Projection Geometry

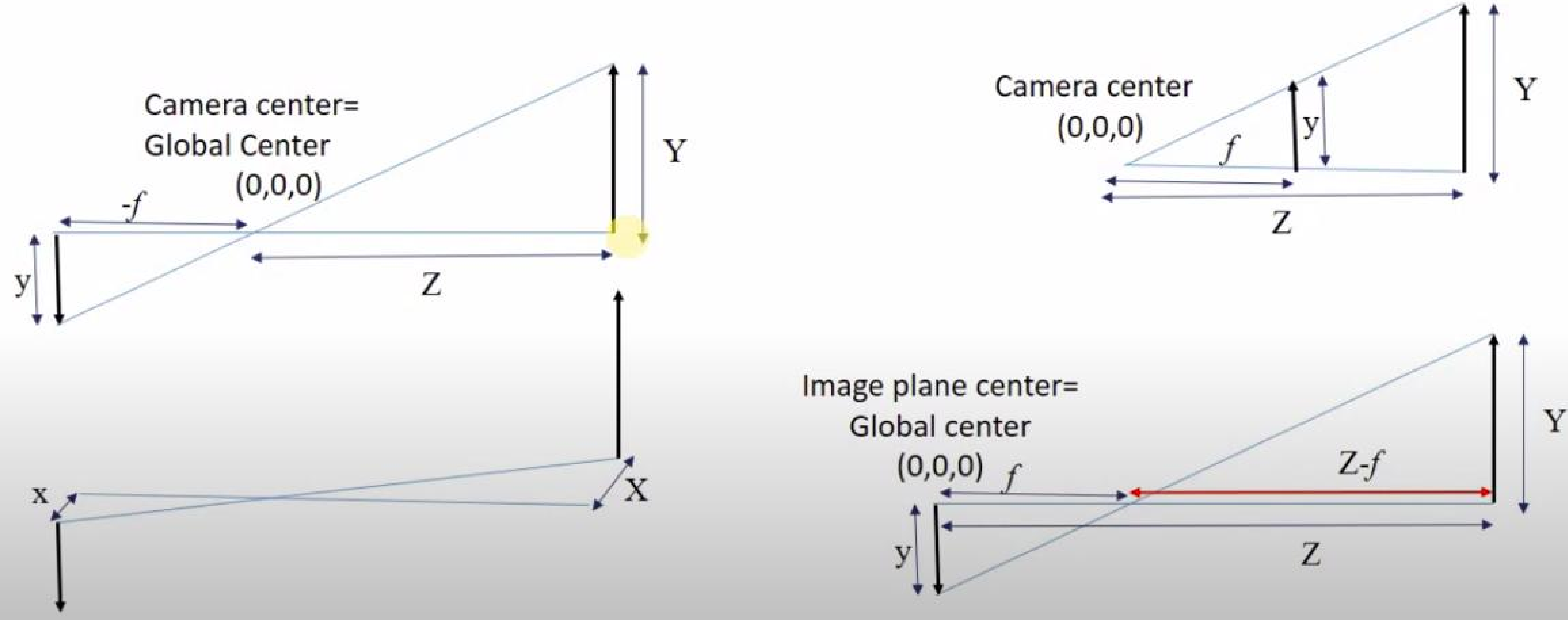
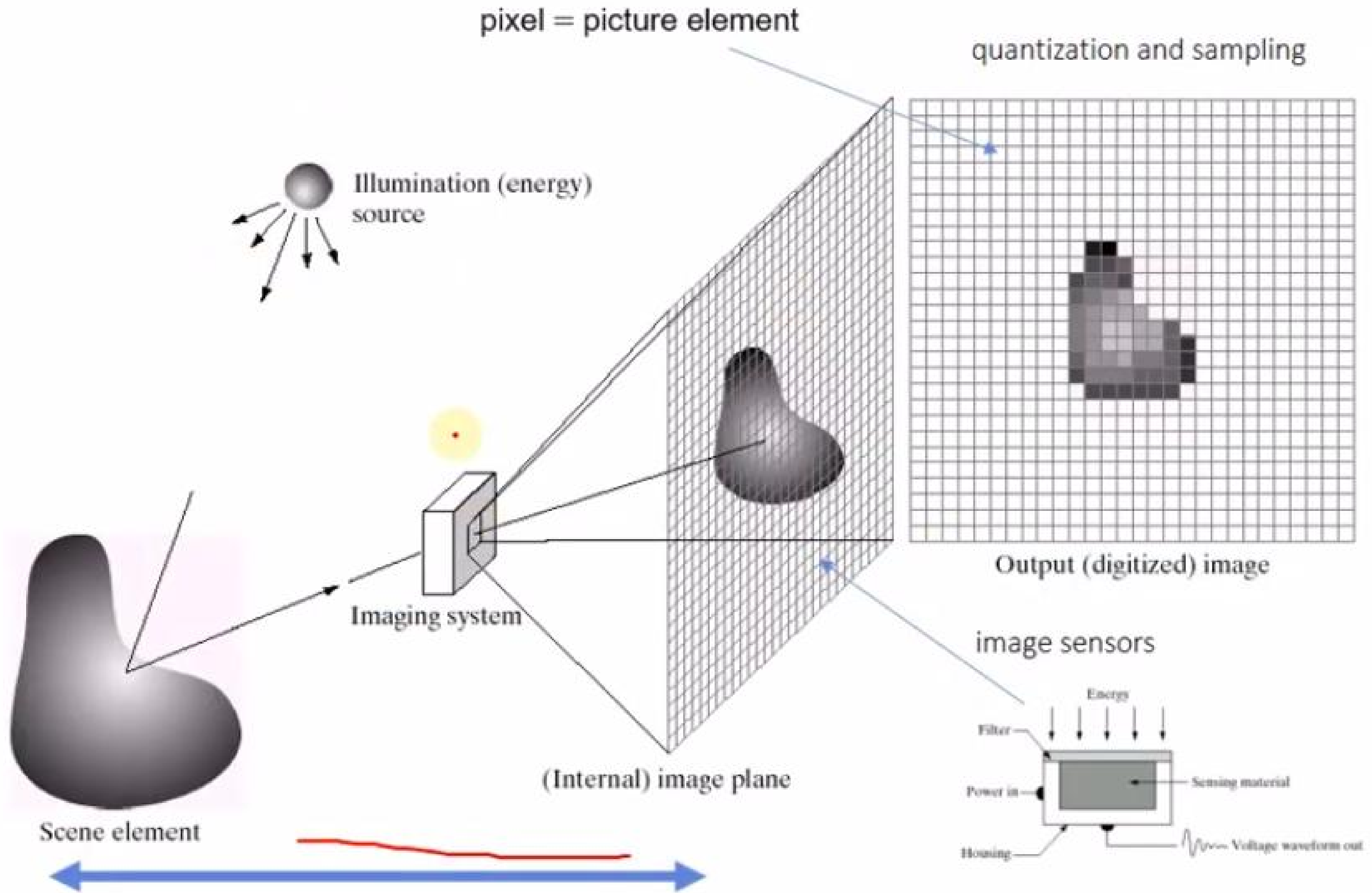
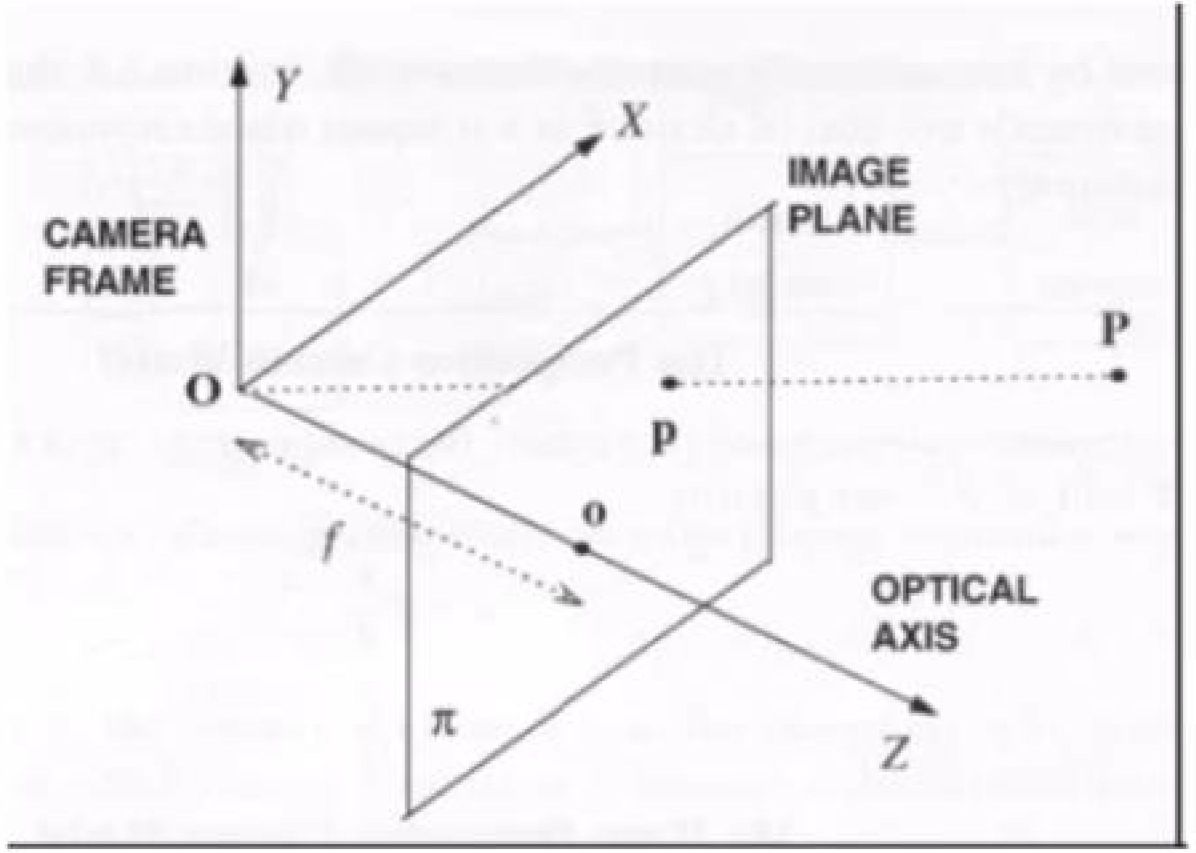


