① x_{n+m} is m-tail seq^n of x_n.

② x_n is convergent. let c be limit pt. of x_n

for any $\varepsilon > 0$, $\exists\, K(\varepsilon) \in \mathbb{N} \ni |x_n - c| < \varepsilon \quad \forall\, n \geq K(\varepsilon)$

① $y_1, y_2, \ldots, y_n$ is m tail seq^n x_n

$x_{m+1}, x_{m+2}, \ldots, x_{m+n}$

Assume $n > m$

$x_n = y_{n-m}$

$|x_n - c| < \varepsilon \quad \forall\, n \geq K(\varepsilon)$

$|y_{n-m} - c| < \varepsilon \quad \forall\, n \geq K(\varepsilon)$

$|y_{n'} - c| < \varepsilon \quad \forall\, n' + m \geq K(\varepsilon)$

$n' \geq K(\varepsilon) - m$

$n' = n - m \Rightarrow n = n' + m$

$|y_{n'} - c| < \varepsilon \quad \forall\, n' \geq K'(\varepsilon)$

---

for any $\varepsilon > 0$ $\exists\, K(\varepsilon) \in \mathbb{N}$, $|x_n - c| < \varepsilon \quad \forall\, n \geq K(\varepsilon)$

$\Rightarrow x_K, x_{K+1}, \ldots x_{K+n}, \ldots \in (c-\varepsilon, c+\varepsilon)$

By the concept of tail seq^n

$\Rightarrow y_{K-m}, y_{K+1-m}, \ldots y_{K+n-m} \ldots \in (c-\varepsilon, c+\varepsilon)$

$\Rightarrow y_n \in (c-\varepsilon, c+\varepsilon) \quad \forall\, n \geq \underbrace{K-m}_{K(\varepsilon)}$

$$|x_n - x| \leq C \cdot a_n$$

$a_n \to 0$, for any $\varepsilon > 0$ $\exists K(\varepsilon) \in \mathbb{N} \ni |a_n - 0| < \varepsilon/C \ \forall n \geq K(\varepsilon)$
$\ni \varepsilon/C > 0$ $a_n < \varepsilon/C$

$$C a_n < \cancel{C} \cdot \varepsilon/\cancel{C}$$

$$C \cdot a_n < \varepsilon \qquad \forall \ n \geq K(\varepsilon)$$

for $n \geq m$ $|x_n - x| < C \cdot a_n$

Let $K_1(\varepsilon) = \max(K(\varepsilon), m)$

for $\varepsilon > 0$ $\exists K_1(\varepsilon) \in \mathbb{N}$. $\ni \forall n \geq K_1(\varepsilon)$

$$|x_n - x| < C \cdot a_n < \varepsilon$$

$$|x_n - x| < \varepsilon \qquad \forall n \geq K_1(\varepsilon)$$

$$\Rightarrow \quad x_n \to x$$

* Convergent seq^n of Real No.s is always bounded

→ Let $x_n \to x$
① for any $\varepsilon > 0$ $\exists K(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x| < \varepsilon$ $\forall n \geq K(\varepsilon)$
② To prove: $|x_n| \leq m$ $\forall n \in \mathbb{N}$

$$|x_n - x| < \varepsilon \qquad \forall n \geq K(\varepsilon)$$

$$x - \varepsilon < x_n < x + \varepsilon \qquad \forall n \geq K(\varepsilon)$$

let $m = \max\{x+\varepsilon, |x_1|, |x_2|, \ldots |x_K|\}$

$$|x_n| < m \qquad \forall n \in \mathbb{N}$$

① $x_n \to x$, $y_n \to y$, $\Rightarrow x_n + y_n \to x + y$

For $\varepsilon > 0$, $\frac{\varepsilon}{2} > 0$, $\exists K_1(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x| < \varepsilon/2$ $\forall n \geq K_1(\varepsilon)$

$\frac{\varepsilon}{2} > 0$ $\exists K_2(\varepsilon) \in \mathbb{N}$, $\ni |y_n - y| < \varepsilon/2$ $\forall n \geq K_2(\varepsilon)$

for $n \geq K(\varepsilon) = \max(K_1(\varepsilon), K_2(\varepsilon))$

$$|(x_n + y_n) - (x + y)| = |x_n - x + y_n - y|$$
$$\leq |x_n - x| + |y_n - y|$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$\leq \varepsilon$$

