Exercises on Topology

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1 Topological Spaces

1.1 Topology

1. Let $X = \{a, b, c, d, e, f\}$.
   (a) $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, f\}, \{b, f\}, \{a, b, f\}\}$ is not a topology on X because $\{a, f\}, \{b, f\} \in \mathcal{T}_1$ but
   $$\{a, f\} \cap \{b, f\} = \{f\} \notin \mathcal{T}_1.$$
   (b) $\mathcal{T}_2 = \{X, \emptyset, \{a, b, f\}, \{a, b, d\}, \{a, b, d, f\}\}$ is not a topology on X because $\{a, b, f\}, \{a, b, d\} \in \mathcal{T}_2$ but
   $$\{a, b, f\} \cap \{a, b, d\} = \{a, b\} \notin \mathcal{T}_2.$$
   (c) $\mathcal{T}_3 = \{X, \emptyset, \{f\}, \{e, f\}, \{a, f\}\}$ is not a topology on X because $\{e, f\}, \{a, f\} \in \mathcal{T}_3$ but
   $$\{e, f\} \cap \{a, f\} = \{a, e, f\} \notin \mathcal{T}_3.$$

2. Let $X = \{a, b, c, d, e, f\}$.
   (a) $\mathcal{T}_1 = \{X, \emptyset, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$ is not a topology on X because $\{c\}, \{b\} \in \mathcal{T}_1$ but
   $$\{c\} \cap \{b\} = \{b, c\} \notin \mathcal{T}_1.$$
   (b) $\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}\}$ is not a topology on X because $\{b, d, e\}, \{a, b, d\} \in \mathcal{T}_2$ but
   $$\{b, d, e\} \cap \{a, b, d\} = \{b, d\} \notin \mathcal{T}_2.$$
   (c) $\mathcal{T}_3 = \{X, \emptyset, \{b\}, \{a, b, c\}, \{d, e, f\}, \{b, d, e, f\}\}$ is a topology on X.

3. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T}$ the discrete topology on X. The following statements are true: (a), (d), (g), (i), (l), (k).

4. The case for the intersection of two subsets is taken care by the axioms of a topology. Assume that $\bigcup_{i=0}^{n} A_i = B \in \mathcal{T}$, then
   $$\bigcup_{i=0}^{n+1} A_i = \left(\bigcup_{i=0}^{n} A_i\right) \cup A_{n+1} = B \cup A_{n+1} \in \mathcal{T}.$$

5. Let $X = \mathbb{R}$.
   (a) $\mathcal{T}_1 = \{\mathbb{R}, \emptyset\} \cup \{(-n, n) : n \in \mathbb{N}\}$.
      i. $\mathbb{R}, \emptyset \in \mathcal{T}_1$
      ii. Let $n, m \in \mathbb{N}$ then $(-n, n) \cap (-m, m) = (-\min(n, m), \min(n, m)) \in \mathcal{T}_1$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} (-n_i, n_i) = (-\sup(n_i), \sup(n_i)) \in \mathcal{T}_1.$$
   (b) $\mathcal{T}_2 = \{\mathbb{R}, \emptyset\} \cup \{[-n, n] : n \in \mathbb{N}\}$.
      i. $\mathbb{R}, \emptyset \in \mathcal{T}_2$
      ii. Let $n, m \in \mathbb{N}$ then $[-n, n] \cap [-m, m] = [-\min(n, m), \min(n, m)] \in \mathcal{T}_2$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} [-n_i, n_i] = [-\sup(n_i), \sup(n_i)] \in \mathcal{T}_2.$$
   (c) $\mathcal{T}_3 = \{\mathbb{R}, \emptyset\} \cup \{[n, \infty) : n \in \mathbb{N}\}$.
      i. $\mathbb{R}, \emptyset \in \mathcal{T}_3$
      ii. Let $n, m \in \mathbb{N}$ then $[n, \infty) \cap [m, \infty) = [\max(n, m), \infty) \in \mathcal{T}_3$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} [n_i, \infty) = [\inf(n, m), \infty) \in \mathcal{T}_3.$$
6. Let $X = \mathbb{N}$.
   (a) Let $S_n = \{1, \ldots, n\}$ and $\mathcal{T}_1 = \{\mathbb{N}, \emptyset\} \cup \{S_n : n \in \mathbb{N}\}$.
      i. $\mathbb{N}, \emptyset \in \mathcal{T}_1$
      ii. Let $n, m \in \mathbb{N}$ then $S_n \cap S_m = S_{\min(n,m)} \in \mathcal{T}_1$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} S_{n_i} = S_{\sup(n_i)} \in \mathcal{T}_1.$$
   (b) Let $S_n = \{n, n+1, \ldots\}$ and $\mathcal{T}_2 = \{\mathbb{N}, \emptyset\} \cup \{S_n : n \in \mathbb{N}\}$.
      i. $\mathbb{N}, \emptyset \in \mathcal{T}_2$
      ii. Let $n, m \in \mathbb{N}$ then $S_n \cap S_m = S_{\max(n,m)} \in \mathcal{T}_2$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} S_{n_i} = S_{\inf(n_i)} \in \mathcal{T}_2.$$
7. Number of topologies goes brrrr.

8. Let us have an infinite set X and topology $\mathcal{T}$ on X such that every infinite subset of X is open. $\mathcal{T}$ is the discrete topology if every finite subset of $K \subseteq X$ is also open. Start by taking two disjunct infinite subsets of $A, B \subseteq X$. By definition these are open. Thus the sets $A \cup K$ and $B \cup K$ are also open. Hence

$$(A \cup K) \cap (B \cup K) = K \in \mathcal{T}$$

for any finite set K, so $\mathcal{T}$ is the discrete topology on X.

