Image Modelling

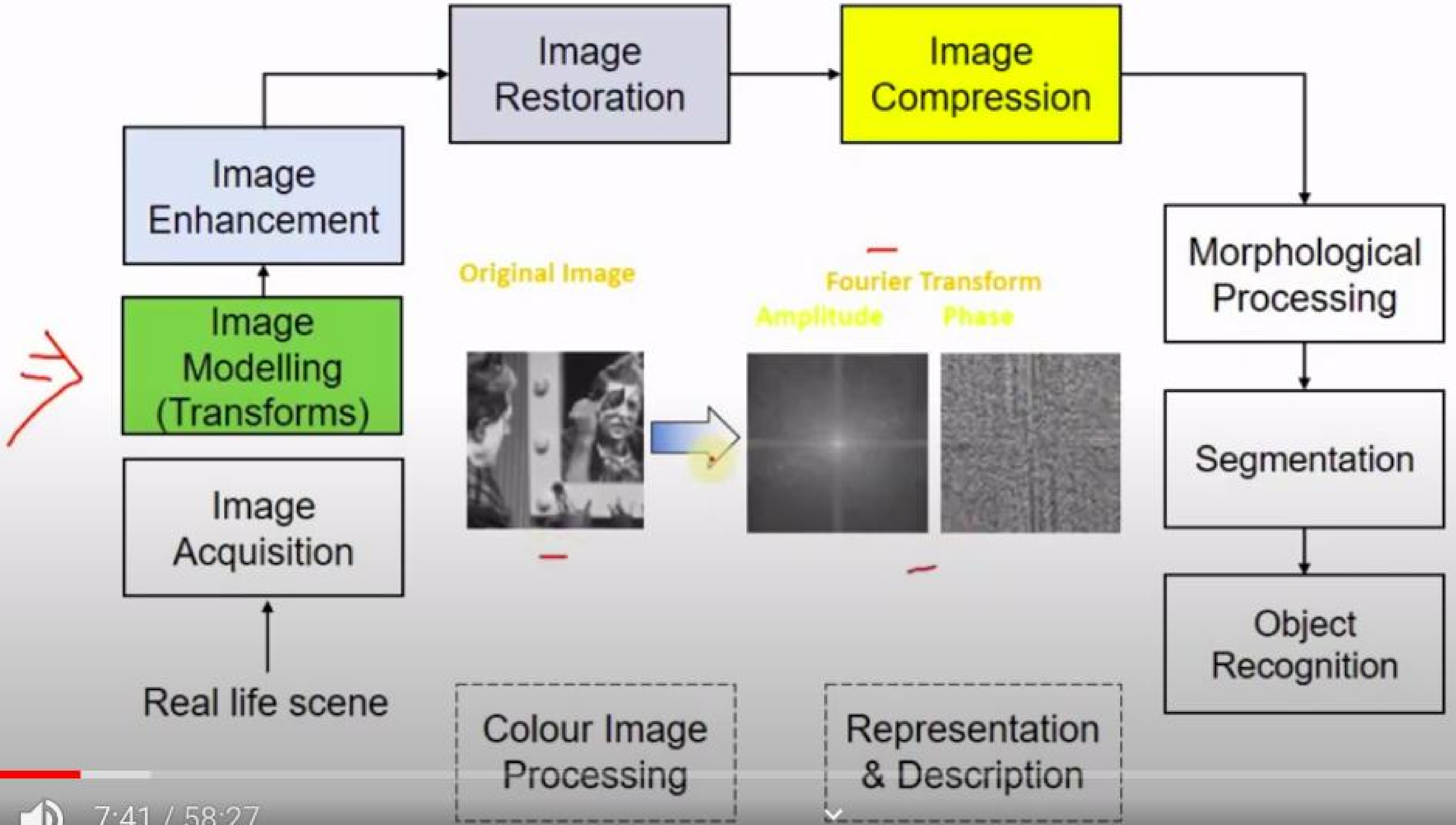

Image Transforms

Why?

- To extract information from the image
- To 'process' the image 'easily'

e.g. For continuous functions, orthogonal series expansions provide series coefficients which can be used for any further processing/analyses.