$x_n \to x$ for any $\varepsilon > 0$, $|y| > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ $\ni |x_n - x| \leq \varepsilon/2|y|$ $\forall n \geq K(\varepsilon)$

for any $\varepsilon > 0$, $M > 0$ $\exists K_2(\varepsilon) \in \mathbb{N}$ $\ni |y_n - y| \leq \varepsilon/2M$ $\forall n \geq K_2(\varepsilon)$

$$|x_n \cdot y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$$
$$= |x_n(y_n - y) + y(x_n - x)|$$
$$\leq |x_n||y_n - y| + |y||x_n - x|$$
$$\leq |x_n| \frac{\varepsilon}{2M} + |y| \frac{\varepsilon}{2|y|}$$

$|x_n| \leq M$

$\frac{|x_n|}{M} \leq 1$.

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$\leq \varepsilon$$

$x_n \to x$, $z_n \to z$, $z_n \neq 0$ $\forall n$, $z \neq 0$

$$\frac{x_n}{z_n} \to \frac{x}{z}$$

For any $\varepsilon > 0$, $\varepsilon > 0$, $\exists K_1(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x| < \varepsilon$ $\forall n \geq K_1(\varepsilon)$

$\varepsilon |z| m > 0$ $\exists K_2(\varepsilon) \in \mathbb{N}$, $\ni |z_n - z| < \varepsilon m |z|$ $\forall n \geq K_2(\varepsilon)$

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \left|\frac{z - z_n}{z_n}\right| = \frac{|z_n - z|}{|z_n||z|} \leq \frac{\varepsilon \cancel{m|z|}}{\cancel{m|z|}}$$

$z_n \neq 0$, $z \neq 0$, $z_n \to z$. $m \leq |z_n| \leq M$ $\qquad m > 0$

$$\frac{1}{Z_n} \to \frac{1}{Z}$$

let $y_n = 1/Z_n$ $\quad y = 1/Z$

we have $x_n \to x,\ y_n \to y \Rightarrow x_n y_n \to xy$

$$x_n \cdot y_n = \frac{x_n}{Z_n} \to xy = \frac{x}{Z}$$

$x_n \geq 0 \ \forall n$

$x_n \to x$

$x \geq 0$

① For any $\varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x| < \varepsilon \quad \forall n \geq K(\varepsilon)$

$x - \varepsilon < x_n < x + \varepsilon \quad \forall n \geq K(\varepsilon)$

contradiction

$x < 0$

let $\varepsilon = -x > 0$

$x - (-x) < x_n < x + (-x)$

$\cancel{2x} < x_n < 0 \quad \forall n > K(\varepsilon)$

① $x_n \geq 0 \ \forall n \in \mathbb{N}$ & if x_n is convergent, $\lim x_n \geq 0$

② $x_n \to x$, $y_n \to y$ $\quad x_n \leq y_n \ \forall n \in \mathbb{N}$

To prove :- $\lim x_n \leq \lim y_n$

let $Z_n = y_n - x_n$

$Z_n \geq 0 \quad \lim Z_n \geq 0 \quad$ by ①

$\lim (y_n - x_n) \geq 0$

$\lim y_n \geq \lim x_n$

$x_n \to x$ & $\underline{a \leq x_n \leq b}$ $\Rightarrow$ $a \leq \lim x_n \leq b$

① $x_n \to x$, $y_n \to y$, $x_n \leq y_n$ $\Rightarrow$ $x \leq y$

let $a_n = a \to a$

$b_n = b \to b$

$a \leq x_n \leq b \quad \forall n$

$\underbrace{a_n \leq x_n} \leq b_n \quad \forall n$

$\lim a_n \leq \lim x_n \leq \lim b_n$

$a \leq x \leq b$

* $x_n, y_n, z_n \in \mathbb{R}$ $\quad x_n \leq y_n \leq z_n \quad \forall n$

$\lim x_n = \lim z_n = w$

for any $\varepsilon > 0$, $\exists K_1(\varepsilon) \in \mathbb{N}$, $\ni |x_n - w| < \varepsilon \quad \forall n \geq K_1(\varepsilon)$