9. Let $X = \mathbb{R}$.

(a) $\mathcal{T}_1 = \{\mathbb{R}, \emptyset\} \cup \left\{(a,b) \in \mathbb{R}^2 : a < b\right\}$ is not a topology on $\mathbb{R}$. Notice that $(0,1) \in \mathcal{T}_1$ and $(2,3) \in \mathcal{T}_1$ but

$$(0,1) \cup (2,3) \notin \mathcal{T}.$$

(b) $\mathcal{T}_2 = \{\mathbb{R}, \emptyset\} \cup \{(-r,r) : r \in \mathbb{R}\}$ is a topology on $\mathbb{R}$.

i. $\mathbb{R}, \emptyset \in \mathcal{T}_2$

ii. Let $r_1, r_2 \in \mathbb{R}$ then $(-r_1, r_1) \cap (-r_2, r_2) = (-\min(r_1, r_2), \min(r_1, r_2)) \in \mathcal{T}_2$

iii. Let there be a family of subsets $(-r_i, r_i)$, with real numbers r_i for $i \in I$ arbitrary indices. Then

$$\bigcup_{i \in I} (-r_i, r_i) = (-\sup(r_i), \sup(r_i)) \in \mathcal{T}_2.$$

We take the chance to prove that $\bigcup_{i \in I} (-r_i, r_i) = (-\sup(r_i), \sup(r_i))$.

(c) $\mathcal{T}_3 = \{\mathbb{R}, \emptyset\} \cup \{(-r,r) : r \in \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left(-\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^n\right) \in \mathcal{T}_3$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-e, e) \notin \mathcal{T}_3.$$

(d) $\mathcal{T}_4 = \{\mathbb{R}, \emptyset\} \cup \{[-r,r] : r \in \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left[-\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^n\right] \in \mathcal{T}_4$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = [-e, e] \notin \mathcal{T}_4.$$

(e) $\mathcal{T}_5 = \{\mathbb{R}, \emptyset\} \cup \{(-r, r) : r \notin \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left(-\frac{\pi}{n} - 1, \frac{\pi}{n} + 1\right) \in \mathcal{T}_5$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1) \notin \mathcal{T}_5.$$

(f) $\mathcal{T}_6 = \{\mathbb{R}, \emptyset\} \cup \{[-r, r] : r \notin \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left[-\frac{\pi}{n} - 1, \frac{\pi}{n} + 1\right] \in \mathcal{T}_6$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = [-1, 1] \notin \mathcal{T}_6.$$

(g) $\mathcal{T}_7 = \{\mathbb{R}, \emptyset\} \cup \{[-r, r) : r \in \mathbb{R}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right) \in \mathcal{T}_7$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1) \notin \mathcal{T}_7.$$

(h) $\mathcal{T}_8 = \{\mathbb{R}, \emptyset\} \cup \{(-r, r] : r \in \mathbb{R}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] \in \mathcal{T}_8$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1) \notin \mathcal{T}_8.$$

(i) $\mathcal{T}_9$ is a topology on $\mathbb{R}$.

(j) $\mathcal{T}_{10}$ is a topology on $\mathbb{R}$.

1.2 Open, Closed and Clopen Sets

1. Number of subsets goes brrrrrr.

2. Let $(X, \mathcal{T})$ be a topological space such that every subset $A \subseteq X$ is closed. Then the subset $X \setminus A$ is closed. By the definition of a closed subspace

$$X \setminus (X \setminus A) = A$$

is open. Because A is arbitrary, it follows that $\mathcal{P}(X) \subseteq \mathcal{T}$, so $\mathcal{T}$ is the discrete topology on X.

3. Let $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{c, d\}\}$ be a topology on X. It is easy to check that every set of $\mathcal{T}$ is open and closed.

4. Let X be and infinite set and $\mathcal{T}$ be a topology on X such that every infinite subset of X is closed. Let $A \subseteq X$. If $X \setminus A$ is infinite, then it is closed, and thus

$$X \setminus (X \setminus A) = A$$

is open. If $X \setminus A$ is finite, then A is infinite, and as such closed. So by definition $X \setminus A$ is open. Because A is arbitrary, we conclude that $\mathcal{T}$ is the discrete topology on X.

5. No. Let $\mathcal{T} = \{X, \emptyset, \{x\}\}$ for some element $x \in X$ be a topology on X. Then $\mathcal{T}$ is not the indiscrete topology on X but the only infinite open subset of X is X.

6. Let X be a nonempty set and $\mathcal{T} = \{X, \emptyset, A, B\}$ such that A and B are nonempty distinct proper subsets of X.
   (a) If none of the conditions apply, we have two possible cases, $A \cap B = \emptyset$ and $A \cap B \neq \emptyset$. If A and B are disjunct, then their union will not be contained in A nor B, but will also not be X, else condition (a) would apply. As such, the union is not open. If they are not disjunct, but are distinct and condition (b) and (c) do not apply, then their intersection will not be open. As such, we have that some condition must apply. But the conditions are obviously exlusive, so we conclude that $\mathcal{T}$is a topology on X if one (and only one) of the conditions holds.
   (b) Topologies on X go brrrrrrr.

7. Let X_n be a finite space such that $|X|$=n. Let us denote the elements of X_n by x_i for $i \in N$. Let M_n be the set of all topologies on X_n.
   (a) For $X_{n+1} = \{x_1, \ldots, x_n\} \cup \{x_{n+1}\}$ we have atleast $|M_n|$ topologies, that only contain the element x_{n+1} in the subset X_{n+1}. Because the famility $\{X_{n+1}, \emptyset, \{x_{n+1}\}\} \in M_{n+1}$ it follows that $|M_{n+1}| > |M_n|$.
   (b) It is clear that if $\mathcal{T}$ is a topology on X_n, then if we replace x_i by x_{n+1} it is clear that $\mathcal{T}$ is a topology on X_{n+1}. As such, for each topology on X_n, there are atleast n topologies on X_{n+1}. Therefore we conclude that X_n has at least $(n-1)!$ topologies.
   (c) Let now X be an infinite set. Then for any subset $A \subseteq X$ we have that the topology

$$\mathcal{T} = \{X, \emptyset, A\}$$

is a topology on X of cardinatily $\aleph$. As such, we have that there are at least $|\mathcal{P}(X)|$ topologies on X. For a set of N elements we have that

$$|\mathcal{P}(X_N)| = \sum_{i=0}^{N} \binom{N}{i} = 2^N.$$

As such, there are at least $2^{\aleph}$ topologies defined on X.

