Infinite Series :-

$\sum x_n$ $\quad \sum_{i=1}^{\infty} x_n = ?$

$S_1 = x_1$

$S_2 = x_1 + x_2$

$S_n = \sum_{i=1}^{n} x_i$

$S_n \to S$

$S_n \to S$

$S_{n+1} \to S$

$\lim_n S_n - S_{n-1} = 0$

$\lim_n \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i = 0$

$\therefore \lim_n x_n = 0$

$\sum \frac{1}{n}$ $\quad \sum \frac{(-1)^n}{n}$ $\quad \sum x^n = \frac{1}{1-x}$ $\quad |x|<1$

$= \sum \frac{1}{n}$

$= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \dots$

$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}$

$\geq 1 + \frac{1}{2}\left[1 + 1 + 1 + \dots\right.$

$\geq \infty$

$\sum \frac{(-1)^{n+1}}{n}$

$= \underbrace{1 - \frac{1}{2}} + \underbrace{\frac{1}{3} - \frac{1}{4}} + \underbrace{\frac{1}{5} - \frac{1}{6}}$

$S_n = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \dots$

$= 1 - \left(\underbrace{\frac{1}{2 \cdot 3}}_{>0}\right) - \left(\underbrace{\frac{1}{4 \cdot 5}}_{>0}\right)$

$S_n \downarrow$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots\dots + \cancel{\frac{1}{n}} - \underline{\frac{1}{n+1}} \longrightarrow 0$$

$$\Rightarrow 1$$

$\sum \frac{(-1)^n}{n}$ cgt $\quad \frac{(-1)^n}{n} \to 0$

$\sum r^n$ $\quad |r|<1$ cgt $\quad r^n \to 0 \quad 0.5^2$

$\sum \frac{1}{n^2}$ cgt $\quad \frac{1}{n^2} \to 0$

$\sum \frac{1}{n}$ dvge $\quad \frac{1}{n} \to 0$

Cauchy Criterion for convergence of series

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for any $\varepsilon > 0 \ \exists\ K(\varepsilon) \in \mathbb{N} \ni |S_n - S_m| < \varepsilon \quad \forall\ n \geq m \geq K(\varepsilon)$

$$\Rightarrow \left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} x_i \right| < \varepsilon \quad \forall\ n \geq m \geq K(\varepsilon)$$

$$\Rightarrow \left| \sum_{i=m+1}^{n} x_i \right| < \varepsilon$$

Let xn be a sequence of non-negative real numbers then the series P \sum xn converges if and only if the sequence Sk of partial sum is bounded.

① Let $\sum_{n=1}^{\infty} x_n$ converges , $\sum_{n=1}^{\infty} x_n = s$

$\Rightarrow S_n = \sum_{i=1}^{n} x_i$ = seq^n of partial sums

$\Rightarrow$ S_n is convergent to s.

$\Rightarrow$ Every convergent seq^n is bounded

$\Rightarrow$ S_n is bounded.

② S_n is bounded $\Rightarrow$ To prove S_n is cgt.

$S_{n+1} = \sum_{i=1}^{n+1} x_i = S_n + x_{n+1}$ as $x_{n+1} \geq 0$

$S_{n+1} \geq S_n$

S_n is monotonically $\uparrow$ & bounded

by MCT it is cgt.

Show that $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots = \frac{1}{1-r}$ if $|r| < 1$

$S_{n+1} = \sum_{i=1}^{n+1} r^i = 1 + r + r^2 + \quad + r^{n-1} + r^n$

$S_n = \sum_{i=1}^{n} r^i = 1 + r + r^2 + \ldots + r^{n-1}$

$r S_n = r + r^2 + \ldots + r^n = S_{n+1} - 1$

$1 + r S_n = S_{n+1}$?

$\lim (1 + r \cdot S_n) = \lim S_{n+1}$

$1 + r \cdot S = S \Rightarrow S = 1/(1-r)$?