$\exists K_2(\varepsilon) \in \mathbb{N}$, $\ni |z_n - w| < \varepsilon \quad \forall n \geq K_2(\varepsilon)$

$\forall n \geq \max(K_1(\varepsilon), K_2(\varepsilon)) = K(\varepsilon)$

$|x_n - \omega| < \varepsilon$ & $|z_n - \omega| < \varepsilon$

$\underline{\omega - \varepsilon < x_n} < \omega + \varepsilon$ & $\omega - \varepsilon < \underline{z_n < \omega + \varepsilon}$

$x_n \leq y_n \leq z_n \quad \forall n$

$\omega - \varepsilon \leq x_n \leq y_n \leq z_n \leq \omega + \varepsilon \quad \forall n \geq K(\varepsilon)$

$\omega - \varepsilon \leq y_n \leq \omega + \varepsilon \quad \forall n \geq K(\varepsilon)$

$|y_n - \omega| < \varepsilon \quad \forall n \geq K(\varepsilon)$

$\Rightarrow y_n$ is convergent and $\lim y_n = \omega = \lim x_n = \lim z_n$

* $x_n \to x \Rightarrow |x_n| \to |x|$

$\{$ for any $\varepsilon>0$ $\exists K(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x| < \varepsilon$ $\forall n \geq K(\varepsilon) \}$

$||a|-|b|| \leq |a-b|$

$$||x_n|-|x|| \leq |x_n - x| < \varepsilon \quad \forall n \geq K(\varepsilon)$$

$$||x_n|-|x|| < \varepsilon \quad \forall n \geq K(\varepsilon)$$

$\Rightarrow |x_n| \to |x|$

$(a+b)(a-b) = a^2 - b^2$

$(\sqrt{x_n}+\sqrt{x})(\sqrt{x_n}-\sqrt{x}) = x_n - x$ $< \varepsilon$

$|\sqrt{x_n}-\sqrt{x}| < \varepsilon$

$\sqrt{x_n} \to \sqrt{x}$

(without loss of generality assume $x_n \geq 0$ $x>0$)

for any $\varepsilon > 0$, $\varepsilon\sqrt{x} > 0$, $\exists K(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x| < \varepsilon\sqrt{x}$ $\forall n \geq K(\varepsilon)$

$$|\sqrt{x_n} - \sqrt{x}| = \left| \frac{(\sqrt{x_n}-\sqrt{x})(\sqrt{x_n}+\sqrt{x})}{\sqrt{x_n}+\sqrt{x}} \right|$$

$$= \left| \frac{x_n - x}{\sqrt{x_n}+\sqrt{x}} \right|$$

$$= \frac{|x_n - x|}{\sqrt{x_n}+\sqrt{x}} \qquad \frac{1}{a+b} < \frac{1}{a}$$

$$\leq \frac{|x_n - x|}{\sqrt{x}} \qquad \sqrt{x} > 0$$

$$\leq \frac{\varepsilon \cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}$$

$x_n \to x$ To prove $\sqrt{x_n} \to \sqrt{x}$

For any $\varepsilon > 0$, $\varepsilon\sqrt{M} > 0$, $\exists\ K(\varepsilon) \in \mathbb{N}$

$|x_n - x| < \varepsilon\sqrt{M} \quad \forall\ n \geq K(\varepsilon)$

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})|}{|\sqrt{x_n} + \sqrt{x}|}$$

$$= \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} \leq \frac{|x_n - x|}{|\sqrt{x_n}|}$$

① $x_n \to x$, x_n is convergent hence bounded

$|x_n| \leq M$, $M \in \mathbb{R}$

$|\sqrt{x_n}| \leq \sqrt{M}$

$$\leq \frac{|x_n - x|}{\sqrt{M}}$$

$$\leq \frac{\varepsilon \cdot \cancel{\sqrt{M}}}{\cancel{\sqrt{M}}}$$

$$\leq \varepsilon$$

Monotone Convergence Theo.

let x_n be monotonically increasing seq^n ($x_n \uparrow$)

* x_n is convergent iff it is bounded.

