Chapter 1

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1.1 Basis of Topology

Definition 1. Let X be a nonempty set and $B \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is the basis of some topology on X if and only if:

1. $X = \bigcup_{B_i \in \mathcal{B}} B_i$
2. for all $B_i, B_j \in \mathcal{B}$ implies that $B_i \cap B_j \in \mathcal{B}$.

Proof. Let X be a nonempty set. Suppose that $\mathcal{B}$ is a basis of a topology $\mathcal{T}$ on X. Then X is a union of members of B, hence $X = \bigcup_{B \in \mathcal{B}} B$. Additionally let $B_i, B_j \in \mathcal{B} \subseteq \mathcal{T}$ then the intersection is in $\mathcal{T}$ and therefore a union of members of $\mathcal{B}$. Now assume that $\mathcal{B}$ generates some topology $\mathcal{T}$, that $X = \bigcup_{B \in \mathcal{B}} B$ and the union of subsets of $\mathcal{B}$ is in $\mathcal{B}$. Then

1. X, $\emptyset$ are a union of subsets of $\mathcal{B}$, hence in $\mathcal{T}$.
2. Let $A, B \in \mathcal{T}$. We have that for some families of elements of $\mathcal{B}$ that $A = \bigcup_{i \in I} B_i$ and $B = \bigcup_{j \in J} B_j$. Then
$$A \cap B = \left(\bigcup_{i \in I} B_i \right) \cap \left(\bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \bigcup_{j \in J} (B_i \cap B_j) \in \mathcal{T}$$
3. The union of subsets of $\mathcal{T}$ translates to the union of unions of subsets of $\mathcal{B}$, and therefore a union of subsets of $\mathcal{B}$.

$\square$

Definition 2. Let $\mathcal{B}$ be the set of rectangles of the form $(a_1, b_1) \times (a_2, b_2)$, with $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$ for $i \in \{1, 2\}$.

Remark. Notice that the union of all these rectangles is the $\mathbb{R}$ plane and that the intersection of rectangles is either empty or also a rectangle. Therefore $\mathcal{B}$ is the basis for some topology in $\mathbb{R}^2$. This is the **euclidean topology** in $\mathbb{R}^2$.

This procedure is analogous to generate the euclidean topology in $\mathbb{R}^n$.

Definition 3. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \in \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis of $\mathcal{T}$ if and only if

1. $B \subseteq \mathcal{T}$
2. for all $A \in \mathcal{T}$ and $a \in A$, there exists some $B \in \mathcal{B}$ such that $a \in B \subseteq A$.

Proof. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subseteq \mathcal{P}(X)$. Assume that $\mathcal{B}$ is a basis of $\mathcal{T}$. Then obviously $B \subseteq \mathcal{T}$. Every set $A \in \mathcal{T}$ is a union of elements of $\mathcal{B}$. Thus for an element $a \in A$ there exists some B_j in $\mathcal{B}$ such that $a \in B_j \subseteq A$.

Now assume that $B \subseteq \mathcal{T}$ and that for every element $a \in A$, there exists some $B \in \mathcal{B}$ such that $a \in B \subseteq A$. We must show that every element of $\mathcal{T}$ is a union of members of B. Take any subset $A \in \mathcal{T}$. Then for every element $a \in A$ we find a subset $B_a \in \mathcal{B}$ contained in A that contains a. Hence $A = \bigcup_{a \in A} a \subseteq \bigcup_{a \in A} B_a \subseteq A$, therefore $A = \bigcup_{a \in A} B_a$. □

Definition 4. Let X be a nonempty set. Let $\mathcal{B}$ and $\mathcal{B}'$ be basis of X. Then we say $\mathcal{B}$ and $\mathcal{B}'$ are **equivalent basis** if they generate the same topology.

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis of topologies on a nonempty set. These are equivalent if and only if

1. for all $B \in \mathcal{B}$ and element $b \in B$ there exists some $B' \in \mathcal{B}'$ such that $b \in B' \subseteq B$
2. for all $B' \in \mathcal{B}'$ and element $b' \in B'$ there exists some $B \in \mathcal{B}$ such that $b' \in B \subseteq B'$.

Proof. The direct consequence follows from the application of definition 3. Assume now that $\mathcal{B}$ and $\mathcal{B}'$ generate topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Take an element $a \in A \subseteq \mathcal{T}$, then $b \in B_a$ for some $B_a \in \mathcal{B}$. From the the aforementioned conditions if follows that there exists some B'_a such that $a \in B'_a \subseteq B_a$. Then we have that

$$A = \bigcup_{a \in A} a \subseteq \bigcup_{a \in A} B'_a \subseteq \bigcup_{a \in A} B_a = A$$

which implies that any subset $A \in \mathcal{T}$ is a union of subsets $B' \in \mathrm{B}'$, that is, $\mathcal{T} \subseteq \mathcal{T}'$. Applying the same argument switching $\mathcal{T}$ and $\mathcal{T}'$ we conclude the proof. □

Remark. It is important to remember that to show that some family of subsets $\mathcal{B}$ is a basis a given topological space, then one must check that $\mathcal{B}$ is a basis of a topology, and then that that topology is the one topology one is analyzing.

