

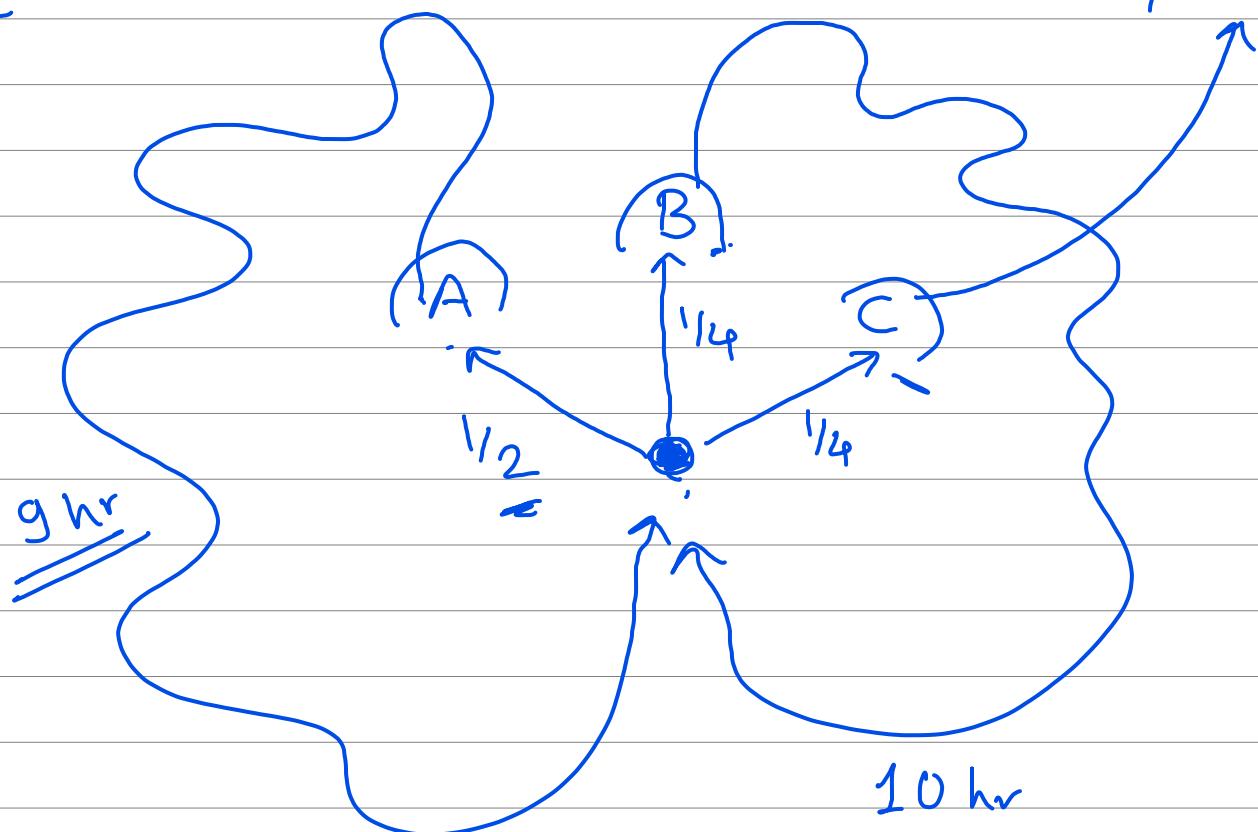
① Distributions Revision

Discrete

Rat

Rat Cave

Continuous
outside
7 hr



A $\frac{1}{2}$ \rightarrow 9 hr return

B $\frac{1}{4}$ \rightarrow 10 hr return

C $\frac{1}{4}$ \rightarrow 7 hr leave

$E(X) = ?$

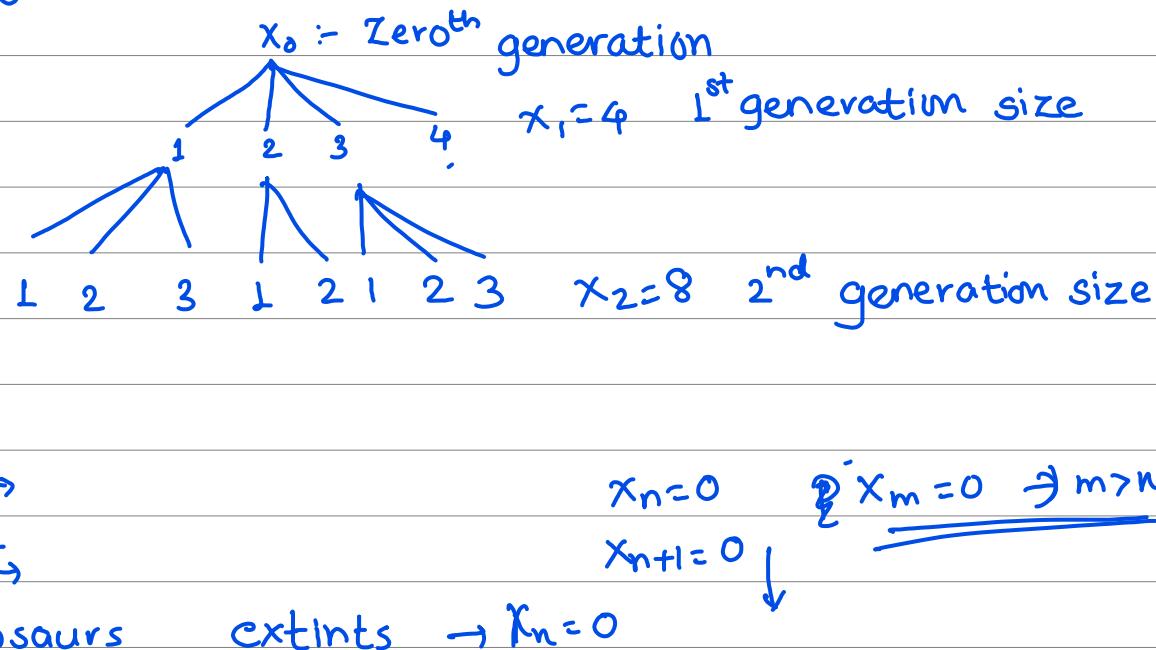
Assume

$$\begin{array}{l} A \frac{1}{2} \rightarrow \frac{g+\mu}{2} \\ B \frac{1}{4} \rightarrow \frac{10+\mu}{4} \\ C \frac{1}{4} \rightarrow 7 \end{array}$$

$$\mu = \frac{1}{2} \underline{(g+\mu)} + \frac{1}{4} \underline{(10+\mu)} + \underline{\mu}$$

=

Offspring Distribution:-



$$\underline{x_0 = 0} \quad x_1 = 3, \quad x_2 = 5$$

$$\lim_{n \rightarrow \infty} x_n = 0$$

Sum of expectations

$$y = \sum_{i=1}^n x_i \leftarrow \begin{array}{l} \text{fixed} \\ \text{Random} \end{array} \Rightarrow E(y) = n \cdot E(x)$$

$$= \sum_{i=1}^n E(x_i)$$

$$\text{Random sum of Random number } v(y) = \sum_{i=1}^n V(x_i) + \sum_{i,j} \sum_{i \neq j} Cov(x_i, x_j)$$

↳ compound distributions

$$Y \quad \textcircled{1} \quad N \sim \text{Bino} \rightarrow y \quad \text{Comp. Bino.-distribution} \quad (K, P) \quad E(y) Kp \cdot E(x)$$

$$Y \quad \textcircled{2} \quad \sim \text{Pois} \sim \text{Comp. Pois. distribution} \quad E(y) \lambda \cdot E(x)$$

likewise Comp. Geometric, Negative Binomial

$$\underline{E(Y)} = E\left(\sum_{i=1}^n x_i\right) = E_N\left(E_{Y|N}\left(\frac{\sum_{i=1}^n x_i}{N=n}\right)\right)$$

$$v(x_i) = v(t) \quad E(x_i) = \underline{u} \quad E(x_i) = E(x) = \underline{u}$$

$$E(x_i) = E(x) = \underline{u}$$

$$v(y) = v\left(\sum_{i=1}^n x_i\right) = E_N\left(\sum_{i=1}^n E(x_i)\right) = E_N(n \cdot \underline{E(x)}) = \underline{E(x)} \cdot \underline{E(N)}.$$

$$= E(v(\sum_{i=1}^n x_i)) + v(E(\sum_{i=1}^n x_i))$$

$$\checkmark = E_N(N v(x)) + v(N \cdot E(x)) \Rightarrow \underline{E(N)} v(x) + [\underline{E(x)}]^2 v(N)$$

$$\text{Comp. Bino.} \Rightarrow V(Y) = kp \cdot V(x) + [E(x)]^2 (kpq) \quad \Rightarrow \text{Comp. Poisson} \Rightarrow V(Y) = \lambda [V(x) + E(x)^2]$$

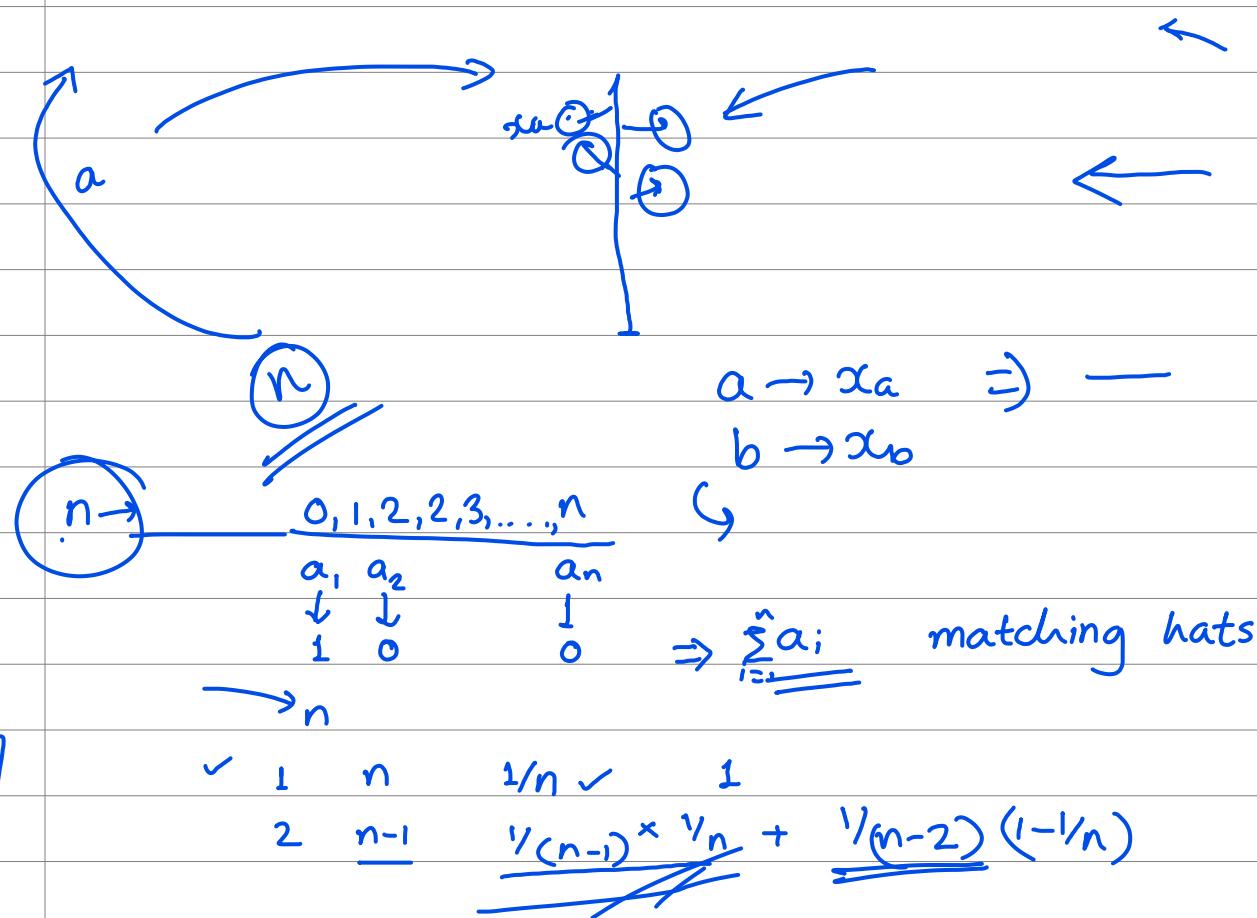
$$\Rightarrow \lambda \cdot E(x^2)$$

Manoj C Patil

$$V(x) = E(x^2) - [E(x)]^2$$

Lecture:

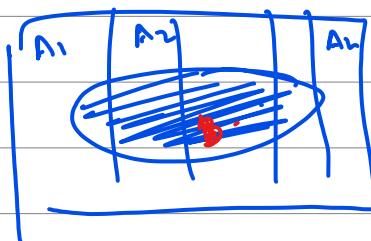
Matching hat problem



Bayes' Rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A_i|B) = \frac{P(B|A_i)}{\sum_{i=1}^n P(B|A_i)}$$



$$\sum A_i$$

$$\cup A_i = \Omega$$

mutually exclusive & exhaustive

e.g. Family 2 children

$$X_i = 0, \text{ if } i \text{ is female} \quad \frac{1}{2}$$

$$= 1 \text{ if } i \text{ is male} \quad \frac{1}{2}$$

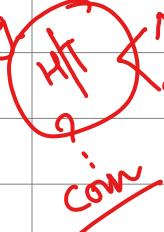
$$P\left[\sum_{i=1}^2 X_i = 2 / \sum_{i=1}^2 X_i \geq 1\right] = ?$$

$$\text{E } \neg = \{ BB, BG, GB, GG \}$$

$$\text{B } A = \{ BB, BG, GB \} \checkmark$$

$$B = \{ BB \} \checkmark$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

e.g.  Maths $\rightarrow A \rightarrow \frac{1}{2}$
 Stats $\rightarrow A \rightarrow \frac{1}{3}$

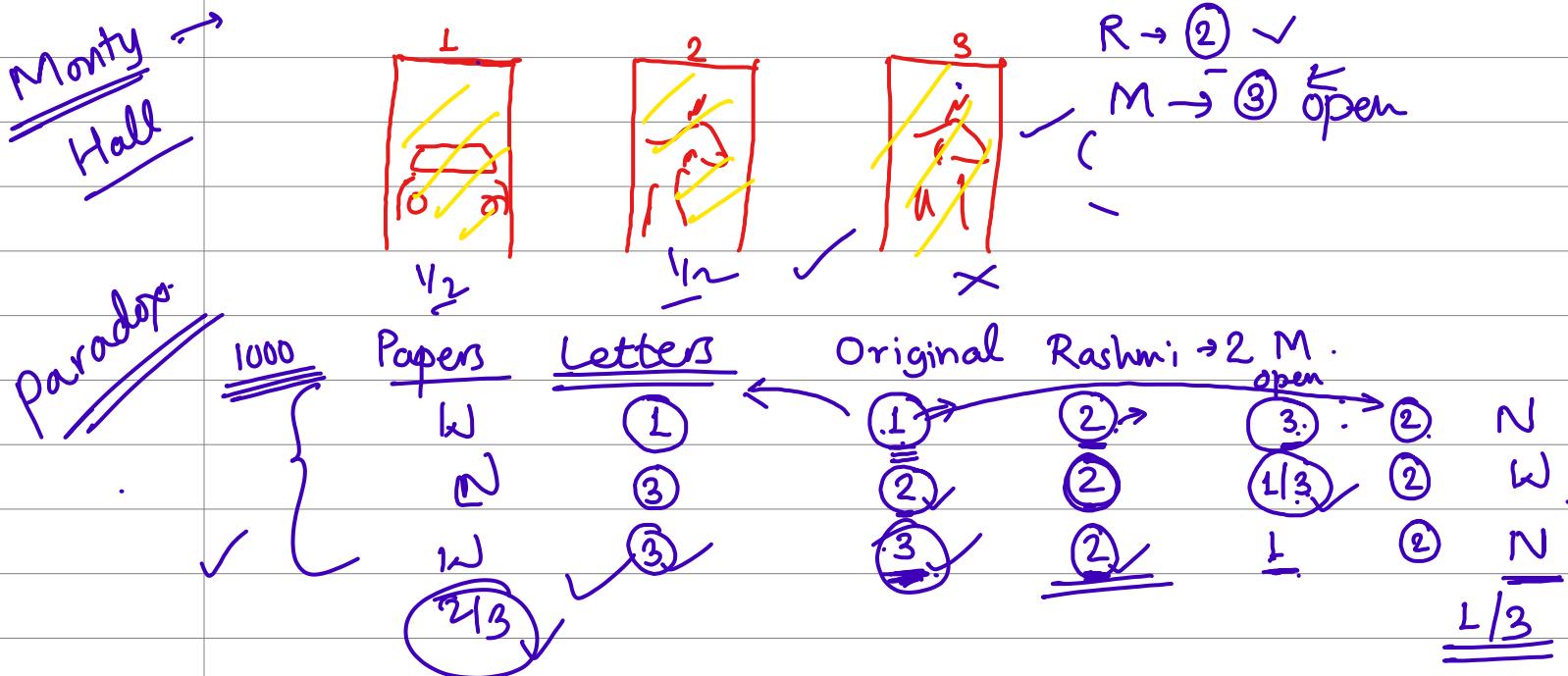
$$P[A/\text{Stats}] = \frac{1}{3} \quad P[A/\text{Maths}] = \frac{1}{2}$$

$$P[\text{Stats}] = \frac{1}{2} \quad P[\text{Maths}] = \frac{1}{2}$$

$$P[\text{Stats}/A] = \frac{P[\text{Stats}, A]}{P[\text{Stats}, A] + P[\text{Maths}, A]}$$

$$P[\text{Stats}/A] = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{2}} = \frac{2}{5}$$

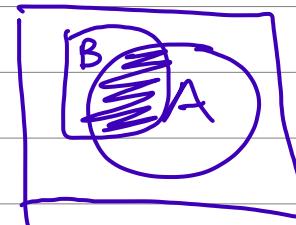
Monty Hall Problem ? Paradox ? Show 80



$$E(X|Y)$$

$$= V(X|Y) \cdot E(X) + E(X|Y) \cdot V(X)$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$