① Let x_n is convergent. let $\lim x_n = x$

for any $\varepsilon > 0$ $\exists\ K(\varepsilon) \in \mathbb{N}$, $\ni$ $|x_n - x| < \varepsilon \quad \forall\ n \geq K(\varepsilon)$.

$\Rightarrow \quad x - \varepsilon \leq x_n \leq x + \varepsilon \quad \forall\ n \geq K(\varepsilon)$

let $M = \max(x_1, x_2, \ldots, x_{K(\varepsilon)}, x + \varepsilon)$.

$\Rightarrow \quad |x_n| \leq M \quad \forall \ n \in \mathbb{N}.$

$\Rightarrow \quad x_n$ is bounded.

② let $x_n \uparrow$ and bounded.

$S = \{x_n, n \in \mathbb{N}\}$, Set S is also bounded.

$\therefore$ by completeness property it has supremum.

let $M = \text{Sup}\{x_n, n \in \mathbb{N}\}$

let $\varepsilon > 0$, $M - \varepsilon$ cannot be upperbound/Sup of S.

$\Rightarrow \quad M - \varepsilon < x_K \quad$ for some $K \in \mathbb{N}$

$\Rightarrow \quad M - \varepsilon < x_K \leq x_{K+1} \leq x_{K+2} \ldots\ldots \leq M$

$\Rightarrow \quad M - \varepsilon < x_n \leq M < M + \varepsilon \quad \forall \ n \geq K$