1.3 Finite-Closed Topology

1. Let $f : X \to Y$ be a function between two sets X, Y.

(a) We aim to prove that

$$f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}\left(B_j\right).$$

Let $x \in \bigcup_{j\in J} B_j$, so $y = f^{-1}(x) \in f^{-1}\left(\bigcup_{j\in J} B_j\right)$. Then $x \in B_k$ for some $k \in J$. Thus

$$y \in f^{-1}\left(B_k\right) \subseteq \bigcup_{j\in J} f^{-1}\left(B_j\right).$$

Now let $y \in \bigcup_{j\in J} f^{-1}(B_j)$. Then $y \in f^{-1}(B_k)$ for some $k \in J$. As such, $x \in B_k \subseteq \bigcup_{j\in J} B_j$, hence

$$y \in f^{-1}\left(\bigcup_{j\in J} B_j\right)$$

and that concludes the proof.

(b) We want to prove that for any $B_j \subseteq Y$ with $j \in J$,

$$f^{-1}\left(B_1 \cap B_2\right) = f^{-1}\left(B_1\right) \cap f^{-1}\left(B_2\right).$$

Let $x \in B_1 \cap B_2$, then $x \in B_1$ and $x \in B_2$. As such

$$y = f^{-1}(x) \in f^{-1}\left(B_1\right) \quad \text{and} \quad y \in f^{-1}\left(B_2\right)$$

thus $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Now assume $y \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $y \in f^{-1}(B_1)$ and $y \in f^{-1}(B_2)$, thus $x \in B_1$and $x \in B_2$. We may then conclude that

$$x \in B_1 \cap B_2 \Longrightarrow y \in f^{-1}\left(B_1 \cap B_2\right).$$

(c) Let $X = Y = \mathbb{R}$ and

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} x+1 & \text{if } x < 1 \\ x-1 & \text{if } x \geq 1 \end{cases}. \end{aligned}$$

Let $A = (0,1)$ and $B = (2,3)$. Then $A, B \subseteq X$ but $f^{-1}(A \cap B) = \emptyset$ but

$$f^{-1}(A) \cap f^{-1}(B) = (1,2).$$

2. No. For example, the subset

$$A = \{3, 4, \cdots\} \cup \{1\}$$

is not open in the final segment topology, but it is open in the cofinite topology, because $\mathbb{N} \setminus A = \{2\}$ is finite.

3. The every singleton set $\{x\}$ in a topological space $(X, \mathcal{T})$ is a closed subset if $(X, \mathcal{T})$ is a discrete space of an infinite space with the cofinite topology.

4. Let $\mathcal{T}$ be a cofinite topology on a set X. Assume that $\mathcal{T}$ is discrete. As we showed before, every set the discrete topology is clopen. As such, any $A \in \mathcal{T}$, we have that $X \setminus A$ is open. But $X \setminus A \in \mathcal{T}$ because it is discrete, hence $X \setminus (X \setminus A) = A$ is open. Thus, A is finite. Because A is arbitrary, X is finite.

5. Let $(X, \mathcal{T})$ be a T_1 topological space.
   (a) Because $(X, \mathcal{T})$ is T_1, for every pair $a, b \in X$ there exist open sets A, B such that A contains a but not b, and B contains b but not a. As such, a set that contains one point but not the other, for example, A. Thus, every T_1 is T_0.
   (b) We have that
      1. The discrete space is contains every singleton subset of X, which trivially contains a point but no other. As such, a discrete space is T_1 and therefore T_0.
      2. The only open subsets of an indiscrete space with at least two points are the empty set and X. As such, every open set containing a point a contains every other point. Thus the indiscrete space with at least two points is not T_0.
      3. The cofinite topology contains every sigleton subset of X, thus the same argument of (i) applies.
      4. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then every set containing c also contains d, thus $(X, \mathcal{T})$ is not T_0.

6. Let X be an infinite set and

$$\mathcal{T} = \{X \setminus A \subseteq X : A \subseteq X \text{ is countable}\} \cup \{\emptyset\}.$$

   (a) By definition $X, \emptyset \in \mathcal{T}$.
   (b) Let $A, B \in \mathcal{T}$. Then

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) \in \mathcal{T}$$

   because the union of countable sets is countable.
   (c) Let $A_i \in \mathcal{T}$ for $i \in I$ be a family of open sets. Then

$$X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \in \mathcal{T}$$

   because the arbitrary intersection of countable sets is countable.

7. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.
   (a) The union $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$ is not necesseraly a topology. Take $X = \{a, b, c\}$, $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$. Then $\mathcal{T}_3 = \{X, \emptyset, \{a\}, \{b\}\}$, which is not a topology because

$$\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_3.$$

   (b) The intersection of topologies $\mathcal{T}_4 = \mathcal{T}_1 \cap \mathcal{T}_2$, is a topology.
      i. We have that $\{X, \emptyset\} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$, hence $X, \emptyset \in \mathcal{T}_4$
      ii. Let $A, B \in \mathcal{T}_4$. Then $A \cap B \in \mathcal{T}_1$ and $A \cap B \in \mathcal{T}_2$, thus $A \cap B \in \mathcal{T}_4$
      iii. Let $A_i \in \mathcal{T}_4$ for $i \in I$ be a family of open sets. The $\bigcup_{i \in I} A_i$ is still contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$. Thus $\bigcup_{i \in I} A_i \in \mathcal{T}_4$.

(c) We assume that for all distinct points $a, b \in X$ there exists open sets $A_1, B_1 \in \mathcal{T}_1$ and $A_2, B_2 \in \mathcal{T}_2$ such that A_i contains a but not b and B_i contains b but not a. Thus $A_1 \cap A_2$ contains a but not b and $B_1 \cap B_2$ contains b but not a. We conclude that $\mathcal{T}_4$ is T_1.

(d) Let $X = \{a, b\}$, $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are T_0 spaces but

$$\mathcal{T}_4 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{X, \emptyset\}$$

is not, as there is not subset of $\mathcal{T}_4$ that contains a but not b.

(e) The case for $n = 2$ was handled above. Let us assume that $\mathcal{T} = \bigcap_{i=1}^{n} \mathcal{T}_i$ is a topology on X. Then

$$\bigcap_{i=1}^{n+1} \mathcal{T}_i = \mathcal{T}_{n+1} \cap \mathcal{T}$$

is a topology on X, because it is the intersection of two topologies. Thus the proof follows by induction.