X, Y ind //

x/y	1	2	3
prob	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

x	1	2	3
1			
2			
3			

$$E(X/S)$$

$$\underline{S=3} \rightarrow (x=1) \frac{1}{4}, (x=2) \frac{1}{2}, (x=3) \frac{1}{4}$$

x	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
y	1	2	3
1	<u><u>$\frac{1}{4}$</u></u>	<u><u>$\frac{1}{2}$</u></u>	<u><u>$\frac{1}{4}$</u></u>
2	<u><u>$\frac{1}{2}$</u></u>	<u><u>$\frac{1}{4}$</u></u>	<u><u>$\frac{1}{4}$</u></u>
3	<u><u>$\frac{1}{4}$</u></u>	<u><u>$\frac{1}{4}$</u></u>	<u><u>$\frac{1}{4}$</u></u>

$$P(S) = \frac{2}{16} = \frac{3}{16} - \frac{\frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4}}{\frac{1}{16} + \frac{1}{4} + \frac{1}{16}}$$

$$= \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = \frac{2}{16} + \frac{1}{4} = \frac{3}{8}$$

$$S = \frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{1}{16}$$

$$x = 1, 2, 3, 4, 5$$

$$x=1 \quad 1 \quad \frac{1}{2} \quad \parallel \quad \frac{1}{6}$$

$$x=2 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{2}{3}$$

$$\frac{P(X=1, S=3)}{P(S=3)} = \frac{P(X=1, Y=2)}{P(X=1) P(Y=2)} = \frac{1/4}{1/2} = 1/2$$

$$= \frac{1/4 \times 1/2}{3/8} \approx 1/2$$

$$\cancel{\frac{P(X=2, S=3)}{P(S=3)}}$$

$$\frac{P(X=1, S=4)}{P(S=4)} = \frac{1/4 \times 1/4}{3/8} = \frac{3/8}{3/8} = \frac{1/16}{3/8}$$

$$\frac{P(X=1, S=4)}{P(S=4)} = \frac{P(X=1), P(Y=3)}{3/8} = \frac{\frac{1}{4} \times \frac{1}{4} - 2}{3/8} = \frac{\frac{1}{3} \times \frac{1}{2}}{3/8} = \frac{1}{6}$$

$$\frac{P(X=2, S=4)}{P(S=4)} = \frac{P(X=2) \cdot P(Y=2)}{P(S=4)} = \frac{\frac{1}{2} \times \frac{1}{2}}{3/8} = \frac{\frac{1}{4} \times 1}{3/8} = \frac{2}{3}$$

$$\frac{1}{6} + \frac{2}{3} = \frac{1+4}{6} = \frac{5}{6} \rightarrow P(X=3|S=4)^{1/6}$$

$$E(X|S=2) = 1$$

$$E(X|S=3) = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5$$

$$E(X|S=4) = \underline{\quad}$$

↗ $E(X) = E[\cancel{E[X|Y]}] = E_Y [E_{X|Y}(X|Y)]$

MGF of compound distributions $Y = \sum_{i=1}^N X_i$

$$M_Y(t) = E(e^{tY}) = E_N(E_{Y|N}(e^{tY}|N=n))$$

$$= E_N(E(e^{t \sum_{i=1}^N X_i}|N=n))$$

X_i = i.i.d.

$$= E_N(\prod_{i=1}^N \underline{E(e^{tX_i})})$$

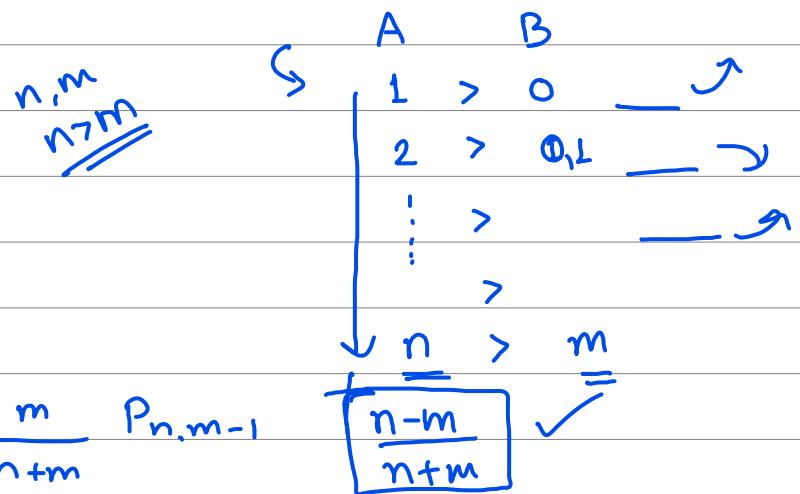
$$= E_N(\underline{[E(e^{tX})]^N})$$

$E(e^{tx})$
 $E(x)$
 $e^{\log t}$

$$M_Y(t) = E_N([M_X(t)]^N) = P_N(M_X(t))$$

$$= E_N(e^{N \cdot \log M_X(t)}) = M_N(\log M_X(t))$$

Ballot Problem



$$P_{n,m} = \frac{n}{n+m} P_{n-1,m} + \frac{m}{n+m} P_{n,m-1}$$

$$= \frac{n}{n+m} \cdot \frac{(n-1)-m}{(n-1)+m} + \frac{m}{n+m} \cdot \frac{(n)-(m-1)}{n+m-1}$$

$$= \frac{n^2 - n - mn + mn - m^2 + m}{(n+m)(m+n-1)} = \frac{n^2 - n - m^2 + m}{(n+m)(m+n-1)}$$

$$= \frac{(n^2 - m^2) - (n-m)}{(n+m)(n+m-1)} = \frac{(n-m)(n+m) - (n-m)}{(n+m)(n+m-1)}$$

$$= \frac{(n-m)}{(n+m)} \frac{(n+m)}{(n+m-1)}$$

Exercise 1.8 $X_1 \sim \text{Poi}(\lambda_1)$ $X_2 \sim \text{Poi}(\lambda_2)$ $X_1 + X_2 \sim ?$

$$Z = X_1 + X_2$$

$$M_Z(t) = M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \\ = \frac{(e^{\lambda_1+e^{\lambda_2}})(e^t-1)}{e^t}$$

$$Z \sim \text{Poi}(\lambda_1 + \lambda_2)$$

Conditional distribution of $X_1 / X_1 + X_2$

$$P(X_1=x / X_1 + X_2) = \frac{P(X_1=x, X_1 + X_2 = z)}{P(X_1 + X_2 = z)} = \frac{P(X_1=x, X_2=z-x)}{P(z)}$$

$$= \frac{P(X_1=x) \cdot P(X_2=z-x)}{P(z)}$$

$$= \frac{e^{-\lambda_1} \frac{\lambda_1^x}{x!}}{x!} \cdot \frac{e^{-\lambda_2} \frac{\lambda_2^{z-x}}{(z-x)!}}{(z-x)!} \Bigg/ \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^z}{z!}$$

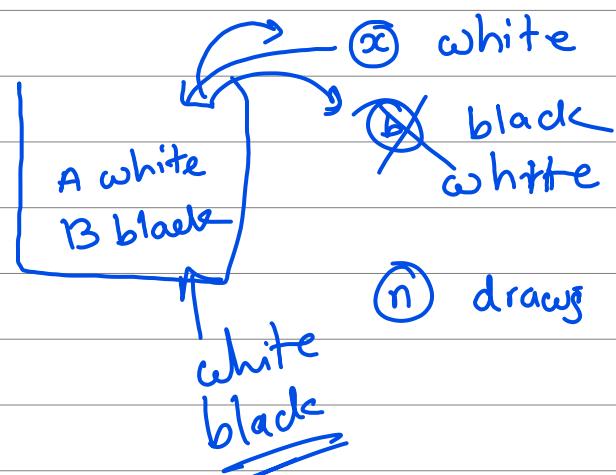
$$= \frac{z!}{x!(z-x)!} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^x \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{z-x}$$

$$= Z_{Cx}$$

$$X_1 / X_1 + X_2 \sim \text{Bino} \left(z, \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)$$

1.19 exercise

Urn

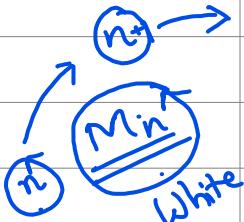


$$a \quad b \quad M_0 = a \quad \rightarrow$$

$$\begin{aligned} 1. \quad & \cancel{WB} \quad \cancel{W} \uparrow \quad \cancel{B} \uparrow \\ & M_1 = \underline{\underline{a}} \times \underline{\underline{\frac{a}{a+b}}} + (a+1) \underline{\underline{\frac{b}{a+b}}} = \frac{a(a+b)}{(a+b)} + \frac{b}{(a+b)} \\ & M \end{aligned}$$

 $M_0 = a$ white ball

$$M_1 = \underline{\underline{a}} \left(\frac{a}{a+b} \right) + (a+1) \left(\frac{b}{a+b} \right) \quad \rightarrow \underline{\underline{\bar{z} + \gamma}}$$



$$\begin{aligned} M_{n+1} &= \underline{\underline{M_n}} \left(\frac{?}{a+b} \right) + (M_n + 1) \left(\frac{(a+b) - ?}{a+b} \right) \\ &= M_n \cdot \left(\frac{M_n}{a+b} \right) + (M_n + 1) \left(\frac{(a+b) - M_n}{a+b} \right) \quad \leftarrow 1 - \frac{M_n}{a+b} \\ &= \cancel{\frac{M_n^2}{(a+b)}} + (M_n + 1) - \cancel{\frac{M_n^2}{(a+b)}} - \frac{M_n}{(a+b)} \end{aligned}$$

$$\textcircled{1} \quad \underline{M_{n+1}} = 1 + M_n \left(1 - \frac{1}{a+b} \right) \quad \textcircled{2} \quad M_0 = a$$

$$M_n = a + b - b \left(1 - \frac{1}{a+b} \right)^n$$

$$M_1 = 1 + a \left(1 - \frac{1}{a+b} \right) = 1 + ax$$

$$M_2 = 1 + (1+ax) \cdot x = 1 + x + ax^2$$

$$\begin{aligned} M_3 &= 1 + (1+x+ax^2)x = 1 + x + x^2 + ax^3 \\ &= \underline{1+x+x^2+x^3} + (a-1)x^3 \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} \\ \sum_{n=0}^j x^n &= \frac{1-x^{j+1}}{1-x} \\ M_n &= \frac{(1-x^{n+1})}{1-x} + (a-1)x^n \\ x &= 1 - \frac{1}{a+b} \Rightarrow \frac{1}{1-x} = (a+b) \end{aligned}$$

$$\begin{aligned} M_n &= (a+b) - \frac{x^{n+1}}{1-x} + (a-1)x^n \\ &= (a+b) - (a+b)x^{n+1} + (a-1)x^n \\ &= (a+b) - x^n [(a+b) \cdot x - a+1] \\ &= (a+b) - \left(1 - \frac{1}{a+b} \right)^n \left[(a+b) \left(1 - \frac{1}{a+b} \right) - a+1 \right] \end{aligned}$$



Sheldon Ross

$$= (a+b) - b \left(1 - \frac{1}{a+b} \right)^n$$

 $(a+b) - x - a + x$

J. Medhi ↴

Stochastic Process

Seqⁿ of Random Variable

✓ X_n = fortune of gambler after n^{th} game

$$X_n = X_{n-1} + Z_n$$

$$Z_n = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$$

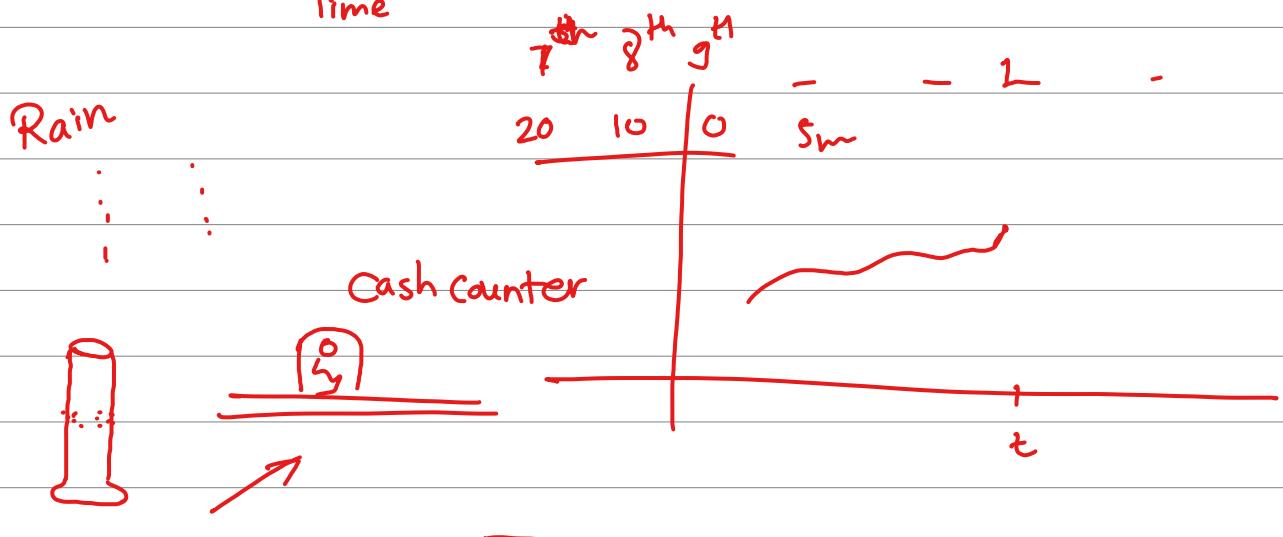
100
99
101

$x_0 \ x_1 \ x_2 \dots$

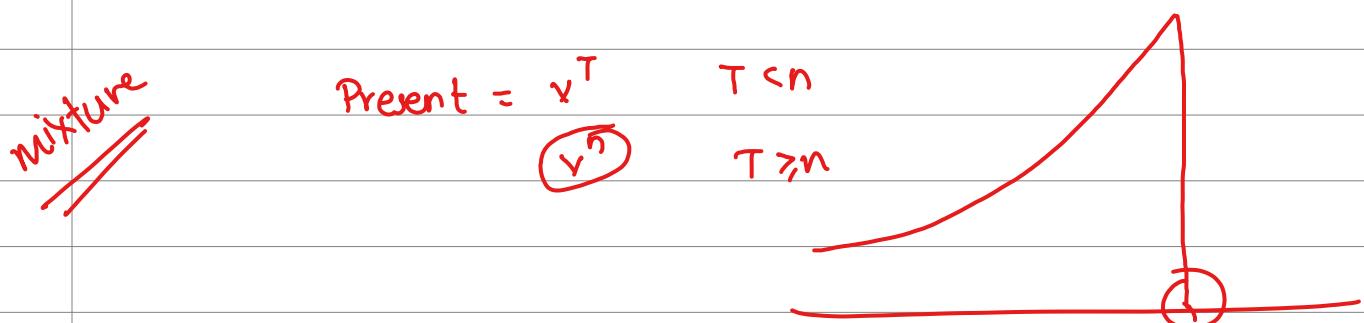
$x_n \dots$

Discrete Seqⁿ of Random Variables
 $\{x_i\}_{i=1}^{\infty}$

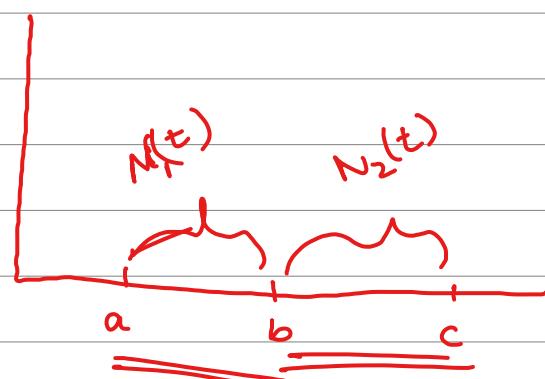
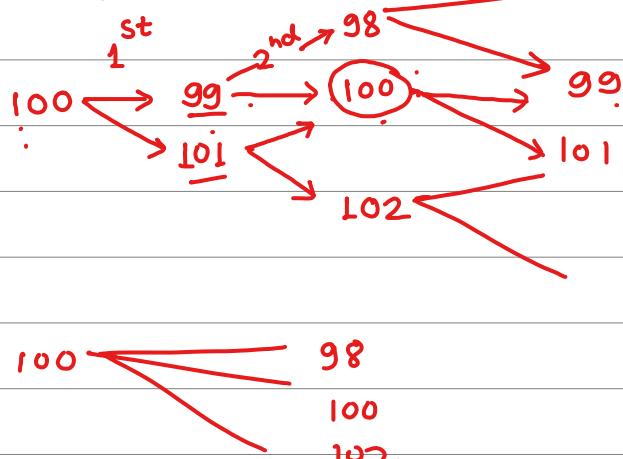
9.00 → 9.12, 9.12 → 9.15 9.30 3.30



	Statespace	Time domain	Example
1	Discrete	Discrete	* No. of covid patients on n^{th} day * No. of customers on a shop - ..
2	Discrete	Conti	* Pop ⁿ size \rightarrow continuous time discrete pop * No. of accidents up to time t
3	Continuous	Discrete	* Milk by cow on n^{th} day * Min ^m /Max ^m temp on n^{th} day
4	Continuous	Contin.	* Fever - body temp on t time * Blood conc up to time t
5	Mixture	Discrete Cont	* Speed of internet * Rainfall up to time t



Independent Increments



disjoint time interval
 $N_1(t)$ & $N_2(t)$
distⁿ indep

independent increments

$$\begin{matrix} t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n \\ \rightarrow \underline{x_{t_0}}, \underline{x_{t_1}}, \underline{x_{t_2}}, \dots, \underline{x_{t_{n-1}}}, \underline{x_{t_n}} \end{matrix}$$

$x_{t_1} - x_{t_0}, x_{t_2} - x_{t_1}, \dots, x_{t_n} - x_{t_{n-1}}$ indep. inc.

