Unit - I - Real Numbers $\mathbb{R}$

Groups Mal

$(\mathbb{R}, +, \cdot)$

* Algebraic Properties of $\mathbb{R}$:-

Add^n Mult^n

On set of $\mathbb{R}$ there are two binary operators $+$ & $\cdot$

These two operations follows few properties :-

add^n

A1 > Commutative property of addition

$$a+b = b+a \quad \forall\, a,b \in \mathbb{R}$$

A2 > Associative property of add^n

$$(a+b)+c = a+(b+c) \quad \forall\, a,b,c \in \mathbb{R}$$

A3) Existence of Zero element (Additive Identity)

$$a+0 = 0+a = a \quad \forall\, a \in \mathbb{R}$$

A4) Existence of negative element (Additive Inverse)

$$a+(-a) = (-a)+a = 0 \quad \forall\, a \in \mathbb{R}$$

$a + -a = 0$

$-a + -(-a) = 0$

a

matric

$A \cdot B \neq B \cdot A$

M1) Commutative property of multiplication

$$a \cdot b = b \cdot a \quad \forall\, a,b \in \mathbb{R}$$

M2) Associative p of multi.

$$(a \cdot b) c = a (b \cdot c) \quad \forall\, a,b,c \in \mathbb{R}$$

M3) Existence of unit element (Multiplicative identity)

$$a \cdot 1 = 1 \cdot a = a \quad \forall\, a \in \mathbb{R}$$

Extended Real

$\infty \leftarrow \mathbb{R}$

M4) Existence of multiplicative inverse / Reciprocals

$$a \cdot \left(\frac{1}{a}\right) = \frac{1}{a} \cdot a = 1 \quad \forall\, a \in \mathbb{R} - \{0\}$$

D) Distributive property of multiplication over add^n

$$\left.\begin{aligned} a \cdot (b+c) &= a \cdot b + a \cdot c \\ (b+c) \cdot a &= b \cdot a + c \cdot a \\ &= a \cdot b + a \cdot c \end{aligned}\right] \quad \forall\, a,b,c \in \mathbb{R}$$

* Order Properties of $\mathbb{R}$

Why do we study Real Analysis?

CLT $\rightarrow \dfrac{\Sigma x_i - \mu}{\sigma} \rightarrow$ ____

$\bar{x} \rightarrow \mu$

seq^n of fun^c Convergence of Series of fun^{c}s

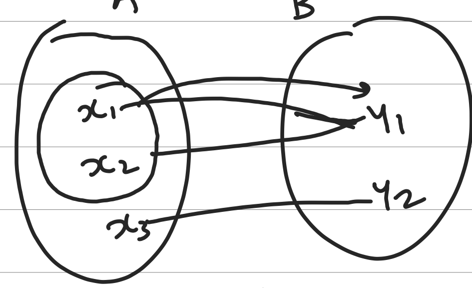
$\Sigma f_n(x) = \Sigma \dfrac{x_i}{n} \longrightarrow$ ____

$\dfrac{\Sigma x_i}{n} \longrightarrow \mu$

A B

Set Fun^c

Fun^c



input
element $\rightarrow$ set

Classi

Prob. $\dfrac{n(A)}{n(S)}$ $\in \mathbb{N}$

Input Set $\longrightarrow$ ____ ~~Real~~

$x^3 + 2x + 3 = 0$

$x = ?$

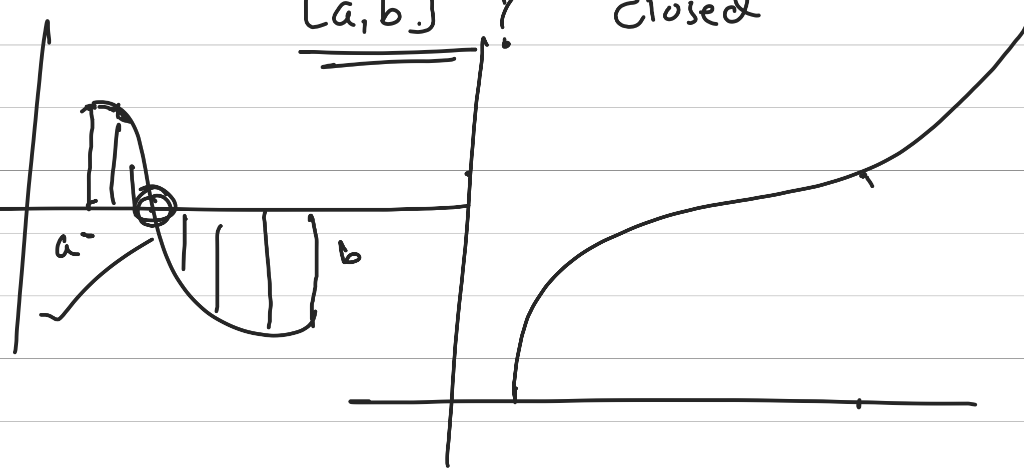
ST-201 (A) Borel fun^c

0 1 $\mathbb{R}$

[a, b] ? Closed

ST-PO5

Numerical Methods



Cont?

Serier $\longrightarrow$ $P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$E(X) = \sum_x \frac{e^{-\lambda}\lambda^x}{x!}x$$
$$= e^{-\lambda} \sum_x \frac{\lambda^x}{x!}x$$
$$= e^{-\lambda}\lambda \sum_x \frac{\lambda^{x-1}}{(x-1)!}$$ Series
$$= \cancel{e^{-\lambda}}\lambda\ \cancel{e^{\lambda}} \quad ?$$

Geometric $P(X=x) = pq^{x-1}$

Real Analysis

$$E(X) = \sum x \cdot pq^{x-1}$$
$$= p \cdot \sum x \cdot q^{x-1} \quad ?$$
$$= p \frac{1}{(1-q)^2}$$
$$= \cancel{p}\frac{1}{p^{\cancel{2}}} = \frac{1}{p}$$
$$= 1/p$$

$$\sum x^n = \frac{1}{1-x} \qquad |x|<1$$

$$? \quad \frac{d}{dx}\sum x^n = \frac{d}{dx}\frac{1}{1-x}$$
$$\sum n x^{n-1} = \frac{1}{(1-x)^2}$$

Inverse fun^c $\longleftarrow$ X Random Variable

ST-201 Probability

ST-301 Asymptotic $\longrightarrow$ as $n \to \infty$

$\bar{x} \to \mu \qquad \frac{1}{n}\sum x_i \to \mu$

1. Unbiased
2. Suff
3. Eff
4. Consistent $\to$ as $n \to \infty$

$X_{(n)}$ # C.E. $\quad X \sim U(0,\theta)$

$2\bar{x} \longrightarrow$

* Order Properties of $\mathbb{R}$

$\mathbb{P}$ set of +ve nos $\in \mathbb{R}$, $\mathbb{P} \subseteq \mathbb{R}$, $\mathbb{R}^+$

$\mathbb{R}^+$ satisfies following properties

① if $a, b \in \mathbb{R}^+ \Rightarrow a+b \in \mathbb{R}^+$

② if $a, b \in \mathbb{R}^+ \Rightarrow a \cdot b \in \mathbb{R}^+$

③ if $a \in \mathbb{R}$ then exactly one of the following is true:-

$\underline{a \in \mathbb{R}^+}$, / $\underline{a = 0}$, / $-\underline{a} \in \mathbb{R}^+$

× OR × OR ✓

-4.5

Law of Trichotomy

$\underline{\underline{a \in \mathbb{R}}}$ >0 <0 $=0$

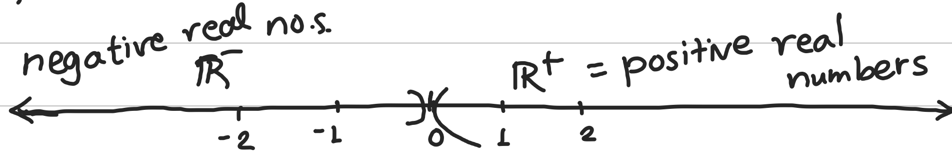
$a \in \mathbb{R}^+$ $a \in \mathbb{R}^-$

$-a \in \mathbb{R}^+$

Let $a, b \in \mathbb{R}$

ⓐ If $\underline{a}-b \in \mathbb{R}^+ \Rightarrow a > b$ or $b < a$

ⓑ If $a-b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a \geq b$ or $b \leq a$



✓ $\mathbb{R}^+ = \{x \,/\, x > 0, x \in \mathbb{R}\}$

$\mathbb{R}^- = \{x \,/\, x < 0, x \in \mathbb{R}\}$

Proof :- ⓐ $a, b \in \mathbb{R}$, $a-b \in \mathbb{R}^+ \Rightarrow a-b > 0 \Rightarrow a > b$

ⓑ $a, b \in \mathbb{R}$ $a-b \in \mathbb{R}^+ \underline{\underline{\cup \{0\}}} \Rightarrow a-b \geq 0 \Rightarrow a \geq b$

Theo:- Let $a, b, \underline{c} \in \mathbb{R}$

① If $a > b$ & $b > c \Rightarrow a > c$ (Transitivity)

② If $a > b \Rightarrow a+c > b+c$

③ If $a > b$, $c > 0 \Rightarrow a \cdot c > b \cdot c$

If $a > b$, $\underline{\underline{c < 0}} \Rightarrow a \cdot c < b \cdot c$

① If $a>b$, & $b>c$

$a-b>0$ & $b-c>0$

$\Rightarrow$ $a-b \in \mathbb{R}^+$ & $b-c \in \mathbb{R}^+$

$\Rightarrow$ $(a-b)+(b-c) \in \mathbb{R}^+$ (by order properties)

