

Real Analysis

Manoj C Patil

November 8, 2021

Contents

1	Introduction to Real Analysis	7
1.1	The Algebraic Properties of \mathbb{R}	7
1.2	The Order Properties of \mathbb{R}	9
1.3	Absolute Value and Real Line	12
1.4	Triangular Inequality	15
1.5	Completeness Property	28
2	Sets Operations	33

2.1	Set Operations	33
2.2	Distributive Law	38
2.3	Basic Notatioons Theory	44
2.4	Archemedian Property	64
2.5	Cauchy Schwartz Inequality	73
3	Elements of Point Set Topology	77
3.1	Terminology and Notations	77
3.2	Compact Set	96
3.3	Heine Borel theorem	98
4	Sequence and Series	101
4.1	Squeeze Theorem	116
4.2	Monotone Sequence	120

4.3 Cauchy Sequence	126
4.4 Infinite Series	129
4.5 Establish the converges/divergence of series	141
4.6 Test for Non-Absolute Convergence	152
5 Function and Continuity	159
5.1 Continuous Function	169
5.2 Continuous function on Interval	174
5.3 Continuity	183
5.4 Continuity And Gauges	190
6 Differentiation	197
6.1 Derivative	197
6.2 Chain Rule	205

6.3 Taylor's Theorem	218
6.4 Maximum or Minimum for function of two variables	221
7 Sequence and Series of Function	225
7.1 Sequence of Function	225
7.2 Cauchy Criteria for Uniform Convergence	234
7.3 Series of Function	236
8 Riemann Integral	243
8.1 Introduction	243
8.2 Some Properties of Integral	247
8.3 Fundamental theorem of Integral calculus	254
8.4 Indefinite Integral	255
8.5 Examples	256

Introduction to Real Analysis

1.1 The Algebraic Properties of \mathbb{R}

Algebraic Properties of \mathbb{R} On the set \mathbb{R} of real numbers there are two binary operations, denoted by $+$ and \cdot and called addition and multiplication, respectively. These operations satisfy the following properties :

- (A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$ (commutative property of addition);
- (A2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$ (associative property of addition) ;

- (A3) there exists an element $0 \in \mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all $a \in \mathbb{R}$ (existence of a zero element) ;
- (A4) for each $a \in \mathbb{R}$ there exists an element $a \in \mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (existence of negative elements) ;
- (M1) $ab = ba$ for all $a, b \in \mathbb{R}$ (commutative property of multiplication) ;
- (M2) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$ (associative property of multiplication) ;
- (M3) there exists an element $1 \in \mathbb{R}$ distinct from 0 such that $1a = a$ and $a1 = a$ for all $a \in \mathbb{R}$ (existence of a unit element) ;
- (M4) for each $a \in \mathbb{R} - \{0\}$ there exists an element $1/a \in \mathbb{R}$ such that $a(1/a) = (1/a)a = a$ and
- (D) $a(b + c) = (ab) + (ac)$ and $(b + c)a = (ba) + (ca)$ for all $a, b, c \in \mathbb{R}$ (distributive property of multiplication over addition).

1.2 The Order Properties of \mathbb{R}

There is a nonempty subset \mathbb{R}^+ of \mathbb{R} , called the set of positive real numbers, that satisfies the following properties :

1. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}$.
 2. If $a, b \in \mathbb{R}^+$, then ab belongs to \mathbb{R} .
 3. If $a \in \mathbb{R}$, then exactly one of the following holds : $a \in \mathbb{R}^+$ OR $a = 0$ OR $(-a) \in \mathbb{R}^+$.
1. Let $a, b, c \in \mathbb{R}$

if $a > b$ and $b > c$ then $a > c$

Given that,

$$a > b \text{ and } b > c$$

$$\therefore a - b > 0 \text{ and } b - c > 0 \cdots \text{i.e } (a - b), (b - c) \in \mathbb{R}^+$$

$$\therefore (a - b) + (b - c) > 0 \cdots (1^{st} \text{ order prop})$$

$$\therefore a - c > 0 \Rightarrow a > c$$

2. If $a > b$ then $a + c > b + c$

Given that,

$$a > b \text{ i.e. } a - b > 0$$

$$\therefore a - b \in \mathbb{R}^+$$

$$\therefore a + c - c - b > 0$$

$$\therefore (a + c) - (b + c) > 0$$

$$\therefore a + c > b + c$$

3. If $a > b$ and $c > 0$ then, $ca > cb$

Given that, $a > b$ & $c > 0 \therefore (a - b) > 0 \& c > 0$

$$\text{i.e. } (a - b), c \in \mathbb{R}^+$$

$$\therefore (a - b) \cdot c \in \mathbb{R}^+ \dots (2^{\text{nd}} \text{ order prop})$$

$$\therefore (a - b) \cdot c > 0 \Rightarrow a \cdot c - bc > 0 \Rightarrow ac > bc$$

4. If $a > b$ and $c < 0$ then, $ca < cb$

Given that, $a > b$ & $c < 0$

$\therefore (a - b) \in \mathbb{R}^+ \& -c \in \mathbb{R}^+ \dots (3^{\text{rd}} \text{ order prop})$

$\therefore -c(a - b) > 0$

$\therefore -ca + cb > 0$

$\therefore cb > ca$

$\therefore ca < cb$

1.3 Absolute Value and Real Line

Absolute value and Real line

Absolute value:-

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Theorem 1.3.1. For $a, b \in \mathbb{R}$

a) $|ab| = |a|.|b| \forall a, b \in \mathbb{R}$

b) $|a|^2 = a^2 \forall a \in \mathbb{R}$

c) if $c > 0$ then $|a| \leq c$ iff $-c \leq a \leq c$

d) $-|a| \leq a \leq |a| \forall a \in \mathbb{R}$

Proof. a) $|ab| = |a| \cdot |b| \forall a, b \in \mathbb{R}$

- if $a = 0$ or $b = 0 \Rightarrow ab = 0 = |ab| = |a| \cdot |b|$

- if $a > 0$ or $b > 0 \Rightarrow ab > 0$

$$|ab| = a \cdot b = |a| \cdot |b|$$

- if $a > 0$ or $b < 0 \Rightarrow ab < 0$

$$\therefore |ab| = -a \cdot b = (-a) \cdot b = a \cdot (-b) = |a| \cdot |b|$$

- if $a < 0$ or $b > 0 \Rightarrow ab < 0$

$$\therefore |ab| = -ab = |a| \cdot |b|$$

- if $a < 0$ or $b < 0 \Rightarrow ab > 0$

$$\therefore |ab| = ab = |a| \cdot |b|$$

- Hence proved -

b) $|a|^2 = a^2 \forall a \in \mathbb{R}$

$$\forall a \in \mathbb{R}, a^2 \in \mathbb{R} \text{ i.e } a^2 \geq 0$$

$$\text{let } |a^2|^2 = |a| \cdot |a| = a \cdot a = a^2, \text{ if } a > 0$$

$$(-a) \cdot (-a) = a^2, \text{ if } a < 0$$

$$\text{Hence, } |a|^2 = a^2$$

c) if $c > 0$ then $|a| \leq c$ iff $-c \leq a \leq c$

Given that,

$$c > 0 \text{ & } |a| \leq c$$

i) To show $-c \leq a \leq c$

$$\text{Now, } |a| = \max(a, -a) \leq c$$

$$\Rightarrow a \leq c \& -a \leq c$$

$$\Rightarrow a \leq c \& a \geq -c$$

$$-c \leq a \leq c$$

ii) Given that, $-c \leq a \leq c$ & To show:- $|a| \leq c$

$$\Rightarrow a \leq c \& -a \leq -c$$

$$\therefore |a| \leq c \dots (|a| = \max(a, -a))$$

d) For $a \neq 0 \in \mathbb{R}$, $|a| > 0 \dots |a| = \max(a, -a)$

Put $c = |a| > 0$ in c)

$$\therefore -c \leq a \leq c \Rightarrow -|a| \leq a \leq |a|$$

□

1.4 Triangular Inequality

Triangular Inequality:-

Theorem 1.4.1. If $a, b \in \mathbb{R}$ then $|a + b| \leq |a| + |b|$

Proof. if $a, b \in \mathbb{R}$ then

$$-|a| \leq a \leq |a|$$

+

$$-|b| \leq b \leq |b|$$

$$-(|a| + |b|) \leq a + b \leq (|a| + |b|) \dots (\text{Theorem:-1.1.1-d}))$$

let $|a| + |b| = c$

$$\therefore -c \leq a + b \leq c$$

$$\Rightarrow |a + b| \leq c \dots \dots (Th^m - 1.1.1 - c)$$

$$\therefore |a + b| \leq |a| + |b|$$

□

Corollary 1.4.1.1. If $a, b \in \mathbb{R}$

a) $||a| - |b|| \leq |a - b|$

b) $|a - b| \leq |a| + |b|$

Proof. a) We know that, $a, b \in \mathbb{R}$

$$a = a - b + b$$

$$\therefore |a| = |a - b + b| \leq |a - b| + |b|$$

$$\Rightarrow |a| - |b| \leq |a - b| \quad (1.1)$$

Also, $b = b - a + a$

$$|b| = |b - a + a| \leq |b - a| + |a|$$

$$\therefore |b| - |a| \leq |a - b| \quad (1.2)$$

from equation (1.1) & (1.2)

$$||a| - |b|| \leq |a - b|$$

b)

if $a, b, c \in \mathbb{R}$

$$\therefore |a + c| \leq |a| + |c|$$

Put $c = -b$, $|c| = |-b| = |b|$

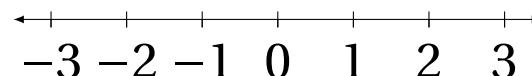
$$\therefore |a + (-b)| \leq |a| + |-b|$$

$$\therefore |a - b| \leq |a| + |b|$$

□

Corollary 1.4.1.2. If $a_1, a_2 \dots a_n$ are any real no then $|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$

Definition 1.4.1 (Real line): A convenient and Familiar interpretation of real no system is the real line.



Definition 1.4.2 (ϵ -Neighbourhood:-): let $a \in \mathbb{R}$ & $\epsilon > 0$, then ϵ - neighbourhood of a is the set

$$V_\epsilon(a) = \{x | x \in \mathbb{R}, |x - a| < \epsilon\} \dots 0 \leq |x - a| < \epsilon$$

$$\therefore V_\epsilon(a) = (a - \epsilon, a + \epsilon) \subseteq \mathbb{R}$$

Since $|x - a| < \epsilon \Rightarrow -\epsilon < x - a < \epsilon \Rightarrow a - \epsilon < x < a + \epsilon$

Definition 1.4.3 (Deleted- ϵ -Neighbourhood:-): $\delta_\epsilon(a) = v_\epsilon(a) - \{a\}$

$$= (a - \epsilon, a + \epsilon) - \{a\}$$

$$i.e. 0 < |x - a| < \epsilon$$

Example 1:

If $a, b \in \mathbb{R}$. Show that $|a + b| = |a| + |b|$ if and only if $ab \geq 0$

Proof. i) Given that $ab \geq 0$, To prove- $|a + b| = |a| + |b|$

if $ab \geq 0 \Rightarrow a \geq 0, b \geq 0$ or $a \leq 0, b \leq 0$

$$a + b \geq 0$$

$$\therefore |a| = a, |b| = b$$

$$|a + b| = a + b$$

$$= |a| + |b|$$

$$a + b \leq 0$$

$$\therefore |a| = -a, |b| = -b$$

$$|a + b| = -(a + b)$$

$$= -a - b$$

$$= |a| + |b|$$

ii) Given that $|a + b| = |a| + |b|$, To prove $ab \geq 0$

$$|a + b|^2 = (|a| + |b|)^2$$

$$\therefore a^2 + b^2 + 2ab = |a|^2 + |b|^2 + 2 \cdot |a| \cdot |b|$$

$$\therefore 2ab = 2|a|.|b| \dots (\because |a|^2 = a^2)$$

$$ab = |a| \cdot |b|$$

$$ab = |ab| \dots (\text{Theorem:- 1.1.1-a}))$$

$$\therefore ab \geq 0$$

□

Example 2:

Show that if $a, b \in \mathbb{R}$ then

i) $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|), \min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$

ii) $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$

Proof. i) let $a > b \Rightarrow |a - b| = a - b$

$$\max(a, b) = a \quad (1.3)$$

Consider, RHS

$$\begin{aligned} &= \frac{1}{2}(a + b - |a - b|) \\ &= \frac{1}{2}(a + b + a - b) \dots \text{from (1.3)} (a - b) \geq 0 \\ &= a \\ &= \text{LHS} \end{aligned}$$

$$\text{let } \min(a, b) = b \quad (1.4)$$

Consider, RHS = $\frac{1}{2}(a + b - |a - b|)$

$$= \frac{1}{2}(a + b - (a - b))$$

$$= b$$

$$= \text{LHS}$$

ii) Suppose, $a > b > c$

$$\text{LHS} = \min\{a, b, c\} = c$$

$$\text{RHS} = \min\{\min\{a, b\}, c\} = \min\{b, c\}$$

$$\text{RHS} = C$$

$$= \text{LHS}$$

$$\text{Hence, } \min\{a, b, c\} = \min\{\min\{a, b\}, c\}$$



Example 3:

If $x, y, z \in \mathbb{R}$ & $x \leq z$, Show that $x \leq y \leq z$ if and only if $|x - y| + |y - z| = |x - z|$

$$x \leq z \Rightarrow x - z \leq 0 \therefore |x - z| = z - x$$

Proof. i) Given that $x \leq y \leq z, x, y, z \in \mathbb{R}$

$$\therefore |x - y| = y - x \& |y - z| = z - y$$

To show $|x - y| + |y - z| = |x - z|$

$$\text{Consider, LHS} = |x - y| + |y - z|$$

$$= y - x + z - y$$

$$= z - x$$

$$= |x - z|$$

$$= \text{RHS}$$

ii) Given that $|x - y| + |y - z| = |x - z|$

To show, $x \leq y \leq z$

$$\text{let } a = (x - y), b = (y - z)$$

$$\therefore |(x - y) + (y - z)| = |x - y| + |y - z|$$

$$\Rightarrow (x - y)(y - z) \geq 0 \dots (\because \text{if } |a + b| = |a| + |b| \Leftrightarrow ab \geq 0)$$

$$\therefore a, b \geq 0$$

$$\text{i.e. } (x - y), (y - z) \geq 0$$

$$x \geq y, y \geq z$$

$$\therefore x \geq y \geq z$$

which is not possible Since $x \leq z$ -(given)

$$a, b \leq 0$$

$$(x - y), (y - z) \leq 0$$

$$\therefore x \leq y, y \leq z$$

$$\therefore x \leq y \leq z$$



Example 4:

If $a < x < b, a < y < b$. Show that $|x - y| < b - a$.

Proof. Given that,

$$a < x < b, a < y < b$$

$$0 < x - a < b - a \tag{1.5}$$

$$0 < y - a < b - a \tag{1.6}$$

multiplying by (-1) to (1.6) and add in (1.5)

$$\begin{array}{r} 0 \leq -a \leq b - a \\ + \quad -(b - a) \leq a - y \leq b - a \\ \hline -(b - a) \leq x - y \leq (b - a) \Rightarrow |x - y| < (b - a) \end{array}$$

□

Definition 1.4.4 (Upper bound): Let $S \neq \emptyset \subseteq \mathbb{R}$, the set s is said to be bounded above if $\exists a \in \mathbb{R} \stackrel{s.t}{\Rightarrow} x \leq a \forall x \in S$ Each such ' a ' is called as upper bound of S .

Definition 1.4.5 (Lower bound): Let $S \neq \emptyset \subseteq \mathbb{R}$. The set S is said to be bounded below if $\exists b \in \mathbb{R} \stackrel{s.t}{\Rightarrow} x \geq b \forall x \in S$ Each such b is called as lower bound of S .

Definition 1.4.6 (Bounded Set): If both lower and upper bound exist.

Definition 1.4.7 (Unbounded set): If set S is not bounded.

Definition 1.4.8 (Supremum & Infimum): Let S be a non-empty subset of \mathbb{R} if S is bounded above/below then a no u is said to be supremum/Infimum (least upper bound or greatest lower bound) of S if it satisfies the conditions:-

- i) u is an upper(lower) bound of S .
- ii) if v is any upper(lower) bound of S then $u \leq v (u \geq v)$.

1.5 Completeness Property

Statement:-If set is bounded below then its infimum must be exists and if set is bounded above then its supremum must be exists this property is known as completeness property.

let $\mathbb{N} = 1, 2, \dots$ bounded below

Unbounded= $\{\infty\}$ = Supremum

Lower bound= $\{\infty, \dots, -1, 0, 1\}$ = Infimum = 1

Example 5:

Let $A \subseteq B$ then Prove that,

I) $\inf A \geq \inf B$

II) $\sup A \leq \sup B$

Proof. I) Given that, $A \subseteq B, x \in A \Rightarrow x \in B$

also, $\inf A = u$ and $\inf B = v \dots$ (assume)

if u is $\inf A$ then, by definition,

i) u is lower bound, $x \geq u \forall x \in A$

ii) if u_1 is another lower bound, then $u_1 < u \forall u_1$. Assume that, $\inf B \geq \inf A$

Assume that, $\inf B \geq \inf A$

i.e $v \geq u$

i.e $x \geq v \geq u, \forall x \in B$

$\therefore x \geq v \geq u, \forall x \in A$

\Rightarrow if u is \inf , we can not have lower bound greater than u .

So, our assumption is wrong.

Hence, $u \geq V$ i.e $\inf A \geq \inf B$

II) let $\sup A = u$ and $\sup B = v$

if u is supremum of A then, by definition

- i) u is upper bound of A i.e $x \leq u, \forall x \in A$
- ii) if u_1 is any other upper bound then $u \leq u_1 \forall u_1$

Assume that, $\text{Sup } A \geq \text{sup } B$

$$u \geq v$$

i.e $v \leq u$

$$x \leq v \leq u, \forall x \in B$$

$$x \leq v \leq u, \forall x \in A$$

\Rightarrow if u is sup of A then we can not have upper bound less than u . So assumption is wrong.

Hence, $u \leq V$ i.e $\text{sup } A \leq \text{sup } B$



Example 6:

$S = 1 - \frac{(-1)^n}{n}, n \in \mathbb{N}$. Find infimum & suptemum

$$S = \{2, 1/2, 1 + 1/3, 1 - 1/4, 1 + 1/5, 1 - 1/6, \dots\}$$

$$\therefore \inf s = 1/2 \text{ of } \sup s = 2$$

Example 7:

$$S = \frac{(-1)^n}{n}, n \in \mathbb{N}$$

$$S = \{-1, 1/2, -1/3, 1/4, -1/5, \dots\}$$

$$LB = (-\infty, -1] \Rightarrow \text{Inf} = -1 \in S,$$

$$UB = [1/2, \infty) \Rightarrow \sup = 1/2 \in S$$

Example 8:

$$S = \left\{ \frac{1}{m} - \frac{1}{n}, m, n \in \mathbb{N} \right\}$$

$$S = \{0, 1/2, -1/2, 1 - 1/3, -2/3, 1, -1, \dots\}$$

$$LB = (-\infty, -1] \Rightarrow \inf S = -1 \notin S,$$

$$UB = (1, \infty) \Rightarrow \sup S = 1 \notin S$$

Example 9:

$$S = \left\{ \frac{n-1}{n}, n \in \mathbb{N} \right\} = \left\{ 1 - \frac{1}{n}, n \in \mathbb{N} \right\}$$

$$S = \{0, 1 - 1/2, 1 - 1/3, 1 - 1/4, \dots\}$$

$$LB = (-\infty, 0] \Rightarrow \inf S = 0 \in S,$$

$$UB = [1, \infty) \Rightarrow \sup S = 1 \notin S$$

Sets Operations

2.1 Set Operations

1. Union $A \cup B = \{x / x \in A \text{ or } x \in B\}$
2. Intersection $A \cap B = \{x / x \in A \text{ and } x \in B\}$
3. Complement $A^c = \{x / x \in A, x \in \Omega\}$
4. Substraction $A - B = A \setminus B = A \cap B^c = \{x / x \in A \text{ but } x \notin B\}$

Theorem 2.1.1. if A, B, C are sets then,

$$a) A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

$$b) A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)^C$$

Proof. To Prove:-

$$\text{i) } A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

$$\text{ii) } (A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$$

$$\text{i) } A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

$$\text{i) let } x \in A \setminus (B \cup C) \text{ i.e } x \in A \cap (B \cup C)^C$$

$$\Rightarrow x \in A \text{ and } x \in (B \cup C)^C$$

$$\Rightarrow x \in A \text{ and } x \notin (B \cup C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ and } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \cap B^C \text{ and } x \in A \cap C^C$$

$$\Rightarrow x \in (A \setminus B) \cap (A \setminus C)$$

$$\therefore A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C) \quad (2.1)$$

ii) $x \in (A \setminus B) \cap (A \setminus C)$

$$\Rightarrow x \in A \setminus B \text{ and } x \in A \setminus C$$

$$\Rightarrow x \in (A \cap B^C) \text{ and } x \in (A \cap C^C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ and } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \in (B \cup C)^C)$$

$$\Rightarrow x \in A \cap (B \cup C)$$

$$A \setminus B \cap A \setminus C \subseteq A \setminus (B \cup C) \quad (2.2)$$

from (2.1) & (2.2)

$$A \setminus (B \cup C) = A \setminus B \cap A \setminus C$$

ii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

To Prove:-

i) $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$

ii) $A \setminus B \cup A \setminus C \subseteq A \setminus (B \cap C)$

i) let $x \in A \setminus (B \cap C)$ i.e $x \in A \cap (B \cap C)^C$

$$\Rightarrow x \in A \text{ and } x \in (B \cap C)^C$$

$$\Rightarrow x \in A \text{ and } x \notin (B \cap C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ or } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \cap B^C \text{ or } x \in A \cap C^C$$

$$\Rightarrow x \in (A \setminus B) \cup (A \setminus C)$$

$$\therefore A \setminus (B \cap C) \subseteq A \setminus B \cup A \setminus C \quad (2.3)$$

ii) $x \in A \setminus B \cup A \setminus C$

$$\Rightarrow x \in A \setminus B \text{ or } x \in A \setminus C$$

$$\Rightarrow x \in (A \cap B^C) \text{ or } x \in (A \cap C^C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ or } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \in (B \cap C)^C)$$

$$\Rightarrow x \in A \cap (B \cap C)^C$$

$$\Rightarrow x \in A \setminus (B \cap C)$$

$$A \setminus B \cup A \setminus C \subseteq A \setminus (B \cap C) \quad (2.4)$$

from (2.3) & (2.4)

$$A \setminus (B \cap C) = A \setminus B \cup A \setminus C$$

-Hence Proved- □

2.2 Distributive Law

Distributive Law:-

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. a) To Prove:-

$$\text{i) } A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

let $x \in A \cup (B \cap C)$

$$\Rightarrow x \in A \text{ or } x \in B \cap C$$

$$\Rightarrow x \in A \text{ or } x \in B \text{ and } x \in C$$

$$\Rightarrow x \in A \text{ or } x \in B \text{ and } x \in A \text{ or } x \in C$$

$$\Rightarrow x \in A \cup B \text{ and } x \in A \cup C$$

$$\Rightarrow x \in (A \cup B) \cap (A \cup C)$$

$$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \quad (2.5)$$

$$\text{ii) } (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

let $x \in (A \cup B) \cap (A \cup C)$

$$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$$

$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$

$$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$$

$$\Rightarrow x \in A \cup (B \cap C)$$

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \quad (2.6)$$

from (2.5) & (2.6)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

b) To Prove:-

i) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A \cap B)$ or $(x \in A \cap C)$

$\Rightarrow x \in (A \cup B) \cup (A \cap C)$

$$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad (2.7)$$

ii) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

let $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$$\Rightarrow x \in A \text{ and } (x \in B \cup C)$$

$$\Rightarrow x \in A \cap (B \cup C)$$

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \quad (2.8)$$

from (2.7) & (2.8)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

□

Theorem 2.2.1. If A & B are sets, Show that $A \subseteq B$ if and only if $A \cap B = A$

Proof. i) Assume that $A \subseteq B$ to Prove that $A \cap B = A$

$$\text{let } x \in A \Rightarrow x \in B$$

$$\therefore x \in B \dots (\because A \subseteq B)$$

$$\Rightarrow x \in A \cap B$$

$$\therefore A \subseteq A \cap B \quad (2.9)$$

Also, by definition,

$$A \cap B \subseteq A \quad (2.10)$$

from (2.9) and (2.10)

$$A = A \cap B \quad (2.11)$$

ii) Assume that $A \cap B = A$, to prove $A \subseteq B$

We know that, $A \cap B \subseteq B$

$$\Rightarrow A \subseteq B \quad (2.12)$$

from (2.11) and (2.12)

$$A \subseteq B \text{ iff } A = A \cap B \quad (2.13)$$

-Hence Proved-

□

2.3 Basic Notations Theory

Definition 2.3.1 (Cartesian Product): let $A \& B$ be two sets,

$A = \langle 2, 3, 4 \rangle \& \langle 1, 5, 6 \rangle$ then cartesian product is given by

$$A \times B = \{\langle 2, 1 \rangle, \langle 2, 5 \rangle, \langle 2, 6 \rangle, \langle 3, 5 \rangle, \langle 3, 6 \rangle, \langle 4, 1 \rangle, \langle 4, 5 \rangle, \langle 4, 6 \rangle\}$$

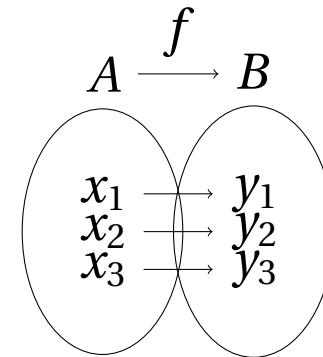
Definition 2.3.2 (Function): Let $A \& B$ be sets then a function from A to B is a set f of ordered

pairs in $A \times B$ such that for each $a \in A$ then there exists a unique $b \in B$ with $(a, b) \in f$.