12-2

$$\frac{\underline{12-1}}{12.30 - 1.30}$$



$$\begin{matrix} \text{Joint same} & x_{t_0+u} & x_{t_1+u} & x_{t_2+u} \\ \downarrow & t_0 & t_1 & t_2 \\ \text{Joint} & (x_{t_0}, x_{t_1}, x_{t_2}, & & \\ & x_{t_1}-x_{t_0} & x_{t_2}-x_{t_1}, & \\ & & & x_{t_{n-1}}, x_{t_n}) \\ & & & x_{t_n}-x_{t_{n-1}} \end{matrix}$$

$$\begin{matrix} \text{independent} & & \text{Stationarity} & \text{Strictly} \\ \text{increment } \underline{(s,t)} & \frac{x(t) - x(s)}{s < t} & \xrightarrow{s \geq 0} & \text{No. of students} \\ \checkmark \text{interval length } (t-s) & & & \xrightarrow{12-2} \\ & (s+u, t+u] & \frac{x(t+u) - x(s+u)}{s+u < t+u} & \xrightarrow{7-9} \\ \checkmark \text{interval length } (t-s) & & & \end{matrix}$$

x_{t_0}

$$\text{weak stationary} \quad \text{if} \quad \left\{ \begin{array}{l} E(x(t)) = \underline{\mu} \quad \text{constant } \forall t \in I \\ \text{Cov}(x(t), x(s)) = \text{fun}(t-s) \end{array} \right.$$



Find the p.g.f. of the sum $S_n = X_1 + \dots + X_n$ of n independent and identical zero-truncated Poisson variates. Find $E(S_n)$ and $\Pr\{S_n = m\}$, $m = n, n+1, n+2\dots$

$X_i \sim \text{Zero-truncated poisson}$

$$P[X=x] = \frac{e^{-\lambda} \lambda^x}{x!} / (1-e^{-\lambda}) \rightarrow E(X)$$

$$S_n = \sum_{i=1}^n X_i \geq E(S_n) = n \cdot E(X) = n \cdot \frac{\lambda}{1-e^{-\lambda}}$$

$$P_s(t) = E(t^S) = E(t^{\sum_{i=1}^n X_i}) = [P_X(t)]^n$$

$$\begin{aligned} P_X(t) &= \frac{1}{1-e^{-\lambda}} e^{-\lambda} \sum_{x=1}^{\infty} \frac{(\lambda t)^x}{x!} \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left[e^{\lambda t} \left(\sum_{x=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^x}{x!} \right) - e^{-\lambda t} \right] \end{aligned}$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} [e^{\lambda t} - e^{-\lambda t}]$$

Markov Chains:-

discrete statespace & discrete time domain stochastic process following Markov property.

Markov Property :-

Future depends on present not on past

Notation

$$P_{ij}^{(n)} = P[X_n=j / X_0=i]$$



Starting from state i
process reaches to
state j in n -steps

Stationary \rightarrow Time-homogeneous process

Non-stationary

$P_{ij}^{(m,n)}$ = Starting from i process reaches to state j from m to n steps (No. of steps - $n-m$)

$\checkmark P_{ij}^{(m,n+m)} = P[X_{n+m}=j / X_m=i]$

statespace = $S = \{1, 0\} \Rightarrow 1$ if it rains today 0 o.w.

0 0
1 2 3

1 1
8 9

1
15

0
30

$\nearrow P_{10}^{(15)}$
 $i j$

$\checkmark P_{10}^{(15,30)}$

$\cancel{\text{if } P_{ij}}$ $P[X_{n+1}=j / X_n=i] = P_{ij}^+ = P_{ij}$

X_{n+1}

$P[X_{n+1}=j / X_n=i, X_{n-1}=i_{n-1}, \dots, X_1=i_1, X_0=i_0]$

$P[X_{30}=1 / \underline{X_{29}=1}, \underline{X_{28}=1}, \dots, \underline{X_0=0}] = P[X_{30}=1 / \underline{X_{29}=1}]$

$P_{ij} = P[X_{n+1}=j | X_n=i, X_{n-1}=i_{n-1}, \dots, X_0=i_0] = P[X_{n+1}=j | X_n=i]$

↳ $\{X_n, n \geq 0\}$ as Markov chain.

one-step transition prob. matrix

0	1	← future state
Current state	0	$\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$
1		

$P_{00} \Rightarrow P[X_{n+1}=0 | X_n=0]$

$P_{10} \Rightarrow P[X_{n+1}=0 | X_n=1]$

starting ending

$P^{(2)}$ = Two step transition prob. matrix

↳

n-step

$S = \{0, 1\} \rightarrow S = \{1, 2, \dots, K\}$

$P^{(2)}$

current state

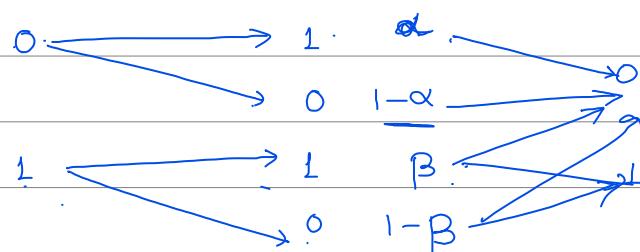
Future

0	$P_{00}^{(2)}$	$P_{01}^{(2)}$
1	$P_{10}^{(2)}$	$P_{11}^{(2)}$

$P^{(n)}$

$S = \{0, 1\} P^{(n)} = \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} \\ P_{10}^{(n)} & P_{11}^{(n)} \end{bmatrix}$

Yesterday Today Tomorrow

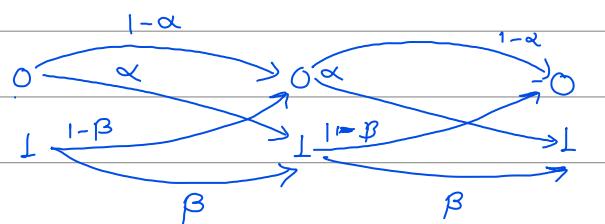


one-step

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix}$$

Yesterday Today Tomorrow

Two-step

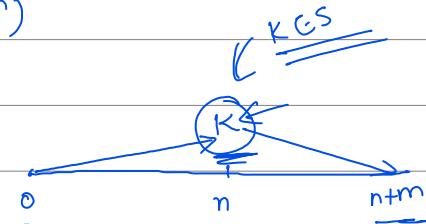


$$\begin{array}{c}
 \text{Today} \\
 \begin{matrix} 0 & 1 \\ 0 & P_{00}^{(2)} & P_{01}^{(2)} \\ 1 & P_{10}^{(2)} & P_{11}^{(2)} \end{matrix}
 \end{array}
 = P_{00} \cdot P_{00} + P_{01} P_{10} \\
 = (1-\alpha)^2 + \alpha(1-\beta)$$

n-step transition probabilities

Chapman-Kolmogorov's - (CK-equation)

$$P_{ij}^{(n+m)} = \sum_{k \in S} P_{ik}^{(n)} P_{kj}^{(m)}$$



$\{X_n, n \geq 0\}$ is Markov chain?

$$\begin{aligned}
 P_{ij}^{(n+m)} &= P[X_{n+m}=j / X_0=i] \\
 &= \sum_{k \in S} \underbrace{P[X_{n+m}=j / X_m=k]}_{\text{future}} \cdot \underbrace{P[X_m=k / X_0=i]}_{\text{present past}} \\
 &= \sum_{k \in S} \underbrace{P[X_{n+m}=j / X_m=k]}_{\downarrow} \cdot \underbrace{P[X_m=k / X_0=i]}_{\text{stationary processes}}
 \end{aligned}$$

$$= \sum_{k \in S} \underbrace{P[X_m=j / X_0=k]}_{\text{past}} \cdot \underbrace{P[X_m=k / X_0=i]}_{\text{present}}$$

$$= \sum_{k \in S} P_{ik}^{(n)} \cdot P_{kj}^{(m)}$$

if we have used ~~it~~ Matrix form

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

if $n=m=1$,

$$P^{(1+1)} = P^{(2)} = P^{(1)} \cdot P^{(1)}$$

$$= \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix}$$

$$= \begin{bmatrix} (1-\alpha)^2 + \alpha(1-\beta) & \alpha(1-\alpha) + \beta\alpha \\ (1-\alpha)(1-\beta) + \beta(1-\beta) & (1-\alpha)\beta + \beta^2 \end{bmatrix}$$

$\{x_n, n \geq 0\}$ denotes whether it rains on n^{th} day or not
 $x_n = \begin{cases} 0 & \text{No rain} \\ 1 & \text{Rain} \end{cases}$

onestep transition
 $x_{n+1} = \begin{cases} 0 & \text{Monday} \\ 1 & \text{Tuesday} \\ 2 & \text{Wednesday} \\ 3 & \text{Thursday} \\ 4 & \text{Friday} \\ 5 & \text{Saturday} \end{cases}$

Monday $\Rightarrow 0$
 $\begin{array}{c|cc} & \text{Monday} & \dots \\ \hline & 1 & \dots \\ \text{Tuesday} & \vdots & \ddots \\ & 5 & \end{array}$

fivestep $P^{(5)}$

$$P[x_5 = 1 | x_0 = 1]$$

$$\begin{aligned} &= P^{(5)} = P^{(4)} \cdot P \\ &= P^{(3)} \cdot P^{(2)} \\ &= P^{(3)} \cdot \left[\begin{array}{cc} 2/3 & 1/3 \\ 1/2 & 1/2 \end{array} \right] \left[\begin{array}{cc} 2/3 & 1/3 \\ 1/2 & 1/2 \end{array} \right] \end{aligned}$$

$$\begin{aligned} &= P^{(3)} \cdot \frac{1}{6} \left[\begin{array}{cc} 4 & 2 \\ 3 & 3 \end{array} \right] \cdot \frac{1}{6} \left[\begin{array}{cc} 4 & 2 \\ 3 & 3 \end{array} \right] \\ &= \frac{1}{6^2} P^{(3)} \left[\begin{array}{cc} 22 & 14 \\ 21 & 15 \end{array} \right] \end{aligned}$$

$$= \frac{1}{6^2} P \cdot P^{(2)} \cdot \left[\begin{array}{cc} 22 & 14 \\ 21 & 15 \end{array} \right]$$

$$\begin{aligned} &= \frac{1}{6^4} P \cdot \left[\begin{array}{cc} 22 & 14 \\ 21 & 15 \end{array} \right] \left[\begin{array}{cc} 22 & 14 \\ 21 & 15 \end{array} \right] \\ &= \frac{1}{6^4} P \left[\begin{array}{cc} 778 & 518 \\ 777 & 519 \end{array} \right] \xrightarrow{\text{Row sum}} \begin{array}{l} = 1 \\ \hline 1296 \end{array} \end{aligned}$$

$$= \frac{1}{6^4} \left[\begin{array}{cc} 778 & 518 \\ 777 & 519 \end{array} \right] \times \frac{1}{6} \left[\begin{array}{cc} 4 & 2 \\ 3 & 3 \end{array} \right]$$

$$= \frac{1}{6^5} \left[\begin{array}{cc} 4666 & 3110 \\ 4665 & 3111 \end{array} \right]$$

5-step
transition
prob-matrix

$$P[x_5 = 1 | x_0 = 1] = \frac{3111}{6^5} = \underline{0.4}$$

MC $\{X_n, n \geq 0\}$ Markov chain with statespace $S = \{0, 1, 2\}$

One-step TPM

$$P = \begin{bmatrix} 0 & 1/2 & 1/3 & 1/6 \\ 1 & 0 & 1/3 & 2/3 \\ 2 & 1/2 & 0 & 1/2 \end{bmatrix}$$

Initial prob:

$$\textcircled{1} \quad P[X_0=0] = P[X_0=1] = 1/2 \quad ?$$

$$\textcircled{2} \quad X_1=?$$

$$P[X_3=0]$$

$$P[X_3=1]$$

$$P[X_3=2]$$

$$E[X_3] = ?$$

$$P[X_1=1] = ? = P[X_1=1/X_0=0] \cdot P[X_0=0] + \left\{ \begin{array}{l} P[X_1=1/X_0=1] \cdot P[X_0=1] + \\ P[X_1=1/X_0=2] \cdot P[X_0=2] \end{array} \right\} \begin{array}{l} 1/3 \times 1/2 + \\ 1/3 \times 1/2 + \\ 1/2 \times 0 \end{array}$$

$$= \sum_k P[X_1=1/X_0=k] \cdot \underline{P[X_0=k]}$$

$$= \frac{1}{3}$$

$$P[X_1=?] = \underline{\alpha \cdot P}$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

$$P[X_1=?] = \frac{1}{2} [3 \ 4 \ 5] \cdot \underbrace{\frac{P[X_1=0]}{P[X_1=1]} \frac{P[X_1=2]}{4/12}}_{= 3/12} \underbrace{P[X_1=2]}_{= 5/12}$$

$$P[X_3=?] \Rightarrow [P(X_3=0) \ P(X_3=1) \ P(X_3=2)] = \underline{\alpha \cdot P^3}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} \cdot P^2 = \frac{1}{12} \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} P^2$$

$$= \frac{1}{12} \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} P$$

$$\frac{9}{15} \quad \frac{6}{8} \quad \cancel{3+16+15}$$

$$= \frac{1}{72} \begin{bmatrix} 24 & 14 & 34 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

$$+ \frac{24}{56}$$

X_3

$$= \frac{1}{432} \begin{bmatrix} 174 & 76 & 182 \end{bmatrix} \checkmark$$

$$E(X_3) = \frac{1}{432} [0 \times 174 + 1 \times 76 + 2 \times 182]$$

$$= \frac{1}{432} [76 + 364] = \frac{1}{432} [440] = 1.0885$$

Let the TPM of two state markov chain is $P = \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix}$

Show that, by mathematical induction

$$P^{(n)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2P-1)^n & \frac{1}{2} - \frac{1}{2}(2P-1)^n \\ \frac{1}{2} - \frac{1}{2}(2P-1)^n & \frac{1}{2} + \frac{1}{2}(2P-1)^n \end{bmatrix}$$

Put $n=1$.

$$P^{(1)} = P = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2P-1)^1 & \frac{1}{2} - \frac{1}{2}(2P-1)^1 \\ \frac{1}{2} - \frac{1}{2}(2P-1)^1 & \frac{1}{2} + \frac{1}{2}(2P-1)^1 \end{bmatrix} = \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix}$$

Assume that it is true for $n=k$

$$P^{(k)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2P-1)^k & \frac{1}{2} - \frac{1}{2}(2P-1)^k \\ \frac{1}{2} - \frac{1}{2}(2P-1)^k & \frac{1}{2} + \frac{1}{2}(2P-1)^k \end{bmatrix}$$

$$\begin{aligned} P^{(k+1)} &= P^{(k)} \cdot P = \left[\begin{array}{cc} & \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix} \\ = & \begin{bmatrix} P(\frac{1}{2} + \frac{1}{2}(2P-1)^k) + (1-P)(\frac{1}{2} - \frac{1}{2}(2P-1)^k) & (1-P)(\frac{1}{2} + \frac{1}{2}(2P-1)^k) + P(\frac{1}{2} - \frac{1}{2}(2P-1)^k) \\ P(\frac{1}{2} - \frac{1}{2}(2P-1)^k) + (1-P)(\frac{1}{2} + \frac{1}{2}(2P-1)^k) & (1-P)(\frac{1}{2} - \frac{1}{2}(2P-1)^k) + P(\frac{1}{2} + \frac{1}{2}(2P-1)^k) \end{array} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2P-1)^{k+1} & \frac{1}{2} - \frac{1}{2}(2P-1)^{k+1} \\ \frac{1}{2} - \frac{1}{2}(2P-1)^{k+1} & \frac{1}{2} + \frac{1}{2}(2P-1)^{k+1} \end{bmatrix}$$

$$S = \{1, 2\} \quad P = \frac{1}{2} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad P_{12}^{(3)-} = ?$$

$$\begin{aligned} P^{(3)} &= P^{(2)} \cdot P = \frac{1}{3^2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} P \\ &= \frac{1}{3^2} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{3^3} \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} \end{aligned}$$

$$(P^{(3)})_{12} = \frac{13}{27} = 0.48$$

$$P = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 7 & 3 \\ 7 & 3 \\ 5 & 5 \end{bmatrix}, P_{11}^{(4)} = ?$$

$$P^{(2)} = \frac{1}{10^2} \begin{bmatrix} 7 & 3 \\ 7 & 3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 7 & 3 \\ 5 & 5 \end{bmatrix} = \frac{1}{10^2} \begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}$$

$$\begin{aligned} P^{(4)} &= P^{(2)} \cdot P^{(2)} = \frac{1}{10^4} \begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix} \begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix} \\ &= \frac{1}{10^4} \begin{bmatrix} 6256 & 3744 \\ 6240 & \boxed{3760} \end{bmatrix}. \end{aligned}$$

$$P_{11}^{(4)} = \frac{\boxed{3760}}{10^4}$$

$$P = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \stackrel{P^4}{=} \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_1$$

$$\frac{1}{10^2} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 7 & 3 \\ 5 & 5 \end{bmatrix} P^2 \begin{bmatrix} 7 & 3 \\ 7 & 3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_1$$

$$\Rightarrow \frac{1}{10^2} \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} P^2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{5}{10^4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 7 & 3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{5}{10^4} [12 \ 8] \begin{bmatrix} 36 \\ 40 \end{bmatrix}$$

$$= \frac{5}{10^4} [752] = \frac{3760}{10^4}$$

$$P = \frac{1}{6} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \times \frac{1}{6}$$

$P_{2,3}^{(3)}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= [3 \ 2 \ 2] P \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6^3} [3 \ 2 \ 2] \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6^3} [14 \ 17 \ 11] \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6^3} [76] = \frac{76}{216}$$

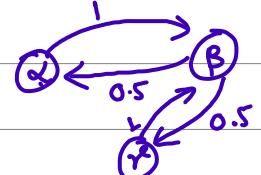
$C(\omega) = \{\alpha, \beta, \gamma\}$ Irreducible

$\alpha \neq \beta \neq \gamma$

$$\textcircled{a} \quad [0.25 \ 0.25 \ 0.5]$$

$$\xrightarrow{x_0} = \frac{1}{4} [1 \ 1 \ 2]$$

$$\alpha \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$



$$\textcircled{b} \quad P[X_4 = \gamma] = ? \Rightarrow \frac{1}{4} [1 \ 1 \ 2] P^4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

✓

P₁₄

$$P[(\tilde{X}_1 = 1) \cap (\tilde{X}_2 = 2)] \Rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} P^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \checkmark$$

Classification

✓ Periodicity is class property ie. $i \leftrightarrow j \Rightarrow d(i) = d(j)$

$\rightarrow i \rightarrow j \quad p_{ij}^n > 0 \dots$ for some n

$j \rightarrow i \quad p_{ji}^m > 0 \quad$ for some m

$$d(i) = \underline{\text{gcd}} \{ t / p_{ii}^t > 0 \} \quad d(j) = \underline{\text{gcd}} \{ t / p_{jj}^t > 0 \}$$

$$\begin{aligned} p_{ii}^{(n+m)} &= \sum_{k \in S} p_{ik}^n p_{ki}^m \\ &\geq p_{ij}^n p_{ji}^m \\ &> 0 \end{aligned}$$

$$\begin{aligned} p_{jj}^{(n+m)} &= \sum_{k \in S} p_{jk}^m p_{kj}^n \\ &\geq p_{ji}^m p_{ij}^n \\ &> 0 \end{aligned}$$

$\Rightarrow \underline{n+m}$ is multiple of $d(i) \Rightarrow n+m = \underline{d(i)} \times \text{something}$

Similarly, $p_{jj}^{(n+m)} > 0 \Rightarrow n+m$ is multiple of $d(j) \Rightarrow n+m = \underline{d(j)} \times \text{something}$

✓ for some s, $p_{ii}^{(s)} > 0 \Rightarrow s = \underline{d(i)} \times \text{something}$ —①

$$\begin{aligned} p_{jj}^{(n+m+s)} &\geq p_{ji}^{(m)} p_{ii}^{(s)} p_{ij}^{(n)} \\ &> 0 \quad > 0 \quad > 0 \\ &> 0 \end{aligned}$$

$\Rightarrow \underline{n+m+s} \neq \underline{d(j)} \times \text{something}$

$\Rightarrow s = \underline{d(j)} \times \text{something}$ —② (as $d(j)$ divides $n+m$)

From ① & ②

$$\Rightarrow d(i) = d(j)$$

Transitivity :- $i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$

$i \leftrightarrow j \quad P_{ij}^n > 0, P_{ji}^m > 0 \quad \text{for some } n, m$

$j \leftrightarrow k \quad P_{jk}^s > 0, P_{kj}^t > 0 \quad \text{for some } s, t$

To show $i \leftrightarrow k \Rightarrow$

$$P_{ik}^{n+s} \geq P_{ij}^n P_{jk}^s \\ > 0 \quad > 0$$

$$P_{ki}^{t+m} \geq P_{kj}^t P_{ji}^m \\ > 0 \quad > 0$$

$$\begin{matrix} i \rightarrow k \\ \vdots \\ \rightarrow \end{matrix} \quad i \leftrightarrow k$$

$$k \rightarrow i$$

Recurrent

Starting from state i ,

Process reenters in the same state with probability 1

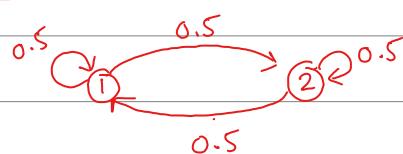
$$\begin{array}{l} \text{① } F_i = 1 \\ \text{② } \sum P_{ii} < 0 \end{array}$$

For any state i , F_i denote the probability that the process, starting from state i , will ever reaches state i .

