Infinite Series :-

$\sum x_n$ $\quad \sum_{i=1}^{\infty} x_n = ?$

$S_1 = x_1$

$S_2 = x_1 + x_2$

$S_n \to S$

$S_{n+1} \to S$

$S_n = \sum_{i=1}^{n} x_i$

$S_n \to S$

$\lim_n S_n - S_{n-1} = 0$

$\lim_n \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i = 0$

$\therefore$

$\lim_n x_n = 0$

$\sum \frac{1}{n}$ $\quad \sum \frac{(-1)^n}{n}$ $\quad \sum x^n = \frac{1}{1-x} \quad |x|<1$

$= \sum \frac{1}{n}$

$= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \dots$

$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}$

$\geq 1 + \frac{1}{2}\left[1 + 1 + 1 + \dots\right.$

$\geq \infty$

$\sum \frac{(-1)^{n+1}}{n}$

$= \underbrace{1 - \frac{1}{2}} + \underbrace{\frac{1}{3} - \frac{1}{4}} + \underbrace{\frac{1}{5} - \frac{1}{6}}$

$S_n = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \dots$

$= 1 - \underbrace{\left(\frac{1}{2\cdot 3}\right)}_{>0} - \underbrace{\left(\frac{1}{4\cdot 5}\right)}_{>0}$

$S_n \downarrow$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots\dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} \longrightarrow 0$$

$$\Rightarrow 1$$

$\sum \frac{(-1)^n}{n}$ cgt $\quad \frac{(-1)^n}{n} \to 0$

$\sum r^n$ $\quad |r|<1$ cgt $\quad r^n \to 0 \quad 0.5^2$

$\sum \frac{1}{n^2}$ cgt $\quad \frac{1}{n^2} \to 0$

$\sum \frac{1}{n}$ dvge $\quad \frac{1}{n} \to 0$

Cauchy Criterion for convergence of series

*

for any $\varepsilon > 0 \ \exists \ K(\varepsilon) \in \mathbb{N} \ \ni \ |S_n - S_m| < \varepsilon \quad \forall \ n \geq m \geq K(\varepsilon)$

$$\Rightarrow \left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} x_i \right| < \varepsilon \quad \forall \ n \geq m \geq K(\varepsilon)$$

$$\Rightarrow \left| \sum_{i=m+1}^{n} x_i \right| < \varepsilon$$

Let xn be a sequence of non-negative real numbers then the series P \sum xn converges if and only if the sequence Sk of partial sum is bounded.

① Let $\sum_{n=1}^{\infty} x_n$ converges, $\sum_{n=1}^{\infty} x_n = s$

$\Rightarrow$ $S_n = \sum_{i=1}^{n} x_i$ = seq^n of partial sums

$\Rightarrow$ S_n is convergent to s.

$\Rightarrow$ Every convergent seq^n is bounded

$\Rightarrow$ S_n is bounded.

② S_n is bounded $\Rightarrow$ To prove S_n is cgt.

$S_{n+1} = \sum_{i=1}^{n+1} x_i = S_n + x_{n+1}$ as $x_{n+1} \geq 0$

$S_{n+1} \geq S_n$

S_n is monotonically $\uparrow$ & bounded

by MCT it is cgt.

Show that $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots = \frac{1}{1-r}$ if $|r| < 1$

$S_{n+1} = \sum_{i=1}^{n+1} r^i = 1 + r + r^2 + \quad + r^{n-1} + r^n$

$S_n = \sum_{i=1}^{n} r^i = 1 + r + r^2 + \ldots + r^{n-1}$

$r S_n = r + r^2 + \ldots + r^n = S_{n+1} - 1$

$1 + r S_n = S_{n+1}$?

$\lim (1 + r \cdot S_n) = \lim S_{n+1}$

$1 + r \cdot S = S \Rightarrow S = 1/(1-r)$?

$p>1$

$$\sum_n \frac{1}{n^p} = 1+\underbrace{\frac{1}{2^p}+\frac{1}{3^p}}+\underbrace{\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}}+\underbrace{\frac{1}{8^p}}+\dots$$

$$\leq 1+\underbrace{\frac{1}{2^p}+\frac{1}{2^p}}+\underbrace{\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}}+\underbrace{\frac{1}{8^p}}+\dots$$

$$\leq 1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\dots$$

$$\leq 1+\frac{2}{2^p}+\frac{2^2}{2^{2p}}+\frac{2^3}{2^{3p}}+\dots$$

$$\leq 1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+\dots$$

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \leq \sum_{n=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^n.$$

as $p>1$, $\frac{1}{2^{p-1}}<1$ $\Rightarrow$ $\sum r^n \rightarrow \frac{1}{1-r}$ cgt.

$\Rightarrow$ $\sum \frac{1}{n^p}$ is also cgt.

Comparison Test :- $K\in\mathbb{N}$ $\ni$ $0\leq x_n\leq y_n$ $\forall\, n\geq K$
$\sum y_n$ cgt. $\Rightarrow$ $\sum x_n$ cgt

by Cauchy criterion for convergence of series
for any $\varepsilon>0$ $\exists\, M(\varepsilon)\in\mathbb{N}$. $|y_{m+1}+y_{m+2}+\dots+y_n|<\varepsilon$ $\forall\, n>m>M(\varepsilon)$

$K'(\varepsilon) = \max(M(\varepsilon), K)$

for $n, m \geq K'(\varepsilon)$

$$|y_{m+1} + \dots + y_n| < \varepsilon$$

by Δ inequality

$$|x_{m+1} + x_{m+2} + \dots + x_n| < |y_{m+1} + \dots + y_n| < \varepsilon$$

by Cauchy criterion Σx converges.

$\Sigma \frac{1}{n^2+n}$ Σx_n $\Sigma y_n = \Sigma \frac{1}{n^2}$ $\rightarrow$ cgt

$$r = \lim \frac{x_n}{y_n} = \lim \frac{1}{n^2+n} \times n^2 = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

$$r \neq 0$$

$0 < x_n,\ 0 < y_n \ \forall n$

$$r = \lim \frac{x_n}{y_n}$$

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$ $\ni \left|\frac{x_n}{y_n} - r\right| < \varepsilon$

$$r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$$

$$y_n(r-\varepsilon) \leq x_n \leq y_n(r+\varepsilon)$$

$x_n \leq y_n$, Σy_n cgt $\Rightarrow$ Σx_n cgt if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt

if $\sum x_n$ cgt $\Rightarrow$ $\sum y_n$ cgt

if $r=0$, $r-\varepsilon < \frac{x_n}{y_n} < r+\varepsilon$

$$-\varepsilon < \frac{x_n}{y_n} < \varepsilon$$

$$-\varepsilon < 0 < \frac{x_n}{y_n} < \varepsilon$$

$$x_n \leq y_n \cdot \varepsilon$$

$\Rightarrow$ if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt.

$\sum \frac{(-1)^n}{n}$ cgt $\qquad$ $\sum \frac{1}{n}$ divergent

Absolute: $\sum x_n$ abs. cvgt if $\sum |x_n|$ is cvgt.

Conditional Convergence $\sum x_n$ is cgt but $\sum |x_n|$ is not cgt.

$\sum \frac{(-1)^n}{n^2}$ abs cgt $\qquad$ $\sum \frac{1}{n^2}$ cgt

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$.

$|x_{m+1} + \ldots + x_n| \leq ||x_{m+1}| + |x_{m+2}| + \ldots + |x_n|| < \varepsilon$ $\quad \forall$ ~~$m > n >$~~ $n > m > K(\varepsilon)$

$\sum |x_n|$

Root test $\quad r = \lim |x_n|^{1/n}$ exists.

① $r < 1$, $\exists$ $r < r_1 < 1$

for some $K(\varepsilon) \in \mathbb{N}$.

$$|x_n|^{1/n} \le r_1 \qquad \forall n \ge K(\varepsilon)$$

$$\Rightarrow \quad |x_n| \le r_1^n \qquad (r_1 < 1)$$

by Comparison Test

$$\underset{\text{cgt.}}{\sum |x_n|} \le \underset{\text{cgt.}}{\sum r_1^n}$$

② $r > 1$, $\exists$ $1 < r_1 < r$

Root

$$1 \le r_1 \le |x_n|^{1/n}$$

$$1^n \le r_1^n \le |x_n|$$

$$\underset{\infty}{\sum 1} \le \sum r_1^n \le \sum |x_n| \quad \text{dvgt.}$$

Ratio $\quad r = \lim \left| \dfrac{x_{n+1}}{x_n} \right|$

$r < 1$ cgt. $\quad r > 1$ dvgt

$r = 1$ Test fails

① $r < 1$

$$\left| \frac{x_{n+1}}{x_n} \right| < r \qquad \text{for } n \ge K(\varepsilon)$$

$$|x_{n+1}| < r \cdot |x_n|$$
$$\leq r \cdot r \cdot |x_{n-1}| = r^2 |x_{n-1}|$$
$$\vdots$$
$$\leq r^n \cdot |x_1|$$

$\sum \frac{1}{(n+1)(n+2)}$ $x^{\cancel{1/x}\ 1/x}$

Ratio Test :- $r = \lim \left| \frac{x_{n+1}}{x_n} \right|$

$$= \lim \left| \frac{(n+1)\cancel{(n+2)}}{\cancel{(n+2)}(n+3)} \right|$$

$$= \lim \left| \frac{1+1/n}{1+3/n} \right|$$

$$= 1 \quad \text{Test fail}$$

Limit Comparison Test $\sum y_n = \sum \frac{1}{n^2}$

$$r = \lim \left| \frac{x_n}{y_n} \right|$$

$$= \lim \left| \frac{n^2}{(n+1)(n+2)} \right|$$

$$= \lim \left| \frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})} \right|$$

$$= 1 \neq 0 \qquad \sum |y_n| \text{ cgt}$$

$$\Rightarrow \sum |x_n| \text{ cgt.}$$

$$\frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

$\sum \frac{1}{n^2} \to$ cgt

cgt.

$$\sum \frac{1}{(n+1)(n+2)} \Rightarrow \quad x_n = \frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

$$S_1 = x_1 = 1 - \frac{1}{2}$$

$$S_2 = S_1 + x_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3}.$$

$$S_3 = S_2 + x_3 = 1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

$\sum x_n$ cgt if $S_n -$ cgt.

$$\lim S_n = \lim 1 - \frac{1}{n+1}$$

$S_n \to 1$ $\qquad = 1$

$\sum 2^n$ Ratio $\Rightarrow$ $\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \left|\frac{2^{n+1}}{2}\right|$

$= 2 > 1$ dgt.

$\sum 2^r$ Root test

$$\lim |x_n|^{1/n} = \lim |2^n|^{1/n}$$

$= 2 > 1$ dgt.

$\sum 2^{-1/n}$

Ratio $\lim_{n} \frac{2^{-1/(n+1)}}{2^{-1/n}}$

$\lim \quad 2^{\frac{-1}{n+1} + \frac{1}{n}}$

$\lim_{n\to\infty} \quad 2^{\frac{1}{n^2+n}} \quad \to 1$

Test fails

$\sum 2^{-1/n}$ $\quad \sum x_n$ cgt $\Rightarrow \lim x_n \Rightarrow 0$

$\lim x_n \neq 0 \Rightarrow \sum x_n$ not cgt.

$\lim 2^{-1/n} = 1 \quad \Rightarrow \quad \sum 2^{-1/n}$ dvgt.

* $x_n = n/2^n$

Ratio Test $\quad r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} \right|$

$= \lim \left| \frac{n+1}{n} \right| \cdot \lim \left| \frac{2^n}{2^{n+1}} \right|$

$= \frac{1}{2} < 1$

$\Rightarrow \sum x_n$ cgt.

Root test:-

$r = \lim |x_n|^{1/n} = \lim \left| \frac{n}{2^n} \right|^{1/n} = \frac{1}{2} \underbrace{\lim n^{1/n}}_{\to 1}$

$= \frac{1}{2} < 1$

$$\sum \frac{3^n}{n!} \ ?$$

Ratio Test $\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n} \right|$

$= \lim \frac{3}{n}$

$= 0 < 1$ cgt.

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \dots$$

$$x_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$\frac{1}{\sqrt{1\cdot2}} + \frac{1}{\sqrt{2\cdot3}} + \frac{1}{\sqrt{3\cdot4}}$$

$$x_n = \frac{1}{\sqrt{n(n+1)}}$$

$$\frac{1}{n} < \frac{1}{\sqrt{n(n+1)}} < \cancel{\frac{1}{n}}$$

limit Comparison $\lim \left| \frac{n}{\sqrt{n(n+1)}} \right|$

$= \lim \left| \frac{1}{\sqrt{1(1+1/n)}} \right| = 1.$

$\Rightarrow \sum y_n$ dvgt $\Rightarrow \sum x_n$ dvgt.

Ratio :- $x_n = \dfrac{1}{\sqrt{n(n+1)}}$

$$\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \frac{\sqrt{n\cancel{(n+1)}}}{\sqrt{\cancel{(n+1)}(n+2)}} = \lim \sqrt{\frac{1}{1+2/n}}$$

$= 1$ Test fails.

