

## Infinite Series:-

$$\underline{\sum x_n}$$

$$\sum_{i=1}^{\infty} x_n = ?$$

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

$$S_n \rightarrow S$$

$$S_{n+1} \rightarrow S$$

$$\lim_{n \rightarrow \infty} S_n - S_{n-1} = 0$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i - \sum_{i=1}^{n-1} x_i = 0$$

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$$\lim_{n \rightarrow \infty} x_n = 0$$

$$(S_n) = \sum_{i=1}^n x_i$$

$$\underline{S_n \rightarrow S}$$

$$\sum \frac{1}{n}$$

$$\sum \frac{(-1)^n}{n}$$

$$\sum x^n = \frac{1}{1-x} \quad |x| < 1$$

$$\checkmark \sum \frac{1}{n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\checkmark} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\checkmark}$$

$$\geq 1 + \frac{1}{2} [1 + 1 + 1 + \dots]$$

$$\geq \infty$$

$$\begin{aligned} & \sum \frac{(-1)^{n+1}}{n} \\ &= \underbrace{1 - \frac{1}{2}}_{\checkmark} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{\checkmark} + \underbrace{\frac{1}{5} - \frac{1}{6}}_{\checkmark} \end{aligned}$$

$$\begin{aligned} S_n &= 1 - \left( \frac{1}{2} - \frac{1}{3} \right) - \left( \frac{1}{4} - \frac{1}{5} \right) - \dots \\ &= 1 - \left( \frac{1}{2 \cdot 3} \right) - \left( \frac{1}{4 \cdot 5} \right) \end{aligned}$$

$$\underline{S_n \downarrow}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} \rightarrow 0$$

→ 1

$$\sum_n \frac{(-1)^n}{n} \quad \text{cgt} \quad \frac{(-1)^n}{n} \rightarrow 0$$

$\sum r^n$        $|r| < 1$       cgt       $r^n \rightarrow 0$        $0.5^2$

$$\sum \frac{1}{n^2} \quad \text{cgt} \quad \frac{1}{n^2} \rightarrow 0$$

$\sum \frac{1}{n}$

$\text{duge}$        $\frac{1}{n} \rightarrow 0$

Cauchy Criterion for convergence of series  
\*\*

for any  $\epsilon > 0 \exists K(\epsilon) \in \mathbb{N}, \exists |S_n - S_m| < \epsilon \Rightarrow n > m > K(\epsilon)$

$$\Rightarrow \left| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right| < \epsilon \quad \Rightarrow n > m > K(\epsilon)$$

$$\Rightarrow \left| \sum_{i=m+1}^n x_i \right| < \epsilon$$

Let  $x_n$  be a sequence of non-negative real numbers then the series P  $\sum x_n$  converges if and only if the sequence  $S_k$  of partial sum is bounded.

① Let  $\sum_{n=1}^{\infty} x_n$  converges ,  $\sum_{n=1}^{\infty} x_n = s$

$\Rightarrow S_n = \sum_{i=1}^n x_i = \text{seq of partial sums}$

$\Rightarrow S_n$  is convergent to  $s$ .

$\Rightarrow$  Every convergent seq<sup>n</sup> is bounded

$\Rightarrow S_n$  is bounded.

②  $S_n$  is bounded  $\Rightarrow$  To prove  $S_n$  is cgt.

$$S_{n+1} = \sum_{i=1}^{n+1} x_i = S_n + \underline{x_{n+1}} \quad \text{as } x_{n+1} > 0$$

$$S_{n+1} \geq S_n$$

$S_n$  is monotonically  $\uparrow$  & bounded  
by MCT it is cgt.

Show that  $\sum_{n=0}^{\infty} r^n = 1+r+r^2+\dots = \frac{1}{1-r}$  if  $|r| < 1$

$$\left| \begin{array}{l} S_{n+1} = \sum_{i=1}^{n+1} r^i = 1+r+r^2+\dots+r^{n-1}+r^n \\ S_n = \sum_{i=1}^n r^i = 1+r+r^2+\dots+r^{n-1} \\ rS_n = r+r^2+\dots+r^n = S_{n+1}-1 \end{array} \right.$$

$$1+rS_n = S_{n+1}$$

$$\lim_{n \rightarrow \infty} (1+r \cdot S_n) = \lim_{n \rightarrow \infty} S_{n+1} \quad ?$$

$$1+r \cdot \overline{S} = S \quad \Rightarrow \quad S = \frac{1}{1-r} \quad ?$$

$$\begin{aligned}
 p &> 1 \\
 \sum \frac{1}{n^p} &= 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{\leq 2^p} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}}_{\leq 4^p} + \frac{1}{8^p} + \dots \\
 &\leq 1 + \underbrace{\frac{1}{2^p} + \frac{1}{2^p}}_{\leq 2^p} + \underbrace{\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}}_{\leq 4^p} + \frac{1}{8^p} + \dots \\
 &\leq 1 + \frac{2}{2^p} + \frac{4}{2^{2p}} + \frac{8}{2^{3p}} + \dots \\
 &\leq 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \dots \\
 \sum_{n=1}^{\infty} \frac{1}{n^p} &\leq \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \\
 \text{as } p &> 1, \quad \frac{1}{2^{p-1}} < 1 \quad \Rightarrow \quad \sum r^n = \frac{1}{1-r} \quad \text{cgt.} \\
 \Rightarrow \sum \frac{1}{n^p} &\text{ is also cgt.}
 \end{aligned}$$

Comparison Test :-  $K \in \mathbb{N} \Rightarrow \frac{0 \leq x_n \leq y_n}{\sum y_n \text{ cgt.}} \Rightarrow n \geq K \Rightarrow \sum x_n \text{ cgt.}$

by Cauchy criterion for convergence of series  
for any  $\epsilon > 0 \exists M(\epsilon) \in \mathbb{N}. |y_{m+1} + y_{m+2} + \dots + y_n| < \epsilon \Rightarrow n > m > M(\epsilon)$

$$K'(\epsilon) = \max(M(\epsilon), K)$$

for  $\forall m \geq K(\epsilon)$

$$|y_{m+1} + \dots + y_n| < \epsilon$$

by  $\Delta$  inequality

$$\underline{|x_{m+1} + x_{m+2} + \dots + x_n|} < |y_{m+1} + \dots + y_n| < \epsilon$$

by Cauchy criterion  $\sum x$  converges.

$$\sum \frac{1}{n^2+n}$$

$$\sum x_n$$

$$\sum y_n = \sum \frac{1}{n}$$

$\rightarrow$  cgt

$$r = \lim \frac{x_n}{y_n} = \lim \frac{1}{n^2+n} \times n^2 = \lim_{n \rightarrow \infty} \frac{1}{1+1/n} = 1$$

$$r \neq 0$$

$$0 < x_n, 0 < y_n \rightarrow n$$

$$r = \lim \frac{x_n}{y_n}$$

for any  $\epsilon > 0 \exists K(\epsilon) \in \mathbb{N}, \exists \left| \frac{x_n}{y_n} - r \right| < \epsilon$

$$r - \epsilon < \frac{x_n}{y_n} < r + \epsilon$$

$$y_n(r - \epsilon) \leq x_n \leq y_n(r + \epsilon)$$

$$x_n \leq y_n, \sum y_n \text{ cgt} \Rightarrow \sum x_n \text{ cgt}$$

if  $\sum y_n \text{ cgt} \Rightarrow \sum x_n \text{ cgt}$

if  $\sum x_n \text{ cgt} \Rightarrow \sum y_n \text{ cgt}$

if  $r=0,$

$$r-\varepsilon < \frac{x_n}{y_n} < r+\varepsilon$$

$$-\varepsilon < \frac{x_n}{y_n} < \varepsilon$$

$$-\varepsilon < 0 < \frac{x_n}{y_n} < \varepsilon$$

$$x_n \leq y_n \cdot \varepsilon$$

$\Rightarrow$  if  $\sum y_n \text{ cgt} \Rightarrow \sum x_n \text{ cgt}.$

$$\sum \frac{(-1)^n}{n} \xrightarrow{\text{cgt}} \sum \frac{1}{n} \text{ divergent}$$

Absolute:  $\sum |x_n| \text{ abs. cvgt if } \sum |x_n| \text{ is cvgt.}$

Conditional Convergence  $\sum x_n$  is cgt but  $\sum |x_n|$  is not cgt.

$$\sum \frac{(-1)^n}{n^2} \text{ abs cgt}$$

$$\sum \frac{1}{n^2} \text{ cgt}$$

for any  $\varepsilon > 0 \exists K(\varepsilon) \in \mathbb{N}.$

$$\frac{|x_{m+1} + \dots + x_n|}{\sum |x_n|} \leq |x_{m+1}| + |x_{m+2}| + \dots + |x_n| \leq \varepsilon \quad \begin{matrix} \rightarrow n \rightarrow \\ n \rightarrow m > K(\varepsilon) \end{matrix}$$

Root test       $r = \lim |x_n|^{1/n}$  exists.

$$\textcircled{1} \quad r \leq 1, \exists r_1 \leq r < 1$$

for some  $K(\epsilon) \in \mathbb{N}$ .

$$|x_n|^{1/n} \leq r, \quad \nexists n \geq K(\epsilon)$$

$$\Rightarrow |x_n| \leq r_1^n \quad (r_1 < 1)$$

by Comparison Test

$$\sum |x_n| \leq \sum r_1^n \quad \begin{matrix} \uparrow \\ \text{cgt.} \end{matrix} \quad \begin{matrix} \nearrow \\ \text{cgt.} \end{matrix}$$

$$\textcircled{2} \quad r > 1, \exists r_1 < r$$

~~Root~~

$$\begin{aligned} & r \leq r_1 \leq |x_n|^{1/n} \\ & 1^n \leq r_1^n \leq |x_n| \\ & \sum_{\infty}^1 \leq \sum r_1^n \leq \sum |x_n| \quad \text{dvgt.} \end{aligned}$$

Ratio

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right|$$

$$\textcircled{1} \quad r < 1 \quad \text{cgt.}$$

$r > 1$       dvgt

$r = 1$       Test fails

$$\textcircled{2} \quad r \leq 1$$

$$\left| \frac{x_{n+1}}{x_n} \right| < r$$

for  $n \geq K(\epsilon)$

$$\begin{aligned}
 |x_{n+1}| &< r \cdot |x_n| \\
 &\leq r \cdot r \cdot |x_{n-1}| = r^2 |x_{n-1}| \\
 &\vdots \\
 &\leq r^n \cdot |x_1|
 \end{aligned}$$

$$\begin{aligned}
 \sum \frac{1}{(n+1)(n+2)} & \\
 \text{Ratio Test} : - \quad r &= \lim \left| \frac{x_{n+1}}{x_n} \right| \\
 &= \lim \left| \frac{(n+1)(n+2)}{(n+2)(n+3)} \right| \\
 &= \lim \left| \frac{1 + \frac{1}{n}}{1 + \frac{3}{n}} \right| \\
 &= 1 \quad \text{Test fail}
 \end{aligned}$$

$x > 1/x$

Unit Comparison Test  $\sum y_n = \sum \frac{1}{n^2}$

$$\begin{aligned}
 r &= \lim \left| \frac{x_n}{y_n} \right| \\
 &= \lim \left| \frac{n^2}{(n+1)(n+2)} \right| \\
 &= \lim \left| \frac{1}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} \right| \\
 &= 1 \neq 0 \quad \sum |y_n| \text{ cgt}
 \end{aligned}$$

$$\Rightarrow \sum |x_n| \text{ cgt.}$$

$$\sum \frac{1}{n^2} > \sum \frac{1}{(n+1)(n+2)}$$

cgt.

$$\sum \frac{1}{(n+1)(n+2)} \geq x_n = \frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

$$S_1 = x_1 = 1 - \frac{1}{2}$$

$$S_2 = S_1 + x_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$$

$$S_3 = S_2 + x_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

$\sum x_n$  cgt if  $S_n$  cgt.

$$\lim S_n = \lim 1 - \frac{1}{n+1}$$

$$S_n \rightarrow 1 = 1$$

$$\sum 2^n \quad \text{Ratio} \Rightarrow \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{2^{n+1}}{2^n} \right| = 2 > 1 \quad \text{dgt.}$$

$\sum 2^n$  Root test

$$\lim |x_n|^{1/n} = \lim [2^n]^{1/n} = 2 > 1 \quad \text{dgt.}$$

$$\sum 2^{-1/n}$$

Ratio  $\lim_{n \rightarrow \infty} \frac{2^{-1/(n+1)}}{2^{-1/n}}$

$$\lim_{n \rightarrow \infty} \frac{2^{\frac{-1}{n+1}} + \frac{1}{n}}{2^{\frac{-1}{n+1}}} \rightarrow 1$$

Test fails

$$\sum 2^{-1/n}$$

$$\sum x_n \text{ cgt} \Rightarrow \lim x_n = 0$$

$$\lim x_n \neq 0 \Rightarrow \sum x_n \text{ not cgt.}$$

$$\lim 2^{-1/n} = 1$$

$$\Rightarrow \sum 2^{-1/n} \text{ diverges.}$$

\*  $x_n = n/2^n$

$$\text{Ratio Test } r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} \right|$$

$$= \lim \left| \frac{n+1}{n} \right| \cdot \lim \left| \frac{2^n}{2^{n+1}} \right|$$

$$= \frac{1}{2} < 1$$

$$\Rightarrow \sum x_n \text{ cgt.}$$

Root test :-

$$r = \lim |x_n|^{1/n} = \lim \left| \frac{n}{2^n} \right|^{1/n}$$

$$= \frac{1}{2} \lim \frac{n^{1/n}}{2} \rightarrow 1$$

$$= \frac{1}{2} < 1$$

$$\underline{\sum \frac{3^n}{n!}} ?$$

Ratio Test

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left( \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n} \right)$$

$$= \lim \cancel{\frac{3}{n+1}} \frac{3}{n}$$

$$= 0 < 1 \text{ cgt.}$$

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \dots$$

$$x_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$\frac{1}{\sqrt{1 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 3}} + \frac{1}{\sqrt{3 \cdot 4}}$$

$$x_n = \frac{1}{\sqrt{n(n+1)}}$$

limit  
Comparison

$$\lim \left| \frac{n}{\sqrt{n(n+1)}} \right|$$

$$= \lim \left| \frac{1}{\sqrt{1 + 1/n}} \right|$$

$$= 1$$

$$\Rightarrow \sum y_n \text{ diverges} \Rightarrow \sum x_n \text{ diverges}$$

Ratio:  $x_n = \frac{1}{\sqrt{n(n+1)}}$

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \frac{\sqrt{n(n+1)}}{\sqrt{(n+1)(n+2)}} = \lim \sqrt{\frac{1}{1 + 2/n}}$$

$$= 1 \quad \text{Test fails.}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots$$

$$x_n = \frac{1}{n^n}$$

Root test  $\lim |x_n|^{1/n} = \lim \left| \frac{1}{n^n} \right|^{1/n} = \lim \left| \frac{1}{n} \right| = 0 < 1$

$$\sum \frac{n^2-1}{n^2+1} \quad \text{by necessary cond'} \quad \lim \frac{n^2-1}{n^2+1} = 1 \neq 0$$