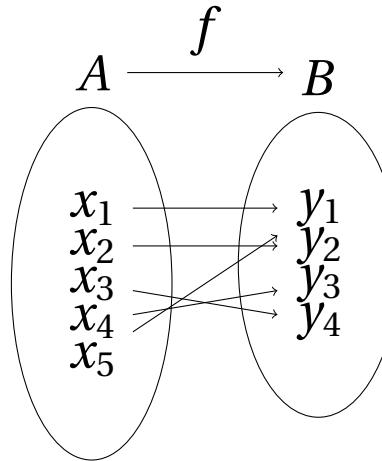
In other words if $\langle a, b \rangle \in f \& \langle a, b' \rangle \in f \Rightarrow b = b'$

Types of Function

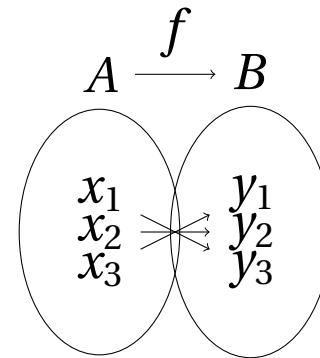
Definition 2.3.3 (One-One (Injective) Function): *The Function f is said to be injective (or One-One) if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.*



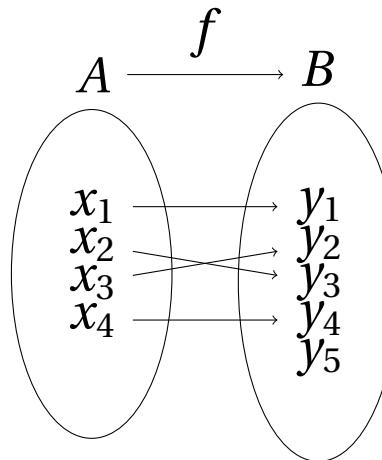
Definition 2.3.4 (Onto (Surjective) Function): *The function f is said to be Surjective if $f(A) = B$ i.e if the range $R(f) = B$.*



Definition 2.3.5 (One-One & Onto (Bijective) Function): *The Function f is both one-one and onto then it is said to be bijective.*



Definition 2.3.6 (Into Function): *If f is not onto then it is called as into function.*



Definition 2.3.7 (Composite Function): If $f : A \rightarrow B$ and $g : A \rightarrow C$ and if $R(f) \subseteq D(g) = B$ then the composite function $g \circ f$ is the function from $A \rightarrow C$
 $g \circ f : A \rightarrow C$ is composite function if $g \circ f(x) = g(f(x))$ $x \in A$

Example 10:

$$f : \mathbb{R} \rightarrow \mathbb{R}, g : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 2x, g(y) = 3y^2 - 1$$

Proof. Given that, $f(x) = 2x, g(y) = 3y^2 - 1$

$$g \circ f(x) = g(f(x))$$

$$= g(2x)$$

$$= 3(2x)^2 - 1$$

$$= 12x^2 - 1$$

$$f \circ g(y) = f(g(y))$$

$$= f(3y^2 - 1)$$

$$= 2(3y^2 - 1)$$

$$= 6y^2 - 2$$

$$\therefore g \circ f \neq f \circ g$$

□

Example 11:

Show that if $f : A \rightarrow B$ then, E, F are subsets of A then,

a) $f(E \cup F) = f(E) \cup f(F)$ and

b) $f(E \cap F) \subseteq f(E) \cap f(F)$

Proof. a) $f : A \rightarrow B, E, F \subseteq A$

$$f(E) = \{y / y = f(x), x \in E \subseteq A\} \subseteq B$$

$$f(F) = \{y / y = f(x), x \in F \subseteq A\} \subseteq B$$

$$f(E \cup F) = \{y / y = f(x), x \in E \cup F\}$$

To Prove,

i) $f(E \cup F) \subseteq f(E) \cup f(F)$

ii) $f(E) \cup f(F) \subseteq f(E \cup F)$

let $y \in f(E \cup F)$

$$\Leftrightarrow y = f(x), x \in E \cup F$$

$$\Leftrightarrow y = f(x), x \in E \text{ or } x \in F$$

$$\Leftrightarrow y = f(x), x \in E \subseteq A \text{ or } y = f(x), x \in F \subseteq A$$

$$\Leftrightarrow y \in f(E) \text{ or } y \in f(F)$$

$$\Leftrightarrow y \in f(E) \cup f(F)$$

$$\therefore f(E \cup F) \subseteq f(E) \cup f(F) \& f(E) \cup f(F) \subseteq f(E \cup F)$$

$$f(E \cup F) = f(E) \cup f(F)$$

To Prove,

b) $f(E \cap F) \subseteq f(E) \cap f(F)$

let $y \in f(E \cap F)$

$$\Rightarrow y = f(x), x \in E \cap F$$

$$\Rightarrow y = f(x), x \in E \text{ and } x \in F$$

$$\Rightarrow y = f(x), x \in E \text{ and } y = f(x), x \in F$$

$$\Rightarrow y \in f(E) \text{ and } y \in f(F)$$

$$\Rightarrow y \in f(E) \cap f(F)$$

$$\therefore f(E \cap F) \subseteq f(E) \cap f(F)$$

□

Example 12:

Example for $f(E) \cap f(F) \subsetneq f(E \cap F)$

let $f(x) = x^2$

$$E = \{1, 2\}, f(E) = \{1, 4\}$$

$$F = \{-2, 4\}, f(F) = \{4, 16\}$$

$$E \cap F = \{\phi\}, f(E) \cap f(F) = \{4\}$$

$$f(E \cap F) = \{\phi\}$$

$$f(E) \cap f(F) \subsetneq f(E \cap F)$$

Example 13:

Show that if $f : A \rightarrow B$ and G, H are subsets of B then,

a) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and

b) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

Proof. a) $f : A \rightarrow B$

$$f^{-1}(G) = \{x / f(x) \in G\} \subseteq A$$

$$f^{-1}(H) = \{x / f(x) \in H\} \subseteq A$$

$$\text{let } x \in f^{-1}(G \cup H)$$

$$\Leftrightarrow f(x) \in G \cup H$$

$$\Leftrightarrow f(x) \in G \text{ or } f(x) \in H$$

$$\Leftrightarrow x \in f^{-1}(G) \text{ or } x \in f^{-1}(H)$$

$$\Leftrightarrow x \in f^{-1}(G) \cup f^{-1}(H)$$

$$\therefore f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$$

-Hence Proved-

b) let $x \in f^{-1}(G \cap H)$

$$\Leftrightarrow f(x) \in G \cap H$$

$$\Leftrightarrow f(x) \in G \text{ and } f(x) \in H$$

$$\Leftrightarrow x \in f^{-1}(G) \text{ and } x \in f^{-1}(H)$$

$$\Leftrightarrow x \in f^{-1}(G) \cap f^{-1}(H)$$

$$\therefore f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$$

-Hence Proved-

**Example 14:**

Show that if $f : A \rightarrow B$ is injective & $E \subseteq A$ then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if f is not injective.

Proof. Given that, $f : A \rightarrow B$ is injective

i.e if $x \neq y \Rightarrow f(x) \neq f(y) \forall x, y \in A$

$$E = \{x / x \in E, f(x) \in B\} \subseteq A$$

$$f(E) = \{y / y = f(x) \in f(E), x \in A\} \subseteq B$$

To prove $f^{-1}(f(E)) = E$

$$\text{let } x \in f^{-1}(f(E))$$

$$\Rightarrow f(x) \in f(E)$$

$\Rightarrow x \in E \dots (\because f \text{ is one-one function})$

$$f^{-1}(f(E)) \subseteq E \quad (2.14)$$

Now, let $x \in E$

$$\Rightarrow f(x) \in f(E) \quad f^{-1}(H) = \{x / f(x) \in H, x \in A\}$$

$$x \in f^{-1}(f(E))$$

$$E \subseteq f^{-1}(f(E)) \quad (2.15)$$

from (2.14) & (2.15)

$$f^{-1}(f(E)) = E$$

□

Example 15:

$$\text{let } f(x) = x^2$$

$$E\{1, 2\} \Rightarrow f(E)\{1, 4\}$$

$$f^{-1}(f(E)) = \{(1, -2, -2)\}$$

$$f^{-1}(f(E)) \neq E$$

Example 16:

Show that if $f : A \rightarrow B$ is surjective and $E \subseteq A$ then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.

Proof. $f : A \rightarrow B, H \subseteq B$ and f is surjective i.e every element in B has inverse image in A

To prove: $f(f^{-1}(H)) = H$

$$\text{let } y \in f(f^{-1}(H))$$

$$\Rightarrow f(x) \in f(f^{-1}(H))$$

$$\Rightarrow x \in f^{-1}(H)$$

$$\Rightarrow y = f(x) \in H$$

$$\therefore f(f^{-1}(H)) \subseteq H \quad (2.16)$$

let $y \in H$ then

$\exists x \in A$ such that,

$$y = f(x) \in H \dots (\because f \text{ is onto})$$

$$\Rightarrow x \in f^{-1}(H)$$

$$\Rightarrow f(x) \in f(f^{-1}(H)) \dots (x \in E \Rightarrow f(x) \in f(E))$$

$$\Rightarrow y \in f^{-1}(H)$$



$$\therefore H \subseteq f(f^{-1}(H)) \quad (2.17)$$

from (2.16) & (2.17)

$$f(f^{-1}(H)) = H$$

ϕ

□

- Definition 2.3.8** (Finite & Infinite Sets):
- 1. The empty set ϕ is said to have zero elements.
 - 2. If $n \in \mathbb{N}$, a set S is said to have n elements if there exists a bijection from set $\mathbb{N} = \{1, 2, \dots, n\}$ onto S .
 - 3. A set S is said to be finite if it is either empty or it has n elements for some $n \in \mathbb{N}$.
 - 4. A set S is said to be infinite if it is not finite.

Theorem 2.3.1 (Uniqueness Theorem). If S is finite set, then the number of elements in S is

unique number in \mathbb{N} .

fixed

The set \mathbb{N} of natural numbers is an infinite set.

Theorem 2.3.2. Suppose that S & T are sets and $T \subseteq S$

subsets of \mathbb{R}

- $S, T \subseteq \mathbb{R}$, & $T \subseteq S$.

a) If S is finite Set, then T is a finite Set.

b) If T is an infinite set then S is an infinite Set.

Proof. a) $T \subseteq S$ and S is finite Set

i) Suppose $S = \phi \Rightarrow T = \phi \Rightarrow T$ is finite

ii) When $S \neq \phi$ then there are two possibilities.

1) $T = \phi \Rightarrow T$ is a finite Set **or**

2) $T \neq \phi$

We will prove this by method of mathematical induction.

- $\#(S) = 1$ and as $T \neq \phi \Rightarrow S = T$

Hence as S is finite $\Rightarrow T$ is finite

- Now assume that this statement is true for $\#(S) = k$

i.e $\#(S) = k \& T \subseteq S \Rightarrow T$ is finite set.