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots$$

$$x_n = \frac{1}{n^n}$$

Root test $\lim |x_n|^{1/n} = \lim \left|\frac{1}{n^n}\right|^{1/n} = \lim \left|\frac{1}{n}\right| = 0$

< 1

$$\sum \frac{n^2-1}{n^2+1}$$ by necessary condn $\lim \frac{n^2-1}{n^2+1} = 1 \neq 0$

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \dots$$

$$\left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \dots\right)$$

$\sum (1/3^2)^n$

$$\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots$$

$$x_n=\frac{1}{n\cdot 2^n}$$

$$\frac{1^2\,2^2}{1!}+\frac{2^2\,3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\cdots$$

$$x_n=\frac{n^2(n+1)^2}{n!}$$

$$r=\lim\left|\frac{x_{n+1}}{x_n}\right|=\lim\frac{\cancel{(n+1)^2}(n+2)^2}{(n+1)!}\;\frac{n!}{\cancel{(n+1)^2}(n^2)}$$

$$=\lim\frac{1}{(n+1)}\cdot(1+2/n)^2$$

$$=0$$

$$x_n=\frac{1\cdot 2\cdot 3\cdots n}{7\cdot 10\cdots(3n+4)}\qquad \lim\left|\frac{x_{n+1}}{x_n}\right|$$

$$\frac{1\cdot 2\cdots\; n\cdot(n+1)}{7\cdot 10\cdots(3n+4)(3(n+1)+4)}\qquad =\lim\left|\frac{n+1}{(3(n+1)+4)}\right|$$

$$=\lim\left|\frac{1}{3+4/(n+1)}\right|$$

$$=\frac{1}{3}<1$$

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{1}{n} \qquad ?$$

$$\lim\left|\frac{x_{n+1}}{x_n}\right| = \lim\left|\frac{2(n+1)-1}{2(n+1)}\left(\frac{n}{n+1}\right)\right| = 1$$

$$x_n \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{\underline{2^n}\cdot\underline{n!}}\cdot\frac{1}{n}$$

$$\lim|x_n|^{1/n} = \frac{1}{2}\left(\lim\frac{1^{1/n}\,3^{1/n}\cdots(2n-1)^{1/n}}{(n!)^{1/n}}\cdot\frac{1}{n^{1/n}}\right)$$

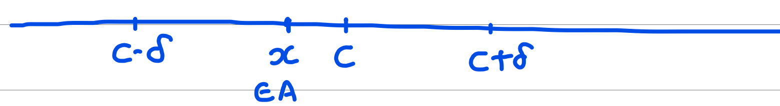
$$= \frac{1}{2} < 1$$

Functions and continuity

limit / Cluster Point :- $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ for every $\delta > 0$

$\exists\, x \in A$, $x \neq c$, $\ni |x-c| < \delta$

for any $\overset{\delta}{\varepsilon} > 0$, $\delta_\varepsilon(c) \cap A \neq \phi$



① c cluster, $\exists\, a_n \in A$ $a_n \to c$

$\Rightarrow$ c cluster pt. of A

for any $\frac{1}{n} > 0$ $\exists\, x_n \in A \ni |x_n - c| < 1/n$

$n \in \mathbb{N}$ $\Rightarrow c - 1/n < x_n < c + 1/n$

$\Rightarrow x_n \to c$

Suppose we have a seq^n $a_n \in A$, $\ni a_n \to c$.

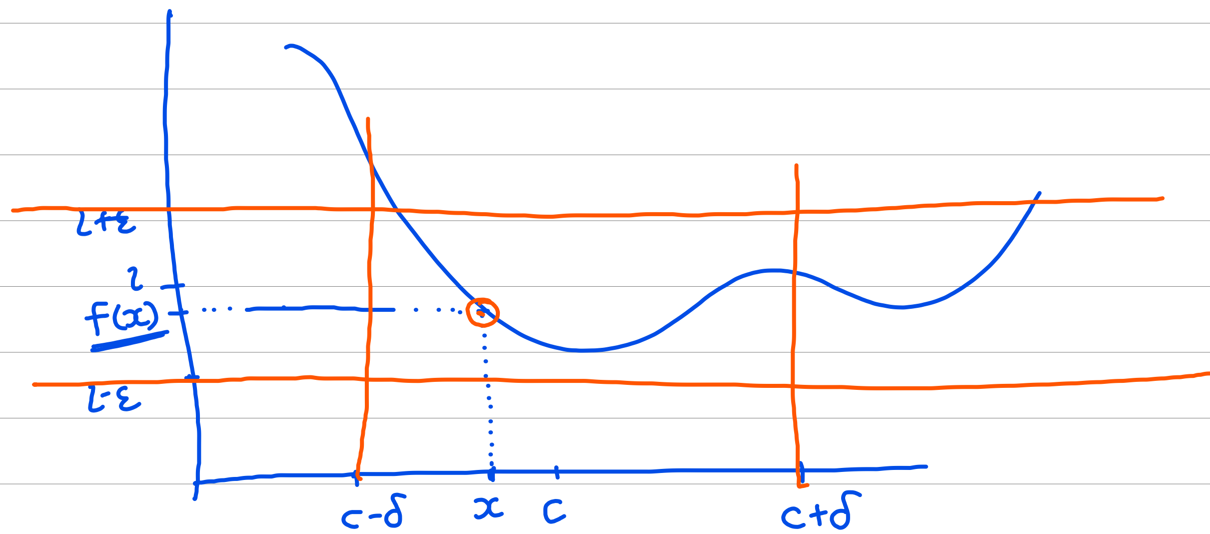
To show that :- c cluster pt. of A

$a_n \to c$

for any $\varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$, $\ni |a_n - c| < \varepsilon$ $\forall n \geq K(\varepsilon)$

$\Rightarrow$ $|a_n - c| < \varepsilon$ $a_n \in A \cap \delta_\varepsilon(c)$

$\Rightarrow$ c is also cluster pt. of A



Limit of fun^c :

$f : A \to \mathbb{R}$, c cluster of A, $L \in \mathbb{R}$

for any $\varepsilon > 0$ $\exists \delta_\varepsilon > 0$ $\ni |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\ni x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(L)$

$\therefore$ $f : A \to \mathbb{R}$, c cluster pt. of A. $\Rightarrow$ f can have only one limit pt. at c.

Contradiction, L & L'

for $\varepsilon > 0$ $\begin{cases} \frac{\varepsilon}{2} > 0 & \exists \delta'_\varepsilon > 0 \ni |x - c| < \delta'_\varepsilon \Rightarrow |f(x) - L| < \varepsilon/2 \\ \frac{\varepsilon}{2} > 0 & \exists \delta''_\varepsilon > 0 \ni |x - c| < \delta''_\varepsilon \Rightarrow |f(x) - L'| < \varepsilon/2 \end{cases}$

$$|L-L'| = |L-f(x)+f(x)-L'|$$

$$\leq |L-f(x)| + |f(x)-L'|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

$$\leq \varepsilon$$

$$L = L'$$ QED

$x_n \to c$
$f(x_n) \to L.$ } seq Criteria

def seq convergent seq^n & def limit pt. of fun^c at some c

$f: A \to \mathbb{R}$, c is cluster pt. of A

① for any $\varepsilon > 0$ $\exists\ \delta_\varepsilon > 0 \ni$ if $|x-c| < \delta_\varepsilon \Rightarrow |f(x)-L| < \varepsilon$
$x \in A$

② $f: A \to \mathbb{R}$ c A
$\exists\ x_n \to c \Rightarrow f(x_n) \to L$

$f(x)$ converges to L at c

① $|x-c| < \delta_\varepsilon$ $|f(x)-L| > \varepsilon_0$
② $x_n \to c$ but $f(x_n) \not\to L$

does not converges $f(x_n)$ at c

$$\boxed{\lim_{x \to c} f(x) = L}$$

$$f(x) = \operatorname{sgn}(x) = \begin{cases} +1 & x>0 \\ -1 & x<0 \\ 0 & x=0 \end{cases}$$

$f :- \mathbb{R} \to \mathbb{R}$

$(-\delta, 0)$ $\quad f(x) = \underline{-1}$ $\qquad (0, \delta)$ $\quad f(x) = \underline{+1}$

$f : A \to \mathbb{R}$, c cluster pt. of A

$f :-$ limit exists at pt. c. $\quad c \in \mathbb{R}$

for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ if $|x - c| < \delta_\varepsilon \Rightarrow \underline{|f(x) - L| < \varepsilon}$

$$\boxed{\Rightarrow L - \varepsilon < f(x) < L + \varepsilon \quad \forall\, x \in V_{\delta_\varepsilon}(c)}$$

Completeness

$M = \operatorname{Sup}\{f(x), x \in V_{\delta_\varepsilon}(c)\} \Rightarrow |f(x)| \le M$ for some $\underline{M > 0}$

$f : A \to \mathbb{R}$, c cluster pt. of A $\quad \lim_{x \to c} f(x) = L$

$g : A \to \mathbb{R}$ $\quad \lim_{x \to c} g(x) = M$

$(f+g) : A \to \mathbb{R}$ $\quad \lim_{x \to c} (f+g)(x) = L+M$

$\lim_{x \to c} f(x) = L$ for $\varepsilon/2 > 0$, $\exists\, \delta_\varepsilon' > 0 \ni \underline{|x-c| < \delta_\varepsilon'} \Rightarrow |f(x) - L| < \varepsilon/2$

for $\varepsilon/2 > 0$ $\exists\, \delta_\varepsilon'' > 0 \ni \underline{|x-c| < \delta_\varepsilon''} \Rightarrow |g(x) - M| < \varepsilon/2$

$\delta_\varepsilon = \min(\delta_\varepsilon', \delta_\varepsilon'')$ $\qquad |x-c| < \delta_\varepsilon$

$|(f+g)(x) - (L+M)|$

$= |(f(x) + g(x)) - (L+M)|$

$= |\underline{f(x) - L} + \underline{g(x) - M}| \leq |f(x) - L| + |g(x) - M|$

$\leq \varepsilon/2 + \varepsilon/2$

$\leq \varepsilon$

$|(f \cdot g)(x) - L \cdot M|$

$$= |f(x)\cdot g(x) - L\cdot g(x) + L\cdot g(x) - LM|$$

$$\leq \underline{|g(x)|}\,|f(x)-L| + |L|\,|g(x)-M|$$

$$\leq M\cdot \frac{\varepsilon}{2M} + \cancel{|L|}\frac{\varepsilon}{2\cancel{|L|}}$$

$f: A \to \mathbb{R}$, c cluster pt. of A

$a \leq f(x) \leq b$, f limit exists at c.

$$\left[\text{for any } \varepsilon > 0 \ \exists\ \delta_\varepsilon > 0 \ni |x-c| < \delta_\varepsilon \Rightarrow |f(x)-L| < \varepsilon\right]$$

seq^n criter

$\exists\, x_n \in A$ $\quad x_n \to c \quad \Rightarrow \quad f(x_n) \to L$

$a \leq f(x) \leq b \qquad \forall\ x \in A$

$a \leq f(x_n) \leq b \qquad \forall\ x_n$

by squeeze theo.

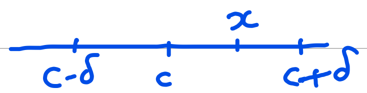
$$a \leq \lim_{x_n \to c} f(x_n) \leq b$$

$$a \leq L \leq b$$

c cluster of A iff $\exists\ a_n \in A$ $a_n \to c$

① c cluster pt. of A.

for every $\delta > 0$ $\exists$ $x \in A$ $|x - c| < \delta$



$\forall\ n \in \mathbb{N}$, $\frac{1}{n} > 0$ $\exists$ $x_n \in A$ $\Rightarrow$ $|x_n - c| < 1/n$ $\ni$ $c - 1/n < x_n < c + 1/n$

$x_n \to c$

② $a_n \in A$, $a_n \to c$

for any $\varepsilon > 0$ $\exists\ K(\varepsilon) \in \mathbb{N}$, $\ni$ $|a_n - c| < \varepsilon$ $\forall\ n \geq K(\varepsilon)$

✓ infinite $a_n \in A$

$\Rightarrow$ c is cluster pt. of A.

$f, g, h : A \to \mathbb{R}$, c cluster pt. of A

$f(x) \leq g(x) \leq h(x)$ $\forall\ x \in A$ $x \neq c$

✓ $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = L$

To show $\lim\limits_{x \to c} g(x) = L$

for any $\varepsilon > 0$ $\exists\ \delta'_\varepsilon > 0$ $\ni$ $|x - c| < \delta'_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$

$\Rightarrow$ $L - \varepsilon < f(x) < L + \varepsilon$ —

-,,- $\delta''_\varepsilon > 0$ $\ni$ $|x - c| < \delta''_\varepsilon$ $\Rightarrow$ $|h(x) - L| < \varepsilon$

$\Rightarrow$ $L - \varepsilon < h(x) < L + \varepsilon$ —

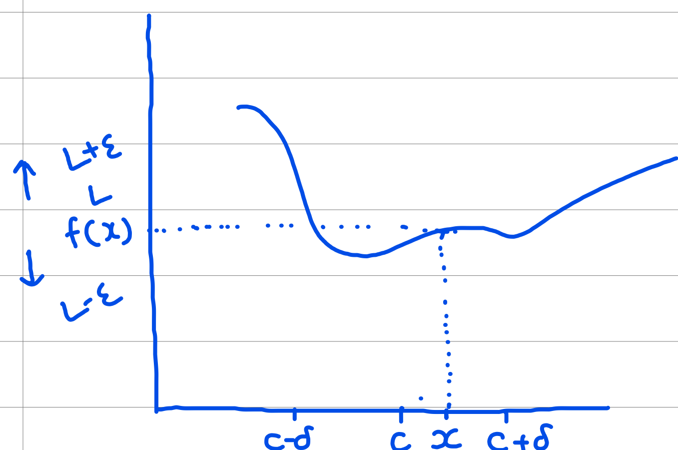
$$f(x) \leq g(x) \leq h(x)$$

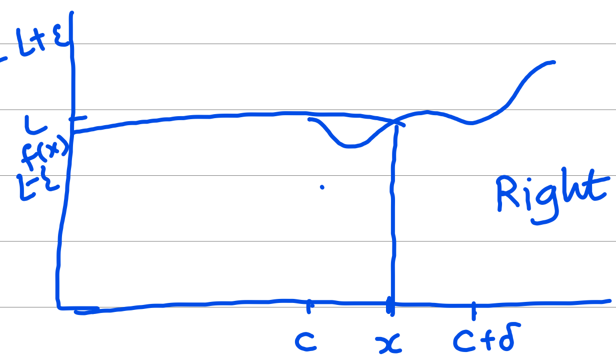
$$\Rightarrow L-\varepsilon < f(x) \leq \underline{g(x)} \leq h(x) < L+\varepsilon$$