$\Rightarrow$ $(a-c) \in \mathbb{R}^+$

$\Rightarrow$ $a-c>0$

$\Rightarrow$ $a>c$ QED

② $a-b \in \mathbb{R}^+$

$a-b+c-c \in \mathbb{R}^+$

$(a+c)-(b+c) \in \mathbb{R}^+$

$a+c > b+c$

③ $a>b$, $c>0$ to prove $a.c > b.c$

$\underline{\underline{a>b}}$, $a-b>0$

$a-b \in \mathbb{R}^+$

$c(a-b) = c.a - cb$ as $c>0$

$ca-cb>0$

$ca>cb$

Similarly $a>b$, $c<0$ $\Rightarrow$ $ac<bc$

Theo:- If $a \in \mathbb{R}$, such that $0 \leq a < \varepsilon$ $\forall$ $\varepsilon \in \mathbb{R}$, $\varepsilon>0$

$\Rightarrow$ $a=0$

$\varepsilon = 0.005$ 08

$\varepsilon = 0.00000$

$a \to 0$

By method of contradiction,

Assume $a > 0$ (as $0 \leq a < \varepsilon$) $\Rightarrow$ a is positive

$\Rightarrow \frac{a}{2}$ is positive

we can assume $\varepsilon_0 = a/2 < a$

for any $\varepsilon > 0$, $0 \leq a < \varepsilon$ but for $\varepsilon_0 \Rightarrow 0 < \varepsilon_0 < a$

which contradicts to our assumption. $\boxed{\varepsilon}$

Rational Nos. :- $Q = \{x \,/\, x = p/q,\ p, q \in \mathbb{I},\ q \neq 0\}$

* Theo. There doesnot exist any rational no. $r \in Q$
$r^2 = 2$

$\Rightarrow$ Assume that $\sqrt{2}$ is rational no.

common divisor of p, q is 1

$(p, q) = 1$ ✓

$$\sqrt{2} = \frac{p}{q}$$

$\frac{4}{6} \Rightarrow \frac{2}{3}$

$$\Rightarrow \quad 2 = \frac{p^2}{q^2}$$

$p = 4$
$p^2 = 16$

$$\Rightarrow \quad 2q^2 = p^2 = p \cdot p$$

$\rightarrow$ $p = 3$
$p^2 = 9$

$\Rightarrow$ p is divisible by 2 $\Rightarrow$ $p = 2 \cdot m$

$\Rightarrow p^2 = 2^2 \cdot m^2 = 4m^2$

p = even

$$\Rightarrow \quad 2q^2 = \cancel{4}^{\,2} m^2$$

$$\Rightarrow \quad q^2 = 2m^2$$

$\Rightarrow$ q is divisible by 2 —— (**)

$(p,q)=1$ but here $(p,q)=2$

which contradicts to our assumption.

even^2

$(2\cdot n)^2$

$= 2(2n^2)$

even

odd^2 $\rightarrow$ $(2n+1)^2$

$= 4n^2+2n+1$

$= 2(2n^2+n)+1$.

$=$ odd

Theo:- If $ab>0$ then either ① $a>0, b>0$

② $a<0, b<0$

* ① If $a,b \in \mathbb{R}$ show that $a^2+b^2=0 \Leftrightarrow a=0, b=0$ ✓

② If $0<c<1$ show that $0<c^2<c<1$ ✓

③ If $x,y \in \mathbb{Q}$, $x+y \in \mathbb{Q}$, $x\cdot y \in \mathbb{Q}$ ✓

If $x \in \mathbb{Q}$, $y \in \mathbb{Q}^c$, $x+y \in \mathbb{Q}^c$

①

$(a+b)^2 = a^2+b^2+2ab = 0 \rightarrow$

$a^2+b^2=0 \Rightarrow$

$2ab=0$

$\Rightarrow a\cdot b=0$

$\Rightarrow a=0$ or $b=0$

(if $a=0$ & $b\neq 0 \Rightarrow a^2+b^2>0$

$a, b \in \mathbb{R}^+$, $c>0$, $\qquad a>b \Rightarrow c\cdot a > c\cdot b$

$0<c<1$, as $c>0$

$c<1$

$c\cdot c < 1\cdot c$

$0<c^2<c<1$

$x, y \in Q \quad \Rightarrow \quad x+y \in Q$

$x = \frac{p_1}{q_1}$, $y = \frac{p_2}{q_2} \quad \Rightarrow \quad x+y = \frac{p_1}{q_1} + \frac{p_2}{q_2}$

$= \frac{p_1 q_2 + p_2 q_1}{q_1 \cdot q_2}$

as $p_1 q_1, p_2 q_2 \in \mathbb{I}$, $\qquad p_1 q_2 \in \mathbb{I}$, $p_2 \cdot q_2 \in \mathbb{I}$, $q_1 \cdot q_2 \in \mathbb{I}$

$p_1 q_1 + p_2 q_2 \in \mathbb{I}$

$= \frac{p^*}{q^*} \in Q$

Absolute values, $\forall\, a \in \mathbb{R}$ $\quad |a| = \begin{cases} +a & \text{if } a>0 \\ -a & \text{if } a<0 \\ 0 & \text{if } a=0 \end{cases}$

$\boxed{|a| = \max(a,-a)}$

Theo:- (a) $|ab| = |a|\cdot|b| \quad \forall\, a,b \in \mathbb{R}$ ✓

(b) $|a|^2 = a^2 \quad \forall\, a \in \mathbb{R}$ $\qquad \underline{-c \le a+b \le c} \Rightarrow |a+b| \le c$

(c) If $c \geq 0$ then $\underline{|a| \le c}$ iff $-c \le a \le c$

(d) $-|a| \le a \le |a|$

Proof:- (a) If $a,b>0 \Rightarrow ab>0 \Rightarrow |a|=a,\ |b|=b,\ |a\cdot b| = ab = |a|\cdot|b|$

$a>0,\ \underline{b<0} \Rightarrow \underline{ab<0} \Rightarrow |a|=a,\ |b|=-b,\ |ab| = -ab = a\cdot(-b) = |a||b|$

$a<0,\ b>0 \Rightarrow ab<0 \Rightarrow$ Simillarly

$a,b<0 \Rightarrow ab>0 \Rightarrow |a|=-a,\ |b|=-b,\ |ab| = ab = (-a)\cdot(-b)$

$= |a|\cdot|b|$ ✓

(b) $|a|^2 = a^2$

If $a>0$, $|a| = a \Rightarrow |a|^2 = a^2$ ✓

$a<0 \quad |a| = -a \Rightarrow |a| = (-a)^2 = a^2$ ✓

$a=0 \quad |a| = a = 0 \Rightarrow |a|^2 = a^2 = 0$ ✓

(c) If $c \geq 0$ then

$|a| \le c$ iff $-c \le a \le c$

i) $|a| \le c \iff |a| = \max(a,-a) \le c$

$\iff a \le c \ \&\ -a \le c$

$\iff a \le c \ \&\ a \geq -c$

$\iff -c \le a \le c$ ✓

(d) If $c \geq 0$, then $\underline{|a| \le c}$ iff $-c \le a \le c$ ✓

now assume $c = |a| \geq 0$, $\quad |a| \le |a|$

$\Rightarrow -|a| \le a \le |a|$

* Triangular inequality :- If $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$

proof :- If $a, b \in \mathbb{R}$, then

$$\begin{array}{rl} & -|a| \leq a \leq |a| \\ + & -|b| \leq b \leq |b| \\ \hline & -|a|-|b| \leq a+b \leq |a|+|b| \end{array}$$

put $c = |a| + |b|$

$\Rightarrow \quad -(|a|+|b|) \leq a+b \leq |a|+|b|$

$\Rightarrow \quad -c \leq a+b \leq c$

$\Rightarrow \quad |a+b| \leq c$

$\Rightarrow \quad |a+b| \leq |a| + |b|$

$|a| = |-a|$

* Corollary : If $a, b \in \mathbb{R}$, then (a) $||a|-|b|| \leq |a-b|$
(b) $|a-b| \leq |a| + |b|$

(a) If $a, b \in \mathbb{R}$

$a = a-b+b$

$|a| = |a-b+b| \leq |a-b| + |b| \Rightarrow$ — (*) (by triangular inequality)

$b = b-a+a$

$|b| = |b-a+a| \leq |b-a| + |a| = |a-b| + |a|$ (by tri. inequality)

— (**)

From (*) $|a| - |b| \leq |a-b|$.

(**) $|b| - |a| \leq |a-b| \Rightarrow |a| - |b| \geq -|a-b|$

(*)

$\boxed{-c \leq a \leq c \quad \Rightarrow \quad |a| \leq c}$

$\Rightarrow \quad -|a-b| \leq |a| - |b| \leq |a-b|$

put $c = |a-b| \Rightarrow ||a|-|b|| \leq c \Rightarrow ||a|-|b|| \leq |a-b|$

$$\underset{c}{-|a-b|} \le |a|-|b| \le \underset{c}{|a-b|} \qquad \Rightarrow \big||a|-|b|\big| \le |a-b|$$

$$\boxed{-c \le a \le c \quad \Rightarrow \quad |a| \le c}$$

(b) $|a-b| \le |a|+|b|$

We have triangular inequality $|a+b| \le |a|+|b|$

replace b by $-b$

$$|a+(-b)| \le |a|+|(-b)|$$

$$\Rightarrow \quad |a-b| \le |a|+|-b| \qquad (|b|=|-b|)$$

$$\Rightarrow \quad |a-b| \le |a|+|b|$$

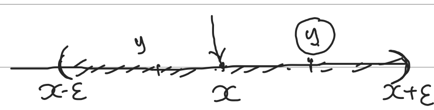
$$a-b \le a+b \quad ? \qquad a,b \in \mathbb{R}$$

Triangular inequality $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$

* Real line Extended Real Numbers

$\underline{\underline{\varepsilon\text{-nbhd.}}}$ $V_\varepsilon(x) = \{y \,/\, y \in (x-\varepsilon, x+\varepsilon)\} \subseteq \mathbb{R}$