- Now, let's prove it for $\#(S) = k + 1$

As S is finite, it has bijection with N_{k+1}

$$S = \{f(1), f(2), \dots, f(k+1)\} \quad (2.18)$$

let's define, $S_1 = S - f(k+1)$

$$\therefore \#(S)_1 = k \text{ and } T_1 = T - f(k+1) \quad \#(S_1)$$

Now, if $f(k+1) \notin T \Rightarrow T_1 = T \subseteq S_1$

and as $\#(S)_1 = k \& T \subseteq S_1 \subseteq T$ is finite \Rightarrow

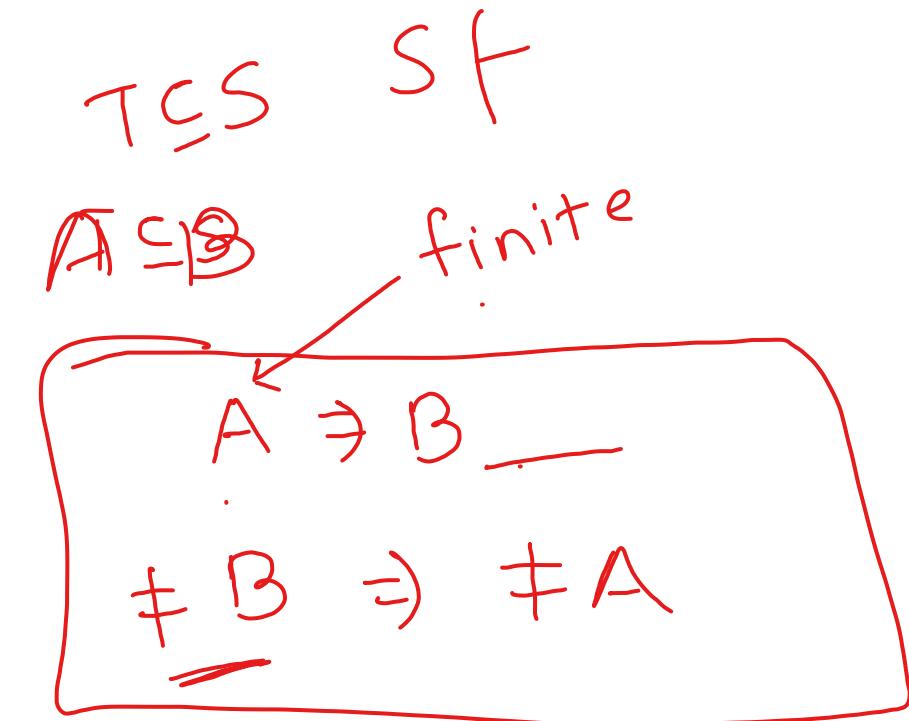
if $f(k+1) \in T_1 \Rightarrow T_1 = T - f(k+1) \subseteq S_1$

$\therefore T_1 \subseteq S_1, \#(S_1) = k \Rightarrow T_1$ is finite $\Rightarrow T$ is finite.

-Hence Proved-

b) (b) is a contrapositive statement to (a).

Hence, if T is infinite $\Rightarrow S$ is also infinite.





enumerable

Definition 2.3.9 (Countably Infinite): A set is said to be denumerable or countably infinite if there exists bijection of \mathbb{N} onto S .

Definition 2.3.10 (Countable Set): A set S is said to be countable if it is either finite or denumerable.

Definition 2.3.11 (Uncountable Set): A set S is said to be uncountable if it is not countable.

The following statements are equivalent :—

1. S is a countable set.

2. \exists surjection of \mathbb{N} onto $\underline{\underline{S}}$.

3. \exists injection of S onto \mathbb{N}

$$\{x_1, x_2, \dots\} \xrightarrow{\text{Bij}} \{x_n, x_{n+1}, \dots\} \xrightarrow{\text{Surj}} \{x_{n+2}, \dots\}$$

Example

1. Set of even/odd numbers are denumerable .
2. Set of all integers(denumerable).
3. The union of two disjoint denumerable sets is again denumerable .
4. The sets \mathbb{N} , \mathbb{N}^2 , \mathbb{N}^n are denumerable .

Theorem 2.3.3. Suppose that S & T are sets and $T \subseteq S$

a) If S is countable, then T is a countable set.

b) If T is an uncountable then S is an uncountable Set.

Theorem 2.3.4. The Set \mathbb{Q} of rational numbers is denumerable.

Proof. lets prove it for \mathbb{Q}^+ first.

$$\mathbb{Q} = \left\{ \frac{p}{q}, q \neq 0 \right\}, \mathbb{Q}^+ = \left\{ 1, \frac{1}{2}, \dots, \frac{2}{1}, \frac{2}{2}, \frac{2}{3} \right\}$$

We can map \mathbb{Q}^+ with \mathbb{N}^2 however, mapping will not be injection as

$$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} \dots \text{ or } \frac{1}{1} = \frac{2}{4} = \frac{3}{6} \dots$$

To proceed $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countable.

lets define, $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}^+$ is mapping of ordered pairs $< m, n >$ into rational no $\frac{m}{n}$

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 3 & 4 \end{array} \dots$$

$$\begin{array}{cccc} 2 & 2 & 2 & 2 \\ \hline 1 & 2 & 3 & 4 \end{array} \dots$$

$$\begin{array}{cccc} 3 & 3 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 \end{array} \dots$$

:

:

$\Rightarrow \mathbb{Q}^+$ is countable

Similarly, \mathbb{Q}^- is also countable

So, $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$ is countable.... (\because Union of two disjoint denumerable sets is again denumerable)



- Countable union of countable sets again countable.

2.4 Archimedean Property

If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ subject to $x < n_x$.

Proof. By method of contradiction,

$$x \in \mathbb{R}, n_x < x \forall n_x \in \mathbb{N}$$

$\therefore x$ is upper bound for set \mathbb{N}

By completeness property, the set which has upper bound must have supremum (says)

$$n_x < u \quad n_x \in \mathbb{N}$$

$$n_{x+1} \leq u \quad \forall n_x$$

$$n_x \leq u - 1 \quad \forall n_x$$

$\therefore u - 1$ is also upper bound $< u$ (by definition)

But we know that, Supremum is the least upper bound i.e there exists no other upper bound which is less than u .

So our assumption is wrong.

Hence, $x < n_x, x \in \mathbb{R}$

Corollary 2.4.0.1. If $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ then $\inf S = 0$

Proof. $S \neq \emptyset$ and 0 is lower bound of S .

\therefore By completeness Property, set S has infimum (v)

Let, $\varepsilon \in \mathbb{R}, \frac{1}{\varepsilon} > 0 \Rightarrow \frac{1}{\varepsilon} \in \mathbb{R}$

\therefore By archimedean property

$\exists n \in \mathbb{N}, 0 < \frac{1}{\varepsilon} < n \Rightarrow 0 < \frac{1}{n} < \varepsilon \Rightarrow 0 \text{ is inf } (S)$

□

Corollary 2.4.0.2. If $t > 0$, $\exists n_t \in \mathbb{N} \Rightarrow 0 < \frac{1}{n_t} < t$

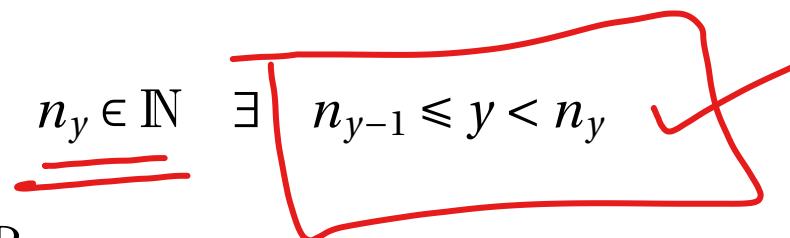
Proof. $t > 0, \frac{1}{t} > 0 \Rightarrow \frac{1}{t} \in \mathbb{R}$

\therefore By archimedean property, $\exists n \in \mathbb{N}$ subject to $\frac{1}{t} < n, \exists n_t \in \mathbb{N}$

$$\Rightarrow 0 < \frac{1}{n_t} < t$$

□

Corollary 2.4.0.3. If $y > 0$, $\exists n_y \in \mathbb{N}$



Proof. Given that $y > 0$ i.e $y \in \mathbb{R}$

$y < n_y, \exists n_y \in \mathbb{N}$...By archimedean property

$$E_y = \{n \mid y < n, n \in \mathbb{N}\}$$

$\Rightarrow y$ is lower Bound of E_y

\Rightarrow least element of E_y is $\inf(n_y)$

$\Rightarrow \underline{n_{y-1}} \leq y < \underline{n_y}$

□

Theorem 2.4.1 (Density Theorem). If x & y are any real numbers with $x < y$, then \exists a rational numbers $r \in \mathbb{Q}$ such that $x < r < y$

Proof. assume $x > 0, x \in \mathbb{R}$

Given, $x > y \Rightarrow y - x > 0, y - x \in \mathbb{R}$

$\exists n \in \mathbb{N}, \frac{1}{n} < y - x \dots$ (corollary 2.4.0.2)

$$x, y \in \mathbb{R}, \quad \underline{x < y} \Rightarrow y - x > 0$$

$$[\Rightarrow \exists r \in \mathbb{Q} \quad \underline{x < r < y}]$$

$$y - x > 0, y - x \in \mathbb{R}$$

$$\frac{1}{n} < y - x \quad n \in \mathbb{N}$$

$$\underline{\underline{1 < ny - nx}}$$

$$\underline{\underline{1 < ny - n_* x}}$$

(2.19)

~~Assume~~ $\underline{\underline{n_x > 0}}$

Also, $x > 0 \Rightarrow \underline{\underline{n_x > 0}}$ then $\exists m \in \mathbb{N}$ such that $\underline{\underline{m-1}} \leq \underline{\underline{n_x}} < \underline{\underline{m}}$... (corollary 2.4.0.3)

$$\underline{\underline{nx + 1 < ny}}$$

from (2.19)

$$\boxed{n_x < m \leq n_{x+1} < n_y}$$

$$\Rightarrow n_x < m < n_y$$

$$\Rightarrow x < \frac{m}{n} < y$$

$$\Rightarrow x < r < y, \text{ where } r = \frac{m}{n} = \text{rational number}$$

-Hence Proved-

$$m-1 \leq nx < m$$

$$\underline{\underline{nx \leq m}} \leq \underline{\underline{nx+1}} \leq \underline{\underline{ny}}$$

□

Corollary 2.4.1.1. If x and y are any real numbers with $x < y$ then \exists an irrational number

$$\underline{\underline{r \in \mathbb{Q}^c}} \exists x < r < y$$

Proof. By density theorem,

If $x < y$ then $\exists r_1 \in \mathbb{Q} \exists x < r_1 < y$. Here $x < y$

$$x, y \in \mathbb{R} \quad \underline{x < y}. \quad \exists r \in \mathbb{Q} \quad x < r < y$$

$$x, y \in \mathbb{R} \quad x < r <$$

$$\underline{\underline{\sqrt{2}}} \quad \underline{\underline{\sqrt{2}}} \quad \underline{\underline{\frac{x}{\sqrt{2}}}}$$

$$\underline{\underline{\sqrt{2}}} < \underline{\underline{\sqrt{2}}} \in \mathbb{R}$$

$$\underline{\underline{\frac{x}{\sqrt{2}}}} \in \mathbb{R}, \underline{\underline{\frac{y}{\sqrt{2}}}} \in \mathbb{R}$$

$$\therefore \sqrt{2}x < \sqrt{2}y$$

$r/\sqrt{2}$ irr

$$\sqrt{2}x < r_1 < \sqrt{2}y$$

$$x < \frac{r_1}{\sqrt{2}} < y$$

$$x < r < y \quad \text{where} \quad r = \frac{r_1}{\sqrt{2}} = \text{irrational number}$$

-Hence Proved-

□

Intervals:-

- $\underline{[a, b]} = \{x / a \leq x \leq b\} = \underline{\text{Closed}}$



- $\underline{(a, b)} = \{x / a < x < b\} = \underline{\text{Open}}$



- $\underline{\underline{[a, b)}} = \{x / a \leq x < b\} = \underline{\underline{\text{Half Closed- Half Open}}}$



- $\underline{(a, b]} = \{x / a < x \leq b\} = \underline{\underline{\text{Half Closed- Half Open}}}$

Intersection:-

Finite :- $\bigcap_{i=1}^n \left[0, \frac{1}{n} \right] = \left[0, \frac{1}{n} \right]$



Arbitrary:-

$$x \in \mathbb{R} \quad \text{such that } \frac{1}{n} < x$$

$$\bigcap_{i=1}^{\infty} \left[0, \frac{1}{n_i} \right] = \{0\} = [0, 0.\underline{\underline{0000}}1]$$

$$= \{0\}$$

$$x < n_x$$

$$\bigcap_{i=1}^{\infty} \left(0, \frac{1}{n} \right) = \{0\}$$

$$\bigcap_{n=1}^{\infty} (n, \infty) = \emptyset$$

$$\bigcup_{n=1}^{\infty} (-n, n) = \{-\infty, \infty\}$$

$$\bigcap_{n=1}^{\infty} (-n, n) = \{-1, 1\}$$

$$i=1 \quad \left[0, \frac{1}{1} \right]$$

$$i=2 \quad \left[0, \frac{1}{2} \right]$$

$$[0, 1] \cap [0, 1/2] = [0, 1/2]$$

$$[0, 1] \cap [0, 1/2] \cap [0, 1/3] = [0, 1/3]$$

$$\bigcap_{i=1}^n \left[0, \frac{1}{n} \right] = \left[0, \frac{1}{n} \right]$$

$$\bigcap_{n=1}^{\infty} \left[-1, 1 + \frac{1}{n} \right] = [-1, 1] \quad \checkmark$$

$$\bigcup_{n=1}^{\infty} \left[-1, 1 - \frac{1}{n} \right] = [-1, 1) \quad \checkmark$$

$$\bigcap_{n=1}^{\infty} [-n, n] = [-1, 1]$$

~~$\bigcap_{n=1}^{\infty}$~~

$$\bigcap_{n=1}^{\infty} [-n, n] = (-\infty, \infty)$$

Theorem 2.4.2. \mathbb{R} is uncountable.