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \dots$$

$$\underbrace{\left( \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots \right)} + \left( \frac{1}{3^2} + \frac{1}{3^4} + \dots \right)$$

$$\underbrace{\sum \left( \frac{1}{3^2} \right)^n}_{=}$$

$$\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \dots$$

$$x_n = \frac{1}{n \cdot 2^n}$$

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \dots$$

$$x_n = \frac{n^2(n+1)^2}{n!}$$

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \frac{(n+1)^2(n+2)^2}{(n+1)!} \cdot \frac{n!}{(n+1)^2(n^2)}$$

$$= \lim \frac{1}{(n+1)} \cdot (1 + 2/n)^2 \\ = 0$$

$$x_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{7 \cdot 10 \cdots (3n+4)}$$

$$\lim \left| \frac{x_{n+1}}{x_n} \right|$$

$$\frac{1 \cdot 2 \cdots n \cdot (n+1)}{7 \cdot 10 \cdots (3n+4)(3n+1+4)} = \lim \left| \frac{\frac{n+1}{(3(n+1)+4)}}{\frac{n+1}{(3(n+1)+4)}} \right|$$

$$\frac{1}{3+4/(n+1)} = \lim \left| \frac{1}{3+4/(n+1)} \right|$$

$$= \frac{1}{3} < 1$$

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{1}{n} ?$$

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{2(n+1)-1}{2(n+1)} \left( \frac{1}{n+1} \right) \right| = 1$$

$$x_n \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} \cdot \frac{1}{n}$$

$$\lim |x_n|^{\frac{1}{n}} = \frac{1}{2} \left( \lim \frac{1^{\frac{1}{n}} 3^{\frac{1}{n}} \cdots (2n-1)^{\frac{1}{n}}}{(n!)^{\frac{1}{n}}} \cdot \frac{1}{n^{\frac{1}{n}}} \right)$$

$$= \frac{1}{2} < 1$$









## Functions and continuity

limit / Cluster Point :-  $\underline{A \subseteq \mathbb{R}}$ ,  $c \in \mathbb{R}$  for every  $\delta > 0$   
 $\exists \underline{x \in A, x \neq c, \exists |x - c| < \delta}$

for any  $\underline{\delta > 0}$ ,  $\underline{\delta_\epsilon(c) \cap A \neq \emptyset}$



①  $c$  cluster,  $\exists \underline{a_n \in A} \quad a_n \rightarrow c$

$\Rightarrow c$  cluster pt. of A

$$\left\{ \begin{array}{l} \text{for any } \frac{1}{n} > 0 \quad \exists x_n \in A \quad \Rightarrow \quad |x_n - c| < \frac{1}{n} \\ \quad \uparrow n \in \mathbb{N} \end{array} \right. \Rightarrow c - \frac{1}{n} < x_n < c + \frac{1}{n}$$

$$\Rightarrow x_n \rightarrow c$$

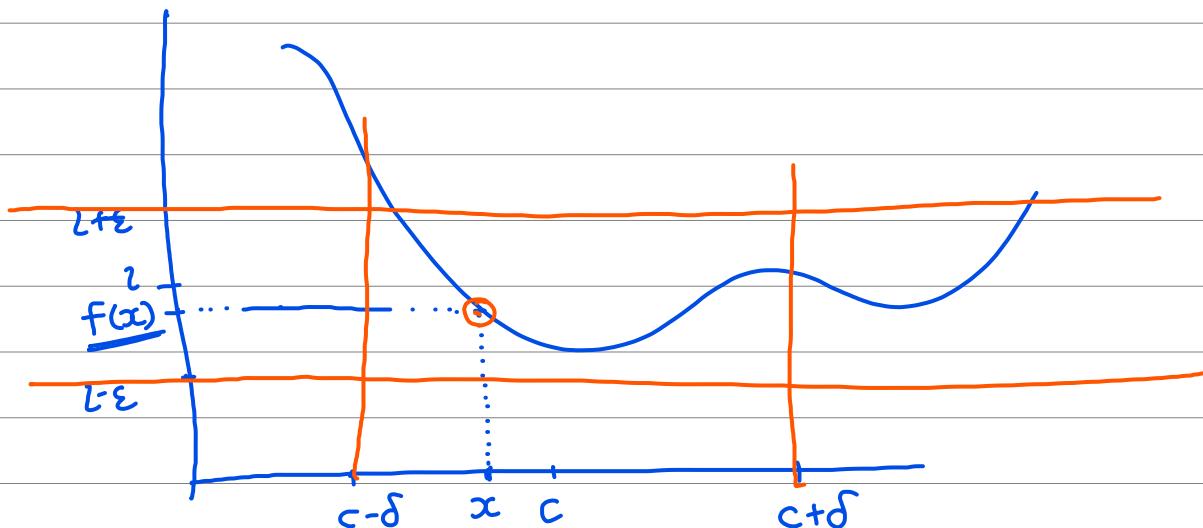
Suppose we have a seq<sup>n</sup>  $a_n \in A, \exists a_n \rightarrow c$ .  
To show that :- c cluster pt. of A

✓  $a_n \rightarrow c$

for any  $\epsilon > 0$ ,  $\exists K(\epsilon) \in \mathbb{N}$ ,  $\exists |a_n - c| < \epsilon$   $\Rightarrow n > K(\epsilon)$

$\Rightarrow |a_n - c| < \epsilon$   $a_n \in A \cap S_\epsilon(c)$

$\Rightarrow c$  is also cluster pt. of A



Limit of func :

$f: A \rightarrow \mathbb{R}$ ,  $c$  cluster of  $A$ ,  $L \in \mathbb{R}$

for any  $\epsilon > 0$   $\exists \delta_\epsilon > 0$   $\exists |x - c| < \delta_\epsilon \Rightarrow |f(x) - L| < \epsilon$   
 $\Rightarrow x \in V_\delta(c) \Rightarrow f(x) \in V_\epsilon(L)$

$\therefore f: A \rightarrow \mathbb{R}$ ,  $c$  cluster pt. of  $A$ .  $\Rightarrow f$  can have only one limit pt. at  $C$ .

Contradiction .  $L$  &  $L'$

for  $\epsilon > 0$   $\left\{ \begin{array}{l} \frac{\epsilon}{2} > 0 \quad \exists \delta'_\epsilon > 0 \quad \exists |x - c| < \delta'_\epsilon \Rightarrow |f(x) - L| < \epsilon/2 \\ \frac{\epsilon}{2} > 0 \quad \exists \delta''_\epsilon > 0 \quad \exists |x - c| < \delta''_\epsilon \Rightarrow |f(x) - L'| < \epsilon/2 \end{array} \right.$

$$\begin{aligned}
 |L - L'| &= \underline{|L - f(x)|} + \underline{|f(x) - L'|} \\
 &\leq |L - f(x)| + |f(x) - L'| \\
 &\leq \varepsilon_1 + \varepsilon_2 \\
 &\leq \varepsilon
 \end{aligned}$$

$L = L'$

QED

seq Criteria

def seq convergent seqn & def limit pt. of func at some  $c$

$f: A \rightarrow \mathbb{R}$ ,  $c$  is cluster pt. of  $A$

① for any  $\varepsilon > 0 \exists \delta_\varepsilon > 0 \ni \text{if } \underline{|x - c| < \delta_\varepsilon} \Rightarrow \underline{|f(x) - L| < \varepsilon} \quad x \in A$

②  $f: A \rightarrow \mathbb{R} \quad c$   $A \quad f(x) \text{ converges to } L \text{ at } c$

$\underline{\exists x_n \rightarrow c} \Rightarrow \underline{f(x_n) \rightarrow L}$

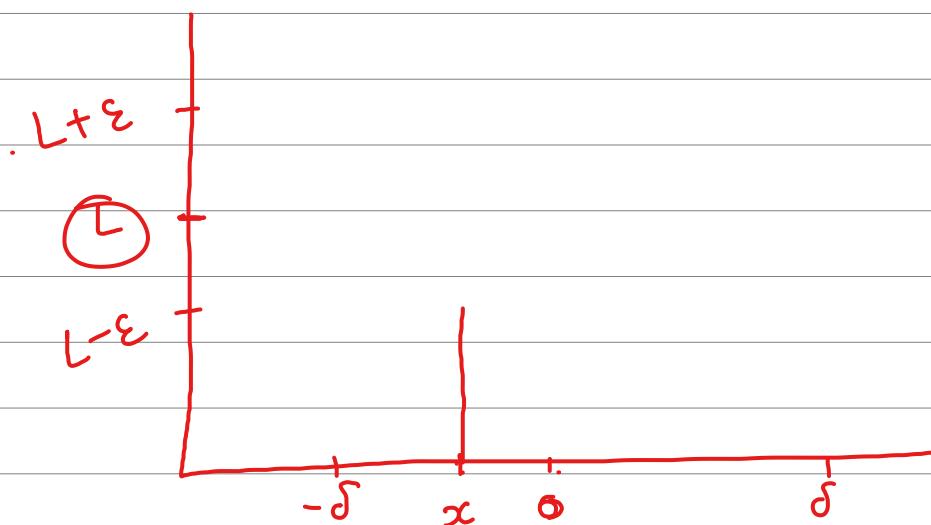
①  $|x - c| < \delta_\varepsilon \quad |f(x) - L| > \varepsilon_0$  does not converges

②  $x_n \rightarrow c \quad \text{but} \quad f(x_n) \not\rightarrow L$   $f(x_n) \rightarrow \text{out } c$

$$\boxed{\lim_{x \rightarrow c} f(x) = L}$$

$$f(x) = \operatorname{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$



$$(-\delta, 0) \quad f(x) = -1 \quad (0, \delta) \quad f(x) = +1$$

$f: A \rightarrow \mathbb{R}$ ,  $c$  cluster pt. of  $A$

if limit exists at pt.  $c$ ,  $c \in \mathbb{R}$   
 for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni$  if  $|x - c| < \delta_\epsilon \Rightarrow |f(x) - L| < \epsilon$

$$\Rightarrow (L - \epsilon < f(x) < L + \epsilon) \Rightarrow x \in V_{\delta_\epsilon}(c)$$

Completeness

$$M = \sup \{ f(x), x \in V_{\delta_\epsilon}(c) \}$$

$$\Rightarrow |f(x)| \leq M$$

for some  $M > 0$

$$\begin{array}{l}
 f: A \rightarrow \mathbb{R}, c \text{ cluster pt. of } A \quad \left. \begin{array}{l} \lim_{x \rightarrow c} f(x) = L \\ \lim_{x \rightarrow c} g(x) = M \end{array} \right\} \\
 g: A \rightarrow \mathbb{R} \\
 \downarrow \\
 (f+g): A \rightarrow \mathbb{R} \\
 \left. \begin{array}{l} \lim_{x \rightarrow c} (f+g)(x) = L+M \end{array} \right\}
 \end{array}$$

$$\lim_{x \rightarrow c} f(x) = L \quad \text{for } \varepsilon_1 > 0, \exists \delta'_\varepsilon > 0 \ni |x - c| < \delta'_\varepsilon \Rightarrow |f(x) - L| < \varepsilon_1$$

$$\text{for } \varepsilon_2 > 0, \exists \delta''_\varepsilon > 0 \ni |x - c| < \delta''_\varepsilon \Rightarrow |g(x) - M| < \varepsilon_2$$

$$\delta_\varepsilon = \min(\delta'_\varepsilon, \delta''_\varepsilon) \quad |x - c| < \delta_\varepsilon$$

$$|(f+g)(x) - (L+M)|$$

$$= |(f(x) + g(x)) - (L+M)|$$

$$= |\underline{f(x)-L} + \underline{g(x)-M}| \leq |f(x)-L| + |g(x)-M|$$

$$\leq \varepsilon_1 + \varepsilon_2$$

$$\leq \varepsilon$$

$$|(f+g)(x) - (L+M)|$$

$$= |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - LM|$$

$$\leq |g(x)| |f(x)-L| + |L| |g(x)-M|$$

$$\leq M \cdot \frac{\epsilon}{2M} + L \cdot \frac{\epsilon}{2L}$$

$f: A \rightarrow \mathbb{R}$ ,  $c$  cluster pt. of  $A$   
 $a \leq f(x) \leq b$ ,  $f$  limit exists at  $c$ .

for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |x - c| < \delta_\epsilon \Rightarrow |f(x) - L| < \epsilon$

seq Criter  $\exists x_n \in A$   $x_n \rightarrow c \Rightarrow f(x_n) \rightarrow L$

$a \leq f(x) \leq b \Rightarrow x \in A$   
 $a \leq f(x_n) \leq b \Rightarrow x_n$

by squeeze theo.

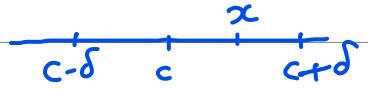
$$a \leq \lim_{x_n \rightarrow c} f(x_n) \leq b$$

$$a \leq L \leq b$$

$c$  cluster of  $A$  iff  $\exists a_n \in A \quad a_n \rightarrow c$

i)  $c$  cluster pt. of  $A$ .

for every  $\delta > 0 \quad \exists x \in A \quad |x - c| < \delta$



$\nexists n \in \mathbb{N}, \frac{1}{n} > 0 \quad \exists x_n \in A \rightarrow |x_n - c| < \frac{1}{n} \Rightarrow c - \frac{1}{n} < x_n < c + \frac{1}{n}$

$x_n \rightarrow c$

ii)  $a_n \in A, a_n \rightarrow c$

for any  $\varepsilon > 0 \quad \exists k(\varepsilon) \in \mathbb{N}, \quad \exists \frac{|a_n - c| < \varepsilon}{\text{infinite}} \quad \nexists n, k(\varepsilon)$

$\Rightarrow c$  is cluster pt. of  $A$ .

