Chapter 1

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1.1 Euclidean Topology in $\mathbb{R}$

Definition 1. A set $A \subseteq \mathbb{R}$ is open in the Euclidean topology if $\forall a \in A, \exists b, c \in \mathbb{R}$ such that $a \in (b, c) \subseteq A$.

To see that this is a topology on $\mathbb{R}$ notice that:

1. $\mathbb{R}, \emptyset \in \mathcal{T}$
2. Let A, B be open sets and a an element of $A \cap B$. Then there exist $b_1, c_1 \in A$ and $b_2, c_2 \in B$ such that $a \in (b_1, c_1) \subseteq A$ and $a \in (b_2, c_2) \subseteq B$. Thus $a \in (\max(b_1, b_2), \min(c_1, c_2)) \subseteq A \cap B$.
3. Let $A_i \in \mathcal{T}$ for $i \in I$. Then $a \in \bigcup_{i \in I} A_i \Rightarrow a \in A_j$ for some $j \in I$. As such, there exist $b, c \in A_j$ such that $a \in (b, c) \subseteq A_j \subseteq \bigcup_{i \in I} A_i$.

In this topology we have that, for $a < b \in \mathbb{R}$

- (a, b) is an open
- $(-\infty, a)$ and (a, ∞) are open
- $[a, b]$ is closed, because
$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cap (b, \infty) \in \mathcal{T}$$
- $(-\infty, a]$ and $[a, \infty)$ are closed
- Any singular subset $\{a\}$ is closed
- $[a, b]$ is not open.

  Proof. Supposed $[a, b]$ is open and let $c \in [a, b]$. Then there exist $d, e \in [a, b]$ such that $c \in (d, e)$. Taking $c = a$, we analyse the lower bound
  $$a \leq d < c = a \Longrightarrow a < a$$
  which constitutes a contradiction. The same could be done for the upper bound. In this way, any finitely bounded closed interval is not open. □

- $\mathbb{Z}$ is closed but not open.
- $\mathbb{Q}$ is closed but not open.

Theorem 1. *Let $A \subseteq \mathbb{R}$. Then A is open in the Euclidean topology if and only if it is an union of open sets.*

Proof. Assume A is open. Then $\forall a \in A$ there exists $(b_a, c_a) \subseteq A$ such that $a \in (b_a, c_a)$. As such

$$A = \bigcup_{a \in A} (b_a, c_a).$$

Now assume A is the union of open sets. Then A is open too. $\square$

Definition 2. Let $(X, \mathcal{T})$ be a topological space. Then $B \subseteq \mathcal{T}$ is a base of if any open set is an union of members of B.

Example 1. The open sets constitute a base of the Euclidean topology.

Example 2. The singular sets constitute a base of the discrete topology.

Definition 3. A family of subsets $B \in \mathcal{P}(X)$ constitutes a base of a topology in X if and only if X is an union of members of B, and the intersection of members of B is in B.

Example 3. Let $X = \{a, b, c\}$ and $B = \{\{a\}, \{b\}\}$. Then B is not the base of any topology on X because $X \in \mathcal{T}$ but X is not generated by B.