~~we will prove this by method of contradiction~~
 Proof. Assume that \mathbb{R} is countable so does $(0, 1)$ is countable

We can write one-one correspondence with \mathbb{N} as,

$$b_1 = 0.a_{11}a_{12}a_{13}\dots \neq C$$

$$b_2 = 0.a_{21}a_{22}a_{23}\dots \neq C$$

$$b_3 = 0.a_{31}a_{32}a_{33}\dots \neq C$$

 \vdots
 \vdots
 \vdots
 \vdots

$$a \dots \underline{\quad} \quad \dots$$

O.C

$$C \neq a$$

$$C_2 \neq a_{22}$$

$$C_i \neq a_{ii}$$

$T \subseteq S$, S is countable $\Rightarrow T$ is also countable
 T is uncountable $\Rightarrow S$ is also uncountable

$(0, 1) \subseteq \mathbb{R}$
 uncountable

Assume $\underline{(0, 1)}$ countable ✓

~~Finite~~ or denumerable ✓

onto
one-one \mathbb{N}

$$S = \underline{(0, 1)} = \{b_1, b_2, b_3, \dots\}$$

\uparrow \downarrow \uparrow \downarrow
 $1, 2, 3, \dots$

$$b_i = 0.a_{i1}a_{i2}a_{i3}\dots a_{ii} \neq C$$

$$b_i = 0.C_1C_2C_3\dots \in (0, 1)$$

$$C_1 \neq a_{11}$$

$$C_2 \neq a_{22}$$

$$C_3 \neq a_{33}$$

:

:

$$C_i \neq a_{ii}$$

As $C_i \neq a_i$ there does not exists any $C_i \neq C$

⇒ Our counting Scheme is wrong.

⇒ Our assumption is wrong.

⇒ $(0, 1)$ must be uncountable .

⇒ \mathbb{R} is uncountable. □

2.5 Cauchy Schwartz Inequality

Let $a_i, b_i \in \mathbb{R} \forall i$ then

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \stackrel{=} {\downarrow} \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) \quad \checkmark$$

Proof. let $x \in \mathbb{R}$ then,

$$a_i x + b_i \in \mathbb{R} \dots \because a_i, b_i \in \mathbb{R}$$

$$\therefore (a_i x + b_i)^2 \geq 0 \quad \checkmark$$

$$a_i^2 x^2 + 2a_i x b_i + b_i^2 \geq 0$$

$$\frac{(a_i x + b_i)^2 \geq 0}{\sum (a_i x + b_i)^2 = 0} = 0$$

$$x = \frac{-B \pm \sqrt{B^2 - 2A}}{2}$$

$$\Rightarrow \left(\sum_{i=1}^n a_i^2 \right) x^2 + 2 \left(\sum_{i=1}^n a_i b_i \right) x + \sum_{i=1}^n b_i^2 \geq 0$$

$$Ax^2 + 2Bx + C \geq 0 \quad : \quad (2.20)$$

where,

$$A = \sum_{i=1}^n a_i^2, B = \sum_{i=1}^n a_i b_i, C = \sum_{i=1}^n b_i^2$$

let $x = \frac{-B}{A}$

\therefore from (2.20)

$$A\left(\frac{B}{A}\right)^2 + 2B\left(\frac{-B}{A}\right) + C \geq 0 \Rightarrow \frac{B^2}{A} - \frac{2B^2}{A} + C \geq 0$$

$$\Rightarrow \frac{-B^2}{A} + C \geq 0$$

$$\Rightarrow C \geq \frac{B^2}{A}$$

$$\Rightarrow A \cdot C \geq B^2$$

$$\Rightarrow B^2 \geq A \cdot C$$

$$\therefore \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

-Hence Proved-

□

Note:-

Equality hold if a_i and b_i is equal to zero.

If $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ then $Ax^2 + 2Bx + C = 0$

$$a_i = b_i \quad \checkmark$$

how to
(prove it?)

?

A

$$Ax^2 + 2Bx + C = 0$$

$$\frac{A(-B + \sqrt{B^2 - 4AC})^2}{2A^2} + 2B \left(\frac{-B + \sqrt{B^2 - 4AC}}{2A} \right) + C = 0$$

$$\frac{B^2 - 2B\sqrt{B^2-4AC}}{4A} + \frac{B^2-4AC}{A} + \frac{-B^2 + B\sqrt{B^2-4AC}}{A} + C = 0$$

$$\frac{2B^2 - 2By - 4AC - 4B^2 + 4By}{4A} + C = 0$$

$$\frac{2By - 2B^2 - 4AC}{4A} + C = 0$$

✓

$$\frac{B\sqrt{B^2-4AC} - B^2 - 2AC}{2A} + C = 0 \Rightarrow ?$$

=

Chapter 3

Elements of Point Set Topology

$$A \subseteq B, A \neq B \Rightarrow A \subset B$$

↑
proper

3.1 Terminology and Notations

Definition 3.1.1 (Member of a set): *If an element x is in a set A , we write $x \in A$ and say that x is a member of A , or that x belongs to A . If x is not in A , we write $x \notin A$*

Definition 3.1.2 (Subset): *If every element of a set A also belongs to a set B , we say that A is a subset of B and write $A \subseteq B$ or $B \supseteq A$.*

Definition 3.1.3 (Proper Subset): *We say that a set A is a proper subset of a set B if $A \subset B$, but there is at least one element of B that is not in A .*

$$A \subset B \quad \text{if } A \subseteq B \text{ but } A \neq B$$

Definition 3.1.4 (Equal Sets): Two sets A and B are said to be equal, and we write $\underline{A = B}$, if they contain the same elements. i.e. $A \subseteq B$ and $B \supseteq A$.

$$A = B \text{ iff } \begin{matrix} A \subseteq B, B \subseteq A \\ \wedge \end{matrix}$$

A set is normally defined by either listing its elements explicitly, or by specifying a property that determines the elements of the set.

- The set of natural numbers $\mathbb{N} := \{1, 2, 3, \dots\}$ lists
- The set of integers $\mathbb{Z} := \{0, 1, -1, 2, -2, 3, -3, \dots\}$ Rule
- The set of rational numbers $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$,
- The set of real numbers \mathbb{R} .

$$\frac{m}{n}$$

Definition 3.1.5 (Open Set): A subset G of \mathbb{R} is open in \mathbb{R} if for each $x \in G$ there exists a neighbourhood $\forall v$ of x such that $v \subseteq G$.

Definition 3.1.6 (Closed set): A subset f of \mathbb{R} is closed in \mathbb{R} if the complement f^C is open in \mathbb{R} ,

G is open iff for $x \in G \exists \epsilon > 0$

$$x \in (x - \epsilon_x, x + \epsilon_x) \subseteq G$$

e.g. $(-\infty, \infty) = \mathbb{R}$ - open as well as closed

$(0, 1)$ - open

(a, b) - open

$[a, \infty)$ - not open but closed ?

$[a, b]$ - not open but closed

\emptyset - open and closed

$[a, b)$ - neither open nor closed

$(a, b]$ - neither open nor closed

\mathbb{Q} - not closed not open

\mathbb{N} - closed but not open

\mathbb{I} - closed but not open

Definition 3.1.7 (Interior point): For some $x \in s$ if \exists open interval $I_x \ni x \in I_x \subseteq S$ then x is called interior point of set S .

Definition 3.1.8 (Interior of Set): Collection of all interior point is called interior of set (S_i) .

example $S = \{[0, 1], [0, 1), (0, 1]\}, S_i(0, 1)$

S^i

Theorem 3.1.1. Finite union of open sets is open.

Proof. let A and B be two finite open sets.

Claim- $A \cup B$ is open set.

$\therefore A$ & B be two open set.

$\Rightarrow \forall x \in A, \exists I_x \subseteq A$ and $\forall x \in B \exists I_x \subseteq B$

let $x \in A \cup B$

$x \in A$ or $x \in B$

$\therefore x \in I_x \subseteq A$ or $x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A \cup B$

$\Rightarrow A \cup B$ is open set.



Theorem 3.1.2. *Finite intersection of open set is open.*

Proof. let A & B be two open sets.

claim- $A \cap B$ is open.

let $x \in A \cap B$

$\therefore x \in A$ or $x \in B$

$\Rightarrow \exists I_x \ni x \in I_x \subseteq A$ and $x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A \cap B$

$\Rightarrow A \cap B$ is open set.



Theorem 3.1.3. *Arbitrary union of open sets is open.*

Proof. let $\{A_i\}_{i=1}^{\infty}$ be collection of open sets.

claim- $\bigcup_{i=1}^{\infty} A_i$ is open set



$$\text{let } x \in \bigcup_{i=1}^{\infty} A_i$$

$\Rightarrow x \in A_j, \text{ for some } j \in I$

$\Rightarrow x \in I_x \subseteq A_j, \text{ for some } j \in I$

$\Rightarrow x \in I_x \subseteq A_j \subseteq \bigcup_{i=1}^{\infty} A_i$

$\therefore \bigcup_{i=1}^{\infty} A_i$ is open set. □

Theorem 3.1.4. *Arbitrary intersection of open sets may or may not be open set.*

Proof. Set $S_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

$\bigcap_{n=1}^{\infty} S_n = \{1\}$ which is not open set. □

Theorem 3.1.5. *Finite union of two closed set is closed.*

Proof. let A & B closed set.

Claim- $A \cup B$ is closed set.

Since, A^C & B^C are open sets.

$\Rightarrow A^C \cap B^C$ is open set.

$\Rightarrow (A \cup B)^C$ is open set

$\Rightarrow A \cup B$ is closed set

□

Theorem 3.1.6. *Finte intersection of two closed set is closed.*

Proof. let A & B two closed set.

$\Rightarrow A^C$ & B^C are two open sets.

$\Rightarrow A^C \cup B^C$ is again open set.

$\Rightarrow (A \cap B)^C$ is open set

$\Rightarrow A \cap B$ is closed set

□

Theorem 3.1.7. *Arbitrary union of closed sets may not be closed.*

Example 17:

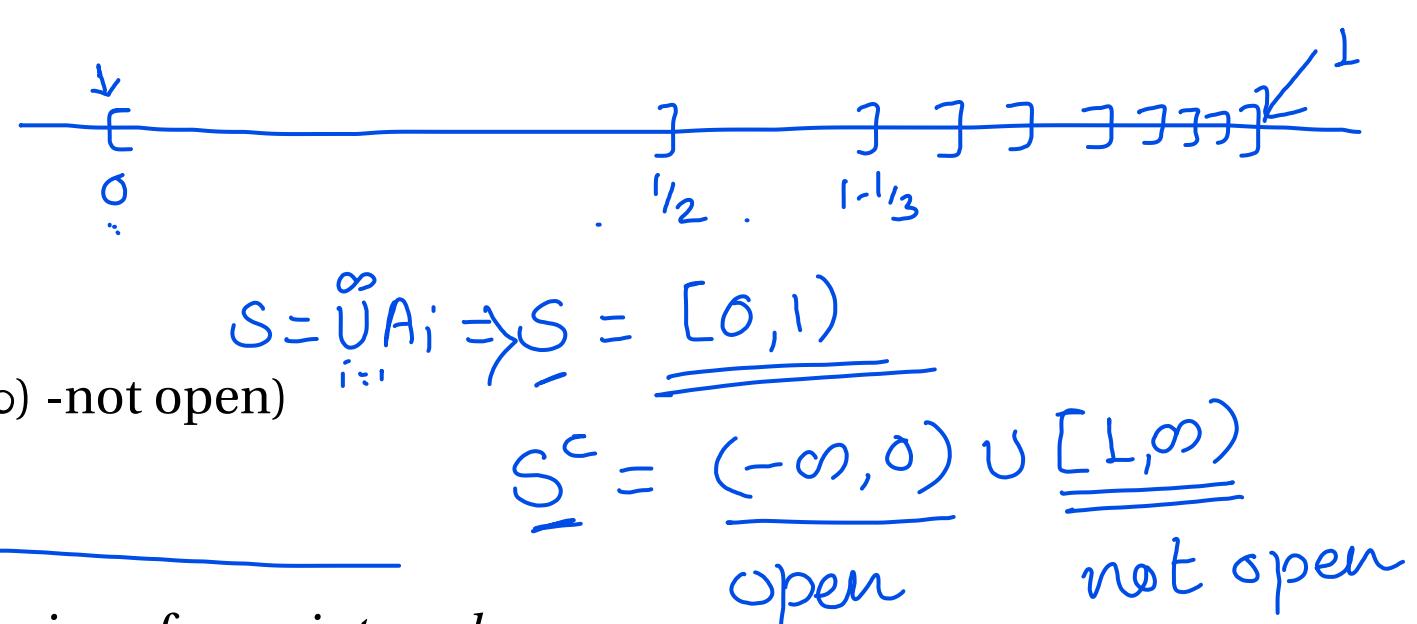
[Counter example] $A_n = [0, \underline{n}] \cup A_n$ $[0, \infty)$ -closed

$$A_n = \left[0, 1 - \frac{1}{n} \right]$$

$$A_1 = \overline{\{0\}}$$

$$A_2 = \left[0, \frac{1}{2}\right] \cup A_n [0, 1) \text{ -not closed}$$

$$A_3 = \left[0, 1 - \frac{1}{3} \right] \dots (\because (-\infty, 0) \cup [1, \infty) \text{-not open})$$



Theorem 3.1.8. *Every open set is union of open intervals*

Proof. Suppose $S = \{x_1, x_2, x_3\}$

let S be an open set, $S = \{x_1, x_2, x_3 \dots\} = \{x_i\}$

for each $x_i \in I_{x_i} \subseteq S$

$$\{x_i\} \subseteq I_{x_i} \subseteq S$$

$$S = \cup\{x_i\} \subseteq \cup_{i \in I} \subseteq I_{x_i} \subseteq S$$

Hence, Every open set is union of open intervals

Theorem 3.1.9. *Interior of set is open set.*

Proof. Given that, Let S^i is interior.