$(x-\varepsilon, x+\varepsilon) = V_\varepsilon(x)$



interval length $\Rightarrow 2\varepsilon$

$|x-y| \le \varepsilon \quad \not\Rightarrow y \in V_\varepsilon(x)$

$$V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$$

$$y \in V_\varepsilon(x) \Rightarrow |x-y| \le \varepsilon$$

$$\Rightarrow x-\varepsilon \le y \le x+\varepsilon$$

$$\delta_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) - \{x\} \qquad \text{deleted nbhd of } x$$

e.g. we know $|a+b| \leq |a|+|b|$ but if $|a+b| = |a|+|b|$ iff $ab \geq 0$

If ab≥0 to prove $|a+b| = |a|+|b|$

i> If $a>0, b>0, a+b>0 \Rightarrow |a+b| = a+b = |a|+|b|$

$a<0, b<0, a+b<0 \Rightarrow |a+b| = -(a+b) = (-a)+(-b) = |a|+|b|$

~~$a>0, b<0$~~

If $|a+b| = |a|+|b|$ to prove $ab \geq 0$

$$|a+b|^2 = (|a|+|b|)^2$$

$$(a+b)^2 = |a|^2+|b|^2+2|a|\cdot|b|$$

$$\Rightarrow \cancel{a^2}+\cancel{b^2}+\cancel{2}ab = \cancel{a^2}+\cancel{b^2}+\cancel{2}|a||b|$$

$$\Rightarrow ab = |a|\cdot|b| = |a\cdot b|$$

$$\Rightarrow ab \geq 0$$

e.g. If $x,y,z \in \mathbb{R}$ & $x \leq z$ show that

$x \leq y \leq z$ iff $|x-y|+|y-z| = |x-z|$

i> Let $x \leq y \leq z$. $x-y$ & $y-z$ negative.

LHS $= |x-y|+|y-z|$

$= (y-x)+(z-y)$

$= z-x$

$= |x-z|$ ✓

$=$ R.H.S.

$x \leq y \Rightarrow x-y \leq 0$, $y \leq z \Rightarrow y-z \leq 0$

$|x-y| = -(x-y) = y-x$

$|y-z| = -(y-z) = z-y$.

$x \leq z \Rightarrow x-z \leq 0$

$|x-z| = z-x$

$|x-y| \neq (x-y)$

ii) Let $|x-y|+|y-z| = |x-z|$, $x \leq z$

To prove $x \leq y \leq z$

$x \leq z \Rightarrow x-z \leq 0$, $z-x \geq 0$

$x-z = x-y+y+z$

$$x - z = \underline{x - y} + \underline{y - z}$$

$$|x - z| = |(x-y) + (y-z)| \leq |x-y| + |y-z| \checkmark$$

but $|x-z| = |x-y| + |y-z|$ so. we can use

$$\Big[\ |a+b| = |a| + |b| \quad \text{iff} \quad \underline{\underline{ab \geq 0}}\ \Big]$$

$$(x-y)(y-z) \geq 0$$

$x - y \geq 0$

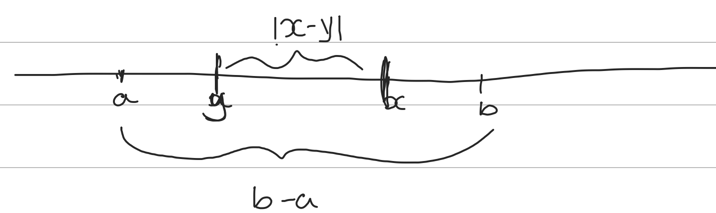
$x \geq y$

Only two possibilities
$\begin{cases} (x-y),(y-z) \geq 0 & \Rightarrow \quad x \geq y, \ y \geq z \quad \Rightarrow \quad \underline{x \geq y \geq z}, \text{ which is not possible} \\ (x-y)(y-z) \leq 0 & \Rightarrow \quad x \leq y, \ y \leq z \quad \Rightarrow \quad \underline{x \leq y \leq z} \end{cases}$

if $a < x < b$

$a < y < b$

$|x-y| \leq b - a$



?

$$-a > -x > -b \quad \Rightarrow \quad -b < -x < -a$$

$$a < y < b \qquad + \quad \underline{a < y < b}$$

$$-(b-a) < y - x < b - a$$

$$\Rightarrow \quad |x-y| \leq b - a$$

e.g. $|x-1| > |x+1|$ $\qquad x \in ?$ $\qquad \underline{\underline{x < 0}}$

$\underline{\underline{-3}}$ $\qquad \underline{\underline{<0}}$ $\qquad -0.01$

$$|x+1| + |x-2| = 7 \qquad\qquad x = -3 \qquad x = 4$$

e.g. $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$ $\quad$ $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$

→ (1) Let $\underline{a<b}$, $\min(a,b) = a$ $\quad$ $\max(a,b) = b$

$|a-b| = b-a$

$$\text{RHS} = \frac{1}{2}(a+b+|a-b|) \qquad \text{RHS} = \frac{1}{2}(a+b-|a-b|)$$

$$= \frac{1}{2}(\cancel{a}+b+b-\cancel{a}) \qquad = \frac{1}{2}(a+b-(b-a))$$

$$= \frac{2b}{2} = \max(a,b) \qquad = \frac{2a}{2} = \min(a,b)$$

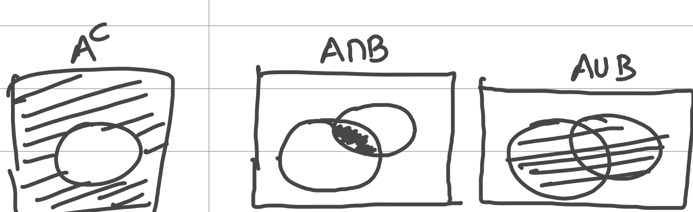
github.com/manojcpatil/Lecture-scribbles

Absent: Aditya, Dipali, Anushree, Aakash, Lalit, Vishal, Bhavesh, Nikita, Sopan, Nikhil, Pranit, Bhanuja

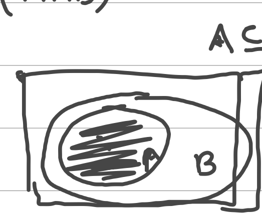
* Set Operations, $A, B \subseteq \mathbb{R}$

- Union: $A \cup B = \{x / x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x / x \in A \ \& \ x \in B\}$
- Complement: $A^c = \{x / x \notin A, x \in \mathbb{R}\}$
- Substract: $A - B = A \cap B^c = A \setminus B$

$= A - (A \cap B)$



(1) Subset $A \subseteq B$: $x \in A \Rightarrow x \in B$ $\quad \not\Rightarrow x \in A$



Theo:-
(1) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
(2) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

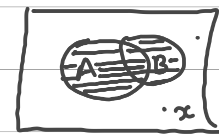
$A=B \Rightarrow A \subseteq B$ & $B \subseteq A$? Can we say that?

$(A \cup B)^c = A^c \cap B^c$ Can you prove this?

To prove ① $(A \cup B)^c \subseteq A^c \cap B^c$

② $A^c \cap B^c \subseteq (A \cup B)^c$

→ Let $x \in (A \cup B)^c$



$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A$ and $x \notin B$

$\Leftrightarrow x \in A^c$ and $x \in B^c$

$\Leftrightarrow x \in A^c \cap B^c$.

$\Rightarrow (A \cup B)^c \subseteq A^c \cap B^c$

$\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c$

$\Rightarrow$ $=$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law)

→To prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ✓

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Let $x \in A \cap (B \cup C)$ $\boxed{\Rightarrow x \in (A \cap B) \cup (A \cap C)}$

$\Leftrightarrow x \in A$ and $x \in B \cup C$

$\Leftrightarrow x \in A$ and ($x \in B$ or C)

$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ → equal $=$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

eg. If A and B are two sets such that
$A \subseteq B$ iff $A \cap B = A$

i) $A \subseteq B \Rightarrow x \in A \Rightarrow x \in B \quad \forall x \in A$
$\Rightarrow$ each and every element of A is is B.
$\Rightarrow A \cap B = A \subseteq B$

ii) $A \cap B = A \Rightarrow$ To prove $A \subseteq B$
$A \cap B = \{x / x \in A \text{ and } x \in B\} \subseteq B$

$A \cap B \subseteq B$
$\Rightarrow A \subseteq B$

$A \setminus (B \cup C) = A \cap (B \cup C)^c$. $\qquad A \setminus B \cap A \setminus C$
$= (A \cap B^c) \cap (A \cap C^c)$

$A \cap (B \cup C)^c \subseteq (A \cap B^c) \cap (A \cap C^c)$
$\supseteq$

Let $x \in A \cap (B \cup C)^c$
$\Leftrightarrow x \in A$ and $x \in (B \cup C)^c$ $\qquad \cup$ - or/and
$\Leftrightarrow x \in A$ and $x \notin B \cup C$
$\Leftrightarrow x \in A$ and $[x \notin B$ and $x \notin C]$

$\Leftrightarrow x \in A$ & $x \notin B$ and $x \in A$ & $x \notin C$
$\Leftrightarrow x \in A \setminus B$ and $x \in A \setminus C$

$\Leftrightarrow \quad x \in A \setminus B \cap A \setminus C$

Similarly we can prove ② part too

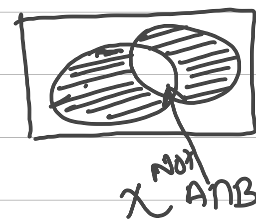
* Symmetric Difference :-

$x \in (A \cup B) \cap (A \cap B)^c$

$A \Delta B = \{x / \underline{x \in A \text{ or } x \in B} \text{ but } \underline{x \notin A \cap B}\} = (A \cup B) \cap (A \cap B)^c$

$= (A \cup B) - (A \cap B)$

$= (A \cup B) \setminus (A \cap B)$



$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

$= (A \cap B^c) \cup (B \cap A^c)$

$= \underline{(A \cap B^c) \cup B} \cap \underline{(A \cap B^c) \cup A^c}$

$= (A \cup B) \cap \underbrace{(B^c \cup B)}_{\mathbb{R}} \cap \underbrace{(A \cup A^c)}_{\mathbb{R}} \cap (B^c \cup A^c)$

B^c

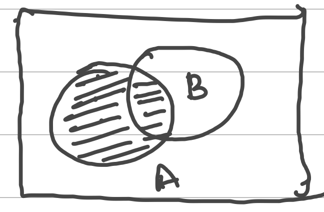
$= (A \cup B) \cap (B^c \cup A^c)$

$= (A \cup B) \cap (A \cap B)^c$

$= (A \cup B) \setminus (A \cap B)$

$A \setminus B$
$= A \cap B^c$

$A \setminus (B \setminus A)$



$A \setminus (B \setminus A) = B \cap A^c$

$A \cap (B \setminus A)^c$

$= A \cap (B \cap A^c)^c$

$= A \cap (A \cup B^c)$

$= \underline{\underline{A}}$

Lower bound :- Upper bound

$S = \{\frac{1}{n}, n \in \mathbb{N}\} \ni$ low $S = 0 \notin S$ Upper bound $(S) = 1 \in S$

Bounded below. Bounded Above.

