$$P[N_1(t) = n_1] = \frac{e^{-\lambda_1 t}(\lambda_1 t)^{n_1}}{n_1!} \qquad n_1 = 0, 1, 2, \ldots$$

$$P[N_2(t) = n_2] = \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n_2}}{n_2!} \qquad n_2 = 0, 1, 2, \ldots$$

$$M_{N_1}(t) = e^{\lambda_1 (e^t - 1)} \qquad M_{N_2}(t) = e^{\lambda_2 (e^t - 1)}$$

$$M_N(t) = M_{N_1+N_2}(t) = M_{N_1}(t) \cdot M_{N_2}(t)$$

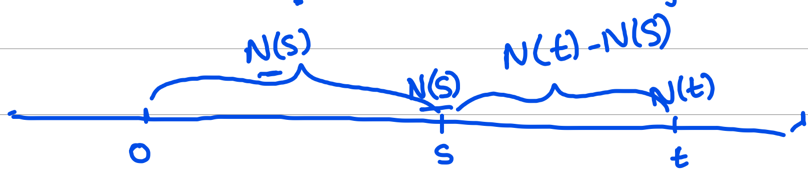
$$= e^{(\lambda_1+\lambda_2)(e^t - 1)}$$

$\{N(t), t \geq 0\}$ is also Poisson process will be $\lambda_1 + \lambda_2$.

* Let $\{N(t), t \geq 0\}$ be a poisson process with rate λ. Calculate $Cov(N(S), N(t))$. $s<t$.

→ $\{N(t), t \geq 0\}$ is a Poisson process,

⇒ It follows indep. & stationary increment. ✓



$$cov(N(S), N(t)) = cov(N(S), N(t) + N(S) - N(S))$$

$$= cov(\underline{N(S)}, N(S) + \underline{(N(t) - N(S))})$$

$$= cov(N(S), N(S)) + cov(N(S), N(t) - N(S))$$

$$= V(N(S)) + 0$$

$$cov(N(s), N(t)) = \lambda s \qquad \underline{s<t}$$

for any s, t $\quad cov(N(S), N(t)) = \lambda \min(s, t)$

Let $\{N_i(t), t \geq 0\}$ represents no. of type-i events that occured by time t with rate $\lambda_i p$ & $\lambda_i(1-p)$ $i = 1:2$ & $N_1(t)$ & $N_2(t)$ are indep. Poisson processes.

$N_1(t)$:- male with prob. p

$N_2(t)$:- female $1-p$

$$P[N_1(t)=m, N_2(t)=n] = \sum_{K=} P[N_1(t)=m, N_2(t)=n / N(t)=K] \cdot P[N(t)=K]$$

$$N(t) \sim \lambda_1 p + \lambda_2(1-p)$$

$$= \sum_K \frac{\frac{\cancel{e^{-\lambda_1 p}} (\lambda_1 p)^m}{m!} \cdot \frac{\cancel{e^{-\lambda_2(1-p)}} (\lambda_2(1-p))^n}{n!}}{\frac{\cancel{e^{-(\lambda_1 p + \lambda_2(1-p))}} \cdot (\lambda_1 p + \lambda_2 - \lambda_2 p)^{m+n}}{(m+n)!}}$$

✓

If $\lambda_1 = \lambda_2 = \lambda$

$$= \binom{m+n}{m} p^m (1-p)^n / (\cancel{p+1-p})^{m+n}$$

$$= \binom{m+n}{n} p^m (1-p)^n \quad \text{Binomial if } \lambda_1 = \lambda_2 = \lambda$$

* Compound Poisson Process :-

$\{S(t), t \geq 0\}$

$$S(t) = \sum_{i=1}^{N(t)} x_i$$

where $\{N(t), t \geq 0\}$ is Poisson Process and x_i & $N(t)$ are independent

Imagine Bank Cashier $S(t)$ = counpounded cash counter

i^{th} custumer arrives he/she deposits x_i amounts no. of customers

$$\underline{\underline{S(t)}} = \sum_{i=1}^{N(t)} x_i$$

$N(t)$ Poisson Process λ

$\{x_i\}$ family of some F_x distⁿ i.i.d.

$\{x_i\}$ & $N(t)$ indep.

