# Exercises on Topology

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## 1 Topological Spaces

### 1.1 Topology

1. Let $X = \{a, b, c, d, e, f\}$.
   (a) $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, f\}, \{b, f\}, \{a, b, f\}\}$ is not a topology on $X$ because $\{a, f\}, \{b, f\} \in \mathcal{T}_1$ but
   $$\{a, f\} \cap \{b, f\} = \{f\} \notin \mathcal{T}_1.$$
   (b) $\mathcal{T}_2 = \{X, \emptyset, \{a, b, f\}, \{a, b, d\}, \{a, b, d, f\}\}$ is not a topology on $X$ because $\{a, b, f\}, \{a, b, d\} \in \mathcal{T}_2$ but
   $$\{a, b, f\} \cap \{a, b, d\} = \{a, b\} \notin \mathcal{T}_2.$$
   (c) $\mathcal{T}_3 = \{X, \emptyset, \{f\}, \{e, f\}, \{a, f\}\}$ is not a topology on $X$ because $\{e, f\}, \{a, f\} \in \mathcal{T}_3$ but
   $$\{e, f\} \cap \{a, f\} = \{a, e, f\} \notin \mathcal{T}_3.$$

2. Let $X = \{a, b, c, d, e, f\}$.
   (a) $\mathcal{T}_1 = \{X, \emptyset, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$ is not a topology on $X$ because $\{c\}, \{b\} \in \mathcal{T}_1$ but
   $$\{c\} \cap \{b\} = \{b, c\} \notin \mathcal{T}_1.$$
   (b) $\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}\}$ is not a topology on $X$ because $\{b, d, e\}, \{a, b, d\} \in \mathcal{T}_2$ but
   $$\{b, d, e\} \cap \{a, b, d\} = \{b, d\} \notin \mathcal{T}_2.$$
   (c) $\mathcal{T}_3 = \{X, \emptyset, \{b\}, \{a, b, c\}, \{d, e, f\}, \{b, d, e, f\}\}$ is a topology on $X$.

3. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T}$ the discrete topology on $X$. The following statements are true: (a), (d), (g), (i), (l), (k).

4. The case for the intersection of two subsets is taken care by the axioms of a topology. Assume that $\bigcup_{i=0}^{n} A_i = B \in \mathcal{T}$, then
   $$\bigcup_{i=0}^{n+1} A_i = \left(\bigcup_{i=0}^{n} A_i\right) \cup A_{n+1} = B \cup A_{n+1} \in \mathcal{T}.$$

5. Let $X = \mathbb{R}$.
   (a) $\mathcal{T}_1 = \{\mathbb{R}, \emptyset\} \cup \{(-n, n) : n \in \mathbb{N}\}$.
      i. $\mathbb{R}, \emptyset \in \mathcal{T}_1$
      ii. Let $n, m \in \mathbb{N}$ then $(-n, n) \cap (-m, m) = (-\min(n, m), \min(n, m)) \in \mathcal{T}_1$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} (-n_i, n_i) = (-\sup(n_i), \sup(n_i)) \in \mathcal{T}_1.$$
   (b) $\mathcal{T}_2 = \{\mathbb{R}, \emptyset\} \cup \{[-n, n] : n \in \mathbb{N}\}$.
      i. $\mathbb{R}, \emptyset \in \mathcal{T}_2$
      ii. Let $n, m \in \mathbb{N}$ then $[-n, n] \cap [-m, m] = [-\min(n, m), \min(n, m)] \in \mathcal{T}_2$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} [-n_i, n_i] = [-\sup(n_i), \sup(n_i)] \in \mathcal{T}_2.$$
   (c) $\mathcal{T}_3 = \{\mathbb{R}, \emptyset\} \cup \{[n, \infty) : n \in \mathbb{N}\}$.
      i. $\mathbb{R}, \emptyset \in \mathcal{T}_3$
      ii. Let $n, m \in \mathbb{N}$ then $[n, \infty) \cap [m, \infty) = [\max(n, m), \infty) \in \mathcal{T}_3$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} [n_i, \infty) = [\inf(n, m), \infty) \in \mathcal{T}_3.$$
6. Let $X = \mathbb{N}$.
   (a) Let $S_n = \{1, \ldots, n\}$ and $\mathcal{T}_1 = \{\mathbb{N}, \emptyset\} \cup \{S_n : n \in \mathbb{N}\}$.
      i. $\mathbb{N}, \emptyset \in \mathcal{T}_1$
      ii. Let $n, m \in \mathbb{N}$ then $S_n \cap S_m = S_{\min(n,m)} \in \mathcal{T}_1$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} S_{n_i} = S_{\sup(n_i)} \in \mathcal{T}_1.$$
   (b) Let $S_n = \{n, n+1, \ldots\}$ and $\mathcal{T}_2 = \{\mathbb{N}, \emptyset\} \cup \{S_n : n \in \mathbb{N}\}$.
      i. $\mathbb{N}, \emptyset \in \mathcal{T}_2$
      ii. Let $n, m \in \mathbb{N}$ then $S_n \cap S_m = S_{\max(n,m)} \in \mathcal{T}_2$
      iii. Let us have a family of natural numbers $n_i \in \mathbb{N}$. Then
$$\bigcup_{n_i \in \mathbb{N}} S_{n_i} = S_{\inf(n_i)} \in \mathcal{T}_2.$$
7. Number of topologies goes brrrr.

8. Let us have an infinite set $X$ and topology $\mathcal{T}$ on $X$ such that every infinite subset of $X$ is open. $\mathcal{T}$ is the discrete topology if every finite subset of $K \subseteq X$ is also open. Start by taking two disjunct infinite subsets of $A, B \subseteq X$. By definition these are open. Thus the sets $A \cup K$ and $B \cup K$ are also open. Hence

$$(A \cup K) \cap (B \cup K) = K \in \mathcal{T}$$

for any finite set $K$, so $\mathcal{T}$ is the discrete topology on $X$.

9. Let $X = \mathbb{R}$.

(a) $\mathcal{T}_1 = \{\mathbb{R}, \emptyset\} \cup \left\{(a, b) \in \mathbb{R}^2 : a < b\right\}$ is not a topology on $\mathbb{R}$. Notice that $(0,1) \in \mathcal{T}_1$ and $(2,3) \in \mathcal{T}_1$ but

$$(0,1) \cup (2,3) \notin \mathcal{T}.$$

(b) $\mathcal{T}_2 = \{\mathbb{R}, \emptyset\} \cup \{(-r, r) : r \in \mathbb{R}\}$ is a topology on $\mathbb{R}$.

i. $\mathbb{R}, \emptyset \in \mathcal{T}_2$

ii. Let $r_1, r_2 \in \mathbb{R}$ then $(-r_1, r_1) \cap (-r_2, r_2) = (-\min(r_1, r_2), \min(r_1, r_2)) \in \mathcal{T}_2$

iii. Let there be a family of subsets $(-r_i, r_i)$, with real numbers $r_i$ for $i \in I$ arbitrary indices. Then

$$\bigcup_{i \in I} (-r_i, r_i) = (-\sup(r_i), \sup(r_i)) \in \mathcal{T}_2.$$

We take the chance to prove that $\bigcup_{i \in I} (-r_i, r_i) = (-\sup(r_i), \sup(r_i))$.

(c) $\mathcal{T}_3 = \{\mathbb{R}, \emptyset\} \cup \{(-r, r) : r \in \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left(-\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^n\right) \in \mathcal{T}_3$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-e, e) \notin \mathcal{T}_3.$$

(d) $\mathcal{T}_4 = \{\mathbb{R}, \emptyset\} \cup \{[-r, r] : r \in \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left[-\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^n\right] \in \mathcal{T}_4$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = [-e, e] \notin \mathcal{T}_4.$$

(e) $\mathcal{T}_5 = \{\mathbb{R}, \emptyset\} \cup \{(-r, r) : r \notin \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left(-\frac{\pi}{n} - 1, \frac{\pi}{n} + 1\right) \in \mathcal{T}_5$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1) \notin \mathcal{T}_5.$$

(f) $\mathcal{T}_6 = \{\mathbb{R}, \emptyset\} \cup \{[-r, r] : r \notin \mathbb{Q}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left[-\frac{\pi}{n} - 1, \frac{\pi}{n} + 1\right] \in \mathcal{T}_6$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = [-1, 1] \notin \mathcal{T}_6.$$

(g) $\mathcal{T}_7 = \{\mathbb{R}, \emptyset\} \cup \{[-r, r) : r \in \mathbb{R}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right) \in \mathcal{T}_7$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1) \notin \mathcal{T}_7.$$

(h) $\mathcal{T}_8 = \{\mathbb{R}, \emptyset\} \cup \{(-r, r] : r \in \mathbb{R}\}$ is not a topology on $\mathbb{R}$. Notice that for $n \in \mathbb{N}$,

$$A_n = \left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] \in \mathcal{T}_8$$

therefore

$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1) \notin \mathcal{T}_8.$$

(i) $\mathcal{T}_9$ is a topology on $\mathbb{R}$.