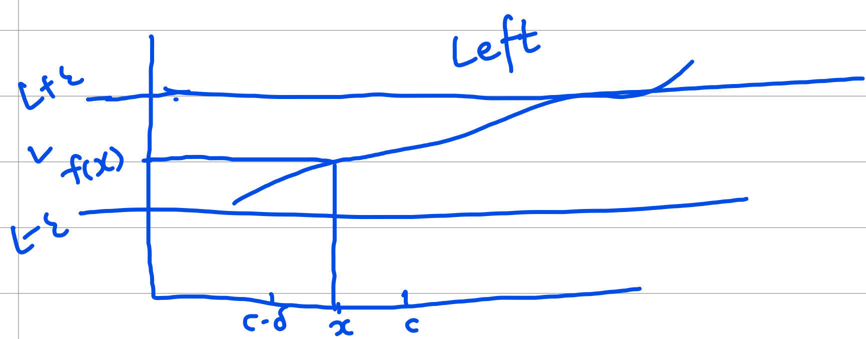
$$\Rightarrow \quad L-\varepsilon < g(x) < L+\varepsilon$$

$$\Rightarrow \quad |g(x) - L| < \varepsilon$$

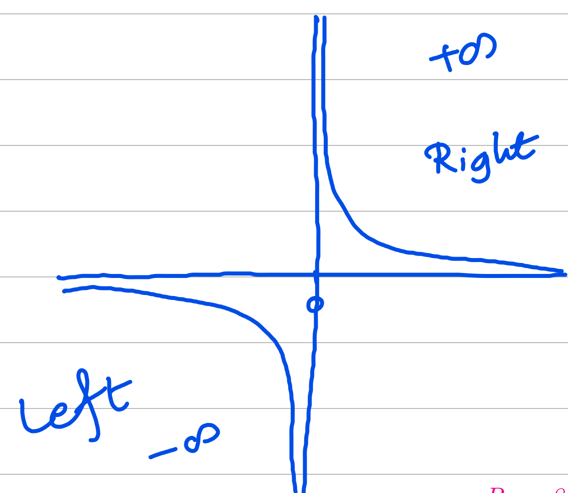
$$\Rightarrow \quad g(x) \to L$$







$f(x) = 1/x$



$(c-\delta, c)$ $\quad \underline{(-\delta, 0)}$

$(c, c+\delta)$ $\quad (0, \delta)$

$$\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$$

$(c-\delta, c+\delta)$ $(c, c+\delta)$ $(c-\delta, c)$

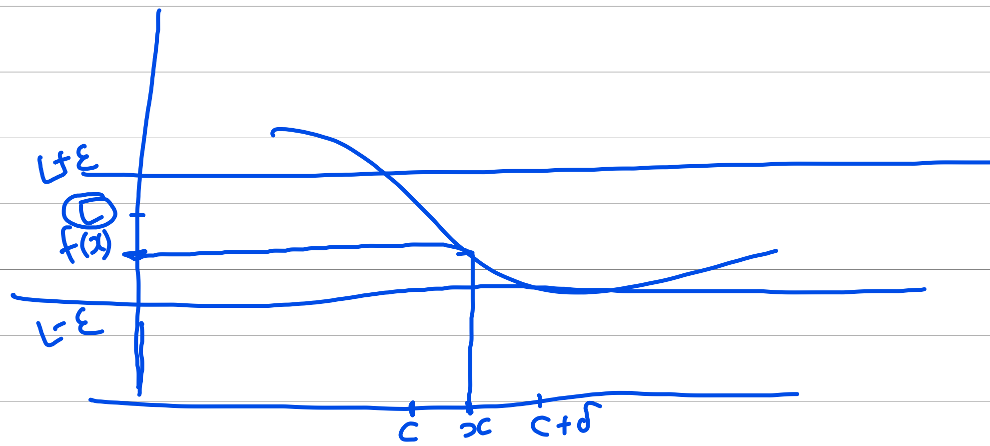
$\lim_{x\to c^+} f(x)$ for any $\varepsilon>0$ $\exists$ $\delta_\varepsilon>0$ $\ni$ $x\in(c, c+\delta)$

$\ni$ $|f(x)-L|<\varepsilon$

$|x-c|<\delta_\varepsilon$

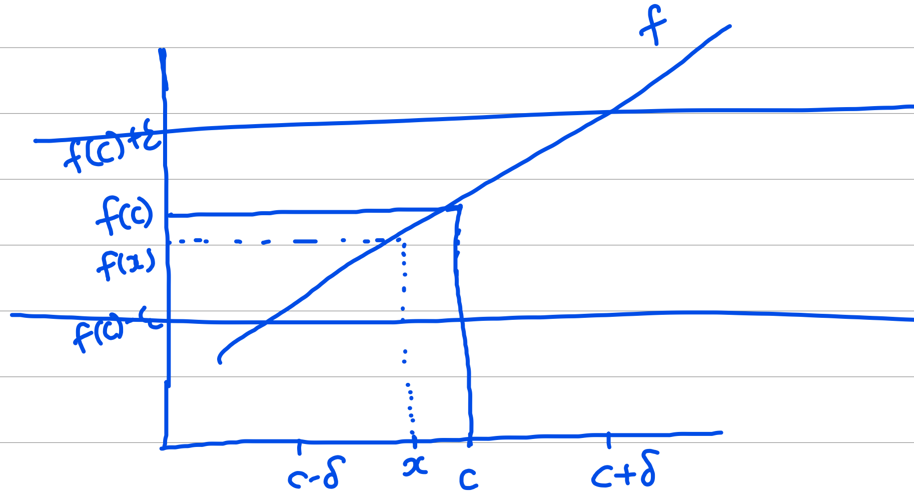
$c-\delta_\varepsilon < x < c+\delta_\varepsilon$

$(c-\delta_\varepsilon, c)\cup(c, c+\delta_\varepsilon)$



$\lim_{x\to c} f(x) = L = f(c)$

f cont. at pt. c.



Right

$\lim_{x\to c^+} f(x) = f(c)$

$\lim_{x\to c^-} f(x) = f(c)$

$\lim_{x\to c} f(x) = f(c)$

$$f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$$

(i) for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|x - c| < \delta_\varepsilon$ & $x \in A$

$x \in V_\delta(c)$ & $x \in A$ $\ni$ $\underline{x \in V_\delta(c) \cap A}$

$\ni$ $|f(x) - f(c)| < \varepsilon$

$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

$\underline{f(x) \in V_\varepsilon(f(c))}$

$$\ni \quad f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$$

Sequential Criteria for continuity

$x_n \to c$

$f(x_n) \to f(c)$

if $x_n \to c \Rightarrow f(x_n) \to f(c)$

f, g cont. at c $\qquad$ $\underline{f+g}$ cont at c.

for any $\frac{\varepsilon}{2} > 0$ $\exists \delta_\varepsilon > 0$ $\ni$ $|x - c| < \delta_\varepsilon$ $\Rightarrow$ $|f(x) - f(c)| < \varepsilon/2$

$\Rightarrow$ $|g(x) - g(c)| < \varepsilon/2$

$$|f(x) + g(x) - f(c) - g(c)| = |f(x) - f(c)| + |g(x) - g(c)|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

$$\leq \varepsilon$$

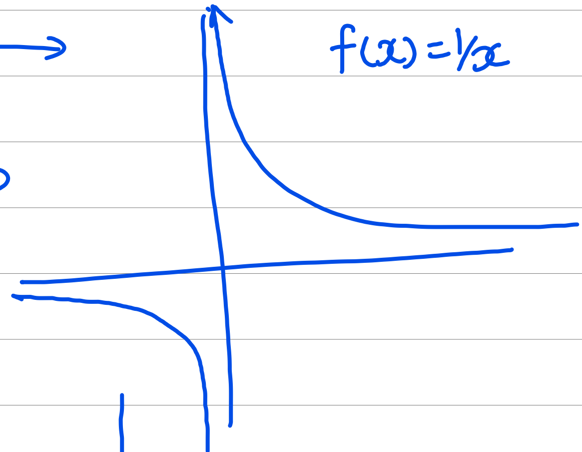
(c cluster pt. A $\longrightarrow$ $f: A \to \mathbb{R}$, $\lim\limits_{x \to c} f(x) = f(c)$) ✓

Conti. on set B, $\forall$ $y \in B$ $\lim\limits_{x \to y} f(x) = f(y)$

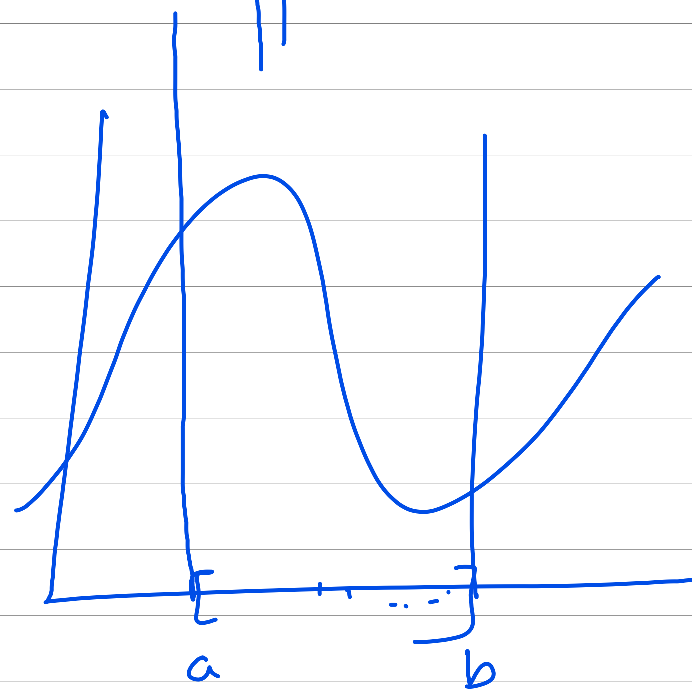
$\Rightarrow$ conti at every point of B.

$B = [-1, 1]$ $\longrightarrow$ $f(x) = 1/x$

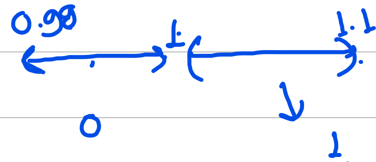
$$\begin{cases} \lim\limits_{x \to 0^+} f(x) = +\infty \\ \lim\limits_{x \to 0^-} f(x) = -\infty \end{cases}$$



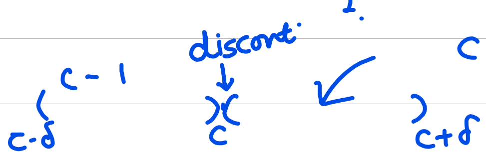
$f(x) = 2x^2 + 3$ ✓



$f(x) = [x]$



$[x]$

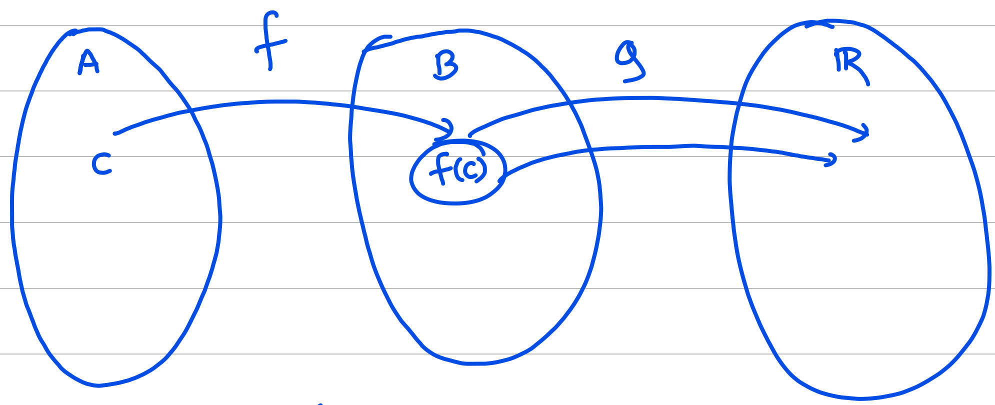


f cont. at pt. c.

for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|x_1 - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

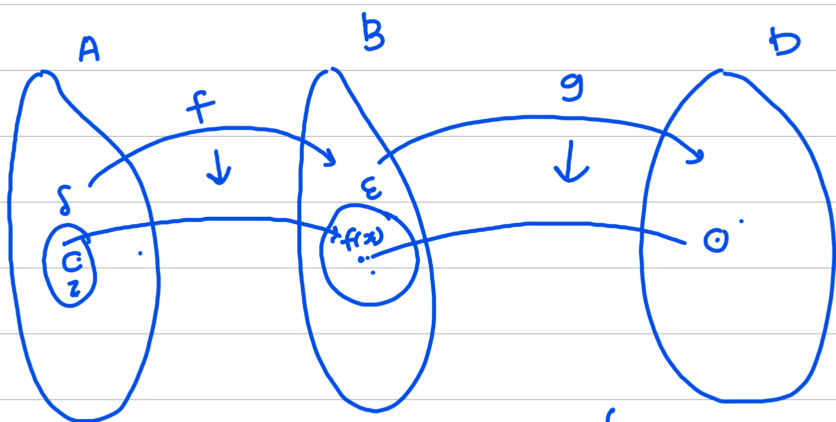
$$\big|\, |f(x)| - |f(c)| \,\big| \le |f(x) - f(c)| < \varepsilon$$

$\Rightarrow$ $|f|$ is cont. at c.



$g \circ f$ $\qquad$ $\underline{g(f(x))}$.

Let A,B R & let f : A R & g : B R be function such that f (A) B if f is continuous at point c A and g is continuous at b = f (c) B then the composition g f : A R is continuous at c.