(f) Let $\mathcal{T}_i$ for $i \in I$ be an arbitrary family of topologies on X. Then

i. $X, \emptyset \in \mathcal{T}_i$ for all $i \in I$, thus $X, \emptyset \in \bigcap_{i \in I} \mathcal{T}_i$

ii. Let $A, B \in \bigcap_{i \in I} \mathcal{T}_i$. Then both A and B are contained in each $\mathcal{T}_i$. Thus their intersection is too and $A \cap B \in \bigcap_{i \in I} \mathcal{T}_i$

iii. Let $A_j \in \bigcap_{i \in I} \mathcal{T}_i$ for $j \in J$ be an arbitrary familiy of open sets. Thus A_j is contained in each $\mathcal{T}_i$ for all $j \in J$. Because each $\mathcal{T}_i$ is a topology, we have that $\bigcup_{j \in J} A_j \in \mathcal{T}_i$ for all $i \in I$. Hence

$$\bigcup_{j \in J} A_j \in \bigcap_{i \in I} \mathcal{T}_i.$$

8. Let X_n denote a space of cardinality $|X_n| = n$. Let M_n denote the number of possible T_0 topologies on X_n. Because we may construct every topology of M_n using the first n elements of X_{n+1} (and replacing X_n with X_{n+1}), we have that $M_{n+1} \geq M_n$. Notice now that the discrete topology on X_{n+1} is not counted in M_n, because $\mathcal{P}(X_n) < \mathcal{P}(X_{n+1})$ and is T_0 on X_{n+1}. We conclude that $M_{n+1} > M_n$.

9. Let $(X, \mathcal{T})$ be a topological space.

(a) In the discrete topology on X, every subset of X is clopen, thus a discrete space is a door space.

(b) If $|X| \geq 1$, the indiscrete topology is not a door space, for the only open or closed sets are X and $\emptyset$.

(c) Let X be infinite. Then the cofinite topology on X is not necesseraly a door space. For example, let $X = \mathbb{Z}$. Then $2\mathbb{Z}$, the set of all even intergers, is not open in $\mathbb{Z}$ with the cofinite topology, for $\mathbb{Z} \setminus 2\mathbb{Z}$ is infinite, and is also not closed, and $\mathbb{Z} \setminus 2\mathbb{Z}$ is also not open.

(d) The only door topology on X is the discrete one. Let $\mathcal{T}$ be a topology on X. Consider the singular sets. If they are open, then $\mathcal{T}$ is the discrete topology. If they are closed, then all subsets of the form $\{x, y, z\}$ for $x, y, z \in X$ must be open. But so must their intersections, which imply that $\mathcal{T}$ is the discrete topology on X.

10. Let $(X, \mathcal{T})$ be a topological space.

(a) Let $A \in \mathcal{T}$. Then $A = A \cap A$, so A is saturated.

(b) Let $(X, \mathcal{T})$ be T_1. Then every singular subset of X is closed. Writing $A = \bigcup_{x_i \in A} \{x_i\}$ we have that

$$X \setminus \bigcup_{x_i \in A} \{x_i\} = \bigcap_{x_i \in A} (X \setminus \{x_i\})$$

so A is the intersection of open sets and thus saturated.

(c) Let $X = \{a, b\}$ with the indiscrete topology. Then the subset $A = \{a\}$ is not saturated.

(d) Let $a, b \in X$ be distinct. Because $\{a\}$ is saturated, we have that $\{a\} = \bigcap_{i \in I} U_i$ for some family of open sets U_i. Thus there exists some $k \in I$ such that $a \in U_k$ but $b \notin U_k$. By the application of the same argument to the saturated set $\{b\}$, it follows that $(X, \mathcal{T})$ is T_1.

2 Euclidean Topology

2.1 The Euclidean Topology on $\mathbb{R}$

1. Let $a, b \in \mathbb{R}$ such that $a < b$. Assume that $[a, b)$ is open in the euclidean topology. Therefore we have that for all $x \in [a, b)$ there exist $c, d \in \mathbb{R}$ such that $c < d \in [a, b)$ and $c < x < d$. Thus taking $x = a$, it follows that $c < a$ and $c \in [a, b)$ so

$$c < a \leq c$$

which is a contradiction. The proof for $(a, b]$ is analogous.
Assume now that $[a, b)$ is closed. Then $\mathbb{R} \setminus [a, b) = (-\infty, a) \cup [b, \infty)$ is open. But by an analogous argument, now considering b, this is also a contradiction. The same applies to $(a, b]$. Thus half closed intervals are neither open nor closed in the euclidean topology on $\mathbb{R}$.

2. Let $a \in \mathbb{R}$. We have that $[a, \infty)$ is closed if $\mathbb{R} \setminus [a, \infty) = (-\infty, a)$ is open. Notice that for all $x \in (-\infty, a)$ there exist $b, c \in \mathbb{R}$ such that $x \in (b, c) \subseteq (-\infty, a)$. For example, take

$$\begin{cases} b = & x - 1 \\ c = & \frac{x+a}{2} \end{cases}.$$

Hence $[a, \infty)$ is closed in the euclidean topology on $\mathbb{R}$. The argument for $(-\infty, a]$ is analogous.

3. Consider the union

$$A = \bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] = (-1, 1)$$

which is clearly not closed in the euclidean topology on $\mathbb{R}$.

4. Consider $\mathbb{R}$ with the euclidean topology.

(a) Assume $\mathbb{Z}$ is open. Then for any $n \in \mathbb{Z}$ there exist $a, b \in \mathbb{R}$ such that $a < b$ and $n \in (a, b) \subseteq \mathbb{Z}$. But $\mathbb{Z}$ contains no intervals, because for every two elements m_1, m_2 in $\mathbb{Z}$ such that $m_1 < m_2$ we have that

$$m_1 < m_1 + \frac{1}{2} < m_2$$

for some $m_1 + \frac{1}{2} \notin \mathbb{Z}$. Therefore $\mathbb{Z}$ is not open.