S is open set.

Claim- $x \in S^i, \exists I_x \in S^i \ni x \in I_x \subseteq S^i$

let $x \in S^i$

$\Rightarrow x$ is interior point of S

$\Rightarrow x \in I_x \subseteq S^i$

let $y \in I_x \Rightarrow y \in S \Rightarrow y \in I_x \subseteq S$

$\Rightarrow y \in S^i, y \in I_x$

$\therefore y$ is also interior point of S

this is true for all $y \in I_x$

$\therefore I_x \subseteq S^i \Rightarrow x \in I_x \subseteq S^i$

$\Rightarrow S^i$ is open set. □

Theorem 3.1.10. *Interior of set is largest open subset of set.*

Proof. let $S \subseteq \mathbb{R}$, S^i is interior set of S .

Claim:- $S^i \subseteq S$ is largest open set.

We prove this by method of contradiction

Assume that, T is largest open subset of set S .

(S^i is not largest) i.e $S^i \subseteq T \subseteq S$

S^i is proper subset of T

Since, $S^i \in T$

\exists some $x \in T, x \notin S^i$

Now, $x \in T \subseteq S \Rightarrow x$ is interior point of S

This contradicts to our assumption that $x \notin S^i$

\therefore Our assumption is wrong.

Hence, Interior of set is largest open subset. □

Definition 3.1.9 (Limit point of set): Let c be the limit point of set S iff for any $\varepsilon > 0$, $\exists x \in S \exists$

$$0 < |x - c| < \varepsilon$$

$$\text{i.e } -\varepsilon < x - c < \varepsilon$$

$$\text{i.e } c - \varepsilon < x < c + \varepsilon$$

$$\text{i.e } x \in \delta_\varepsilon(c)$$

$$\Rightarrow \#(\delta_\varepsilon \cap A) \neq 0$$

example- $S = \left\{ \frac{1}{n}, n \in \mathbb{R} \right\}$, 0 is limit point of S .

Correct it

Definition 3.1.10 (Derived Set): The set of all limit points of Set S is called the derived set of S and denoted by S' .

$$S' = \{c \mid c \text{ is limit point of } S\}$$

Definition 3.1.11 (Closed Set): The set S is said to be closed set if it contains all of its limit points (i.e $S' \subseteq S$)

Definition 3.1.12 (Closure Set): $S = S \cup S'$

Example 18:

$$1. S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}, S = \{0\} \notin S \text{ [Neither open nor closed]}$$

$$\bar{S} = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \cup \{0\}$$

$$2. S = \mathbb{Q}, S' = \mathbb{R} \bar{S} = \mathbb{R}$$

$$3. S = \mathbb{I}, S' = \phi \bar{S} = \mathbb{I}$$

$$4. S = \mathbb{N}, S' = \phi \bar{S} = \mathbb{N}$$

Note:- If S is closed then $S = \bar{S}$

Theorem 3.1.11. Let $S \subseteq T$ then $S' \subseteq T'$

Proof. Let $c \in S'$

$\Rightarrow c$ is limit point of S

for any $\varepsilon > 0, \delta_\varepsilon(c) \cap S \neq \emptyset$

$\Rightarrow \delta_\varepsilon(c) \cap T \neq \emptyset$ as $S \subseteq T$

$\Rightarrow c$ is limit point of T

$\Rightarrow c \in T'$

$\therefore S' \subseteq T'$

□

Theorem 3.1.12. Show that $(S \cup T)' = S' \cup T'$

Proof. To prove, $(S \cup T)' = S' \cup T'$

i.e

a) $(S \cup T)' \subseteq S' \cup T'$

b) $S' \cup T' \subseteq (S \cup T)'$

- first we prove part b)

$$S \subseteq S \cup T \Rightarrow S' \subseteq (S \cup T)'$$

$$T \subseteq S \cup T \Rightarrow T' \subseteq (S \cup T)'$$

$$\Rightarrow S' \cup T' \subseteq (S \cup T)' \tag{3.1}$$

- a) let $c \in (S \cup T)'$

$\Rightarrow c$ is limit points of $S \cup T$

$\Rightarrow \exists S \cup T \exists x \in \delta_\varepsilon(c)$

$\Rightarrow x \in S \exists x \in \delta_\varepsilon(c)$ or $x \in T \exists x \in \delta_\varepsilon(c)$

$\Rightarrow c$ is limit point of S or c is limit point of T

$\Rightarrow c \in S'$ or $c \in T'$

$\Rightarrow c \in S' \cup T'$

$$(S \cup T)' \subseteq S' \cup T' \quad (3.2)$$

from (3.1) and (3.2)

$$(S \cup T)' = S' \cup T'$$

□

Theorem 3.1.13. *Finite intersection of two closed set is closed.*

Proof. let S & T be two closed sets.

$$\therefore S' \subseteq S \text{ and } T' \subseteq T$$

Claim: $S \cap T$ is closed

i.e $(S \cap T)' \subseteq (S \cap T)$

We know,

$$S \cap T \subseteq S \Rightarrow (S \cap T)' \subseteq S' \subseteq S$$

$$S \cap T \subseteq T \Rightarrow (S \cap T)' \subseteq T' \subseteq T$$

$$(S \cap T)' \subseteq (S \cap T)$$

$\therefore S \cap T$ is closed set. □

Theorem 3.1.14. let S & T be subsets of \mathbb{R} , $S' \cap T'$ may or may not be subset of $S \cap T'$

Proof. $\therefore S' = [1, 2], T' = [2, 3]$

$$(S \cap T) = \emptyset \text{ & } S' \cap T' = \{2\}$$

$$\Rightarrow (S' \cap T')' = \emptyset$$

$$\therefore S' \cap T' \not\subseteq (S' \cap T')'$$

Definition 3.1.13 (Dense Set): A Subset $A \subseteq \mathbb{R}$ is said to be dense set in \mathbb{R} if every point of \mathbb{R} is point of A or limit point of \mathbb{R} or equivalently if closure of A is \mathbb{R}

$$\overline{A} = A' \cup A = \mathbb{R}$$

- A set A is said to be dense in itself if $\overline{A} = A$
- A set A is said to be nowhere dense relative to \mathbb{R} if no neighborhood of \mathbb{R} is contained in the closure of A
- A set is said to be perfect if it is identical with its derived set or equivalently a set which is closed and dense in itself.

Theorem 3.1.15. Set is closed if and only if its complement is open.

Proof. a) let S be closed set

To prove- S^c is open.

let $x \in S^c$

$\Rightarrow x$ is not limit point of $S(\bar{S} = S)$

for some $\varepsilon > 0, V_\varepsilon(x) \cap S = \emptyset$

$(x - \varepsilon, x + \varepsilon) \subseteq S^c$

$\therefore S^c$ is open.

b) let S^c is open set

To prove- S is closed set

By method of contradiction,

Assume that S is not closed.

$\therefore \exists$ some limit point of x of $S \ni x \notin S$

$\Rightarrow x \in S^c$

for some $\varepsilon > 0 \exists (x - \varepsilon, x + \varepsilon) \subseteq S^c \dots (\because S^c$ is open set)

$\therefore V_\varepsilon(x) \cap S = \emptyset$

which is not possible as x is limit point

⇒ Our Assumption is that $x \in S$ is wrong

⇒ All limit point of S are in S

⇒ is closed set.

□

Theorem 3.1.16. *Derived set of set is closed.*

Proof. let $S \subseteq \mathbb{R}$, S' is derived set of S .

To prove- S' is closed i.e $(S')' \subseteq S' = S''$

let $c \in S'' \Rightarrow c$ is limit point of S'

i.e every $\varepsilon - neighborhood$ v of c contains atleast one point x of $S' \ni x \neq c$

i.e $x \in S' \Rightarrow x$ is limit point of set S .

\therefore Every ε neighborhood v of x contains atleast one point of S .

As $x \in v$, v is also a ε neighborhood of x

$\therefore v$ contains atleast one point of S .

In this way, we can prove that, every ε neighborhood v of c contains infinitely many points of S .

$\therefore C$ is limit point of set S .

Also $c \in S'$

As $c \in S'' \Rightarrow c \in S'$, $S'' \subseteq S' \Rightarrow S'$ is closed set when $S'' = \phi$

then $S'' \subseteq S' \Rightarrow S'$ is closed set. □

3.2 Compact Set

Definition 3.2.1 (Open Cover): *Let A be a subset of \mathbb{R} . An open cover of A is a collection*

$G = \{G_\alpha\}$ of open sets in \mathbb{R} whose union contains A i.e

$$A \subseteq \bigcup_\alpha G_\alpha$$

Definition 3.2.2 (Subcover): if G' is subcollection of sets from G such that the union of sets in G' also contains A then G' is called a subcover of G

Definition 3.2.3 (Finite Subcover): A subset k of \mathbb{R} is said to be compact if every open cover of \mathbb{R} has finite subcover.

Example 19:

$$1. S = (0, 1), G_i = \left(0, 1 - \frac{1}{i}\right)$$

$$\cap G_i = (0, 1) \supseteq (0, 1)$$

$$\cap G_i = \left(0, 1 - \frac{1}{n}\right) \not\subseteq (0, 1)$$

$\therefore (0, 1)$ is not compact

2. \mathbb{N} is not compact

3.3 Heine Borel theorem

Theorem 3.3.1 (Heine Borel theorem). *The set k is compact set if and only if it is closed & bounded.*

Proof. Given that, k is compact set.

i.e Every open cover exists finite subcover.

claim- k is bounded & closed.

1. k is bounded

$$G_i = (-i, i), G = \mathbb{R}$$

$$\cup_{i=1}^n G_i = (-n, n), k \subseteq (-n, n)$$

$\therefore k$ is bounded

2. k is closed i.e k^c is open

$$\text{let } x \in k^c$$

$$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$

$$G_1 = (-\infty, x - 1) \cup (x + 1, \infty)$$

$$G_2 = \left(-\infty, x - \frac{1}{2}\right) \cup \left(x + \frac{1}{2}, \infty\right)$$

...

...

...

$$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$

$\therefore k$ is closed.

Hence, from a) and b),

k is compact if and only if it is closed and bounded.

□

Sequence and Series

Definition 4.0.1 (Sequence and Series): *A sequence of real numbers is function defined on the set \mathbb{N} whose range is contained in the set $\mathbb{R}(x : \mathbb{N} \rightarrow \mathbb{R})$*

Denoted by $x, (x_n), (x_n, n \in \mathbb{N})$

example $\frac{1}{n}, \frac{1}{n^2}, 2n, n^2 + 1, n^2 - n$

- *Constant Sequence- $x_n = x, \forall n \in \mathbb{N}$*
- *Increasing Sequence- $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$*

- Strictly increasing sequence- $x_n < x_{n+1} \forall n \in \mathbb{N}$

$$x_n \rightarrow x$$

- Decreasing Sequence- $x_n \geq x_{n+1}, \forall n \in \mathbb{N}$

- Strictly Decreasing Sequence- $x_n > x_{n+1}, \forall n \in \mathbb{N}$

$$\varepsilon = 0.1 \\ |(-1)^n - 1| > \varepsilon \\ \uparrow \\ n > k(\varepsilon)$$

Definition 4.0.2 (Fibonacci Sequence): $x_1, x_2, x_{n+2} = x_{n+1} + x_n$ for some odd n

- Limit of Sequence- A Sequence $(x_n) \in \mathbb{R}$ is said to be converge to $x \in \mathbb{R}$ or x is said to be limit of (x_n) if for every $\varepsilon > 0 \exists > 0 k(\varepsilon) \in \mathbb{N}$ such that $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

$$|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon) \quad \times$$

If sequence has limit, we say that sequence is convergent. If it has no limit has no limit, we say that is divergent.

$$\lim(x_n) = x \text{ or } x_n \rightarrow x$$

- Oscillating Sequence: $(x_n) = (-1)^n, n \in \mathbb{N}$ - (non convergent)

$$\underline{(x_n) = \frac{(-1)^n}{n}, n \in \mathbb{N}} \quad \rightarrow \circ \quad \text{convergent}$$

Definition 4.0.3 (Uniqueness of limit point): A sequence in \mathbb{R} have atmost limit point one.