$\Rightarrow \quad |x_n - M| \leq \varepsilon \quad \forall \ n \geq K$

$\Rightarrow \quad M = \lim x_n = \text{Sup}\{x_n, n \in \mathbb{N}\}$

Similarly we can prove that $x_n \downarrow \ \inf\{x_n, n \in \mathbb{N}\}$

MCT :- If x_n is monotone

it is convergent iff it is bounded

$x_n \uparrow \qquad \lim x_n = \sup\{x_n, n \in \mathbb{N}\}$

$x_n \downarrow \qquad \lim x_n = \inf\{x_n, n \in \mathbb{N}\}$

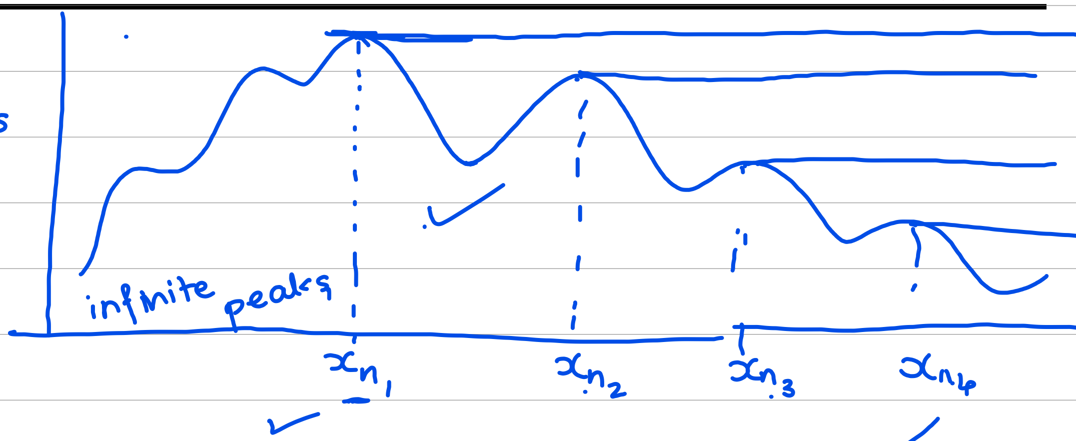
Monotone Subseq^n theo :-

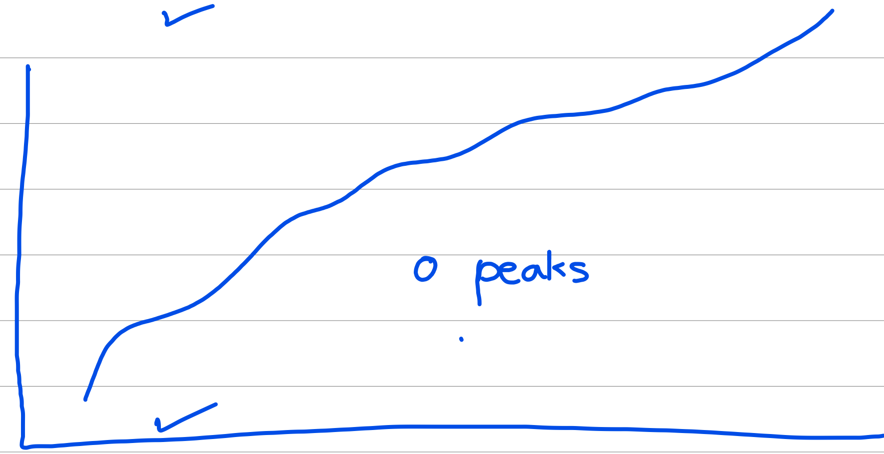
(Monotone Subsequence theorem). If xn is sequence of real numbers then there is subsequence of xn that is monotone.