State i is said to recurrent if $F_i = 1$, & transient if $F_i < 1$

$f_i^{(n)}$ \Rightarrow Prob. that starting from state i process reaches to state i in n steps first time

$$F_i = \sum_n f_i^n$$



$$f_1^1 = 0.5$$

$$f_1^2 = 0.5^2$$

$$f_1^3 = 0.5^3$$

$$\Rightarrow f_1^n = 0.5^n$$

~~1-1-1~~

~~1-2-1~~

~~1-2-2-1~~

$$F_i = \sum_n f_i^n \Rightarrow = 0.5 + 0.5^2 + \dots$$

$$= 0.5 (1 + 0.5 + 0.5^2 + \dots)$$

$$= 0.5 \left(\frac{1}{1-0.5} \right) = 0.5 \cdot \frac{1}{0.5} = 1$$

1 & 2 recurrent



$$f_1^{(1)} = 0$$

$$f_1^{(2)} = 1$$

$$f_1^{(3)} = 0$$

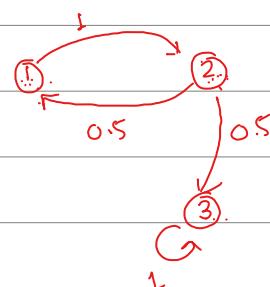
$$F_1 = f_1^{(1)} + f_1^{(2)} + 0 \dots + 0$$

$$= 0 + 1 + 0$$

$$= 1$$

$\Rightarrow 1 \& 2$ recurrent

Reducible



$$C(1) = \{1, 2\} = C(2)$$

$$C(3) = \{3\}$$

$$d(1) = \gcd\{2, 4, 6, \dots\}$$

$$d(3) = 1$$

$$= 2$$

$$= d(2)$$

(Periodicity is class property)

$$\begin{aligned} f_2^1 &= 0, & f_2^2 &= 0.5, & f_2^3 &= 0, & f_2^4 &= 0 \dots \\ f_1^1 &= 0 & f_1^2 &= 0.5 & f_1^3 &= 0 & f_1^4 &= 0 \dots \end{aligned}$$

$F_2 = 0.5 < 1 \Rightarrow 1 \& 2$ are transient states

$$F_3 = f_3^{(1)} + f_3^{(2)} + f_3^{(3)} + \dots = 1 + 0 + 0 + \dots = 1 \Rightarrow 3 \text{ is recurrent.}$$

persistent

$P_{ii}^{(n)}$ \rightarrow Starting from i, process reaches to state i, in n steps

To prove

i recurrent iff $\sum_n P_{ii}^{(n)} = \infty$

Transient iff $\sum_n P_{ii}^{(n)} < \infty$

$C(1) = \{1, 2\}$
 $C(2) = \{2\}$
Irreducible MC

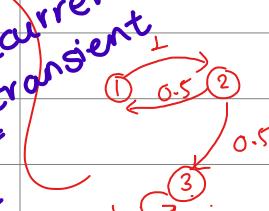


$$\begin{aligned} \sum_n P_{11}^{(n)} &= P_{11}^1 + P_{11}^2 + P_{11}^3 + \dots \\ &= 0 + 1 + 0 + 1 + \\ &= \infty \end{aligned}$$

$$= \sum_n P_{11}^{2n}$$

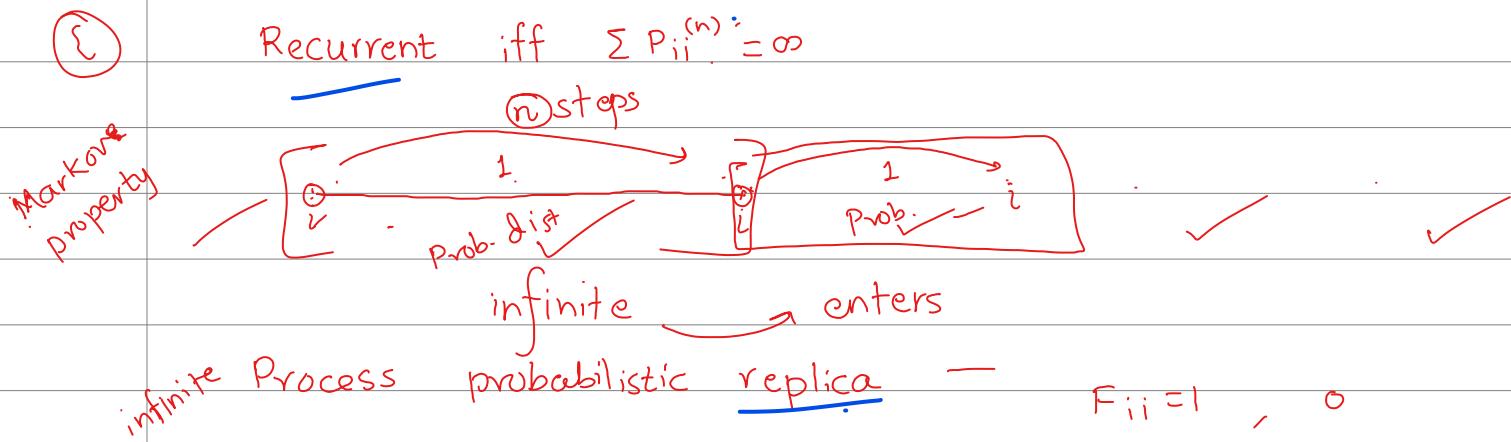
infinite - Recurrent

$C(1) = \{1, 2\}$
 $C(2) = \{3\}$
Transient
Reducible MC



$$\begin{aligned} \sum_n P_{11}^{(n)} &= P_{11}^1 + P_{11}^2 + P_{11}^3 + P_{11}^4 + \dots \\ &= 0 + 0.5 + 0 + 0.5^2 + \dots \\ &= \frac{1}{1-0.5} = \frac{1}{0.5} = 2 < \infty \end{aligned}$$

Finite \Rightarrow Transient



i state transient, $\left\{ \begin{array}{l} \text{Starting from } i \text{ stat process reaches to} \\ \text{state } i \text{ w.p. } < 1 \end{array} \right.$

$F_{ii} < 1$

$1 - F_{ii} > 0$ it will never return to state i

No. visits Finite no.
~ Geo

$$E(\text{No. of visits}) \Rightarrow \frac{1}{1 - F_{ii}}$$

- State i is recurrent if, with probability 1, a process starting from state i , will eventually return.
- However, by Markovian Property, the process probabilistically restarts itself upon returning to state i . Hence with prob. 1 it will return to i .
- Repeating this argument, with probability 1, the no. of visits to state i will be infinite & will thus have infinite expectation.~

On the other hand, if state i is transient, there is positive prob., $1 - F_{ii} > 0$, that it will never return again. Hence the no. of visits is Geometric with $E(n) = \frac{1}{1 - F_{ii}}$

\therefore State i is recurrent iff $E(\text{no. of visits to } i / X_0 = i) = \infty$

$$\text{Let } I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases} \quad \left| \begin{array}{l} E(I_n) = P(I_n = 1 / X_0 = i) \\ = P(X_n = i / X_0 = i) \\ = P_{ii}^n \end{array} \right.$$

$\therefore \sum_n I_n$ denotes no. of visits to i . \checkmark

$$\begin{aligned} E\left(\sum_n I_n / X_0 = i\right) &= \sum_n E(I_n / X_0 = i) \\ &= \sum_n P_{ii}^n \end{aligned}$$

$\Rightarrow i$ is recurrent iff $\sum_n P_{ii}^n = \infty \checkmark$

Theo

Recurrence is class property so does transience

Recurrence

$$\hookrightarrow F_{ii} = 1 \text{ or/and } \sum_n P_{ii}^n = \infty$$

Transience

$$F_{ii} < 1$$

$$\sum_n P_{ii}^n < \infty$$

$i \leftrightarrow j$ \Rightarrow i is recurrent \Rightarrow j is also recurrent

\rightarrow $i \leftrightarrow j$, $\exists i \rightarrow j$ for some s , $P_{ij}^s > 0$
 $j \rightarrow i$ for some t , $P_{ji}^t > 0$

i is recurrent $\Rightarrow \sum_n P_{ii}^n = \infty$

$$P_{jj}^{s+t+n} = \sum_{k \in S} P_{jk}^t P_{kk}^n P_{kj}^s$$

$$< \sum_{n=0}^{\infty} P_{jj}^n = \infty$$

$$\geq P_{ji}^t P_{ii}^n P_{ij}^s$$

$$\sum_n P_{jj}^{s+t+n} \geq \sum_n P_{ji}^t P_{ii}^n P_{ij}^s$$

$$> P_{ji}^t \left(\sum_n P_{ii}^n \right) P_{ij}^s \\ > 0 \quad \infty \quad > 0$$

$$\Rightarrow \sum_n P_{jj}^n \geq \sum_n P_{jj}^{s+t+n} \geq \infty$$

~~$\leq \infty$~~

$$\Rightarrow \sum_n P_{jj}^n = \infty$$

Transience is class property.

$i \leftrightarrow j$, if i is transient $\Rightarrow j$ is also transient.

$$i \leftrightarrow j \cdot i \rightarrow j, \quad 1 \geq P_{ij}^s > 0 \\ j \rightarrow i, \quad 1 \geq P_{ji}^t > 0$$

i is transient $\Rightarrow \sum_n P_{ii}^n < \infty$

$$P_{jj}^{s+t+n} \geq P_{ji}^t P_{ii}^n P_{ij}^s$$

$$\sum_n P_{jj}^{s+t+n} \geq \boxed{P_{ji}^t \left(\sum_n P_{ii}^n \right) P_{ij}^s} \\ \leq 1 \quad < \infty \quad \leq 1$$

Recurrence
 $i \leftrightarrow j$ i transient
 $\Rightarrow j$ transient

By method of contradiction
 i transient but j is recurrent

$$\sum_n P_{jj}^{s+t+n} \leq \infty$$

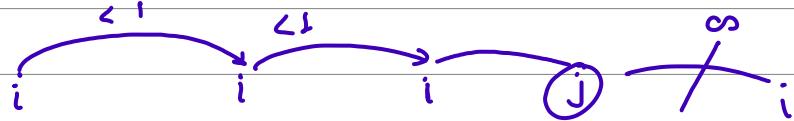
j recurrent $\Rightarrow i$ recurrent

which is contradiction

Irreducible MC :- Only one class , All states are communicating with each other .

Ergodic //
Recurrent
 $\circ f_i = 1$
 $\circ \sum p_{ii} = 0$

let μ_{jj} denotes the expected no. of transitions need to return to state j



$$\mu_{jj} = \begin{cases} \infty & \text{if } j \text{ is transient.} \\ \sum n f_i^n & \text{if } j \text{ is recurrent.} \\ \infty & \text{null recurrent} \\ \mu_{jj} < \infty & \text{tve recurrent} \end{cases}$$

First time ↓

$$\begin{array}{c}
 \alpha \quad \beta \quad \gamma^* \\
 \alpha \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right] \\
 \beta \\
 \gamma^*
 \end{array}$$
$$\begin{aligned}
 \mu_{\alpha\alpha} &= \sum_n n \cdot f_\alpha^n \\
 &= 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 0 + \dots \\
 &= 3
 \end{aligned}$$

$\mu_{\alpha\alpha} = 3 < \infty$

$c(\alpha) = \{\alpha, \beta, \gamma^*\} \Rightarrow$ Irreducible

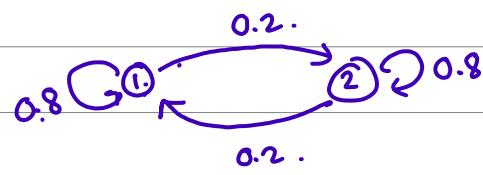
$$F_\alpha = f_\alpha^1 + f_\alpha^2 + f_\alpha^3 + f_\alpha^4$$

$$= 0 + 0 + 1 + 0 + \dots$$

= 1 $\Rightarrow \alpha, \beta, \gamma^*$ all are recurrent

$\mu_{\alpha\alpha} = 3 < \infty \Rightarrow \alpha$ is tve recurrent

$$P = \begin{bmatrix} 1 & 2 \\ 0.8 & 0.2 \\ 2 & 0.2 & 0.8 \end{bmatrix}$$



$C(L) = \{1, 2\} \Rightarrow$ Irreducible

$1 \rightarrow 2 \rightarrow 2 \rightarrow 1$

$$F_1 = f_1 + f_2^2 + f_3^3 + f_4^4 + \dots$$

$$= 0.8 + 0.2^2 + 0.2^2 \cdot 0.8^1 + 0.2^2 \cdot 0.8^2 + \dots + \underline{0.2^2 (0.8)^{n-2}} + \dots$$

$$= \underline{0.8} + 0.2^2 [1 + \underline{0.8 + 0.8^2 + \dots}] < \infty$$

$0.8 < 1$

$$\left\{ \begin{array}{l} = 0.8 + 0.2^2 \left[\frac{1}{1-0.8} \right] \\ = 0.8 + 0.2 \cdot \\ = 1 \end{array} \right.$$

$\Rightarrow 1, 2$ are recurrent states

$$\mu_{11} = \sum_n n \cdot f_1^n = 1 \cdot 0.8 + 2 \cdot \underline{0.2^2} + 3 \cdot \underline{0.2^2 \cdot 0.8} + \dots + n \cdot 0.2^2 \cdot 0.8^{n-2} + \dots$$

$$= 0.8 + 0.2^2 [2 + 3 \cdot 0.8 + \dots + n \cdot 0.8^{n-2} + \dots]$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \underline{\frac{d}{dx} \sum_{n=0}^{\infty} x^n} = \frac{d}{dx} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2}$$

$$\begin{aligned} &= 0.8 + 0.2^2 \left[\sum_n (n+1) x^{n-1} \right] = 0.8 + 0.2^2 \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1} + 0.2^2 \sum_{n=0}^{\infty} x^n \\ &= 0.8 + (0.2^2) \cdot \frac{1}{(1-0.8)^2} + 0.2^2 \sum_{n=0}^{\infty} x^n = 0.8 + 1 + 0.2^2 \frac{n!}{(n-1)!} \end{aligned}$$

1, 2 +ve recurrent.

- * Ergodic state: Aperiodic, positive recurrent
in ergodic state $\Rightarrow d(i) = 1, \bar{F}_i = 1$ or $\sum P_{ii}^n = \infty$
 $\mu_{ii} < \infty$

- * Ergodic MC \Rightarrow All states are ergodic.

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \quad C(1) = \{1, 2\} \quad 1, 2, \text{ positive recurrent}$$

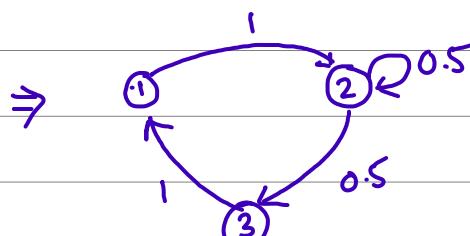
$$d(1) = \gcd(1, 2, \dots) = 1 = d(2)$$

1, 2 positive recurrent, aperiodic \Rightarrow ergodic

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0.5 & 0.5 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

States:
 Aperiodic Recurrent Transient
 tre null
 Ergodic

\Rightarrow MC Ergodic ?



$C(1) = \{1, 2, 3\} \Rightarrow$ Irreducible

$d(2) = \gcd\{1, 2, \dots\} = 1 \Rightarrow$ Aperiodic

Periodicity is class property \Rightarrow

All states are aperiodic.

$$\begin{aligned}
 F_2 &= f_2^1 + f_2^2 + \dots = 0.5 + 0 + 0.5 \cdot 1 \cdot 1 + 0 + \dots \\
 &= 0.5 + 0.5 \\
 &= 1 \quad \Rightarrow \text{Recurrent}
 \end{aligned}$$

\therefore Recurrence is class property. \Rightarrow All states are recurrent.

$$\mu_2 = 1 \cdot f_2^1 + 2 \cdot f_2^2 + 3 \cdot f_2^3 + 4 \cdot f_2^4 + \dots$$

$$\begin{aligned}
 &= 1 \cdot 0.5 + 0 + 3 \cdot 0.5 + 0 + \dots \\
 &= 0.5 + 1.5 \\
 &= 2 < \infty
 \end{aligned}$$

\Rightarrow Positive Recurrent

All states are positive recurrent & aperiodic, i.e. all states are ergodic \therefore MC is ergodic MC.