Unitary Transform: 1D

For a one-dimensional sequence $\{ u(n), 0 \le n \le N-1 \}$, represented as a vector $\mathbf{u}$ of size N, a general transformation is written as

$$\mathbf{v} = \mathbf{A}\mathbf{u} \quad \Rightarrow v(k) = \sum_{n=0}^{N-1} a(k,n)u(n), \quad 0 \le k \le N-1$$

and

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$$

In a special case, when $\mathbf{A}^{-1} = \mathbf{A}^{*\mathbf{T}}$, i.e. $\mathbf{A}$ is unitary matrix, we called the transform as unitary transform and can be written as

$$\mathbf{u} = \mathbf{A}^{*\mathbf{T}}\mathbf{v} \quad \Rightarrow u(n) = \sum_{k=0}^{N-1} v(k)a^*(k,n), \quad 0 \le n \le N-1$$

The columns of $\mathbf{A}^{*\mathbf{T}}$, that is the vectors $\{\mathbf{a}_{\mathbf{k}}^* = a^*(k,n), 0 \le n \le N-1\}^T$ are called basis vectors of $\mathbf{A}$.

2-D orthogonal and unitary transforms

- Orthogonal series expansion for an $N\times N$ image $u(m, n)$

$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}u(m,n)a_{k,l}(m,n) \qquad 0\le k,l\le N-1$$

$$u(m,n)=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}v(k,l)a^{*}_{k,l}(m,n) \qquad 0\le m,n\le N-1$$

  - $v(k, l)$'s are the transform coefficients, $\mathbf{V}\equiv\{v(k,l)\}$ represents the transformed image
  - $\{a_{k,l}(m,n)\}$ is a set of orthonormal functions, representing the image transform

Orthonormality and completeness

- $\{a_{k,l}(m,n)\}$ must satisfies

$$orthonormality: \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{k,l}(m,n)a^*_{k',l'}(m,n) = \delta(k-k', l-l')$$

$$completeness: \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_{k,l}(m,n)a^*_{k,l}(m',n') = \delta(m-m', n-n')$$

2-D orthogonal and unitary transforms

$$v(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} u(m,n) a_{k,l}(m,n) \qquad 0 \le k,l \le N-1$$

Complexity?

2-D orthogonal and unitary transforms

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Complexity? $O(N^4)$!!

a_k

b_l

Separable unitary transforms

$$a_{k,l}(m,n)=a_k(m)b_l(n)=a(k,m)b(l,n)$$

Where $\{a_k(m), k=0,\ldots,N-1\}$ and $\{b_l(n), l=0,\ldots,N-1\}$ are one-dimensional complete orthonormal sets of basis vectors.

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$$a_{k,l}(m,n)=a_k(m)b_l(n)=a\underbrace{(k,m)}_{A}b\underbrace{(l,n)}_{B}$$

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$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n}^{N-1}a(k,m)u(m,n)b(l,n)\leftrightarrow \mathbf{V}=\mathbf{AUB}^{\mathrm{T}}$$

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$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}a(k,m)u(m,n)b(l,n)\leftrightarrow \mathbf{V}=\mathbf{A}\mathbf{U}\mathbf{B}^{\mathrm{T}}$$

$$u(m,n)=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}a^{*}(k,m)v(k,l)b^{*}(l,n)\leftrightarrow \mathbf{U}=\mathbf{A}^{*\mathrm{T}}\mathbf{V}\mathbf{B}^{*\mathrm{T}}$$

Complexity?

$O(N^3)$!!

Basis Images

Let $\mathbf{a}_k^*$ denote the kth column of $\mathbf{A}^{*T}$. Then we can define matrices

$$\mathbf{A}_{k,l}^* = \mathbf{a}_k^* \mathbf{a}_l^{*T} \quad \leftrightarrow a_{k,l}^*(m,n) = A^*_{k,l}(m,n)$$

- $< \mathbf{F}, \mathbf{G} > = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) g^*(m,n)$ is the matrix inner product

$$\begin{cases} v(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} u(m,n) a_{k,l}(m,n) \\ u(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v(k,l) a_{k,l}^*(m,n) \end{cases} \Rightarrow \quad ??$$

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$$\begin{cases} v(k,l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u(m,n)a_{k,l}(m,n) \\ u(m,n) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l)a_{k,l}^*(m,n) \end{cases} \Rightarrow \begin{cases} \mathbf{U} = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l)\mathbf{A}_{k,l}^* \\ v(k,l) = <\mathbf{U},\mathbf{A}_{k,l}^*> \end{cases}$$