We will prove this by method of contradiction

let x_1 & x_2 be two limit points of x_n

\therefore for any $\varepsilon > 0 \forall n \geq k_1(\varepsilon) \& |x_n - x_1| < \varepsilon$

$\exists k_1(\varepsilon) \in \mathbb{N} \exists |x_n - x_1| < \varepsilon, \forall n \geq k_1(\varepsilon)$

$\exists k_2(\varepsilon) \in \mathbb{N} \exists |x_n - x_2| < \varepsilon, \forall n \geq k_2(\varepsilon)$

$k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\forall n \in \mathbb{N} \exists n \geq k(\varepsilon)$

$$|x_1 - x_2|$$

$$= |x_1 - x_n + x_n - x_2|$$

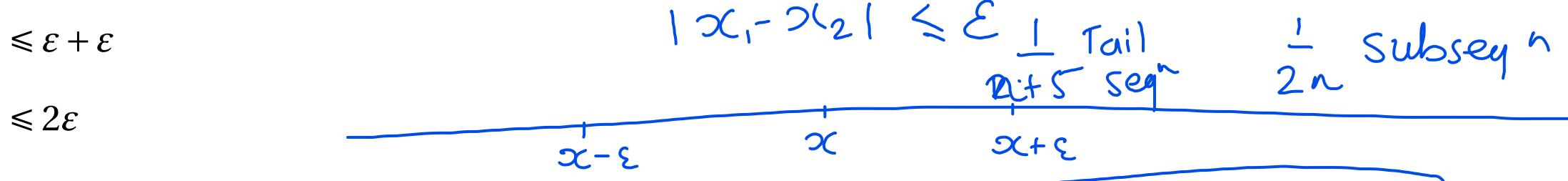
$$\leq |x_n - x_1| + |x_n - x_2|$$

$$\begin{aligned} x_n &\rightarrow x_1 & x_n &\rightarrow x_2 && > 10 \\ \text{any } \varepsilon_1 &> 0 \exists K_1(\varepsilon) \ni |x_n - x_1| < \varepsilon_1/2 \nexists n > K_1(\varepsilon) \\ \text{any } \varepsilon_2 &> 0 \exists K_2(\varepsilon) \ni |x_n - x_2| < \varepsilon_2/2 \nexists n > K_2(\varepsilon) && && > 15 \\ && \xrightarrow{\quad \quad \quad} && \xleftarrow{\quad \quad \quad} & \\ && x_1 - \varepsilon_1/2 && x_1 & \\ && && & \\ && && & x_1 + \varepsilon_1/2 \end{aligned}$$

$$K(\varepsilon) = \max(K_1(\varepsilon_1), K_2(\varepsilon_2))$$

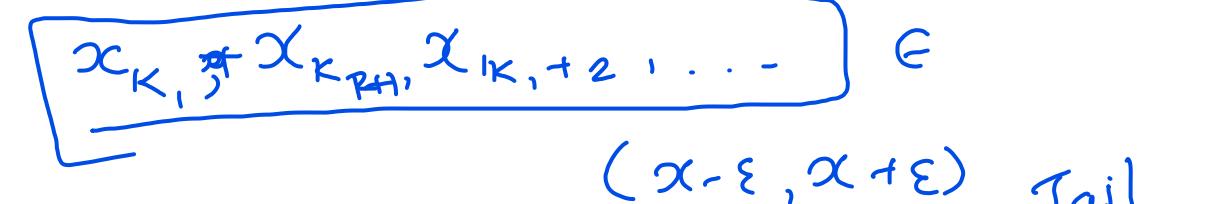
$$|x_1 - x_2| = |x_1 - x_n + x_n - x_2|$$

$$\begin{aligned} &\leq |x_1 - x_n| + |x_n - x_2| && \Delta \text{ inequ} \\ &\leq \varepsilon_1/2 + \varepsilon_2/2 \end{aligned}$$



As this statement is true for any $\varepsilon > 0$, $x_1 = \underline{x_2}$

Hence, Sequence have atmost one limit point.



Definition 4.0.4 (Tail Sequence): If $\{x_1, x_2, \dots\}$ is sequence of real numbers and if m is given natural number then m -tail of x_n is sequence

$$x_m = \{x_{m+n} / x_{m+1}, x_{m+2}, \dots\}$$

x_n

tail

x_{n+k}

Tail seq
1/n
 $\frac{1}{n+5}$

Theorem 4.0.1. Let x_n be sequence of real numbers and let $m \in \mathbb{N}$ then m -tail x_m of x_n converges if & only if x_n converges.

$$\frac{1}{6}, \frac{1}{7}, \frac{1}{8}$$

Proof. Let $x_n \rightarrow x$ i.e. $\lim_{n \rightarrow \infty} x_n = x$

\Rightarrow for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon$, $\forall n \geq k(\varepsilon)$

$\Rightarrow x - \varepsilon < x_n < x + \varepsilon$, $\forall n \geq k(\varepsilon)$

$$\Rightarrow x - \varepsilon < x_k, x_{k+1}, \dots < x + \varepsilon$$

$$\text{let } y_n = x_{m+n}, n$$

$$\Rightarrow x - \varepsilon < y_{k-m}, y_{k+1-m}, \dots < x + \varepsilon$$

$$\Rightarrow x - \varepsilon < y_n < x + \varepsilon \forall n \geq k(\varepsilon) - m = \underline{k_1(\varepsilon)}$$

$$\Rightarrow |y_n - x| < \varepsilon \forall n \geq k_1(\varepsilon)$$

$$y_n \rightarrow x$$

-Hence proved-

$$\left\{ \begin{array}{l} x_n \in \mathbb{R}, x \in \mathbb{R} \\ a_n > 0 \in \mathbb{R} \\ c > 0 \\ m \in \mathbb{N} \\ a_n \rightarrow 0 \\ |x_n - x| \leq c \cdot a_n \\ \lim x_n = x \end{array} \right.$$

$\nexists n > m$

Theorem 4.0.2. Let x_n be a sequence of real numbers and $x \in \mathbb{R}$ if a_n is sequence of positive real numbers with $\lim a_n = 0$ and if for some constant $c > 0$ and some $m \in \mathbb{N}$, we have $|x_n - x| \leq ca_n, \forall n \geq m$ then it follows that $\lim x_n = x$

Proof. Given that $\lim a_n = 0$

$$\text{i.e } a_n \rightarrow 0$$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{c} (\because c > 0)$

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - 0| < \frac{\varepsilon}{c}$$

$$a_n < \frac{\varepsilon}{c} \dots (\because a_n > 0)$$

let $k_1(\varepsilon) = \max(m_1 k_1(\varepsilon))$

$\forall n \geq k_1(\varepsilon)$

$$|x_n - x|$$

$$\leq c a_n$$

$$\leq c(\varepsilon/c)$$

$$\leq \varepsilon, \forall n \geq k_1(\varepsilon)$$

$$\therefore x_n \rightarrow x$$

□

Definition 4.0.5 (Bounded Sequence): A Sequence of real numbers x_n is said to be bounded

if $\exists m > 0$ such that $|x_n| \leq m, \forall n \in \mathbb{N}$

Theorem 4.0.3. The Convergent sequence of real numbers is bounded.

Proof. let $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geq k(\varepsilon)$

let

$$M = \max\{|x_1|, |x_2|, \dots, |x_k|, x + \varepsilon\}$$

$$\therefore |x_n| \leq M, \forall n$$

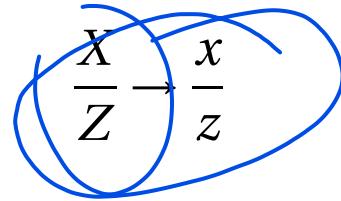
$\Rightarrow x_n$ is bounded.

-Hence Proved-

□

Theorem 4.0.4. a) Let x_n and y_n be sequence of real numbers that converges to x and y respectively and let $c \in \mathbb{R}$ then, the sequence $X + Y, X - Y, XY$ and CX converges to $x + y, x - y, xy$ and cx

b) If $x_n \rightarrow x$ and z_n is sequence of non-zero real numbers that converges to z and if $z \neq 0$ then



Proof. a) given that $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2} > 0$

$\exists k_1(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon/2, \forall n \geq k_1(\varepsilon)$$

also, $y_n \rightarrow y$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2} > 0$

$\exists k_2(\varepsilon) \in \mathbb{N}$ such that

$$|y_n - y| < \varepsilon/2, \forall n \geq k_2(\varepsilon)$$

$$\text{let } k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$$

$\therefore \forall n \geq k(\varepsilon)$

i) $|x_n + y_n - (x + y)| = |x_n - x + y_n - y|$

$\leq |x_n - x| + |y_n - y| \dots$ (triangular inequality)

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$\therefore (x_n + y_n) \rightarrow (x + y)$

ii) $|(x_n - y_n) - (x - y)| = |x_n - x - y_n + y|$

$\leq |x_n - x| + |y_n - y| \dots$ (triangular inequality)

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$\therefore (x_n - y_n) \rightarrow (x - y)$

iii) $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2M} > 0, \dots$ ($\because M > 0$)

$\exists k_1(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon/2M, \forall n \geq k_1(\varepsilon)$$

also, $y_n \rightarrow y$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2|x|} > 0, \dots (\because |x| > 0)$

$\exists k_2(\varepsilon) \in \mathbb{N}$ such that

$$|y_n - y| < \varepsilon/2|x|, \forall n \geq k_2(\varepsilon)$$

let $k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\therefore \forall n \geq k(\varepsilon)$

$$\begin{aligned} |(x_n y_n) - (xy)| &= |x_n y_n - xy_n + xy_n - xy| \\ &\leq |y_n||x_n - x| + |x_n||y_n - y| \dots \text{(triangular inequality)} \\ &\leq M \frac{\varepsilon}{2M} + |x|_2 \frac{\varepsilon}{2} \end{aligned}$$

$$\leq \varepsilon$$

$$\therefore x_n y_n \rightarrow xy$$

iv) $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{|c|} > 0, \dots (\because |c| > 0)$

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{|c|}, \forall n \geq k(\varepsilon)$$

$$|(cx_n - cx)| = |c|. |x_n - x|$$

$$\leq |c| \cdot \frac{\varepsilon}{|c|}$$

$$\leq \varepsilon$$

$$\therefore cx_n \rightarrow cx$$

b) $x_n \rightarrow x$ and $z_n \rightarrow z$

\therefore by definition, for any $\varepsilon > 0, \exists |z|.m > 0$

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon. |z|.m, \forall n \geq k(\varepsilon)$$

$$\text{let } y_n = \frac{1}{z_n}$$

$$\text{consider, } |(y_n - \cancel{x})| = \left| \frac{1}{z_n} - \frac{1}{z} \right| \\ = \frac{|z - z_n|}{|z_n.z|}$$

$$\leq \frac{\varepsilon. |z|. m}{|z_n|. |z|}$$

$$\leq \frac{\varepsilon. m}{|z_n|}$$

$$\leq \varepsilon \dots (\because z_n \text{ is bounded } \underline{m} < z_n < m)$$

$$\therefore \frac{1}{x_n} \rightarrow \frac{1}{z}$$

$$\therefore y_n \rightarrow y$$

we know that, $x_n y_n \rightarrow xy \dots (\because \text{if } x_n \rightarrow x \text{ & } y_n \rightarrow y \text{ then } x_n y_n \rightarrow xy)$

$$\therefore \frac{x_n}{z_n} \rightarrow \frac{x}{y}$$

-Hence Proved-



Theorem 4.0.5. If $x_n \rightarrow x$ and if $x_n \geq 0, \forall n \in \mathbb{N}$ then $x = \lim x_n \geq 0$

Proof. Given that, $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$

$\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

we will prove this by method of contradiction.

let if possible $x < 0$

$$\therefore -x > 0$$

Assume, $0 < \varepsilon < -x$

$$\therefore x - \varepsilon < 0 \text{ and } x + \varepsilon < 0 \text{ &}$$

$$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geq k(\varepsilon)$$

$$\therefore x_n < 0$$

which contradicts to given statement that $x_n \geq 0$

\therefore Our assumption is wrong.

$$\therefore x = \lim x_n \geq 0$$

-Hence Proved-



Theorem 4.0.6. If $x_n \rightarrow x$, $y_n \rightarrow y$ are convergent sequence of real numbers and if $x_n \leq y_n, \forall n \in \mathbb{N}$ then $\lim x_n \leq \lim y_n$

Proof. Given that, $x_n \rightarrow x$, and $y_n \rightarrow y$ also, $x_n \leq y_n, \forall n$

$$\Rightarrow y_n - x_n \geq 0$$

$$\Rightarrow z_n \geq 0$$

Now, $y_n - x_n \rightarrow y - x$ (say z)

As, $z_n \geq 0, z_n \rightarrow z$

$\therefore z \geq 0 \dots$ (by above theorem)

$$\therefore y - x \geq 0$$

$$\therefore y \geq x$$

$$\therefore x \leq y$$

-Hence Proved □

Theorem 4.0.7. If x_n is convergent to some $x \in \mathbb{R}$ and $a \leq x_n \leq b, \forall n$ then $a \leq x \leq b$

Proof. Given that, $x_n \rightarrow x$ and $a \leq x_n \leq b$

$$\text{let } a_n = a \& b_n = b$$

$$\text{i.e } a_n \rightarrow a \& b_n \rightarrow b$$

$$\therefore a_n \leq x_n \leq b_n$$

i.e $a_n \leq x_n \& x_n \leq b_n$

$\lim a_n \leq \lim x_n \& \lim x_n \leq b_n \dots$ (by above theorem)

$a \leq x$ and $x \leq b \therefore a \leq x \leq b$

-Hence Proved-



4.1 Squeeze Theorem

Theorem 4.1.1. Suppose x_n , y_n and z_n are sequence of real numbers $\exists x_n \leq y_n \leq z_n, \forall n \in \mathbb{N}$

$\lim x_n = \lim z_n$
 and $\lim x_n \leq \lim y_n$ then y_n is convergent and $\lim x_n = \lim y_n = \lim z_n$.