$f: A \to B$ cont $\qquad g: B \to D$ cont $\qquad g \circ f : A \to D$ cont

for any $\varepsilon > 0 \ \exists\ \delta_\varepsilon > 0 \ni |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

$x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$

for any $\varepsilon > 0 \ \exists\ \delta_\varepsilon > 0 \ni |y - c'| < \delta \Rightarrow |g(y) - g(c')| < \varepsilon'$

$y \in V_\delta(c') \Rightarrow g(y) \in V_\varepsilon(g(c'))$

As we have assumed that $f(A) \subseteq B$

$\exists$ some $c' = f(c)$

cont def cont

$\Big[$ for any $\varepsilon > 0 \ \exists\ \delta_\varepsilon > 0 \ni |z - c| < \delta \Rightarrow \Big]$

$|(g \circ f)(z) - (g \circ f)(c)| = |g(f(z)) - g(f(c))|$

$(\ |z - c| < \delta \Rightarrow |f(z) - f(c)| < \varepsilon,\ \varepsilon > 0\ )$

$|y - c'| < \delta'$ $\cancel{\Rightarrow |g(y) - g(c')| < \varepsilon}$

$\boxed{\Rightarrow |g(f(z)) - g(f(c))| < \varepsilon}$

Boundedness Theo.

$I = [a, b]$ closed bounded interval.

$f : I \to \mathbb{R}$ cont.

To show: f bounded. $|f(x)| \le M$

Assume $f(x)$ is not ~~bounded on~~ I

$\exists$ some $n \in \mathbb{N}$ $|f(x_n)| > M$

[bounded seq^n cgt subseq^n]

- I closed bounded interval..

Let $\exists$ some seq^n $x_n \in I$,

I bounded $\Rightarrow$ x_n is bounded.

By Bolzano Weierstrass theo. $\exists$ subseq^n $x_{n_k} \in I$ which is convergent.

$$x_{n_k} \to x^* \quad \text{(say)}$$

if x_{n_k} is subseq^n in I & I is closed

$\Rightarrow x^* \in I$

$\therefore$ f is cont.

$$x_{n_k} \to x^* \Rightarrow f(x_{n_k}) \to f(x^*)$$

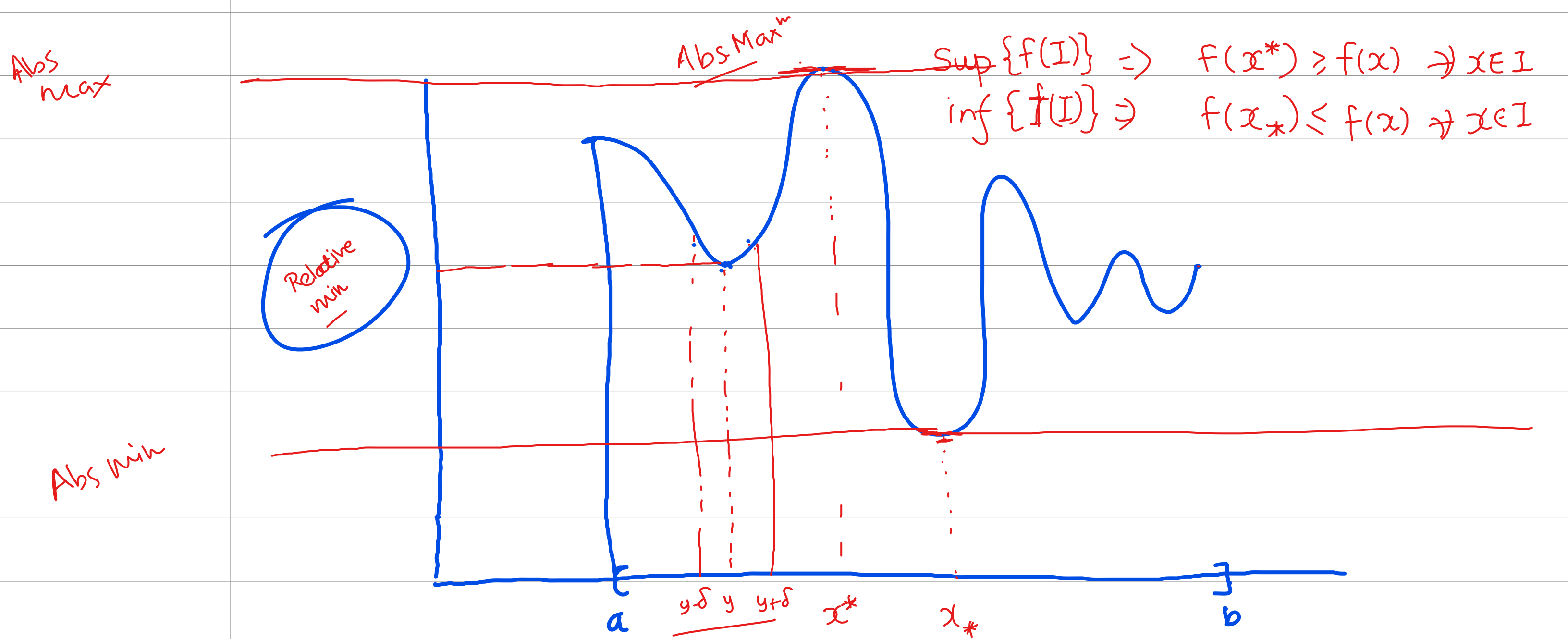
$\Rightarrow f(x_{n_k})$ is convergent subseq^n

$\Rightarrow$ bounded seq^n

$\Rightarrow$ Our assumption is wrong.

Absolute Extremum — Relative extremum

max^m min max^m min



Abs. min	$f(x_*) \leq f(x)$	$\forall x \in I$	Abs. max^m	$f(x^*) \geq f(x)$	$\forall x \in I$
Rel. min	$f(y) \leq f(x)$	$\forall x \in (y-\delta, y+\delta)$	Rel. min	$f(y) \geq f(x)$	$\forall x \in V_\delta(y)$

Max^m & Min^m the

$f : I \to \mathbb{R}$, I closed bounded, f cont.

To show :- $\exists$ x^*, x_* as abs. max^m & abs. min resp.

$\because$ I closed bounded, f cont. $f : I \to \mathbb{R}$

by Boundedness Theo. $|f(x)| \leq M$ $\forall x \in I$

$f(I) = \{f(x), x \in I\}$ bounded.

by Completeness property.

$\exists$ $s^* = \operatorname{Sup}\{f(I)\}$ and $s_* = \operatorname{Inf}\{f(I)\}$

If s^* is sup. then for $n \in \mathbb{N}$. $s^* - \frac{1}{n}$ can't be sup.

$$\checkmark \left[\quad s^* - \frac{1}{n} < f(x_n) < s^* \qquad \forall n \in \mathbb{N} \right]$$

So we got seq^n $x_n \in I$, I closed bounded
by Bolzano weierstrass $\exists$ $x_{n_k} \to x^*$
and as $x_{n_k} \in I$, $\ni$ I closed $\Rightarrow x^* \in I$

Now $x_{n_k} \to x^*$
but f is cont on I, by seq^n criteria

$$f(x_{n_k}) \to f(x^*)$$

$$s^* - \frac{1}{n_k} < f(x_{n_k}) < s^*$$

by squeez theo.

$$\lim s^* - \frac{1}{n_k} < \lim f(x_{n_k}) \leq \lim s^*$$

$$s^* \leq f(x^*) \leq s^*$$

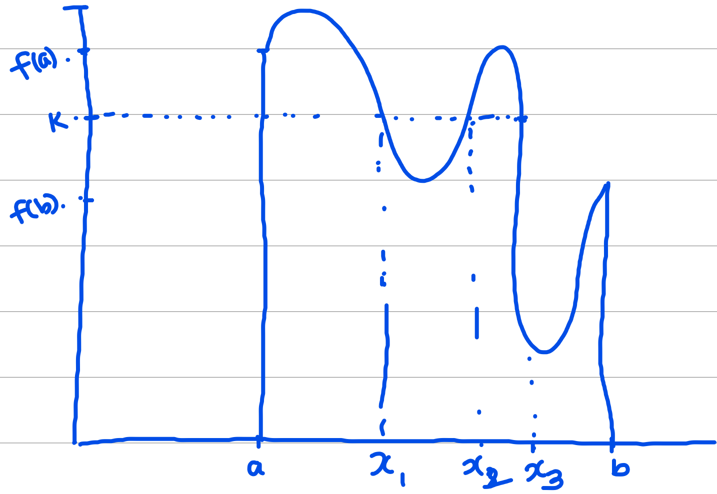
$$\Rightarrow \quad s^* = f(x^*)$$

$\Rightarrow$ $s^* = \sup\{f(I)\}$ $\Rightarrow$ $f(x^*) \geq f(x)$ $\forall x \in I$

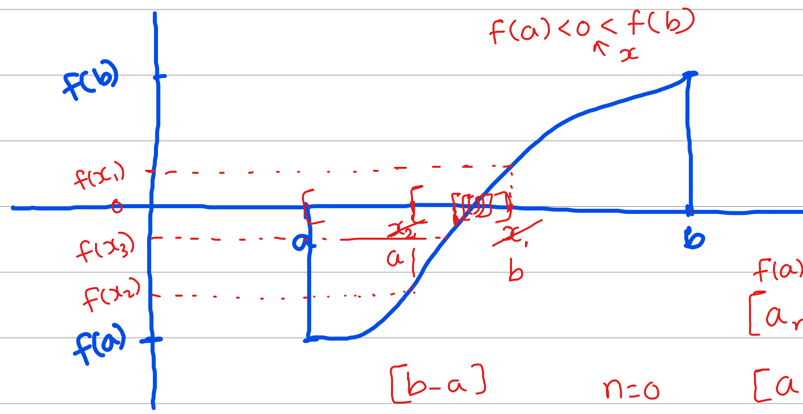
Simillarly we can obtain x_* $\ni$ $f(x_*) \leq f(x)$ $\forall x \in I$.
$x^*, x_* \in I$

* Location of Root

Bolzano (Intermediate) Value Theo.



$k = f(x)$



$F(a) < f(b)$

$f(x) \simeq 0$?
$x = ?$

$f(a) < 0$ $f(b) > 0$

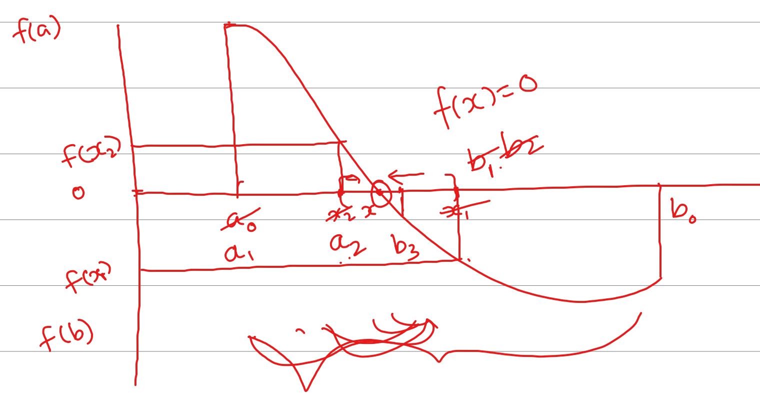
$[a_n , b_n]$

$[b-a]$ $n=0$ $[a_0, b_0] = [a, b]$

$x = \frac{a+b}{2}$ $\frac{[b-a]}{2}$ $n=1$ $[a_1 = a_0, b_1 = x]$ $f(a_1) < 0$ $f(b) > 0$

$n=2$ $x = \frac{a+b}{2}$ $\frac{[b-a]}{2^2}$ $n=2$ $[a_2 = x, b_2 = b_1]$

$n=3$ $x = \frac{a+b}{2}$ $\frac{[b-a]}{2^n}$ $n=3$ $[a_3 = x, b_2 = b_1]$



$f(x) = 0$

Bolzano Intermediate value Theo.

~~Existence of Root~~ $f:[a,b]\to\mathbb{R}$ cont.

assume $f(a)<K<f(b)$ or $f(b)<K<f(a)$

$\exists\ c\in[a,b] \ni f(c)=K$

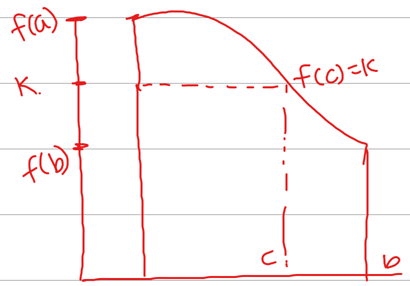
$f: I \to \mathbb{R}$ cont.

$$\boxed{\begin{array}{c} f(a)<0<f(b) \\ \exists c\in[a,b]\ f(c)=0 \end{array}}$$

$a<b,\quad [a,b]\subseteq I$

$f:[a,b]\to\mathbb{R}$ cont. Assume $f(a)<K<f(b)$

$g(x)=f(x)-K \Rightarrow g:[a,b]\to\mathbb{R}$ cont.