(j) $\mathcal{T}_{10}$ is a topology on $\mathbb{R}$.

## 1.2 Open, Closed and Clopen Sets

1. Number of subsets goes brrrrrr.

2. Let $(X, \mathcal{T})$ be a topological space such that every subset $A \subseteq X$ is closed. Then the subset $X \setminus A$ is closed. By the definition of a closed subspace

$$X \setminus (X \setminus A) = A$$

is open. Because $A$ is arbitrary, it follows that $\mathcal{P}(X) \subseteq \mathcal{T}$, so $\mathcal{T}$ is the discrete topology on $X$.

3. Let $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{c, d\}\}$ be a topology on $X$. It is easy to check that every set of $\mathcal{T}$ is open and closed.

4. Let $X$ be and infinite set and $\mathcal{T}$ be a topology on $X$ such that every infinite subset of $X$ is closed. Let $A \subseteq X$. If $X \setminus A$ is infinite, then it is closed, and thus

$$X \setminus (X \setminus A) = A$$

is open. If $X \setminus A$ is finite, then $A$ is infinite, and as such closed. So by definition $X \setminus A$ is open. Because $A$ is arbitrary, we conclude that $\mathcal{T}$ is the discrete topology on $X$.

5. No. Let $\mathcal{T} = \{X, \emptyset, \{x\}\}$ for some element $x \in X$ be a topology on $X$. Then $\mathcal{T}$ is not the indiscrete topology on $X$ but the only infinite open subset of $X$ is $X$.

6. Let $X$ be a nonempty set and $\mathcal{T} = \{X, \emptyset, A, B\}$ such that $A$ and $B$ are nonempty distinct proper subsets of $X$.
   (a) If none of the conditions apply, we have two possible cases, $A \cap B = \emptyset$ and $A \cap B \neq \emptyset$. If $A$ and $B$ are disjunct, then their union will not be contained in $A$ nor $B$, but will also not be $X$, else condition (a) would apply. As such, the union is not open. If they are not disjunct, but are distinct and condition (b) and (c) do not apply, then their intersection will not be open. As such, we have that some condition must apply. But the conditions are obviously exlusive, so we conclude that $\mathcal{T}$is a topology on $X$ if one (and only one) of the conditions holds.
   (b) Topologies on $X$ go brrrrrrr.

7. Let $X_n$ be a finite space such that $|X|$=n. Let us denote the elements of $X_n$ by $x_i$ for $i \in N$. Let $M_n$ be the set of all topologies on $X_n$.
   (a) For $X_{n+1} = \{x_1, \ldots, x_n\} \cup \{x_{n+1}\}$ we have atleast $|M_n|$ topologies, that only contain the element $x_{n+1}$ in the subset $X_{n+1}$. Because the family $\{X_{n+1}, \emptyset, \{x_{n+1}\}\} \in M_{n+1}$ it follows that $|M_{n+1}| > |M_n|$.
   (b) It is clear that if $\mathcal{T}$ is a topology on $X_n$, then if we replace $x_i$ by $x_{n+1}$ it is clear that $\mathcal{T}$ is a topology on $X_{n+1}$. As such, for each topology on $X_n$, there are atleast $n$ topologies on $X_{n+1}$. Therefore we conclude that $X_n$ has at least $(n-1)!$ topologies.
   (c) Let now $X$ be an infinite set. Then for any subset $A \subseteq X$ we have that the topology

   $$\mathcal{T} = \{X, \emptyset, A\}$$

   is a topology on $X$ of cardinatily $\aleph$. As such, we have that there are at least $|\mathcal{P}(X)|$ topologies on $X$. For a set of $N$ elements we have that

   $$|\mathcal{P}(X_N)| = \sum_{i=0}^{N} \binom{N}{i} = 2^N.$$

   As such, there are at least $2^{\aleph}$ topologies defined on $X$.

### 1.3 Finite-Closed Topology

1. Let $f : X \to Y$ be a function between two sets $X, Y$.

(a) We aim to prove that

$$f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j).$$

Let $x \in \bigcup_{j\in J} B_j$, so $y = f^{-1}(x) \in f^{-1}\left(\bigcup_{j\in J} B_j\right)$. Then $x \in B_k$ for some $k \in J$. Thus

$$y \in f^{-1}(B_k) \subseteq \bigcup_{j\in J} f^{-1}(B_j).$$

Now let $y \in \bigcup_{j\in J} f^{-1}(B_j)$. Then $y \in f^{-1}(B_k)$ for some $k \in J$. As such, $x \in B_k \subseteq \bigcup_{j\in J} B_j$, hence

$$y \in f^{-1}\left(\bigcup_{j\in J} B_j\right)$$

and that concludes the proof.

(b) We want to prove that for any $B_j \subseteq Y$ with $j \in J$,

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2).$$

Let $x \in B_1 \cap B_2$, then $x \in B_1$ and $x \in B_2$. As such

$$y = f^{-1}(x) \in f^{-1}(B_1) \quad \text{and} \quad y \in f^{-1}(B_2)$$

thus $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Now assume $y \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $y \in f^{-1}(B_1)$ and $y \in f^{-1}(B_2)$, thus $x \in B_1$and $x \in B_2$. We may then conclude that

$$x \in B_1 \cap B_2 \Longrightarrow y \in f^{-1}(B_1 \cap B_2).$$

(c) Let $X = Y = \mathbb{R}$ and

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} x+1 & \text{if } x < 1 \\ x-1 & \text{if } x \geq 1 \end{cases}. \end{aligned}$$

Let $A = (0,1)$ and $B = (2,3)$. Then $A, B \subseteq X$ but $f^{-1}(A \cap B) = \emptyset$ but

$$f^{-1}(A) \cap f^{-1}(B) = (1,2).$$

2. No. For example, the subset

$$A = \{3, 4, \cdots\} \cup \{1\}$$

is not open in the final segment topology, but it is open in the cofinite topology, because $\mathbb{N} \setminus A = \{2\}$ is finite.

3. The every singleton set $\{x\}$ in a topological space $(X, \mathcal{T})$ is a closed subset if $(X, \mathcal{T})$ is a discrete space of an infinite space with the cofinite topology.

4. Let $\mathcal{T}$ be a cofinite topology on a set $X$. Assume that $\mathcal{T}$ is discrete. As we showed before, every set the discrete topology is clopen. As such, any $A \in \mathcal{T}$, we have that $X \setminus A$ is open. But $X \setminus A \in \mathcal{T}$ because it is discrete, hence $X \setminus (X \setminus A) = A$ is open. Thus, $A$ is finite. Because $A$ is arbitrary, $X$ is finite.

5. Let $(X, \mathcal{T})$ be a $\mathrm{T}_1$ topological space.
   (a) Because $(X, \mathcal{T})$ is $\mathrm{T}_1$, for every pair $a, b \in X$ there exist open sets $A, B$ such that $A$ contains $a$ but not $b$, and $B$ contains $b$ but not $a$. As such, a set that contains one point but not the other, for example, $A$. Thus, every $\mathrm{T}_1$ is $\mathrm{T}_0$.
   (b) We have that
      1. The discrete space is contains every singleton subset of $X$, which trivially contains a point but no other. As such, a discrete space is $\mathrm{T}_1$ and therefore $\mathrm{T}_0$.
      2. The only open subsets of an indiscrete space with at least two points are the empty set and $X$. As such, every open set containing a point $a$ contains every other point. Thus the indiscrete space with at least two points is not $\mathrm{T}_0$.
      3. The cofinite topology contains every sigleton subset of $X$, thus the same argument of (i) applies.
      4. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then every set containing $c$ also contains $d$, thus $(X, \mathcal{T})$ is not $\mathrm{T}_0$.

6. Let $X$ be an infinite set and

$$\mathcal{T} = \{X \setminus A \subseteq X : A \subseteq X \text{ is countable}\} \cup \{\emptyset\}.$$

   (a) By definition $X, \emptyset \in \mathcal{T}$.
   (b) Let $A, B \in \mathcal{T}$. Then

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) \in \mathcal{T}$$

   because the union of countable sets is countable.
   (c) Let $A_i \in \mathcal{T}$ for $i \in I$ be a family of open sets. Then

$$X \setminus \left( \bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \in \mathcal{T}$$

   because the arbitrary intersection of countable sets is countable.

7. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on $X$.
   (a) The union $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$ is not necesseraly a topology. Take $X = \{a, b, c\}$, $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$. Then $\mathcal{T}_3 = \{X, \emptyset, \{a\}, \{b\}\}$, which is not a topology because

$$\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_3.$$

   $\mathbb{N}$

   (b) The intersection of topologies $\mathcal{T}_4 = \mathcal{T}_1 \cap \mathcal{T}_2$, is a topology.
      i. We have that $\{X, \emptyset\} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$, hence $X, \emptyset \in \mathcal{T}_4$
      ii. Let $A, B \in \mathcal{T}_4$. Then $A \cap B \in \mathcal{T}_1$ and $A \cap B \in \mathcal{T}_2$, thus $A \cap B \in \mathcal{T}_4$

iii. Let $A_i \in \mathcal{T}_4$ for $i \in I$ be a family of open sets. The $\bigcup_{i\in I} A_i$ is still contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$. Thus $\bigcup_{i\in I} A_i \in \mathcal{T}_4$.