$f, g, h : A \rightarrow \mathbb{R}, \quad c$  cluster pt. of  $A$

$f(x) \leq g(x) \leq h(x) \quad \Rightarrow x \in A \quad x \neq c$

$$\checkmark \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$$

To show  $\lim_{x \rightarrow c} g(x) = L$

for any  $\varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \exists |x - c| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$

$$\begin{aligned} \dots \quad \delta_\varepsilon > 0 \quad \exists |x - c| < \delta_\varepsilon \Rightarrow & |h(x) - L| < \varepsilon \\ & \Rightarrow L - \varepsilon < h(x) < L + \varepsilon \end{aligned}$$

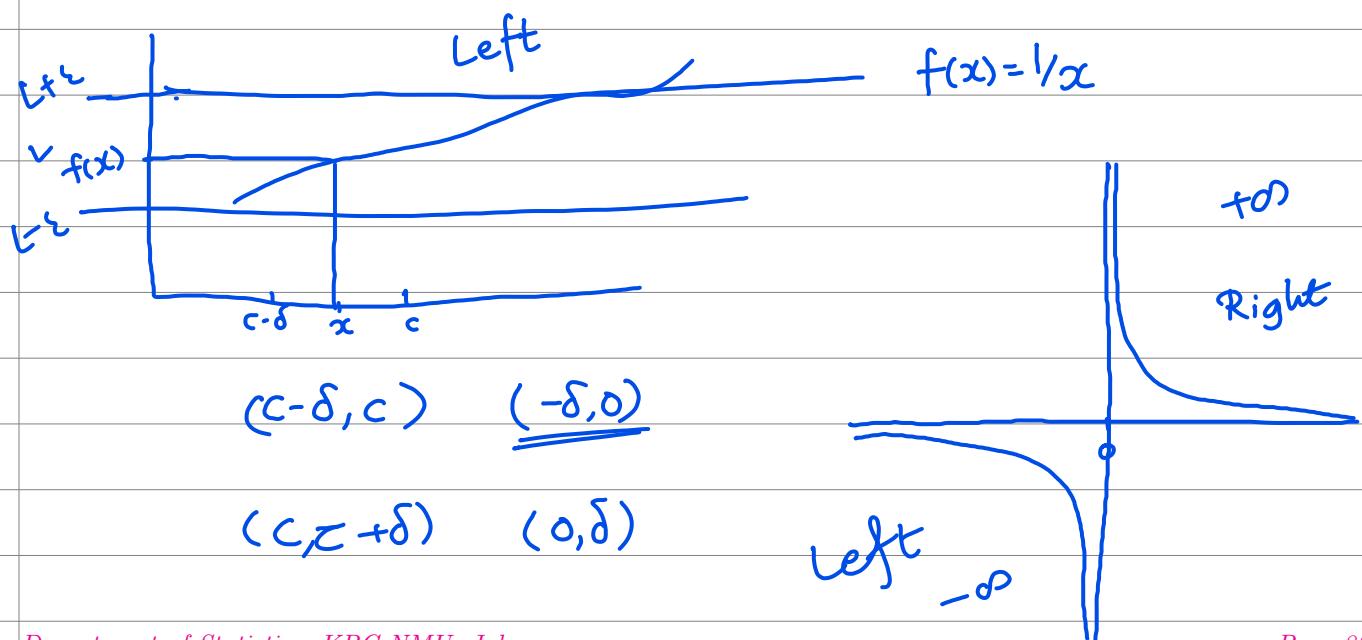
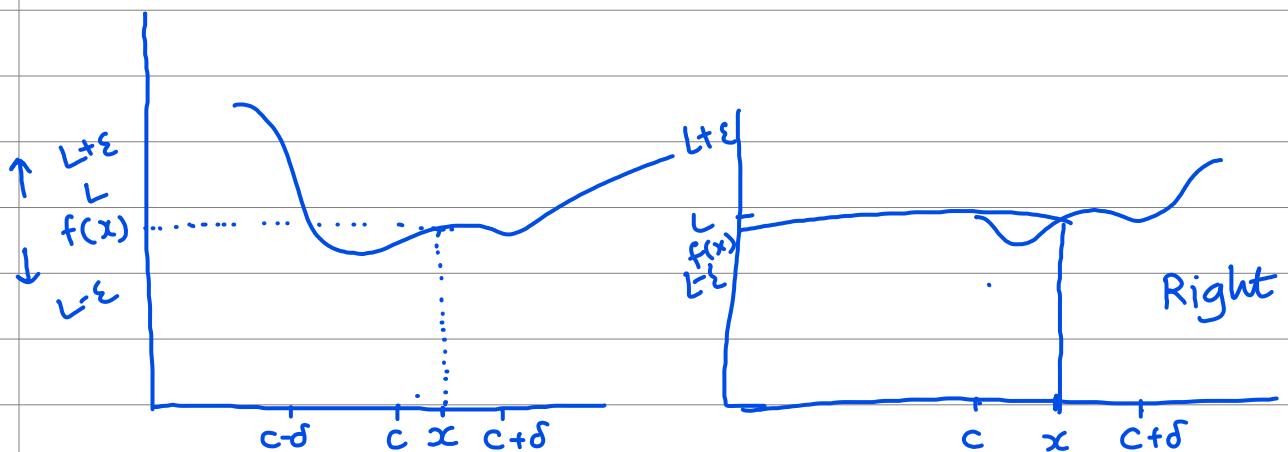
$$f(x) \leq g(x) \leq h(x)$$

$$\Rightarrow L - \varepsilon < f(x) \leq \underline{g(x)} \leq h(x) < L + \varepsilon$$

$$\Rightarrow L - \varepsilon < g(x) < L + \varepsilon$$

$$\Rightarrow |g(x) - L| < \varepsilon$$

$$\Rightarrow g(x) \rightarrow L$$



$$\checkmark \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x)$$

$(c-\delta, c+\delta)$                              $(c, c+\delta)$                              $(c-\delta, c)$

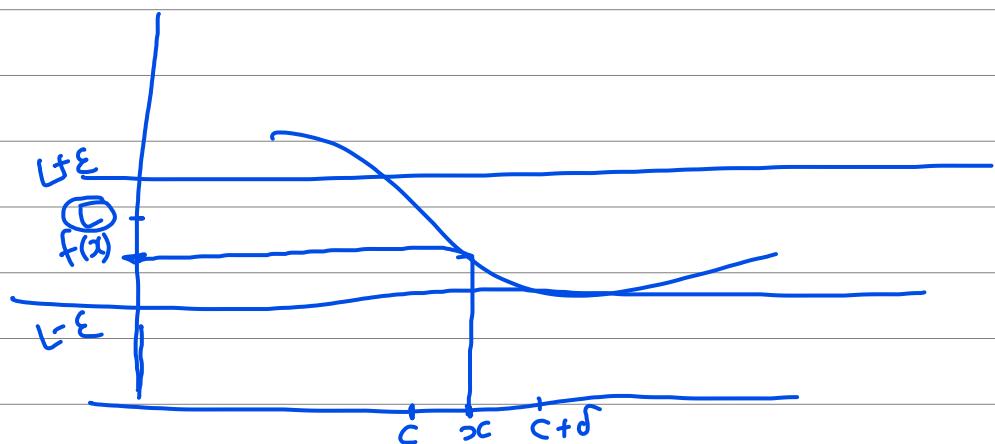
$\lim_{x \rightarrow c^+} f(x)$  for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \Rightarrow x \in (c, c+\delta) \ni$

$$\ni |f(x) - L| < \epsilon$$

$$(x-c) < \delta_\epsilon$$

$$x < c + \delta_\epsilon$$

$$(c - \delta_\epsilon, c) \cup (c, c + \delta_\epsilon)$$



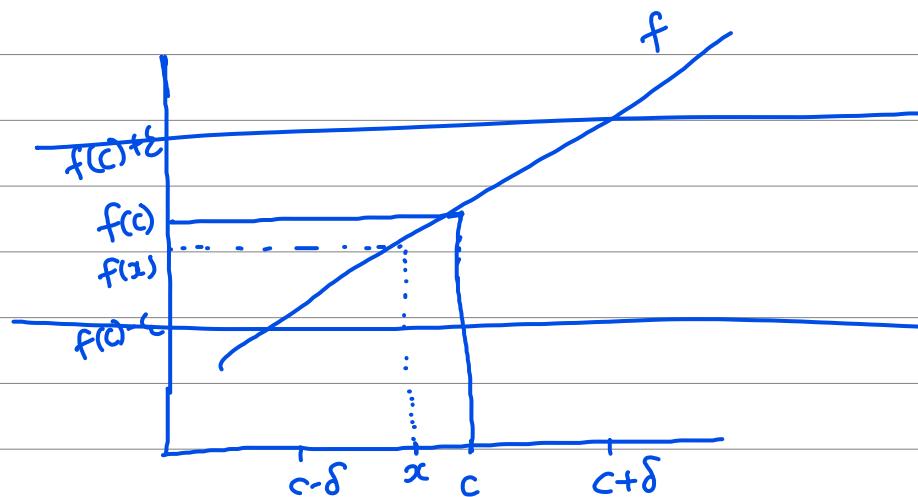
$$\lim_{x \rightarrow c} f(x) = L = \underline{f(c)}$$

f cont. at pt. c.

Right

$$\checkmark \lim_{x \rightarrow c^+} f(x) = f(c)$$

$$\checkmark \lim_{x \rightarrow c^-} f(x) = f(c)$$



$$\lim_{x \rightarrow c} f(x) = \underline{f(c)}$$

$$\checkmark f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c)) \quad \checkmark$$

i) for any  $\varepsilon > 0 \exists \delta_\varepsilon > 0 \ni |x - c| < \delta_\varepsilon \& x \in A$   
 $x \in V_\delta(c) \& x \in A \Rightarrow x \in V_\delta(c) \cap A$

$$\Rightarrow |f(x) - f(c)| < \varepsilon$$

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

$$\underline{f(x) \in V_\varepsilon(f(c))}$$

$$\checkmark \Rightarrow f(\underline{A \cap V_\delta(c)}) \subseteq V_\varepsilon(f(c))$$

Sequential Criteria  
for continuity

$$\begin{cases} x_n \rightarrow c \\ f(x_n) \rightarrow f(c) \end{cases}$$

$$\text{if } x_n \rightarrow c \Rightarrow f(x_n) \rightarrow f(c)$$

f, g cont. at c      f+g cont at c.

for any  $\varepsilon_2 > 0 \exists \delta_\varepsilon > 0 \ni |x - c| < \delta_\varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon_1$   
 $\Rightarrow |g(x) - g(c)| < \varepsilon_2$

$$\begin{aligned} |f(x) + g(x) - f(c) - g(c)| &= |f(x) - f(c)| + |g(x) - g(c)| \\ &\leq \varepsilon_1 + \varepsilon_2 \\ &\leq \varepsilon \end{aligned}$$

( $c$  cluster pt. A)  $\rightarrow f: A \rightarrow \mathbb{R}$   
 $\lim_{x \rightarrow c} f(x) = f(c)$ ) ✓

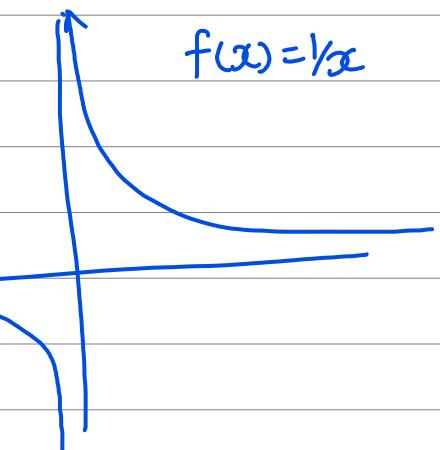
Conti. on set  $B$ ,  $\Rightarrow \forall y \in B$   $\lim_{x \rightarrow y} f(x) = f(y)$  ]

$\Rightarrow$  conti at every point of  $B$ .

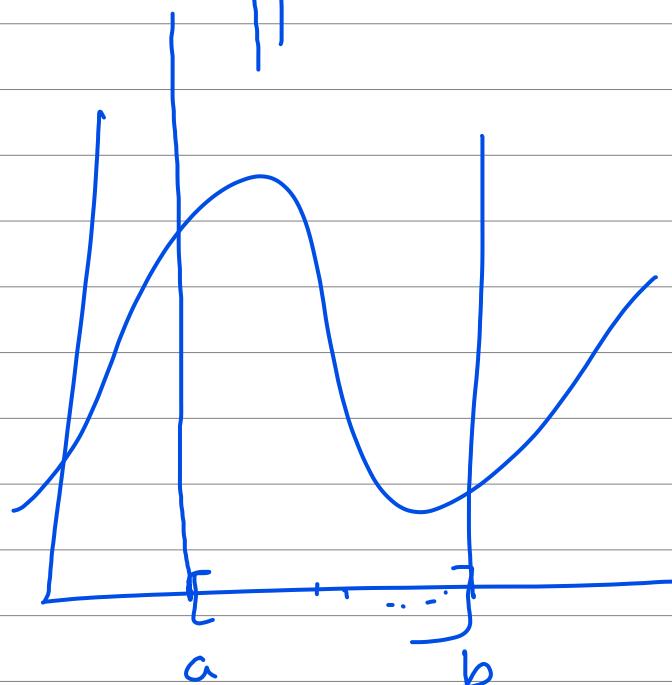
$$\underline{B = [-1, 1]}$$

$$f(x) = 1/x$$

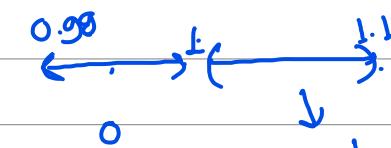
$$\begin{cases} \lim_{x \rightarrow 0^+} f(x) = +\infty \\ \lim_{x \rightarrow 0^-} f(x) = -\infty \end{cases}$$



$$f(x) = 2x^2 + 3 \checkmark$$



$$f(x) = \underline{\underline{x}}$$



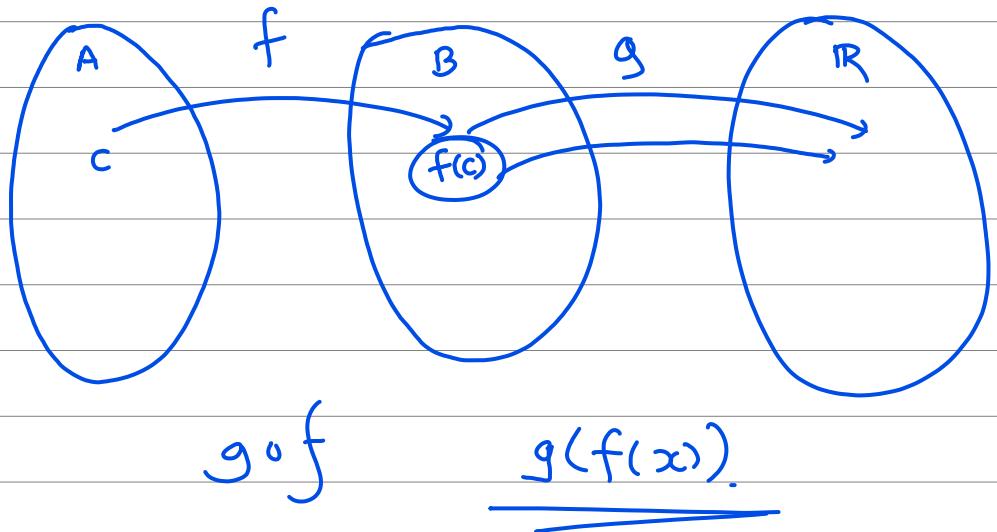
$c-1$   $c$   $c+\delta$   
 $c-\delta$   $c$   $c+\delta$   
 $\downarrow$  discontin.

$f$  cont. at pt.  $c$ .