$\underline{\underline{\mathbb{N}}}, \mathbb{Z}, \underline{\underline{\{\frac{1}{n}, n \in \mathbb{N}\}}}$ $\longrightarrow$ $\mathbb{I}^-, \mathbb{Z}^-, \mathbb{R}^-$

Bounded set bounded below & Above

$\{(-1)^{2n}, n \in \mathbb{N}\} = \{1\}$ singleton set

<u>Unique?</u>

$\{(-1)^n, n \in \mathbb{N}\} = \{-1, 1\}$ lower bound $= -1 \in S$

Upper bound $= +1 \in S$

$\{-10000, -10, \underline{\underline{-0.5}}\}$

$\underline{\underline{S = (0,1]}}$ Lower bound $= 0 \notin S$

$x \geq 0 \quad \forall x \in S$ Upper bound $= 1 \in S$

$x \leq 1 \quad \forall x \in S$

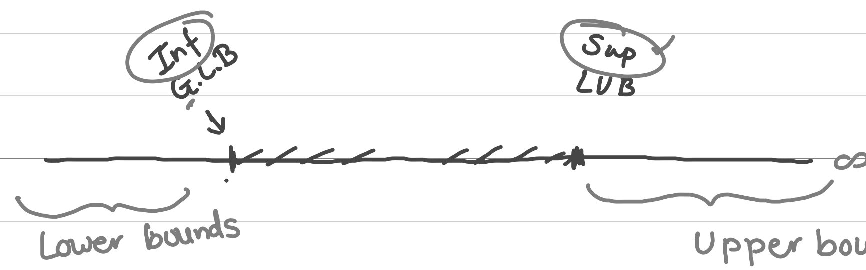
$\underline{u}$ l.b. of $S \Rightarrow \forall x \in S, \ \underline{x \geq u}$

Sup :- u is sup of S if ① u is upper bound
② v is any other upper bound of S then $v \geq u$.

u must be least upper bound.

Inf :- if v is inf of S if ① v is lower bound
② u is any other lower bound of S then $v \geq u$

v must be greatest lower bound.



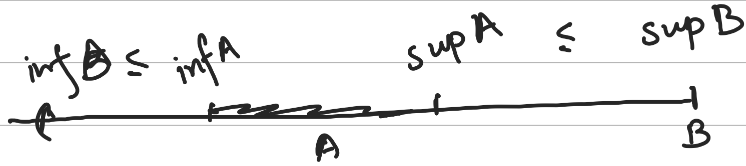
No upper bound / Supremum :- $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{I}, \mathbb{Z} \ldots$

No lower bound / Inf :- $-\mathbb{N}, \mathbb{Q}^-, \mathbb{R}^-, \mathbb{I}^-, \mathbb{Z}^-$

$S = \{1,2,3,4\}$. $\sup(S) = 4$ set of Upper bounds $= [4, \infty)$
$\inf(S) = 1$ set of lower bounds $= (-\infty, 1]$

* Completeness property :- Lower bound exists $\Rightarrow$ Inf exists
Upper Sup

$A \subseteq B \Rightarrow \inf A \geq \inf B$



Infⁿr Suppose

Absent:- Adity. Dipali, Anushree, Pradip, Aakash, Lalit
vishal, Komal, Samiya, Govinda, Atul, Chetan,
Yogita.

Completeness Property :-

Lower bound → inf exists $\quad \mathbb{N} \Rightarrow$ 0 lower - 1 inf
Upper → sup exists $\quad \mathbb{R}^- \Rightarrow$ 1 upper, 0 sup

Supremum — Infimum
least upper bound — Greatest Lower bound

LUB :- If u is ~~up~~ least upper bound of set S then
① u is upper bound ✓
② if v is any other upper bound of S the $v \geq u$.

GLB :-

Theo :- If sup exists, then it is unique.

proof :- We will prove this by method of contradiction.
Assume for set S $\exists$ u_1 & u_2 be two sup.

by def^n u_1 is sup

① u_1 is upper bound of S

② If $\exists$ u_2 as any other u.b. then $u_2 \geq u_1$ ✓

u_2 is sup.

① u_2 is upper bound of S

② If $\exists$ u_1 as any other then $u_1 \geq u_2$ ✓

$$\Rightarrow \qquad u_1 = u_2$$

$\Rightarrow$ Sup is unique

Theo :- If $A, B \subseteq \mathbb{R}$ and $A \subseteq B$ then if inf & Sup of B exists then

① $\inf(A) \geq \inf(B)$

② $\sup(A) \leq \sup(B)$

Proof :- ① Assume $\inf(A) = u_A$ $\inf(B) = u_B$

$A \subseteq B$ to prove $\inf(A) \geq \inf(B)$

$u_A \geq u_B$.

We will prove this by method of contradiction

$$\boxed{u_A < u_B}$$

by def of infimum

① u_B is lower bound of B.

② u_x any lower bound of B $\Rightarrow$ $u_x \leq u_B$ (Greatest Lower bound)

u_B Lower bound of $B \Rightarrow \forall x \in B,\ u_B \leq x$

$A \subseteq B \Rightarrow \forall x \in A \Rightarrow x \in B$

u_B is also lower bound of A as $\forall x \in A \subseteq B,\ u_B \leq x$

u_A is inf of $A \Rightarrow u_A$ is also lower bound of A & if $\exists$ some other lower bound of A then u_A is greater than that lower bound

$\Rightarrow u_B \leq u_A$

which contradicts to our assumption.

$A \subseteq B \quad \inf A \geq \inf B$

Assume $u_A \leq u_B$.

Sup?

① u_B inf $B \Rightarrow$ u_B lower bound $\rightarrow x \geq u_B \ \forall x \in B$

u_x lower bound of B, $\Rightarrow u_x \leq u_B$ GLB

$x \geq u_B \quad \forall x \in B$

$\Rightarrow x \geq u_B. \quad \forall x \in A \subseteq B$

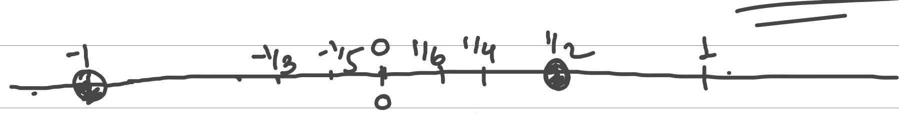
$\Rightarrow u_B$ is lower bound of A.

u_A is $\inf(A)$

1) u_A l.b.

2) u_y l.b. of A, $u_y \leq u_A$

$\Rightarrow u_B \leq u_A$



$S = \left\{ \frac{(-1)^n}{n},\ n \in \mathbb{N} \right\}$

Set of Lower bounds $= (-\infty, -1]$

Upper bounds $= [1/2, \infty)$

$S = \left\{ \frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \dots \right\}$

$\inf = -1 \in S$

$\sup = 1/2 \in S$

$S = \left\{1 - \frac{(-1)^n}{n},\ n \in \mathbb{N}\right\}$ $\inf S = 1/2$

set of LBs = $\sup S = 2$

UBs =

$S = \left\{1 - \frac{(-1)^1}{1},\ 1 - \frac{(-1)^2}{2}, \ldots\right\} = \left\{2, \frac{1}{2}, \frac{4}{3}, \ldots\right\}$

$S = \left\{\frac{1}{m} + \frac{1}{n},\ m, n \in \mathbb{N}\right\}$

2 sup   inf 0

? $\left\{\frac{(-1)^m}{m} + \frac{(-1)^n}{n},\ m, n \in \mathbb{N}\right\}$? -2 1

* Cartesian Product

$A \times B = \{ \langle x, y \rangle \,/\, x \in A,\ y \in B \}$

$A = \{2, 3, 4\}$ $B = \{1, 5\}$.