(c) We assume that for all distinct points $a, b \in X$ there exists open sets $A_1, B_1 \in \mathcal{T}_1$ and $A_2, B_2 \in \mathcal{T}_2$ such that $A_i$ contains $a$ but not $b$ and $B_i$ contains $b$ but not $a$. Thus $A_1 \cap A_2$ contains $a$ but not $b$ and $B_1 \cap B_2$ contains $b$ but not $a$. We conclude that $\mathcal{T}_4$ is $\mathrm{T}_1$.

(d) Let $X = \{a, b\}$, $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathrm{T}_0$ spaces but

$$\mathcal{T}_4 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{X, \emptyset\}$$

is not, as there is not subset of $\mathcal{T}_4$ that contains $a$ but not $b$.

(e) The case for $n = 2$ was handled above. Let us assume that $\mathcal{T} = \bigcap_{i=1}^{n} \mathcal{T}_i$ is a topology on $X$. Then

$$\bigcap_{i=1}^{n+1} \mathcal{T}_i = \mathcal{T}_{n+1} \cap \mathcal{T}$$

is a topology on $X$, because it is the intersection of two topologies. Thus the proof follows by induction.

(f) Let $\mathcal{T}_i$ for $i \in I$ be an arbitrary family of topologies on $X$. Then

i. $X, \emptyset \in \mathcal{T}_i$ for all $i \in I$, thus $X, \emptyset \in \bigcap_{i\in I} \mathcal{T}_i$

ii. Let $A, B \in \bigcap_{i\in I} \mathcal{T}_i$. Then both $A$ and $B$ are contained in each $\mathcal{T}_i$. Thus their intersection is too and $A \cap B \in \bigcap_{i\in I} \mathcal{T}_i$

iii. Let $A_j \in \bigcap_{i\in I} \mathcal{T}_i$ for $j \in J$ be an arbitrary familiy of open sets. Thus $A_j$ is contained in each $\mathcal{T}_i$ for all $j \in J$. Because each $\mathcal{T}_i$ is a topology, we have that $\bigcup_{j\in J} A_j \in \mathcal{T}_i$ for all $i \in I$. Hence

$$\bigcup_{j\in J} A_j \in \bigcap_{i\in I} \mathcal{T}_i.$$

8. Let $X_n$ denote a space of cardinality $|X_n| = n$. Let $M_n$ denote the number of possible $\mathrm{T}_0$ topologies on $X_n$. Because we may construct every topology of $M_n$ using the first $n$ elements of $X_{n+1}$ (and replacing $X_n$ with $X_{n+1}$), we have that $M_{n+1} \geq M_n$. Notice now that the discrete topology on $X_{n+1}$ is not counted in $M_n$, because $\mathcal{P}(X_n) < \mathcal{P}(X_{n+1})$ and is $\mathrm{T}_0$ on $X_{n+1}$. We conclude that $M_{n+1} > M_n$.

9. Let $(X, \mathcal{T})$ be a topological space.

(a) In the discrete topology on $X$, every subset of $X$ is clopen, thus a discrete space is a door space.

(b) If $|X| \geq 1$, the indiscrete topology is not a door space, for the only open or closed sets are $X$ and $\emptyset$.

(c) Let $X$ be infinite. Then the cofinite topology on $X$ is not necesseraly a door space. For example, let $X = \mathbb{Z}$. Then $2\mathbb{Z}$, the set of all even intergers, is not open in $\mathbb{Z}$ with the cofinite topology, for $\mathbb{Z} \setminus 2\mathbb{Z}$ is infinite, and is also not closed, and $\mathbb{Z} \setminus 2\mathbb{Z}$ is also not open.

(d) The only door topology on $X$ is the discrete one. Let $\mathcal{T}$ be a topology on $X$. Consider the singular sets. If they are open, then $\mathcal{T}$ is the discrete topology. If they are closed, then all subsets of the form $\{x, y, z\}$ for $x, y, z \in X$ must be open. But so must their intersections, which imply that $\mathcal{T}$ is the discrete topology on $X$.

10. Let $(X, \mathcal{T})$ be a topological space.

(a) Let $A \in \mathcal{T}$. Then $A = A \cap A$, so $A$ is saturated.

(b) Let $(X, \mathcal{T})$ be $\mathrm{T}_1$. Then every singular subset of $X$ is closed. Writing $A = \bigcup_{x_i \in A} \{x_i\}$ we have that

$$X \setminus \bigcup_{x_i \in A} \{x_i\} = \bigcap_{x_i \in A} (X \setminus \{x_i\})$$

so $A$ is the intersection of open sets and thus saturated.

(c) Let $X = \{a, b\}$ with the indiscrete topology. Then the subset $A = \{a\}$ is not saturated.

(d) Let $a, b \in X$ be distinct. Because $\{a\}$ is saturated, we have that $\{a\} = \bigcap_{i \in I} U_i$ for some family of open sets $U_i$. Thus there exists some $k \in I$ such that $a \in U_k$ but $b \notin U_k$. By the application of the same argument to the saturated set $\{b\}$, it follows that $(X, \mathcal{T})$ is $\mathrm{T}_1$.

# 2 Euclidean Topology

## 2.1 The Euclidean Topology on $\mathbb{R}$

1. Let $a, b \in \mathbb{R}$ such that $a < b$. Assume that $[a, b)$ is open in the euclidean topology. Therefore we have that for all $x \in [a, b)$ there exist $c, d \in \mathbb{R}$ such that $c < d \in [a, b)$ and $c < x < d$. Thus taking $x = a$, it follows that $c < a$ and $c \in [a, b)$ so
$$c < a \leq c$$
which is a contradiction. The proof for $(a, b]$ is analogous.
Assume now that $[a, b)$ is closed. Then $\mathbb{R} \setminus [a, b) = (-\infty, a) \cup [b, \infty)$ is open. But by an analogous argument, now considering $b$, this is also a contradiction. The same applies to $(a, b]$. Thus half closed intervals are neither open nor closed in the euclidean topology on $\mathbb{R}$.

2. Let $a \in \mathbb{R}$. We have that $[a, \infty)$ is closed if $\mathbb{R} \setminus [a, \infty) = (-\infty, a)$ is open. Notice that for all $x \in (-\infty, a)$ there exist $b, c \in \mathbb{R}$ such that $x \in (b, c) \subseteq (-\infty, a)$. For example, take
$$\begin{cases} b = & x - 1 \\ c = & \frac{x+a}{2} \end{cases}.$$
Hence $[a, \infty)$ is closed in the euclidean topology on $\mathbb{R}$. The argument for $(-\infty, a]$ is analogous.

3. Consider the union
$$A = \bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] = (-1, 1)$$
which is clearly not closed in the euclidean topology on $\mathbb{R}$.

4. Consider $\mathbb{R}$ with the euclidean topology.

   (a) Assume $\mathbb{Z}$ is open. Then for any $n \in \mathbb{Z}$ there exist $a, b \in \mathbb{R}$ such that $a < b$ and $n \in (a, b) \subseteq \mathbb{Z}$. But $\mathbb{Z}$ contains no intervals, because for every two elements $m_1, m_2$ in $\mathbb{Z}$ such that $m_1 < m_2$ we have that
   $$m_1 < m_1 + \frac{1}{2} < m_2$$
   for some $m_1 + \frac{1}{2} \notin \mathbb{Z}$. Therefore $\mathbb{Z}$ is not open.

(b) Let $p_n$ for $n \in \mathbb{N}$ denote the sequence of primes in increasing order. Then

$$\mathbb{R} \setminus \left( \bigcup_{n \in \mathbb{N}} \{p_n\} \right) = \bigcap_{n \in \mathbb{N}} (\mathbb{R} \setminus \{p_n\}) = (-\infty, p_1) \cup \left[ \bigcup_{n \in \mathbb{N}} (p_n, p_{n+1}) \right]$$

which is the union of open sets and therefore open.

(c) We have that $\mathbb{I}$ contains no intervals, as between every irrational number we find rational numbers. By an analogous argument to (a) we conclude that it is not open. Now consider $\mathbb{R} \setminus \mathbb{I} = \mathbb{Q}$. But the aforementioned argument applies to $\mathbb{Q}$ so it is not open either. As such, $\mathbb{I}$ is neither open not closed.

5. Suppose that $F$ is a nonempty finite subset of $\mathbb{R}$. Because $F$ is finite is contains no intervals, thus it is not open. Now let us write $F = \{f_1, \ldots, f_n\}$. Then

$$\mathbb{R} \setminus F = (-\infty, f_1) \cup \left[ \bigcup_{i=1}^{n-1} (f_i, f_{i+1}) \right] \cup (f_n, \infty)$$

is the union for open intervals and therefore open. We conclude that $F$ is closed but not open.