$p>1$

$$\sum_n \frac{1}{n^p} = 1+\underbrace{\frac{1}{2^p}+\frac{1}{3^p}}+\underbrace{\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}}+\underbrace{\frac{1}{8^p}}+\dots$$

$$\leq 1+\underbrace{\frac{1}{2^p}+\frac{1}{2^p}}+\underbrace{\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}}+\underbrace{\frac{1}{8^p}}+\dots$$

$$\leq 1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\dots$$

$$\leq 1+\frac{2}{2^p}+\frac{2^2}{2^{2p}}+\frac{2^3}{2^{3p}}+\dots$$

$$\leq 1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+\dots$$

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \leq \sum_{n=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^n$$

as $p>1$, $\frac{1}{2^{p-1}}<1 \Rightarrow \sum r^n \to \frac{1}{1-r}$ cgt.

$\Rightarrow \sum \frac{1}{n^p}$ is also cgt.

Comparison Test :- $K\in\mathbb{N} \Rightarrow \underline{0\leq x_n\leq y_n} \;\forall\, n\geq K$

$\sum y_n$ cgt. $\Rightarrow \sum x_n$ cgt

by Cauchy criterion for convergence of series

for any $\varepsilon>0$ $\exists\, M(\varepsilon)\in\mathbb{N}$. $|y_{m+1}+y_{m+2}+\dots+y_n|<\varepsilon \;\forall\, n>m>M(\varepsilon)$

$K'(\varepsilon)=\max(M(\varepsilon),K)$

for $n, m \geq K'(\varepsilon)$

$$|y_{m+1} + \dots + y_n| < \varepsilon$$

by Δ inequality

$$|x_{m+1} + x_{m+2} + \dots + x_n| < |y_{m+1} + \dots + y_n| < \varepsilon$$

by Cauchy criterion Σx converges.

$\Sigma \frac{1}{n^2+n}$ Σx_n $\Sigma y_n = \Sigma \frac{1}{n^2} \to$ cgt

$$r = \lim \frac{x_n}{y_n} = \lim \frac{1}{n^2+n} \times n^2 = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

$$r \neq 0$$

$0 < x_n$, $0 < y_n$ $\forall n$

$$r = \lim \frac{x_n}{y_n}$$

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$ $\ni \left| \frac{x_n}{y_n} - r \right| < \varepsilon$

$$r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$$

$$y_n(r-\varepsilon) \leq x_n \leq y_n(r+\varepsilon)$$

$x_n \leq y_n$, Σy_n cgt $\Rightarrow$ Σx_n cgt if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt

if $\sum x_n$ cgt $\Rightarrow$ $\sum y_n$ cgt

if $r = 0$, $r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$

$$-\varepsilon < \frac{x_n}{y_n} < \varepsilon$$

$$-\varepsilon < 0 < \frac{x_n}{y_n} < \varepsilon$$

$$x_n \leq y_n \cdot \varepsilon$$

$\Rightarrow$ if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt.

$\sum \frac{(-1)^n}{n}$ cgt $\qquad \sum \frac{1}{n}$ divergent

Absolute: $\sum x_n$ abs. cvgt if $\sum |x_n|$ is cvgt.

Conditional Convergence $\sum x_n$ is cgt but $\sum |x_n|$ is not cgt.

$\sum \frac{(-1)^n}{n^2}$ abs cgt $\qquad \sum \frac{1}{n^2}$ cgt

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$.

$$\underbrace{|x_{m+1} + \dots + x_n|}_{\sum |x_n|} \leq ||x_{m+1}| + |x_{m+2}| + \dots + |x_n|| < \varepsilon \qquad \forall\ \cancel{m > n} \ n > m > K(\varepsilon)$$

Root test $\quad r = \lim |x_n|^{1/n}$ exists.

① $r < 1$, $\exists\ r < r_1 < 1$

for some $K(\varepsilon) \in \mathbb{N}$.

$$|x_n|^{1/n} \leq r_1 \qquad \forall n \geq K(\varepsilon)$$

$$\Rightarrow \quad |x_n| \leq r_1^n \qquad (r_1 < 1)$$

by Comparison Test

$$\sum |x_n| \leq \sum r_1^n$$

cgt. cgt.