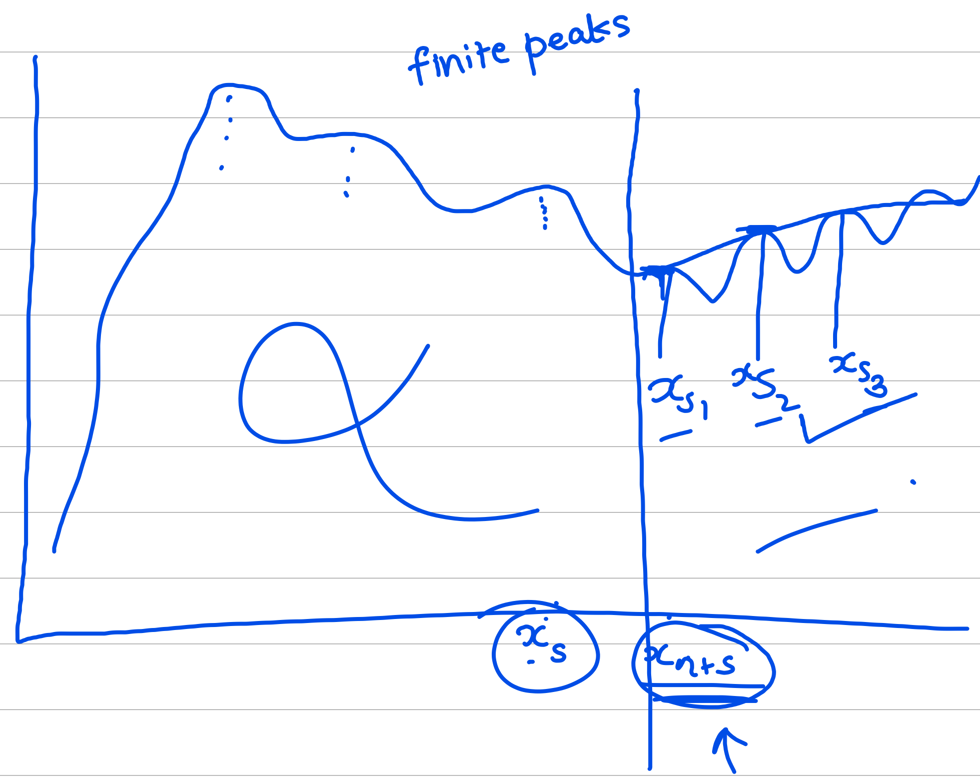
x_n is Real Seq^n

Peak

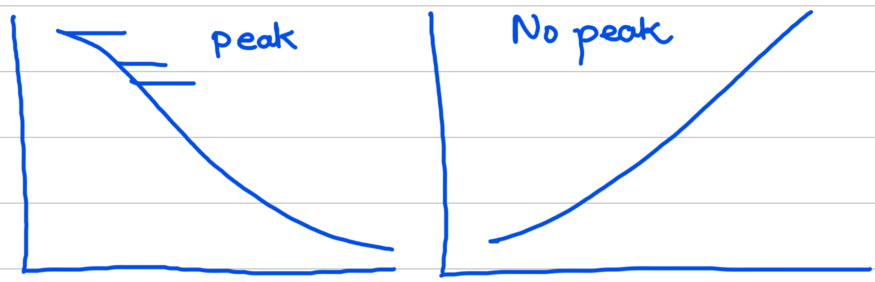
$x_n \geq x_{n+s} \ \forall s$



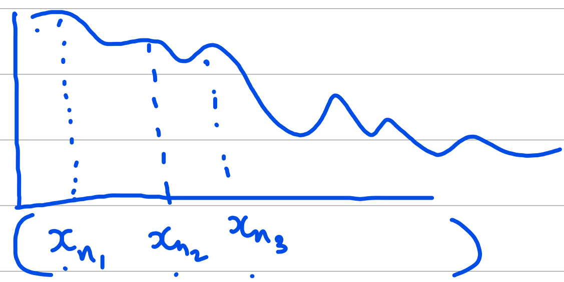


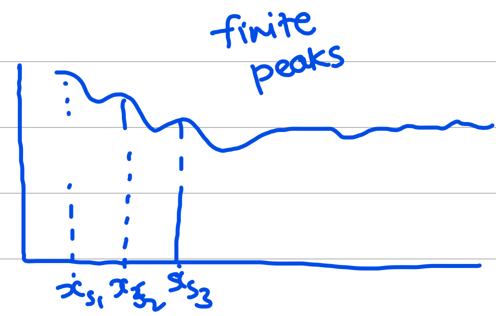


x_n



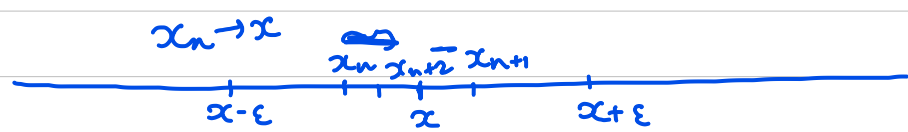
subseq^n





Cauchy seq^n

for any $\varepsilon > 0$, $\exists\ K(\varepsilon) \in \mathbb{N} \ni |x_n - x_m| < \varepsilon \quad \forall\ n, m \geq K(\varepsilon)$



$\frac{1}{n} \to 0$ $\quad \varepsilon = 0.2$

Every convergent seq^n is Cauchy seq^n.

let x_n be cvgt seq^n $\quad x_n \to x$.

limit of seq^n

For any $\varepsilon > 0$, $\exists\ K(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x| < \varepsilon/2 \ \forall\ n \geq K(\varepsilon)$

For $m, n \geq K(\varepsilon)$

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x - x_m| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

✓ For any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$ $\ni$ $|x_n - x_m| < \varepsilon$ $\forall n, m \geq K(\varepsilon)$

$\Rightarrow$ x_n is Cauchy

Cauchy Convergence Criterion

Cauchy $\Leftrightarrow$ Convergent

To prove Cauchy to Convergent

let x_n be Cauchy seq^n

✓ For any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$, $\ni$ $|x_n - x_m| < \varepsilon$ $\forall n, m \geq K(\varepsilon)$

To prove x_n is convergent.

$$\big||x_n| - |x_m|\big| \leq |x_n - x_m| \leq \varepsilon$$

$$M - \varepsilon \leq |x_m| - \varepsilon \leq |x_n| \leq |x_m| + \varepsilon \leq M + \varepsilon$$

$$|x_n - x_m| \leq \varepsilon \quad \forall n, m \geq K(\varepsilon)$$

Bounded x_n $x_{n_k} \to x$ $|x_{n_k} - x| < \varepsilon/2$

$\forall n, m \geq K_1(\varepsilon)$ $\forall n_k \geq K_2(\varepsilon)$

$|x_n - x_m| \leq \varepsilon/2$ B $|x_{n_k} - x| < \varepsilon/2$

$\forall n \geq K(\varepsilon)$

$= \max(K_1(\varepsilon), K_2(\varepsilon))$

$$\begin{aligned}
|x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\
&\leq |x_n - x_{n_k}| + |x_{n_k} - x| \\
&\leq \frac{\varepsilon}{2} + \varepsilon/2 \\
&< \varepsilon.
\end{aligned}$$

Cauchy Convergence Criterion

① Every convergent seq^n is Cauchy Seq^n (Use previous)

② Every cauchy seq^n is bounded. (- " -)

$\therefore$ by Bolzano Weierstrass theo for real seq^ns.

every bounded seq^n has convergent $subseq^n$.

let x_{n_k} be that $subseq^n$ and $x_{n_k} \to x$ (say)

To prove :- x_n also converges to x.