6. Let $F \subset \mathbb{R}$ be countable. Then $F$ contains no intervals so it is not open. However, $F$ may or may not be closed. Notice, for example, that

   (a) $\mathbb{Z}$ is countable and closed in $\mathbb{R}$

   (b) Let us have the succession $s_n = \frac{1}{n}$. Then $A = \bigcup_{i=1}^{n} \{s_n\} = (0, 1]$ which is not closed.'

7. Consider $\mathbb{R}$ with the euclidean topology.

   (a) Let $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \right\}$. If $s_n = \frac{1}{n}$ we have that

$$S = \{0\} \cup \bigcup_{i=1}^{n} \{s_n\} = \{0\} \cup (0, 1] = [0, 1]$$

   so it is closed.

   (b) Yes, as seen before.

   (c) Yes. Let $\mathbb{Z}\left(\sqrt{2}\right) = \left\{n\sqrt{2} : n \in \mathbb{Z}\right\}$. Then

$$\mathbb{R} \setminus \mathbb{Z}\left(\sqrt{2}\right) = \bigcup_{n \in \mathbb{Z}} \left(n\sqrt{2}, (n+1)\sqrt{2}\right)$$

   which is the union of open intervals and therefore open. Hence $\mathbb{Z}\left(\sqrt{2}\right)$ is closed.

8. Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$. $S$ is said to be an $F_\sigma$-set if it is the coutable union of closed sets. $S$ is said to be $G_\delta$-set if it is the countable intersection of open sets.

   (a) Notice that

      i. $(a, b) = \bigcup_{i=1}^{n} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$

      ii. $[a, b] = [a, b] \cup [a, b]$

   (b) Notice that

i. $(a,b) = (a,b) \cap (a,b)$

ii. $[a,b] = \bigcap_{i=1}^{n} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$

(c) We have that $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ so $\mathbb{Q}$ is the countable union of closed sets.

(d) Let $A_i$ for $i \in \mathbb{N}$ be a countable famility of open sets and $B_i$ for $i \in \mathbb{N}$ be a countable family of closed sets. Then

i. $X \setminus \left(\bigcup_{i \in \mathbb{N}} B_i\right) = \bigcap_{i \in \mathbb{N}} (X \setminus B_i)$ which is the countable intersection of open sets.

ii. $X \setminus \left(\bigcap_{i \in \mathbb{N}} A_i\right) = \bigcup_{i \in \mathbb{N}} (X \setminus A_i)$ which is the countable intersection of closed sets, because if $X \setminus (X \setminus A_i) = A_i$ is open for all $i \in \mathbb{N}$.

## 2.2 Basis for a Topology

1. Consider the disc $D = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$.

(a) Let $R_{(a,b)} = \left(a \pm \frac{1-r}{8}, b \pm \frac{1-r}{8}\right)$. Notice that

$$\left|x^2 + y^2\right| \leq |x|^2 + |y|^2 \leq |a|^2 + |b|^2 + ab\left(\frac{1-r}{8}\right) + \frac{(1-r)^2}{32}.$$

Because $ab$ is maximum when $a = b = r/\sqrt{2}$ we have

$$\left|x^2 + y^2\right| \leq r^2 + r^2\left(\frac{1-r}{16}\right) + \frac{1 - 2r + r^2}{32} = \frac{1}{32}\left[-2r^3 + 3r^2 - 2r + 1\right].$$

We seek to understand if $\left|x^2 + y^2\right|$ is ever greater than 1. Notice that if for $r < 1$, we have that $-2r^3 + 3r^2 < r$, so the above inequality becomes

$$\left|x^2 + y^2\right| \leq \frac{1}{32}[1 - r] \leq 1$$

as we wanted to show.

(b) We verified that for any point $(a,b) \in D$ we can always find a rectangle $R_{(a,b)} \subseteq D$ containing the point $(a,b)$. As such, it follows that

$$D = \bigcup_{(a,b) \in D} R_{(a,b)}.$$

(c) As $R_{(a,b)}$ is open in the euclidean topology on $\mathbb{R}^2$, we have that $D$ is the union of open sets and therefore open.

(d) By the same argument, we have that any disc centered in an arbitrary point is the union of open rectangles, and therefore an open set in the euclidean topology of $\mathbb{R}^2$.

2. Consider the euclidean topology in $\mathbb{R}^2$.

(a) Let $D_1$ and $D_2$ be disks in $\mathbb{R}^2$ with radius $a_1$ and $a_2$ respectively, centered at the points $(c_1, d_1)$ and $(c_2, d_2)$. Let $(a, b) \in D_1 \cap D_2$. Then we have that $(a, b)$ is contained in both $D_1$ and $D_2$. Let $r_1$ and $r_2$ be the distance to the center of each disk, that is

$$\begin{cases} r_1 = & \sqrt{(a - c_1)^2 + (b - d_1)^2} \\ r_2 = & \sqrt{(a - c_2)^2 + (b - d_2)^2} \end{cases}.$$

Taking $k = \min(a_1 - r_1, a_2 - r_2)$ we have that the disk centered at $(a, b)$ of radius $k$ is contained in $D_1 \cap D_2$.

(b) The union of the afore mentioned disks on each point $(a, b) \in D_1 \cap D_2$ give the whole intersection $D_1 \cap D_2$.

(c) We have that the intersection of disks is the union os disks and that the union of disks generates the whole space $\mathbb{R}^2$, so disks are a basis for the euclidean topology on $\mathbb{R}^2$.

3. Consider the intervals $(a, b)$ such that $a < b \in \mathbb{Q}$. We start by proving that these form a basis for some topology on $\mathbb{R}^2$. Notice that $\bigcup_{n \in \mathbb{Z}} (n - 1, n + 1) = \mathbb{R}$. Now let $(a_1, b_1)$ and $(a_2, b_2)$ be intervals with rational limits. The intersection

$$(a_1, b_1) \cap (a_2, b_2) = \begin{cases} \emptyset & \text{if disjoint} \\ (\min(a_1, a_2), \max(b_1, b_2)) & \text{otherwise} \end{cases}$$

has rational limits, so it is the basis of some topology. Now take an arbitrary open $A \subseteq \mathbb{R}$. Let $a \in A$, then there exists $(c, d) \in A$ such that $c < a < d$. But because because between two irrationals we may always find a rational number, even if $c, d$ are irrational, we find $e, f \in \mathbb{Q}$ such that

$$c < e < a < f < d$$

and as such we that an interval $a \in (e, f) \in \mathcal{B} \subseteq A$ so $\mathcal{B}$ is the basis of the euclidean topology on $\mathbb{R}$.

4. A topological space satisfies the second axiom of countability if it is generated by a countable basis.

(a) $\mathbb{R}$ with the euclidean topology admits a basis

$$\mathcal{B} = \{(a, b) \in \mathbb{R} : a < b \in \mathbb{Q}\}$$

so it second countable.

(b) Let $\mathcal{B}$ be a basis of an uncountable topological space $(X, \mathcal{T})$. For $\mathcal{B}$ to be a basis we require every intersection of elements of $\mathcal{B}$ be the union of elements of $\mathcal{B}$. The proof revolves around noticing all the singular sets must belong to $\mathcal{B}$. If that were not the case, then set sets $\{x_i\} \in \mathcal{T}$ would not be contained in any subset of $\mathcal{B}$. As such, because $X$ is uncountable, we have that $\mathcal{B}$ is uncountable, thus $X$ is not second countable.

(c) The proof proceeds analogously to the ones performed before, but now with n-cubes of rational lengths.

(d) Yes. A finite subset of a countable set is necessarily countable. Hence, for any finite subset $A \subseteq X$ we have that $X \setminus A$ is open and the union of all these is $X$. These constitute all the sets in the topology, hence $(X, \mathcal{T})$ is second countable.

5. We consider the statements.

(a) Let $m, c \in \mathbb{R}$. We define the line $L = \{(x,y) \in \mathbb{R}^2 : y = mx + c\}$. Then $\mathbb{R} \backslash L = \underline{\{(x,y) \in \mathbb{R}^2 : y \neq mx + c\}}$. Then either $y < mx + c$ or $ymx + c$. Let us define these two subsets

$$\begin{cases} Q_1 = & \{(x,y) \in \mathbb{R}^2 : y < mx + c\} \\ Q_2 = & \{(x,y) \in \mathbb{R}^2 : y > mx + c\} \end{cases}.$$

We will prove that these are disjoint open sets whose union is $\mathbb{R} \setminus L$. That $Q_1$ and $Q_2$ are disjoint is obvious. Now assume $(u,v) \in Q_1$. Then the distance to the line is given by

$$d = \frac{|mu + c - v|}{\sqrt{1 + m^2}}.$$

As such we may always find an open disk $D_d$ of radius $d$ centered at $(u,v)$ such that $(u,v) \in D \subseteq Q_1$. Then $Q_1 = \bigcup_{(u,v) \in \mathbb{R}^2} D_d(u,v)$ is the union of open sets, therefore open. The same argument applies to $Q_2$. We conclude that $Q_1 \cup Q_2$ is the union of open sets and therefore open, and $\mathbb{R} \setminus L$ is closed in $\mathbb{R}^2$.

(b) The argument is analogous to (a), but now considering a distance $d = 1 - \sqrt{u^2 + v^2}$.

