Min $z = -x_1 + x_2 + Ma_1 + Ma_2$ Max $z = x_1 - x_2 - Ma_1 - Ma_2$

s.t. $x_1 - 2x_2 - x_3 = 1$ $x_1 - 2x_2 - x_3 + a_1 = 1$

$-x_1 + 2x_2 - x_4 = 1$ $-x_1 + 2x_2 - x_4 + a_2 = 1$

$x_i \geq 0 \ \forall i$ $\underline{x_i \geq 0}$ $a_i \geq 0 \ \forall i$

	Basic	1	-1	0	0	-M	-M		
		x_1	x_2	x_3	x_4	a_1	a_2	b	θ
-M	a_1	1	-2	-1	0	1	0	1	1
-M	a_2	-1	2	0	-1	0	1	1	—
	z_j	0	0	M	M	-M	-M		
	$c_j - z_j$	1 ↑	-1	-M	-M	0	0		
1	x_1	1	-2	-1	0		0	1	—
$R_2 + R_1$ -M	a_2	0	0	-1	-1		1	2	—
	z_j	1	-2	-1+M	M		-M		
	$c_j - z_j$	0	$\underline{1}$ ↑	1-M	-M		0		

Unbounded solⁿ

min $z = -x_1 + x_2$

$x_1 - 2x_2 - x_3 = 1$ $x_1 - 2x_2 \geq 1$

$-x_1 + 2x_2 - \underline{x_4} = 1$ $-x_1 + 2x_2 \geq 1$

$x_1 - 2x_2 \leq -1$

Dual

$3y_1 + 5y_2 \geq 6$

$2y_1 + 6y_2 \geq 5$

Min $18y_1 + 25y_2$

Max $6x_1 + 5x_2$

y_1 $\quad 3x_1 + 2x_2 \leq 18$ $\quad \times 3$ $\quad \times 1$ $\quad 0$

y_2 $\quad 5x_1 + 6x_2 \leq 25$ $\quad \times 0$ $\quad \times 1$ $\quad \times 6/5$

$x_1, x_2 \geq 0$

$6x_1 + 5x_2 \leq 9x_1 + 6x_2 \leq 54$ $\qquad z \leq 54$

$\leq 8x_1 + 8x_2 \leq 43$ $\qquad z \leq 43$

$\frac{6}{5}x_1 + \frac{36}{5}x_2 \leq 30$ $\qquad z \leq 30$

$$\left[\begin{array}{l} \text{Max } z = c'x \\ Ax \leq b \\ x \geq 0 \end{array}\right. \qquad \left.\begin{array}{l} \text{Min } w = b'y \\ A'y \geq c \\ y \geq 0 \end{array}\right]$$

SLP DSLP

		SLP	DSLP
1	constraint	$A_{m \times n}$	$A'_{n \times m}$
2	constant	b	c
3	No. of variables	n	m
	No. of constraints	m	n

④ Constraints $\leq$ Variable ≥ 0

$\geq$ ≤ 0

$=$ unrestricted

$y_1 \leftarrow x_1+x_2=2$

$x_1+x_2 \leq 2 \rightarrow y_1' \rightarrow y_1$

$x_1+x_2 \geq 2 \rightarrow -y_1''$

$$y_1 = \underset{\geq 0}{y_1'} - \underset{\geq 0}{y_1''}.$$

⑤ Variable $x_i \geq 0$ Constraint $\geq$

$x_i \leq 0$ $\leq$

unrestricted x_i $=$

Dual of dual is primal.

SLP Max $z = \underline{c}'\underline{x}$

$A\underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

DSLP Min $w = \underline{b}'\underline{y}$

$A'\underline{y} \geq \underline{c}$ $\underline{y} \geq 0$

D(DSLP) Max $v = c'\underline{u}$

$(A')'\underline{u} \leq (b')' \ni A\underline{u} \leq \underline{b}$

$\underline{u} \geq \underline{0}$

Hence Proved.

SLP Max $z = \underline{c}'\underline{x} = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$A\underline{x} \leq \underline{b}$

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \longrightarrow y_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \longrightarrow y_2$

$\vdots$

$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n = b_k$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \longrightarrow y_m$

convert this equality in less than type.