Limit Theo. :-

If j is transient, then $\sum_n P_{jj}^n < \infty$

$$P_{jj}^n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

We have only finite no. of transitions returning to j .

$$\underline{\pi_j} = \lim_{n \rightarrow \infty} P_{jj}^n$$

$$\underline{\pi_{ij}} = \lim_{n \rightarrow \infty} P_{ij}^n$$

✓ tve recurrent state $\pi_j > 0$

✗ null recurrent $\pi_j = 0 ?$ —

Stationary

CK eqⁿ

$$P^{n+1} = P^n \cdot P$$

$$\lim_{n \rightarrow \infty} P_{ij}^{(n+1)} = \lim_{n \rightarrow \infty} \sum_{k \in S} P_{ik}^{(n)} \cdot P_{kj}^{(1)}$$

$$\textcircled{e} \quad \underline{\underline{\pi_{ij}}} = \sum_{k \in S} \underline{\pi_{ik}} P_{kj}$$

$$P^{n+1} = P^n \cdot P$$

$$\left(\lim_{n \rightarrow \infty} P^{n+1} \right) = \left(\lim_{n \rightarrow \infty} P^n \right) \cdot P$$

$$\underline{\underline{\pi}} = \underline{\underline{\pi}} \cdot P$$

In Matrix form

Recurrent limiting $\rightarrow \underline{\underline{\pi}} = \text{Stationary dist}^\wedge$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 0.5 & 0.5 \\ 3 & 1 & 0 & 0 \end{bmatrix} \quad \underline{\pi} = [\pi_1, \pi_2, \pi_3]'$$

$$\underline{\pi}' = \underline{\pi}' P$$

$$[\pi_1, \pi_2, \pi_3] = [\pi_1, \pi_2, \pi_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\pi_1 = \pi_3$$

$$\pi_2 = \pi_1 + \pi_2/2 \Rightarrow \pi_1 = \pi_2/2$$

$$\pi_3 = \pi_2/2$$

$$\underline{\pi} = \underline{\left[\frac{\pi_2}{2}, \pi_2, \frac{\pi_2}{2} \right]} \Rightarrow \sum_{i=1}^3 \pi_i = 1 \Rightarrow$$

$$\Rightarrow \frac{\pi_2}{2} + \pi_2 + \frac{\pi_2}{2} = 2\pi_2 = 1$$

$$\pi_2 = 1/2$$

$$\checkmark \quad \underline{\pi}_r = [1/4, 1/2, 1/4]$$

$$P = \frac{1}{10} \begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix}$$

$$\pi = ?$$

$$\underline{\pi} = \underline{\pi} P$$

$$[\pi_1, \pi_2, \pi_3] = \left[\begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix} \frac{1}{10} \right]$$

$$10\pi_1 = 3\pi_1 + 4\pi_2 + 3\pi_3 \Rightarrow 7\pi_1 = 4\pi_2 + 3\pi_3 \Rightarrow 7\pi_1 = 4\pi_2 + \frac{3}{2}(\pi_1 + \pi_2)$$

$$10\pi_2 = 4\pi_1 + 3\pi_2 + 3\pi_3 \Rightarrow 7\pi_2 = 4\pi_1 + 3\pi_3$$

$$10\pi_3 = 3\pi_1 + 3\pi_2 + 4\pi_3 \Rightarrow 6\pi_3 = 3\pi_1 + 3\pi_2 \Rightarrow 2\pi_3 = \pi_1 + \pi_2$$

$$11\pi_2 + \pi_1 = 8\pi_2 + 3\pi_1 + 3\pi_2 \Rightarrow \pi_1 = \pi_2$$

$$\underline{\pi} = [\pi_1, \pi_2, \pi_3] = [\pi_1, \pi_1, \pi_1]$$

$$\sum \pi_i = 1, \quad \pi_1 + \pi_1 + \pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{3}$$

$$\pi = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$$

① $L = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 6 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix} \Rightarrow ? \quad [x, x, y, y] \quad \begin{array}{l} x, y \in (0, 1) \\ x+y = 1 \end{array}$

Stationary distribution may or may not
be unique.

② $\frac{1}{6} \begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$

Transition Graph

$C(1) = \{1, 2\}, \quad C(3) = \{3\}, \quad C(4) = \{4\} \Rightarrow$ Reducible MC

$d(1) = 1 \quad d(3) = 1 \quad d(4) = 1 \Rightarrow$ aperiodic

$$F_1 = f_1^1 + f_1^2 + f_1^n + \dots$$

$$= \frac{1}{2} + \frac{1}{2} \cdot 1 + 0 + \dots$$

$\Rightarrow F_1 = 1 \Rightarrow 1, 2$ are recurrent

$$F_3 = f_3^1 + f_3^2 + f_3^3 + \dots$$

$$= \frac{2}{3} + 0 + 0 + \dots$$

$= \frac{2}{3} < 1 \Rightarrow$ State 3 is transient

$$F_4 = f_4^1 + f_4^2 + f_4^3 + \dots$$

$$= \frac{1}{3} + 0 + \dots$$

$= \frac{1}{3} < 1 \Rightarrow$ State 4 is transient

$$\mu_{11} = \sum_n n f_1^n = 1 \cdot f_1^1 + 2 \cdot f_1^2 + 0 = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5 < \infty$$



Recurrence
is class
property

$$\begin{aligned}
 F_2 &= f_2^1 + f_2^2 + f_2^3 + f_2^4 + \dots + f_2^n + \dots \\
 &= 0 + \frac{1}{2} + \frac{1 \cdot 1 \cdot \frac{1}{2}}{\underline{\underline{\frac{1}{2}}}} + \frac{1}{2} \cdot 1 \cdot \left(\frac{1}{2}\right)^{n-2} = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right) \\
 &= \frac{1}{2} \cdot \left(\frac{1}{1 - \frac{1}{2}}\right) = 1 \Rightarrow 2 \text{ is recurrent.}
 \end{aligned}$$

$$\begin{aligned}
 \mu_2 &= \sum n f_2^n \Rightarrow 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2} + \dots + n \cdot \frac{1}{2^{n-1}} + (n+1) \frac{1}{2^n} \\
 &\Rightarrow \sum (n+1) \frac{1}{2^n} = \sum n \cdot \frac{1}{2^n} + \underline{\underline{\frac{1}{2^n}}} < \infty \\
 &\Rightarrow 2 \text{ is positive recurrent}
 \end{aligned}$$

1,2, aperiodic, positive recurrent \Rightarrow Ergodic States

Lets find out limiting distribution / Stationary

$$[\pi_1, \pi_2, \pi_3, \pi_4] = [\pi_1, \pi_2, \pi_3, \pi_4] \underset{-G}{\perp} \begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$6\pi_1 = 3\pi_1 + \underline{\underline{6\pi_2}} + 2\pi_3 \rightarrow 6\pi_1 = 3\pi_1 + 3\pi_2 \Rightarrow \pi_1 = 2\pi_2 -$$

$$6\pi_2 = 3\pi_1 \Rightarrow \pi_2 = 0$$

$$6\pi_3 = 4\pi_3 + 4\pi_4 \Rightarrow \pi_3 = 0$$

$$\underline{6\pi_4} = 2\pi_4 \Rightarrow \pi_4 = 0 \Rightarrow 4\pi_4 = 0 \Rightarrow \pi_4 = 0$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \Rightarrow 2\pi_2 + \pi_2 + 0 + 0 = 1 \Rightarrow \pi_2 = 1/3$$

$$\Rightarrow \pi_1 = 2/3$$

Stationary
Distribution

$$\begin{bmatrix} 2/3 & 1/3 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \underline{\underline{\pi}} = [\pi_1, \pi_1, \pi_3, \pi_3]$$

$$\pi_1 + \pi_1 + \pi_3 + \pi_3 = 1$$

$$\Rightarrow \underline{\pi_1 + \pi_3 = 1}$$

$\pi_1 = x$

$[x \ x \ 1-x \ 1-x]$

$$\lim_{n \rightarrow \infty} p_{ij}^n \quad i \rightarrow j \text{ in } n \text{ steps}$$

π_j ✓ $i \rightarrow \text{doesn't matter}$

$$\pi' = \pi P \quad \text{eq}^n \rightarrow \checkmark$$

$$\pi' = \pi P$$

$$I \underline{\pi} = P \underline{\pi}$$

$$\Rightarrow (P - I) \underline{\pi} = 0$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$[1 \ 1 \ 1] \pi_1 = 1$$

$$\pi_3$$

$$A = [P - I] \quad \text{comes}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

No limiting distribution exists

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$$

$$\begin{array}{c}
 \text{G} \quad \overbrace{\pi' = \overbrace{\pi P}^{\text{initial}}}^{\text{II}} \quad \overbrace{\pi}^{\text{II}} \\
 \overbrace{\pi_2 = \overbrace{\pi P P^2}^{\text{P(X}_0=x\text{)}}}^{\text{III}} \quad \lim_{n \rightarrow \infty} \overbrace{P(X_n=x)}^{\text{P(X}_n=x\text{)}} \rightarrow \overbrace{\pi}^{\text{II}}
 \end{array}$$

$$\frac{\alpha P}{\pi P} = P$$

Theo.:- An irreducible aperiodic MC belongs to one of the following classes :-

j transient following classes :- *i es*

① Either the states are all transient or all null recurrent
in this case, $P_{ij}^n \rightarrow 0 \forall i, j$ and \exists no stationary dist.

① Either the states are all transient or all null recurrent
 in this case, $P_{ij}^n \rightarrow 0 \forall i, j$ and \exists no stationary dist.
 ② Or else, all states are positive recurrent, that is $F_j = 1, \mu_{jj} < \infty$

unique stationary dist $T_{ij} = \lim_{n \rightarrow \infty} P_{ij}^n > 0$ \nwarrow limiting dist

In this case, $\{T_{ij}, j=0,1,2,\dots\}$ is stationary dist &

3 no other stationary distribution.

Proof: We will prove (ii)

$$S = \{0, 1, 2, \dots\} = \{0, 1, 2, \dots, M, \underline{\dots}\}$$

for some $M_{\mathcal{S}}$

$$\sum_{j=0}^n p_{ij} \leq 1 \quad \text{for all } i$$

$$\sum_{j=0}^M P_{ij} \leq \sum_{j=0}^{\infty} P_{ij} = 1.$$

⇒ all MES

$$\sum_{j=0}^m p_{ij} \leq 1$$

Letting $n \rightarrow \infty$ yields

$$\sum_{j=0}^{\infty} \lim_{n \rightarrow \infty} P_{ij} \leq 1$$

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$$\sum_{j=0}^M \pi_{ij} \leq 1$$

$\Rightarrow M \in S$

π_{ij}

$M \rightarrow \infty$

$S = \{0, 1, \dots\}$

implying that

$$\sum_{j=0}^{\infty} \pi_{ij} \leq 1$$

? $S = \{0, 1, \dots, M, \dots\}$

Now

$$P_{ij}^{n+1} = \sum_{k=0}^M P_{ik}^n P_{kj}^n \geq \sum_{k=0}^M P_{ik}^n P_{kj} \Rightarrow M \in S$$

C-K eqⁿ

letting $n \rightarrow \infty$ yields

$$\lim_{n \rightarrow \infty} P_{ij}^{n+1} \geq \lim_{n \rightarrow \infty} \sum_{k=0}^M P_{ik}^n P_{kj} \Rightarrow M \in S$$

$$\lim_{n \rightarrow \infty} P_{ik}^n \Rightarrow \pi_{ik}$$

implying that

$$\pi_{ij} \geq \sum_{k=0}^M (\lim_{n \rightarrow \infty} P_{ik}^n) P_{kj} \Rightarrow M \in S$$

$$\pi_{ij} \geq \sum_{k=0}^M \pi_{ik} P_{kj} \Rightarrow M \in S$$

$$\Rightarrow \pi_{ij} \geq \sum_{k=0}^{\infty} \pi_{ik} P_{kj} \Rightarrow j \geq 0$$

To prove equality. assume strict inequality for some j

$$\Rightarrow \pi_{ij} > \sum_{k=0}^{\infty} \pi_{ik} P_{kj}$$

$k \rightarrow j$

$j = 0, 1, \dots, \infty$

summation

$$\Rightarrow \sum_{j=0}^{\infty} \pi_{ij} > \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \pi_{ik} P_{kj} = \sum_{k=0}^{\infty} \pi_{ik} \sum_{j=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \pi_{ik} \cdot 1$$

$$\Rightarrow \sum_{j=0}^{\infty} \pi_{ij} > \sum_{k=0}^{\infty} \pi_{ik}$$

[which is contradiction]

$$\begin{aligned} P^{(2)} &= P \cdot P \\ \pi^{(2)} &= \alpha \cdot \pi \\ \pi &= \pi \cdot P \end{aligned}$$

 \Rightarrow

$$\pi_{lj} = \sum_{k=0}^{\infty} \pi_{lk} p_{kj}$$

 $\Rightarrow j \in S = \{0, 1, 2, \dots\}$

Putting $P_j = \frac{\pi_{lj}}{\sum \pi_{lk}}$ $\Rightarrow P_j$ is stationary dist^{*}

\therefore At least one stationary dist^{*} exists.

Now let $\{P_j, j=0, 1, 2, \dots\}$ be any stationary distribution.

Then if $\{P_j, j=0, 1, 2, \dots\}$ is the prob. dist^{*} of X_0 (Initial Prob. dist)

$$\begin{aligned} x_i &\sim \alpha \cdot P \\ x_i &\sim \pi \cdot P \\ x_i &\sim \pi_i \cdot P \\ x_i &\sim \pi_i \end{aligned}$$

$$P_j = P\{X_n=j\}$$

$$= \sum_{i=0}^{\infty} P\{X_n=j | X_0=i\} P\{X_0=i\}$$

$$P_j = \sum_{i=0}^{\infty} P_{ij}^n P_i$$

$$\Rightarrow P_j \geq \sum_{i=0}^M P_{ij}^n P_i \quad \Rightarrow \text{MES}$$

Let $n \rightarrow \infty$ and $M \rightarrow \infty$

$$P_j \geq \sum_{j=0}^{\infty} \left(\lim_{n \rightarrow \infty} P_{ij}^n \right) \cdot P_i \quad \begin{matrix} \downarrow \\ \pi_j \end{matrix} \quad \begin{matrix} \downarrow \\ \text{limiting} \end{matrix} \quad \text{--- ②}$$

$$\stackrel{\text{stationary}}{=} P_j \geq \sum_{i=0}^{\infty} \pi_{ij} P_i = \pi_j \left(\sum_{i=0}^{\infty} P_i \right) = \pi_j$$

Lets prove other side,

CK

$$P_j = \sum_{i=0}^{\infty} P_{ij}^n P_i = \sum_{i=0}^M P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_{ij}^n P_i$$

$$\leq \sum_{i=0}^M P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_i \quad (\text{as } P_{ij}^n \leq 1)$$

$$\lim_{n \rightarrow \infty} P_j \leq \sum_{i=0}^M \left(\lim_{n \rightarrow \infty} P_{ij}^n \right) \cdot P_i + \sum_{i=M+1}^{\infty} P_i$$

$$\underline{P_j} \leq \sum_{i=0}^M \pi_i \cdot p_i + \sum_{i=M+1}^{\infty} p_i$$

$$\leq \pi_i \left(\sum_{i=0}^M p_i \right) + \sum_{i=M+1}^{\infty} p_i$$

letting $M \rightarrow \infty$

$$\underline{P_j} \leq \pi_i \left(\sum_{i=0}^{\infty} p_i \right) + 0$$

$$\underline{P_j} \leq \pi_i$$

—③

from ② & ③ $\Rightarrow P_j = \pi_i \quad j = 0, 1, 2, \dots$, is ^{only} _{stationary} dist.

If all states are transient or null recurrent and $\{P_j, j=0, 1, 2, \dots\}$ is stationary dist, then ① eqn holds i.e.

$$\underline{P_j} = \sum_{i=0}^{\infty} P_{ij} \pi_i$$

$\because \sum_{i=0}^{\infty} \pi_i = 1 \rightarrow 0$

and $P_{ij} \rightarrow 0$, which is impossible.

Thus for case ①, no stationary dist exists.

Note:-

* In the irreducible, positive recurrent, we have that $\pi_i, i \geq 0$ are unique non-negative solution of

$$\pi_i = \sum_j \pi_j P_{ij} \quad \& \quad \sum_j \pi_j = 1$$

i.e. In long run, π_i is the proportion of time that the MC

is in state j .

$$\Rightarrow \pi_j = \frac{1}{\mu_{jj}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} P_{jj}^{n.d(j)} = \frac{d}{\mu_{jj}} = d \cdot \pi_j \quad (d = d(j) \nrightarrow j)$$

Prob. Dist: $\{P_j, j \geq 0\}$ is said to be stationary for MC if

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij} \quad j \geq 0$$

* Random Walk

X_0 = initial position

$$X_0 = 100$$

$$Z_i = \begin{cases} +1 & \text{with } p \\ -1 & \text{with } 1-p \end{cases}$$

$$X_1 = X_0 + Z_1 \leftarrow \text{outcome of 1st trial/game}$$

$$X_2 = X_1 + Z_2 \dots$$

$$X_n = X_{n-1} + Z_n \leftarrow \text{outcome of } n^{\text{th}} \text{ game/trial}$$

$$\left| \begin{array}{l} X_1 = X_0 + Z_1 \\ X_2 = X_0 + \sum_{i=1}^2 Z_i \\ \vdots \\ X_n = X_0 + \sum_{i=1}^n Z_i \end{array} \right.$$

$$\{X_i, i \in I\} \Rightarrow S = \overset{\text{Statespace}}{\{ -\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, \infty \}}$$

↑ Random Walk

$$R.W. \quad X_n = X_0 + \sum_{i=1}^n Z_i$$

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if Z_i takes ± 1 values

$$Z_i = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$$

\hookrightarrow Simple ~~R.W.~~ R.W.

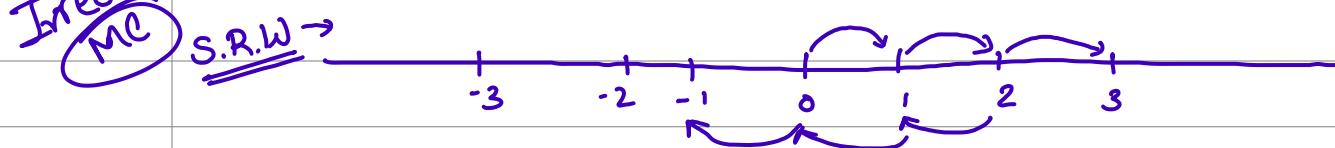
$$\text{If } Z_i = \begin{cases} +\omega & \text{w.p. } 1/2 \\ -\omega & \text{w.p. } 1/2 \end{cases}$$

\hookrightarrow symmetric r.w.

$$Z_i = \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

Simple Symmetric R.W.