② $r > 1$, $\exists\ 1 < r_1 < r$

$$1 \leq r_1 \leq |x_n|^{1/n}$$

$$1^n \leq r_1^n \leq |x_n|$$

Root

$$\sum 1 \leq \sum r_1^n \leq \sum |x_n|$$

∞ dvgt.

Ratio $\quad r = \lim \left|\dfrac{x_{n+1}}{x_n}\right|$

$r < 1$ cgt. $\qquad r > 1$ dvgt

$r = 1$ Test fails

① $r < 1$

$$\left|\frac{x_{n+1}}{x_n}\right| < r \qquad \text{for } n \geq K(\varepsilon)$$

$$|x_{n+1}| < r \cdot |x_n|$$
$$\leq r \cdot r \cdot |x_{n-1}| = r^2 |x_{n-1}|$$
$$\vdots$$
$$\leq r^n \cdot |x_1|$$

$$\sum \frac{1}{(n+1)(n+2)}$$

$x^{\cancel{1/x}}$

Ratio Test :- $r = \lim \left| \frac{x_{n+1}}{x_n} \right|$

$$= \lim \left| \frac{(n+1)\cancel{(n+2)}}{\cancel{(n+2)}(n+3)} \right|$$

$$= \lim \left| \frac{1 + 1/n}{1 + 3/n} \right|$$

$= 1$ Test fail

Limit Comparison Test $\sum y_n = \sum \frac{1}{n^2}$

$$r = \lim \left| \frac{x_n}{y_n} \right|$$

$$= \lim \left| \frac{n^2}{(n+1)(n+2)} \right|$$

$$= \lim \left| \frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})} \right|$$

$= 1 \neq 0$ $\sum |y_n|$ cgt

$\Rightarrow \sum |x_n|$ cgt.

$$\frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

$$\sum \frac{1}{n^2} \searrow \text{cgt}$$

cgt.

$$\sum \frac{1}{(n+1)(n+2)} \Rightarrow \quad x_n = \frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

$$S_1 = x_1 = 1 - \frac{1}{2}$$

$$S_2 = S_1 + x_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3}.$$

$$S_3 = S_2 + x_3 = 1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4}.$$

$$S_n = 1 - \frac{1}{n+1}$$

$\sum x_n$ cgt if S_n – cgt.

$$\lim S_n = \lim 1 - \frac{1}{n+1}$$

$$S_n \to 1 \qquad = 1$$

$\sum 2^n$ Ratio $\Rightarrow$

$$\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \left|\frac{2^{n+1}}{2}\right|$$

$$= 2 > 1 \quad \text{dgt.}$$

$\sum 2^r$ Root test

$$\lim |x_n|^{1/n} = \lim |2^n|^{1/n}$$

$$= 2 > 1 \quad \text{dgt.}$$

$\sum 2^{-1/n}$

Ratio $\lim_{n} \frac{2^{-1/(n+1)}}{2^{-1/n}}$

$\lim \quad 2^{\frac{-1}{n+1}+\frac{1}{n}}$

$\lim_{n\to\infty} \quad 2^{\frac{1}{n^2+n}} \quad \to 1$

Test fails

$\sum 2^{-1/n}$ $\quad \sum x_n$ cgt $\Rightarrow \lim x_n \Rightarrow 0$

$\lim x_n \neq 0 \Rightarrow \sum x_n$ not cgt.

$\lim 2^{-1/n} = 1 \quad \Rightarrow \sum 2^{-1/n}$ dvgt.

* $x_n = n/2^n$

Ratio Test $\quad r = \lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \left|\frac{n+1}{2^{n+1}} \times \frac{2^n}{n}\right|$

$= \lim \left|\frac{n+1}{n}\right| \cdot \lim \left|\frac{2^n}{2^{n+1}}\right|$

$= \frac{1}{2} < 1$

$\Rightarrow \sum x_n$ cgt.

