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Unit - I - Real Numbers $\mathbb{R}$

Groups Mat

$(\mathbb{R}, +, \cdot)$

* Algebraic Properties of $\mathbb{R}$:-

Add^n Mult^n

On set of $\mathbb{R}$ there are two binary operators $+$ & $\cdot$

These two operations follows few properties :-

add^n

A1 > Commutative property of addition

$$a+b = b+a \quad \forall\ a,b \in \mathbb{R}$$

A2 > Associative property of add^n

$$(a+b)+c = a+(b+c) \quad \forall\ a,b,c \in \mathbb{R}$$

A3) Existence of zero element (Additive Identity)

$$a+0 = 0+a = a \quad \forall\ a \in \mathbb{R}$$

A4) Existence of negative element (Additive Inverse)

$$a+(-a) = (-a)+a = 0 \quad \forall\ a \in \mathbb{R}$$

$a + -a = 0$

$-a + -(-a) = 0$

a

matric

$A \cdot B \neq B \cdot A$

M1) Commutative property of multiplication

$$a \cdot b = b \cdot a \quad \forall\ a,b \in \mathbb{R}$$

M2) Associative p of multi.

$$(a \cdot b) c = a (b \cdot c) \quad \forall\ a,b,c \in \mathbb{R}$$

M3) Existence of unit element (Multiplicative identity)

$$a \cdot 1 = 1 \cdot a = a \quad \forall\ a \in \mathbb{R}$$

Extended Real

$\infty \leftarrow \mathbb{R}$

M4) Existence of multiplicative inverse / Reciprocals

$$a \cdot \left(\frac{1}{a}\right) = \frac{1}{a} \cdot a = 1 \quad \forall\ a \in \mathbb{R} - \{0\}$$

D) Distributive property of multiplication over add^n

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a \quad \forall\ a,b,c \in \mathbb{R}$$

$$= a \cdot b + a \cdot c$$

* Order Properties of $\mathbb{R}$

Why do we study Real Analysis?

CLT $\to \dfrac{\sum x_i - \mu}{\sigma} \to$ ____

$\bar{x} \to \mu$

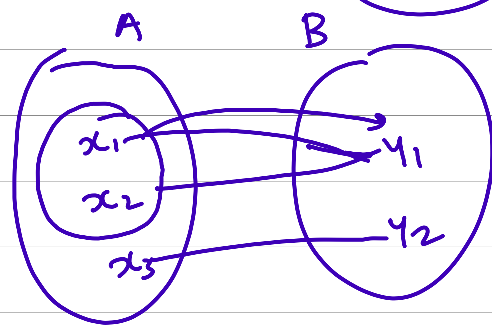
seq^n of fun^c — Convergence of Series of fun^{c}s

$\sum f_n(x) = \sum \dfrac{x_i}{n} \to$ ____

$\dfrac{\sum x_i}{n} \to \mu$

Set Fun^c

Fun^c



Prob.

input element $\to$ set

$\dfrac{n(A)}{n(S)}$ Classi $\in \mathbb{N}$

input Set $\to$ ____ ~~Real~~

$x^3 + 2x + 3 = 0$

$x = ?$

0 1 $\mathbb{R}$

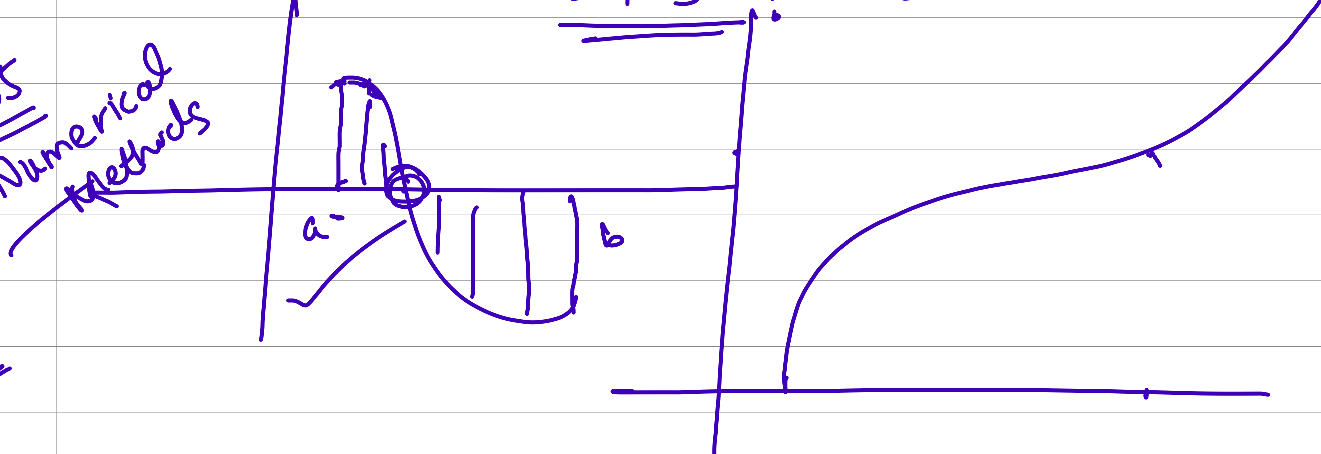
ST-201

A

Borel fun^c

$[a, b]$? Closed

ST-205 Numerical Methods



Cont?