Irreducible C(0) = {0, 1, -1, 2, -2, ...} Single communicating class



Does random walk follows markov property?

Random Walk

$$X_n = X_0 + \sum_{i=1}^n Z_i$$

$Z_i \sim i.i.d.$

$$\left[\begin{aligned} & P[X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0] \\ & = P[X_{n+1} = j / X_n = i] \end{aligned} \right]$$

$$\rightarrow P[X_0 + \sum_{i=1}^n Z_i = j / X_0 + \sum_{i=1}^{n-1} Z_i = i, \dots]$$

$$= P[\underbrace{X_0 + \sum_{i=1}^{n-1} Z_i}_{\text{constant}} + Z_n = j / \underbrace{X_0 + \sum_{i=1}^{n-1} Z_i}_{\text{constant}} = i, \dots]$$

$$= P[Z_n = j - i / X_0 + \sum Z_i = i] \dots$$

as Z_i are i.i.d

X_n is MC

$$X_n = X_0 + \sum_{i=1}^n Z_i \quad \text{or} \quad X_n = X_{n-1} + Z_i$$

* TPM for random walk

	:	-3	-2	-1	0	1	2	3
-3		0	P					
-2		p	0	P				
-1			q	0	P			
0				q	0	P		
1					q	0	P	
2						q	0	P
3							q	0

* Gambler's Ruin Problem

- X_0 = initial amount

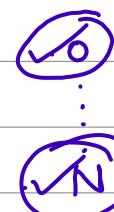
$$Z_i = \begin{cases} +1 & p \\ -1 & q \end{cases}$$

X_n = Gambler's Fortune at n^{th} game

Markov chain $S = \{0, 1, 2, \dots, N\}$

Interest $\underline{X_n \rightarrow 0}$ ^{Gambler Ruined}

A B
 x_0 $N-x_0$



L +L
-1

Absorbing State $\rightarrow 0$

$$\begin{matrix} 0 & 1 & 2 & \cdots & N-1 & N \\ \downarrow & & & & & \\ \frac{1}{2} & 0 & & & & \\ \frac{1}{2} & q & 0 & p & & \\ 2 & & q & 0 & p & \\ & & & & & \\ N-1 & & & q & 0 & p \\ -N & & & & 1 & \end{matrix}$$

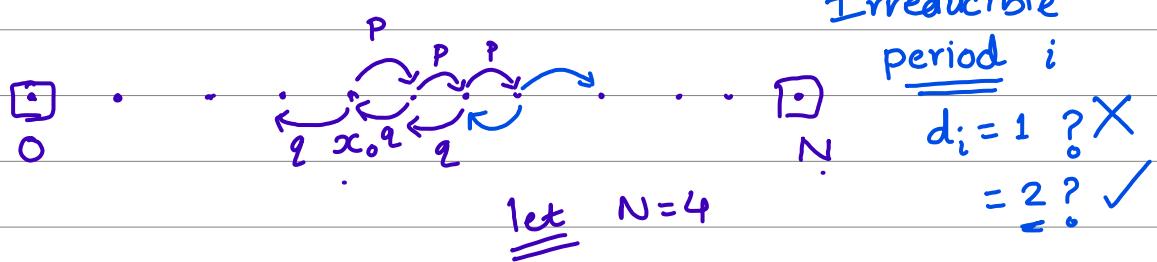
Absorbing State

$$d(i) = \gcd\{n / P_{ii}^n > 0\}$$

Lecture:

Manoj C Patil

Transition graph



✓ A B

$x_0 \quad N-x_0$

+ $x_0+1 \quad N-x_0-1$

-1 $x_0 \quad N-x_0$

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & q & 0 & p & 0 \\ 3 & 0 & q & 0 & p \\ 4 & 0 & 0 & 0 & 1 \end{matrix}$$



Classes

$\{0\}, \{N\}, \{1, 2, \dots, N-1\}$

↓ Recurrent

↓ Transient

Long run probabilities

$$P_{ij} \quad i \rightarrow j$$

$$j = i-1, i+1$$

$$q \quad p$$

✓ $P_{00} = 1 = P_{NN}$

✓ $P_{i,i+1} = p = 1 - P_{i,i-1} \quad i = 1, 2, \dots, N-1$

Let f_i denote the prob. that starting from i gambler's fortune reaches to state N before reaching 0.

$$\begin{aligned}
 & f_i \quad i \geq N \\
 & \cancel{i \geq N} \\
 & \underset{\cancel{i \geq N}}{i \rightarrow N} \quad \underset{i+1 \rightarrow N}{i+1 \rightarrow N} \quad \underset{i-1 \rightarrow N}{i-1 \rightarrow N} \\
 & - f_i = p \cdot f_{i+1} + q \cdot f_{i-1} \quad i=1,2,\dots,N-1 \\
 & \cancel{p+q=1} \\
 & (p+q) \cdot f_i = p \cdot f_{i+1} + q \cdot f_{i-1} \\
 & p+q+r=1 \\
 & \cancel{r=0} \\
 & \cancel{\text{tie}}
 \end{aligned}$$

$$\begin{aligned}
 & q(f_i - f_{i-1}) = p(f_{i+1} - f_i) \\
 & \cancel{(f_{i+1} - f_i)} = \frac{q}{p}(f_i - f_{i-1}) \quad i=1,2,\dots,N-1 \\
 & \left[\begin{array}{l} f_1 = ? \\ f_0 = 0 \end{array} \right] \\
 & \left\{ \begin{array}{l} i=1, f_2 - f_1 = \frac{q}{p}(f_1 - f_0) = \frac{q}{p}(f_1) \\ i=2, f_3 - f_2 = \frac{q}{p}(f_2 - f_1) = \frac{q}{p}\left(\frac{q}{p}f_1\right) = \left(\frac{q}{p}\right)^2 f_1 \\ \vdots \quad f_{j+1} - f_j = \left(\frac{q}{p}\right)^j f_1 \\ i=N-1 \quad f_N - f_{N-1} = \left(\frac{q}{p}\right)^{N-1} f_1 \end{array} \right.
 \end{aligned}$$

sum of first
2 terms

$$\underline{f_3 - f_2 + f_2 - f_1} = \frac{q}{p} f_1 + \left(\frac{q}{p}\right)^2 f_1$$

j terms

$$\underline{f_{j+1} - f_1} = f_1 \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^j \right]$$

$$f_{j+1} = f_1 \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^j \right]$$

$\frac{q}{p} = 1$
 $q=p=1$

$$f_{j+1} = f_1 [1 + 1 + 1 + \dots + 1] = (j+1)f_1 \Rightarrow \underline{f_j = j \cdot f_1}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + x^{n+1} + \dots \quad x < 1$$

if $\frac{q}{p} \neq 1 < 1$

$$\frac{1}{1-x} = \text{sum?} + x^{n+1} [1 + x + \dots]$$

$$\frac{1}{1-x} = \text{sum?} + \frac{x^{n+1}}{p-x}$$

$$\text{sum?} = \frac{1-x^{n+1}}{1-x} \quad \checkmark$$

$$f_j = f_1 \left[1 + \frac{q}{p} + \dots + \left(\frac{q}{p}\right)^{j-1} \right] = \begin{cases} f_1, & \frac{1 - \left(\frac{q}{p}\right)^j}{1 - q/p} \\ j \cdot f_1, & \frac{q}{p} = 1 \end{cases}$$

$$f_N = 1, \quad j=N \Rightarrow \quad f_N = \begin{cases} f_1, & \frac{1 - \left(\frac{q}{p}\right)^N}{1 - q/p} \\ N \cdot f_1, & \frac{q}{p} = 1 \end{cases}$$

But we are not going to discuss ✓

	0	1	2	3
0	1			
1	q	r	p	
2	q	r	p	
3				1

$r > 0$ ~~$p+q+r=1$~~

$$1 = \begin{cases} f_1, & \frac{1 - \left(\frac{q}{p}\right)^N}{1 - q/p} \\ f_1, & N \end{cases}$$

$$\Rightarrow f_1 = \begin{cases} \frac{1 - q/p}{1 - \left(\frac{q}{p}\right)^N}, & \frac{q}{p} < 1 \\ 1/N, & \frac{q}{p} = 1 \end{cases}$$

$$f_j = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N} \cdot \frac{1-(q/p)^j}{1-(q/p)} & q/p < 1 \\ j \cdot \frac{1}{N} & q/p = 1 \end{cases}$$

Gambler's Ruin prob.

$$1 - f_j = \begin{cases} 1 - \frac{1-(q/p)^j}{1-(q/p)^N} & q/p < 1 \\ 1 - \frac{j}{N} & q/p = 1 \end{cases}$$

limitting $N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} f_j = \begin{cases} \lim_{N \rightarrow \infty} \frac{1-(q/p)^j}{1-(q/p)^N} = 1-(q/p)^j & q/p < 1 \\ \lim_{N \rightarrow \infty} \frac{j}{N} = 0 & (q/p)^N \rightarrow 0 \end{cases}$$

Thoughtful

Determine the expected number of steps/games that gambler makes, starting from state i , before reaching 0 or N .

\uparrow \uparrow



✓ $B = \min \{ m \mid \sum_{i=1}^m x_i = -i \text{ or } N-i \}$

$x_i = +1 \quad \text{w.p. } P$

$$\begin{aligned} E(X_i) &= +1(p) + (-1)(1-p) \\ &= p - 1 + p \\ &= 2p - 1 \end{aligned}$$

by Wald's eq's

$$E\left(\sum_{i=1}^B X_i\right) = E(B) \cdot E(X_i) = (2p-1) \cdot E(B)$$

started from i , to reach N , $f_i = \frac{1 - (q/p)^i}{1 - (q/p)^N}$ -
 i , to reach 0 , $1 - f_i$.

$$\sum_{j=1}^B X_j = \begin{cases} -i & 1 - f_i \\ n-i & f_i \end{cases}$$

$$E\left(\sum_{j=1}^B X_j\right) = -i \cdot (1 - f_i) + (n-i) \cdot f_i = -i + i f_i + N f_i - i f_i$$

$$(2p-1) \cdot E(B) = -i + N \left(\frac{1 - (q/p)^i}{1 - (q/p)^N} \right)$$

$$E(B) = \frac{1}{(2p-1)} \left[-i + N \left(\frac{1 - (q/p)^i}{1 - (q/p)^N} \right) \right]$$

$$\text{at } N=4, \quad i=2 \quad p=0.6 \quad q=0.4 \quad \frac{q}{p} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$f_2 = \frac{1 - (q/p)^2}{1 - (q/p)^4} = \frac{1 - (2/3)^2}{1 - (2/3)^4} = \frac{3^2 - 2^2}{3^4 - 2^4} \times \frac{3^4 - 3^2}{3^2}$$

winning

$$= \frac{9-4}{81-16} \times 9 = \frac{45}{65} = \frac{9}{13}$$

Irreducible
MC

$$\text{P} = \left\{ \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.4 & 0.1 \\ 2 & 0.5 & 0.5 & 0 \\ 3 & 0 & 0 & 1 \end{array} \right.$$

Reduces

3- absorbing state

stationary / limiting dist $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$$C(1) = \{1, 2\} = C(2), \quad C(3) = \{3\}$$

absorbing state

After sufficiently large no. of steps process $\xrightarrow{\text{is in}}$ 3

$$P = \begin{matrix} S-A & \xrightarrow{S-A} & A \\ \circled{S-A} & \left[\begin{array}{c|cc} N & B \\ \hline 0 & I \end{array} \right] & A \end{matrix}$$

$$N = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

$$\underline{0} = [0 \ 0]$$

~~A = {3}~~

$j \in A$

$$P[X_n=j, X_{n-1} \notin A, X_{n-2} \notin A, \dots] = \underline{\alpha}' N^{n-1} B_j$$

\uparrow j^{th} column of B

$$\underline{\alpha} = [1 \ 0 \ 0]$$

$$P[X_n=3, X_{n-1} \neq 3, X_{n-2} \neq 3, \dots, X_0=1] = [1 \ 0 \ 0] \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} [0.1 \ 0]$$

* Simulation of RW

$$Z_i = \begin{cases} +1 & P \\ -1 & q \end{cases} \quad \text{Simulate?}$$

$$X_n = X_0 + \sum_{i=1}^n Z_i = X_{n-1} + Z_n$$

$$X_0 = 100$$

$n=7$ simulate?

$$\cdot X_n = X_{n-1} + \underline{Z_n}$$

$n=7$

$$X_n = X_0 + \sum Z_i$$

$$Z_i = \begin{cases} +1 & P = \\ -1 & q \end{cases}$$

\downarrow

$$Z_1 = \text{rbinom}(1, p)$$

$$Z_2 =$$

=

=

=

$$X_0 = 100$$

$$X_1 = X_0 + Z_1$$

$$X_2 = X_1 + \underline{Z_2}$$

⑦ R.S.

Sample

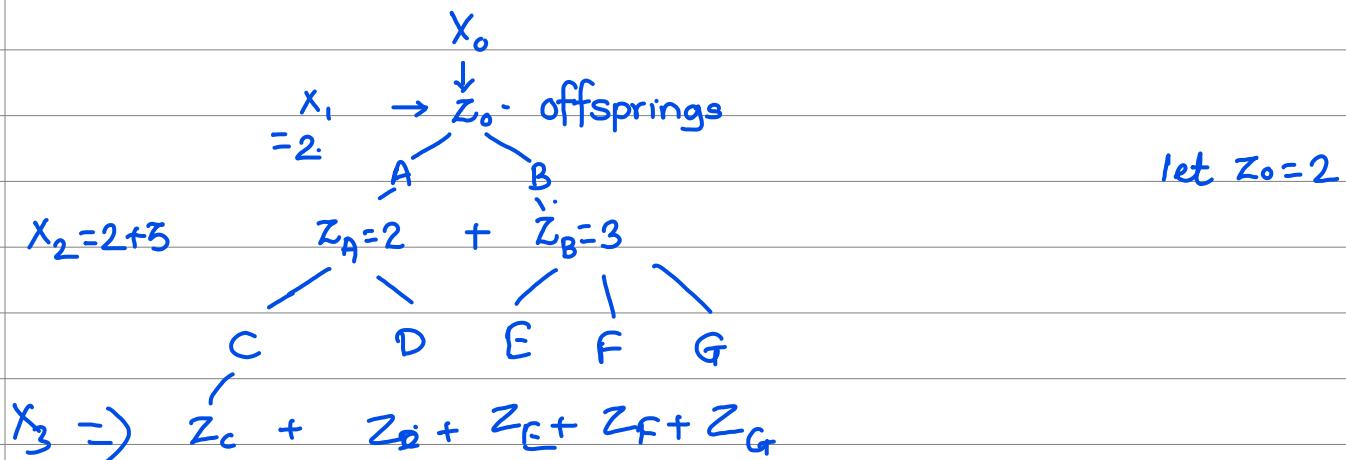
?

$$Z_i = \begin{cases} +2 & 0.25 \\ -1 & 0.35 \\ 0 & 0.2 \\ 1 & 0.2 \end{cases}$$

General
R.W

Absent: 2001, 3, 6, 10, 16, 17, 22, 23, 33, 35, 38, 39, 43, 44, 45, 47, 50, 51, 55

Branching Process (Galton-Watson Branching process)



Consider the popⁿ consisting of individuals able to produce offspring of the same kind.

~~$Z_i \sim \text{i.i.d discrete}$~~ Suppose that each individual have produced j offspring with prob. p_j , $j \geq 0$, independent of each other.

X_0 = size of zeroth generation

X_n = size of n^{th} generation.

let $X_n = 10$

$X_{n+1} = \sum_{i=1}^{X_n} Z_i$ = size of $(n+1)^{\text{st}}$ generation



✓ $X_n = 3$

$$\underline{X_{n+1}} = \underline{Z_A + Z_B + Z_C}$$

$$X_{n+1} = \sum_{i=1}^{3=X_n} Z_i \quad \checkmark$$

$X_n = 0$

$Z_0 = 0, 1, 2, \dots$

$$X_{n+1} = ? \Rightarrow X_{n+1} = 0$$

$$\begin{aligned}
 X_n &= \sum_{i=1}^{X_{n-1}} Z_i \\
 E(X_n) &= E_{X_{n-1}} \left(E_Z \left(\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1}=k \right) \right) \\
 &= E_{X_{n-1}} \left(\sum_{i=1}^k E(Z_i) \mid X_{n-1}=k \right) \quad \left\{ \begin{array}{l} \text{as } Z_i \text{ are i.i.d} \\ E(Z)=\mu \\ V(Z)=\sigma^2 \end{array} \right. \\
 &= E_{X_{n-1}} (k \cdot E(Z) \mid X_{n-1}=k) \\
 &= \mu \cdot E_{X_{n-1}} (X_{n-1})
 \end{aligned}$$

$$\begin{aligned}
 E(X_n) &= \mu \cdot E(X_{n-1}) \\
 E(X_{n-1}) &= \mu \cdot E(X_{n-2}) \\
 E(X_n) &= \mu \cdot \mu \cdot \dots \cdot E(X_0) = \mu^n E(X_0)
 \end{aligned}$$

Assume $E(X_0)=1$

$$\underline{E(X_n)=\mu^n}$$

$$V(X_n) = V \left(\sum_{i=1}^{X_{n-1}} Z_i \right)$$

$$\begin{aligned}
 &= E_{X_{n-1}} \left(V_Z \left(\sum_{i=1}^k Z_i \mid X_{n-1}=k \right) \right) + V_{X_{n-1}} \left(E_Z \left(\sum_{i=1}^k Z_i \mid X_{n-1}=k \right) \right) \\
 &= E_{X_{n-1}} (k \cdot \sigma^2) + V_{X_{n-1}} (k \mu)
 \end{aligned}$$

$$V(X_n) = \sigma^2 \cdot E(X_{n-1}) + \mu^2 V(X_{n-1})$$

$$\begin{array}{ll}
 \textcircled{X_0=1} & \underline{\underline{V(X_1)}} = \sigma^2 \underline{\underline{E(X_0)}} + \mu^2 \underline{\underline{V(X_0)}} = \sigma^2 \\
 & = 1 + 0 = 0
 \end{array}$$

$$\begin{aligned} V(X_2) &= \delta^2 \cdot E(X_1) + \mu^2 \cdot V(X_1) \\ &= \delta^2 \mu + \mu^2 \delta^2 \\ &= \mu \delta^2 (1 + \mu) \end{aligned}$$

$$\begin{aligned} V(X_3) &= \delta^2 (E(X_2)) + \mu^2 \cdot V(X_2) \\ &= \delta^2 \mu^2 + \mu^2 \cdot \delta^2 (\mu + \mu^2) \\ &= \delta^2 (\mu^2 + \mu^3 + \mu^4) \\ &= \mu^2 \delta^2 (1 + \mu + \mu^2) \end{aligned}$$

$$V(X_n) = \mu^{n-1} \delta^2 (1 + \mu + \mu^2 + \dots + \mu^{n-1})$$

↳ Geometric Series.