10:00 ⇒ ______ 3:00

amount deposited or withdrawn by those customers

$$E(S(t)) = E\left(\sum_{i=1}^{N(t)} x_i\right)$$

$$= E_N\left(E_x \sum_{i=1}^{N} x_i\right)$$

$$= E_N\left(\sum_{i=1}^{N} E_x(x)\right)$$

$E(x) = \mu$

$$= E_N(N\mu)$$

$$= \mu \cdot E(N(t))$$

$$= \mu \cdot \lambda t$$

$$V(S(t)) = V\left(\sum_{i=1}^{N(t)} x_i\right)$$

$$= V_N\left(E_x \sum_{i=1}^{N} x_i\right) + E_N\left(V_x \sum_{i=1}^{N} x_i\right)$$

$$= V_N(N \cdot E(x)) + E_N(N \cdot V(x))$$

$$= [E(x)]^2 \cdot \underline{\underline{V(N)}} + V(x) \cdot \underline{\underline{E(N)}}$$

$$= \left([E(x)]^2 + V(x)\right) \cdot E(N)$$

$$= E(x^2) \cdot E(N)$$

assume $t < z$

$$\mathrm{cov}(S(t), S(z)) = \mathrm{cov}\left(S(t),\ S(t) + (S(z) - S(t))\right)$$

$$= V(S(t)) + \mathrm{cov}\left(S(t), \sum_{i=N(s)+1}^{N(t)} x_i\right) \quad \text{as all } x_i\text{'s are i.i.d.}$$

$\downarrow$ $\qquad\qquad$ $\downarrow$ 0

$$= V(S(t))$$

e.g. $\lambda = 1/\text{minute}$ exp $\qquad t = 15$ mins $\qquad$ unifor

$x_i \sim N(1000, \sigma^2 = 400)$

$$S(t) = \sum_{i=1}^{N(t)} x_i$$

$n = 30$

~~ist~~ exp(~~a~~, 1)

arr $\qquad$ >15 $\qquad$ cumsum

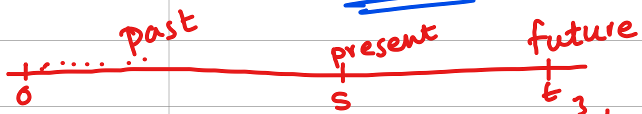
13 arrivals $\rightarrow$ 0-15 $\qquad$ 11:00 - 11:15 $\qquad$ 13

$$S(t) = \sum_{i=1}^{13} x_i$$

$x_i = \text{rnorm}(13, \_\_, \_\_)$ ✓

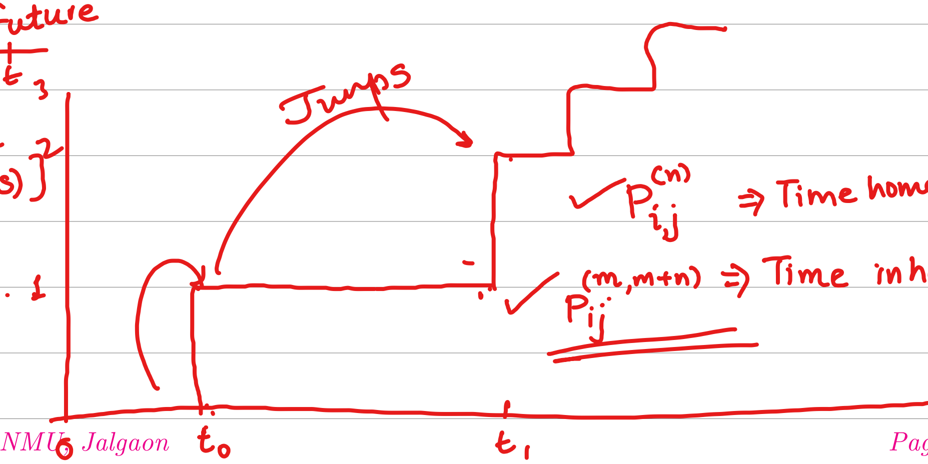
sum(x) $\Rightarrow S(t)$

$$p^{(m, m+n)}_{\ (n)} \times p^{(s, s+n)}_{\ (n)} = p^{(n)}$$



$P[X_t = x / X_s = y,\ x_u = i_u,\ u \in (0, s)]$

$= P[X_t = x / X_s = y]$



✓ $P_{ij}^{(n)} \Rightarrow$ Time homo

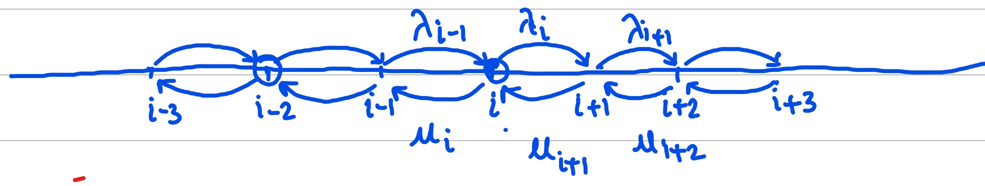
✓ $P_{ij}^{(m, m+n)} \Rightarrow$ Time inhomo

* Continuous time Markov ^Jump Processes (CTMP)