(b) Let p_n for $n \in \mathbb{N}$ denote the sequence of primes in increasing order. Then

$$\mathbb{R} \setminus \left(\bigcup_{n\in\mathbb{N}} \{p_n\} \right) = \bigcap_{n\in\mathbb{N}} (\mathbb{R} \setminus \{p_n\}) = (-\infty, p_1) \cup \left[\bigcup_{n\in\mathbb{N}} (p_n, p_{n+1}) \right]$$

which is the union of open sets and therefore open.

(c) We have that $\mathbb{I}$ contains no intervals, as between every irrational number we find rational numbers. By an analogous argument to (a) we conclude that it is not open. Now consider $\mathbb{R}\setminus\mathbb{I} = \mathbb{Q}$. But the aforementioned argument applies to $\mathbb{Q}$ so it is not open either. As such, $\mathbb{I}$ is neither open not closed.

5. Suppose that F is a nonempty finite subset of $\mathbb{R}$. Because F is finite is contains no intervals, thus it is not open. Now let us write $F = \{f_1, \dots, f_n\}$. Then

$$\mathbb{R} \setminus F = (-\infty, f_1) \cup \left[\bigcup_{i=1}^{n-1} (f_i, f_{i+1}) \right] \cup (f_n, \infty)$$

is the union for open intervals and therefore open. We conclude that F is closed but not open.

6. Let $F \subset \mathbb{R}$ be countable. Then F contains no intervals so it is not open. However, F may or may not be closed. Notice, for example, that

   (a) $\mathbb{Z}$ is countable and closed in $\mathbb{R}$

   (b) Let us have the succession $s_n = \frac{1}{n}$. Then $A = \bigcup_{i=1}^{n} \{s_n\} = (0, 1]$ which is not closed.'

7. Consider $\mathbb{R}$ with the euclidean topology.

   (a) Let $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots \right\}$. If $s_n = \frac{1}{n}$ we have that

$$S = \{0\} \cup \bigcup_{i=1}^{n} \{s_n\} = \{0\} \cup (0, 1] = [0, 1]$$

   so it is closed.

   (b) Yes, as seen before.

   (c) Yes. Let $\mathbb{Z}\left(\sqrt{2}\right) = \left\{n\sqrt{2} : n \in \mathbb{Z}\right\}$. Then

$$\mathbb{R} \setminus \mathbb{Z}\left(\sqrt{2}\right) = \bigcup_{n\in\mathbb{Z}} \left(n\sqrt{2}, (n+1)\sqrt{2}\right)$$

   which is the union of open intervals and therefore open. Hence $\mathbb{Z}\left(\sqrt{2}\right)$ is closed.

8. Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$. S is said to be an F_σ-set if it is the coutable union of closed sets. S is said to be G_δ-set if it is the countable intersection of open sets.

   (a) Notice that

      i. $(a, b) = \bigcup_{i=1}^{n} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$

      ii. $[a, b]$=$[a, b] \cup [a, b]$

   (b) Notice that

i. $(a, b) = (a, b) \cap (a, b)$

ii. $[a, b] = \bigcap_{i=1}^{n} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$

(c) We have that $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ so $\mathbb{Q}$ is the countable union of closed sets.

(d) Let A_i for $i \in \mathbb{N}$ be a countable famility of open sets and B_i for $i \in \mathbb{N}$ be a countable family of closed sets. Then

i. $X \setminus \left(\bigcup_{i \in \mathbb{N}} B_i\right) = \bigcap_{i \in \mathbb{N}} (X \setminus B_i)$ which is the countable intersection of open sets.

ii. $X \setminus \left(\bigcap_{i \in \mathbb{N}} A_i\right) = \bigcup_{i \in \mathbb{N}} (X \setminus A_i)$ which is the countable intersection of closed sets, because if $X \setminus (X \setminus A_i) = A_i$ is open for all $i \in \mathbb{N}$.

2.2 Basis for a Topology

1. Consider the disc $D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$.

(a) Let $R_{(a,b)} = \left(a \pm \frac{1-r}{8}, b \pm \frac{1-r}{8}\right)$. Notice that

$$\left|x^2 + y^2\right| \leq |x|^2 + |y|^2 \leq |a|^2 + |b|^2 + ab\left(\frac{1-r}{8}\right) + \frac{(1-r)^2}{32}.$$

Because ab is maximum when $a = b = r/\sqrt{2}$ we have

$$\left|x^2 + y^2\right| \leq r^2 + r^2\left(\frac{1-r}{16}\right) + \frac{1 - 2r + r^2}{32} = \frac{1}{32}\left[-2r^3 + 3r^2 - 2r + 1\right].$$

We seek to understand if $\left|x^2 + y^2\right|$ is ever greater than 1. Notice that if for $r < 1$, we have that $-2r^3 + 3r^2 < r$, so the above inequality becomes

$$\left|x^2 + y^2\right| \leq \frac{1}{32}[1 - r] \leq 1$$

as we wanted to show.

(b) We verified that for any point $(a, b) \in D$ we can always find a rectangle $R_{(a,b)} \subseteq D$ containing the point (a, b). As such, it follows that

$$D = \bigcup_{(a,b) \in D} R_{(a,b)}.$$

(c) As $R_{(a,b)}$ is open in the euclidean topology on $\mathbb{R}^2$, we have that D is the union of open sets and therefore open.

(d) By the same argument, we have that any disc centered in an arbitrary point is the union of open rectangles, and therefore an open set in the euclidean topology of $\mathbb{R}^2$.

2. Consider the euclidean topology in $\mathbb{R}^2$.

(a) Let D_1 and D_2 be disks in $\mathbb{R}^2$ with radius a_1 and a_2 respectively, centered at the points (c_1, d_1) and (c_2, d_2). Let $(a, b) \in D_1 \cap D_2$. Then we have that (a, b) is contained in both D_1 and D_2. Let r_1 and r_2 be the distance to the center of each disk, that is

$$\begin{cases} r_1 = & \sqrt{(a - c_1)^2 + (b - d_1)^2} \\ r_2 = & \sqrt{(a - c_2)^2 + (b - d_2)^2} \end{cases}.$$

Taking $k = \min(a_1 - r_1, a_2 - r_2)$ we have that the disk centered at (a, b) of radius k is contained in $D_1 \cap D_2$.