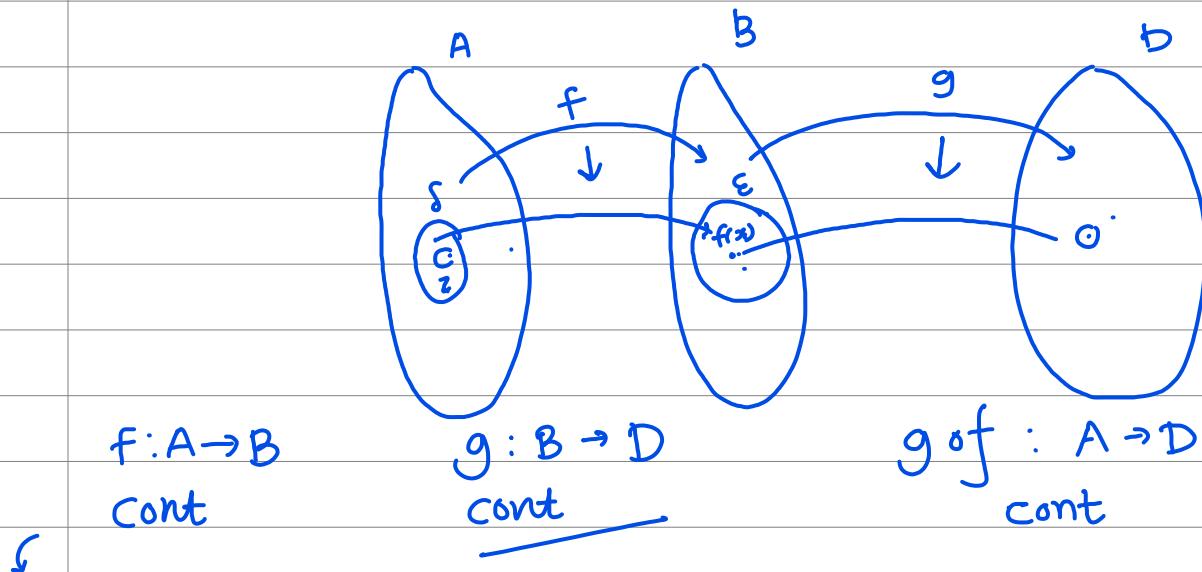
for any  $\varepsilon > 0$   $\exists \delta_\varepsilon > 0 \Rightarrow |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

$$|f(x) - f(c)| \leq |f(x) - f(c)| < \varepsilon$$

$\Rightarrow |f|$  is cont. at  $c$ .



Let  $A, B \subseteq \mathbb{R}$  & let  $f : A \rightarrow B$  &  $g : B \rightarrow D$  be functions such that  $f(A) \subseteq B$  if  $f$  is continuous at point  $c \in A$  and  $g$  is continuous at  $b = f(c) \in B$  then the composition  $g \circ f : A \rightarrow D$  is continuous at  $c$ .



for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$   
 $x \in V_\delta(c) \Rightarrow f(x) \in V_\epsilon(f(c))$

for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |y - c'| < \delta \Rightarrow |g(y) - g(c')| < \epsilon$   
 $y \in V_\delta(c') \Rightarrow g(y) \in V_\epsilon(g(c'))$

As we have assumed that  $f(A) \subseteq B$

$\exists$  some  $c' = f(c)$

[ for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |z - c| < \delta \Rightarrow$  ]

cont  
det  
cont

$$|(g \circ f)(z) - (g \circ f)(c)| = |g(f(z)) - g(f(c))|$$

$$( |z - c| < \delta \Rightarrow |f(z) - f(c)| < \epsilon, \epsilon > 0 )$$

$$|y - c'| < \delta \Rightarrow |g(y) - g(c')| < \epsilon$$

$$\Rightarrow |g(f(z)) - g(f(c))| < \epsilon$$

## Boundedness Theo.

$I = [a, b]$  closed bounded interval -

$f: I \rightarrow \mathbb{R}$  cont. ✓

To show:  $f$  bounded.

$$\underline{|f(x)| \leq M}$$

{ Assume  $f(x)$  is not bounded on  $I$

$\exists$  some  $n \in \mathbb{N}$   $|f(x_n)| > M$  ✓

-  $I$  closed bounded interval..

Let  $\exists$  some seq  $\underline{x_n} \in I$ ,  
 $I$  bounded  $\Rightarrow \underline{x_n}$  is bounded.

By Bolzano Weierstrass Theo.  $\exists$  subseq  $\underline{x_{n_k}} \in I$   
 which is convergent.

$$x_{n_k} \downarrow \rightarrow x^* \quad (\text{say})$$

if  $x_{n_k}$  is subseq in  $I$  &  $I$  is closed  
 $\Rightarrow x^* \in I$

$\therefore f$  is cont.

$$x_{n_k} \rightarrow x^* \Rightarrow \underline{\underline{f(x_{n_k}) \rightarrow f(x^*)}}$$

$\Rightarrow f(x_{n_k})$  is convergent subseq

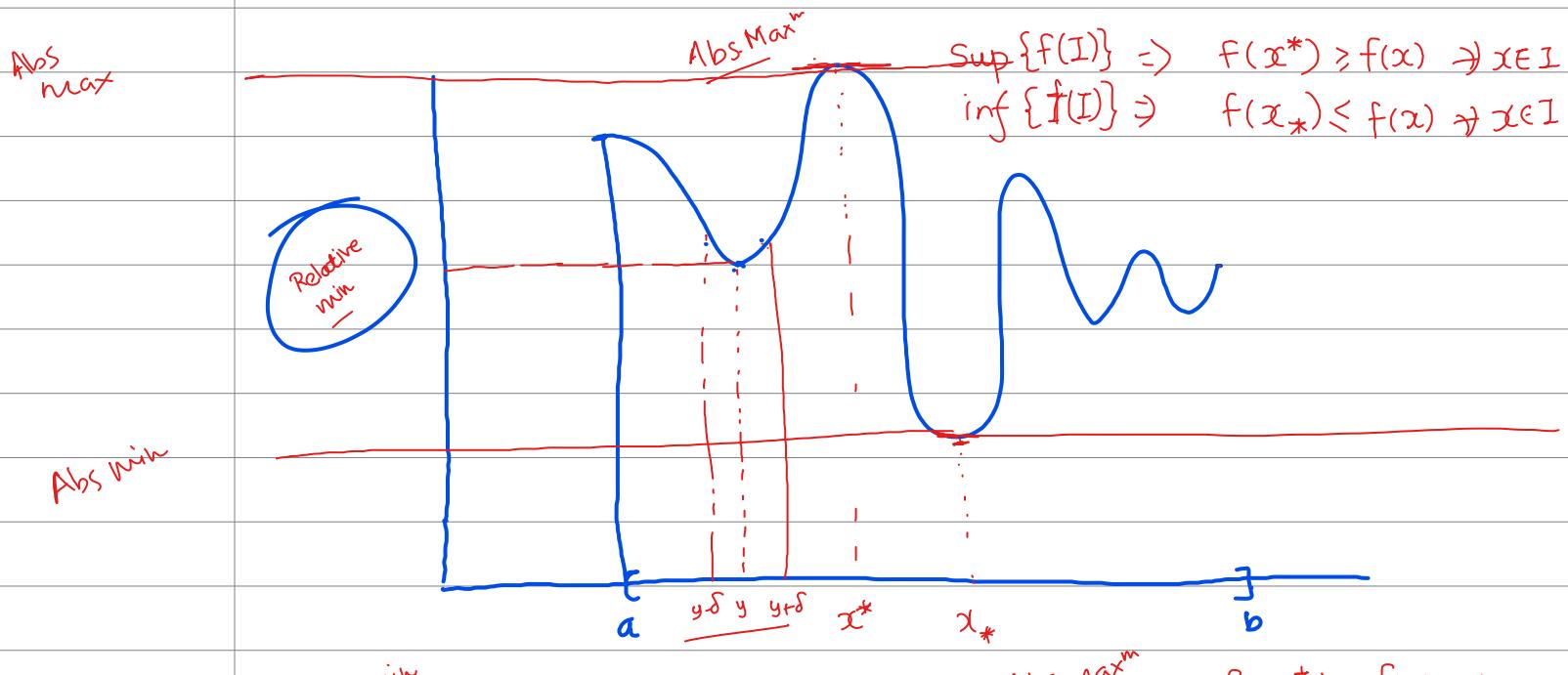
$\Rightarrow$  bounded seq

$\Rightarrow$  Our assumption is wrong.

## Absolute Extremum

max<sup>m</sup> min

## Relative extremum

max<sup>m</sup> minAbs. min  $f(x^*) \leq f(x) \Rightarrow x \in \underline{I}$ rel. min  $f(y) \leq f(x) \Rightarrow x \in (y-\delta, y+\delta)$ Abs. max<sup>m</sup>

Rel. min

 $f(x^*) > f(x) \Rightarrow x \in I$  $f(y) > f(x) \Rightarrow x \in Vg(y)$ Max<sup>m</sup> & Min<sup>m</sup> the $f: I \rightarrow \mathbb{R}$ , I closed bounded, f cont.To show:  $\exists x^*, x_*$  as abs. max<sup>m</sup> & abs. min resp. $\therefore I$  closed bounded, f cont.  $f: I \rightarrow \mathbb{R}$ by Boundedness Theo.  $|f(x)| \leq M \Rightarrow x \in I$  $f(I) = \{f(x), x \in I\}$  bounded.

by Completeness property.

 $\exists S^* = \sup \{f(I)\}$  and  $S_* = \inf \{f(I)\}$ If  $S^*$  is sup. then for  $n \in \mathbb{N}$ .  $S^* - \frac{1}{n}$  can't be sup.

$$\checkmark \left[ S^* - \frac{1}{n} < f(x_n) < S^* \quad \forall n \in \mathbb{N} \right]$$

So we got seq<sup>n</sup>  $x_n \in I$ ,  $I$  closed bounded by Bolzano weierstrass  $\Rightarrow x_{n_k} \rightarrow x^*$   
 and as  $x_{n_k} \in I$ ,  $I$  closed  $\Rightarrow x^* \in I$

Now  $x_{n_k} \rightarrow x^*$

but  $f$  is cont on  $I$ , by seq<sup>n</sup> criteria

$$f(x_{n_k}) \rightarrow f(x^*)$$

$$s^* - \frac{1}{n_k} < f(x_{n_k}) < s^*$$

by squeez theo.

$$\lim s^* - \frac{1}{n_k} < \lim f(x_{n_k}) \leq \lim s^*$$

$$s^* \leq f(x^*) \leq s^*$$

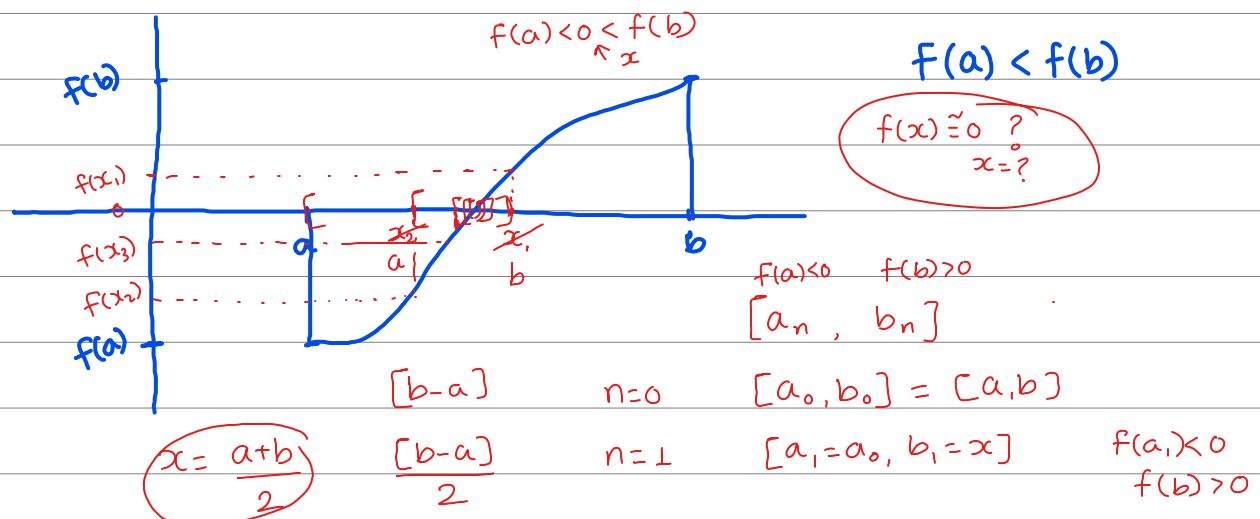
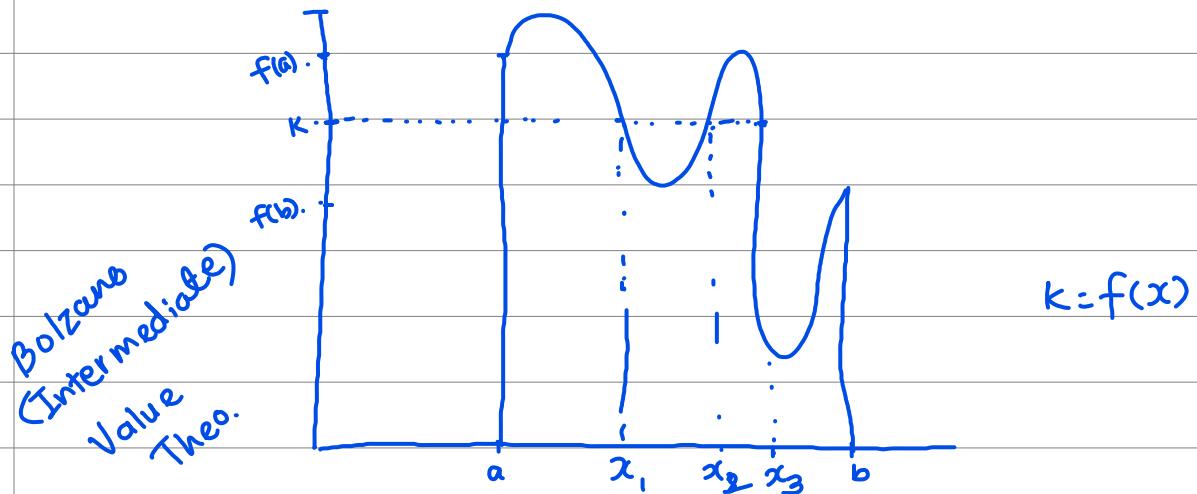
$$\Rightarrow s^* = f(x^*)$$

$$\Rightarrow \cancel{s^*} = \sup\{f(I)\} \Rightarrow f(x^*) \geq f(x) \Rightarrow x \in I$$

Simillarly we can obtain  $x_* \ni f(x_*) \leq f(x) \Rightarrow x \in I$ .