$$= \mu^{n-1} \delta^2 \left(\frac{1 - \mu^n}{1 - \mu} \right)$$

✓ ~~(\mu < 1)~~

* Probability of extinction $\rightarrow x_n=0$ for some $n \in \mathbb{N}$

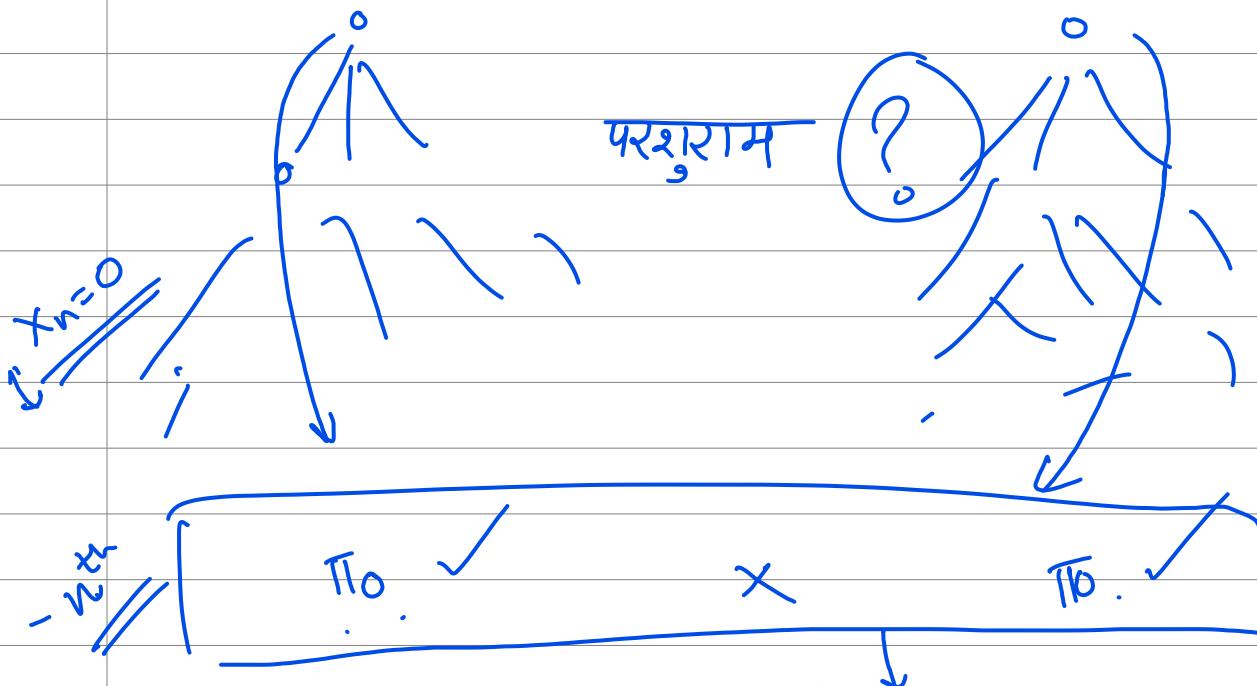
starting from single individual ($x_0=1$) but $x_n=0$ _____

$$\pi_0 = \lim_{n \rightarrow \infty} P(x_n=0) ?$$

=

$\xrightarrow{x_0=1} \pi_0 = P(\text{Pop}^n \text{ ever dies out})$ ← single individual

$$\begin{aligned} \pi_0 &= \sum_{j=0}^{\infty} \left(\text{Pop}^n \text{ ever dies out} / x_i=j \right) \cdot P_j \\ &= \sum_{j=0}^{\infty} \pi_0^j \cdot P_j \end{aligned}$$



$x_n=0$

$x_{n+m}=0$

$x \in \mathbb{N}$

$$P_X(t) = E(t^X) = \sum_x t^x P_X \quad \checkmark$$

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j \cdot p_j = P_Z(\pi_0)$$

PGF of Z

$$\pi_0 = P_Z(\pi_0)$$

by solving this eq we can get π_0 -

e.g. $Z_i \sim \text{Bin}(n=2, p=1/3)$

$$\mu \Rightarrow n \cdot p = 2 \times 1/3 < 1$$

$$\Pi_0 = P_Z(\Pi_0)$$

$$\checkmark t = P_Z(t) \Rightarrow t = \left(\frac{2}{3} + \frac{1}{3} \cdot t \right)^2$$

$$9t = 4 + 4t + t^2$$

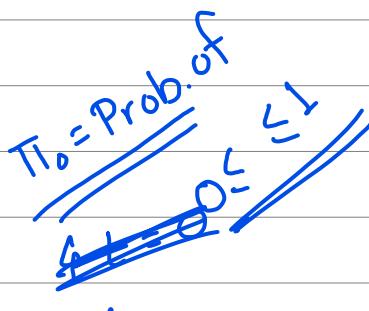
$$0 = 4 - 5t + t^2$$

$$0 \Rightarrow 4 - 4t = t + t^2$$

$$0 = 4(1-t) - t(1-t)$$

$$0 \Rightarrow (4-t)(1-t)$$

$$\underline{\underline{t=4}} \quad \text{or} \quad \underline{\underline{t=1}}$$



minⁿ value of Π_0 obtained by solving this

$$Z = \begin{cases} 0 & 0.3 \\ 1 & 0.3 \\ 2 & 0.4 \end{cases}$$

$$t = \sum t^j \cdot p_j = t^0 \cdot 0.3 + t^1 \cdot 0.3 + t^2 \cdot 0.4$$

$$t = 0.3 + 0.3t + 0.4t^2$$

$$10t = 3 + 3t + 4t^2$$

$$4t^2 - 7t + 3 = 0$$

$$\Rightarrow 4t^2 - 4t - 3t + 3 = 0$$

$$\Rightarrow 4t(t-1) - 3(t-1) = 0$$

$$\Rightarrow t = \frac{3}{4} \text{ or } t = 1$$

$$\Rightarrow \underline{\underline{\Pi_0 = 3/4}}$$

Absu

2001, 4, 6, 10, 15, 16, 22, 25, 30, 35, 38, 39, 43, 44, 45, 47, 50, 55

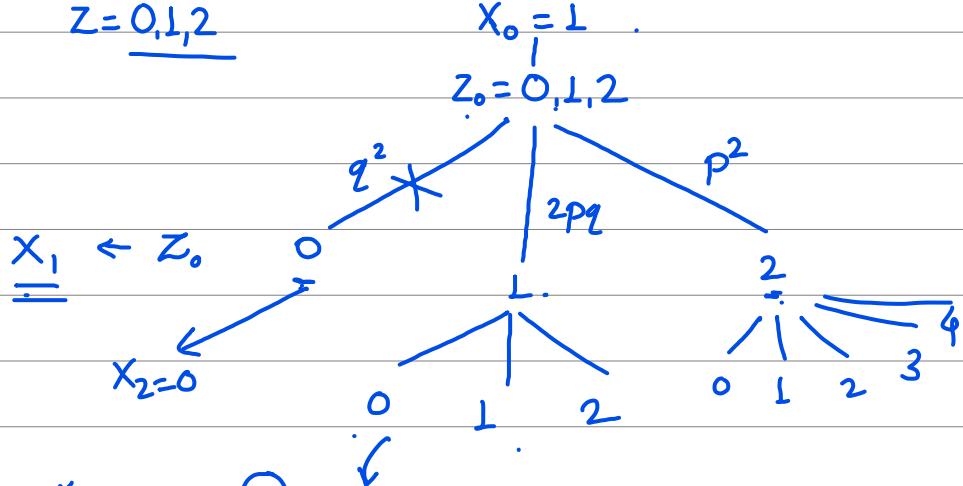
- * Calculate the probability that the popⁿ become extinct for the first time in 3rd generation, consider $x_0=1$, $\underline{z \sim B(2,p)}$
- So we have to find out $P(x_3=0/x_2 \neq 0, x_1 \neq 0, x_0=1)$

$$\begin{aligned} Z &\sim \text{Bin}(2, p) \\ P(Z=0) &= q^2 \\ P(Z=1) &= 2pq \\ P(Z=2) &= p^2 \end{aligned}$$

$$Z = 0, 1, 2$$

$$x_0 = 1$$

$$z_0 = 0, 1, 2$$



Hint?

$$x_1 = 2 \Rightarrow x_2 = \sum_{i=1}^{x_1} z_i = \sum_{i=1}^2 z_i$$

$$z_i \sim B(2, p)$$

$$\underline{\underline{x_2 \sim B(4, p)}}$$

Obtain
it?

$$\checkmark P(x_3=0/x_2 \neq 0, x_1 \neq 0, x_0=1) = ?$$

$$x_0 = 1$$

$$x_1 \sim \text{Bin}(2, p)$$

$$x_2/x_1 \neq 0 \sim \text{Bin}(2 \times x_1, p) \quad \checkmark$$

$$\checkmark x_3/x_2 \neq 0, x_1 \neq 0 \sim \text{Bin}(2 \times x_2, p)$$

Random Walk

$$x_n = x_0 + \sum_{i=1}^n z_i$$

$$x_n = x_{n-1} + z_n$$

$z_i - i.i.d.$

$r.w.$

indep.
increments

$$z_n = \underline{\underline{x_n - x_{n-1}}}$$

* Counting Process



Statespace

discrete

Time domain

cont

$$I = [a, b], [0, \infty) \text{ etc.}$$

state discrete
time cont.

$$\{N(t), t \in I\}$$

no. of events occurred during $(0, t]$
no. of customers arrived in bank/barber
Ticket count

e.g.

no. of deaths occurred due to Covid-19

- . ① $N(t) \geq 0$
- . ② $N(t)$ integer valued
- . ③ $s \leq t \Rightarrow N(s) \leq N(t)$

$$10^{-3} \leq 10^{-5}$$

- ✓ ④ for $s < t$, $N(t) - N(s)$ increment $[s, t]$

* Counting Process with independent increments

for $s < t$, $N(t) - N(s)$ is increment,

Do you remember any Stoc. process? independent increment if $N(s)$ & $N(t) - N(s)$ are indepen-



* Counting Process with stationary increments

$s \leq t$ $N(t) - N(s)$ & $N(t+u) - N(s+u)$ if both have same dist'

if dist' of no. of events occurred in an interval depends on length of interval only

$2 \leq 3$ $\frac{N(3) - N(2)}{\text{same dist'}}$ & $\frac{N(13) - N(12)}{(2,3] \rightarrow 1}$ if both have

length of interval only

$(12, 13] \rightarrow 1$.

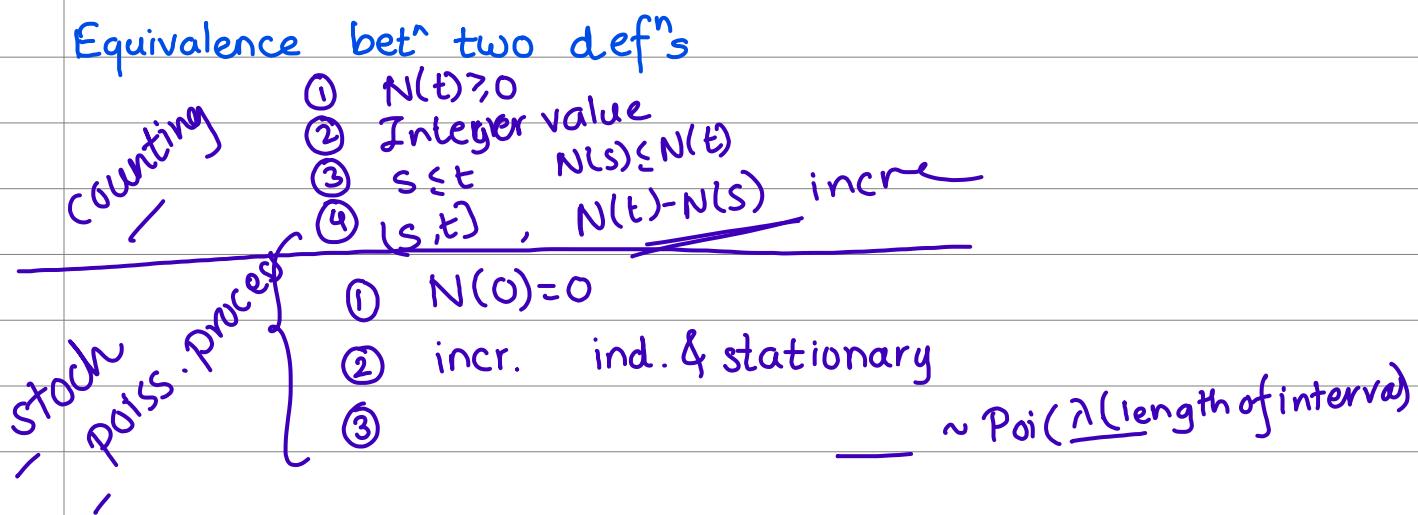
* Poisson Process :-

Defⁿ ① The counting process $\{N(t), t \geq 0\}$ is said to be a poisson process having rate $\lambda, \lambda > 0$ if

① $N(0) = 0$

② The process has indep. & stationary increments

③ The no. of events in ^{any} interval of length t . is poisson distributed with parameter λt
 $(0, t) \quad N(t) \sim \text{Poi}(\lambda t), \quad (s, s+t], \quad N(t+s) - N(s) \sim \text{Poi}(\lambda t)$

Defⁿ ②

Poisson process example

$\{N(t), t \geq 0\}$ poiss $N(0) = 0$ $\lambda = 2/\text{day}$?

$\underline{(0, 5]} \quad N(5) = ?$
 $\sim \text{Poi}(\lambda \cdot 5) = \text{Poi}(10)$

$\underline{P(N(5))} = \underline{\text{P}N(5)} \sim \text{Poi}(10)$
 $0, 1, 2, \dots$
 $E(N(5)) = 10$

Defⁿ ② w^h $\{N(t), t \geq 0\}$ is ^{be} a counting process is said to be Poisson process with rate λ , $\lambda > 0$ if

- ✓ ① $N(0) = 0$
- ✓ ② indep. & stationary increments.
- ③ $P(N(h) = 1) = P_1(h) = \lambda h + o(h)$

$\underline{o(h)} \rightarrow 0$
 as $h \rightarrow 0$
 h small

$$\textcircled{4} \quad P(N(h) \geq 2) = o(h) = \sum_{k=2}^{\infty} P_k(h)$$

equivalence

Def ① \rightarrow Def ②

for first two conditions no need to prove

To Prove Def ② \rightarrow ③ cond. $P_1(h) = \lambda h + o(h)$

$$P_1(h) = P(N(h) = 1) \quad (0, h) \quad N(h) \sim \text{Poi}(\lambda h) \quad \checkmark$$

$$P_n(t) = P(N(t) = n)$$

$$\begin{aligned} &= e^{-\lambda h} \cdot \lambda h \quad \checkmark \\ &= \lambda h \left[1 - \frac{\lambda h + (\lambda h)^2}{2!} - \frac{(\lambda h)^3}{3!} + \dots \right] \quad o(h) = o(h) + o(h) \\ &= \lambda h \underbrace{+}_{o(h)} \end{aligned}$$

$$\frac{P_2(h)}{\checkmark} = \sum_{k=2}^{\infty} P_k(h) = \sum_{k=2}^{\infty} \frac{e^{-\lambda h} (\lambda h)^k}{k!} \quad \checkmark$$

$$P(N(h) \geq 2) = 1 - P_1(h) - P_0(h)$$

$$\begin{aligned} &= 1 - \underline{e^{-\lambda h}} \underline{\lambda h} - \underline{e^{-\lambda h}} \\ &= 1 - \underline{\epsilon(\lambda h + 1)} \cdot e^{-\lambda h} \\ &= 1 - (\lambda h + 1) \left[1 - \lambda h + \frac{(\lambda h)^2}{2} + \dots \right] \end{aligned}$$

$$= 1 - \lambda h \left[1 - \lambda h + \frac{(\lambda h)^2}{2!} + \dots \right] - \left[1 - \lambda h + \frac{(\lambda h)^2}{2!} + \dots \right]$$

$$= o(h)$$

Defⁿ ② → Defⁿ ①

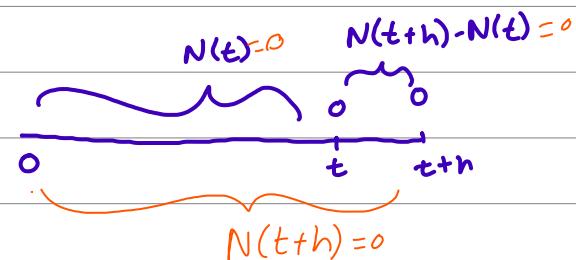
Defⁿ ②
 ① $N(0)=0$ ② Stationary & indep.

$$\begin{cases} \textcircled{3} \quad P_1(h) = \lambda h + o(h) \\ \textcircled{4} \quad \sum_{k=2}^{\infty} P_k(h) = o(h) \quad P(N(h) \geq 2) \end{cases}$$