* Birth and death Process :-

$i \to i+1$ w.rate λ_i $i = 0,1,2,\ldots$

$i \to i-1$ w.rate μ_i $i = 1,2,\ldots$



Transition graph with respective rates

$i-2, i-1 = \lambda_{i-2}$ $\qquad q_{i,i+1} = \lambda_i$

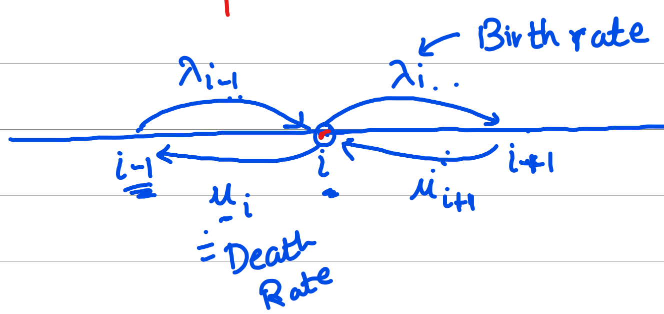
$i-2, i-3 = \mu_{i-2}$ $\qquad q_{i,i-1} = \mu_i$

$S = \{0,1,2,\ldots\}$

P_n :- Pop^n size n prob.

$i \to i+1$ λ_i

$i \to i-1$ μ_i



$i-1 \xrightarrow{+1} i$ λ_{i-1}

$i+1 \xrightarrow{-1} i$ μ_{i+1}

P_i = Prob. that after sufficiently large time point Pop^n will be of size i.

state i, entry $\qquad$ exist

$$P_{i-1}\lambda_{i-1} + P_{i+1}\mu_{i+1} = P_i\lambda_i + P_i\mu_i$$

$i=0$

$$\underbrace{P_{-1}\lambda_{-1}}_{0} + P_1\mu_1 = P_0\lambda_0 + \underbrace{P_0\mu_0}_{0}.$$

$$P_1\mu_1 = P_0\lambda_0 \Rightarrow P_1 = P_0 \cdot \frac{\lambda_0}{\mu_1}$$

$i=1$

$$\cancel{P_0\lambda_0} + P_2\mu_2 = P_1\lambda_1 + \cancel{P_1\mu_1} \Rightarrow P_2 = P_1\frac{\lambda_1}{\mu_2} = P_0 \cdot \frac{\lambda_0\lambda_1}{\mu_1\mu_2}.$$

$i=n$

$$P_n = P_0 \cdot \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} = P_0\left[\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right]$$

Now $\sum_{i=0}^{\infty} P_i = 1$

$$P_0 + P_0\cdot\frac{\lambda_0}{\mu_1} + P_0\cdot\frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + P_0\frac{\lambda_0\cdots\lambda_{n-1}}{\mu_1\cdots\mu_n} + \cdots = 1$$

$$P_0 = \left[1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\lambda_0\cdots\lambda_{n-1}}{\mu_1\cdots\mu_n}\right]^{-1}$$

$$P_n = P_0\cdot\left[\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right] \text{ where } P_0 = \left[1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right]^{-1}$$

Now assume $\lambda_i = \lambda$ & $\mu_i = \mu$ $\forall i$

$$P_n = P_0\left[\prod_{i=1}^{n}\frac{\lambda}{\mu}\right] = P_0\left(\frac{\lambda}{\mu}\right)^n.$$

$$P_0 = \left[1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda}{\mu}\right]^{-1}$$

$$= \left[1 + \sum_{n=1}^{\infty}(\lambda/\mu)^n\right]^{-1}$$

$$= \left[\sum_{n=0}^{\infty}(\lambda/\mu)^n\right]^{-1}$$

$$= \frac{1}{1-\lambda/\mu} \quad \text{iff } \frac{\lambda}{\mu} < 1$$

$$= \frac{\mu}{\mu-\lambda}$$

Abs:- 2001, 5, 6, 10, 12, 13, 15, 16, 20, 22, 25, 27, 34.35, 37, 39, 43-45, 50, 55-57

$$P_n = \frac{\mu}{\mu-\lambda}(\lambda/\mu)^n \quad ?$$

Entry Exit

$$P_{i-1}\lambda_{i-1} + P_{i+1}\mu_{i+1} = P_i\lambda_i + P_i\mu_i$$

$i=0,$ $\cancel{P_{-1}\lambda_{-1}} + P_1\mu_1 = P_0\lambda_0 + \cancel{P_0\mu_0}$

$$\boxed{P_1\mu_1 = P_0\lambda_0}$$

$$P_1 = P_0\frac{\lambda_0}{\mu_1}$$

$i=1,$ $\cancel{P_0\lambda_0} + P_2\mu_2 = P_1\lambda_1 + \cancel{P_1\mu_1}$

$$P_2 = P_1\frac{\lambda_1}{\mu_2}$$

$$P_2 = P_0\frac{\lambda_0\lambda_1}{\mu_1\mu_2}$$

Similarly $P_n = P_0\dfrac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ \mu_n} = P_0\prod_{i=1}^{n}\dfrac{\lambda_{i-1}}{\mu_i}$