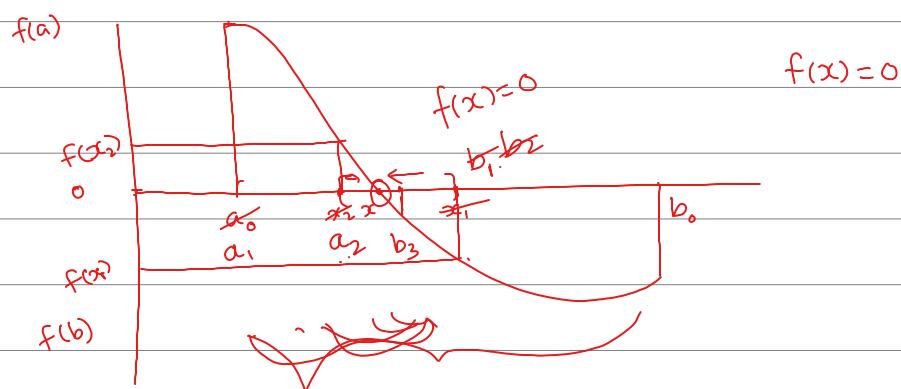
$$x^*, x_* \in I$$

## \* Location of Root



$$n=2 \quad x = \frac{a+b}{2} \quad \frac{[b-a]}{2^2}$$

$$n=3 \quad x = \frac{a+b}{2} \quad \frac{[b-a]}{2^n}$$



~~Bolzano Intermediate Value Theo.~~  
~~Existence of Root~~  $f: [a,b] \rightarrow \mathbb{R}$  cont.  
assume  $f(a) < k < f(b)$  or  $f(b) < k < f(a)$   
 $\exists c \in [a,b] \ni f(c) = k$

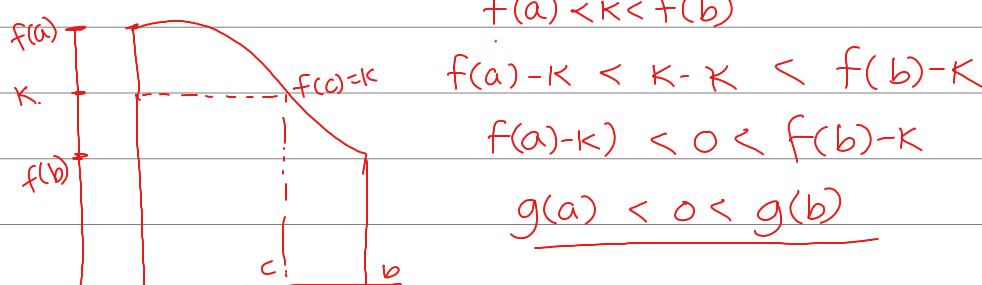
$f: I \rightarrow \mathbb{R}$  cont.

$a < b, [a,b] \subseteq I$

$$\boxed{\begin{array}{l} f(a) < 0 < f(b) \\ \exists c \in [a,b] \ni f(c) = 0 \end{array}}$$

$f: [a,b] \rightarrow \mathbb{R}$  cont. Assume  $f(a) < k < f(b)$

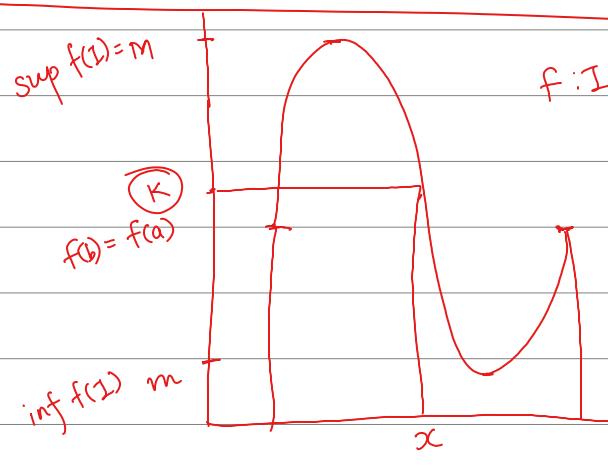
$$g(x) = f(x) - k \Rightarrow g: [a,b] \rightarrow \mathbb{R} \text{ cont.}$$



$\therefore$  By existence of Root Theo.  $\exists c \in [a,b] \ni g(c) = 0$

$$\begin{aligned} g(c) &= 0 \\ \Rightarrow f(c) - k &= 0 \\ \Rightarrow f(c) &= k \end{aligned}$$

(QED)



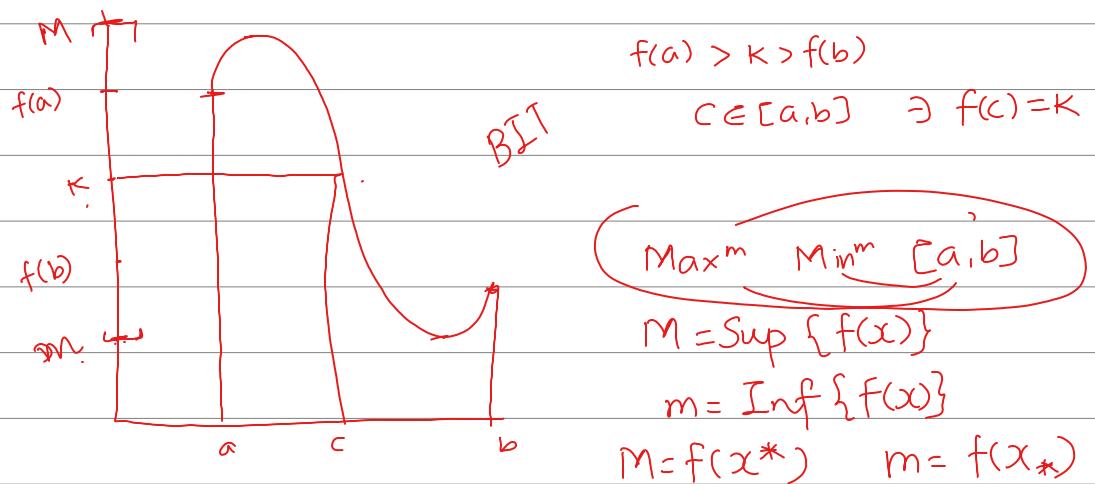
$f: I \rightarrow \mathbb{R}$   $f: [a,b] \rightarrow \mathbb{R}$  cont.

$\Rightarrow$  Bounded  $\Rightarrow$  Max<sup>m</sup> Min<sup>M</sup>

$$\underline{f(I)} = \{f(x), x \in I\}$$

$$= [m, M] \checkmark$$

$x_* \quad x^*$



c f

$$\lim_{x \rightarrow c} f(x) = L$$

Cont. at pt. c

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Cont. on set

 $\nexists c \in B$ 

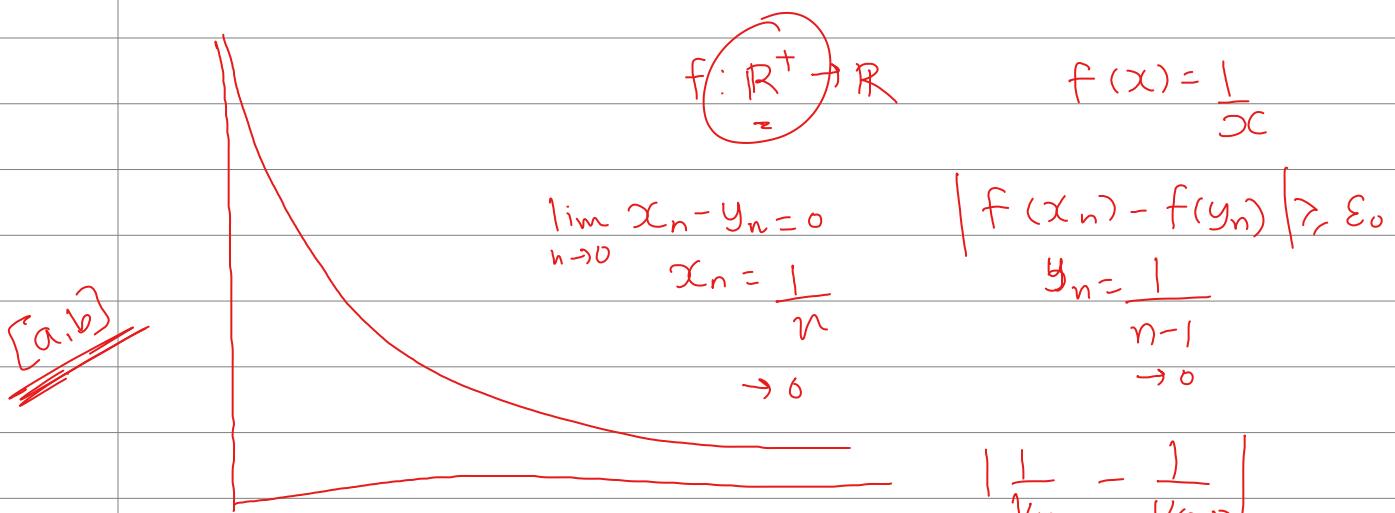
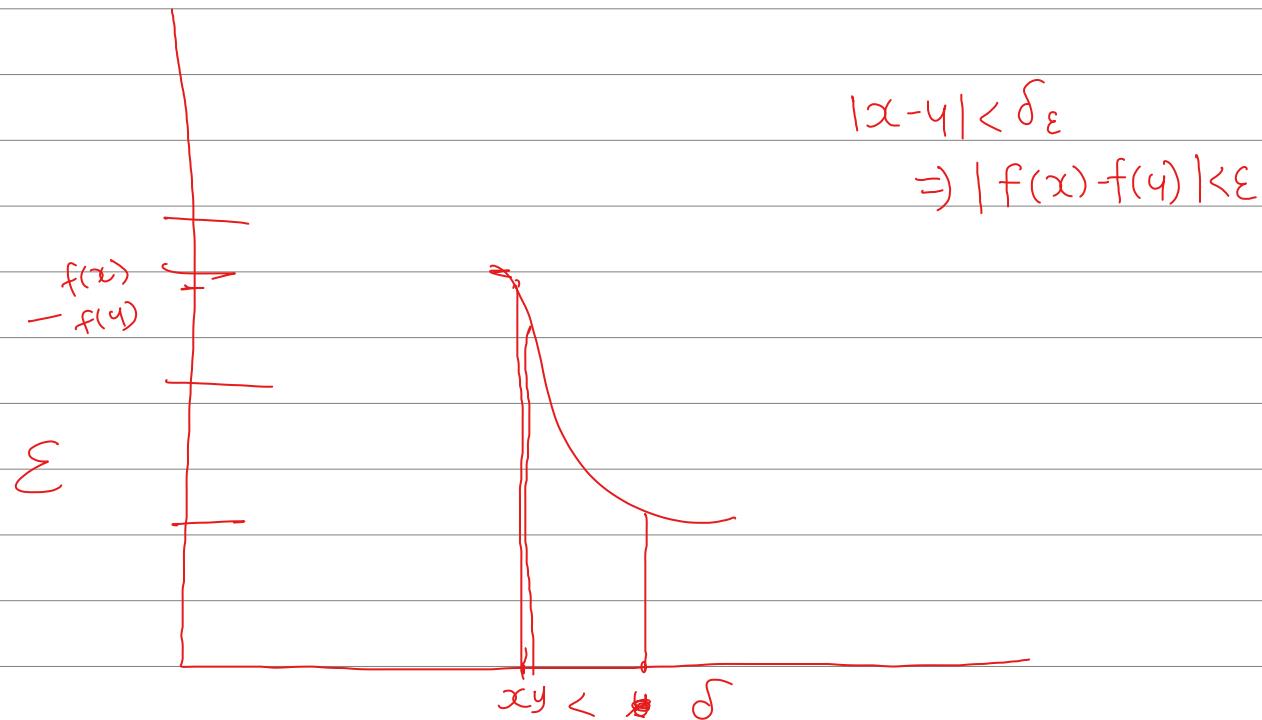
$$\lim_{x \rightarrow c} f(x) = f(c)$$

Uniform Conti

for any  $\varepsilon > 0 \exists \delta_\varepsilon > 0 \ni |x - c| < \delta_\varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon$ 

$$\exists |x - y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$$





$1/x$  is cont. but not uniformly cont on  $\mathbb{R}^+ \setminus (0, 0)$

$= |n - (n-1)|$

$= 1 > \varepsilon_0$

$$\checkmark \quad |f(x) - f(y)| \leq K|x-y| \quad x, y \in I$$

/ Lipschitz fun<sup>c</sup>

cont.? Uniformly cont.?



$$f: I \rightarrow \mathbb{R}, \text{ for any } \varepsilon > 0 \exists \delta_\varepsilon > 0 \ni |x-y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$$

Now for Lipschitz fun<sup>c</sup>

$$\text{Assume}^{\leftarrow ?^0} \text{ for any } \varepsilon > 0 \exists \delta_\varepsilon = \varepsilon/K > 0$$

$$|x-y| < \delta_\varepsilon = \varepsilon/K$$

$$\text{but } |f(x) - f(y)| < K \underline{|x-y|}$$

$$\leq K \cdot \delta_\varepsilon$$

$$\leq K \cdot \varepsilon/K$$

$$\leq \varepsilon \quad \checkmark$$

Lipschitz fun is uniformly cont.

$f: A \rightarrow \mathbb{R}, A \subseteq \mathbb{R}$ , f uniformly cont.

$x_n$  Cauchy seq<sup>n</sup>

To prove,  $f(x_n)$  Cauchy.

$f: A \rightarrow \mathbb{R}$  uniformly cont.

for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |x-y| < \delta_\epsilon \Rightarrow |f(x)-f(y)| < \epsilon$

②  $x_n \in A$ ,  $x_n$  is cauchy seq<sup>n</sup>

for any  $\delta_\epsilon > 0 \exists K(\epsilon) \in \mathbb{N} \ni |x_n - x_m| < \delta_\epsilon \forall n, m > K(\epsilon)$

$|x_n - x_m| < \delta_\epsilon \Rightarrow |f(x_n) - f(x_m)| < \epsilon$

$\Rightarrow f(x_n)$  is cauchy seq<sup>n</sup>

\* ✓ Continuous Extension Theorem: A function  $f$  is uniformly continuous on the interval  $(a, b)$  if and only if it can be defined at the endpoints  $a$  and  $b$  such that the extended function is continuous on  $[a, b]$ .

$f: (a, b) \rightarrow \mathbb{R}$  uniformly cont.

$f(a) = L$ ,  $f(b) = M$   $\Rightarrow$  extended

$f: [a, b] \rightarrow \mathbb{R}$  cont on  $[a, b]$

$f: (a, b) \rightarrow \mathbb{R}$  uniform cont.