$f(a)<K<f(b)$

$f(a)-K < K-K < f(b)-K$

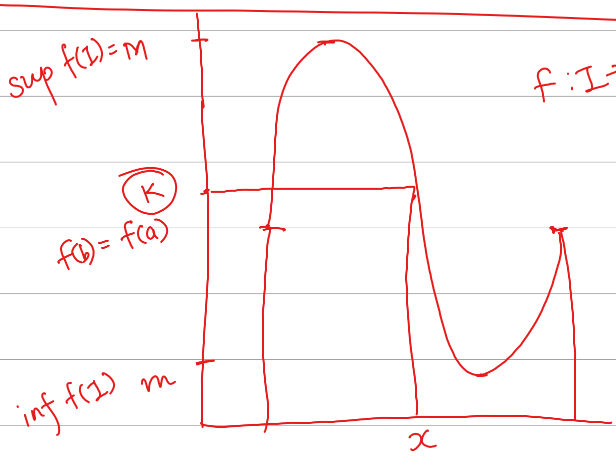
$f(a)-K < 0 < f(b)-K$

$g(a) < 0 < g(b)$

$\therefore$ By existance of Root Theo. $\exists\ c\in[a,b] \ni g(c)=0$

$$g(c)=0$$
$$\Rightarrow f(c)-K=0$$
$$\Rightarrow f(c)=K$$

(QED)



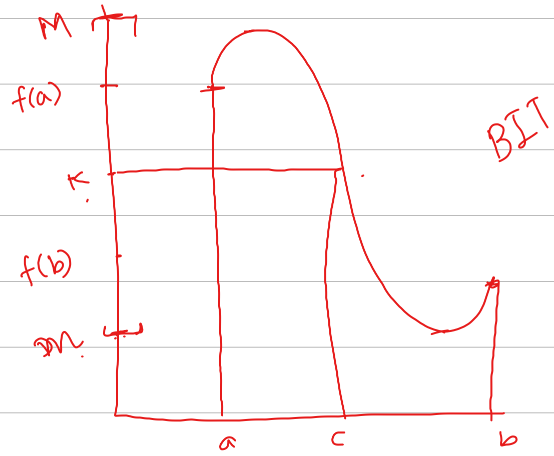
$f: I\to\mathbb{R}$ $f:[a,b]\to\mathbb{R}$ cont.

$\Rightarrow$ Bounded $\Rightarrow$ Max^m Min^m

$f(I) = \{f(x), x\in I\}$

$= [m, M]$

$x_*\quad x^*$



BIT

$f(a) > K > f(b)$

$c \in [a,b] \ni f(c) = K$

$Max^m \quad Min^m \quad [a,b]$

$M = Sup\{f(x)\}$

$m = Inf\{f(x)\}$

$M = f(x^*) \qquad m = f(x_*)$

$c \quad f \qquad \lim_{x \to c} f(x) = L$

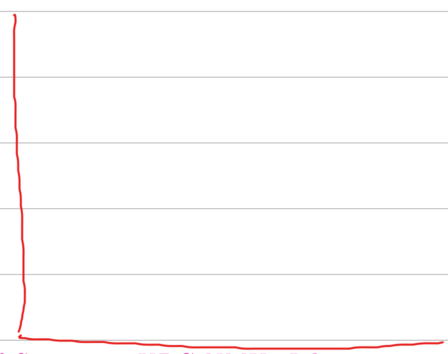
Cont. at pt. c $\qquad \lim_{x \to x} f(x) = f(c)$

Cont. on ~~set~~ $\quad \forall\, c \in B \qquad \lim_{x \to c} f(x) = f(c)$

Uniform Conti

for any $\varepsilon > 0$ $\exists\, \delta_\varepsilon > 0 \ni |x - c| < \delta_\varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon$

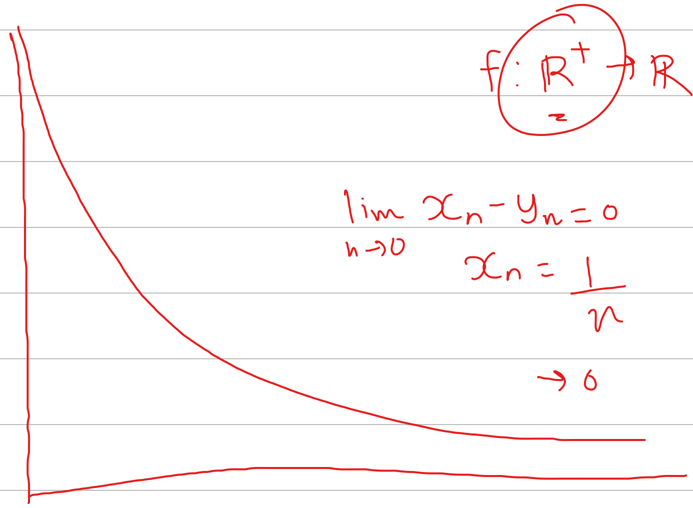
$\ni |x - y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$



$|x-y|<\delta_\varepsilon$

$\Rightarrow |f(x)-f(y)|<\varepsilon$

$f: \mathbb{R}^+ \to \mathbb{R}$ $\quad f(x)=\dfrac{1}{x}$



[a,b]

$\lim\limits_{n\to 0} x_n - y_n = 0$ $\quad |f(x_n)-f(y_n)| \geq \varepsilon_0$

$x_n = \dfrac{1}{n} \to 0$ $\quad y_n = \dfrac{1}{n-1} \to 0$

$\left| \dfrac{1}{1/n} - \dfrac{1}{1/(n-1)} \right|$

$= |n-(n-1)|$

$= 1 > \varepsilon_0$

$1/x$ is cont. but not uniformly cont on $\mathbb{R}^+$ $(0,\infty)$

$|f(x)-f(y)| \le K|x-y|$ $\quad x, y \in I$

Lipschitz fun^c

cont.? uniformly cont.?

$f: I \to \mathbb{R}$, for any $\varepsilon>0$ $\exists$ $\delta_\varepsilon>0$ $\ni$ $|x-y|<\delta_\varepsilon \Rightarrow |f(x)-f(y)|<\varepsilon$

Now for Lipschitz fun^c

Assume $K>0$ for any $\varepsilon>0$ $\exists$ $\delta_\varepsilon = \varepsilon/K > 0$

$|x-y| < \delta_\varepsilon = \varepsilon/K$

but $|f(x)-f(y)| < K|x-y|$

$$\le K \cdot \delta_\varepsilon$$

$$\le \cancel{K} \cdot \varepsilon/\cancel{K}$$

$$\le \varepsilon$$

Liptchitz fun is uniformly cont.

$f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, f uniformly cont.

x_n cauchy seq^n

To prove, $f(x_n)$ cauchy.

① $f: A \to \mathbb{R}$ uniformly cont.

def: For any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni |x-y| < \delta_\varepsilon \Rightarrow |f(x)-f(y)| < \varepsilon$

② $x_n \in A$, x_n is cauchy seq^n

def: for any $\delta_\varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ $\ni |x_n - x_m| < \delta_\varepsilon$ $\forall\, n, m \geq K(\varepsilon)$

$$|x_n - x_m| < \delta_\varepsilon \Rightarrow |f(x_n) - f(x_m)| < \varepsilon .$$

$$\Rightarrow f(x_n) \text{ is cauchy seq}^n$$

* Continuous Extension Theorem: A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on [a, b].

$f : (a,b) \to \mathbb{R}$ uniformly cont. $\Rightarrow$ extended

$f(a) = L$, $f(b) = M$

$f : [a,b] \to \mathbb{R}$ cont on $[a,b]$

$\Leftarrow$

$f : (a,b) \to \mathbb{R}$ uniform cont. $\Rightarrow$

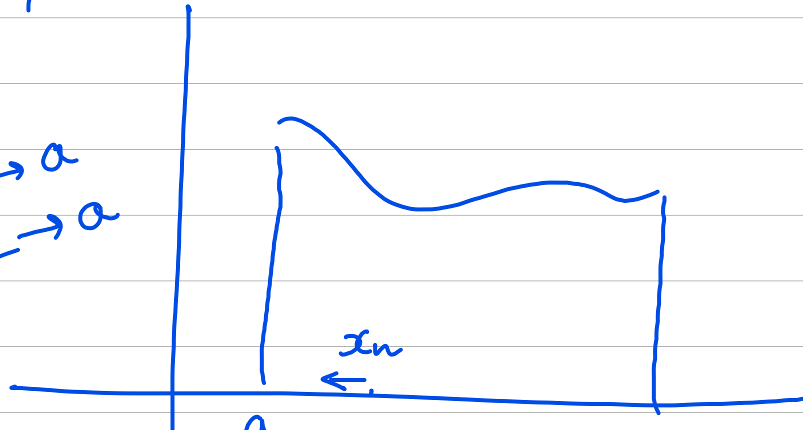
$f(a) = L$, $f(b) = M$

$x_n \to a$

$x_n = a + 1/n \to a$

$y_n = a + 1/n^2 \to a$

(a,b)



~~lim $x_n - y_n$~~ $= 0$

$|x_n - y_n| < \delta_\varepsilon$ (a,b) $\Rightarrow |f(x_n) - f(y_n)| < \varepsilon$

$x_n \to a$ for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|x_n - a| < \delta_\varepsilon \Rightarrow |f(x_n) - L| < \varepsilon/2$

$y_n \to a$ $\quad$ " $\quad$ $|y_n - a| < \delta_\varepsilon \Rightarrow |f(y_n) - L| < \varepsilon/2$

$|x_n - y_n| < \delta_\varepsilon$

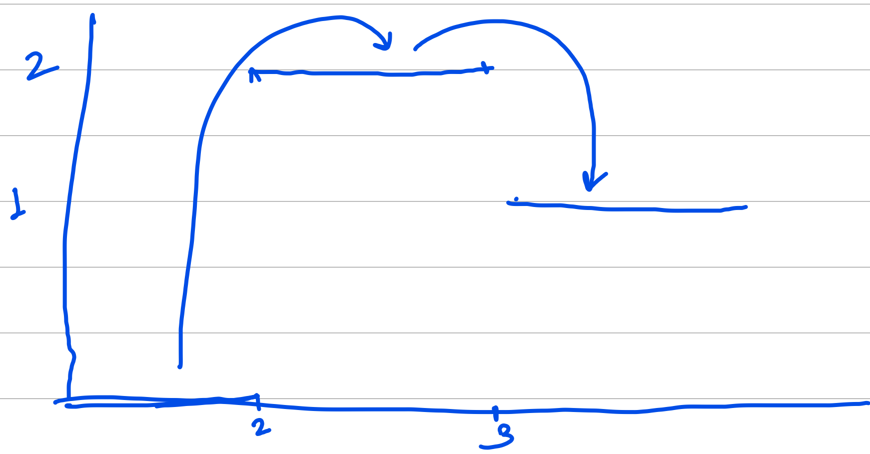
$|x_n - a| + |y_n - a| < \delta_\varepsilon \Rightarrow$

$$
\begin{aligned}
|f(x_n) - f(y_n)| &= |f(x_n) - L + L - f(y_n)| \\
&\leq |f(x_n) - L| + |L - f(y_n)| \\
&\leq \varepsilon/2 + \varepsilon/2 \\
&\leq \varepsilon
\end{aligned}
$$

$\underline{f(a)} = \lim f(x_n) = \lim f(y_n) = L$

Step

$$
f(x) = \begin{cases} 0 & 0 \leq x < 2 \\ 2x & 2 \leq x < 3 \\ 1 & 3 \leq x \leq 5 \end{cases}
$$



$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+} \qquad = \lim_{x\to c^-}$$

$$(c, c+\delta) \qquad (c, c-\delta)$$

$$f(x) = |x| = \begin{cases} x & x>0 \\ 0 & x=0 \\ -x & x<0 \end{cases}$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x}$$

$$= +1$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{f(x)-f(c)}{x-c} \neq \lim_{x\to 0^-} \frac{f(x)-f(c)}{x-c}$$

Derivable $\Rightarrow$ Conti. | Conti $\not\Rightarrow$ deri $|x|$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$\lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x-c} \qquad g(x) \neq 0 \quad \forall x \in I$$

$$= \lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x-c}$$

$$= \lim_{x\to c} \frac{f(x)\,g(c) - f(c)\cdot g(x)}{(x-c)\,g(x)\cdot g(c)}$$

$$= \lim_{x\to c} \frac{f(x)\cdot g(c) - f(c)\cdot g(c) + f(c)\cdot g(c) - f(c)\cdot g(x)}{(x-c)\cdot g(x)\cdot g(c)}$$

$$= \lim_{x\to c} \frac{g(c)}{g(x)\cdot g(c)} \frac{f(x) - f(c)}{x-c} - \frac{f(c)}{g(x)\cdot g(c)} \frac{-g(c) + g(x)}{x-c}$$

$$= \frac{g(c)\,f'(c) - f(c)\,g'(c)}{(g(c))^2}$$

Chain Rule

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

let $f(c)=y$

$$g'(y) = \lim_{x \to y} \frac{g(x)-g(y)}{x-y}$$

$$= \lim_{f(x) \to f(c)} \frac{g(f(x)) - g(f(c))}{f(x)-f(c)}$$

$$g'(y) = \left[\lim_{x \to c} \frac{g(f(x))-g(f(c))}{x-c}\right] \cdot \frac{1}{\frac{f(x)-f(c)}{x-c}}$$