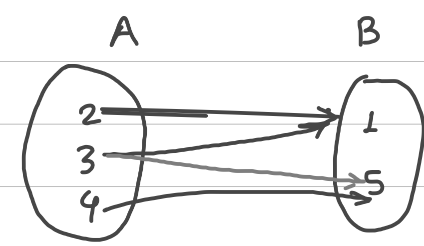
$A \times B = \{ \langle 2,1 \rangle, \langle 2,5 \rangle, \langle 3,1 \rangle, \langle 3,5 \rangle, \langle 4,1 \rangle, \langle 4,5 \rangle \}$

$f = \{ \langle 2,1 \rangle, \langle 3,1 \rangle, \langle 4,5 \rangle \}$

$f(2) = 1$

$f(3) = 1$

$f(4) = 5$

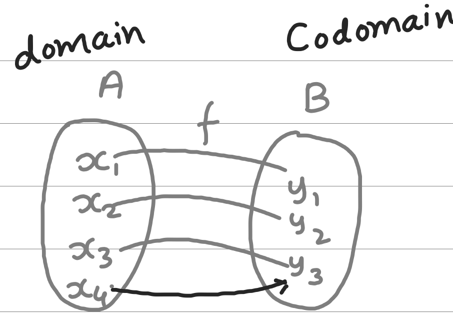


One-one fun^c (Injective)

$x_1, x_2 \in A$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

not one-one　$x_3 \neq x_4 \Rightarrow f(x_3) = f(x_4) = y_3$

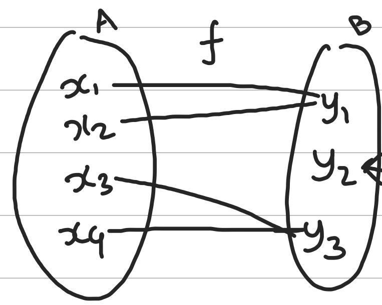


one-one

$f(x) = x^3 \quad \forall x \in \mathbb{I}$　　　$f(x) = x^2 \quad \forall x \in \mathbb{N}$ ✓

*

Into



into

domain $= \{x_1, x_2, x_3, x_4\}$

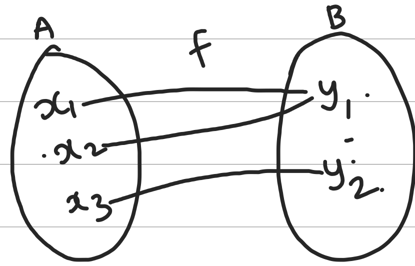
Codomain $= \{y_1, y_2, y_3\}$

Range$(f) = \{y_1, y_3\}$

Range $\subset$ Codomain

Onto

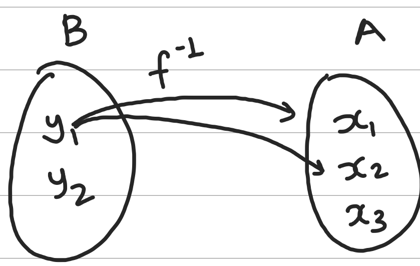
Surjective



Codomain = Range

onto

set fun^c f^{-1}



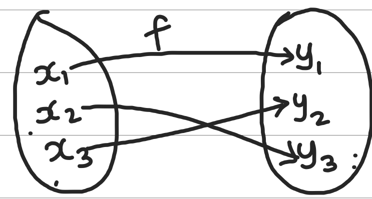
$f^{-1}(y_1) = \{x_1, x_2\}$

$f^{-1}(y_2) = \{x_3\}$.

set fun^c

$$f^{-1}(y) = \{x \,/\, x \in A,\ f(x) = y \in B\}$$ ✓

One-One & onto $\Rightarrow$ (Injective + Surjective) $\Rightarrow$ Bijective fun^c

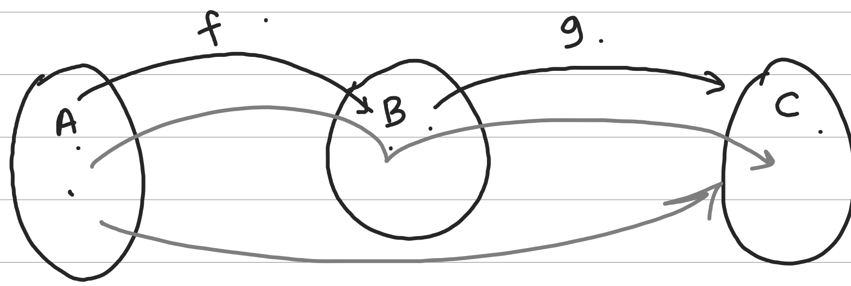


one-one & onto
In Sur
& Bijective

$f(x) = 2x \quad \ni x \in$ [illegible] $, \mathbb{I}$

$f(x) = x^2 \quad \ni x \in \mathbb{N}$

Composite fun^c

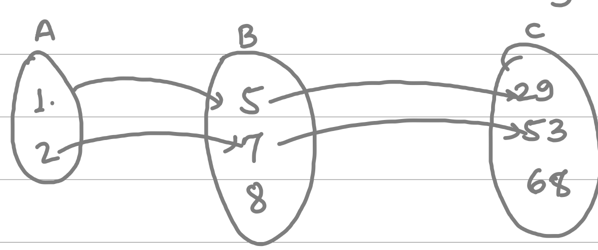


$D(f) = A \qquad R(f) \subseteq B \qquad D(g) = B$

$R(g) \subseteq C$

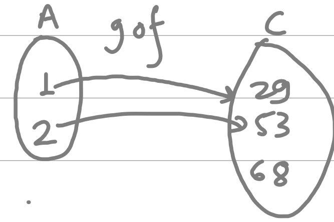
$(2x+3)^2 + 4 \qquad g \circ f$

$f(x) = 2x+3 \qquad g(y) = y^2+4$



$D(f) = A, \quad R(f) \subseteq D(g)$

$g \circ f$



$$g \circ f \overset{\text{may or may not}}{\neq} f \circ g$$

$f: \mathbb{R} \to \mathbb{R}$ & $g: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x+3$ $\quad$ $g(y) = y^2+4$

$$g \circ f(x) = g(f(x)) = g(2x+3) = (2x+3)^2+4$$
$$= 4x^2+12x+13$$

$$f \circ g(x) = f(\underline{g(x)}) = f(x^2+4) = 2(x^2+4)+3$$
$$= 2x^2+11$$

$$g \circ f \neq f \circ g$$

$f: A \to B$, $E, F \subseteq A$

$$\underset{\text{set}}{f(E \cup F)} = \underset{\text{set}}{f(E)} \cup f(F)$$

$f(E) = \{y \,/\, y = f(x), x \in E \subseteq A\}$ ✓

$f(F) = \{y \,/\, y = f(x), x \in F \subseteq A\}$

$\underline{f(E \cup F)} = \{y \,/\, \underline{y = f(x), x \in E \cup F} \subseteq A\}$ ✓

Toprove : $\quad - f(E) \cup f(F) = f(E \cup F)$

$$\subseteq$$
$$\supseteq$$

Let $\quad y \in f(E) \cup f(F)$

$\Leftrightarrow \quad y \in f(E) \quad$ or $\quad y \in f(F)$

$\Leftrightarrow \quad y = f(x) \quad$ and

$\qquad x \in E \quad$ or $\quad x \in F$

$\Leftrightarrow \quad y = f(x) \quad$ and $\quad x \in E$ or F

$\Leftrightarrow \quad y = f(x) \quad$ and $\quad x \in E \cup F$

$\Leftrightarrow \quad y \in f(E \cup F)$

$\Rightarrow$ As $\quad y \in f(E) \cup f(F) \Rightarrow y \in f(E \cup F)$

So $\quad f(E) \cup f(F) \subseteq f(E \cup F)$

Simillarly we can obtain,

$$f(E \cup F) \subseteq f(E) \cup f(F)$$

$$\Rightarrow \quad f(E \cup F) = f(E) \cup f(F)$$

$$f(E \cap F) \subseteq f(E) \cap f(F)$$

let $\quad y \in f(E \cap F)$

$\Rightarrow \quad y = f(x) \quad$ and $\quad x \in E \cap F$

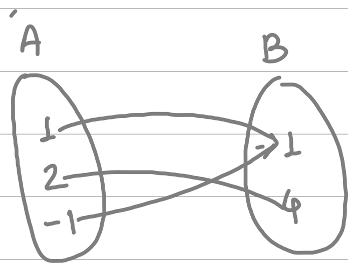
$\Rightarrow \quad \underline{y=f(x)} \;\&\; \underline{x \in E}$ and $y=f(x) \;\&\; x \in F$

$\Rightarrow \quad y \in f(E)$ and $y \in f(F)$

$\Rightarrow \quad y \in f(E) \cap f(F)$

$\Rightarrow \quad f(E \cap F) \subseteq f(E) \cap f(F)$

✓ $f(E \cap F) \not\supseteq f(E) \cap f(F)$



$E = \{1,2\}$ $\quad f(E) = \{1,4\}$
$F = \{-1,2\}$ $\quad f(F) = \{1,4\}$
$\underline{E} \cap F = \{2\}$ $\quad f(E \cap F) = \{4\}$

$f(E) \cap f(F) = \{1,\underline{4}\}$

$\Rightarrow f(E \cap F) \not\supseteq f(E) \cap f(F)$

$f : A \to B \qquad G, H \subseteq B$

i) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

$f^{-1}(G) = \{x \,/\, f\underline{(x)} = y \in \underline{G},\ x \in A\}$
$f^{-1}(H) = \{x \,/\, f(x) = y \in H,\ x \in A\}$
$f^{-1}(G \cup H) = \{x \,/\, \underline{f(x) \in G \cup H},\ x \in A\}$

To prove $f^{-1}(G \cup H) \underset{\supseteq}{\subseteq} f^{-1}(G) \cup f^{-1}(H)$

Let $x \in f^{-1}(G \cup H)$

$\Rightarrow f(x) \in G \cup H \qquad x \in A$

$\Rightarrow f(x) \in G \quad \text{or} \quad f(x) \in H \qquad x \in A$

$\Rightarrow x \in A \;\&\; f(x) \in G \quad \text{or} \quad x \in A \;\&\; f(x) \in H$

$\Rightarrow x \in f^{-1}(G) \quad \text{or} \quad x \in f^{-1}(H)$

$\Rightarrow x \in f^{-1}(G) \cup f^{-1}(H)$

$\Rightarrow f^{-1}(G \cup H) \subseteq f^{-1}(G) \cup f^{-1}(H)$

Simillarly we can obtain $f^{-1}(G \cup H) \supseteq f^{-1}(G) \cup f^{-1}(H)$

Thm. $f : A \to B$ is injective $E \subseteq A$

$f^{-1}(f(E)) = E$

$\longrightarrow$

$\therefore$ $f : A \to B$ as injective fun^c

for $x_1, x_2 \in A$, if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f^{-1}(D) = \{x \,/\, \underline{f(x) \in D}, x \in A\}$ ✓

$f(E) = \{y \,/\, y = f(x), x \in E\}$

Let $x \in f^{-1}(f(E))$

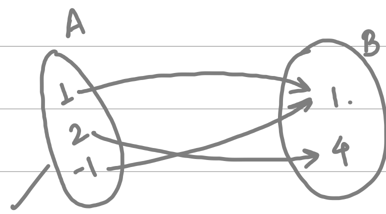
$\Rightarrow y = f(x) \in f(E)$ & $x \in A$, $y \in B$

one-one

$\exists$ some $x \in E$? $(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$

$\Rightarrow f^{-1}(f(E)) \subseteq E$

Not one-one



$E = \{1,2\}$ ✓

$f(E) = \{1,4\}$

$f^{-1}(f(E)) = f^{-1}(\{1,4\})$

$= \{1,-1,2\}$

$E \neq f^{-1}(f(E))$

Finite Set :- $A = \{x_1, x_2, x_3, \dots, x_n\}$

$B = \{1, 2, 3, \quad , n\} \subseteq \mathbb{N}$ n is some fixed $\in \mathbb{N}$

Infinite ✓ $\mathbb{Q}$, $\mathbb{N}$. $\{1,2,3,\dots\dots \_\_\_\_\}$

even odd square ____

complex Real

$S, T \subseteq \mathbb{R}$, $T \subseteq S$

To prove If S is finite then T is also finite.