Root test:-

$r = \lim |x_n|^{1/n} = \lim \left|\frac{n}{2^n}\right|^{1/n} = \frac{1}{2} \underbrace{\lim n^{1/n}}_{\to 1}$

$= \frac{1}{2} < 1$

$$\sum \frac{3^n}{n!} \ ?$$

Ratio Test $\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n} \right|$

$= \lim \frac{3}{n}$

$= 0 < 1$ cgt.

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \cdots$$

$$x_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$\frac{1}{\sqrt{1 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 3}} + \frac{1}{\sqrt{3 \cdot 4}}$$

$$x_n = \frac{1}{\sqrt{n(n+1)}}$$

$$\frac{1}{n} < \frac{1}{\sqrt{n(n+1)}} \ \cancel{< \frac{1}{n}}$$

limit Comparison $\lim \left| \frac{n}{\sqrt{n(n+1)}} \right|$

$= \lim \left| \frac{1}{\sqrt{1(1+1/n)}} \right| = 1.$

$\Rightarrow \quad \Sigma y_n$ dvgt $\Rightarrow \quad \Sigma x_n$ dvgt.

Ratio :- $\quad x_n = \dfrac{1}{\sqrt{n(n+1)}}$

$$\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \frac{\sqrt{n\cancel{(n+1)}}}{\sqrt{\cancel{(n+1)}(n+2)}} = \lim \sqrt{\frac{1}{1+2/n}}$$

$= 1 \qquad$ Test fails.

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots$$

$$x_n = \frac{1}{n^n}$$

Root test $\quad \lim |x_n|^{1/n} = \lim \left|\frac{1}{n^n}\right|^{1/n} = \lim \left|\frac{1}{n}\right| = 0$

< 1

$\Sigma \dfrac{n^2-1}{n^2+1} \qquad$ by necessary condn $\quad \lim \dfrac{n^2-1}{n^2+1} = 1 \neq 0$

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \dots$$

$$\underline{\left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots\right)} + \left(\frac{1}{3^2} + \frac{1}{3^4} + \dots\right)$$

$\uparrow \underline{\Sigma \left(1/3^2\right)^n}$

$$\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots$$

$$x_n = \frac{1}{n\cdot 2^n}$$

$$\frac{1^2\,2^2}{1!}+\frac{2^2\,3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\cdots$$

$$x_n = \frac{n^2(n+1)^2}{n!}$$

$$r = \lim\left|\frac{x_{n+1}}{x_n}\right| = \lim \frac{\cancel{(n+1)^2}(n+2)^2}{(n+1)!}\;\frac{n!}{\cancel{(n+1)^2}(n^2)}$$

$$= \lim \frac{1}{(n+1)}\cdot(1+2/n)^2$$

$$= 0$$

$$x_n = \frac{1\cdot 2\cdot 3\cdots n}{7\cdot 10\cdots(3n+4)} \qquad \lim\left|\frac{x_{n+1}}{x_n}\right|$$

$$\frac{1\cdot 2\cdots\; n\cdot(n+1)}{7\cdot 10\cdots(3n+4)(3(n+1)+4)} \qquad = \lim\left|\frac{n+1}{(3(n+1)+4)}\right|$$

$$= \lim\left|\frac{1}{3+4/(n+1)}\right|$$

$$= \frac{1}{3} < 1$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} \cdot \frac{1}{n} \qquad ?$$

$$\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \left|\frac{2(n+1)-1}{2(n+1)}\left(\frac{n}{n+1}\right)\right| = 1$$

$$x_n: \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n \cdot n!} \cdot \frac{1}{n}$$

$$\lim |x_n|^{1/n} = \frac{1}{2}\left(\lim \frac{1^{1/n}\, 3^{1/n} \cdots (2n-1)^{1/n}}{(n!)^{1/n}} \cdot \frac{1}{n^{1/n}}\right)$$

$$= \frac{1}{2} < 1$$