$$g'(f'(c)) = (g \circ f)'(c)$$

$$f'(c)$$

$$(g \circ f)'(c) = f'(c) \cdot g'(f'(c))$$

$f(x) = 2x$ $\qquad$ $f^{-1}(x) = x/2$. $\qquad$ $1/2$

$$f^{-1}(f(x)) = f^{-1}(2x)$$

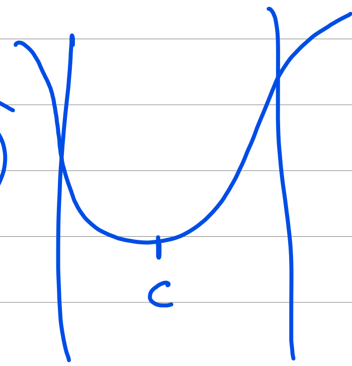
$$= \frac{\cancel{2}x}{\cancel{2}}$$

$$= x$$

even $f(x) = f(-x)$ odd $f(x) = -f(-x)$

$g(c) = f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = -g(-c)$

$f(x) \geq f(c)$



c has ~~relative~~ min^m

$f'(c) > 0$

$\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} > 0$

$\frac{f(x)-f(c)}{x-c} > 0$ $\forall x \in (c-\delta, c)$

$f(x)-f(c) < 0$

$f(x) \leq f(c)$

c relative min $(c-\delta, c+\delta)$ $f(c) \leq f(x)$

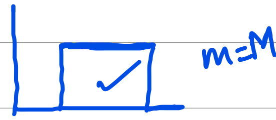
$c, c+\delta$

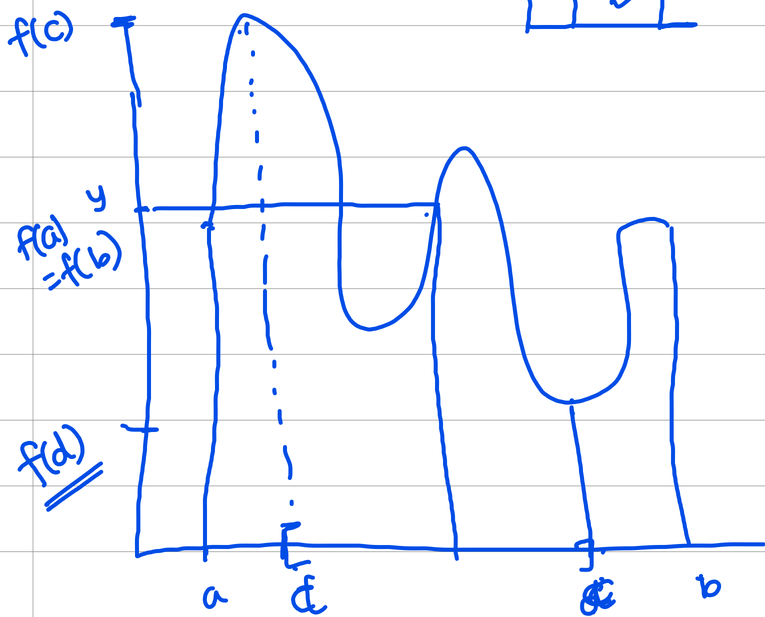
$f'(c) < 0$

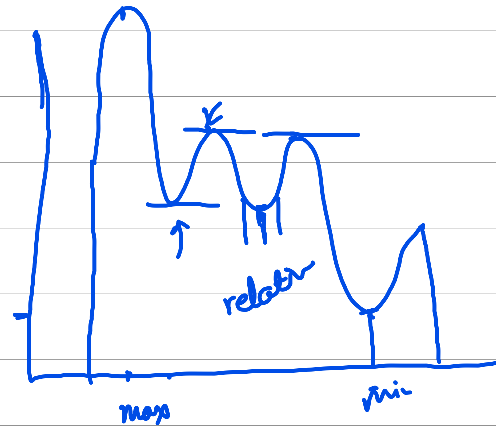
$\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} < 0$

$f(x)-f(c) < 0$ $\forall x \in (c, c+\delta)$

$f(x) \leq f(c)$







$m = \inf f(I)$
$= f(c)$

$M = \sup \{f(I)\}$
$= f(d)$

$f(c), f(d)$

y $[c, d]$

$f'(c) > 0$

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$

$x \in (c - \delta, c)$

$f(x) < f(c)$

$f'(c) < 0$

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} < 0$

$x \in (c, c + \delta)$

$f(x) < f(c)$

$f'(c) = 0$

which contradicts to our assumption that c is min

Rolle's Theo:- $f:[a,b]\to\mathbb{R}$

1. cont. $[a,b]$ ✓
2. deri (a,b) ✓
3. $f(a)=f(b)$? ✓

$\Rightarrow$ $\underline{f'(c)=0}$ for some $c\in(a,b)$

LMVT : $f:[a,b]\to\mathbb{R}$ $\quad \psi:[a,b]\to\mathbb{R}$

1. cont. $[a,b]$ ✓
2. deri (a,b) ✓

$\exists\ c\in(a,b) \ni f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Important step

$\psi(x)=f(x)-A\cdot x$ $\Rightarrow$ cont. $[a,b]$ $\Rightarrow$ deri (a,b)

$\psi(a)=f(a)-A\cdot a = f(b)-A\cdot b = \psi(b)$

$\Rightarrow A(b-a)=f(b)-f(a)$

$A=\dfrac{f(b)-f(a)}{b-a}$ ✓

$$\boxed{\psi(x)=f(x)-\frac{f(b)-f(a)}{b-a}\cdot x}$$

1. cont $[a,b]$ 2. deri (a,b) 3. $\psi(a)=\psi(b)$

by Rolle's theo. $\exists\ c\in(a,b)\ \ni\ \psi'(c)=0$

$$\psi'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}=0$$

$$\Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

CMVT : $f, g : [a,b] \to \mathbb{R}$

① f, g cont. $[a,b]$

② deri (a,b) ③ $g(a) \neq g(b)$ & $g'(x) \neq 0$

$\Rightarrow \exists\, c \in (a,b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\boxed{\psi(x) = f(x) - A \cdot g(x)}$$

$$\psi(a) = f(a) - A \cdot g(a) = f(b) - A \cdot g(b) = \psi(b)$$

$$\Rightarrow A\,(g(b) - g(a)) = f(b) - f(a)$$

$$A = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\psi(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g(x)$$

① ✓ ② ✓ ③ $\psi(a) = \psi(b)$

by Rolle's Theo:- $\exists\, c \in (a,b) \ni \psi'(c) = 0$

$$f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(c) = 0$$

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

QED

$f: I \to \mathbb{R}$ differentiable

① $f \uparrow$ iff $f'(x) \geq 0 \quad \forall x \in I$

② $f \downarrow$ iff $f'(x) \leq 0 \quad \forall x \in I$

① $f \uparrow$, if $x \leq y \Rightarrow f(x) \leq f(y)$

$\forall c \in I$, $f'(x) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

$\forall x \in (c, c+\delta)$, $c \leq x \Rightarrow f(c) \leq f(x)$

$$\frac{f(x)-f(c)}{x-c} \begin{matrix} >0 \\ >0 \end{matrix} > 0$$

$$= \lim_{x \to c^+} f'(x) \geq 0$$

$\forall x \in (c-\delta, c)$, $x \leq c \Rightarrow f(x) \leq f(c)$

$x - c \leq 0 \Rightarrow f(x) - f(c) \leq 0$

$$\frac{f(x)-f(c)}{x-c} \begin{matrix} <0 \\ <0 \end{matrix} > 0$$

$$\lim_{x \to c^-} f'(x) \geq 0$$

$$\boxed{f'(c) \geq 0} \qquad \forall c \in I$$

$$f(x) = f(x_0) + \sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{i!} \cdot (x - x_0)^i$$

$$\left. + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} \right\} \text{Rem}$$

$x_0 = 0$ $\quad \cos(0) = 1 \quad \sin(0) = 0$

$$\sin(x) = \sin(0) + \cos(0)(x) - \frac{\sin(0)}{2!} x^2 - \frac{\cos(0)}{3!} x^3 - \dots$$

$$= 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \log(1+x_0) + \frac{1}{1+x_0} \cdot x + \frac{-1}{(1+x_0)^2} \cdot \frac{x^2}{2!} + \frac{+2}{(1+x_0)^3} \cdot \frac{x^3}{3!} + \cdots$$

$x_0 = 0$

$$= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Sequence and series of fun^{c}s :-

$\frac{1}{n}$ $\quad \sum \frac{1}{n}$ $\quad f(x) = 2x^2+1$

$$f_n(x) = \frac{2x^2+nx}{n^2} = f(n,x)$$

$f_n : \mathbb{R} \to \mathbb{R}$ $\quad x \in \mathbb{R}, \ \forall n \in \mathbb{N}.$

seq^n of fun^{c}s

seq^n of fun^c

$$f_n(x) = \frac{x}{n} \qquad x \in A$$

series of fun^{c}s.

$$\sum_n \frac{x}{n} \qquad g(x) = \sum \frac{x}{n} = \sum_n f_n(x)$$

$\frac{1}{n} \to 0$

$f_n \to f$ — pointwise, uniformly

$x_n \to x$

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$ $\ni$ $|x_n - x| < \varepsilon$ $\forall n \geq K(\varepsilon)$

$f_n(x) \to f(x)$ $\qquad f_n : A \to \mathbb{R}$, $f : A_0 \to \mathbb{R}$

$A_0 \subseteq A$

pointwise convergence

for $x \in A_0$, ✓

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon, x) \in \mathbb{N}$? $\ni$ $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq K(\varepsilon, x)$

$f_n(x) = \frac{x}{n} \to f(x) = 0$ $\qquad x \in [0, a]$

$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| < \varepsilon$

$= \left|\frac{x}{n}\right| < \varepsilon$

$\ni \frac{|x|}{\varepsilon} < n$

$K(\varepsilon, x)$

$\left|\frac{x}{n}\right| \leq \left|\frac{a}{n}\right| < \varepsilon$

$\ni \frac{|a|}{\varepsilon} < n$

? $K(\varepsilon) < n$

uniform conti

$f_n(x) = x^{1/n}$ $\qquad x \geq 0$

$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$

pointwise convergence

$f_n(x) = \frac{x}{n}$ $\qquad f(x) = 0$ $\qquad x \in \mathbb{R}$

$$\left| f_{n_k}(x_k) - f(x_k) \right|$$

$$= \left| \frac{x_k}{n_k} - 0 \right|$$

$x_k = n_k \in \mathbb{R}$

$$\left| \frac{\cancel{n_k}}{\cancel{n_k}} \right| = 1$$

for any $\varepsilon > 0$

0.9

$$= 1 \geq \varepsilon_0$$

$g_n(x) = x^n$ $x \in \mathbb{R}$, $n \in \mathbb{N}$. $x \in (-1, 1]$ $g(x) = 0$

$$\left| g_{n_k}(x_k) - g(x_k) \right| = \left| x_k^{n_k} - 0 \right|$$

$$= \left| \left(\frac{1}{2}\right)^{\frac{1}{k} \times k} - 0 \right|$$

$$\frac{1}{2}$$

not uniformly conti

$$\|f\|_A = \sup \{ |f(x)|, x \in A \}$$

$f(x) \ni$ bounded

For any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$ $\ni$ $|f_n(x) - f(x)| < \varepsilon$ $\forall$ $n \geq K(\varepsilon)$

$$\varphi_n(x) = \left| f_n(x) - f(x) \right|$$

$$\|\varphi_n\| = \sup \{ |f_n(x) - f(x)|, \ x \in A_0 \}$$

$||\psi_n|| < \varepsilon$

$\sup\{|\psi_n(x)|, x \in A_0\} < \varepsilon$

$||f_n - f|| \to 0$

$\sup\{|f_n(x) - f(x)|, x \in A_0\} < \varepsilon$

$\Rightarrow \quad |f_n(x) - f(x)| < \varepsilon \quad \forall x \in A_0$

$\Rightarrow$ uniform conti

$f_n(x) = \frac{x}{n} \quad f(x) = 0 \qquad x \in \mathbb{R}$

$|f_n(x) - f(x)| = |x/n| \qquad x \in \mathbb{R}$

unbounded

$f_n(x) = \frac{x}{n} \qquad x \in (0, a] \rightarrow$

$|f_n(x) - f(x)| = \left|\frac{x}{n}\right| < \frac{a}{n} \qquad a > 0$

$\frac{a}{n} \to 0$

$f_n(x) \to f$ uniformly converges to f on A_0

$||f_n - f|| = ? \to 0$

$f_n(x) = x^n \quad (-1,1]$ $\quad f(x) = \begin{cases} 0 & -1 < x < 1 \\ 1 & x = 1 \end{cases}$

$-1 < x < 1$ $\quad |f_n(x) - f(x)| = |x^n - 0| \leq 1$

$= |x^n| \qquad x < 1$

$f_n(x) = x^n(1-x) \qquad [0,1]$

Cauchy Criterion for uniform convergence

for any $\varepsilon > 0$ $\exists K(\varepsilon) \in \mathbb{N}$. $\ni$ $|x_n - x_m| < \varepsilon$ $\forall n, m \geq K(\varepsilon)$

$\ni |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \geq K(\varepsilon)$

$\boxed{\|f_n - f_m\|_A \leq \varepsilon}$

① Uniform Con $f_n \rightrightarrows f \Rightarrow \|f_n - f_m\|_A < \varepsilon$

for any $\frac{\varepsilon}{2} > 0$ $\exists K(\varepsilon/2) \in \mathbb{N}$ $\ni$ $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq K(\varepsilon/2)$,

$\exists$ $|f_m(x) - f(x)| < \varepsilon$ - " —

For $n, m \geq K(\varepsilon/2)$.

$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\boxed{\sup|f_n(x) - f_m(x)| \leq \varepsilon} \rightarrow$

$\|f_n - f_m\| < \varepsilon \Rightarrow \sup|f_n(x) - f_m(x)| < \varepsilon \Rightarrow |f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in A_0$
$\forall n \geq K(\varepsilon)$

$\Rightarrow \{f_n(x)\}$ is cauchy seq^n.