$f(a) = L$ ,  $f(b) = M$



$$\begin{aligned} x_n &\rightarrow a \\ x_n &\rightarrow a + 1/n \\ y_n &\rightarrow a + 1/n^2 \end{aligned}$$

$$\lim_{n \rightarrow \infty} x_n - y_n = 0$$

$$|x_n - y_n| < \delta_\epsilon$$

$(a, b)$

$\Rightarrow$

$$|f(x_n) - f(y_n)| < \epsilon$$

$x_n \rightarrow a$  for any  $\epsilon > 0$  if  $\delta_\epsilon > 0 \exists |x_n - a| < \delta_\epsilon \Rightarrow |f(x_n) - L| < \epsilon/2$

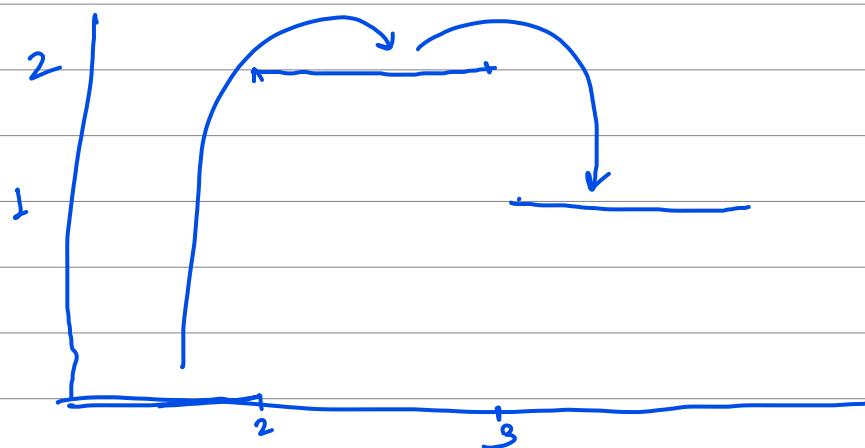
$y_n \rightarrow a$  . . .  
 $|y_n - a| < \delta_\epsilon \Rightarrow |f(y_n) - L| < \epsilon/2$

$$\begin{aligned} |x_n - y_n| &< \delta_\epsilon \\ |x_n - a| + |y_n - a| &< \delta_\epsilon \Rightarrow |f(x_n) - f(y_n)| \\ &\leq |f(x_n) - L| + |L - f(y_n)| \\ &\leq \epsilon_1 + \epsilon_2 \\ &\leq \epsilon \end{aligned}$$

$$\underline{\lim} f(a) = \lim f(x_n) = \lim f(y_n) = L$$

Step

$$f(x) = \begin{cases} 0 & 0 \leq x < 2 \\ 2x & 2 \leq x < 3 \\ 1 & 3 \leq x \leq 5 \end{cases}$$



$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c}$$

$(c, c+\delta)$                                    $(c-\delta, c)$

$$f(x) = |x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(c)}{x - c} \neq \lim_{x \rightarrow 0^-} \frac{f(x) - f(c)}{x - c}$$

Derivable  $\Rightarrow$  Conti.Conti  $\not\Rightarrow$  deri |x|

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$\lim_{x \rightarrow c} \frac{(f/g)(x) - (f/g)(c)}{x - c}$$

 $g(x) \neq 0 \Rightarrow x \in I$ 

$$= \lim_{x \rightarrow c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x)g(c) - f(c)g(x)}{(x - c)g(x) \cdot g(c)}$$

$$= \lim_{x \rightarrow c} \frac{\underbrace{f(x) \cdot g(c)} - \underbrace{f(c) \cdot g(c)} + \underbrace{f(c) \cdot g(c)} - \underbrace{f(c) \cdot g(x)}}{(x - c) \cdot g(x) \cdot g(c)}$$

$$= \lim_{x \rightarrow c} \frac{\cancel{g(c)}}{\cancel{g(x) \cdot g(c)}} \frac{\cancel{f(x) - f(c)}}{x - c} - \frac{\cancel{f(c)}}{\cancel{g(x) \cdot g(c)}} \frac{-\cancel{g(x) + g(x)}}{x - c}$$

$$= \frac{g(c)f'(c)}{(g(c))^2} - \frac{f(c)g'(c)}{(g(c))^2}$$

## Chain Rule

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

let  $f(c) = y$

$$g'(y) = \lim_{x \rightarrow y} \frac{g(x) - g(y)}{x - y}$$

$= \lim_{f(x) \rightarrow f(c)} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$

$\downarrow$

$$g'(y) = \left[ \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} \right] \cdot \frac{1}{\frac{f(x) - f(c)}{x - c}}$$

$$g'(f(c)) = \underline{(g \circ f)'(c)} \cdot \overbrace{f'(c)}$$

$$(g \circ f)'(c) = f'(c) \cdot g'(f(c))$$

$$f(x) = \underline{2x}$$

$$f^{-1}(x) = \underline{\frac{x}{2}}.$$

$$f^{-1}(f(x)) = f^{-1}(2x)$$

$$= \underline{\frac{2x}{2}}$$

$$= x$$

even

$$f(x) = f(-x)$$

odd

$$f(x) = -f(-x)$$

$$\underline{\underline{g(c)}} = f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

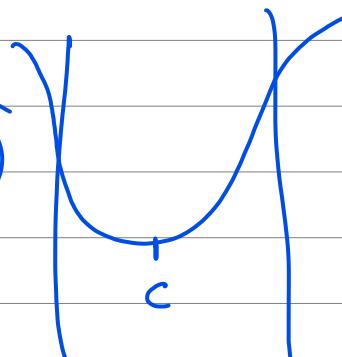
$$= \underline{\underline{-g(-c)}}$$

$c$  has relative min<sup>m</sup>

$$f'(c) > 0$$

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} > 0$$

$$f(x) > f(c)$$



$$\frac{f(x) - f(c)}{x - c} \geq 0$$

$$\Rightarrow x \in (c-\delta, c)$$

$$f(x) - f(c) \leq 0$$

$$f(x) \leq f(c)$$

$c$  relative min

$$f'(c) < 0$$

$$(c-\delta, c+\delta) \\ c, c+\delta$$

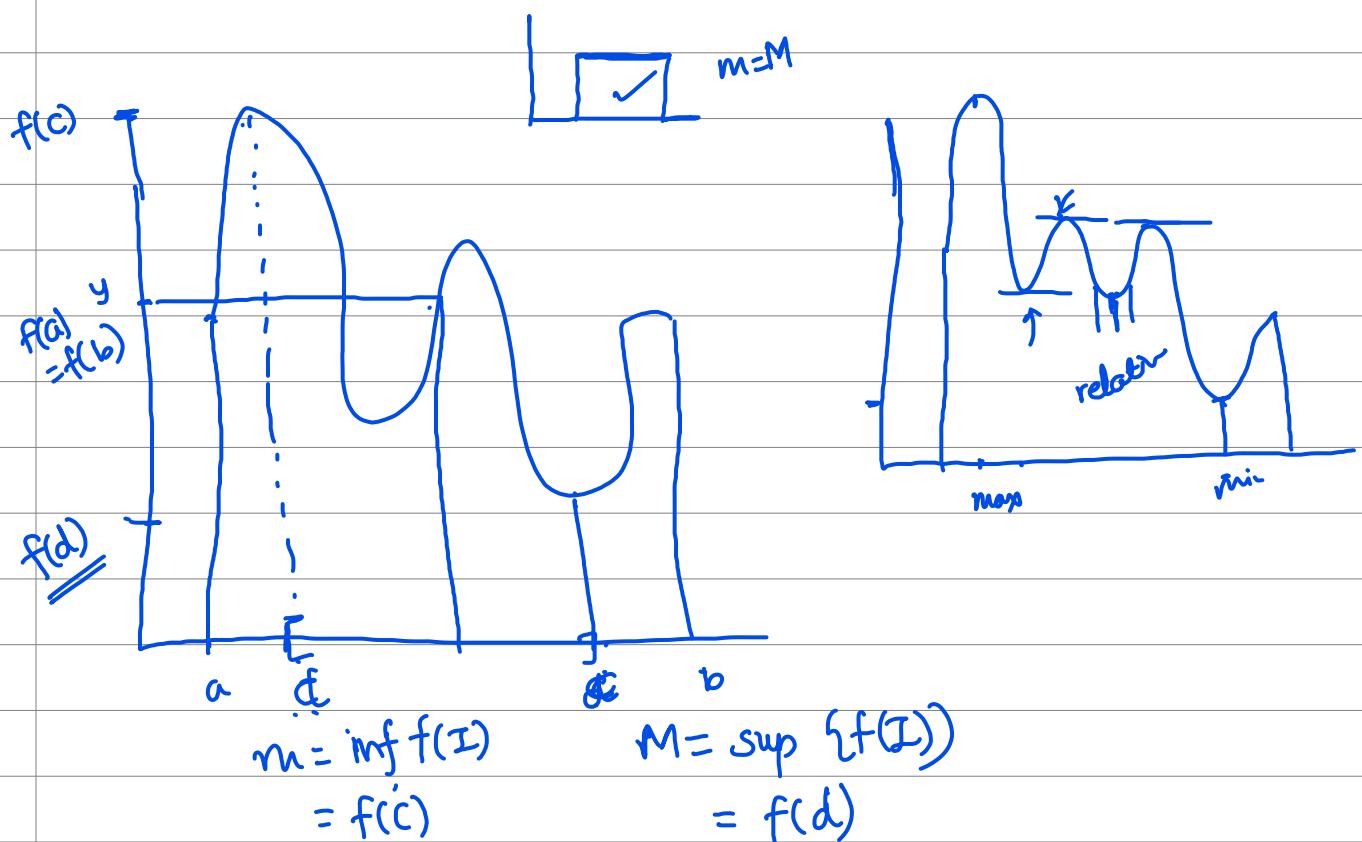
$$\underline{\underline{f(c) \leq f(x)}}$$

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} < 0$$

$$f(x) - f(c) < 0$$

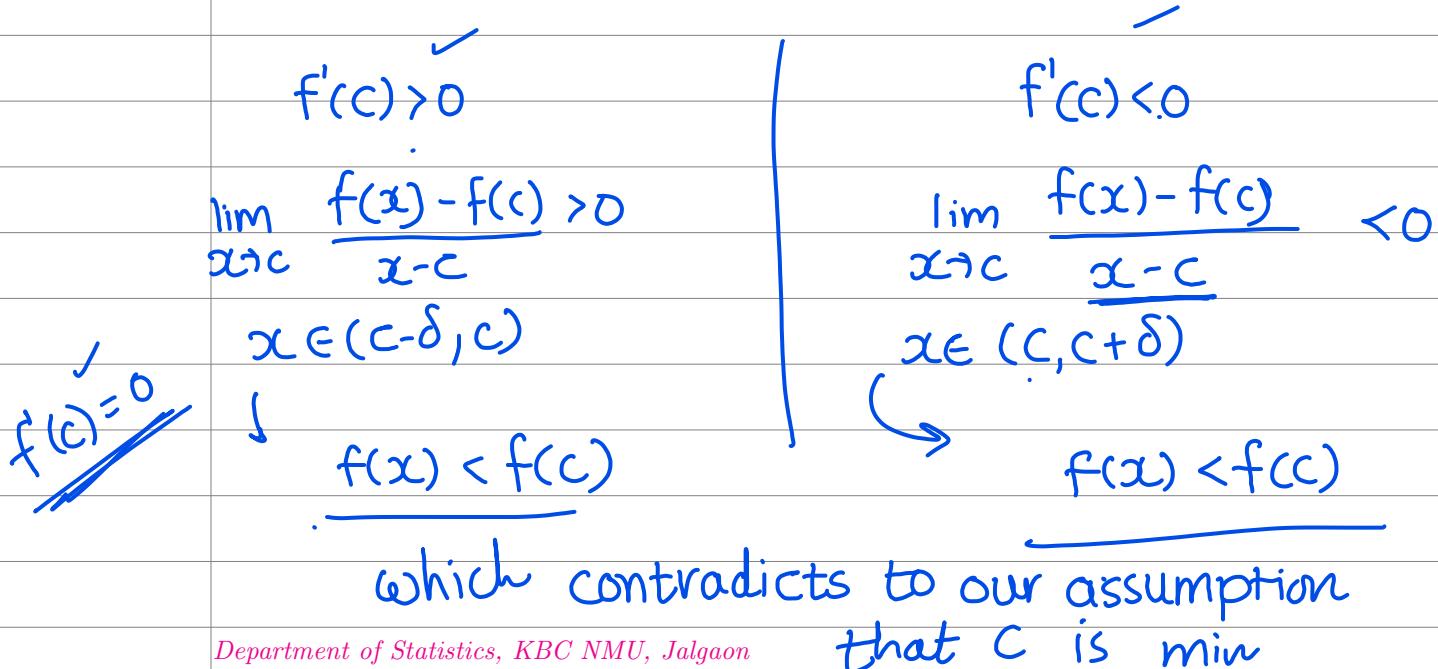
$$\underline{\underline{f(x) \leq f(c)}}$$

$$\Rightarrow x \in (c, c+\delta)$$



$f(c), f(d)$

(y) [c, d]



Rolle's Theo:-  $f: [a,b] \rightarrow \mathbb{R}$

- ① cont.  $[a,b]$  ✓
  - ② deri  $(a,b)$  ✓
  - ③  $f(a) = f(b) ?$  ✓
- $\Rightarrow \underline{f'(c)=0} \text{ for some } c \in (a,b)$

LMYT :  $f: [a,b] \rightarrow \mathbb{R}$        $\psi: [a,b] \rightarrow \mathbb{R}$

- ① cont.  $[a,b]$  ✓
  - ② deri  $(a,b)$  ✓
- $\exists c \in (a,b) \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

Important  
Step

$$\psi(x) = \underline{\underline{f(x) - A \cdot x}}$$

- $\Rightarrow$  cont.  $[a,b]$   
 $\Rightarrow$  deri  $(a,b)$

$$\begin{aligned} \psi(a) &= f(a) - A \cdot a = f(b) - A \cdot b &= \psi(b) \\ \Rightarrow A(b-a) &= f(b) - f(a) \end{aligned}$$

$$A = \frac{f(b)-f(a)}{b-a} \quad \checkmark$$

$$\boxed{\psi(x) = f(x) - \frac{f(b)-f(a)}{b-a} \cdot x}$$

- ① cont  $[a,b]$
  - ② deri  $\underline{(a,b)}$
  - ③  $\psi(a) = \psi(b)$
- by Rolle's theo.  $\exists c \in (a,b) \cdot \exists \psi'(c) = 0$

$$\psi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

CMVT:  $f, g: [a,b] \rightarrow \mathbb{R}$

①  $f, g$  cont.  $[a,b]$

② deri  $(a,b)$

③  $g(a) \neq g(b)$  &  $g'(x) \neq 0$

$\Rightarrow \exists c \in (a,b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\psi(x) = f(x) - A \cdot g(x)$$

$$\psi(a) = f(a) - A \cdot g(a) = f(b) - A \cdot g(b) = \psi(b)$$

$$\Rightarrow A(g(b) - g(a)) = f(b) - f(a)$$

$$A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\psi(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

① ✓      ② ✓      ③  $\psi(a) = \psi(b)$   
by Rolle's Theo:-  $\exists c \in (a,b) \ni \psi'(c) = 0$

$$f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(c) = 0$$

$$\left[ \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \right]$$

QED

$f: I \rightarrow \mathbb{R}$  differentiable ✓

①  $f \uparrow$  iff  $f'(x) > 0 \Rightarrow x \in I$

②  $f \downarrow$  iff  $f'(x) < 0 \Rightarrow x \in I$

①  $f \uparrow$ , if  $x \leq y \Rightarrow f(x) \leq f(y)$

$$\forall c \in I, f'(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$\forall x \in (c, c+\delta)$ ,  $c \leq x \Rightarrow f(c) \leq f(x)$

$$\frac{f(x) - f(c)}{x - c} > 0 \quad > 0$$

$$= \lim_{x \rightarrow c^+} f'(x) > 0$$

$\forall x \in (c-\delta, c), x \leq c \Rightarrow f(x) \leq f(c)$

$$x - c \leq 0 \Rightarrow f(x) - f(c) \leq 0$$

$$\frac{f(x) - f(c)}{x - c} < 0 \quad < 0 \quad > 0$$

$$\lim_{x \rightarrow c^-} f'(x) < 0$$

$$\boxed{f'(c) > 0} \quad \Rightarrow c \in I$$

$$f(x) = f(x_0) + \sum_{i=1}^n \frac{f^{(i)}(x_0)}{i!} \cdot (x - x_0)^i$$

$$+ \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} \quad \left. \right\} \text{Rem}$$

$$\underline{\underline{x_0 = 0}} \quad \cos(0) = 1 \quad \sin(0) = 0$$

$$\sqrt{\sin(x)} = \sin(0) + \cos(0)(x) - \frac{\sin(0)}{2!} x^2 - \frac{\cos(0)x^3}{3!} - \dots$$

$$= 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \log(1+x_0) + \frac{1}{1+x_0} \cdot x + \frac{-1}{(1+x_0)^2} \cdot \frac{x^2}{2!} + \frac{+2}{(1+x_0)^3} \cdot \frac{x^3}{3!} + \dots$$

