Notes on Topology

Chapter 1

22/09/2020

1.1 Introduction

Our aim is to abstract and generalize concepts such as **space**, **distance** and **continuity**.

As en example of the character of this theory, we intend not to distinguish subspaces A (circle) and B (square) of $\mathbb{R}^2$, but to distinguish these from C (ring), as in figure 1.1.

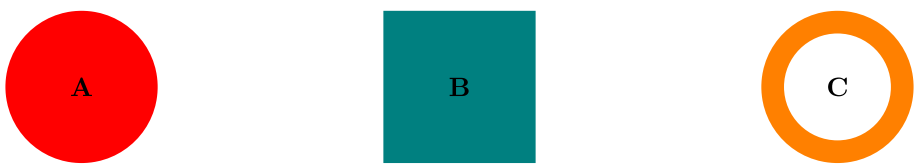

Figure 1.1: Examples of topological subspaces of $\mathbb{R}^2$.

1.2 Topological Spaces

Definition 1. Given a set X, we denote the the set of all subsets of X as $\mathcal{P}(X)$, called the **power set** of X.

In topology, the union of potentially infinite (and uncountable) subsets. We write the union of a family of subsets $\{A_i : i \in I\}$ as $\cup_{i \in I} A_i$.

Definition 2. Given a non-empty set X, we say $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** on X if:

1. $X, \emptyset \in \mathcal{T}$
2. if $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$
3. if $A_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}$.

Then $(X, \mathcal{T})$ is a **topological space**, and the elements of X sometimes denotes by points.

Remark. For a finite family of subsets N, the reunion $\bigcap_{i \in N} A_i \in \mathcal{T}$.

Proof. The case for $n = 2$ is taken care by the second axiom in 2. Assume $\bigcap_{i=1}^{n} A_i = B \in \mathcal{T}$. Then

$$\bigcap_{i=1}^{n+1} A_i = \left(\bigcap_{i=1}^{n} A_i \right) \cap A_{n+1} = B \cap A \in \mathcal{T}$$

so the statement holds by induction. $\square$

Example 1. Let $X_1 = \{a, b, c, d, e, f\}$ and $\mathcal{T}_1 = \{X_1, \emptyset, \{a\}, \{c, b\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then $\mathcal{T}_1$ is a topology on X_1.

Remark. For a finite set X, one does enough to test the union of each two subsets of $\mathcal{T}$, and the finite union follows by induction.

Example 2. Let $X = \{a, b, c, d\}$ and $\mathcal{T}_2 = \{X_2, \emptyset, \{a\}, \{c, d\}, \{a, c\}, \{a, c, d\}\}$. Then $\mathcal{T}_2$ is not a topology on X_2, for $\{a\} \cap \{a, c\} = \{c\} \notin \mathcal{T}$.

Example 3. Let $\mathcal{T}_3 \subseteq \mathcal{P}(\mathbb{N})$ be such that

$$\mathcal{T}_3 = \{\mathbb{N}\} \cup \{A \subseteq \mathbb{N} : A \text{ is finite}\}.$$

$\mathcal{T}_3$ is not a topology on $\mathbb{N}$. Let $B_n = \{2n + 1\}$ for $n \in \mathbb{N}$. Then $B_n \in \mathcal{T}_3$ but

$$\bigcup_{n=1}^{\infty} B_n \notin \mathcal{T}_3.$$

Definition 3. Let X be a non-empty set. Then $\mathcal{T} = \mathcal{P}(X)$ is a topology on X, called the **discrete topology**.

Remark. Let $(X, \mathcal{T})$ be a topological space. Then $\mathcal{T}$ is the discrete topology if and only if $\{x\} \in \mathcal{T}$ for all $x \in X$. These are called the **singular sets** of X.

Proof. Take $A \in \mathcal{P}(X)$ and let x_i for $i \in I$ be elements of X. If A is empty then it is simply the **empty union** of singleton sets. Otherwise, it implies that there does not exist a union such that

$$\bigcup_{i \in I} \{x_i\} = A.$$

Then there exists an element $x_k \in A$ that is not contained in the union. As every element of X is contained in the union of singletons sets of X, A contains an element that is not contained in X. Hence $A \notin \mathcal{P}(X)$, which is contradiction. $\square$

Definition 4. Let X be a non empty set. Then $\mathcal{T} = \{X, \emptyset\}$ is a topology of X, called the **indiscrete topology** ¶.