyset, \{a\}, \{c, b\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then $\mathcal{T}_1$ is a topology on $X_1$.

**Remark.** For a finite set $X$, one does enough to test the union of each two subsets of $\mathcal{T}$, and the finite union follows by induction.

**Example 2.** Let $X = \{a, b, c, d\}$ and $\mathcal{T}_2 = \{X_2, \emptyset, \{a\}, \{c, d\}, \{a, c\}, \{a, c, d\}\}$. Then $\mathcal{T}_2$ is not a topology on $X_2$, for $\{a\} \cap \{a, c\} = \{c\} \notin \mathcal{T}$.

**Example 3.** Let $\mathcal{T}_3 \subseteq \mathcal{P}(\mathbb{N})$ be such that

$$\mathcal{T}_3 = \{\mathbb{N}\} \cup \{A \subseteq \mathbb{N} : A \text{ is finite}\}.$$

$\mathcal{T}_3$ is not a topology on $\mathbb{N}$. Let $B_n = \{2n + 1\}$ for $n \in \mathbb{N}$. Then $B_n \in \mathcal{T}_3$ but

$$\bigcup_{n=1}^{\infty} B_n \notin \mathcal{T}_3.$$

**Definition 3.** Let $X$ be a non-empty set. Then $\mathcal{T} = \mathcal{P}(X)$ is a topology on $X$, called the **discrete topology**.

**Remark.** Let $(X, \mathcal{T})$ be a topological space. Then $\mathcal{T}$ is the discrete topology if and only if $\{x\} \in \mathcal{T}$ for all $x \in X$. These are called the **singular sets** of $X$.

*Proof.* Take $A \in \mathcal{P}(X)$ and let $x_i$ for $i \in I$ be elements of $X$. If $A$ is empty then it is simply the **empty union** of singleton sets. Otherwise, it implies that there does not exist a union such that

$$\bigcup_{i \in I} \{x_i\} = A.$$

Then there exists an element $x_k \in A$ that is not contained in the union. As every element of $X$ is contained in the union of singletons sets of $X$, $A$ contains an element that is not contained in $X$. Hence $A \notin \mathcal{P}(X)$, which is contradiction. □

**Definition 4.** Let $X$ be a non empty set. Then $\mathcal{T} = \{X, \emptyset\}$ is a topology of $X$, called the **indiscrete topology**.

## 1.3 Open and Closed subsets

**Definition 5.** Given a topological space $(X, \mathcal{T})$, the elements of $\mathcal{T}$ are called **open** subsets of $X$.

**Definition 6.** Given a topological space $(X, \mathcal{T})$, a subset $A \in X$ is called **closed** if $X \setminus A \in \mathcal{T}$.

### 1.3.1 Properties of closed subsets

In a topological space $X, \mathcal{T}$:

- $X$ and $\emptyset$ are closed
- the intersection of finitely many closed subsets is closed
- the union of an arbitrary family of closed subsets is closed.

**Remark.** The aforementioned properties of closed subsets are a direct consequence of the axioms in 2.

**Example 4.** Let $X$ be a non-empty set. The family of subsets

$$\mathcal{T} = \{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$$

are a topology on $X$, called the **cofinite topology**. The closed sets of $\mathcal{T}$ are $X$ and its finite subsets.

*Proof.* We have that $\emptyset$ and $X \in \mathcal{T}$.

Let $A, B \in X$. If $A$ or $B$ are empty the intersection is trivial. Otherwise

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

is finite because $X \setminus A$ and $X \setminus B$ are finite, therefore $A \cap B \in \mathcal{T}$.

Let $A_i \in \mathcal{T}$ for $i \in I$. Then

$$X \setminus \left( \bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i)$$

is contained in $\mathcal{T}$, as the intersection of finite sets is finite. □

## 1.4 Functions

**Definition 7.** Let $f : X \to Y$ be a function. Given $A \subseteq Y$, we define **reciprocal imagine** of $A$

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

**Remark.** The reciprocal image allows for the introduction of a topology on $X$ from a given topology on $Y$. Let $f : X \to Y$ be a function with $X \neq \emptyset$. If $\mathcal{T}$ is a topology on $Y$, then

$$\mathcal{T}' = \{f^{-1}(A) : A \in \mathcal{T}\}$$

is a topology on $X$.

*Proof.* Because $\mathcal{T}$ is a topology on $Y$, we know that $\emptyset, Y \in \mathcal{T}$. Then $f^{-1}(\emptyset) \in \mathcal{T}'$ and $f^{-1}(Y) \in \mathcal{T}'$.