$x_0 = 0$

$$= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Sequence and Series of fun's :-

$$\left\{ \frac{1}{n} \quad \sum \frac{1}{n} \quad f(x) = 2x^2 + 1 \right.$$

$$f_n(x) = \frac{2x^2 + nx}{n^2} = f(n, x) \quad \checkmark$$

$$= f_n: \mathbb{R} \rightarrow \mathbb{R} \quad x \in \mathbb{R}, \quad n \in \mathbb{N}$$

seq' of fun's

seq' of fun's

$$f_n(x) = \frac{x^n}{n} \quad x \in A$$

[series  
of  
fun's.]

$$\sum \frac{x}{n}$$

$$g(x) = \sum \frac{x}{n} = \sum f_n(x)$$

$$\frac{1}{n} \rightarrow 0$$

$f_n \rightarrow f$   $\begin{cases} \text{pointwise} \\ \text{uniformly} \end{cases}$

$x_n \rightarrow x$   
 for any  $\epsilon > 0$   $\exists K(\epsilon) \in \mathbb{N}$   $\exists |x_n - x| < \epsilon \Rightarrow n > K(\epsilon)$

$$\underline{f_n(x)} \rightarrow \underline{f(x)}$$

$$f_n: A \rightarrow \mathbb{R}, f: A_0 \rightarrow \mathbb{R}$$

$$A_0 \subseteq A$$

pointwise convergence

for  $x \in A_0$ , ✓  
 for any  $\epsilon > 0$ ,  $\exists K(\epsilon, x) \in \mathbb{N}$   $\underline{\underline{\exists}} ? \Rightarrow |f_n(x) - f(x)| < \epsilon \Rightarrow n > K(\epsilon, x)$

$$f_n(x) = \frac{x}{n} \rightarrow f(x) = 0$$

$$x \in [0, a]$$

$$|f_n(x) - f(x)| = \left| \frac{x}{n} - 0 \right| < \epsilon$$

$$\left| \frac{x}{n} \right| \leq \left| \frac{ax}{n} \right| < \epsilon$$

$$= \left| \frac{x}{n} \right| < \epsilon$$

$$\Rightarrow \frac{|x|}{\epsilon} < n$$

$$\Rightarrow \frac{|x|}{\epsilon} < n$$

$$? \quad K(\epsilon) < n$$

uniform conti

$$\left\{ \begin{array}{l} f_n(x) = x^{1/n} \\ f(x) = \begin{cases} 0 & x > 0 \\ 1 & x = 0 \\ \text{undefined} & x < 0 \end{cases} \end{array} \right. \quad \left. \begin{array}{l} x > 0 \\ x = 0 \\ x < 0 \end{array} \right\} \quad \text{pointwise convergence}$$

$$f_n(x) = \frac{x}{n} \quad f(x) = 0 \quad x \in \mathbb{R}$$

$$\left| f_{n_k}(x_k) - f(x_k) \right| = \left| \frac{x_k}{n_k} - 0 \right|$$

$$x_k = n_k \in \mathbb{R}$$

$$\left| \frac{n_k}{n_k} \right| = 1 \quad \text{for any } \epsilon > 0$$

~~0.g~~

$$\exists i, \epsilon_0$$

$$g_n(x) = x^n \quad x \in \mathbb{R}, n \in \mathbb{N} \quad g(x) = 0$$

$$g(x) = 0$$

$$\begin{aligned} |g_{n_k}(x_k) - g(x_k)| &= |x_k^{n_k} - 0| \\ &= \left| \left(\frac{1}{2}\right)^{k \times k} - 0 \right| \end{aligned}$$

$$\frac{1}{2} \checkmark$$

~~not uniformly  
cont.~~

$$\|f\|_A = \sup \{ |f(x)|, x \in A \} \quad f(x) \ni \text{bounded}$$

For any  $\epsilon > 0 \exists K(\epsilon) \in \mathbb{N} \exists \underline{\underline{|f_n(x) - f(x)| < \epsilon}} \Rightarrow n > K(\epsilon)$

~~$\psi_n(x)$~~

$$\psi_n(x) = (f_n(x) - f(x))$$

$$\|\psi_n\| = \sup \{ |f_n(x) - f(x)|, x \in A_0 \}$$

$$\begin{aligned}
 & \|\psi_n\| < \varepsilon \\
 & \sup \{ |\psi_n(x)|, x \in A_0 \} < \varepsilon \\
 & \sup \{ |f_n(x) - f(x)|, x \in A_0 \} < \varepsilon \\
 \Rightarrow & |f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in A_0 \\
 \Rightarrow & \text{uniform conti}
 \end{aligned}$$

$$f_n(x) = \frac{x}{n} \quad . \quad f(x) = 0 \quad x \in \mathbb{R}$$

$$|f_n(x) - f(x)| \equiv \left| \frac{x}{n} \right| \quad x \in \mathbb{R}$$

unbounded

$$f_n(x) = \frac{x}{n} \quad x \in (0, a] \rightarrow$$

$$|f_n(x) - f(x)| = \left| \frac{x}{n} \right| \leq \frac{a}{n} \quad a > 0$$

$$\frac{a}{n} \rightarrow 0$$

$f_n(x) \rightarrow f$  uniformly converges to  $f$  on  $A_0$

$$\|\psi_n\| = ? \rightarrow 0$$

$$f_n(x) = x^n \quad (-1, 1] \quad \rightsquigarrow \quad f(x) = \begin{cases} 0 & -1 < x < 1 \\ 1 & x=1 \end{cases}$$

~~$-1 < x < 1$~~

$$|f_n(x) - f(x)| = |x^n - 0| \leq 1$$

$$= |x| \quad x < 1$$

~~$f_n(x) = x^n(1-x) \quad [0, 1]$~~

Cauchy Criterion for uniform convergence

for any  $\epsilon > 0 \quad \exists K(\epsilon) \in \mathbb{N}. \quad \exists |x_n - x_m| < \epsilon \Rightarrow n, m > K(\epsilon)$

$$\exists |f_n(x) - f_m(x)| < \epsilon \Rightarrow n, m > K(\epsilon)$$

✓  $\left[ \left| f_n - f_m \right|_A \leq \epsilon \right]$

① Uniform Con  $f_n \rightarrow f \quad \Rightarrow \quad \left| f_n - f_m \right|_A < \epsilon$

for any  $\frac{\epsilon}{2} > 0 \quad \exists K(\frac{\epsilon}{2}) \in \mathbb{N}. \quad \exists |f_n(x) - f(x)| < \frac{\epsilon}{2} \Rightarrow n > K(\frac{\epsilon}{2})$

$$\exists |f_m(x) - f(x)| < \frac{\epsilon}{2}$$

for  $n, m > K(\frac{\epsilon}{2})$ .

$$\left| f_n(x) - f_m(x) \right| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$$

$$\underbrace{|f_n(x) - f_m(x)|}_{\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$$\boxed{\sup |f_n(x) - f_m(x)| \leq \varepsilon} \rightarrow$$

$\|f_n - f_m\| < \varepsilon \Rightarrow \sup |f_n(x) - f_m(x)| < \varepsilon \Rightarrow \|f_n(x) - f_m(x)\| < \varepsilon \Rightarrow x \in A_0 \Rightarrow n > K(\varepsilon)$

$\Rightarrow \{f_n(x)\}$  is cauchy seq<sup>n</sup>.

$\Rightarrow$  Cauchy seq<sup>n</sup>. is convergent & that converges to some  $f(x)$

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

$\Rightarrow x \in A_0$ , for any  $\varepsilon > 0$   $\exists K(\varepsilon) \in \mathbb{N}$ .  $\Rightarrow \|f_n(x) - f(x)\| < \varepsilon \Rightarrow n > K(\varepsilon)$ .

$$\Rightarrow \sup_{x \in A_0} |f_n(x) - f(x)| < \varepsilon$$

$$= \left\| f_n(x) - f(x) \right\|_{A_0} < \varepsilon \quad \forall n > K(\varepsilon)$$

$$\Rightarrow \left\| f_n(x) - f(x) \right\|_{A_0} \rightarrow 0$$

$$\left\| f_m - f_n \right\| \leq \varepsilon$$

$f_n(x)$  uniformly converges to  $f$ .

$f_n$  bounded seq<sup>n</sup> of fun<sup>n</sup>

$$\underline{f_n \rightarrow f}$$

$$\Leftrightarrow \underline{\left\| f_n - f_m \right\| \leq \varepsilon}$$



## Series of functions

$$f_n(x) : A \rightarrow \mathbb{R} \quad \xrightarrow{\uparrow \uparrow} \quad f : A_0 \rightarrow \mathbb{R} \quad A_0 \subseteq \mathbb{R}$$

$\checkmark \quad x \in A_0 \quad \text{for any } \varepsilon > 0 \quad \exists \quad K(\varepsilon, x) \in \mathbb{N}, \quad \exists \quad |f_n(x) - f(x)| < \varepsilon \quad \forall n,$

$K(\varepsilon, x)$

$K(\varepsilon)$

$$\downarrow \\ \|f_n - f\|_{A_0} \rightarrow 0$$

$$\left. \begin{array}{l} S_1(x) = f_1(x) \\ S_2(x) = f_1(x) + f_2(x) = S_1(x) + f_2(x) \\ \vdots \\ S_n(x) = S_{n-1}(x) + f_n(x) = \sum_{i=1}^n f_i(x) \end{array} \right.$$

Cauchy Criterion for convergence of seq<sup>r</sup> of fun<sup>c</sup>s

$$\|S_n(x) - S_m(x)\|_{A_0} \leq \varepsilon$$

$$\Rightarrow \left| \sum_{i=1}^n f_i(x) - \sum_{i=1}^m f_i(x) \right| < \varepsilon$$

$$\stackrel{n>m}{\Rightarrow} \left| f_{m+1}(x) + f_{m+2}(x) + \dots + f_n(x) \right| < \varepsilon$$

~~series~~

Weistrauss M<sub>1</sub> test

~~seq<sup>n</sup> off func~~  $\rightarrow f_n(x) \leq M_n$   $\leftarrow$  Real seq<sup>n</sup>

$$f_n(x) = \frac{x}{n} \quad x \in [0, \underline{a}]$$

$$\frac{x}{n} \leq \frac{a}{n}$$

$$\therefore M_n$$

①  $|f_n(x)| \leq M_n \quad \forall x \in D, \forall n \in \mathbb{N}$ .

②  $\sum M_n$  convergent

To prove  $\rightarrow \sum f_n(x)$  convergent

by Cauchy criterion, for any  $\epsilon > 0 \exists K(\epsilon) \in \mathbb{N} \exists$

$$|a+b| \leq |a| + |b|$$

$$|M_{m+1} + M_{m+2} + \dots + M_n| < \epsilon \quad \not\rightarrow K(\epsilon) < m < n$$

$$\underbrace{|f_{m+1}(x)| + \dots + |f_n(x)|}_{< M_{m+1} + M_{m+2} + \dots + M_n < \epsilon} - \dots -$$

$$\underbrace{|f_{m+1} + f_{m+2} + \dots + f_n|}_{< \epsilon} < \epsilon$$

## Radius of convergence

$$R = \limsup |a_n|^{\frac{1}{n}} \cdot$$

$$\sum a_n (x-c)^n$$

$$R = \begin{cases} \infty \\ \frac{1}{|g|} \\ 0 \end{cases}$$

$$g = 0 \cdot$$

$$g \text{ finite } 0 < g < \infty$$

$$g = \infty$$

$$\sum n^n x^n$$

$$a_n = n^n \quad c = 0$$

$$g = \limsup |n^n|^{\frac{1}{n}} = \infty$$

$$\underline{R = 0}$$

Cauchy Hadmard Theor:

$$\frac{\sum a_n x^n}{\infty} \rightarrow \mathbb{R}$$

↓

$$(-R, R)$$

$$\frac{\sum x^n}{\infty} \quad a_n = 1 \quad c = 0, \quad g = \limsup |a_n|^{1/n}$$

$$\hookrightarrow |x| < 1 \quad \quad \quad = \limsup 1$$

$$= 1$$

$$R = 1/g = 1$$

$$(-R, R) = (-1, 1)$$

$$\frac{\sum n! x^n}{\infty} \quad a_n = n!$$

$$g = \limsup |a_n|^{1/n}$$

$$= \limsup (n!)^{1/n}$$

$$=$$

$$R = 1/g$$

$$(-R, R)$$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0$$

$$x = 0$$

Power Series :-  $\sum a_n(x-c)^n$

$$R = \limsup |a_n|^{1/n}$$

$$R = \begin{cases} 0 & R=0 \\ 1/R & 0 < R < +\infty \\ \infty & R=\infty \end{cases}$$

$$\begin{aligned} R &= \infty \\ 0 &< R < +\infty \\ R &= 0 \end{aligned}$$

Cauchy Hadmard Theo.