Remark. In general, topologies are not countable. For example, the euclidean topology on $\mathbb{R}$ is not countable. But a non countable topology may admit a countable basis. If this is the case, we say that the topological space obeys the **second axioms of countability**.

Example 1. $\mathcal{B} = \{(p, q) : p, q \in \mathbb{Q}\}$ is a countable basis of the euclidean topology on $\mathbb{R}$.

1.2 Subbasis of Topology

Theorem 1. *Let X be a nonempty set. For all $\lambda \in \Lambda$, let $\mathcal{T}_\lambda$ be a topology on X. Then $\bigcap_{\lambda \in \Lambda} \mathcal{T}_\lambda$ is a topology on X.*

Proof. Take X as nonempty set and $\mathcal{T}_\lambda$ a topology on X for every $\lambda \in \Lambda$.

1. $X, \emptyset \in \mathcal{T}_\lambda$ for all $\lambda \in \Lambda$, so they are contained in the intersection.
2. Let $A, B \in \bigcap_{\lambda \in \Lambda} \mathcal{T}_\lambda$. Therefore $A, B \in \mathcal{T}_\lambda$ for all $\lambda \in \Lambda$. As such, the intersection $A \cap B \in \mathcal{T}_\lambda$ for each $\mathcal{T}_\lambda$, hence $A \cap B \in \bigcap_{\lambda \in \Lambda} \mathcal{T}_\lambda$.
3. Take a family open sets $A_i \in \bigcap_{\lambda \in \Lambda} \mathcal{T}_\lambda$ for $i \in I$. Then, each A_i is contained in each $\mathcal{T}_\lambda$ and by the axioms of a topology, the arbitrary union $\bigcup_{i \in I} A_i$ is too. Thus $\bigcup_{i \in I} A_i \in \bigcap_{\lambda \in \Lambda} \mathcal{T}_\lambda$.

□

Definition 5. Let X be a nonempty set and $\mathcal{S} \in \mathcal{P}(X)$. We define $\mathcal{T}_{\mathcal{S}}$ as the intersection of all topologies on X that contain $\mathcal{S}$. This is the smallest topology on X that contains $\mathcal{S}$.

Remark. The discrete topology is always one of these topologies, therefore $\mathcal{T}_\mathcal{S}$ is well defined.

Definition 6. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X such that $\mathcal{T}' \subseteq \mathcal{T}$. Then we say that $\mathcal{T}$ is a **finer** topology on X than $\mathcal{T}'$ or, inversely, that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$.

Additionally, one may say that $\mathcal{T}_\mathcal{S}$ is the topology generated by $\mathcal{S}$ on X.

Definition 7. Let X be a nonempty set and $\mathcal{S} \subseteq \mathcal{P}(X)$. Then

$$\mathcal{B} = \{X\} \cup \left\{ \bigcap_{j=1}^{n} A_j : n \geq 1, A_j \in \mathcal{S} \right\}$$

is a basis for $\mathcal{T}_\mathcal{S}$.

Proof. Obviously, every $A_i \in \mathcal{S} \subseteq \mathcal{T}_\mathcal{S}$, hence $\bigcap_{j=1}^{n} A_j \in \mathcal{T}_\mathcal{S}$. Additionally, $X \in \mathcal{T}_\mathcal{S}$, hence $\mathcal{B} \subseteq \mathcal{T}_\mathcal{S}$. This implies that $\mathcal{B}$ is a basis for some topology on X.

We shall now prove that the topology $\mathcal{T}$ generated by $\mathcal{B}$ is equal to $\mathcal{T}_\mathcal{S}$. By considering one element intersections, we have that $\mathcal{S} \subseteq \mathcal{T}$. Hence the definition of $\mathcal{T}_\mathcal{S}$, it follows that $\mathcal{T}_\mathcal{S} \subseteq \mathcal{T}$. On the other hand, we have that $B \subseteq \mathcal{T}_\mathcal{S}$, so $\mathcal{T} \subseteq \mathcal{T}_\mathcal{S}$. Thus $\mathcal{T} = \mathcal{T}_\mathcal{S}$. $\square$

Definition 8. Let $(X, \mathcal{T})$ be a topological space. We say $\mathcal{S} \in \mathcal{P}(X)$ is a **subbasis** of $\mathcal{T}$ if $\mathcal{T} = \mathcal{T}_\mathcal{S}$.

Example 2. We have that $\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ is a basis for the euclidean topology on $\mathbb{R}$.

Proof. Notice that the intersection of elements $(-\infty, a) \cap (b, \infty) = (a, b)$ which generate the euclidean topology on $\mathbb{R}$. $\square$