$\Rightarrow$ Cauchy seq^n. is convergent & that converges to some $f(x)$

$$\lim_{n\to\infty} f_n(x) = f(x)$$

$\forall x \in A_0$, for any $\varepsilon > 0$ $\exists K(\varepsilon) \in \mathbb{N}$. $\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq K(\varepsilon)$.

$$\Rightarrow \sup_{x\in A_0}|f_n(x) - f(x)| < \varepsilon$$

$$= \|f_n(x) - f(x)\|_{A_0} < \varepsilon \quad \forall n \geq K(\varepsilon)$$

$$\Rightarrow \|f_n(x) - f(x)\|_{A_0} \to 0$$

$\|f_m - f_n\| \leq \varepsilon$

$f_n(x)$ uniformly converges to f.

f_n bounded seq^n of fun^n

$f_n \rightrightarrows f \quad \Longleftrightarrow \quad \|f_n - f_m\| \leq \varepsilon$

Series of functions

$f_n(x) : A \to \mathbb{R}$ $\to$ $f : A_0 \to \mathbb{R}$ $A_0 \subseteq \mathbb{R}$

✓ $x \in A_0$ for any $\varepsilon > 0$ $\exists$ $K(\varepsilon, x) \in \mathbb{N}$, $\ni |f_n(x) - f(x)| < \varepsilon$ $\forall n \geq K(\varepsilon, x)$

$\underline{\underline{K(\varepsilon)}}$ $K(\varepsilon)$

$\|f_n - f\|_{A_0} \to 0$

$S_1(x) = f_1(x)$

$S_2(x) = f_1(x) + f_2(x) = S_1(x) + f_2(x)$

$\vdots$

$S_n(x) = S_{n-1}(x) + f_n(x) = \sum_{i=1}^{n} f_i(x)$

$\vdots$

Cauchy Criterion for convergence of seq^n of fun^{c}s

$\|S_n(x) - S_m(x)\|_{A_0} \leq \varepsilon$

$\Rightarrow \left| \sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{m} f_i(x) \right| < \varepsilon$

$n > m$

$\Rightarrow |f_{m+1}(x) + f_{m+2}(x) + \dots + f_n(x)| < \varepsilon$

$\uparrow$

$\underline{\underline{\text{series}}}$

Weistrauss M_1 test

seq^n of funs $\rightarrow f_n(x) \le M_n$ ← Real seq^n

$f_n(x) = \frac{x}{n}$ $x \in [0,a]$

$\frac{x}{n} \le \frac{a}{n}$

M_n

① $|f_n(x)| \le M_n \quad \forall x \in D, \ \forall n \in \mathbb{N}$.

② $\sum M_n$ convergent

To prove → $\sum f_n(x)$ convergent

by Cauchy criterion, for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$ $\ni$

$|M_{m+1} + M_{m+2} + \ldots + M_n| < \varepsilon \quad \forall \ K(\varepsilon) < m < n$

$|a+b| \le |a| + |b|$

$|f_{m+1}(x)| + \ldots + |f_n(x)| < M_{m+1} + M_{m+2} + \ldots + M_n < \varepsilon$ — " —

$|f_{m+1} + f_{m+2} + \ldots + f_n| < \quad < \varepsilon$

Radius of convergence

$\rho = \lim\sup |a_n|^{1/n}$

$R = \begin{cases} \infty & \rho = 0 \\ 1/\rho & \rho \text{ finite } 0 < \rho < \infty \\ 0 & \rho = \infty \end{cases}$

$\sum a_n (x-c)^n$

$\sum n^n x^n$ $\quad a_n = n^n \quad c = 0$, $\rho = \lim\sup |n^n|^{1/n} = \infty$

$R = 0$

Cauchy Hadmard Theo: $\sum a_n x^n$ R

$(-R, R)$

$\sum x^n$ $a_n = 1$ $c = 0$, $\rho = \lim \sup |a_n|^{1/n}$

$= \lim \sup 1$

$= 1$

$|x| < 1$

$R = 1/\rho = 1$

$(-R, R) = (-1, 1)$

$\sum n! x^n$ $a_n = n!$ $\rho = \lim \sup |a_n|^{1/n}$

$= \lim \sup (n!)^{1/n}$

$=$

$=$

$R = 1/\rho$

$(-R, R)$

$R = \lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n!}{(n+1)!} = \lim_{n} \frac{1}{n+1} = 0$

$x = 0$

Power Series :- $\sum a_n (x-c)^n$

$$\rho = \lim\sup |a_n|^{1/n} \qquad R = \begin{cases} 0 & \rho = \infty \\ 1/\rho & 0 < \rho < +\infty \\ \infty & \rho = 0 \end{cases}$$

Cauchy Hadmard Theo.

$\sum a_n x^n$ converges only if $x \in (-R, R)$

$a_n = \frac{1}{n^n}$ $\qquad \rho = \lim\sup |a_n|^{1/n} = \lim\sup \left|\frac{1}{n^n}\right|^{1/n} = 0$

$S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ $\qquad \inf S = 0 \qquad \sup S = 1$

$\lim \frac{1}{n} \Rightarrow$ $\qquad \lim\inf \frac{1}{n} = 0, \quad \lim\sup \frac{1}{n} = 0$

$\lim (-1)^n \Rightarrow$ $\qquad \lim\inf (-1)^n = -1 \qquad \lim\sup (-1)^n = +1$

$a_n = \frac{n^n}{n!}$ $\qquad R = \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \left| \frac{n^n}{\cancel{n!}} \times \frac{\cancel{(n+1)!}}{(n+1)^{\cancel{n+1}}} \right|$

$$= \lim \left| \frac{n^n}{(n+1)^n} \right|$$

$$= \lim_{n \to \infty} \frac{1}{(1 + 1/n)^n} = \frac{1}{e}$$

$a_n = \frac{(n!)^2}{(2n)!}$

$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$

$= \lim \left| \frac{(n!)^2}{\cancel{(2n)!}} \times \frac{(2n+2)!}{((n+1)!)^2} \right|$

$= \lim \frac{2\cancel{(n+1)}(2n+1)\ \cancel{n!}\ \cancel{n!}}{\cancel{(n+1)!}\ \cancel{(n+1)!}}$

$= \lim \frac{2(2n+1)}{(n+1)}$

$= 4$

$\rightarrow (-4, 4)$

$a_n = (n!)^{1/n}$

$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$

$= \lim \left| \frac{(n!)^{1/n}}{((n+1)!)^{1/(n+1)}} \right|$

$= \lim (n!)^{1/n}$

$\rho = \infty$ $\underline{\underline{R = 0}}$

$\lim (n!)^{1/2n}$ $= \lim \exp(\log (n!)^{1/n})$

$\underline{\underline{\infty}}$ $= \lim \exp\left(\frac{1}{n} \log n!\right)$

?

o

$= \lim \exp\left(\frac{1}{n}(\log 1 + \log 2 + \ldots + \log n)\right)$

$= \lim \exp\left(\frac{1}{n} \int_1^n \log x \, dx \right)$

Riemann Integration / Integral

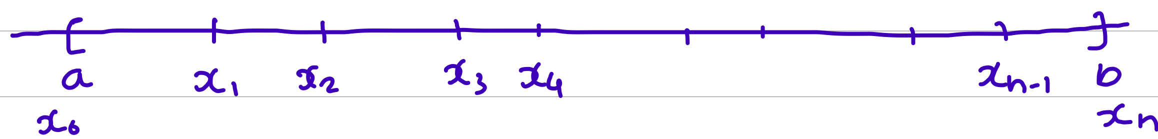
If $I = [a,b]$

partition of I, finite and ordered set

$$P = (x_0, x_1, x_2, \dots, x_n)$$

$$a = x_0 < x_1 < \dots\dots < x_n = b$$

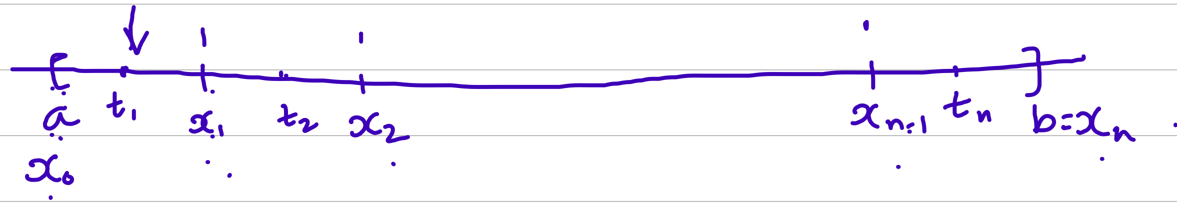
subintervals ✓ $[x_{i-1}, x_i]$ $\quad i = 1, \dots, n$



Partition $\quad P = \{ [x_{i-1}, x_i], \; i = 1:n, \; x_0 = a, x_n = b \}$

Tagged Partition $\quad \dot{P} = \{ ([x_{i-1}, x_i], t_i), \; i = 1:n, \; x_0 = a, x_n = b \}$

$h \in x_i, x_{i+1}$

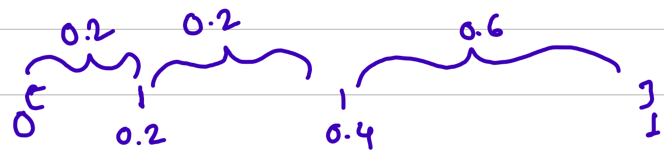


$$t_i \in [x_{i-1}, x_i]$$

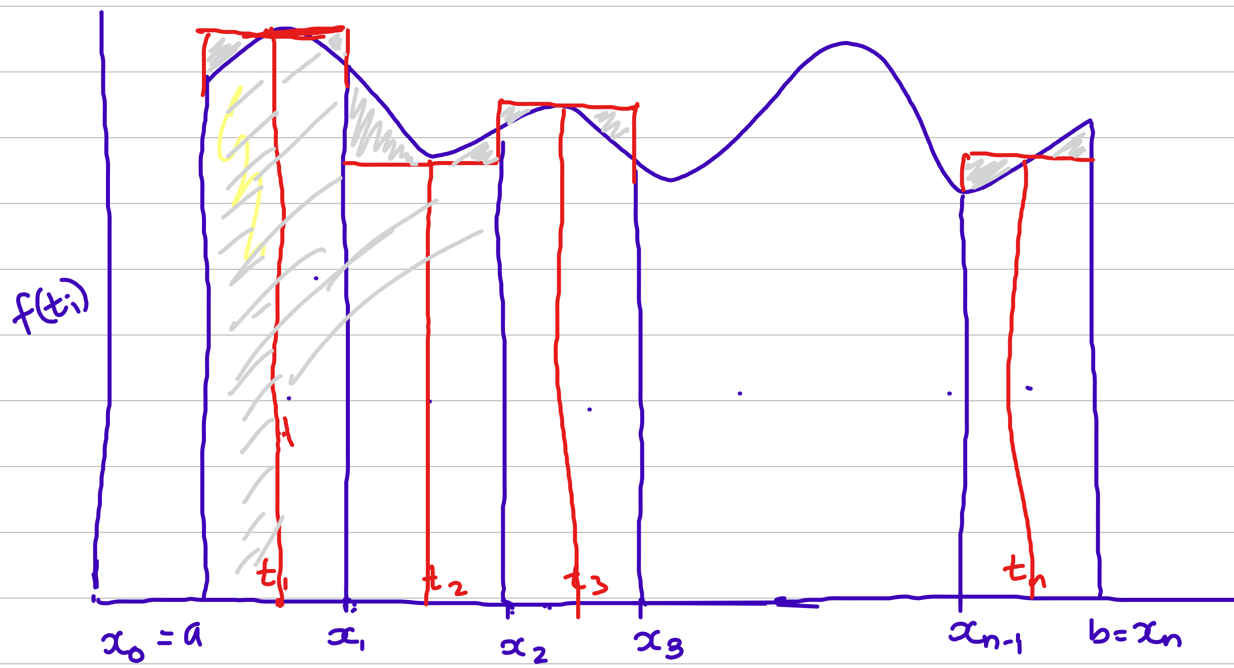
$[x_{i-1}, x_i]$

$$\max\{x_i - x_{i-1},\ i = 1:n\} < \delta$$

δ- fine partition



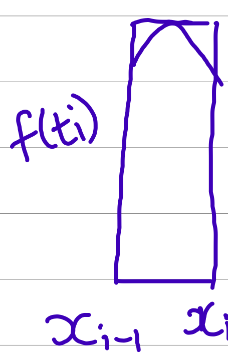
0.6 fine parti·



$\int_a^b f$

$$\dot{P} := \{[x_{i-1}, x_i], t_i\}$$

Riemann Sum



$$S(f;\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) :\simeq \int_a^b f(x)\,dx$$

* norm of partition $||\dot{p}|| = \max\{x_i - x_{i-1},\ i = 1:n\}$

* Riemann integrable fun^c

$f : [a,b] \to \mathbb{R}$. $\exists\ L \in \mathbb{R}$

for any $\varepsilon > 0$ $\exists\ \delta_\varepsilon > 0$, $\exists\ \dot{P} \ni ||\dot{p}|| < \delta_\varepsilon$

$\Rightarrow |S(f;\dot{P}) - L| < \varepsilon$

$f \in \mathcal{R}[a,b]$ ← collection of Riemann integrable fun^{c}s.

$$L \cong \int_a^b f\,dx$$

Uniqueness

L_1 & L_2 limit

for any $\varepsilon_{/2} > 0$ $\exists\ \delta_{\varepsilon_{/2}} > 0 \ni ||\dot{p}_1|| < \delta_{\varepsilon/2}$ ✓

$\Rightarrow |S(f;\dot{P}_1) - L_1| < \varepsilon$

for any $\varepsilon_{/2} > 0$ $\exists\ \delta_{\varepsilon_{/2}} > 0 \ni ||\dot{p}_2|| < \delta_{\varepsilon/2}$ ✓

$\Rightarrow |S(f;\dot{P}_2) - L_2| < \varepsilon/2$

let $\delta_\varepsilon > 0$, $\delta_\varepsilon = \min(\delta^1_{\varepsilon/2}, \delta^2_{\varepsilon/2}) \Rightarrow ||\dot{p}|| < \delta_\varepsilon$

$$|L_1 - L_2| = |L_1 - S(f;\dot{P}) + S(f;\dot{P}) - L_2|$$

$$\leq |L_1 - S(f;\dot{P})| + |S(f;\dot{P}) - L_2|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$\Rightarrow \quad L_1 = L_2$$

$f : [a,b] \to \mathbb{R}$ $\qquad f(x) = c$

$$\dot{P} = \{(\{[x_{i-1}, x_i], t_i\}_{i=1}^{n}\}$$

$$S(f;\dot{P}) = \sum_{i=1}^{n} f(t_i) \cdot [x_i - x_{i-1}]$$

$$= c \sum_{i=1}^{n} [x_i - x_{i-1}]$$

$$= c \cdot [x_n - x_0]$$

$$= c[b-a]$$

Every constant fun^c is Riemann integrable

$$\int_a^b kf = k\int_a^b f$$

$f \in R[a,b]$ ✓ L

$kf \in R[a,b]$

assume $k > 0$

✓ For any $\frac{\varepsilon}{|k|} > 0$ $\exists\ \delta_\varepsilon > 0$ $\ni \|\dot{p}\| < \delta_\varepsilon$