Image acquisition: Geometry



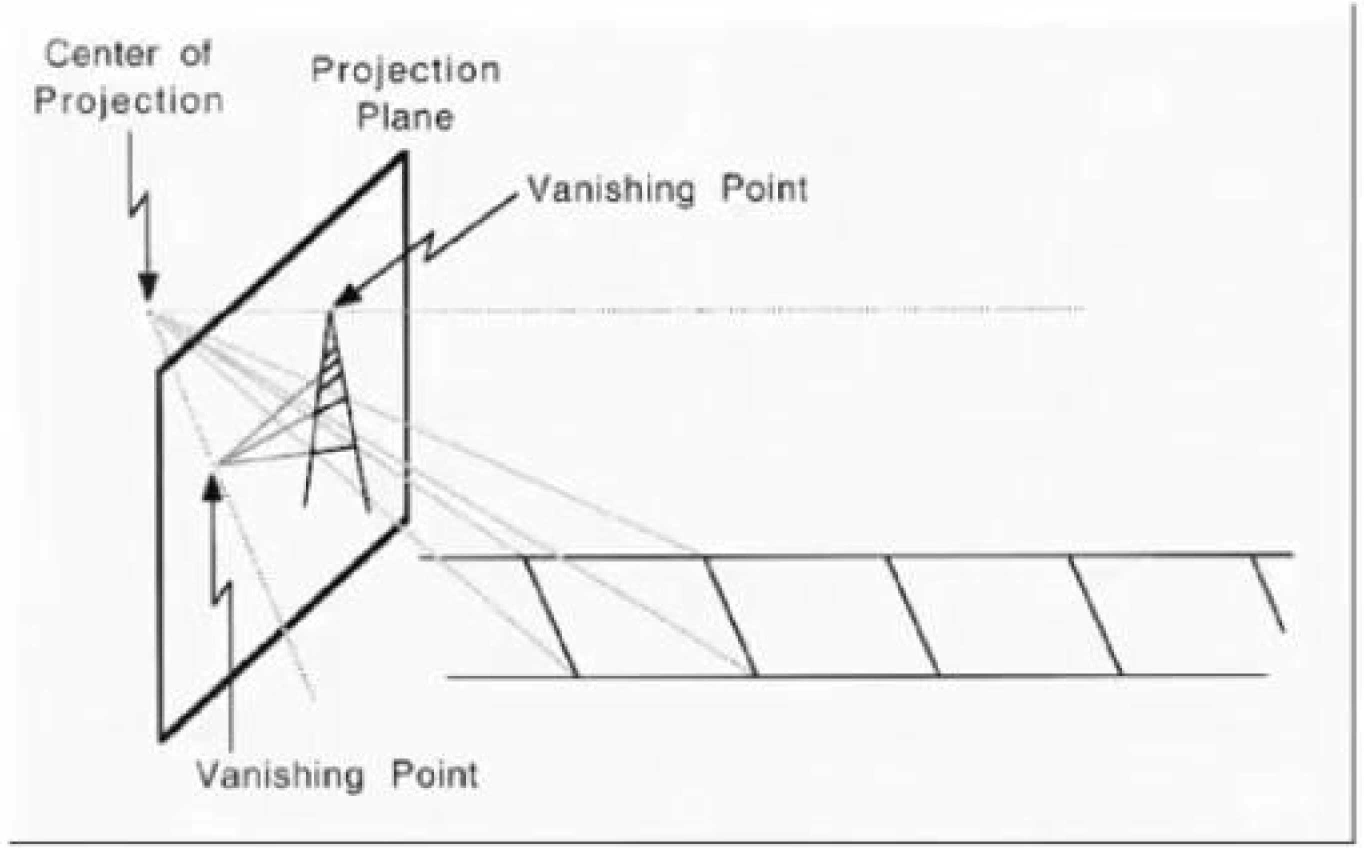
Properties of perspective projection

- Many-to-one mapping
  - The projection of a point is *not* unique
  - Any point on the line *OP* has the same projection