(b) The union of the afore mentioned disks on each point $(a, b) \in D_1 \cap D_2$ give the whole intersection $D_1 \cap D_2$.

(c) We have that the intersection of disks is the union os disks and that the union of disks generates the whole space $\mathbb{R}^2$, so disks are a basis for the euclidean topology on $\mathbb{R}^2$.

3. Consider the intervals (a, b) such that $a < b \in \mathbb{Q}$. We start by proving that these form a basis for some topology on $\mathbb{R}^2$. Notice that $\bigcup_{n \in \mathbb{Z}} (n - 1, n + 1) = \mathbb{R}$. Now let (a_1, b_1) and (a_2, b_2) be intervals with rational limits. The intersection

$$(a_1, b_1) \cap (a_2, b_2) = \begin{cases} \emptyset & \text{if disjoint} \\ (\min(a_1, a_2), \max(b_1, b_2)) & \text{otherwise} \end{cases}$$

has rational limits, so it is the basis of some topology. Now take an arbitrary open $A \subseteq \mathbb{R}$. Let $a \in A$, then there exists $(c, d) \in A$ such that $c < a < d$. But because because between two irrationals we may always find a rational number, even if c, d are irrational, we find $e, f \in \mathbb{Q}$ such that

$$c < e < a < f < d$$

and as such we that an interval $a \in (e, f) \in \mathcal{B} \subseteq A$ so $\mathcal{B}$ is the basis of the euclidean topology on $\mathbb{R}$.

4. A topological space satisfies the second axiom of countability if it is generated by a countable basis.

(a) $\mathbb{R}$ with the euclidean topology admits a basis

$$\mathcal{B} = \{(a, b) \in \mathbb{R} : a < b \in \mathbb{Q}\}$$

so it second countable.

(b) Let $\mathcal{B}$ be a basis of an uncountable topological space $(X, \mathcal{T})$. For $\mathcal{B}$ to be a basis we require every intersection of elements of $\mathcal{B}$ be the union of elements of $\mathcal{B}$. The proof revolves around noticing all the singular sets must belong to $\mathcal{B}$. If that were not the case, then set sets $\{x_i\} \in \mathcal{T}$ would not be contained in any subset of $\mathcal{B}$. As such, because X is uncountable, we have that $\mathcal{B}$ is uncountable, thus X is not second countable.

(c) The proof proceeds analogously to the ones performed before, but now with n-cubes of rational lengths.

(d) Yes. A finite subset of a countable set is necessarily countable. Hence, for any finite subset $A \subseteq X$ we have that $X \setminus A$ is open and the union of all these is X. These constitute all the sets in the topology, hence $(X, \mathcal{T})$ is second countable.

5. We consider the statements.

(a) Let $m, c \in \mathbb{R}$. We define the line $L = \{(x,y) \in \mathbb{R}^2 : y = mx + c\}$. Then $\mathbb{R} \backslash L = \underline{\{(x,y) \in \mathbb{R}^2 : y \neq mx + c\}}$. Then either $y < mx + c$ or $ymx + c$. Let us define these two subsets

$$\begin{cases} Q_1 = & \{(x,y) \in \mathbb{R}^2 : y < mx + c\} \\ Q_2 = & \{(x,y) \in \mathbb{R}^2 : y > mx + c\} \end{cases}.$$

We will prove that these are disjoint open sets whose union is $\mathbb{R} \setminus L$. That Q_1 and Q_2 are disjoint is obvious. Now assume $(u,v) \in Q_1$. Then the distance to the line is given by

$$d = \frac{|mu + c - v|}{\sqrt{1 + m^2}}.$$

As such we may always find an open disk D_d of radius d centered at (u,v) such that $(u,v) \in D \subseteq Q_1$. Then $Q_1 = \bigcup_{(u,v) \in \mathbb{R}^2} D_d(u,v)$ is the union of open sets, therefore open. The same argument applies to Q_2. We conclude that $Q_1 \cup Q_2$ is the union of open sets and therefore open, and $\mathbb{R} \setminus L$ is closed in $\mathbb{R}^2$.

(b) The argument is analogous to (a), but now considering a distance $d = 1 - \sqrt{u^2 + v^2}$.

(c) The argument is analogous to (a), but now considering a distance $d = 1 - \sqrt{\sum_{i=1}^{n} x_i^2}$.

(d) The argument is similar, but now we consider only the outer region of the ball.

(e) The argument is analogous to (a).\

6. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces admiting bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. We definite the product topology by the topology generated by the subsets $B_1 \times B_2 \in \mathcal{B}_1 \times \mathcal{B}_2$. We call this basis $\mathcal{B}$ of the topology $\mathcal{T}$ on $X \times Y$. We check that

   (a) $\mathcal{B}$ trivial generates $\emptyset$, and $\mathcal{B}_1$ and $\mathcal{B}_2$ generate X and Y, hence $\mathcal{B}$ generates $X \times Y$.

   (b) Let $C, D \in B$. Then

   $$C \cap D = (X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \times Y_2) \in \mathcal{B}$$

   for $X_1 \cap X_2$ and $Y_1 \times Y_2$ are open in $X \times Y$.

7. Consider $\mathbb{R}$ with the euclidean topology. Let A be an open subset of $\mathbb{R}$. We showed that $\mathbb{R}$ is second countable, and therefore admits a countable basis $\mathcal{B}$. Any open subset is then the union of elements of $\mathcal{B}$. We also showed that this union is coutable, as any open or closed interval of $\mathbb{R}$ is the countable union of closed and open intervals, respectively. Therefore any open set is the coutable union of open intervals which are themselves the coutable union of closed intervals, thus A is F_σ. On the other hand, any closed subset B is the countable intersection of open intervals, which themselves are a countable union of open intervals. As such, B is G_δ.

2.3 Basis for a Given Topology

1. All are equivalent basis for the euclidean topology on $\mathbb{R}^2$.

2. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on a nonempty set X.