(c) The argument is analogous to (a), but now considering a distance $d = 1 - \sqrt{\sum_{i=1}^n x_i^2}$.

(d) The argument is similar, but now we consider only the outer region of the ball.

(e) The argument is analogous to (a).\

6. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces admiting bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. We definite the product topology by the topology generated by the subsets $B_1 \times B_2 \in \mathcal{B}_1 \times \mathcal{B}_2$. We call this basis $\mathcal{B}$ of the topology $\mathcal{T}$ on $X \times Y$. We check that

   (a) $\mathcal{B}$ trivial generates $\emptyset$, and $\mathcal{B}_1$ and $\mathcal{B}_2$ generate $X$ and $Y$, hence $\mathcal{B}$ generates $X \times Y$.

   (b) Let $C, D \in B$. Then

   $$C \cap D = (X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \times Y_2) \in \mathcal{B}$$

   for $X_1 \cap X_2$ and $Y_1 \times Y_2$ are open in $X \times Y$.

7. Consider $\mathbb{R}$ with the euclidean topology. Let $A$ be an open subset of $\mathbb{R}$. We showed that $\mathbb{R}$ is second countable, and therefore admits a countable basis $\mathcal{B}$. Any open subset is then the union of elements of $\mathcal{B}$. We also showed that this union is coutable, as any open or closed interval of $\mathbb{R}$ is the countable union of closed and open intervals, respectively. Therefore any open set is the coutable union of open intervals which are themselves the coutable union of closed intervals, thus $A$ is $F_\sigma$. On the other hand, any closed subset $B$ is the countable intersection of open intervals, which themselves are a countable union of open intervals. As such, $B$ is $G_\delta$.

## 2.3 Basis for a Given Topology

1. All are equivalent basis for the euclidean topology on $\mathbb{R}^2$.

2. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on a nonempty set $X$.

(a) Let $\mathcal{B}_1$ a collection of subsets of $X$ such that $\mathcal{B} \subseteq \mathcal{B}_1 \subseteq \mathcal{T}$. We start by showing that $\mathcal{B}_1$ is the basis for a given topology on $X$. It is clear that because it contains a basis of $\mathcal{T}$, $\mathcal{B}_1$ generates $X$. Let $B_1, B_2 \in \mathcal{B}_1$. Then we have that $B_1 = \bigcup_{i\in I} B_i$ and $B_2 = \bigcup_{j\in J} B_j$ for families of subsets of $\mathcal{B}$ and indices $i \in I$ and $j \in J$. Thus we have

$$B_1 \cap B_2 = \left(\bigcup_{i\in I} B_i\right) \cap \left(\bigcup_{j\in J} B_j\right) = \bigcup_{i\in I}\bigcup_{j\in J}(B_i \cap B_j)$$

for $B_i \cap B_j \in \mathcal{B} \subseteq \mathcal{B}_1$. Thus every intersection of subsets of $\mathcal{B}_1$ is the union of subsets of $\mathcal{B}_1$, so it is the basis for some topology on $X$. Now it remains to show to show that $\mathcal{B}_1$ and $\mathcal{B}$ generate the same topology. Let $B \in \mathcal{B}$ and $x \in B$. Then $x \in B \subseteq B \in \mathcal{B}_1$ because $\mathcal{B} \subseteq \mathcal{B}_1$. Now consider $B \in \mathcal{B}_1$ and $x \in B$. We have that $B = \bigcup_{i\in I} B_i$ for some $B_i \in \mathcal{B}$. Then $x \in B_k \in \mathcal{B}$ for some $k \in I$ and hence $x \in B_k \subseteq \mathcal{B}$. Thus both basis generate the same topology and are equivalent.

(b) We take as the basis of the euclidean topology $\mathcal{B}$ the set os intervals with rational limits. To this basis one might add any interval with irrational limits, forming $\mathcal{B}_1$ such that $\mathcal{B} \subseteq \mathcal{B}_1$. Then by (a) this is still a basis for the euclidean topology on $\mathbb{R}$. As such, we have as many basis as intervals with irrational limits to add to $\mathcal{B}$ and, as such, uncountably many.

3. Let $\mathcal{B} = \{(a, b] : a < b \in \mathbb{R}\}$. Consider the union

$$\bigcup_{i=1}^{n}\left(a - \frac{1}{n}, a\right] = \{a\} \in \mathcal{T}.$$

This implies that every singular subset of $\mathbb{R}$ belongs to the topology generated by $\mathcal{B}$. As such, $\mathcal{B}$ generates the discrete topology on $\mathbb{R}$. As subset of $\mathbb{R}$ is open in the discrete topology, so is every open interval.

4. Let $\mathrm{C}\,[0, 1]$ be the set of all continuous real-valued function on $[0, 1]$.

(a) Let $\mathcal{M} = \{M(f,\epsilon) : f \in \mathcal{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ and $M(f,\epsilon) = \left\{g : g \in \mathrm{C}[0,1], \int_0^1 |f-g| < \epsilon\right\}$. We have that

$$\mathcal{C}[0,1] = \bigcup_{f \in \mathrm{C}[0,1]} M(f,1).$$

Now consider $f, h \in \mathcal{C}[0,1]$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Then for $k \in M(f,\epsilon_1) \cap M(h,\epsilon_2)$, we write define

$$\begin{cases} d_1 = & \int_0^1 |f-k| \\ d_2 = & \int_0^1 |h-k| \end{cases}.$$

Setting $\epsilon = \min(\epsilon_1 - d_1, \epsilon_2 - d_2)$ we get that for all for any $g \in M(k,\epsilon)$

$$\int_0^1 |k-g| < \epsilon_1 - \int_0^1 |f-k| \Longrightarrow \int_0^1 |f-g| \leq \int_0^1 |f-k| + |k-g| < \epsilon_1$$

$$\int_0^1 |k-g| < \epsilon_2 - \int_0^1 |h-k| \Longrightarrow \int_0^1 |h-g| \leq \int_0^1 |h-k| + |k-g| < \epsilon_2$$

thus $M(k,\epsilon) \subseteq M(f,\epsilon_1)$ and $M(k,\epsilon) \subseteq M(h,\epsilon_2)$. Now the union

$$\bigcup_{k \in M(f,\epsilon_1) \cap M(h,\epsilon_2)} M(k,\epsilon) = M(f,\epsilon_1) \cap M(h,\epsilon_2)$$

and we conclude that $\mathcal{M} = \{M(f,\epsilon) : f \in \mathrm{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ is the basis for some topology on $\mathcal{C}[0,1]$.

(b) Let $\mathcal{U} = \{U(f,\epsilon) : f \in \mathcal{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ and $U(f,\epsilon) = \left\{g : g \in \mathcal{C}[0,1], \sup_{x \in [0,1]} |f-g| < \epsilon\right\}$. We have that

$$\mathcal{C}[0,1] = \bigcup_{f \in \mathcal{C}[0,1]} U(f,1).$$

Now consider $f, h \in \mathcal{C}[0,1]$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Then for $k \in U(f,\epsilon_1) \cap U(h,\epsilon_2)$, we write define

$$\begin{cases} d_1 = & \sup_{x \in [0,1]} |f-k| \\ d_2 = & \sup_{x \in [0,1]} |h-k| \end{cases}.$$

Setting $\epsilon = \min(\epsilon_1 - d_1, \epsilon_2 - d_2)$ we get that for all for any $g \in U(k,\epsilon)$

$$\sup_{x \in [0,1]} |k-g| < \epsilon_1 - \sup_{x \in [0,1]} |f-k| \Longrightarrow \sup_{x \in [0,1]} |f-g| \leq \sup_{x \in [0,1]} |f-k| + \sup_{x \in [0,1]} |k-g| < \epsilon_1$$

$$\sup_{x \in [0,1]} |k-g| < \epsilon_2 - \sup_{x \in [0,1]} |h-k| \Longrightarrow \sup_{x \in [0,1]} |h-g| \leq \sup_{x \in [0,1]} |h-k| + \sup_{x \in [0,1]} |k-g| < \epsilon_2$$

thus $U(k,\epsilon) \subseteq U(f,\epsilon_1)$ and $U(k,\epsilon) \subseteq U(h,\epsilon_2)$. Now the union

$$\bigcup_{k \in U(f,\epsilon_1) \cap U(h,\epsilon_2)} U(k,\epsilon) = U(f,\epsilon_1) \cap U(h,\epsilon_2)$$

and we conclude that $\mathcal{U} = \{U(f,\epsilon) : f \in \mathcal{C}[0,1], \epsilon > 0 \in \mathbb{R}\}$ is the basis for some topology on $\mathcal{C}[0,1]$.