As $x_{n_k} \to x$

for any $\varepsilon > 0$, $\varepsilon/2 > 0$, $\exists K_1(\varepsilon) \in \mathbb{N}$ $\ni |x_{n_k} - x| < \varepsilon \ \forall n \geq K_1(\varepsilon)$

for Cauchy seq^n x_n

for any $\varepsilon > 0$, $\varepsilon/2 > 0$ $\exists K_2(\varepsilon) \in \mathbb{N}$, $\ni |x_n - x_m| < \varepsilon \ \forall \ n, m \geq K_2(\varepsilon)$

Now for $K(\varepsilon) = \max(K_1(\varepsilon), K_2(\varepsilon))$

for $n, m, n_k \geq K(\varepsilon)$

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x|$$
$$\leq |x_n - x_{n_k}| + |x_{n_k} - x|$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow x_n \to x$.

* Contractive Seq^n

$$|x_{n+2} - x_{n+1}| \leq C \cdot |x_{n+1} - x_n| \quad \forall n \qquad 0 < C < 1$$

let $n > m \geq K(\varepsilon)$ $\quad K(\varepsilon) \leq (m+1, m+2, \ldots, n)$

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} + \ldots + x_{m+1} - x_m|$$
$$\leq |x_n - x_{n-1}| + \underline{|x_{n-1} - x_{n-2}|} + \ldots + |x_{m+1} - x_m|$$

$$|x_{m+2} - x_{m+1}| \le c \cdot |x_{m+1} - x_m|$$
$$\le c^2 |x_m - x_{m-1}|$$
$$\le c^m |x_2 - x_1|$$
$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \dots + x_{m+1} - x_m|$$
$$\le [c^{n-2} + c^{n-3} \quad + c^{m-1}] |x_2 - x_1|$$
$$\le c^{m-1} [1 + c + \dots + c^{n-2-m+1}] |x_2 - x_1|$$
$$\le c^{m-1} \left[\frac{1 - c^{n-m}}{1 - c}\right] |x_2 - x_1|$$
$$\le \varepsilon \qquad \text{as } c^n \to 0$$

$0 < c < 1$

c^n

Bounded Seq^n $\to$ Cvgt
$\to$ oscillates finitely

Unbounded seq^n $\to$ diverges to $\to +\infty$
$\to -\infty$

$1 + (-1)^n$ oscillating $0, 2$

$x_{n,m} = m + \frac{1}{n}$ $m, n \in \mathbb{N}$. oscillate $\mathbb{N}$

$\lim \frac{2n-3}{n+1} = 2$ $\lim \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$

$\lim \sqrt[n]{n} = 1$

$$\lim \frac{1+2+\dots+n}{n^2} = \lim \frac{n(n+1)/2}{n^2} = \lim_{n\to} \frac{(n+1)}{2n} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1+3+5+\dots+(2n-1)}{n^2} = \lim_{n\to0} \frac{n^2}{n^2}$$

$$\sqrt{n+1} - \sqrt{n} \to 0$$

$$\lim_{n\to\infty} \frac{2n^2-5}{3n^2+7n} = \lim_{n\to\infty} \frac{2-5/n^2}{3+7/n} = \frac{2}{3}$$

$$x_{n+1} = 2 - \frac{1}{x_n} \qquad \lim x_{n+1} = ?$$

$$\cancel{x_1 = 3/2}, \quad x_2 = 4/3, \quad x_3 = 5/4, \dots \qquad \leq 2$$
$$> \qquad > \qquad >$$

$$\lim x_{n+1} = 2 - \frac{1}{\lim x_n}$$

$$x = 2 - \frac{1}{x}$$

$$x^2 = 2x - 1 \qquad \Rightarrow x^2 - 2x + 1 \Rightarrow 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

$*$ $a_n = \sqrt{a_{n-1}a_{n-2}} \qquad n > 2 \qquad a_n > 0 \ \forall n$

$$a_n \to (a_1 a_2^2)^{1/3} \qquad ?$$

$*$ $S_{n+1} = \sqrt{7+S_n}, \quad S_1 = \sqrt{7} \qquad ?$