(a) Let $\mathcal{B}_1$ a collection of subsets of X such that $\mathcal{B} \subseteq \mathcal{B}_1 \subseteq \mathcal{T}$. We start by showing that $\mathcal{B}_1$ is the basis for a given topology on X. It is clear that because it contains a basis of $\mathcal{T}$, $\mathcal{B}_1$ generates X. Let $B_1, B_2 \in \mathcal{B}_1$. Then we have that $B_1 = \bigcup_{i \in I} B_i$ and $B_2 = \bigcup_{j \in J} B_j$ for families of subsets of $\mathcal{B}$ and indices $i \in I$ and $j \in J$. Thus we have

$$B_1 \cap B_2 = \left(\bigcup_{i \in I} B_i \right) \cap \left(\bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \bigcup_{j \in J} (B_i \cap B_j)$$

for $B_i \cap B_j \in \mathcal{B} \subseteq \mathcal{B}_1$. Thus every intersection of subsets of $\mathcal{B}_1$ is the union of subsets of $\mathcal{B}_1$, so it is the basis for some topology on X. Now it remains to show to show that $\mathcal{B}_1$ and $\mathcal{B}$ generate the same topology. Let $B \in \mathcal{B}$ and $x \in B$. Then $x \in B \subseteq B \in \mathcal{B}_1$ because $\mathcal{B} \subseteq \mathcal{B}_1$. Now consider $B \in \mathcal{B}_1$ and $x \in B$. We have that $B = \bigcup_{i \in I} B_i$ for some $B_i \in \mathcal{B}$. Then $x \in B_k \in \mathcal{B}$ for some $k \in I$ and hence $x \in B_k \subseteq \mathcal{B}$. Thus both basis generate the same topology and are equivalent.

(b) We take as the basis of the euclidean topology $\mathcal{B}$ the set os intervals with rational limits. To this basis one might add any interval with irrational limits, forming $\mathcal{B}_1$ such that $\mathcal{B} \subseteq \mathcal{B}_1$. Then by (a) this is still a basis for the euclidean topology on $\mathbb{R}$. As such, we have as many basis as intervals with irrational limits to add to $\mathcal{B}$ and, as such, uncountably many.

3. Let $\mathcal{B} = \{(a, b] : a < b \in \mathbb{R}\}$. Consider the union

$$\bigcup_{i=1}^{n} \left(a - \frac{1}{n}, a \right] = \{a\} \in \mathcal{T}.$$

This implies that every singular subset of $\mathbb{R}$ belongs to the topology generated by $\mathcal{B}$. As such, $\mathcal{B}$ generates the discrete topology on $\mathbb{R}$. As subset of $\mathbb{R}$ is open in the discrete topology, so is every open interval.

4. Let $\mathrm{C}\,[0, 1]$ be the set of all continuous real-valued function on $[0, 1]$.

(a) Let $\mathcal{M} = \{M(f,\epsilon) : f \in \mathcal{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ and $M(f,\epsilon) = \left\{g : g \in \mathrm{C}[0,1], \int_0^1 |f-g| < \epsilon\right\}$. We have that

$$\mathcal{C}[0,1] = \bigcup_{f \in \mathrm{C}[0,1]} M(f,1).$$

Now consider $f, h \in \mathcal{C}[0,1]$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Then for $k \in M(f,\epsilon_1) \cap M(h,\epsilon_2)$, we write define

$$\begin{cases} d_1 = & \int_0^1 |f-k| \\ d_2 = & \int_0^1 |h-k| \end{cases}.$$

Setting $\epsilon = \min(\epsilon_1 - d_1, \epsilon_2 - d_2)$ we get that for all for any $g \in M(k,\epsilon)$

$$\int_0^1 |k-g| < \epsilon_1 - \int_0^1 |f-k| \Longrightarrow \int_0^1 |f-g| \leq \int_0^1 |f-k| + |k-g| < \epsilon_1$$

$$\int_0^1 |k-g| < \epsilon_2 - \int_0^1 |h-k| \Longrightarrow \int_0^1 |h-g| \leq \int_0^1 |h-k| + |k-g| < \epsilon_2$$

thus $M(k,\epsilon) \subseteq M(f,\epsilon_1)$ and $M(k,\epsilon) \subseteq M(h,\epsilon_2)$. Now the union

$$\bigcup_{k \in M(f,\epsilon_1) \cap M(h,\epsilon_2)} M(k,\epsilon) = M(f,\epsilon_1) \cap M(h,\epsilon_2)$$

and we conclude that $\mathcal{M} = \{M(f,\epsilon) : f \in \mathrm{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ is the basis for some topology on $\mathcal{C}[0,1]$.

(b) Let $\mathcal{U} = \{U(f,\epsilon) : f \in \mathcal{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ and $U(f,\epsilon) = \left\{g : g \in \mathcal{C}[0,1], \sup_{x \in [0,1]} |f-g| < \epsilon\right\}$. We have that

$$\mathcal{C}[0,1] = \bigcup_{f \in \mathcal{C}[0,1]} U(f,1).$$

Now consider $f, h \in \mathcal{C}[0,1]$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Then for $k \in U(f,\epsilon_1) \cap U(h,\epsilon_2)$, we write define

$$\begin{cases} d_1 = & \sup_{x \in [0,1]} |f-k| \\ d_2 = & \sup_{x \in [0,1]} |h-k| \end{cases}.$$

Setting $\epsilon = \min(\epsilon_1 - d_1, \epsilon_2 - d_2)$ we get that for all for any $g \in U(k,\epsilon)$

$$\sup_{x \in [0,1]} |k-g| < \epsilon_1 - \sup_{x \in [0,1]} |f-k| \Longrightarrow \sup_{x \in [0,1]} |f-g| \leq \sup_{x \in [0,1]} |f-k| + \sup_{x \in [0,1]} |k-g| < \epsilon_1$$

$$\sup_{x \in [0,1]} |k-g| < \epsilon_2 - \sup_{x \in [0,1]} |h-k| \Longrightarrow \sup_{x \in [0,1]} |h-g| \leq \sup_{x \in [0,1]} |h-k| + \sup_{x \in [0,1]} |k-g| < \epsilon_2$$

thus $U(k,\epsilon) \subseteq U(f,\epsilon_1)$ and $U(k,\epsilon) \subseteq U(h,\epsilon_2)$. Now the union

$$\bigcup_{k \in U(f,\epsilon_1) \cap U(h,\epsilon_2)} U(k,\epsilon) = U(f,\epsilon_1) \cap U(h,\epsilon_2)$$

and we conclude that $\mathcal{U} = \{U(f,\epsilon) : f \in \mathcal{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ is the basis for some topology on $\mathcal{C}[0,1]$.