$\sum a_n x^n$  converges only if  $x \in (-R, R)$

$$a_n = \frac{1}{n^n} \quad R = \limsup |a_n|^{1/n} = \limsup \left| \frac{1}{n^n} \right|^{1/n}$$


$$S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \quad \inf S = 0 \quad \sup S = 1$$

$$\lim \frac{1}{n} \Rightarrow \liminf \frac{1}{n} = 0, \quad \limsup \frac{1}{n} = 0$$

$$\lim (-1)^n \Rightarrow \liminf (-1)^n = -1, \quad \limsup (-1)^n = +1$$

$$a_n = \frac{n^n}{n!}$$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \left| \frac{\frac{n^n}{n!}}{\frac{(n+1)^{n+1}}{(n+1)!}} \right|$$

$$= \lim \left| \frac{n^n}{(n+1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^n} = \frac{1}{e}$$

$$a_n = \frac{(n!)^2}{(2n)!}$$

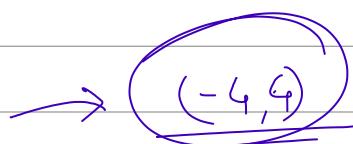
$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim \left| \frac{(n!)^2}{(2n)!} \times \frac{(2n+2)!}{(n+1)!^2} \right|$$

$$= \lim \frac{2(n+1)(2n+1)}{(n+1)!(n+1)!}$$

$$= \lim 2 \frac{(2n+1)}{(n+1)}$$

$$= 4$$



$$a_n = (n!)^{1/n}$$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim \left| \frac{(n!)^{1/n}}{((n+1)!)^{1/(n+1)}} \right|$$

$$= \lim (n!)^{1/n}$$

$\varrho = \infty$        ~~$R = 0$~~

$$\lim (n!)^{1/n}$$

$$= \lim \exp(\log(n!)^{1/n})$$

$\varrho = \infty$

$$\cong = \lim \exp\left(\frac{1}{n} \log n!\right)$$

$$= \lim \exp\left(\frac{1}{n} (\log 1 + \log 2 + \dots + \log n)\right)$$

$$= \lim \exp\left(\frac{1}{n} \int_1^n \log x dx\right)$$

## Riemann Integration / Integral

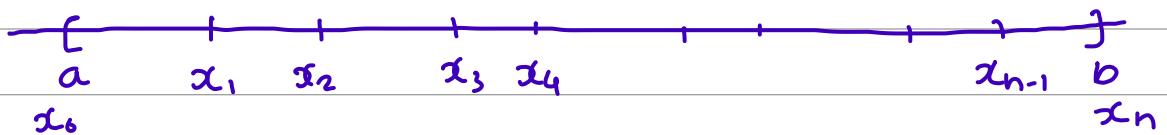
If  $I = [a, b]$

partition of  $I$ , finite and ordered set

$$P = (x_0, x_1, x_2, \dots, x_n)$$

$$a = x_0 < x_1 < \dots < x_n = b$$

Subintervals ✓  $[x_{i-1}, x_i]$   $i = 1, \dots, n$

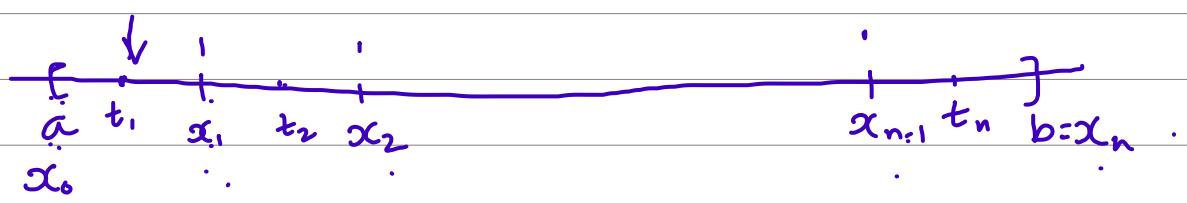


Partition

$$P = \{ [x_{i-1}, x_i] , i=1:n, x_0=a, x_n=b \}$$

Tagged partition

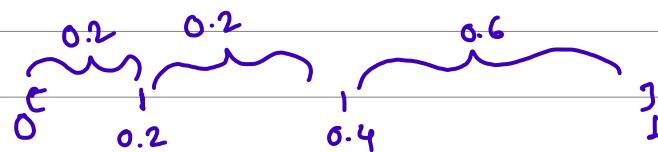
$$\dot{P} = \{ ([x_{i-1}, x_i], t_i) , i=1:n, x_0=a, x_n=b \}$$



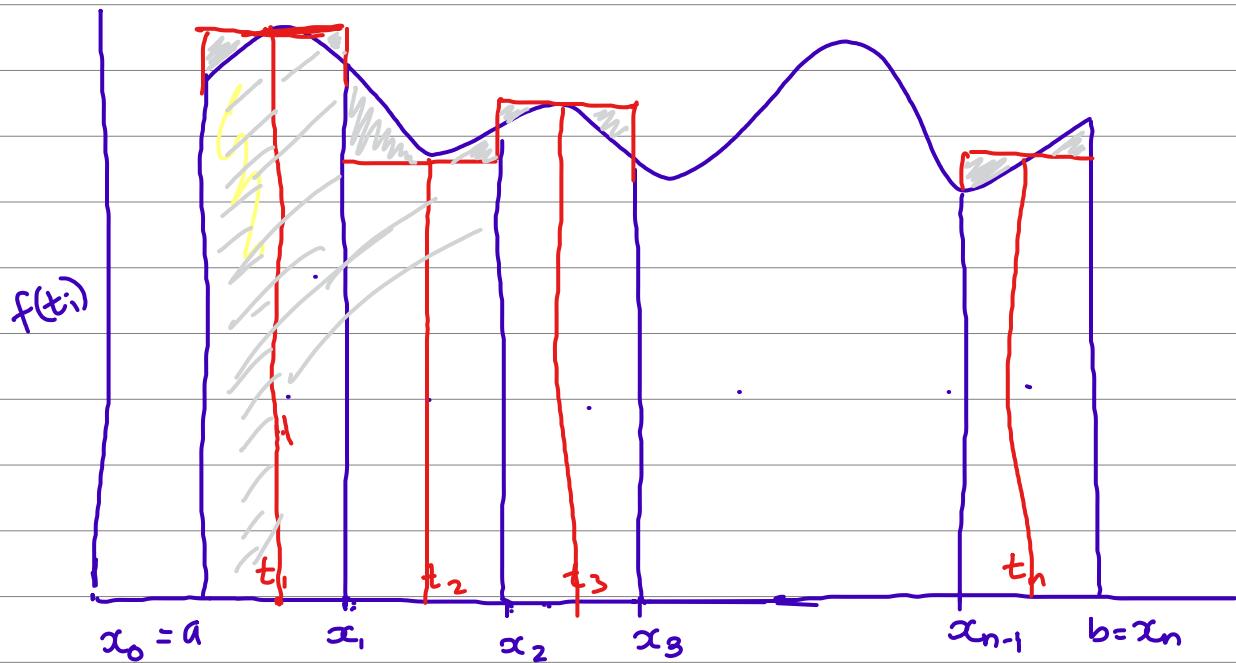
$$t_i \in [x_{i-1}, x_i]$$

$$[x_{i-1}, x_i] \quad \max_{-} \{ x_i - x_{i-1}, i=1:n \} < \delta.$$

$\delta$ - fine partition



$\underline{0.6}$  fine parti.



$$\underline{\dot{P}} := \{ [x_{i-1}, x_i], t_i \}$$

Riemann Sum

$$S(f; \dot{P}) = \sum_{i=1}^n f(t_i) (x_i - x_{i-1}) \underset{-}{\approx} \int_a^b f(x) dx$$

\* norm of partition       $\|P\| = \max \{x_i - x_{i-1}, i=1:n\}$

Lecture : Real Analysis

Manoj C Patil

\* Riemann integrable func

$$f: [a,b] \rightarrow \mathbb{R} . \exists L \in \mathbb{R}$$

for any  $\epsilon > 0 \ \exists \delta_\epsilon > 0, \ \exists \dot{P} \ni \|P\| < \underline{\delta_\epsilon}$

$$\Rightarrow | \underline{s(f; \dot{P})} - L | < \epsilon$$

$f \in R[a,b]$  ← collection of Riemann integrable funcs.

$$L \approx \int_a^b f dx$$

Uniqueness

$L_1, L_2$  limit

for any  $\epsilon_1 > 0 \ \exists \delta_{\epsilon_1} > 0 \ \exists \|P_1\| < \underline{\delta_{\epsilon_1}}$  ✓

$$\Rightarrow | s(f; P_1) - L_1 | < \epsilon$$

for any  $\epsilon_2 > 0 \ \exists \delta_{\epsilon_2} > 0 \ \exists \|P_2\| < \underline{\delta_{\epsilon_2}}$  ✓

$$\Rightarrow | s(f; P_2) - L_2 | < \epsilon_{12}$$

$$\text{let } \delta_\epsilon > 0, \underline{\delta_\epsilon} = \min(\underline{\delta_{\epsilon_1}}, \underline{\delta_{\epsilon_2}}) \Rightarrow \|P\| < \underline{\delta_\epsilon}$$

$$\begin{aligned}
 |L_1 - L_2| &= |L_1 - \underline{S(f; \dot{P})} + \underline{S(f; \dot{P})} - L_2| \\
 &\leq |L_1 - \underline{S(f; \dot{P})}| + |\underline{S(f; \dot{P})} - L_2| \\
 &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &\leq \epsilon
 \end{aligned}$$

$$\Rightarrow L_1 = L_2$$

$$f: [a, b] \rightarrow \mathbb{R} \quad f(x) = c$$

$$\dot{P} = \left\{ \left( \{[x_{i-1}, x_i], t_i\}_{i=1}^n \right) \right\}$$

$$S(f; \dot{P}) = \sum_{i=1}^n f(t_i) \cdot [x_i - x_{i-1}]$$

$$= c \sum_{i=1}^n [x_i - x_{i-1}]$$

$$\begin{aligned}
 &= c \cdot [x_n - x_0] \\
 &= c [b-a]
 \end{aligned}$$

Every constant func is Riemann integrable

$$\int_a^b kf = k \int_a^b f$$

$$f \in R[a,b] \quad \checkmark \quad \square$$

$$kf \in R[a,b]$$

assume  $k > 0$

$\checkmark$  for any  $\varepsilon > 0$   $\exists \delta_\varepsilon > 0 \ni \|p\| < \delta_\varepsilon \Rightarrow |S(f; p) - L| < \frac{\varepsilon}{|k|}$

$$\begin{aligned} & |S(kf; p) - kL| \\ &= \left| \sum_{i=1}^n kf(t_i) \cdot (x_i - x_{i-1}) - kL \right| \\ &= |k| \left| \sum_{i=1}^n f(t_i) (x_i - x_{i-1}) - L \right| \\ &= |k| |S(f; p) - L| \end{aligned}$$

$$\leq |k| \frac{\varepsilon}{|k|} \leq \varepsilon \quad \Rightarrow \quad kf \in R[a,b]$$

$$\int_a^b f + g =$$

$$f, g \in R[a,b]$$

$$S(f; p) \rightarrow \int_a^b f$$

$$\begin{aligned} & f(x) \leq g(x) \quad \forall x \in [a, b] \\ & \int_a^b f \leq \int_a^b g \end{aligned}$$

$$S(g; p) \rightarrow \int_a^b g$$

$$\epsilon > 0 \quad \delta_\epsilon > 0 \quad \Rightarrow \quad |l(p)| < \delta_\epsilon.$$

$$\left. \begin{aligned} S(f; p) &= \sum f(t_i) \cdot (x_i - x_{i-1}) \\ S(g; p) &= \sum g(t_i) (x_i - x_{i-1}) \end{aligned} \right\}$$

$$\left( \begin{array}{c} f(t_i) \leq g(t_i) \quad \forall i = 1:n \\ \int_a^b f \leq \int_a^b g \end{array} \right).$$

$f \in R[a,b]$

bounded

unbounded

$$p = \{ [x_{i-1}, x_i], t_i \} \quad |l(p)| < \delta_\epsilon$$

$$\text{for any } \epsilon > 0 \quad \exists \quad \delta_\epsilon > 0 \quad |l(p)| < \delta_\epsilon \quad \Rightarrow \quad |S(f; p) - L| < \epsilon$$

$$|l(Q)| < \delta_\epsilon$$

$$\Rightarrow |S(f; Q) - L| < \epsilon ?$$

$t_i = x_i \quad i = 1:n \quad \text{except } k$

If  $f$  is not bounded in  $[x_{k-1}, x_k]$



$$\begin{aligned}
 & |S(f; Q) - L| \\
 &= \left| \sum f(t_i) (x_i - x_{i-1}) - L \right| \\
 &= \left| \underbrace{f(t_k) \cdot (x_k - x_{k-1})}_{\cdot} + \sum_{\substack{i=1 \\ i \neq k}}^n f(t_i) (x_i - x_{i-1}) - L \right|
 \end{aligned}$$

$$|f(t_k) \cdot (x_k - x_{k-1})| \geq \underline{|L|} + \left| \sum_{\substack{i=1 \\ i \neq k}}^n f(t_i) (x_i - x_{i-1}) \right| \quad \checkmark$$

$$\text{circled } \underline{S(f; Q)} \geq |f(t_k)(x_k - x_{k-1})| \geq \underline{|L| + \varepsilon} \\
 \geq \varepsilon_0$$

$\therefore$  Our assumption is wrong if  $f \in R[a,b]$   
 $f$  cannot be unbounded.

Cauchy Criterion

$f \in R[a,b]$   
for any  $\varepsilon > 0$   $\exists \delta_\varepsilon > 0 \Rightarrow \|p\| < \delta_\varepsilon, \|q\| < \delta_\varepsilon$

$$\Rightarrow |S(f; p) - S(f; q)| < \varepsilon$$

$f \in R[a,b]$

for any  $\frac{\varepsilon}{2} > 0 \exists \delta_\varepsilon > 0 \ni ||\dot{p}|| < \delta_\varepsilon$

$$|S(f; \dot{p}) - L| < \varepsilon_{1/2}$$

for any  $\frac{\varepsilon}{2} > 0 \exists \delta_\varepsilon > 0 \ni ||\dot{Q}|| < \delta_\varepsilon$

$$|S(f; \dot{Q}) - L| < \varepsilon_{1/2}$$

$$|S(f; \dot{p}) - S(f; \dot{Q})| = |S(f; \dot{p}) - L + L - S(f; \dot{Q})|$$

$$\leq |S(f; \dot{p}) - L| + |L - S(f; \dot{Q})|$$

$$\leq \varepsilon_{1/2} + \varepsilon_{1/2}$$

$$\leq \varepsilon$$

Squeeze Theo.

$$a_n \leq b_n \leq c_n \quad \checkmark$$

$$\lim a_n = \lim c_n = L$$

$$\Rightarrow \lim b_n = L$$

$$f: [a, b] \rightarrow \mathbb{R}, f \in R[a, b]$$

$$(\alpha_\varepsilon, \beta_\varepsilon \in R[a, b]),$$

$$\int_a^b (\beta_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

$$\alpha_\varepsilon(x) \leq f(x) \leq \beta_\varepsilon(x)$$

$$\Rightarrow f \in R[a, b]$$

$\alpha_\varepsilon \in \mathbb{R}[a,b]$      $\beta_\varepsilon \in \mathbb{R}[a,b]$   
 for any  $\varepsilon > 0$   $\exists \delta_\varepsilon > 0 \ni |p| < \delta_\varepsilon$

$$\left| \underline{\int_a^b} s(\alpha_\varepsilon; p) - \int_a^b \alpha_\varepsilon \right| < \varepsilon$$

$$\left| \underline{\int_a^b} s(\beta_\varepsilon; p) - \int_a^b \beta_\varepsilon \right| < \varepsilon$$

$$\underline{\int_a^b} \alpha_\varepsilon - \varepsilon \leq s(\alpha_\varepsilon; p)$$

$$s(\beta_\varepsilon; p) \leq \underline{\int_a^b} \beta_\varepsilon + \varepsilon$$

$$\underline{\int_a^b} \alpha_\varepsilon - \varepsilon \leq s(\alpha_\varepsilon; p) \leq s(f; p) \leq s(\beta_\varepsilon; p) \leq \underline{\int_a^b} \beta_\varepsilon + \varepsilon$$

$$\underline{\int_a^b} \alpha_\varepsilon - \varepsilon \leq s(f; p) \leq \underline{\int_a^b} \beta_\varepsilon + \varepsilon$$

similarly

$$\underline{\int_a^b} \alpha_\varepsilon - \varepsilon \leq s(f; Q) \leq \underline{\int_a^b} \beta_\varepsilon + \varepsilon$$



$$|s(f; p) - s(f; Q)|$$

$$\leq \underline{\int_a^b} \beta_\varepsilon + \varepsilon - \underline{\int_a^b} \alpha_\varepsilon + \varepsilon$$

$$\leq \underline{\int_a^b} (\beta_\varepsilon - \alpha_\varepsilon) + 2\varepsilon = \boxed{3\varepsilon}$$

$$\Rightarrow f \in \underline{\mathbb{R}}[a,b]$$









































































































