(c) For all $f, g \in \mathcal{C}[0,1]$ we have that

$$\int_0^1 |f-g| \leq \sup_{x\in[0,1]} |f-g|.$$

Consider $k : [0,1] \to \mathbb{R}$ such that $k(x) = f(x) + x$. Then we have that

$$\int_0^1 |f-k| = \int_0^1 x\mathrm{d}x = \frac{1}{2} < \epsilon_1$$

so we have that $k \in M(f, \epsilon_1)$ for any $\epsilon_1 > \frac{1}{2}$. Now consider

$$\sup_{x\in[0,1]} |f-k| = \sup_{x\in[0,1]} |x| = 1 < \epsilon_2$$

so $k \in U(f, \epsilon_2)$ only if $\epsilon_2 > 1$. Thus by taking $\epsilon_1 = \frac{2}{3}$, we have that there exists a function $k = f(x) + x$ such that $k \in M(f, \epsilon_1)$ but $k \notin U(f, \epsilon_2)$ for any $\epsilon_2$ such that $U(f, \epsilon_2) \subseteq M(f, \epsilon_1)$. We conclude that $\mathcal{M}$ and $\mathcal{U}$ generate different topologies on $\mathcal{C}[0,1]$.

5. Let $(X, \mathcal{T})$ be a topological space.
   (a) Notice that, for $a < b \in \mathbb{R}$

   $$(-\infty, b) \cap (a, \infty) = (a, b)$$

   which generate the euclidean topology on $\mathbb{R}$. As such, we say that $\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ are a subbasis of $\mathbb{R}$.

   (b) Consider $X = \{a, b, c, d, e, f\}$, $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$ and $\mathcal{S} = \{\{a\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. $\mathcal{S}$ is a subbasis of $\mathcal{T}$ because
      i. $X = \{a, c, d\} \cup \{b, c, d, e, f\}$
      ii. $\{a\} \in \mathcal{S}$
      iii. $\{c, d\} = \{a, c, d\} \cap \{b, c, d, e, f\}$
      iv. $\{a, c, d\} \in \mathcal{S}$
      v. $\{b, c, d, e, f\} \in \mathcal{S}$.

6. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on $\mathbb{R}$ such that closed intervals of the form $[a, b]$ for $a < b$ are contained in $\mathcal{S}$. Then, for any $a \in \mathbb{R}$ we have that for $c < a < b$

   $$\{a\} = [a, b] \cap [c, a]$$

   so every singular set is contained in $\mathcal{T}$, thus it is the discrete topology on $\mathbb{R}$.

7. Every subset in the cofinite topology is of the form $X \setminus A$ for some finite subset $A = \{x_1, \ldots, x_n\}$. Thus

   $$X \setminus A = X \setminus \left(\bigcup_{i=1}^{n} \{x_i\}\right) = \bigcap_{i=1}^{n} (X \setminus \{x_i\})$$

   so every open set $X \setminus A$ is an intersection of sets of the $\mathcal{S}$, so $\mathcal{S}$ is a subbasis of the confinite topology on $X$.

8. Consider the subbasis $\mathcal{S} = \{A \in X : |A| = 2\}$. By intersection of such sets $A$ we get all singleton sets, and therefore generate the discrete topology on $X$.

9. The intersection of straight lines can be either empty, a line, or a single point. As such, we have that all singular points are an intersection of elements of the subbasis, and as such, the topology generated by all straight lines in $\mathbb{R}^2$ is the discrete topology.

10. The open sets in $(X, \mathcal{T})$ are the empty set, straight lines parallel to the x axis, and unions of these lines. By arbitrary unions of these lines, one may create regions of the form

$$S = \left\{(x, y) \in \mathbb{R}^2 : a < y < b\right\}$$

for $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$.

11. The circles in the plane may intersect at only one point. As such, we have the the topology generated by $\mathcal{S}$ is the discrete topology on $\mathbb{R}^2$.

12. Intersections go brrrrr.

# 3 Limit Points

## 3.1 Limit Points and Closure

1. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.
   (a) The limit points are
      i. $\{a\}' = \{f\}$
      ii. $\{b, c\}' = \{b, d, e, f\}$
      iii. $\{a, c, d\}' = \{b, c, d, e, f\}$
      iv. $\{b, c, e, f\}' = \{b, d, e\}$.
   (b) The closures are
      i. $\overline{\{a\}} = \{a, f\}$
      ii. $\overline{\{b, c\}} = \{b, c, d, e\}$
      iii. $\overline{\{a, c, d\}} = X$
      iv. $\overline{\{b, c, e, f\}} = \{b, c, d, e, f\}$.
   (c) Alternatively, we have that the closed sets of $(X, \mathcal{T})$ are $\{X, \emptyset, \{b, c, d, e, f\}, \{a, b, e, f\}, \{a, f\}\}$. We now select the smallest containing such elements to find the closure. Then
      i. $\overline{\{a\}} = \{a, f\}$
      ii. $\overline{\{b, c\}} = \{b, c, d, e, f\}$
      iii. $\overline{\{a, c, d\}} = X$
      iv. $\overline{\{b, c, e, f\}} = \{b, c, d, e, f\}$.

2. Let $(\mathbb{Z}, \mathcal{T})$ be the of integers with the cofinite topology. We have that
   (a) $\overline{\{1, 2, 3, \ldots, 10\}} = \{1, 2, 3, \ldots, 10\}$ as it is the smallest closed set that contains the elements up to 1 to 10. So there are no limit points other than these elements. We need only check if any of these elements are limit points. Let $a \in \{1, \ldots, 10\}$, then the set $\{a\} \cup (11, 12, \ldots)$ is open but intersects $\{1, 2, 3, \ldots, 10\}$ only on $\{a\}$. Hence $\{1, 2, 3, \ldots, 10\}' = \emptyset$.

(b) Let $E = 2\mathbb{Z}$ be the set of even integers. Any open set in $(\mathbb{Z}, \mathcal{T})$ contains infinite even numbers. As such any open set intersects $E$ for some $n \in E$, so $E' = \mathbb{Z}$.

3. Consider $\mathbb{R}$ with the euclidean topology.
Let $x \in (a, b)$. Take any open $A$ set that intersects $(a, b)$ at x. Because $A$ is open, it is the union of open intervals, and as such, $x$ is contained in at least one of them. Without loss of generality, we take this interval to be of the form $(c, d)$. Now consider the set $S = (\max(c, a), \min(d, b))$. We have that $S \subseteq A \cap (c, d)$ and $x \in S$. As such, every open set that that contains $x$ intersects $A$ at other point. Thus every point in $(a, b)$ is a limit point of $(a, b)$.
Now consider the points $a, b$. By an analogous resoning to before, the set $S = [a, \min(d, b))$ is contained in $(c, d)$ and contains $a$, and $S = (\max(c, a), b]$ is contained in $(c, b)$ and contains $b$, both intersecting $A$ at a point different other than $a$ or $b$, respectively. Hence $a$ and $b$ are limit points of $(a, b)$.
The smallest closed set that contains $(a, b)$ is $[a, b]$. As such, we conclude that $(a, b)' = [a, b]$.

4. Consider $\mathbb{R}$ with the euclidean topology.
   (a) The closure of the following sets is
      i. $\overline{\left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}} = \{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$
      ii. $\overline{\mathbb{Z}} = \mathbb{Z}$
      iii. $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$
   (b) Let $S \subseteq \mathbb{R}$ be nonempty and $a \in \mathbb{R}$. Suppose that $a \in \overline{S}$. Then either $a \in S$ or $a \in S'$. If $a \in S$ then by choosing $x_n = a \in S$ we have $|x_n - a| = 0 < \frac{1}{n}$ for all $n \in \mathbb{N}$. If $a \in S'$, then every open set $A$ that intersects $S$ at $a$ also does at a different point. We know $\overline{S}$ is smallest closed set that contains $a$. Then $a \in [c, d]$ for some $c < d \in \mathbb{R}$. If we took $c = d = a$ it then $a \in S$ or $a$ would be an isolated limit point, which is a contradiction in the euclidean topology (for we could find an open $(a - \epsilon, a + \epsilon)$ that would intersect $S$ only at $a$). Let $B = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$. Then for some $k \in \mathbb{N}$, $B \in (c, d)$. Hence for all $x \in S$

$$a - \frac{1}{n} < x < a + \frac{1}{n} \Longrightarrow |x - a| < \frac{1}{n}.$$

   Now suppose that for we have $x_n \in S$ such that $|x_n - a| < \frac{1}{n}$. Then either $a \in S$, so $x_n = a$, or $a \in S'$, because any open subset $K$ contains an open interval of the form $a \in (c, d)$ so if we choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \min(|a - c|, |a - d|)$, the point $a \neq x_n \in S \cap K$ and thus $a \in S'$.