(c) For all $f, g \in \mathcal{C}[0,1]$ we have that

$$\int_0^1 |f-g| \leq \sup_{x\in[0,1]} |f-g|.$$

Consider $k : [0,1] \to \mathbb{R}$ such that $k(x) = f(x) + x$. Then we have that

$$\int_0^1 |f-k| = \int_0^1 x \mathrm{d}x = \frac{1}{2} < \epsilon_1$$

so we have that $k \in M(f, \epsilon_1)$ for any $\epsilon_1 > \frac{1}{2}$. Now consider

$$\sup_{x\in[0,1]} |f-k| = \sup_{x\in[0,1]} |x| = 1 < \epsilon_2$$

so $k \in U(f, \epsilon_2)$ only if $\epsilon_2 > 1$. Thus by taking $\epsilon_1 = \frac{2}{3}$, we have that there exists a function $k = f(x) + x$ such that $k \in M(f, \epsilon_1)$ but $k \notin U(f, \epsilon_2)$ for any ϵ_2 such that $U(f, \epsilon_2) \subseteq M(f, \epsilon_1)$. We conclude that $\mathcal{M}$ and $\mathcal{U}$ generate different topologies on $\mathcal{C}[0,1]$.

5. Let $(X, \mathcal{T})$ be a topological space.

(a) Notice that, for $a < b \in \mathbb{R}$

$$(-\infty, b) \cap (a, \infty) = (a, b)$$

which generate the euclidean topology on $\mathbb{R}$. As such, we say that $\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ are a subbasis of $\mathbb{R}$.

(b) Consider $X = \{a, b, c, d, e, f\}$, $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$ and $\mathcal{S} = \{\{a\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. $\mathcal{S}$ is a subbasis of $\mathcal{T}$ because

i. $X = \{a, c, d\} \cup \{b, c, d, e, f\}$
ii. $\{a\} \in \mathcal{S}$
iii. $\{c, d\} = \{a, c, d\} \cap \{b, c, d, e, f\}$
iv. $\{a, c, d\} \in \mathcal{S}$
v. $\{b, c, d, e, f\} \in \mathcal{S}$.

6. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on $\mathbb{R}$ such that closed intervals of the form $[a, b]$ for $a < b$ are contained in $\mathcal{S}$. Then, for any $a \in \mathbb{R}$ we have that for $c < a < b$

$$\{a\} = [a, b] \cap [c, a]$$

so every singular set is contained in $\mathcal{T}$, thus it is the discrete topology on $\mathbb{R}$.

7. Every subset in the cofinite topology is of the form $X \setminus A$ for some finite subset $A = \{x_1, \ldots, x_n\}$. Thus

$$X \setminus A = X \setminus \left(\bigcup_{i=1}^{n} \{x_i\}\right) = \bigcap_{i=1}^{n} (X \setminus \{x_i\})$$

so every open set $X \setminus A$ is an intersection of sets of the $\mathcal{S}$, so $\mathcal{S}$ is a subbasis of the confinite topology on X.

8. Consider the subbasis $\mathcal{S} = \{A \in X : |A| = 2\}$. By intersection of such sets A we get all singleton sets, and therefore generate the discrete topology on X.

9. The intersection of straight lines can be either empty, a line, or a single point. As such, we have that all singular points are an intersection of elements of the subbasis, and as such, the topology generated by all straight lines in $\mathbb{R}^2$ is the discrete topology.

10. The open sets in $(X, \mathcal{T})$ are the empty set, straight lines parallel to the x axis, and unions of these lines. By arbitrary unions of these lines, one may create regions of the form

$$S = \left\{(x, y) \in \mathbb{R}^2 : a < y < b\right\}$$

for $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$.

11. The circles in the plane may intersect at only one point. As such, we have the the topology generated by $\mathcal{S}$ is the discrete topology on $\mathbb{R}^2$.

12. Intersections go brrrrr.

3 Limit Points

3.1 Limit Points and Closure

1. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.
   (a) The limit points are
      i. $\{a\}' = \{f\}$
      ii. $\{b, c\}' = \{b, d, e, f\}$
      iii. $\{a, c, d\}' = \{b, c, d, e, f\}$
      iv. $\{b, c, e, f\}' = \{b, d, e\}$.
   (b) The closures are
      i. $\overline{\{a\}} = \{a, f\}$
      ii. $\overline{\{b, c\}} = \{b, c, d, e\}$
      iii. $\overline{\{a, c, d\}} = X$
      iv. $\overline{\{b, c, e, f\}} = \{b, c, d, e, f\}$.
   (c) Alternatively, we have that the closed sets of $(X, \mathcal{T})$ are $\{X, \emptyset, \{b, c, d, e, f\}, \{a, b, e, f\}, \{a, f\}\}$. We now select the smallest containing such elements to find the closure. Then
      i. $\overline{\{a\}} = \{a, f\}$
      ii. $\overline{\{b, c\}} = \{b, c, d, e, f\}$
      iii. $\overline{\{a, c, d\}} = X$
      iv. $\overline{\{b, c, e, f\}} = \{b, c, d, e, f\}$.

2. Let $(\mathbb{Z}, \mathcal{T})$ be the of integers with the cofinite topology. We have that
   (a) $\overline{\{1, 2, 3, \ldots, 10\}} = \{1, 2, 3, \ldots, 10\}$ as it is the smallest closed set that contains the elements up to 1 to 10. So there are no limit points other than these elements. We need only check if any of these elements are limit points. Let $a \in \{1, \ldots, 10\}$, then the set $\{a\} \cup (11, 12, \ldots)$ is open but intersects $\{1, 2, 3, \ldots, 10\}$ only on $\{a\}$. Hence $\{1, 2, 3, \ldots, 10\}' = \emptyset$.
   (b) Let $E = 2\mathbb{Z}$ be the set of even integers. Let $a \in E$. Considering the set $\{2n + 1 : n \in \mathbb{Z}\} \cup (a)$ we have an open set that intersects E only at a. Because a is arbitrary, we conclude that $E' = \emptyset$.