Properties of perspective projection (cont'd)

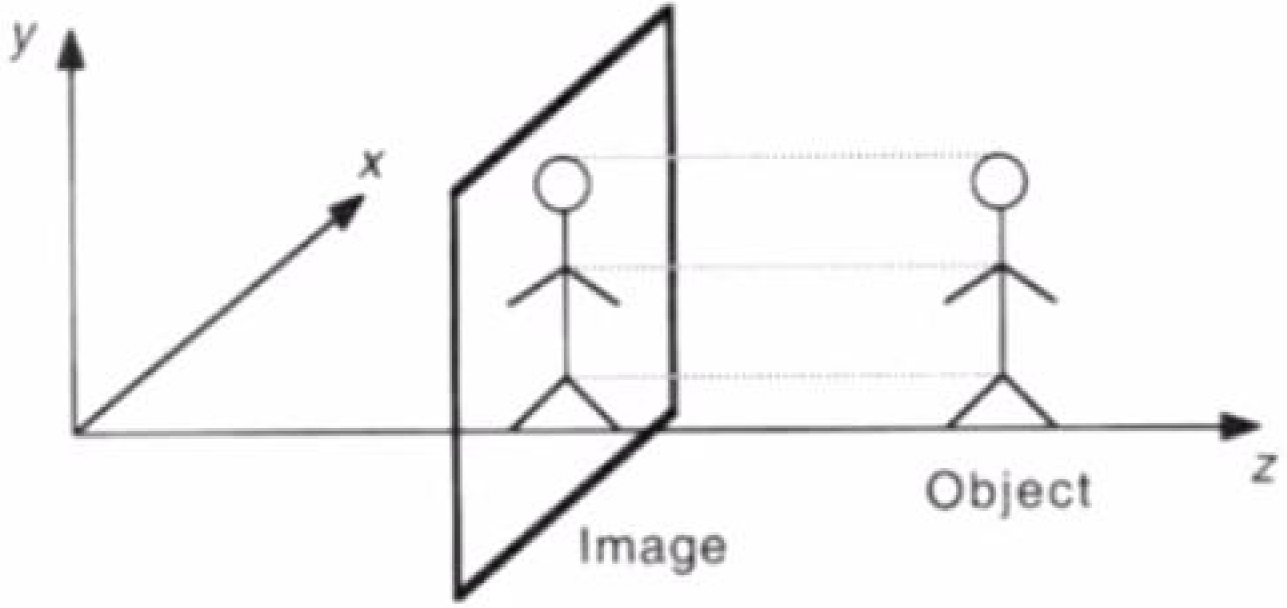
- Vanishing point
  - Parallel lines in space project perspectively onto lines that on extension intersect at a single point in the image plane called *vanishing point* or *point at infinity*.



Warning: vanishing points might lie outside of the image plane!

Orthographic Projection

- The projection of a 3D object onto a plane by a set of parallel rays orthogonal to the image plane.
- It is the limit of perspective projection as $f -> \infty$ (i.e., $f/Z -> 1$)



$$x = \frac{Xf}{Z} \qquad y = \frac{Yf}{Z} \qquad z = f$$

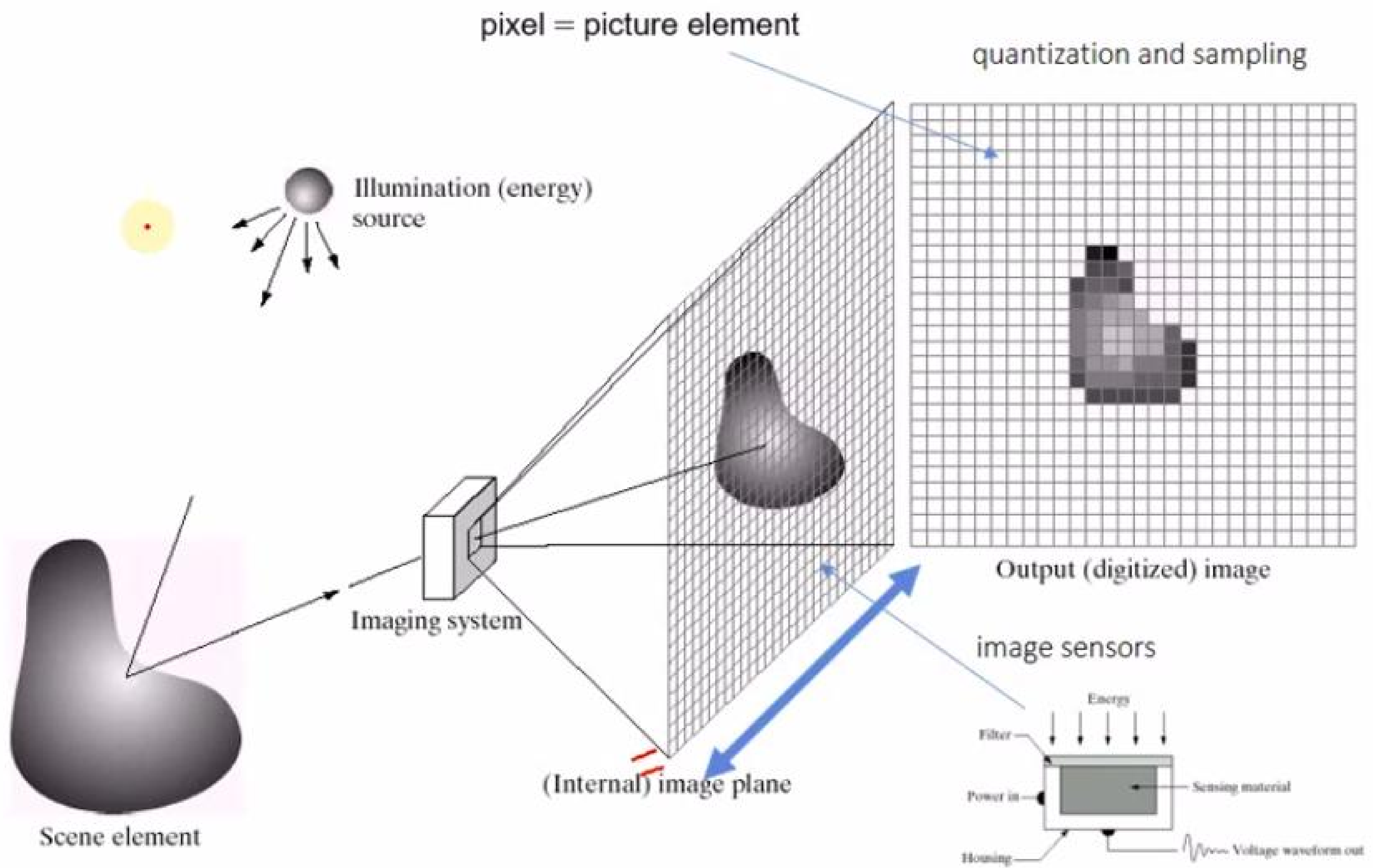
orthographic proj. eqs: $x = X, \quad y = Y \quad \text{(drop } Z\text{)}$