→ S is finite if it is either empty or it has n element

① $S=\phi$ & $T \subseteq S$ $\Rightarrow$ $T=\phi$ $\Rightarrow$ T is finite.

② We will prove this by mathematical induction

S is finite ✓

✓ $\#(S)=1$ & $T \subseteq S$ (2.1) $\#(T)=0$ or $\#(T)=1$

$T=\phi$ or $T=S$

T is finite

✓ $\#(S)=K$ & $T \subseteq S$, $\Rightarrow$ T is finite.

$\#(S)=K+1$ $\Rightarrow$ $S=\{x_1, x_2, \ldots, x_{K+1}\}$

$\{1, 2, \ldots, K+1\}$.

$S_1 = S - \{\overset{x_{K+1}}{f(K+1)}\}$ & $T \subseteq S$

$f(K+1) \in T$ or

$T \nsubseteq S_1$

$T_1 = T - \{f(K+1)\} \subseteq S_1$

$f(K+1) \notin T$

$T \subseteq S_1 \subseteq S$

$\Rightarrow$ T is finite

$T_1 \subseteq S_1$ & $\#(S_1)=K$

$\Rightarrow$ T_1 is finite set

$\#(T_1)+1 = \#(T)$

$\Rightarrow$ T is also finite

* Countable Set?

even $\{2n, n \in \mathbb{N}\} = \{2,4,6,8,10,\dots\}$] Bijection with $\mathbb{N}$
odd $\{1,2,3,4,5,\dots\}$ of subset of $\mathbb{N}$

countable set :- finite or bijection with $\mathbb{N}^m$, $m \in \mathbb{N}$.

$\mathbb{N}^2$
$(1,1)\ (1,2)\ (1,3)\ \dots$
$(2,1)\ (2,2)\ (2,3)\ \dots$
$\vdots$

1, 2, 3, 4, . . .

countable finite/bijection $\mathbb{N}^m$

$\mathbb{N}^3$ $(1,2,2)$

$S, T \subseteq \mathbb{R}$, $T \subseteq S$
① S is countable $\Rightarrow$ T is also countable

· S is countable $\Rightarrow$ S is either finite or bijection with $\mathbb{N}$ (denumerable)

① if S is finite $\Rightarrow$ T is finite $\Rightarrow$ T is countable (by previous theo.)

② if S has bijection with $\mathbb{N}$ (denumerable)
$\mathbb{N} = \{1,2,3,\dots\}$
We can write $S = \{f(1), f(2), \dots\}$
$T = \phi$, $T =$ finite., if T is infinite
$T \subseteq S \Rightarrow T = \{f(n_1), f(n_2), f(n_3) \dots\}$
$B = \{n_1, n_2, n_3, \dots\}$
$\mathbb{N} = \{1, 2, 3, \dots\}$ T denumerable set

countable

$\mathbb{N}^2$ 1, 2, 3, 4, 5, ... countably infinite / denumerable

$\mathbb{N}^2$

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \dots \\ (2,1) & (2,2) & (2,3) & (2,4) & \dots \\ (3,1) & (3,2) & (3,3) & (3,4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

countable

Set of Rational Numbers is denumerable.

$$Q = \left\{ \frac{p}{q},\ p, q \in \mathbb{I},\ q \neq 0 \right\}$$

Lets discuss Q^+

2.5 $\frac{5}{2}$. $\frac{10}{4}$

$\exists$ $\mathbb{N}^2$

$$\begin{matrix} 1/1 & 1/2 & 1/3 & 1/4 & \dots \\ 2/1 & 2/2 & 2/3 & 2/4 & \dots \\ 3/1 & 3/2 & 3/3 & 3/4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

< 5000

countable

Archemedian Property

$$x \in \mathbb{R} \quad \exists\ n_x \in \mathbb{N}, \quad x < n_x$$

$$23.56 \in \mathbb{R} \quad \exists\ 24, 25 \in \mathbb{N}, \quad 23.56 < 24$$

Set of $\mathbb{N}$ is bounded below $\Rightarrow$ It has no upper bound.
$\Rightarrow$ It doesnt have supremum.

We will prove this by method of contradiction

$x \in \mathbb{R}$ $\nexists$ any $n_x \in \mathbb{N}$, $\ni$ $x < n_x$

$\Rightarrow$ $n \leq x$ $\forall$ $n \in \mathbb{N}$

$\Rightarrow$ x is upper bound for $\mathbb{N}$

Completeness $\curvearrowright$ it has some supr let u.

$\Rightarrow$ $n \leq u$ $\forall$ $n \in \mathbb{N}$.

$\Rightarrow$ $n+1 \leq u$ $\forall$ $n \in \mathbb{N}$

$\Rightarrow$ $n \leq u-1$ $\forall$ $n \in \mathbb{N}$ $\Rightarrow$ $u-1$ upper bound of $\mathbb{N}$

by def^n of supremum if u is sup $\Rightarrow$ ① u upper bound
② There exist no upper bound less than u

but $u-1$ is upper bound
u -"- but u is sup.

$\therefore$ This contradicts to our assumption that u is sup.

$\therefore$ $\exists$ $n_x \in \mathbb{N}$, $\ni$ $x < n_x$ (QED)

* If $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ then $\inf S = 0$

let $\varepsilon > 0$, $1/\varepsilon \in \mathbb{R}$ $0 < 1/\varepsilon$

by Archemedian property

$\varepsilon > 0$, $1/\varepsilon \in \mathbb{R}$, $\exists\, n_\varepsilon \in \mathbb{N} \ni 0 < 1/\varepsilon < n_\varepsilon$

Any $\varepsilon > 0$. $\Rightarrow 0 < 1/n_\varepsilon < \varepsilon$

$\Rightarrow$ for set S, 0 is lower bound

$\inf S \geq 0$

Assume $\inf S > 0$, $\varepsilon = \inf S > 0$

$\Rightarrow \exists\, n_\varepsilon, \ni 0 < 1/n_\varepsilon < \varepsilon$

$\boxed{0 < 1/n_\varepsilon < \inf S}$

$\inf S$ is infimum of S $\Rightarrow$ S is Lower bound

$\Rightarrow \inf S \leq 1/n \quad n \in \mathbb{N}$

$\therefore$ Our assumption is wrong $\inf S > 0$

$\Rightarrow \inf S = 0$

If $t > 0$, $\exists\, n_t \in \mathbb{N}, \ni 0 < \frac{1}{n_t} < t$

* Let $y \in \mathbb{R}$, $\exists\, n_y \in \mathbb{N}$, $n_y - 1 \leq y \leq n_y$

$2.34 \in \mathbb{R}$ $\exists\, 3 \in \mathbb{N}$ $3-1 \leq y \leq 3 \Rightarrow 2 \leq y \leq 3$

Let $y \in \mathbb{R}$, $E_y = \{n \mid n > y, n \in \mathbb{N}\}$

if $y = 2.34$ $E_y = \{3, 4, 5, 6, \ldots\}$

$y \leq n \quad \forall\, n \in E_y$

$\Rightarrow$ y is Lower bound of E_y

by completeness property, $\exists$ inf of $E_y = u$

$\therefore \quad u \leq n \quad \forall\, n \in \mathbb{N}$

if u is inf, $u+1$ cannot be lower bound of S

$\exists$ some $n \in \mathbb{N}$, $\quad n < u+1$

$\Rightarrow \quad n-1 < u$.

Density Theo:- $\quad x, y \in \mathbb{R}$, $x < y$, $\exists\, r \in \mathbb{Q}$, $x < r < y$

$x < y$, $y - x > 0$ $\exists\, n \in \mathbb{N}$, $\quad 0 < \frac{1}{n} < y - x$

$\Rightarrow \quad 1 < ny - nx$

$\Rightarrow \quad nx + 1 \leq ny$.