Series → $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$E(X) = \sum_x \frac{e^{-\lambda}\lambda^x}{x!} x$$
$$= e^{-\lambda} \sum_x \frac{\lambda^x}{x!} x$$
$$= e^{-\lambda}\lambda \sum_x \frac{\lambda^{x-1}}{(x-1)!}$$
$$= \cancel{e^{-\lambda}} \lambda \cancel{e^{\lambda}}$$

?

Series

Geometric $P(X=x) = pq^{x-1}$

Real Analysis

$$E(X) = \sum x \cdot pq^{x-1}$$
$$= p \cdot \sum x \cdot q^{x-1}$$

?

$$= p \frac{1}{(1-q)^2}$$
$$= \cancel{p} \frac{1}{p^2} = \frac{1}{p}$$
$$= 1/p$$

$\sum x^n = \frac{1}{1-x}$ $|x|<1$

? $\frac{d}{dx} \sum x^n = \frac{d}{dx} \frac{1}{1-x}$

$\sum n x^{n-1} = \frac{1}{(1-x)^2}$

ST-201 Probability

Inverse fun^c ← X Random Variable

ST-301 Asymptotic → as $n \to \infty$

$\bar{x} \to \mu$ $\frac{1}{n}\sum x_i \to \mu$

① Unbiased
② Suff
③ Eff
④ Consistent → as $n \to \infty$

$X_{(n)}$ # C.E. $X \sim U(0,\theta)$

$2\bar{x}$ →

* Order Properties of $\mathbb{R}$

$\mathbb{P}$ set of +ve nos $\in \mathbb{R}$, $\mathbb{P} \subseteq \mathbb{R}$, $\mathbb{R}^+$

$\mathbb{R}^+$ satisfies following properties

① if $a, b \in \mathbb{R}^+ \Rightarrow a+b \in \mathbb{R}^+$

② if $a, b \in \mathbb{R}^+ \Rightarrow a \cdot b \in \mathbb{R}^+$

③ if $a \in \mathbb{R}$ then exactly one of the following is true:-

$a \in \mathbb{R}^+$, OR $a = 0$, OR $-a \in \mathbb{R}^+$

-4.5

Law of Trichotomy

$a \in \mathbb{R}$ >0 <0 $=0$

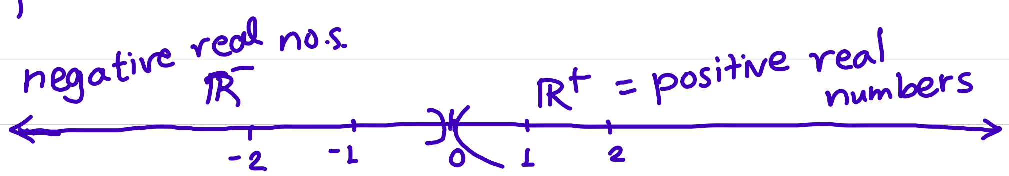
$a \in \mathbb{R}^+$ $a \in \mathbb{R}^-$

$-a \in \mathbb{R}^+$

Let $a, b \in \mathbb{R}$

ⓐ If $a-b \in \mathbb{R}^+ \Rightarrow a > b$ or $b < a$

ⓑ If $a-b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a \geq b$ or $b \leq a$



$\mathbb{R}^+ = \{x / x > 0, x \in \mathbb{R}\}$

$\mathbb{R}^- = \{x / x < 0, x \in \mathbb{R}\}$

Proof :- ⓐ $a, b \in \mathbb{R}$, $a-b \in \mathbb{R}^+ \Rightarrow a-b > 0 \Rightarrow a > b$

ⓑ $a, b \in \mathbb{R}$ $a-b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a-b \geq 0 \Rightarrow a \geq b$

Theo:- Let $a, b, c \in \mathbb{R}$

① If $a > b$ & $b > c \Rightarrow a > c$ (Transitivity)

② If $a > b \Rightarrow a + c > b + c$

③ If $a > b$, $c > 0 \Rightarrow a \cdot c > b \cdot c$

If $a > b$, $c < 0 \Rightarrow a \cdot c < b \cdot c$

① If $a>b$, & $b>c$

$a-b>0$ & $b-c>0$

$\Rightarrow$ $a-b \in \mathbb{R}^+$ & $b-c \in \mathbb{R}^+$

$\Rightarrow$ $(a-b)+(b-c) \in \mathbb{R}^+$ (by order properties)