Extensions

- RDCFace: Radial Distortion Correction for Face Recognition, Zhao et al., CVPR 2020

Extensions

Image acquisition: Sampling



Dirac Delta:

$$1)\ \delta(x) = 0 \ \text{ for } \ x \neq 0$$

$$2)\ \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

for some "sufficiently smooth" function f:

$$\int_{-\infty}^{\infty} \delta(x-t) f(x)\, dx = \int_{t-\epsilon}^{t+\epsilon} \delta(x-t) f(x)\, dx$$

$$= f(t) \int_{t-\epsilon}^{t+\epsilon} \delta(x-t)\, dx = f(t)$$

$f(x)$ is almost constant in this interval

Properties of Fourier Transform:

$$1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega)$$

$$e^{i\omega_0 t} \xrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0)$$

Fourier transform of periodic function

Now consider a periodic function $x(t)$ with period T. Since x is periodic we can write it as a Fourier series

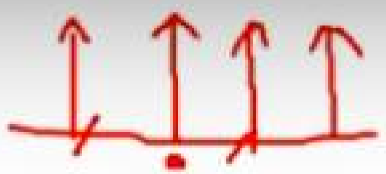
$$x(t) = \sum_{n=-\infty}^{\infty} \hat{x}_n e^{i2\pi nt/T}.$$

Now let's compute the Fourier transform,

$$\tilde{x}(\omega) \equiv \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\, dt$$

$$\text{plug in Fourier series} \qquad = \sum_{n=-\infty}^{\infty} \hat{x}_n \int_{-\infty}^{\infty} dt\, e^{i(2\pi n/T - \omega)t}$$

$$= \sum_{n=-\infty}^{\infty} 2\pi\, \hat{x}_n \delta(\omega - 2\pi n/T).$$



Impulse Train:

Let's consider $it(t) = \sum_{p\in\mathbb{Z}} \delta(t - pT_s)$ a train of T_s-spaced impulsions and let's compute its Fourier transform.

We want to express

$$it(t) = \sum_{n\in\mathbb{Z}} \hat{x}_n e^{i2\pi nt/T_s} = \sum_{n\in\mathbb{Z}} \hat{x}_n e^{i\omega_s nt}$$

$$\hat{x}_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} it(t) e^{-in\omega_s t} = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t) e^{-in\omega_s t} = \frac{1}{T_s}$$

Therefore,

$$it(t) \xrightarrow{\mathcal{F}} \sum_{n\in\mathbb{Z}} \frac{2\pi}{T_s}\delta(\omega - \omega_s n)$$

1-D Sampling Theorem

For any T ; we may sample a CT signal to generate the DT signal

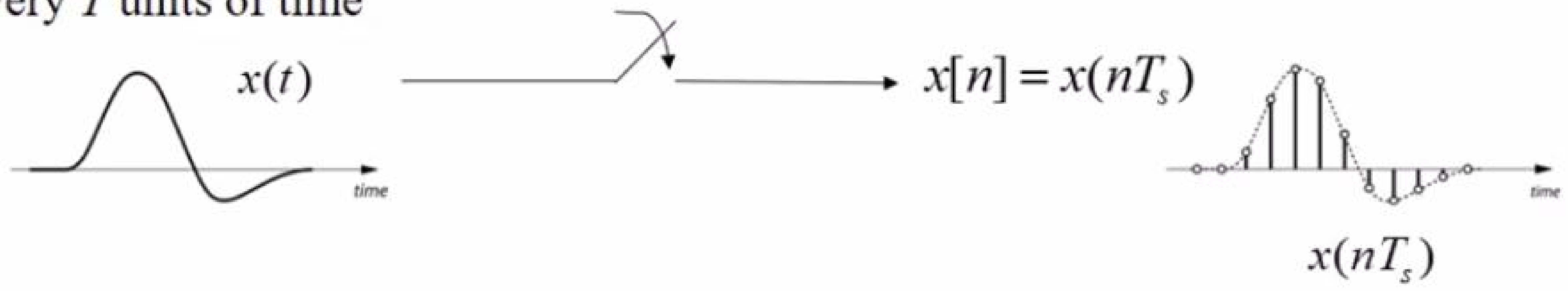
$$x[n] = x(nT_s)$$

This amounts to evaluating $x(t)$ at uniformly spaced points on the t-axis. The number T is the sampling period,

$$f_s = \frac{1}{T_s}$$

is the sampling frequency, and $\omega_s = 2\pi f_s = \dfrac{2\pi}{T_s}$ is the radian sampling frequency.

The process of sampling is sometimes depicted as a switch which closes momentarily every T units of time



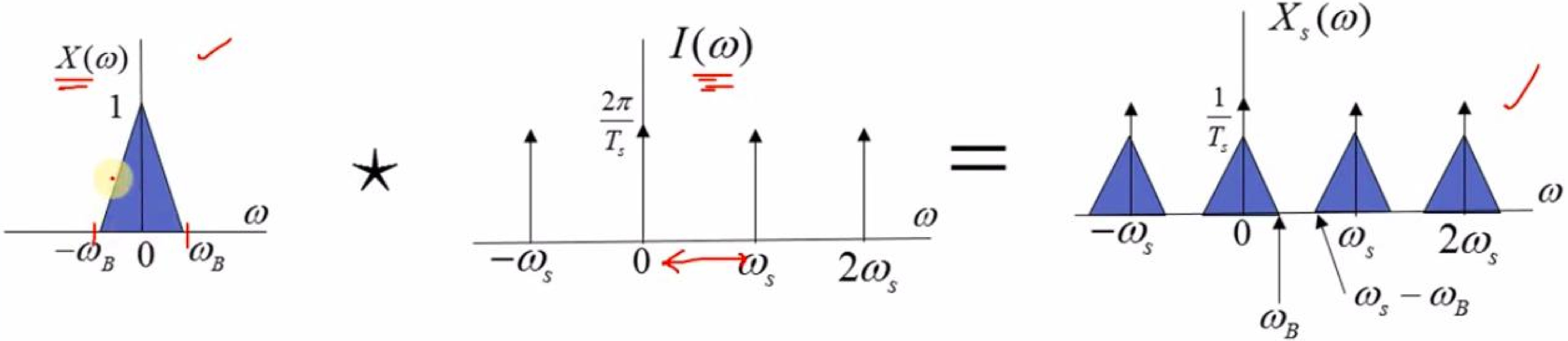
1-D Sampling Theorem

Sampled signal $x_s(t) = x(t)\, it(t) = \sum_{p \in \mathbb{Z}} x(t)\delta(t - pT_s) = \sum_{p \in \mathbb{Z}} x(pT_s)\delta(t - pT_s)$