Defⁿ ① ① ② \checkmark ③ increment $\sim \text{Poi}(\lambda \cdot \text{length})$

$$P_n(t) = P[N(t)=n]$$

$$P_0(t+h) = P[N(t+h)=0]$$



$$= P[N(t)=0, N(t+h)=0]$$

$$= P[\underbrace{N(t)=0}_{(0,t)}, \underbrace{N(t+h)-N(t)=0}_{(t,t+h)}]$$

ind. increment

$$= P[N(t)=0] \cdot P[N(t+h)-N(t)=0]$$

$$= P[N(t)=0] \cdot P_0(h)$$

$$= P[N(t)=0] \cdot [1 - \underbrace{P_1(h)}_{\textcircled{3}} - \underbrace{\sum_{k=2}^{\infty} P_k(h)}_{\textcircled{4}}]$$

$$P[N(t+h)=0] = P[N(t)=0] \cdot [1 - \underbrace{\lambda h}_{+o(h)} + o(h)]$$

$$\rightarrow \lim_{h \rightarrow 0} \frac{P(N(t+h)=0) - P(N(t)=0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-\lambda h P(N(t)=0) + o(h)}{h}$$

$$P'_0(t) = -\lambda P_0(t)$$

$$\frac{1}{P_0(t)} \cdot P'_0(t) = -\lambda$$

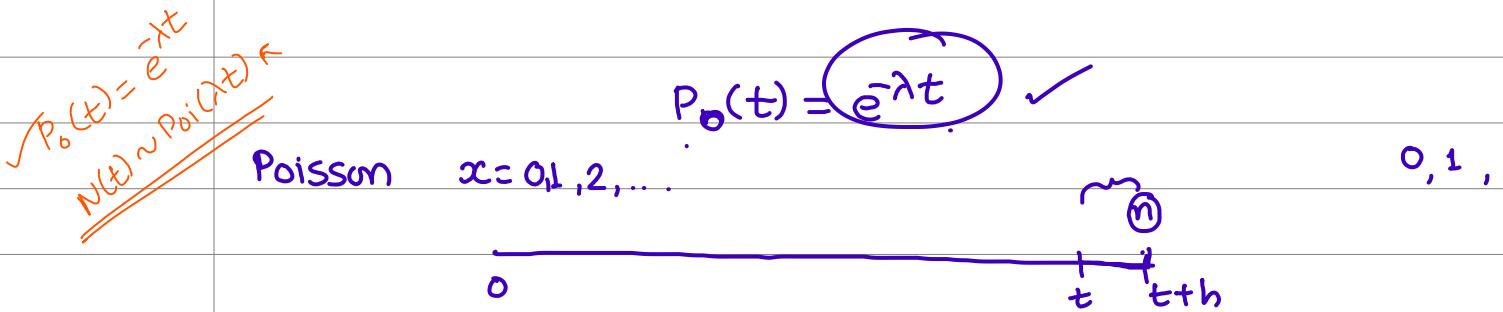
$$\frac{d}{dt} \log P_0(t) = -\lambda$$

$$\int \frac{d}{dt} \log P_0(t) dt = \int_{0}^{t} -\lambda dt$$

$$\log P_0(t) = -\lambda t + C$$

$$P_0(t) = e^{-\lambda t} \cdot e^C$$

$$\textcircled{1} \quad N(0)=0, \Rightarrow P_0(0)=1 \Rightarrow P_0(0) = e^{-\lambda \cdot 0} e^C = e^C = 1 \Rightarrow C=0$$



$$P_n(t+h) = P[N(t+h)=n]$$

$$= P[N(t+h)=n \mid N(t)=n] P[N(t)=n] + \\ P[N(t+h)=n \mid N(t)=n-1] P[N(t)=n-1] + \\ P[N(t+h)=n \mid N(t) \leq n-2] P[N(t) \leq n-2]$$

$$= P_0(h) \cdot P_n(t) + P_1(h) \cdot P_{n-1}(t) + \\ \sum_{k=2}^{\infty} \underline{\underline{P_k(h) \cdot P_{n-k}(t)}}$$

$$P_n(t+h) = [1-\lambda h] \cdot \underline{P_n(t)} + \lambda h P_{n-1}(t) + o(h) + o(h)$$

$$\lim_{h \rightarrow 0} \frac{P_n(t+h) - P_n(t)}{h} = \lim_{h \rightarrow 0} \frac{-\lambda h \cdot P_n(t)}{h} + \frac{\lambda h P_{n-1}(t)}{h} + \frac{o(h)}{h}$$

$$P_n'(t) = -\lambda [P_n(t) - P_{n-1}(t)]$$

$$P_n'(t) + \lambda P_n(t) = -\lambda P_{n-1}(t)$$

$$e^{\lambda t} P_n'(t) + \lambda e^{\lambda t} \underline{P_n(t)} = -\lambda e^{\lambda t} P_{n-1}(t)$$

$$\boxed{\frac{d}{dt} (e^{\lambda t} P_n(t)) = \lambda e^{\lambda t} P_{n-1}(t)} \quad \text{--- } \star$$

$n=1$

$$\frac{d}{dt} (e^{\lambda t} P_1(t)) = \lambda e^{\lambda t} \underline{P_0(t)} = \lambda e^{\lambda t} \underline{e^{-\lambda t}} = \lambda$$

\int_0^t $\int_0^t \lambda$

$$e^{\lambda t} P_1(t) = \underline{\lambda t + C}$$

$$\begin{aligned} N(0) &= 0 \\ N(0) &= 1 \end{aligned}$$

$\omega \cdot P^1$ $\omega \cdot P^0$

put $t=0 \Rightarrow e^{\lambda \cdot 0} P_1(0) = 0 + C$
 $\Rightarrow 0 = C$

$$e^{\lambda t} P_1(t) = \lambda t$$

$$\Rightarrow P_1(t) = e^{-\lambda t} \lambda t$$

Abs. 2001, 5, 6, 9, 10, 14, 15, 16, 27, 33, 35, 39, 41, 43, 44, 45, 51, 54, 55

$$P_0(t) = e^{-\lambda t} \quad P_1(t) = e^{-\lambda t} \lambda t$$

Assume it for $(n-1)$

$$P_{n-1}(t) = \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}$$

from \star

$$\frac{d}{dt} e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

$$\text{from } \star \quad \frac{d}{dt} e^{\lambda t} P_n(t) = \lambda e^{\lambda t} \cancel{e^{-\lambda t}} \cancel{(\lambda t)^{n-1}} = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

$$\int_0^t \frac{d}{ds} e^{\lambda s} P_n(s) ds = \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!} ds$$

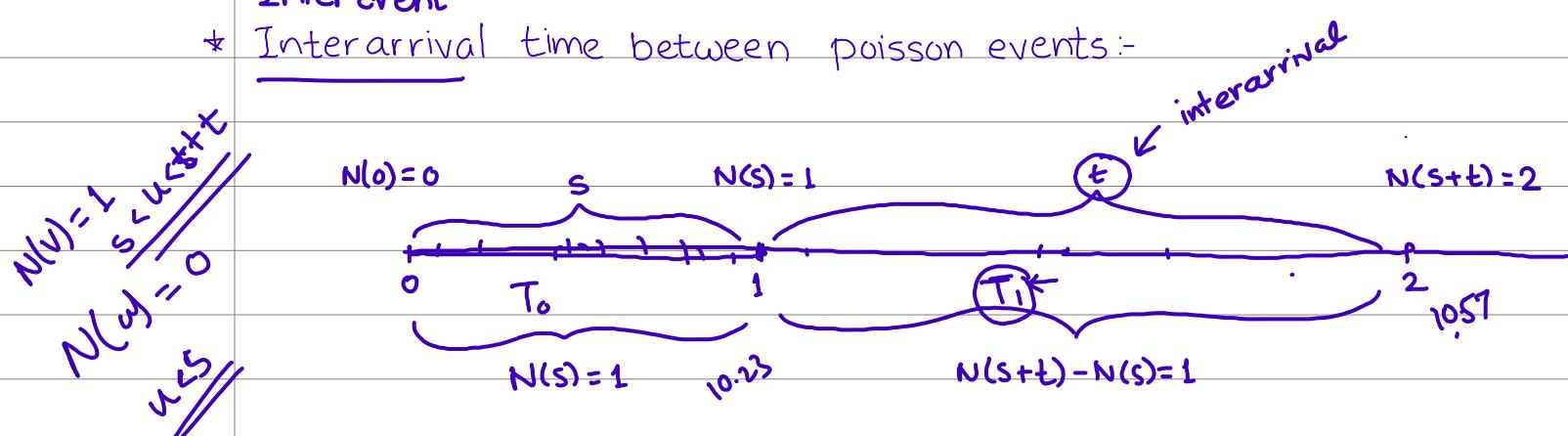
$$e^{\lambda t} P_n(t) = \left(\frac{\lambda^n}{(n-1)!} \frac{s^n}{n} \right)_0^t = \frac{(\lambda t)^n}{n!}$$

$$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

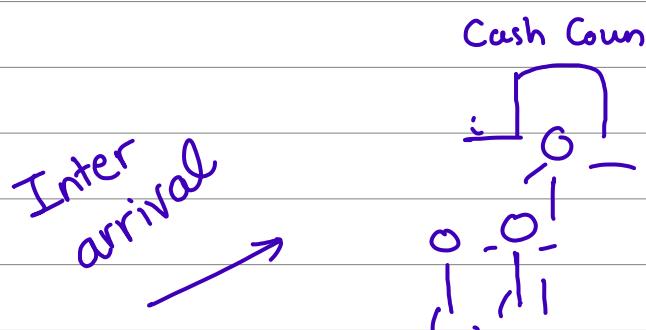
$$N(t) \sim \text{Poi}(\lambda t)$$

Inter event

* Interarrival time between poisson events:-



Bank

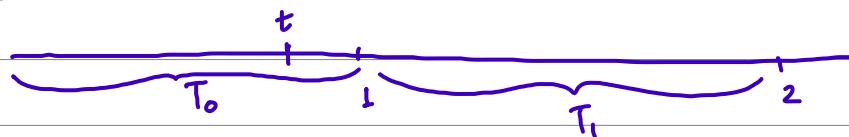


$T_0 \rightarrow$
 $T_1 \rightarrow 2^{\text{nd}}$

T_0, T_1
 interarrival
 $N(T_0) = 1$
 $N(T_0 + T_1) = 2$
 ...

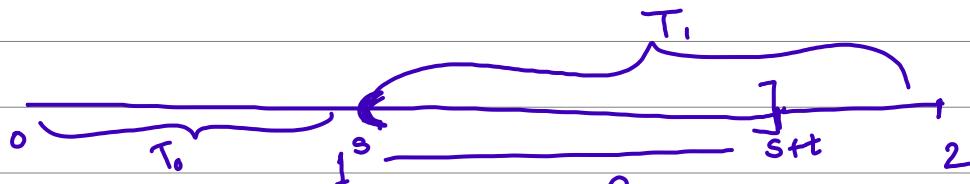
$T_0 + T_1$
 arrival

$N(t) \sim$ follows poisson process



$$\begin{aligned}
 F_{T_0}(t) &= P[T_0 \leq t] = 1 - P[T_0 > t] \\
 &= 1 - P[N(t) = 0] \\
 &= 1 - e^{-\lambda t} \\
 T_0 &\sim \exp(\lambda)
 \end{aligned}$$

$$F_{T_1}(t) = P[T_1 \leq t / T_0 = s] = 1 - P[T_1 > t / T_0 = s]$$



$$\begin{aligned}
 &? \text{ no. of events } = \\
 &= 1 - P[N(t) = 0] \\
 &\quad \cancel{P[N(t+s)]} \quad \cancel{N(s) = 0} \quad \text{Same} \\
 &= 1 - e^{-\lambda t}
 \end{aligned}$$

$$T_i \sim \exp(\lambda)$$

$$\Rightarrow T_0, T_1, \dots \sim \exp(\lambda)$$

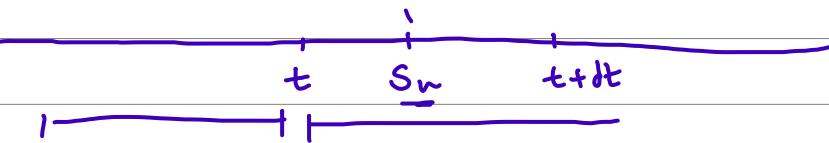
$$E(T_i) = 1/\lambda$$

* Arrival Time

$S_n \Rightarrow$ Arrival time of n^{th} customer/event

$$S_n = \sum_{i=0}^{n-1} T_i$$

T_i 's are i.i.d $\exp(\lambda)$, $N(t) \sim \text{Poi}(\lambda t)$



$$\underline{P(t < S_n < t + \delta t)} = P(N(t) = n-1, N(t + \delta t) - N(t) = 1)$$

$(0, t]$ $(t, t + \delta t]$ ind. incre.

$$= \underline{P(N(t)=n-1)} \cdot P(N(t+\delta t) - N(t) = 1)$$

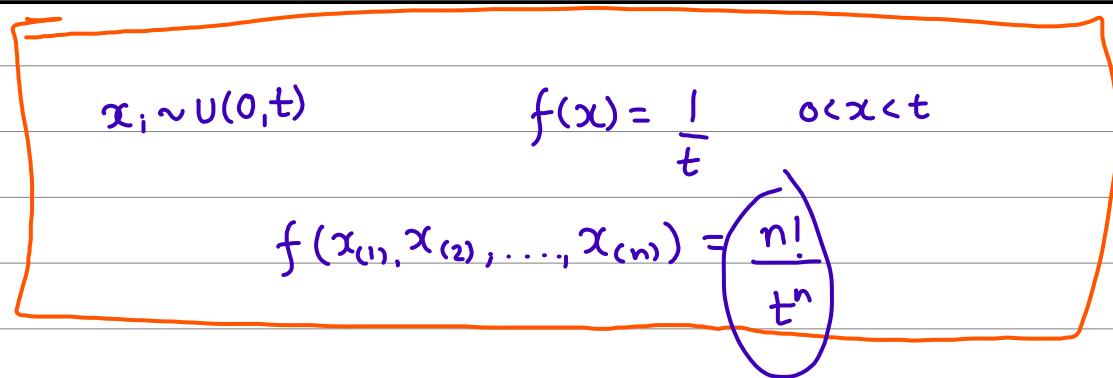
$$F(t+\delta t) - F(t) = \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \cdot e^{-\lambda \delta t} \lambda \delta t$$

$$\begin{aligned}
 f(t) &= \lim_{\delta t \rightarrow 0} \frac{F(t + \delta t) - F(t)}{\delta t} = \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \lim_{\delta t \rightarrow 0} \frac{e^{\lambda \delta t} \lambda \delta t}{\delta t} \\
 &= \frac{e^{-\lambda t}}{(n-1)!} \lambda^n t^{n-1} \\
 &= \frac{\lambda^n}{n!} e^{-\lambda t} t^{n-1}
 \end{aligned}$$

$$S_n \sim \text{Gamma}(n, \lambda)$$

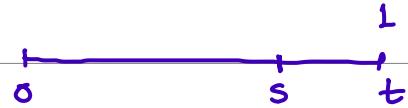
* Conditional distribution of arrival time

- inter arrival
 - exp
 - arrival
- Cond' arrival
 - gamma
 - uniform



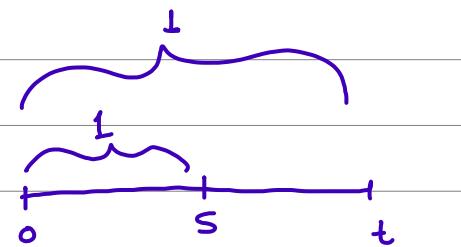
$N(t) = 1$

$S_n = \text{arrival of } n^{\text{th}} = \sum_{i=0}^{n-1} T_i \Rightarrow S_1 = T_0$



For $s \leq t$

$$\begin{aligned} F(s) &= P(S_1 < s \mid N(t) = 1) \\ &= \frac{P(T_0 < s, N(t) = 1)}{P(N(t) = 1)} \\ &= \frac{P[N(S)=1, N(t)-N(s)=0]}{P[N(t)=1]} \end{aligned}$$



ind. incr

$$\begin{aligned} &= \frac{P[N(S)=1] \cdot P[N(t)-N(s)=0]}{P[N(t)=1]} \\ &= e^{-\lambda s} \lambda s \quad e^{-\lambda(t-s)} \\ &\quad \hline e^{-\lambda t} \lambda t \end{aligned}$$

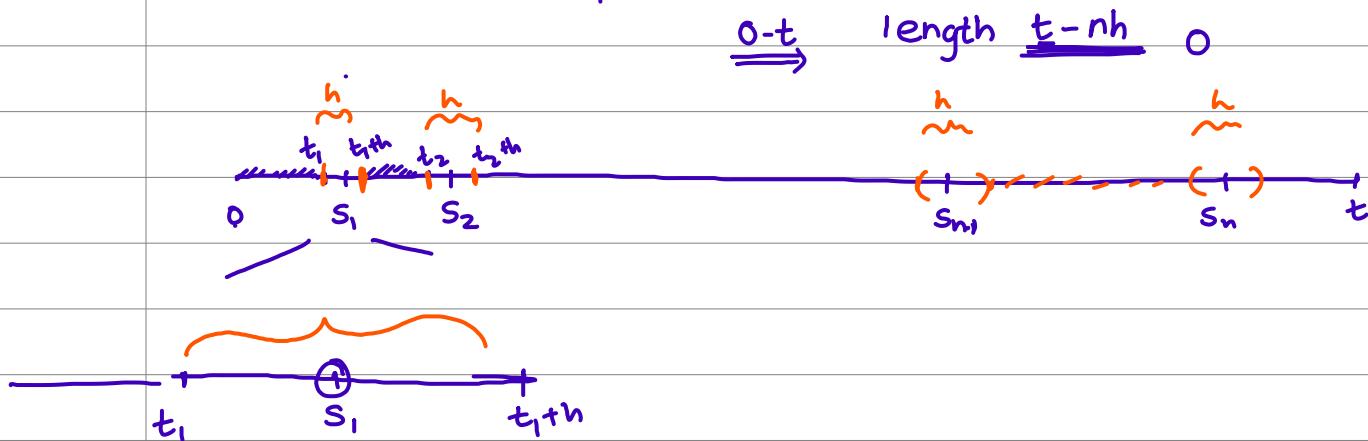
$$= \frac{s}{t}$$

$$F(s) = \frac{s}{t} .$$

$$\underline{U(0,t)} \Rightarrow F(x) = \frac{x}{t}$$

$$S_1 / N(t) = 1 \sim U(0,t) \checkmark$$

- * Given that $N(t) = n$, n arrival times s_1, s_2, \dots, s_n have same distⁿ as of the order statistics of n indep. random variables from $U(0, t)$



let t_i such that $0 < t_i < s_i < t_i + h < t_2 < \dots < t_n < s_n < t_n + h < t$

$$\begin{aligned}
 & P[t_i \leq s_i \leq t_i + h, i=1:n / N(t)=n] \\
 &= P[N(t_i + h) - N(t_i) = 1 \Rightarrow i=1:n, \text{ no event elsewhere in } (0, t)] \\
 &= \frac{P[N(t)=n]}{P[N(t-h)=0]} \\
 &= \frac{[\lambda h e^{-\lambda h}]^n}{\frac{\lambda^n (\lambda t)^n}{n!}} \\
 &= \frac{n!}{\lambda^n} \frac{h^n}{t^n} e^{-\lambda t} e^{\lambda t} \\
 &= \frac{n!}{t^n} h^n
 \end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{1}{h^n} P[t_i \leq s_i \leq t_i + h, i=1:n / N(t)=n] = \frac{n!}{t^n}$$

$$f(s_1, s_2, \dots, s_n / N(t)=n) = \frac{n!}{t^n}$$

Conditional joint pdf of arrival times is equi to joint pdf of order stats $U(0, t)$

Absentee:

2001

6, 9, 10, 12, 15, 16, 22, 23, 31, 33, 35, 39, 43, 44, 45, 47,

Lecture:

Manoj C Patil