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \rightarrow y_1$

$\vdots$

$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \leq b_k \longrightarrow y_k^1$

$\vdots$

$-a_{k1}x_1 - a_{k2}x_2 - \ldots - a_{kn}x_n \leq -b_k \rightarrow y_k^2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \rightarrow y_m$

Dual :-

Min $\omega = b_1y_1 + b_2y_2 + \ldots + b_k(y_k^1 - y_k^2) + \ldots + b_my_m$

$a_{11}y_1 + a_{21}y_2 + \ldots + a_{k1}(y_k^1 - y_k^2) + \ldots + a_{m1}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \ldots + a_{k2}(y_k^1 - y_k^2) + \ldots + a_{m2}y_m \geq c_2$

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{kn}(y_k^1 - y_k^2) + \ldots + a_{mn}y_m \geq c_m$

$y_i \geq 0 \ \forall i \qquad y_k^1, y_k^2 \geq 0$

lets define $y_K = y_K^1 - y_K^2$
as $y_K^1 \geq 0$ & $y_K^2 \geq 0$ $\Rightarrow$ y_K is unrestricted in sign.

$x_p \rightarrow$ unrestricted. $\rightarrow$ p^{th} constraint

Max $c_1x_1 + c_2x_2 + \ldots + c_px_p + \ldots + c_nx_n$

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p + \ldots + a_{1n}x_n \leq b_1$

$\vdots$

$a_{m1}x_1 + \quad + \quad + a_{mp}x_p + \quad a_{mn}x_n \leq b_m$

$x_i \geq 0 \quad \forall i$ except p

x_p unrestricted

let $x_p = x_p^1 - x_p^2$

Max $c_1x_1 + \ldots + c_p(x_p^1 - x_p^2) + \ldots + c_nx_n$

$y_1 -$ $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}(x_p^1 - x_p^2) + \ldots \leq b_1$

$\vdots$

$y_m -$ $a_{m1}x_1 + \ldots + \ldots + a_{mp}(x_p^1 - x_p^2) + \ldots \leq b_m$

$x_i \geq 0 \quad \forall i \qquad x_p^1, x_p^2 \geq 0$

Dual

Min

$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1$

$\vdots$

$a_{1p}y_1 + a_{2p}y_2 + \ldots + a_{mp}y_m \geq c_p$ — *

$-a_{1p}y_1 - a_{2p}y_2 - \ldots - a_{mp}y_m \geq -c_p$

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n$

$\therefore$ from *

- Any feasible solution to primal (SLP) has value z greater than or at least equal to the value v for any feasible solution to dual (DSLP).

SLP Min $z = c'x$, $Ax \geq b$, $x \geq 0$

DSLP Max $v = b'y$, $A'y \leq c$, $y \geq 0$

Let x_0 be any feasible sol^n to primal $\Rightarrow x_0 \geq 0$

let y_0 be any feasible sol^n to dual

$$z_0 = c'x_0 \geq (A'y_0)'x_0 = y_0'Ax_0 \geq y_0'b = b'y_0 = v_0$$

Alternative.

SLP Max $z = c'x$, $Ax \leq b$, $x \geq 0$

DSLP Min $v = b'y$, $A'y \leq c$, $y \geq 0$

Let x_0 & y_0 be any feasible sol^ns to SLP & DSLP resp.

$$z_0 = c'x_0 \leq (A'y_0)'x_0 = y_0'A'x_0 \leq y_0'b = b'y_0 = v_0$$

→ If SLP min $\Rightarrow$ min $z \geq$ max v

if SLP max $\Rightarrow$ min $v \geq$ max z

$\underline{x}_0$ & $\underline{y}_0$ are feasible sol^n to SLP & DSLP resp.
(min) (max)

with $\underline{c}'\underline{x}_0 = \underline{b}'\underline{y}_0$

weak duality then $\Rightarrow$ $\underline{c}'\underline{x} \geq \underline{b}'\underline{y}$ $\forall \underline{x}$ $\forall \underline{y}$.

$\underline{b}'\underline{y} \leq \underline{c}'\underline{x}$

as $\underline{x}_0$ is feasible sol^n

$\Rightarrow$ $\underline{b}'\underline{y} \leq \underline{c}'\underline{x}_0$ $\forall \underline{y}$

$\Rightarrow$ $\max\limits_{\underline{y}} \underline{b}'\underline{y} \leq \underline{c}'\underline{x}_0 = \underline{b}'\underline{y}_0$ $\Rightarrow$ y_0 is optimal sol^n to DSLP

Similarly we can obtain that $\underline{x}_0$ is O.S. to SLP

$\underline{b}'\underline{y} \leq \underline{c}'\underline{x}$ $\forall \underline{y}$ $\forall \underline{x}$

as y_0 is feasible sol^n to DSLP

$\underline{b}'\underline{y}_0 \leq \underline{c}'\underline{x}$ $\forall \underline{x}$

$\underline{c}'\underline{x}_0 = \underline{b}'\underline{y}_0 \leq \min\limits_{\underline{x}} \underline{c}'\underline{x}$ $\Rightarrow$ x_0 is optimal.

min $x_1 - x_2$ $\longrightarrow$ $-\infty$

∞