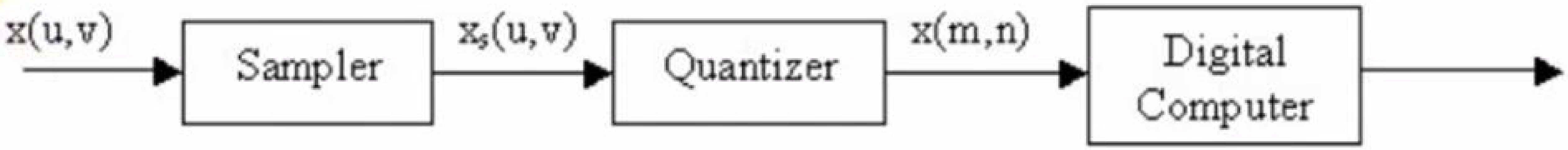
By the multiplication property of FT,

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) \star I(\omega) = \frac{1}{2\pi} X(\omega) \star [\frac{2\pi}{T_s} \sum_n \delta(\omega - n\omega_s)]$$

$$= \frac{1}{T_s} \sum_n X(\omega - n\omega_s)$$



The most basic requirement for computer processing of images is that the images must be available in digital form i.e. arrays of integer numbers. For digitization the given image is sampled on a discrete grid and each sample or pixel is quantized to an integer value representing a gray level. The digitized image can then be processed by the computer.



Definition: An image $x(u, v)$ is called "bandlimited" if its FT $X(\omega_1, \omega_2)$ is zero outside a bounded region in the frequency plane i.e.

$$X(\omega_1, \omega_2) = 0 \quad |\omega_1| > \omega_{10}, |\omega_2| > \omega_{20}$$

ω_{10}, ω_{20}: Bandlimits of the image. If the spectrum is circularly symmetric then the single spatial frequency $[\omega_0 \triangleq \omega_{10} = \omega_{20}]$ is the bandwidth.

Sampling function:

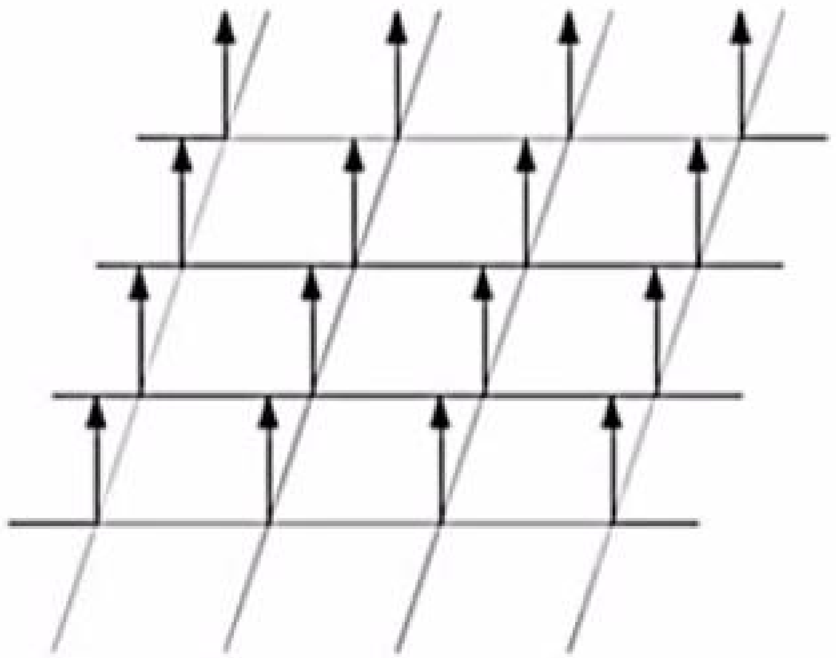
$$s_a(u,v;\Delta u,\Delta v) \triangleq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(u - m\Delta u, v - n\Delta v)$$

with the "sampling intervals" $\Delta u, \Delta v$. The sampled image is

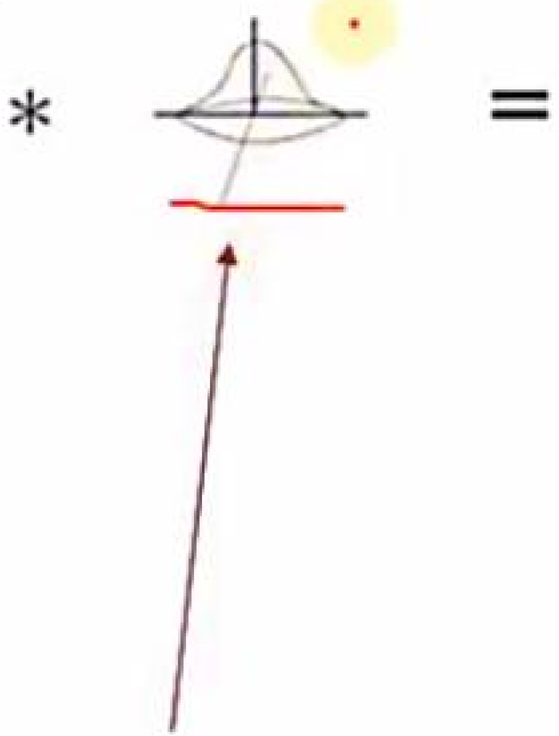
$$\begin{aligned} x_s(u,v) &= x(u,v)s_a(u,v;\Delta u,\Delta v) \\ &= \sum \sum_{m,n=-\infty}^{\infty} x(m\Delta u, n\Delta v)\delta(u - m\Delta u, v - n\Delta v) \end{aligned}$$

It can be shown that the FT of the sampling function with spacing $\Delta u, \Delta v$ is another sampling function with spacing $\frac{2\pi}{\Delta u}, \frac{2\pi}{\Delta v}$. Then, using the convolution in frequency domain we get