Assume

$x > 0$, $nx > 0$ $\quad m \in \mathbb{N}$ $\quad m - 1 \leq nx \leq m$,

$nx \leq m \leq nx + 1 \leq ny$ $\quad$? $\mathbb{Q}$ $\quad r = \frac{m}{n}$

$\Rightarrow \quad nx \leq m \leq ny \quad \Rightarrow x \leq \frac{m}{n} \leq y$

by density theo.

if $x, y \in \mathbb{R}$, $x<y$ then $\exists\ r \in Q$, $x<r<y$

$\sqrt{2}$ is irrational no. so does $x\sqrt{2}$ & $y\sqrt{2}$

$\therefore$ if $x<y$

$$\sqrt{2}x < \sqrt{2}y$$

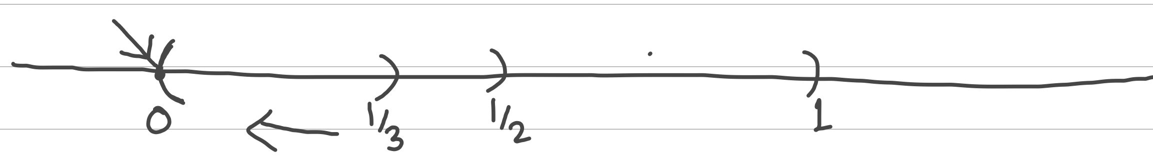
$\because$ by density theo.

$$\sqrt{2}x < r < \sqrt{2}y$$

$$x < \frac{r}{\sqrt{2}} < y$$

Arbitrary intersection

$\bigcap_{n=1}^{k} \left(0, \frac{1}{n}\right)$ finite intersection $\Rightarrow$ $\left(0, \frac{1}{k}\right)$



$$\bigcap_{n=1}^{\infty} (0, 1/n) = \phi$$

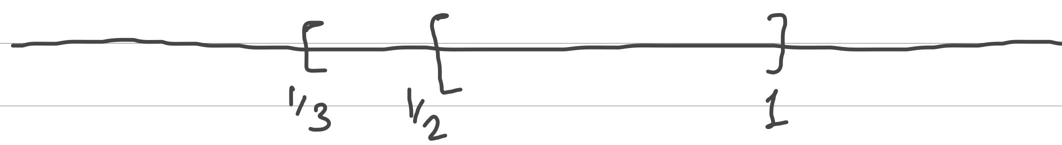
$\bigcap_{n=1}^{\infty} (0,n) = (0,1)$ $\qquad$ $\bigcup_{n=1}^{\infty} (0,n) = (0,\infty)$

$(0,1) \cup (0,2) = (0,2)$

$(0,1) \cup (0,2) \cup (0,\dots) \cup (0,n) = (0,n)$

$\left[\frac{1}{k},1\right] \Leftarrow \bigcup_{n=1}^{k} \left[\frac{1}{n},1\right]$ $\qquad$ $n=1 \Rightarrow \{1\}$ $\qquad$ $n=2 \Rightarrow \left[\frac{1}{2},1\right]$

$n=3 \Rightarrow \left[\frac{1}{3},1\right]$

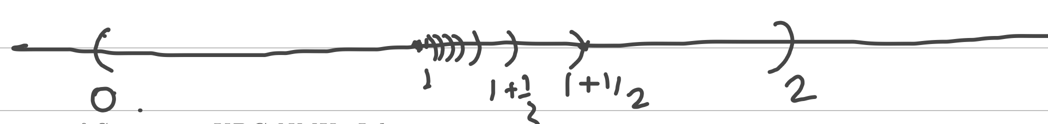


$\bigcup_{n=1}^{\infty} \left[\frac{1}{n},1\right] = (0,1]$

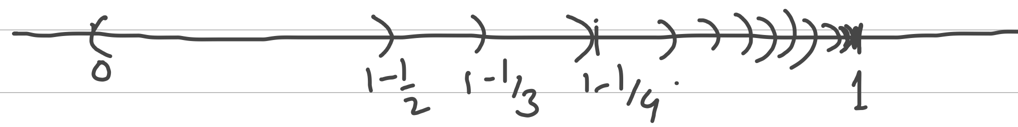
closed

Arbitrary union of closed intervals ~~is~~ may or maynot again closed. interval

$(0,1] = \bigcap_{n=1}^{\infty} (0, 1+1/n)$ $\qquad$ $(0,2) \cap (0, 1+\frac{1}{2}) = (0, 1+\frac{1}{2})$



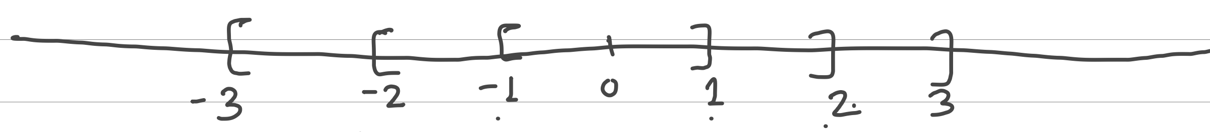
$$\bigcup_{n=1}^{\infty} \left(0, 1 - 1/n\right) = (0, 1-1) = \phi \quad (0, 1 - 1/2)$$



$$= (0, 1)$$

$$\bigcup_{n=1}^{\infty} [0, 1 - 1/n] = [0, 1)$$ half closed half open interval

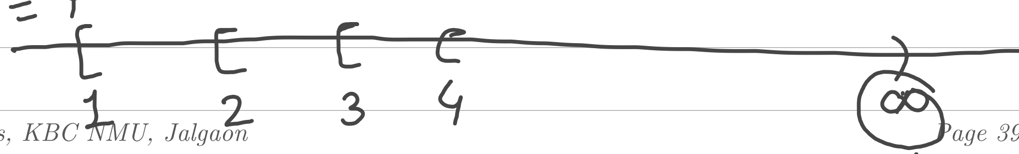
$$\bigcup_{n=1}^{\infty} [-n, n] \checkmark = (-\infty, \infty)$$



$$\bigcup_{n=1}^{K} [-n, n] = [-K, K]$$

$$\bigcap_{n=1}^{\infty} [-n, n] = [-1, 1] \checkmark$$

$$\bigcap_{n=1}^{\infty} [n, \infty) = \phi$$



$\mathbb{R}$ is uncountable
$\Rightarrow$ $(0,1)$ is uncountable

Assume $(0,1)$ is countable.
$\Rightarrow$ $(0,1)$ is denumerable
$\Rightarrow$ $(0,1)$ has one-one & onto ~~cor~~ relation with $\mathbb{N}$
$\Rightarrow$ ~~or~~ $S = (0,1) = \{b_1, b_2, b_3, \dots\}$
$b_i \in (0,1)$ it can be written in the form of $0,1,2,\dots 9$.

$b_1 = 0.\underline{a_{11}}\,a_{12}a_{13}\dots \qquad c_1 \neq a_{11} \qquad 0.121234$
$b_2 = 0.a_{21}\,\underline{a_{22}}\,a_{23}\dots \qquad c_2 \neq a_{22}$
$b_3 = 0.a_{31}\,a_{32}\,a_{33}\dots \qquad c_3 \neq a_{33}$
$\vdots$

✓ $c = 0.c_1c_2c_3\dots$
$c \in (0,1)$ and as $c_i \neq a_{ii} \ \forall i$

$\Rightarrow \quad c \neq b_i \quad \forall i$

but as $c \in (0,1)$ & $c \neq b_i \ \forall i$
$\therefore$ So our assumption that $(0,1)$ is countable is wrong

$\Rightarrow$ $(0,1) \subseteq \mathbb{R}$, $(0,1)$ is uncountable $\Rightarrow$ $\mathbb{R}$ is uncountable.

* Cauch Schwartzs inequality

If $a_i, b_i \in \mathbb{R}$ $\quad i = 1:n$

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

$a_i, b_i \in \mathbb{R}$, some $x \in \mathbb{R}$

$$a_i x + b_i \in \mathbb{R} \qquad \forall i$$

$$\boxed{(a_i x + b_i)^2 \geq 0} \qquad \forall i$$

$$a_i^2 x^2 + 2a_i b_i x + b_i^2 \geq 0 \qquad \forall i$$

$$x^2 \sum_{i=1}^{n} a_i^2 + 2x \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2 \geq 0$$

This statement is true for any $x \in \mathbb{R}$

$$A = \sum a_i^2 \qquad \underline{B = \sum a_i b_i} \qquad C = \sum b_i^2$$

$$A x^2 + 2Bx + C \geq 0$$

Assume $\boxed{x = \frac{-B}{A}}$ ✓

$$A \cdot \frac{B^2}{A^2} + \frac{2B(-B)}{A} + C \geq 0$$

$$\frac{B^2}{A} - \frac{2B^2}{A} + C \geq 0$$

$$\frac{-B^2}{A} + C \geq 0$$

$$C \geq \frac{B^2}{A}$$

$$AC \geq B^2$$

$$B^2 \leq AC$$

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right) \qquad \text{(QED)}$$

$$(\Sigma a_i b_i)^2 = (\Sigma a_i^2)(\Sigma b_i^2)$$

?

a_i, b_i

$\boxed{a_i = b_i}$

$$(\Sigma a_i^2)^2 = (\Sigma a_i^2)(\Sigma a_i^2) \checkmark$$

---

* Set Topology

Open set

(a,b)

$[a,b]$

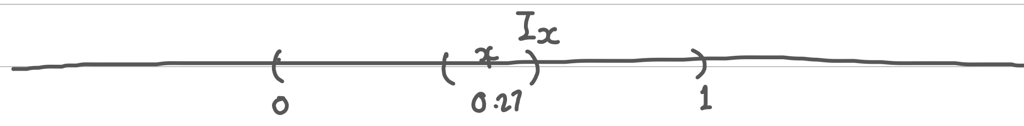
$\boxed{[a,\infty) \quad \text{Open} \mid \text{Closed set}}$ It is actually closed set

$\{1,2,3,4,5\}$

$\mathbb{R}$ $(-\infty,\infty)$ open / Closed both (?)