$\sum_{n=0}^{\infty} P_n = 1 \Rightarrow P_0 + P_1 + P_2 + \ldots + P_n + \ldots = 1$

$$\Rightarrow P_0 + P_0\frac{\lambda_0}{\mu_1} + \ldots + P_0\cdot\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i} + \ldots = 1$$

$$\Rightarrow P_0 = \left[1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right]^{-1}$$

$$\Rightarrow P_n = \frac{\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}}{\left[1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right]}$$

① Birth and death processes

$\lambda_i = \lambda \qquad \mu_i = \mu$

$$P_n = P_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} = P_0 \prod_{i=1}^{n} \frac{\lambda}{\mu} = P_0 \left(\frac{\lambda}{\mu}\right)^n$$

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\right]^{-1}$$

$$= \left[1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n\right]^{-1}$$

$$= \left[\sum_{n=0}^{\infty} (\lambda/\mu)^n\right]^{-1}$$

$$= \left[\frac{1}{1-\lambda/\mu}\right]^{-1} \qquad \text{only if } \frac{\lambda}{\mu} < 1$$

$$= (1-\lambda/\mu)$$

$$P_n = P_0 (\lambda/\mu)^n = (1-\lambda/\mu)(\lambda/\mu)^n \qquad n = 0, 1, 2, \ldots$$

$N \sim \text{Geo.}(1-\lambda/\mu)$

$$E(N) = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

ρ = Traffic intensity $= \underline{\underline{\lambda/\mu}}$

$\lambda_i = \lambda \qquad \mu_i = i \cdot \mu$

$$P_n = P_0 \cdot \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} = P_0 \cdot \prod_{i=1}^{n} \frac{\lambda}{i \cdot \mu} = P_0 \frac{\lambda^n}{n!\,\mu^n} = \frac{(\lambda/\mu)^n}{n!} P_0$$

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\right]^{-1}$$

$$= \left[1 + \sum_{n=1}^{\infty} \frac{(\lambda/\mu)^n}{n!}\right]^{-1}$$

$$= \left[\sum_{n=0}^{\infty} \frac{(\lambda/\mu)^n}{n!}\right]^{-1}.$$

$$\cong e^{-\lambda/\mu}$$

$$P_n = \frac{P_0 (\lambda/\mu)^n}{n!} = \frac{e^{-\lambda/\mu} (\lambda/\mu)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

c = no. of servers

$$\lambda_i = \lambda \qquad \mu_i = \begin{cases} i\mu & i \leq c \\ c\mu & i > c \end{cases}$$

$$P_n = P_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$$

I'

$$= \begin{cases} P_0 \prod_{i=1}^{n} \frac{\lambda}{i\mu} = P_0 \frac{(\lambda/\mu)^n}{n!} & n \leq c \\ P_0 \prod_{i=1}^{c} \frac{\lambda_{i-1}}{\mu_i} \cdot \prod_{i=c}^{n} \frac{\lambda_{i-1}}{\mu_i} = P_0 \frac{(\lambda/\mu)^c}{c!} \cdot \frac{(\lambda/\mu)^{n-c}}{c^{n-c}} & \end{cases}$$

$$P_0 = \left[1 + \sum_{n=1}^{c} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} + \sum_{n=c+1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\right]^{-1}$$

$$= \left[1 + \sum_{n=1}^{c} \frac{(\lambda/\mu)^n}{n!} + \sum_{n=c+1}^{\infty} \frac{(\lambda/\mu)^{n-c}}{c!} \frac{(\lambda/\mu)^{n-c}}{c^{n-c}}\right]^{-1}$$

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n}$$

$+ \; 1$

$+ \; 1 \; 2$

$+ \; 1 \; 2 \; 3$

$\; 1 \; 2 \; 3 \; 4$

$\lambda_i = (i+1)\lambda$

$\mu_i = i \cdot \mu$

Pop size is restricted

$\lambda_n = \lambda \qquad n \le N$

$\mu_n = \mu$

$$P_n = P_0 \cdot \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$$

$$= P_0 \prod_{i=1}^{n} \frac{i\lambda}{i\mu} = P_0 (\lambda/\mu)^n$$

* Machine Repairman Problem

25 $\quad M$ = No. of $\qquad \lambda_i = \qquad \lambda_{25}$

$$S = \{0, 1, 2, \ldots, M\}$$

$\lambda_i = \lambda \qquad 0 \le i < M$

$\mu_i = i\mu \qquad 0 < i \le M$

Two repairman

$$\mu_i = \begin{cases} \mu & i < 2 \\ 2\mu & i \ge 2 \end{cases}$$