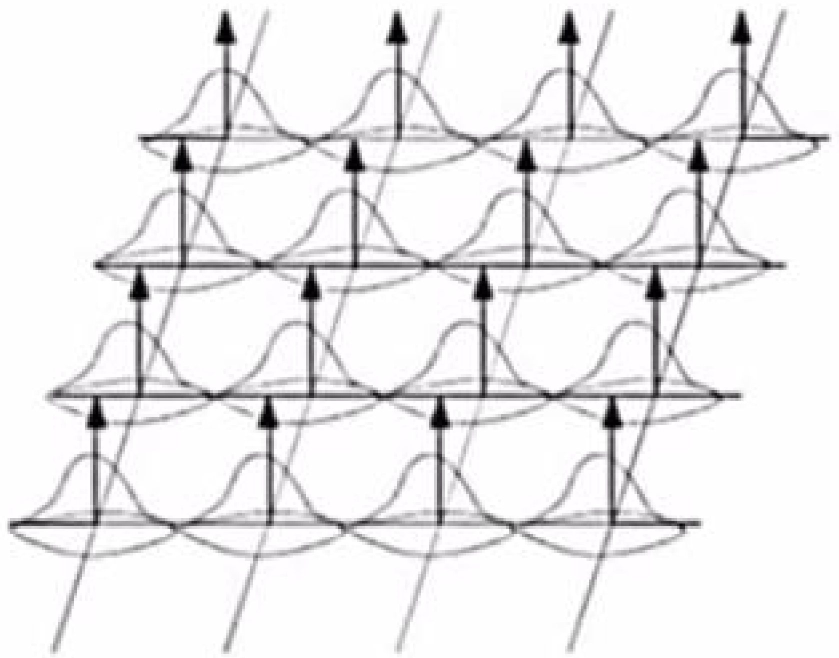
$$X_s(\omega_1, \omega_2) = \frac{1}{\Delta u \Delta v} \sum \sum_{k,l=-\infty}^{\infty} X(\omega_1 - \frac{2\pi k}{\Delta u}, \omega_2 - \frac{2\pi l}{\Delta v})$$



A comb of delta functions

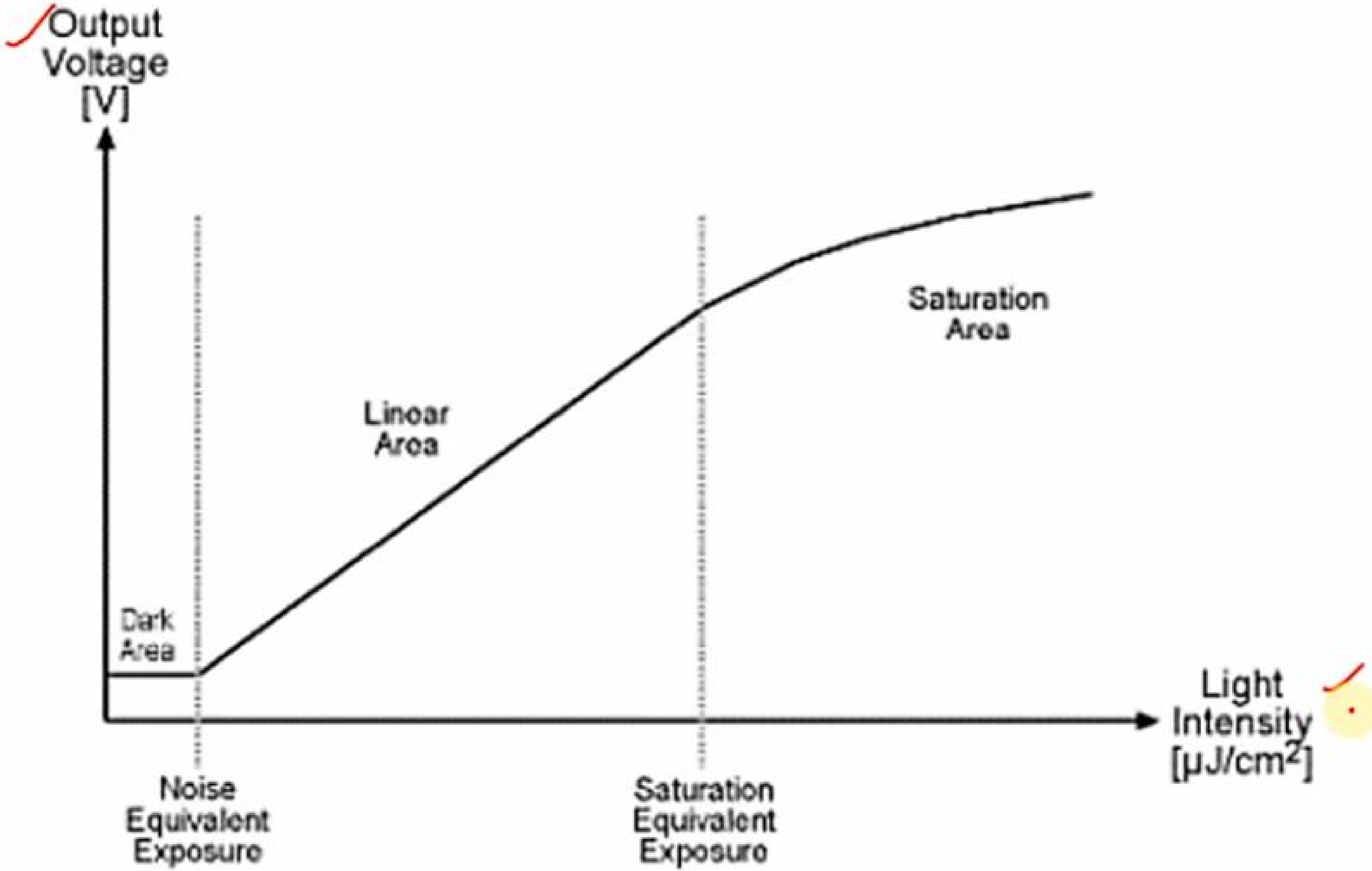


Fourier spectrum of a 2D signal



Fourier spectrum of the sampled 2D signal

Nonlinear Response:



Quantization:

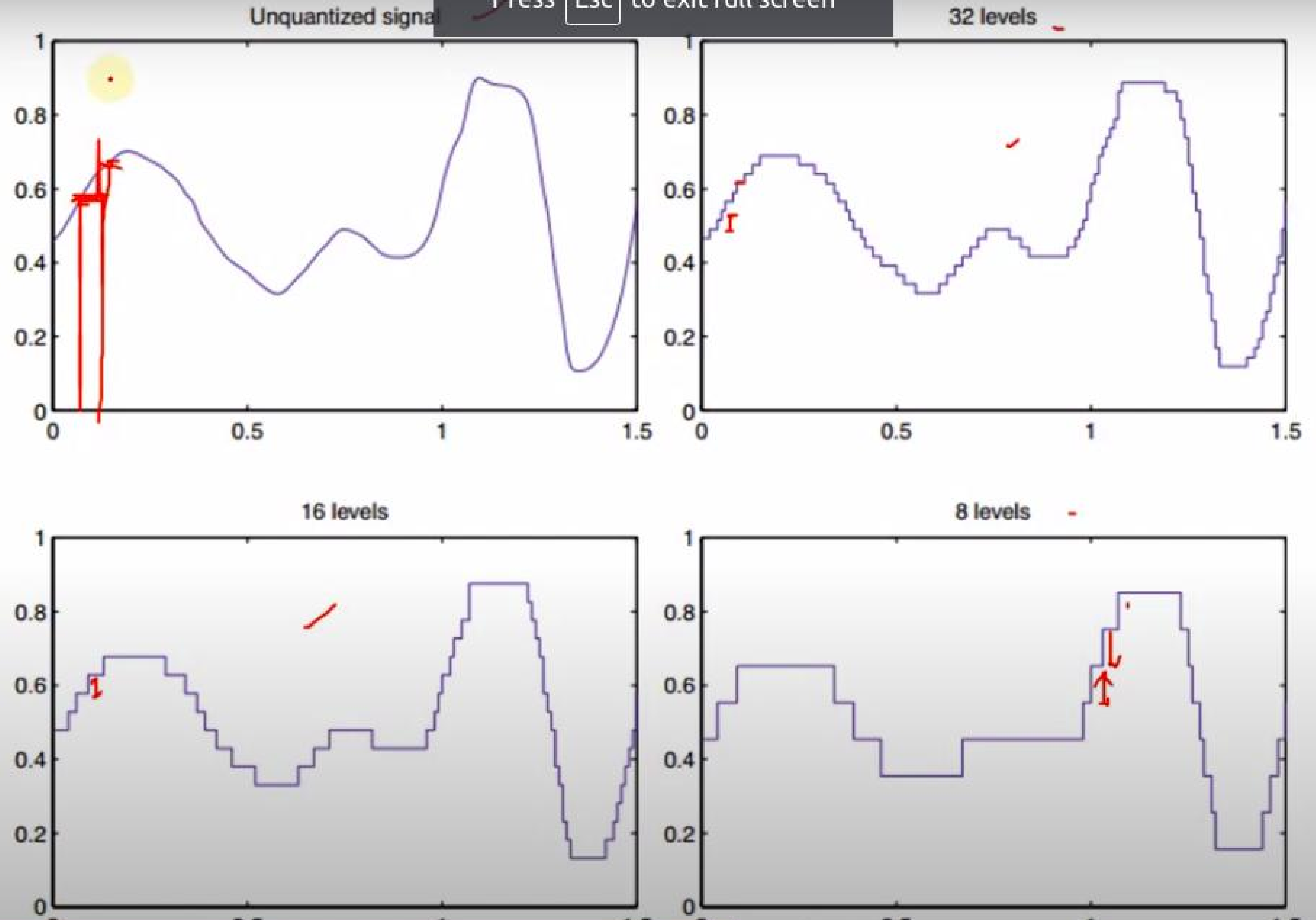
Quantization makes the range of a signal discrete, so that the quantized signal takes on only a discrete, usually finite, set of values.

Quantization is **generally** irreversible and results in loss of information

One of the basic choices in quantization is the number of discrete quantization levels to use. The fundamental tradeoff in this choice is the resulting signal quality versus the amount of data needed to represent each sample.

Digital Image Processing (Autumn 2020-21): Lecture 4

Press Esc to exit full screen



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Zero-memory Quantizer:

The simplest type of quantizers are called ***zero memory* quantizers** in which quantizing a sample is independent of other samples.

$$T : x \rightarrow \{r_1, r_2 \ldots r_L\} \text{ s.t. } \underbrace{t_k}_{\text{Transition level}} \le x < t_{k+1} \rightarrow \underbrace{r_k}_{\text{Reconstruction level}}$$

Digital Image Processing (Autumn 2020-21): Lecture 4 - YouTube - Google Chrome

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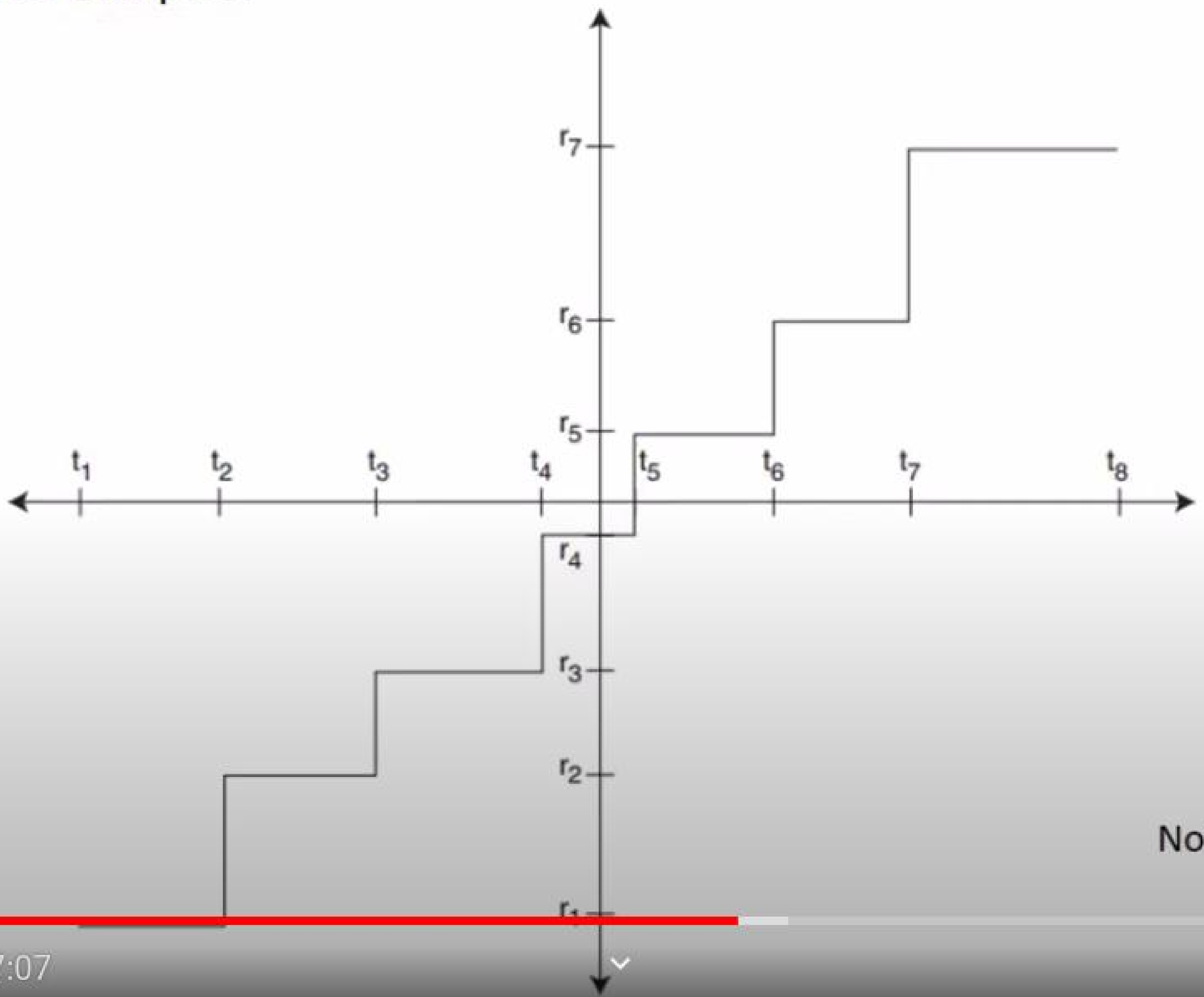