$\Rightarrow |S(f;\dot{p}) - L| < \varepsilon/|k|$ ✓

$|S(kf;\dot{P}) - kL|$

$= \left| \sum_{i=1}^{n} kf(t_i)\cdot(x_i - x_{i-1}) - kL \right|$

$= |k| \left| \sum f(t_i)(x_i - x_{i-1}) - L \right|$

$= |k|\, |S(f;\dot{P}) - L|$

$< \cancel{|k|}\dfrac{\varepsilon}{\cancel{|k|}} \leq \varepsilon$ $\Rightarrow$ $kf \in R[a,b]$

$$\int_a^b f \pm g =$$

$f, g \in R[a,b]$

$S(f;\dot{p}) \to \int_a^b f$

$f(x) \leq g(x)$ $\forall x \in [a,b]$

$S(g;\dot{p}) \to \int_a^b g$

$$\int_a^b f \leq \int_a^b g$$

$\varepsilon > 0 \quad \delta_\varepsilon > 0 \quad \ni \quad \|\dot{p}\| < \delta_\varepsilon$.

$$\left.\begin{aligned} S(f;\dot{p}) &= \sum f(t_i)\cdot(x_i - x_{i-1}) \\ S(g;\dot{p}) &= \sum g(t_i)\,(x_i - x_{i-1}) \end{aligned}\right\}$$

$f(t_i) \le g(t_i) \quad \Rightarrow \quad i = 1:n$

$$\int_a^b f \;\le\; \int_a^b g$$

$f \in R[a,b]$ bounded

unbounded

$\dot{p} = \{[x_{i-1}, x_i], t_i\}$ $\qquad \|\dot{p}\| < \delta_\varepsilon$

for any $\varepsilon > 0 \ \exists \ \delta_\varepsilon > 0 \quad \|\dot{p}\| < \delta_\varepsilon \quad \Rightarrow |S(f;\dot{P}) - L| < \varepsilon$

$\|Q\| < \delta_\varepsilon \qquad \Rightarrow |S(f;Q) - L| < \varepsilon$?

$t_i = x_i \quad i = 1:n$ except k

$|f|$ is not bounded in $[x_{k-1}, x_k]$

$$|S(f;Q)-L|$$

$$= \left|\sum f(t_i)(x_i-x_{i-1}) - L\right|$$

$$= \left|f(t_k)\cdot(x_k-x_{k-1}) + \sum_{\substack{i=1\\ i\neq k}}^{n} f(t_i)(x_i-x_{i-1}) - L\right|$$

$$|f(t_k)\cdot(x_k-x_{k-1})| \geq |L| + \left|\sum_{\substack{i=1\\ i\neq k}}^{n} f(t_i)(x_i-x_{i-1})\right|$$

$$S(f;Q) \geq |f(t_k)(x_k-x_{k-1})| \geq |L|+\varepsilon$$

$$\geq \varepsilon_0$$

$\therefore$ Our assumption is wrong if $f\in R[a,b]$
f cannot be unbounded.

Cauchy Criterion

$f\in R[a,b]$

for any $\varepsilon>0$ $\exists$ $\delta_\varepsilon>0$ $\ni$ $\|\dot{P}\|<\delta_\varepsilon$, $\|\dot{Q}\|<\delta_\varepsilon$

$$\Rightarrow |S(f;\dot{P})-S(f;\dot{Q})|<\varepsilon$$

$f\in R[a,b]$

For any $\frac{\varepsilon}{2} > 0$ $\exists\ \delta_\varepsilon > 0$ $\ni$ $||\dot{P}|| < \delta_\varepsilon$

$$|S(f;\dot{P}) - L| < \varepsilon/2$$

for any $\frac{\varepsilon}{2} > 0$ $\exists\ \delta_\varepsilon > 0$ $\ni$ $||\dot{Q}|| < \delta_\varepsilon$

$$|S(f;\dot{Q}) - L| < \varepsilon/2$$

$$\begin{aligned} |S(f;\dot{P}) - S(f;\dot{Q})| &= |S(f;\dot{P}) - L + L - S(f;\dot{Q})| \\ &\leq |S(f;\dot{P}) - L| + |L - S(f;\dot{Q})| \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &\leq \varepsilon \end{aligned}$$

Squeeze Theo. $a_n \leq b_n \leq c_n$ ✓

$$\lim a_n = \lim c_n = L$$

$$\Rightarrow \lim b_n = L$$

$f : [a,b] \to \mathbb{R}$, $f \in \mathcal{R}[a,b]$

$\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{R}[a,b]$, $\alpha_\varepsilon(x) \leq f(x) \leq \beta_\varepsilon(x)$

$$\int_a^b (\beta_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

$$\Rightarrow f \in \mathcal{R}[a,b]$$

$\alpha_\varepsilon \in R[a,b]$, $\beta_\varepsilon \in R[a,b]$

for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $\|\dot{p}\| < \delta_\varepsilon$

$$\left| S(\alpha_\varepsilon ; \dot{p}) - \int_a^b \alpha_\varepsilon \right| < \varepsilon \qquad \left| S(\beta_\varepsilon ; \dot{p}) - \int_a^b \beta_\varepsilon \right| < \varepsilon$$

$$\int_a^b \alpha_\varepsilon - \varepsilon \leq S(\alpha_\varepsilon ; \dot{p}) \qquad S(\beta_\varepsilon ; \dot{p}) \leq \int_a^b \beta_\varepsilon + \varepsilon$$

$$\int_a^b \alpha_\varepsilon - \varepsilon \leq S(\alpha_\varepsilon ; \dot{p}) \leq S(f ; \dot{p}) \leq S(\beta_\varepsilon ; \dot{p}) \leq \int_a^b \beta_\varepsilon + \varepsilon$$

$$\int_a^b \alpha_\varepsilon - \varepsilon \leq S(f ; \dot{p}) \leq \int_a^b \beta_\varepsilon + \varepsilon$$

similarly

$$\int_a^b \alpha_\varepsilon - \varepsilon \leq S(f ; \dot{Q}) \leq \int_a^b \beta_\varepsilon + \varepsilon$$

$$\left| S(f ; \dot{p}) - S(f ; \dot{Q}) \right|$$

$$\leq \int_a^b \beta_\varepsilon + \varepsilon - \int_a^b \alpha_\varepsilon + \varepsilon$$

$$\leq \int_a^b (\beta_\varepsilon - \alpha_\varepsilon) + 2\varepsilon = 3\varepsilon$$

$$\Rightarrow f \in R[a,b]$$

$f:[a,b]\to\mathbb{R}$ cont.

To show :- $f\in\mathcal{R}[a,b]$. $\int_a^b f \leftarrow S(f;\dot{P})\rightarrow$

$f:[a,b]\to\mathbb{R}$ cont. $\Rightarrow$ As fun^c is cont. on closed bounded interval $\Rightarrow$ f is uniforly cont. on $[a,b]$

$a<b$, $b-a>0$
for any $\frac{\varepsilon}{b-a}>0$, $\exists\ \delta_\varepsilon>0 \ni |x-y|<\delta_\varepsilon \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{b-a}$

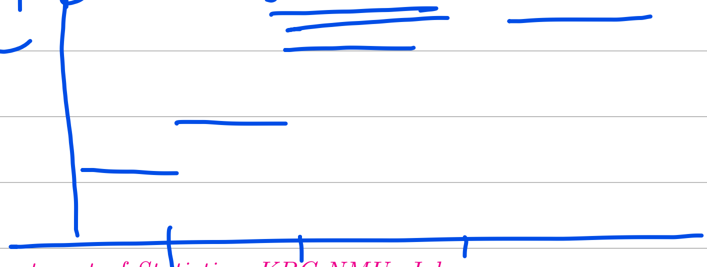
$\dot{P}=\left\{\{[x_{i-1},x_i];t_i\}_{i=1}^{n}\right\}$, $\|\dot{P}\|<\delta_\varepsilon$

f: cont $[a,b]$ $\Rightarrow$ by max^m min^m theo.
inf & sup exists & in $[a,b]$

f cont $[x_{i-1},x_i]$ $\Rightarrow$ Assume that f attend t_i^1 min^m
its max^m at t_i^2

$f(t_i^1)\le f(x)\le f(t_i^2)$ $\forall\ x\in[x_{i-1},x_i]$
$\uparrow$ min $\uparrow$ max

$\alpha_\varepsilon(x)=f(t_i^1)$. $x\in[x_{i-1},x_i]$ $\forall\ i=1:n$
$\beta_\varepsilon(x)=f(t_i^2)$. $x\in[x_{i-1},x_i]$ $\forall\ i=1:n$



$$\left[\ \alpha_\varepsilon(x) \le f(x) \le \beta_\varepsilon(x) \quad \forall x \in [a,b]\right.$$

$$0 \le \int_a^b (\beta_\varepsilon - \alpha_\varepsilon)(x) = \sum_{i=1}^{n} \left[\beta(t_i^2) - \alpha(t_i^1)\right](x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \left(f(t_i^2) - f(t_i^1)\right)(x_i - x_{i-1})$$

$t_i^1, t_i^2 \in [x_{i-1}, x_i]$

$$|t_i^2 - t_i^1| \le |x_i - x_{i-1}| < \delta_\varepsilon$$

by defn of uniform conti.

$$|t_i^2 - t_i^1| < \delta_\varepsilon \Rightarrow |f(t_i^2) - f(t_i^1)| < \frac{\varepsilon}{b-a}$$

$$\le \sum \frac{\varepsilon}{(b-a)} (x_i - x_{i-1})$$

$$\le \frac{\varepsilon}{b-a} \sum (x_i - x_{i-1})$$

$$0 \le \int_a^b (\beta_\varepsilon - \alpha_\varepsilon)(x) \le \varepsilon$$

by squeeze theo $\Rightarrow f \in R[a,b]$

$f : [a,b] \to \mathbb{R}$ monotone

let f is $\uparrow$

$[x_{i-1}, x_i]$ $\quad f(x_{i-1}) \leq f(x) \leq f(x_i)$

$\alpha_\varepsilon(x) = f(x_{i-1}) \quad \forall\, x \in [x_{i-1}, x_i]$

$\beta_\varepsilon(x) = f(x_i) \quad \forall\, x \in [x_{i-1}, x_i]$

$\alpha_\varepsilon(x) \leq f(x) \leq \beta_\varepsilon(x) \quad \forall\, x \in [x_{i-1}, x_i]$

equispaced $h = \frac{b-a}{n}$

$$x_i - x_{i-1} = h = \frac{b-a}{n} \quad i = 1:n.$$

$$S(\alpha_\varepsilon ; \dot{P}) \cong \int_a^b \alpha_\varepsilon = \sum_{i=1}^{n} \alpha_\varepsilon(t_i) \cdot \underline{(x_i - x_{i-1})}$$

$$= \sum_{i=1}^{n} f(x_{i-1}) \cdot \left(\frac{b-a}{n}\right)$$

$$= \frac{(b-a)}{n} \sum_{i=1}^{n} f(x_{i-1}) .$$

$$S(\beta_\varepsilon ; \dot{P}) \cong \int_a^b \beta_\varepsilon = \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f(x_i) .$$

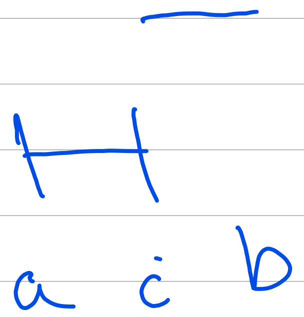
$$0 \le \int_a^b \beta_\varepsilon - \alpha_\varepsilon = \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \underline{f(x_i) - f(x_{i-1})}$$

$$= \left(\frac{b-a}{n}\right) \left(f(x_n) - f(x_0)\right)$$

$$= \left(\frac{b-a}{n}\right) \left(f(b) - f(a)\right)$$

we will choose $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$

$$\underline{0 \le \int_a^b \beta_\varepsilon - \alpha_\varepsilon < \varepsilon} \quad \checkmark \qquad f \in R[a,b]$$



$$\int_a^b f = \int_a^c f + \int_c^b f$$

* Fundamental Theo. of Integral Calculus

$f : [a,b] \to \mathbb{R}$ E finite subset $\subseteq [a,b]$

① F cont $[a,b]$

② $F'(x) = f(x)$ $x \in [a,b] \setminus E$.

③ $f \in \mathcal{R}[a,b]$.

$$\int_a^b f = F(b) - F(a)$$

→ $E = \{a,b\}$

① if $f \in \mathcal{R}[a,b]$,

for any $\varepsilon > 0$ $\exists\, \delta_\varepsilon > 0 \ni \|p\| < \delta_\varepsilon$

$$\Rightarrow \left| S(f;p) - \int_a^b f \right| < \varepsilon$$

$$S(f;p) = \sum_{i=1}^{n} f(t_i) \cdot (x_i - x_{i-1})$$

① $F:[a,b]\to\mathbb{R}$ cont. $[a,b]$ & derivable (a,b)

* by LMVT

$$F'(c)=\frac{F(b)-F(a)}{b-a}, \quad c\in(a,b)$$

f cont $[x_{i-1},x_i]$ deriv (x_{i-1},x_i)

$$F'(t_i)=\frac{F(x_i)-F(x_{i-1})}{(x_i-x_{i-1})}$$

$$F'(t_i)\cdot(x_i-x_{i-1}) = F(x_i)-F(x_{i-1})$$

$$\sum_{i=1}^{n} F'(t_i)\,(x_i-x_{i-1}) = F(b)-F(a)$$

$$\sum_{i=1}^{n} f(t_i)\,(x_i-x_{i-1}) = F(b)-F(a)$$

$$S(f;P) = F(b)-F(a)$$

$$\left| F(b)-F(a) - \int_a^b f \right| < \varepsilon$$