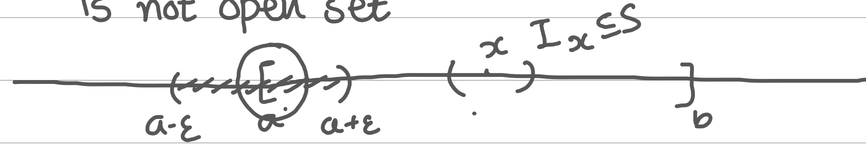
① Open S, $\forall x \in S$ $\exists$ $I_x \subseteq S$ $\ni$ $x \in I_x \subseteq S$

if $x = 0.27$, $I_x = (0.25, 0.30)$ $x \in I_x \subseteq S$



ε-nbhd of x, $I_x = (x-\varepsilon, x+\varepsilon)$ $\varepsilon > 0$

$S = [a,b]$ is not open set $a < x < b$



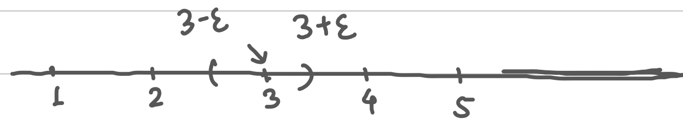
$(a-\varepsilon, a+\varepsilon) \subseteq [a,b]$ $(a-\varepsilon, a)$

open set

$\left[\begin{array}{l} \mathbb{R} \checkmark \\ \phi \end{array}\right.$ $\quad \exists\, x \in \mathbb{R}, \quad I_x \subseteq \mathbb{R} \quad \Rightarrow \quad x \in I_x \subseteq \mathbb{R}$

~~Closed set~~

Set of ① $\mathbb{N}$



$\varepsilon > 0 \qquad (3-\varepsilon, 3+\varepsilon) \nsubseteq \mathbb{N}$

$\mathbb{N}$ is not open, $\mathbb{Z}$,

② $\mathbb{Q}$ is ~~not~~ open

let $x = 0.25$. $\varepsilon = 0.1 > 0$, $I_x = (x-\varepsilon, x+\varepsilon)$

$= (0.15, 0.35)$

$\overset{\mathbb{R}}{(0.15, 0.35)} \nsubseteq \mathbb{Q} \leftarrow$ Set of Rational Nos.

$\left[\text{Density Theo.:- } \exists \text{ any } x, y \in \mathbb{R},\ x<y,\ \exists\ r \in \mathbb{Q} \ni x<r<y \right.$

similarly $\left. \exists\ r_1 \in \mathbb{Q}^c \ni x < r_1 < y \right]$

$x = 0.15$, $y = 0.25 \in \mathbb{R} \ \exists\ r_1 \in \mathbb{Q}^c \Rightarrow$

$0.15 < r_1 < 0.25$

$(0.15, 0.35) \nsubseteq \mathbb{Q}^c$

$\subseteq \mathbb{R}$

$(0.15, 0.35) = \{x \,/\, 0.15 < x < 0.35\} \nsubseteq \mathbb{Q}$

$\exists$ irrational

$[a, \infty)$, $\nexists$ any nbhd of $a \in S$

Interior point $\qquad (a,b)$ for every pt. in (a,b)

$\exists\, x \in (a,b)$, $\exists\ x \in I_x \subseteq (a,b)$

every pt. of (a,b) is interior pt.

$S = [a,b)$, $\therefore$ for $a \in S$, any $\varepsilon > 0$, $(a-\varepsilon, a+\varepsilon) \nsubseteq S$.

$\therefore$ a is not interior pt. of S.

$S_1 = S - \{a\} = (a,b)$

$\uparrow$ every pt. of S_1 is — interior pt.

Interior Set : Collection of Interior points of sets S.

$S \Rightarrow S^i$

$[a,b) \Rightarrow (a,b)$ ← set of interior pts.

$\mathbb{R} \Rightarrow \mathbb{R}$

$\phi \Rightarrow \phi$

$[a,b] \Rightarrow (a,b)$

$[a,b) \Rightarrow (a,b)$

$[a,b] \Rightarrow (a,b)$

$[a,\infty) \Rightarrow (a,\infty)$

$(-\infty,b] \Rightarrow (-\infty,b)$

Finite union of open sets is again open set.

Let A, B as open sets

① If A is open set, then $\forall x \in A$ $\exists I_x \subseteq A$. $\ni x \in I_x \subseteq A$

if $x \in A \cup B$ and $x \in I_x \subseteq A$

$A \subseteq A \cup B$

$\Rightarrow x \in I_x \subseteq A \subseteq A \cup B$

$\Rightarrow A \cup B$ is again open set

$S = [a,b]$ $\qquad S^i$ = interior set of $S = (a,b)$

* Finite intersection of open sets is again open

Let A & B are two open sets.
then $\forall x \in A$ $\exists$ some $I_x \subseteq A \ni x \in I_x \subseteq A$
$\forall x \in B$ $\exists$ some $I_x \subseteq B \ni x \in I_x \subseteq B$

if $x \in A \cap B \Rightarrow x \in A$ and $x \in B$
as A, B are open set $\Rightarrow x \in I_x \subseteq A$ and $x \in I_x \subseteq B$
$\Rightarrow x \in I_x \subseteq A$ & B
$\Rightarrow x \in I_x \subseteq A \cap B$
✓ $\Rightarrow A \cap B$ is also open

(1.3)
* Arbitrary Union of open sets is open

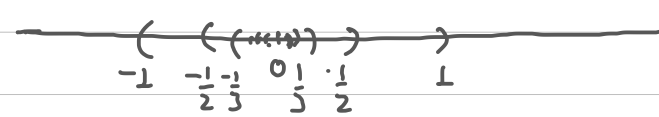
prove:- $\{A_i\}_i$ is collection of open sets, $\bigcup_{i=1}^{\infty} A_i$ is open

Let $x \in \cup A_i$
$\Rightarrow x \in A_i$ for some i $\qquad (0,1)\ (0.5,2)$
$\Rightarrow x \in I_x \subseteq A_i$ as A_i are all open sets $\qquad (0.5,1)$ open
$\Rightarrow x \in I_x \subseteq A_i \subseteq \bigcup_{i=1}^{\infty} A_i \Rightarrow \cup A_i$ is also open set.

* Arbitrary intersection of open sets may or may not be open

counter example

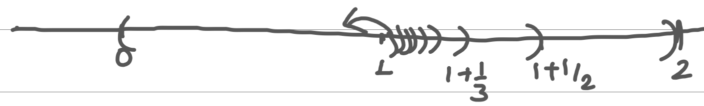
$A_i = \left(\frac{-1}{n}, \frac{1}{n}\right)$



$A_1 = (-1, 1)$
$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\bigcap_{i=1}^{\infty} A_i = \{0\}$ $\qquad 0 \in I_x \nsubseteq \{0\} \Rightarrow$ ~~not~~ $\cap A_i$ is not open here.

$A_n = (0, 1+1/n)$



$A_1 = (0, 2)$

$A_2 = (0, 1+\frac{1}{2})$

$A_3 = (0, 1+\frac{1}{3})$

$\cap A_i = \underline{(0,1]}$ is not open set

Closed Set :- S is closed if S^c is open.

$S = [a, \infty) \Rightarrow S^c = (-\infty, a)$ open $\Rightarrow$ S is closed

$S = (-\infty, b] \Rightarrow S^c = (b, \infty)$ open $\Rightarrow$ S is closed

$S = \{1,2,3\} \Rightarrow S^c = (-\infty,1) \cup (1,2) \cup (2,3) \cup (3,\infty)$:- finite union of open set is again open

$\mathbb{N} = \{1,2,3,\dots\} \Rightarrow S^c = (-\infty, 1) \cup \left[\bigcup_{n=1}^{\infty} (n, n+1)\right]$ $\therefore$ Arbitrary union of open is open

open.

$\mathbb{N}$ is closed set $\Rightarrow \mathbb{I}, \mathbb{Z}, \dots$ closed sets.

$\mathbb{Q} \Rightarrow \mathbb{Q}^c$ = irrational nos. $\Rightarrow \mathbb{Q}^c$ is not open

$\mathbb{Q}$ is not closed

$\mathbb{Q}, \mathbb{Q}^c$ both are not closed sets.

$\mathbb{R}$ open closed $\mathbb{R}^c = \phi \Rightarrow$ open set

$\Rightarrow \mathbb{R}$ is closed.

$\therefore \phi, \mathbb{R}$ ~~is~~ are open and closed

ϕ $\phi^c = \mathbb{R}$ is open $\Rightarrow \phi$ is closed.

* Finite union of closed set is closed
intersecting

limit pt. of set — not open ~~$I_x \subseteq S$~~

$A, B \subseteq \mathbb{R}$ closed, A^c & B^c opensets $\{0\} \cup \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ not closed

To prove $A \cup B$ is closed

$\to (A \cup B)^c$ is open

$(A \cup B)^c = A^c \cap B^c$

A^c, B^c are open sets & Finite intersection of opensets is open

$(A \cup B)^c$ is open

$\Rightarrow A \cup B$ is closed

$(A \cap B)^c = A^c \cup B^c$

A^c, B^c open & Finite union of open sets is open

$\Rightarrow (A \cap B)^c$ is open.

$\frac{1}{n} \to 0$

Theo:- Arbitrary intersection of closed sets is closed.

(We have discussed,

$\to$ Arbitrary union of open sets is open.

$\{A_i\}$ arbitrary collection of closed sets.

$\Rightarrow A_i^c$ is open $\forall i$

$\bigcup_{i=1}^{\infty} A_i^c$ is open set.

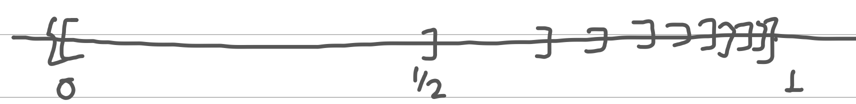
$\bigcup_i A_i^c = \left(\bigcap_{i=1}^{\infty} A_i\right)^c$ open

$\therefore \bigcap_{i=1}^{\infty} A_i$ is closed set

Arbitrary union of closed sets may or may not be closed.

counter examples

$[0, 1-1/n]$ is collection of closed sets



$A_1 = \cancel{[0,0}\ \{0\}$

$A_2 = [0, 1-1/2]$

$A_3 = [0, 1-1/3]$

$S = \bigcup [0, 1-1/n] = [0, 1)$

$S^c = \underbrace{(-\infty, 0)}_{\text{open}} \cup [1, \infty)$ ← not open

$\Rightarrow S^c$ is not open $\Rightarrow$ S is not closed.