Not necessarily uniform

Uniform Quantizer:

The simplest zero memory quantizer is the uniform quantizer

Transition and reconstruction levels are all equally spaced

For example, if the output of an image sensor takes values between 0 and M, and one wants L quantization levels, the uniform quantizer would take

$$r_k = \frac{kM}{L} - \frac{M}{2L}, \quad k = 1, \ldots, L$$

$$t_k = \frac{M(k-1)}{L}, \quad k = 1, \ldots, L+1$$

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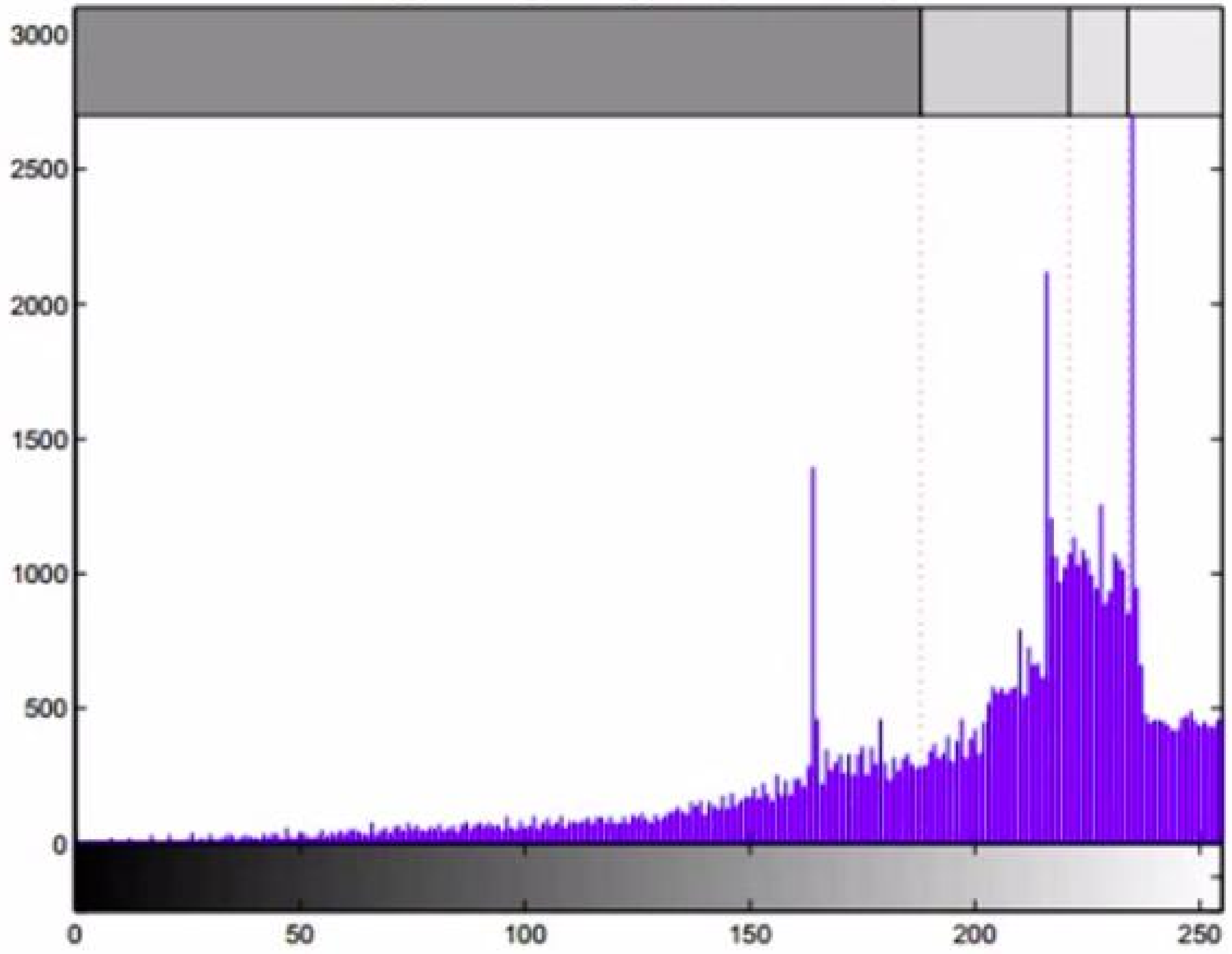
Nonuniform Quantizer:

If an image never takes on a certain range of gray levels then there is no reason to waste quantization levels in this range of gray levels.

Original image, 256 gray levels

Histogram of original image and quantizer breakpoints



Nonuniform Quantizer:

(a)

(b)

(a) Uniformly quantized image. (b) Non-uniformly quantized image

Optimal Quantizer:

This quantizer minimizes the mean squared error for a given number of quantization levels. Let x, with $0 \leq x \leq A$ be a real scalar random variable with a continuous PDF $p_X(x)$. It is desired to find optimum the decision levels t_k and the reconstruction levels r_k for an L-level quantizer such that the mean square error (MSE) (or quantization distortion)

$$\epsilon = E[(x - x_q)^2] = \int_{t_1}^{t_L+1} (x - x_q)^2 p_X(x) dx$$

is minimized. Note that $p_X(x)$ is the PDF of the amplitude of x i.e. $\int_0^A p_X(x) dx = 1$. The MSE can be rewritten as

$$\epsilon = \sum_{i=1}^{L} \int_{t_i}^{t_{i+1}} (x - r_i)^2 p_X(x) dx$$

Optimal Quantizer:

Leibniz Integral Rule considering two terms of the summation i) t_k as upper limit and ii) t_k as lower limit

To minimize ϵ w.r.t. t_k and r_k

$$\frac{\partial \epsilon}{\partial t_k} = 0 \quad \Rightarrow \quad (t_k - r_{k-1})^2 p_X(t_k) - (t_k - r_k)^2 p_X(t_k) = 0$$

$$\frac{\partial \epsilon}{\partial r_k} = 0 \quad \Rightarrow \quad 2\int_{t_k}^{t_{k+1}} (x - r_k) p_X(x) dx = 0, k \in [1, L]$$

where the first derivative is obtained using the fact that $t_k \leq x < t_{k+1} \Rightarrow x_q = r_k$, simplification of the above equations gives

$$t_k = \frac{r_k + r_{k-1}}{2} \quad (1)$$

$$r_k = \frac{\int_{t_k}^{t_{k+1}} x p_X(x) dx}{\int_{t_k}^{t_{k+1}} p_X(x) dx} \quad (2)$$

- The optimum transition levels lie halfway between the optimum reconstruction levels,
- The optimal reconstruction levels lie at the center of mass of the PDF in between the optimum transition levels.

Digital Image!!