5. Let $(X, \mathcal{T})$ be a topological space and $S \subseteq T \subseteq X$ be nonempty.
   (a) Let $p$ be a limit point of $S$. Then for all $A \in \mathcal{T}$ that intersects $S$ at $p$ there exists some $c \in S$ such that $c \in S \cap A$. But $c \in S \subseteq T \Rightarrow c \in T$ hence every such set $A$ intersects $T$ at $c$ so $p \in T'$.
   (b) From (a) we have that $S' \subseteq T'$. As such $S \cup S' \subseteq T \cup T' \Leftrightarrow \overline{S} \subseteq \overline{T}$.
   (c) $S$ is dense in $X$ if $\overline{S} = X$. Then we have $\overline{S} = X \subseteq \overline{T} \subseteq X$ which implies that $T$ is dense in $X$.
   (d) We have that $\overline{\mathbb{Q}} = \mathbb{R}$. So $\mathbb{Q}$ is dense in $\mathbb{R}$. As such, any union of the type $\mathbb{Q} \cup (x)$ for some $x \in \mathbb{R} \setminus \mathbb{Q}$ is dense. Because there are unbountably many such sets, we conclude that there exist uncountably many dense subsets of $\mathbb{R}$.
   (e) The example from (d) shows that there exist an uncountable number of countable dense subsets of $\mathbb{R}$. Now we also have that $\mathbb{I}$ is dense in $\mathbb{R}$. Now for some subset $Q \in \mathbb{Q}$, we have that $\mathbb{I} \cup Q$ is sense in $\mathbb{R}$. Because $\mathbb{Q}$ is countable, let us denote $M$ the number of possible distinct choices of $Q$. The intended result follows by considering the possibities for this choice.

6. Let $A, B$ be subsets of $\mathbb{R}$ with the euclidean topology. Consider the sets $A \cap \overline{B}$, $\overline{A} \cap B$, $\overline{A} \cap \overline{B}$ and $\overline{A \cap B}$.
   (a) Let $A = \mathbb{Q}$ and $B = \mathbb{I}$. Then
      i. $A \cap \overline{B} = \mathbb{Q} \cap \mathbb{R} = \mathbb{Q}$
      ii. $\overline{A} \cap B = \mathbb{R} \cap \mathbb{I} = \mathbb{I}$
      iii. $\overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$
      iv. $\overline{A \cap B} = \overline{\mathbb{Q} \cap \mathbb{I}} = \overline{\emptyset} = \emptyset$.
   (b) Let $A, B$ be open intervals in $\mathbb{R}$. Without loss of generality, we have $A = (a, c)$ and $B = (b, d)$ for $a < c \in \mathbb{R}$ and $b < d \in \mathbb{R}$.
      i. $A \cap \overline{B} = (a, c) \cap [b, d]$
      ii. $\overline{A} \cap B = [a, c] \cap (b, d)$
      iii. $\overline{A} \cap \overline{B} = [a, c] \cap [b, d]$
      iv. $\overline{A \cap B} = \overline{(a, c) \cap (b, d)}$ which is equal to either (i), (ii) or (iii).
   (c) Let $A = (0, 1) \cup (2, 3)$ and $B = (1, 2) \cup (3, 4)$. Then
      i. $A \cap \overline{B} = (0, 4)$
      ii. $\overline{A} \cap B = [0, 2) \cup (2, 4]$
      iii. $\overline{A} \cap \overline{B} = [0, 4]$
      iv. $\overline{A \cap B} = \emptyset$.

## 3.2 Neighbourhoods

1. Let $A$ be a subset of a topological space $(X, \mathcal{T})$. Suppose $A$ is dense in $X$. Then every point $x \in X \setminus A$ is contained in $A'$, so it is a limit point of $A$. As such, every neighbourhood intersects $A$ non-trivially at some point $c$ other than $a$. Now suppose that every neighbourhood of each point $x \in X \setminus A$ intersects $A$ non-trivially. Because every open set containing $x$ is a neighbourhood of $x$, we have that every open set intersecting $A$ at $x$ also does at another point. As such, we have that $A' = X \setminus A$ . Hence $\overline{A} = A \cup A' = X$ so $\overline{A}$ is dense.

2. Let $A, B$ be subsets of a topological space $(X, \mathcal{T})$.
   (a) We have that $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. It follows that $A \cap B \subseteq \overline{A} \cap \overline{B}$. Because $\overline{A \cap B}$ is the smallest closed subset containing $A \cap B$, and the intersection $\overline{A} \cap \overline{B}$ is closed, it follows that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
   (b) Consider $\mathbb{R}$ with the euclidean topology. Taking $A = (0, 1)$ and $B = (1, 2)$ we have that $\overline{A \cap B} = \emptyset$ but $\overline{A} \cap \overline{B} = [0, 2]$.

3. Let $(X, \mathcal{T})$ be a topological space. Suppose that $(X, \mathcal{T})$ is a $\mathrm{T}_1$ space and every infinite subset of $X$ is dense in $X$. Because $(X, \mathcal{T})$ is $\mathrm{T}_1$ it follows that every singleton subset $\{x\}$ for $x \in X$ is closed. Let $A$ be an infinite subset of $X$. We have that $A$ is dense, so the smallest closed set that contains $A$ is $X$. Because $A$ is arbitrary, we have that the only infinite closed set is $X$. From the fact that the finite union of singleton sets is closed, it follows that every finite subset of $X$ is closed. Thus $\mathcal{T}$ is the cofinite topology on $X$.
   Now suppose that $\mathcal{T}$ is the cofinite topology on $X$. Let $A = X \setminus \{x\}$ for some $x \in X$. With $X$ begin finite every finite set is open. If $X$ is infinite, then $A$ is infinite. In any way $A$ is open. From the arbitrariness of $A$ it follows that every singleton set $\{x\}$ is closed, and as such $(X, \mathcal{T})$ is $\mathrm{T}_1$. Now suppose $A$ is an infinite subset of $X$. If $A$ is not dense, then there exists a closed set $B$ such that $A \subseteq B \subseteq X$. As in the cofinite topology the only infinite open set is $X$, we have that $B = X$ so $A$ is dense in $X$.

4. We consider if the following spaces are separable:

(a) $\overline{\mathbb{Q}} = \mathbb{R}$ so $\mathbb{R}$ is separable.

(b) If $X$ is countable then $\overline{X} = X$, so $X$ is separable.

(c) In the cofinite topology any infinite subset is dense. If $X$ is countable then it is separable.

(d) If $X$ is finite, then $\overline{X} = X$ is a dense countable subset of $X$. So $X$ is countable.

(e) Let $\mathcal{T} = (A_1, \ldots, A_n)$. Then taking $a_i \in A_i$ we get $A = \{a_i \in A_i : i \in I\}$ for some finite family. Then $A$ is countable. For any open set $U \in \mathcal{T}$ we have that $U \cap A = A_i \cap A \neq \emptyset$, so $A$ is dense in $(X, \mathcal{T})$.

(f) Consider $\mathbb{R}$ with the discrete topology. Then only dense set is $\mathbb{R}$. Thus it is not a separable space.

(g) In the cofinite topology every infinite subset is dense in $X$. As such, any countable infinite subset is dense, so $X$ is separable.

(h) If $(X, \mathcal{T})$ is second countable, then there exists a countable basis of $(X, \mathcal{T})$. Let this basis be $\mathcal{B}$. We have that $X = \bigcup_{i \in I} B_i$ for a countable family $i \in I$ of $B_i \in \mathcal{B}$. Let $x_i \in B_i$. Then consider $A = \bigcup_{i \in I} \{x_i\}$. Then $A$ is countable. Then let $U \in \mathcal{T} \setminus \emptyset$. Then because $U$ is the union of elements of the basis, we have that $U \cap A \neq 0$, so $A$ is dense in $X$.

5. We consider the following interiors:

(a) In $\mathbb{R}$ with the euclidean topology, then $\operatorname{Int}([0,1]) = (0,1)$ because for $c < d \in (0,1)$ we have $(c,d) \in [0,1]$ and $\bigcup_{c,d \in (0,1)} (c,d) = (0,1)$.

(b) In $\mathbb{R}$ with the euclidean topology, then $\operatorname{Int}((3,4)) = (3,4)$. Proof is analogous to before.

(c) Let $A$ be open. Then the biggest open set containing $A$ is $A$.

(d) In $\mathbb{R}$ with the euclidean topology, then $\operatorname{Int}(\{3\}) = \emptyset$ for $\{3\}$ contains no open sets.

(e) If $X$ is equiped with the indiscrete topology, then for any $A \subset X$ we have that the smallest open set containing $A$ is $\emptyset$, as the only open sets are $\mathcal{T} = \{X, \emptyset\}$.

(f) We have that if a subset of $\mathbb{R}$ is countable, then it is not an interval (nor the union of intervals). As such, it contains no open sets, and thus $\operatorname{Int}(A) = \emptyset$.

6. Let $(X, \mathcal{T})$ is a topological space. We have that $\overline{X \setminus A}$ is the smallest closed set not containing $A$. As such, we have that $X \setminus \left(\overline{X \setminus A}\right)$ is open. We aim to show that it is the biggest such set contained in $A$. Suppose that there exists an open set $B$ such that $a \in X \setminus \left(\overline{X \setminus A}\right) \subseteq B$. Then $X \setminus B \subseteq \overline{X \setminus A}$ is smaller than $\overline{X \setminus A}$ which is a contradiction. As such, $X \setminus \left(\overline{X \setminus A}\right)$ is the biggest open set contained in $A$ and as such $\operatorname{Int}(A) = X \setminus \left(\overline{X \setminus A}\right)$.