Let $A, B \in \mathcal{T}$. Hence $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \in \mathcal{T}'$.

Let $A_i \in \mathcal{T}$ for $i \in I$ be an arbitrary family of subsets of $Y$. We shall now prove that

$$\bigcup_{i \in I} f^{-1}(A_i) = f^{-1} \left( \bigcup_{i \in I} A_i \right) \in \mathcal{T}'$$

to conclude the proof.

Let $y \in \bigcup_{i \in I} A_i$, then $y \in A_k$ for some $k \in I$ and

$$f^{-1}(y) \in f^{-1}(A_k) \subseteq \bigcup_{i \in I} f^{-1}(A_i).$$

Now take $x \in \bigcup_{i \in I} f^{-1}(A_i)$ so, in particular, $x \in f^{-1}(A_k)$ for some $k \in I$. It follows that there exists $y = f(x) \in A_k$ and

$$y \in A_k \subseteq \bigcup_{i \in I} A_i \Rightarrow x \in f^{-1} \left( \bigcup_{i \in I} A_i \right)$$

as we wanted to show. □

**Remark.** Inheriting a topology in the reverse manner, that is, $\mathcal{T}' = \{f(A) : A \in \mathcal{T}\}$ does not work, even if $f$ is surjective.

**Example 5.** Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$ and $f : X \to Y$ defined by

$$a \mapsto 1 \qquad b \mapsto 2 \qquad c \mapsto 3 \qquad d \mapsto 1$$

and $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{c, d\}\}$ on $X$. We notice that $\{1, 2\}, \{1, 3\} \in \mathcal{T}$ and

$$\{1, 2\} \cap \{1, 3\} = \{1\} \notin \mathcal{T}'.$$

## 1.5 Axioms of Separation

**Definition 8.** A topological space $(X, \mathcal{T})$ is $\mathrm{T}_0$ if, given two distinct points $a, b \in X$, there exists an open subset that contains only one of them.

Notice that:

- Every discrete space is $\mathrm{T}_0$.
- A indescrete space is $\mathrm{T}_0$ if and only if $|X| \leq 1$.

**Definition 9.** A topological space $(X, \mathcal{T})$ is $\mathrm{T}_1$ if, given two distinct points $a, b \in X$, there exists an open subset that contains $a$ but not $b$.

**Remark.** Every $\mathrm{T}_1$ topological space is $\mathrm{T}_0$, but not the reverse.

**Example 6.** Let $X = \{a, b\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}\}$. Then $(X, \mathcal{T})$ is $\mathrm{T}_0$ but not $\mathrm{T}_1$, because there does not exist an open subset of $X$ that contains $b$ but not $a$.

**Theorem 1.** *A topological space $(X, \mathcal{T})$ is $\mathrm{T}_1$ if and only if every singular subset of $X$ is closed.*

*Proof.* Assume that every singular subset of $X$ is closed. Take $a, b \in X$. We have that $\{b\}$ is closed, therefore $X \setminus \{b\}$ is open and contains $a$, hence the $(X, \mathcal{T})$ is $\mathrm{T}_1$.

Assume now that $(X, \mathcal{T})$ is $\mathrm{T}_1$. Then for every point $y$ in $X \setminus \{x\}$ there exists an open subset $A_y$ that does not contain $x$. Thus the reunion

$$\bigcup_{y \in X \setminus \{x\}} A_y = X \setminus \{x\}$$

is open and $\{x\}$ is a closed subset of $X$. $\square$

## 1.6 Cardinality

**Definition 10.** Two set $X, Y$ are said to have the same cardinality ($|X| = |Y|$) if there exists a bijection between them.

**Definition 11.** A set is said to be **countable** if $|X| \leq |\mathbb{N}|$.

An equivalent definition is that if $X$ is countable, its elements may be written as a succession $x_1, x_2, \ldots$

**Definition 12.** A set $X$ is said to have **cardinality of the continuum** if $|X| = |\mathbb{R}|$.

### 1.6.1 Properties of Cardinality

- If $I$ is countable and $X_i$ is countable for all $i \in I$, then $\bigcup_{i \in I} X_i$ is countable
- $\mathbb{Q}$ is countable
- If $X_1, \ldots, X_n$ are countable, then $X_1 \times \cdots \times X_n$ is countable.

# Chapter 2

# 24/09/2020

## 2.1 Euclidean Topology in $\mathbf{R}^n$

Text