Chapter 1

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1.1 Accumulation Points

Definition 1. Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. We say that $x \in Y$ is an **accumulation point** of Y if every open subset containing x intersects Y in a a point different to x. We denote the set of accumulation points of Y by Y'.

Example 1. In $\mathbb{R}$ we have that

- $([0,1) \cap \{2\})' = [0,1]$
- $\mathbb{Q} = \mathbb{R}$.

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space and $F \subseteq X$. Then F is closed if and only if $F' \subseteq F$.*

Proof. Suppose that F is closed and that $x \in F'$. We aim to prove that $x \in F$. We have that $X \setminus F$ is open. Suppose $x \notin F$, then $x \in X \setminus F$. But

$$(X \setminus F) \cap F = \emptyset$$

which implies that no intersection of subsets of F intersects with any open sets containing x on any other point, which is a contradiction. Then $F' \subseteq F$.

Suppose now that $F' \subseteq F$. Then if $x \in X \setminus F$ then x is not an accumulation point, so there exists an open subset $A_x \in X \setminus F$ such that $x \in A_x$ and $F \cap A_x = \emptyset$. Then $X \setminus F = \bigcup_{x \in X \setminus F} A_x$ is open, hence F is closed. □

Remark. Let $(X, \mathcal{T})$ be a topological space. Then, for all $Y \subseteq X$, $Y \cup Y'$ is closed.

Proof. Let $x \in X \setminus (Y \cup Y')$. Notice that $X \setminus (Y \cup Y') = (X \setminus Y) \cap (X \setminus Y')$. Let A be an open subset that contains x.

Assume that $A \cap Y' = \emptyset$ but $A \cap Y \neq \emptyset$. Then there exists $z \neq x \in A \cap Y$ so x is an accumulation point, which is a contradiction.

Assume now that $A \cap Y = \emptyset$ but $A \cap Y' \neq \emptyset$. Then there exists $z \neq x \in A \cap Y'$. But because z is an accumulation point, we know that $A \cap Y \neq \emptyset$ which is a contradiction.

Thus we have that $A \cap X \setminus (Y \cup Y') = \emptyset$ and the unions of these sets is open, so $X \setminus (Y \cup Y')$ is closed. □

Definition 2. We call the subset $Y \cup Y'$ the **closure** of Y and write $\bar{Y}$.

Remark. Let $(X, \mathcal{T})$ be a topological space and $x \in X$. Let Y be a subset of X. Then $x \in \bar{Y}$ if every open subset of X containing x intersects Y.

Proof. Let A be an open subset of X such that $x \in A$.

Assume $x \in Y$. Obviously $A \cap Y \neq \emptyset$. Assume now $x \notin Y$. As $x \in Y'$, there exists $z \in A$ such that $z \in A \cap Y$ therefore $A \cap Y \neq \emptyset$. □

Corolary 1.1. *Let $(X, \mathcal{T})$ be a topological space and $F \subseteq X$. Then F is closed if and only if $\bar{F} = F$.*

Proof. Assume F is closed. Then, by theorem 1 it follows that $F' \subseteq F$, thus $\bar{F} = F$.

Assume now that $\bar{F} = F$. Then

$$F \cup F' = F \Longrightarrow F' \subseteq F$$

and by the reverse application of theorem 1, F is closed. □

Corolary 1.2. *$\bar{Y}$ is the smallest closed subset that contains Y.*

Proof. Suppose that F is closed and $Y \subseteq F$. Then $\bar{Y} \subseteq \bar{F}$. But F is closed, so $\bar{Y} \subseteq F$ and $\bar{Y}$ is the smallest. □