7. Let $(X, \mathcal{T})$ is a topological space. Suppose that $\operatorname{Int}(X \setminus A) = \emptyset$. Then

$$\operatorname{Int}(X \setminus A) = X \setminus \overline{A} = \emptyset \Longrightarrow \overline{A} = X$$

so $A$ is dense in $X$. Now suppose that $\operatorname{Int}(X \setminus A) = B \neq \emptyset$. Then $X \setminus \overline{A} = B \Rightarrow \overline{A} = X \setminus B \neq X$ so $A$ is not dense in $X$.

8. Let $(X, \mathcal{T})$ be a topological space and $A_1, A_2 \subseteq X$. Then

(a) $\operatorname{Int}(A_1 \cap A_2) = X \setminus \left[\overline{X \setminus (A_1 \cap A_2)}\right] = X \setminus \left[\overline{(X \setminus A_1) \cup (X \setminus A_2)}\right] = X \setminus \left[\overline{(X \setminus A_1)} \cup \overline{(X \setminus A_2)}\right] = \left[X \setminus \overline{(X \setminus A_1)}\right] \cap \left[X \setminus \overline{(X \setminus A_2)}\right] = \operatorname{Int}(A_1) \cap \operatorname{Int}(A_2)$ by using (c).

(b) Let $X = \mathbb{R}$ with the euclidean topology. Let $A_1 = [0,1]$ and $A_2 = [1,2]$. Then $\operatorname{Int}(A_1 \cup A_2) = (0,2)$ but $\operatorname{Int}(A_1) \cup \operatorname{Int}(A_2) = (0,1) \cup (1,2)$.

(c) We have that $A \subseteq A \cup B \Rightarrow \overline{A} \subseteq \overline{A \cup B}$ and $B \subseteq A \cup B \Rightarrow \overline{B} \subseteq \overline{A \cup B}$ so $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Now notice that $\overline{A \cup B}$ is smallest subset containing $A \cup B$. Because $\overline{A} \cup \overline{B}$ is closed and contains $A$ and $B$, it follows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

9. Let $S$ be a dense subset of a topological space $(X, \mathcal{T})$ and $U \subseteq \mathcal{T}$. We have that $S \cap U \subseteq U \Rightarrow \overline{S \cap U} \subseteq \overline{U}$. Now let $u \in \overline{U}$ and $A \in \mathcal{T}$ such that $u \in A$. Because $U$ is open, then $U \cap A$ is open. Thus $S \cap (U \cap A) \neq 0 \Leftrightarrow (S \cap U) \cap A \neq 0$, that is, for every open set $A$ containing $u$, we have that that the intersection with $S$ contains a point of $S \cap U$. As such, we conclude that $u \in \overline{S \cap U}$. Thus $\overline{U} = \overline{S \cap U}$.

10. By direct application of (9), we have that $\overline{S \cap T} = \overline{S} = X$, and thus $\overline{S \cap T}$ is dense in $(X, \mathcal{T})$.

11. Let $\mathcal{B} = \{[a,b) : a \in \mathbb{R}, b \in \mathbb{Q}, a < b\}$.

(a) We have that $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n-1, n+1)$. Additionaly let have that for $A_1 = [a_1, b_1)$ and $A_2 = [a_2, b_2)$. Then

$$A_1 \cap A_2 = \begin{cases} \emptyset & ; \text{ if disjoint} \\ [\max(a_1, a_2), \min(b_1, b_2)) & ; \text{ otherwise} \end{cases} \in \mathcal{T}_1.$$

Hence $\mathcal{B}$ is the basis of some topology on $\mathbb{R}$.

(b) Let $\mathcal{C}$ be the basis of the euclidean topology on $\mathbb{R}$ composed by open intervals with rational limits. Then for any $(a,b) \in \mathcal{C}$ there exists $[a,b) \in \mathcal{B}$ such that $(a,b) \subseteq [a,b)$. Because $[a,b) \notin \mathcal{T}$ we conclude that $T \subset \mathcal{T}_1$.

(c) Let $A = [a,b)$ for $a < b \in \mathbb{R}$. We have that $\bigcup_{i=1}^{\infty} [a, \lfloor 10^n b \rfloor 10^{-n}) = [a,b)$ so the it is open. Additionally $\mathbb{R} \setminus [a,b) = (-\infty, a) \cup [b, \infty)$ which may be written as a union of the form

$$\left[ \bigcup_{i=1}^{\infty} \left[-n, \lfloor 10^n a \rfloor 10^{-n}\right) \right] \cup \left[ \bigcup_{i=1}^{\infty} [b, n) \right]$$

so $A$ is clopen in $(\mathbb{R}, \mathcal{T}_1)$.

(d) Notice that any open set of the form $[a,b)$ for $a < b$ contains rational elements. As such, similarly to $\mathbb{R}$, we have $\overline{\mathbb{Q}} = \mathbb{R}$ so the set of all rationals is dense.

(e) Because intervals of the form $[a,b)$ for irrational $a$ can not be generated by the union of intervals with rational limits, it follows that sets of form such form are contained in any basis of the topological space. As such, any basis is uncountable so the Sorgenfrey Line is not second countable.

## 3.3 Connectedness

1. Let $S$ be a set of real numbers and $T = \{x : -x \in S\}$.

(a) Let $b = -a$ be the supremum of $T$. Then we have that for all $t \in T$, $t \leq b$. Thus for any $x \in S$ we have that

$$t \leq b \iff -x \leq -a \iff x \geq a$$

that is, $\sup(S) = a$. Now assume that $b$ is not the supremum of $T$. Then $\exists t \in T : t > b \implies \exists x \in S : -x > -a \Leftrightarrow x < a$, that is, $a$ is not the infimum of $S$.

(b) By using the properties discussed in (a), and the fact that any nonempty subset of $S \subset \mathbb{R}$ admits a reciprocal $T = \{x : -x \in S\}$, it follows that any real set $S$ has an infimum element $s$ such that $\forall x \in S : x \geq s$.

2. We find the greatest element and the least upper bound of the following sets:

   (a) $S = \mathbb{R}$. For any $x \in S$ we have that $s + 1 \in S$ so there exist no greatest element nor least upper bound of $S$.

   (b) $S = \mathbb{Z}$. Analogous to (a).

   (c) $S = [9, 10)$. Consider $s_n = 10 - \frac{1}{n}$. Then we have a monotonous increasing sequence of element os $S$. As such, $S$ admits no greater element but has a least upper bound of $x = 10$.

   (d) $S = \left\{1 - \frac{3}{n^2} : n \in \mathbb{N}\right\}$. Analogous to (c), we have that $S$ admits no greatest element, but has a least upper bound of $x = 1$.

   (e) $S = (-\infty, 3]$. $S$ has a greatest element $x = 3$ that coincides with its least upper bound.

3. Let $(X, \mathcal{T})$ be a topological space. Assum there exist two disjoint nonempty sets $A, B \in \mathcal{T}$ such that $A \cup B = X$. As such we have that $X \setminus A = B$ and $X \setminus B = A$ so both $A$ and $B$ are nontrivial clopen sets. Hence $(X, \mathcal{T})$ is not connected. Now supposed that $(X, \mathcal{T})$ is connected but no such subsets exist. From the connectedness property we know there exists a nonempty proper clopen subset $C \subset X$. Because $C$ is closed, we have that $X \setminus C = D \neq \emptyset$ is open. Because $C$ is open it follows that $X \setminus C = D$ is closed. So there exist two nonempty disjoint subsets $C, D$ that are clopen, which is a contradiction.

4. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. We have that $\{a\} \in \mathcal{T}$ and $X \setminus \{a\} = \{b, c, d, e, f\} \in \mathcal{T}$ so $\{a\}$ is a nontrivial clopen set, thus $(X, \mathcal{T})$ is not connected.

5. Let $(X, \mathcal{T})$ be an infinite topological space with the cofinite topology. Then all the nontrivial closed sets $A$ are finite, and the open sets $X \setminus A$ infinite. Consequently, there are no trivial clopen sets, so $(X, \mathcal{T})$ is connected.

6. Consider $X = \mathbb{Z}$ with the countable-closed topology. Then the set $2\mathbb{Z}$ is countable and therefore closed, and $\mathbb{Z} \setminus 2\mathbb{Z}$ is also countable and therefore open. As such, $2\mathbb{Z}$ is a nontrivial clopen set of $(X, \mathcal{T})$, so the topological space is not connected.

7. All of them are connected. None of the sets of the form $(-r, r)$ or $[-r, r]$ are closed, because their reciprocals are of the form $(-\infty, r) \cup (r, \infty)$ or the closed equivalent, which are not open in these topologies.

# 4 Subspaces

1. Let $X = \{a, b, c, d, e\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$. Let $Y = \{a, c, e\}$. Thus

$$\mathcal{T}_Y = \{Y, \emptyset, \{a\}, \{a, c\}, \{a, e\}\}$$

   and analogously for $Z$.

2. Consider $\mathbb{R}$ with the euclidean topology. The topology induces on $\mathbb{N}$ is the discrete topology. Consider $n \in \mathbb{N}$. Then $A = (n - 1, n + 1) \in \mathcal{T}$ and $A \cap \mathbb{N} = \{n\}$, so every singular set is in $\mathcal{T}_{\mathbb{N}}$.