$\Rightarrow$ $(a-c) \in \mathbb{R}^+$

$\Rightarrow$ $a-c>0$

$\Rightarrow$ $a>c$ QED

② $a-b \in \mathbb{R}^+$

$a-b+c-c \in \mathbb{R}^+$

$(a+c)-(b+c) \in \mathbb{R}^+$

$a+c > b+c$

③ $a>b$, $c>0$ to prove $a \cdot c > b \cdot c$

$a>b$, $a-b>0$

$a-b \in \mathbb{R}^+$

$c(a-b) = c \cdot a - cb$ as $c>0$

$ca-cb>0$

$ca>cb$

Similarly $a>b$, $c<0$ $\Rightarrow$ $ac<bc$

Theo:- If $a \in \mathbb{R}$, such that $0 \le a < \varepsilon$ $\forall$ $\varepsilon \in \mathbb{R}$, $\varepsilon>0$

$\Rightarrow$ $a=0$

$\varepsilon=0.005$ 08

$\varepsilon=0.00000$

$a \to 0$

By method of contradiction.

Assume $\underline{\underline{a>0}}$ $\Rightarrow$ (as $\underline{\underline{0 \leq a<\varepsilon}}$) $\Rightarrow$ a is positive

$\Rightarrow$ $\frac{a}{2}$ is positive

we can assume $\varepsilon_0 = a/2 < a$

for any $\varepsilon > 0$, $0 \leq a < \varepsilon$ but for $\varepsilon_0 \Rightarrow$ $0 < \varepsilon_0 < a$

which contradicts to our assumption. $\boxed{\varepsilon}$

Rational Nos. :- $Q = \{x \mid x = p/q, \; p, q \in \mathbb{I}, \; q \neq 0\}$

* Theo. There doesnot exist any rational no. $r \in Q$ $\underline{\underline{r^2 = 2}}$

$\Rightarrow$ Assume that $\sqrt{2}$ is rational no.

$$\sqrt{2} = \frac{p}{q}$$

common divisor of p, q is 1

$(p, q) = 1$ ✓

$\frac{4}{6} \Rightarrow \frac{2}{3}$

$$\Rightarrow \quad 2 = \frac{p^2}{q^2}$$

$p = 4$, $p^2 = 16$

$$\Rightarrow \quad 2q^2 = p^2 = p \cdot p$$

$\rightarrow$ $p = 3$, $p^2 = 9$

$\Rightarrow$ p is divisible by 2 $\Rightarrow$ $p = 2 \cdot m$

$\Rightarrow$ $p^2 = 2^2 \cdot m^2 = 4m^2$

$p =$ even

$$\Rightarrow \quad 2q^2 = \cancel{4}^{\,2} m^2$$

$$\Rightarrow \quad q^2 = 2m^2$$

$\Rightarrow q$ is divisible by 2 ——— (**)

$(p,q)=1$ but here $(p,q)=2$

which contradicts to our assumption.

even^2

$(2.n)^2$

$=2(2n^2)$

even

odd^2

$(2n+1)^2$

$=4n^2+2n+1$

$=2(2n^2+n)+1$

$=$ odd

Theo:- If $ab>0$ then either ① $a>0, b>0$

② $a<0, b<0$

* ① If $a,b \in \mathbb{R}$ show that $a^2+b^2=0 \Leftrightarrow a=0, b=0$ ✓

② If $0<c<1$ show that $0<c^2<c<1$ ✓

③ If $x,y \in Q$, $x+y \in Q$, $x \cdot y \in Q$ ✓

If $x \in Q$, $y \in Q^c$, $x+y \in Q^c$

① $a^2+b^2=0 \Rightarrow (a+b)^2 = a^2+b^2+2ab=0$

$2ab=0$

$\Rightarrow a.b=0$

$\Rightarrow a=0$ or $b=0$

(if $a=0$ & $b\neq 0 \Rightarrow a^2+b^2>0$

$\left(a^2+b^2=0 \Rightarrow b=0 \right.$

$a,b \in \mathbb{R}^+,\ c>0,\quad a>b \Rightarrow c\cdot a > c\cdot b$

$0<c<1$, as $c>0$

$c<1$

$c\cdot c < 1\cdot c$

$0<c^2<c<1$ ✓

$x,y \in Q \Rightarrow x+y \in Q$

$x=\frac{p_1}{q_1},\ y=\frac{p_2}{q_2} \Rightarrow x+y = \frac{p_1}{q_1}+\frac{p_2}{q_2}$

$= \frac{p_1q_2+p_2q_1}{q_1\cdot q_2}$

as $p_1, q_1, p_2, q_2 \in \mathbb{I}$, $\quad p_1q_2 \in \mathbb{I},\ p_2\cdot q_2 \in \mathbb{I},\ q_1\cdot q_2 \in \mathbb{I}$

$p_1q_1+p_2q_2 \in \mathbb{I}$

$= \frac{p^*}{q^*} \in Q$