Proof. Given that, $x_n \leq y_n \leq z_n, \forall n$

let, $\lim x_n = \lim z_n = w$

i.e $x_n \rightarrow w$ and $z_n \rightarrow w$

\therefore by definition, for any $\varepsilon > 0 \ \exists$

$k_1(\varepsilon) \in \mathbb{N}$ and $k_2(\varepsilon) \in \mathbb{N}$ such that

$|x_n - w| < \varepsilon, \forall n \geq k_1(\varepsilon)$ and $|z_n - w| < \varepsilon, \forall n \geq k_2(\varepsilon)$

$\therefore w - \varepsilon \leq x_n \leq w + \varepsilon$ and $w - \varepsilon \leq z_n \leq w + \varepsilon$

$\therefore w - \varepsilon \leq x_n \leq y_n$ and $y_n \leq z_n \leq w + \varepsilon$

$\therefore w - \varepsilon \leq x_n \leq y_n \leq z_n \leq w + \varepsilon$

i.e $w - \varepsilon \leq y_n \leq w + \varepsilon$

i.e $|y_n - w| < \varepsilon, \forall n \in k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\therefore y_n \rightarrow w$

$\therefore \lim x_n = \lim y_n = \lim z_n = w$

□

Theorem 4.1.2. Given that, $x_n \rightarrow x$ then Show that,

a) $|x_n| \rightarrow |x|$

b) $\sqrt{x_n} \rightarrow \sqrt{x}$

Proof. Given that, $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

consider,

$$||x_n| - |x||$$

$\leq |x_n - x| \dots$ (by corollary of triangular inequality)

$$\leq \varepsilon$$

$$|x_n| \rightarrow |x|$$

Given that, $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$, $\sqrt{x} > 0$, $\frac{\varepsilon}{\sqrt{x}} > 0$, $\varepsilon \sqrt{x} > 0$.

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| \leq \underline{\varepsilon \sqrt{x}}, \forall k(\varepsilon) \in \mathbb{N}$$

As, $\sqrt{x} > 0$

$$\therefore 0 < \underline{\sqrt{x}} < \underline{\sqrt{x_n} + \sqrt{x}}$$

$$\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{x_n} + \sqrt{x}} \quad (4.1)$$

$$|\sqrt{x_n} - \sqrt{x}|$$

$$= \frac{|\sqrt{x_n} - \sqrt{x}| \cdot |\sqrt{x_n} + \sqrt{x}|}{|\sqrt{x_n} - \sqrt{x}|}$$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$$

$$\leq \frac{|x_n - x|}{\sqrt{x}} \dots \text{(from 4.1)}$$

$$\leq \frac{\varepsilon \cdot \sqrt{x}}{\sqrt{x}}$$

$$\leq \varepsilon$$

$$\therefore \sqrt{x_n} \rightarrow \sqrt{x}$$

□

4.2 Monotone Sequence

- *Monotone decreasing:* $x_n \geq x_{n+1}, \forall n$

$$x_n > x_{n+1} \quad \frac{1}{n}$$

- *Monotone increasing:* $x_n \leq x_{n+1}, \forall n$

$$x_n \leq x_{n+1} \quad n$$

x_n is called as monotone if it is increasing or decreasing.

Theorem 4.2.1 (Monotone Convergence theorem). A monotone sequence of real numbers is convergent if and only if

- a) If x_n is bounded increasing sequence

$$\lim_{n \rightarrow \infty} (x_n) = \text{Sup} \{x_n, n \in \mathbb{N}\}$$

b) If x_n is bounded decreasing sequence

$$\lim(x_n) = \inf\{x_n, n \in \mathbb{N}\}$$

Proof. We know that, Convergent sequence must be bounded. 

Conversely, let x_n be monotone bounded sequence.

a) Assume x_n is increasing and bounded.

As x_n is bounded $M \in \mathbb{R}, |x_n| \leq M, \forall n$

let, $S = \{x_n, \forall n \in \mathbb{N}\}$

M upper bound of S

\therefore By completeness property, $\exists x^* \in \mathbb{R}$

$\exists x^* = \sup\{x_n, n \in \mathbb{N}\}$

$\therefore x_n \leq x^* \forall \mathbb{N}$

for any $\varepsilon > 0$ $x^* - \varepsilon$ is not supremum of S

$\therefore x^* - \varepsilon < x_k \leq x^*$, for some k

$$\Rightarrow x^* - \varepsilon < x_k \leq x_{k+1} \leq x_{k+2} \leq \dots \leq x^*$$

$$\therefore x^* - \varepsilon < x_n < x^*, \forall n \geq k(\varepsilon)$$

$$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$$

$$\therefore x^* = \lim x_n$$

i.e x_n is convergent sequence.

b) Assume x_n is decreasing and bounded.

As x_n is bounded $M \in \mathbb{R}, |x_n| \leq M, \forall n$

let, $S = \{x_n, \forall n \in \mathbb{N}\}$

$-M$ lower bound of S

\therefore By completeness property, $\exists x^* \in \mathbb{R}$

$$\exists x^* = \inf\{x_n, n \in \mathbb{N}\}$$



$$\therefore x_n \geq x^* \forall \mathbb{N}$$

for any $\varepsilon > 0$ $x^* + \varepsilon$ is not lower bound of S

$\therefore x^* < x_k < x^* + \varepsilon$, for some k

$\Rightarrow x^* < \dots \leq x_{k+2} \leq x_{k+1} \leq x_k < x^* + \varepsilon$

$\therefore x^* < x_n < x^* + \varepsilon$

$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$

$\therefore x^* = \lim x_n$

$\frac{1}{n}$

i.e x_n is convergent sequence.

□

Theorem 4.2.2. If x_n converges to x then any subsequences x_{n_k} of x_n also converges to x .

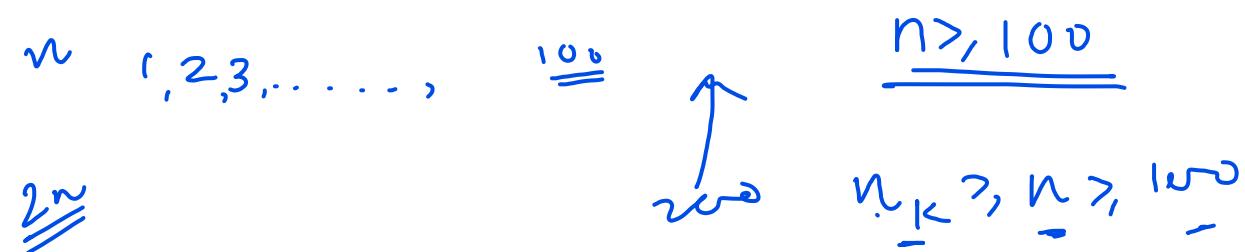
Proof. for any $\varepsilon > 0 \exists k(\varepsilon) \in \mathbb{N}$ such that,

$$|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$$

let subsequence $x_{n_k} = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$

$$\text{As } x_n \rightarrow x \Rightarrow x - \varepsilon < x_n < x + \varepsilon \quad \forall n \geq k(\varepsilon)$$

$$\text{Also, } n_k \geq n \geq k(\varepsilon)$$



$$\rightarrow x - \varepsilon < x_{n_k} < x + \varepsilon, \forall n_k \geq k(\varepsilon)$$

$\therefore x_{n_k} \rightarrow x$

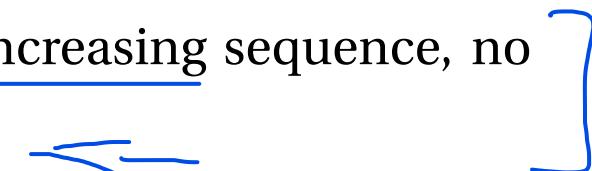
□

Theorem 4.2.3 (Monotone Subsequence theorem). If x_n is sequence of real numbers then there is subsequence of x_n that is monotone.

Proof. We will say that m^{th} term x_m is a peak if $\underline{x_m} \geq \underline{x_n} \forall n \geq m$.

$$x_n > x_{n+s} \rightarrow \text{No}$$

Note that, In a decreasing sequence, every term is peak while in increasing sequence, no term is peak.



Case-1:-

x_n has infinitely many peaks. In this case, we list the peaks by,

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \dots \geq x_{m_k}, \dots$$

— — — — —

\therefore subsequence $\underline{x_{m_k}}$ is decreasing subsequence of x_n .



Case-2:-

x_n has finitely number of peaks.

let these peaks be denoted by,

x_{m_r}

$x_{m_1}, x_{m_2}, x_{m_3} \dots x_{m_r}$

let $S_1 = m_r + 1$ be the first index beyond the last peak since x_{S_1} is not peak $\exists S_2 > S_1$

$\exists x_{S_1} < x_{S_2}$ since x_{S_2} is not peak $\exists S_3 > S_2$

$\exists x_{S_2} < x_{S_3}$ continuing this way, we obtain an increasing sequence. \square

Theorem 4.2.4 (Bozano- Weistress theorem). A bounded sequence of real numbers has convergent subsequence.

Proof. Let x_n be bounded sequence.

\therefore by monotone subsequence theorem,

$\exists x_{n_k}$ subsequence of x_n that is monotone.

As x_n is bounded x_{n_k} is also bounded

\therefore by monotone convergence theorem,

x_{n_k} is monotone and bounded so convergent.

✓ x_n bounded
 $\underline{m} \leq \underline{T} \rightarrow x_{n_k} \uparrow$ or \downarrow

Let $\underline{x_{n_k}} \uparrow$ bounded
 $\hookrightarrow x_{n_k}$ is convergent



$$\begin{aligned} (-1)^{2n+1} &\rightarrow -1 \\ (-1)^{2n} &\rightarrow 1 \end{aligned}$$

\square

4.3 Cauchy Sequence

Definition 4.3.1 (Cauchy Sequence): A sequence of real numbers is said to be cauchy if for

every $\varepsilon > 0$, $\exists H(\varepsilon) \in \mathbb{N}$ such that $|X_n - X_m| < \varepsilon$, $\forall n, m \geq H(\varepsilon)$

Theorem 4.3.1. Every convergent sequence is cauchy.

Proof. let $x_n \rightarrow x$

for any $\frac{\varepsilon}{2} > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

$\exists |X_n - x| < \frac{\varepsilon}{2}$, $\forall n \geq k(\varepsilon)$ let, $k_1, k_2 \in \mathbb{N}$ such that $\forall k_1, k_2 \geq k(\varepsilon)$

$$|X_{k_1} - x_{k_2}|$$

$$\leq |X_{k_1} - x| + |X_{k_2} - x|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

Hence, every convergent sequence is cauchy.

Theorem 4.3.2. *A cauchy sequence of real numbers is bounded.*

Proof. let x_n be cauchy sequence and
let $\varepsilon = 1$ if $H = H(1)$ and $n \geq H$ then $n \geq H$.

$$M = \sup\{|x_1|, |x_2|, |x_3|, \dots, |x_{H-1}|, |x_H| + 1\}$$

then it follows that $|x_n| \leq M \forall n$

∴ cauchy sequence of real numbers is bounded.

Every cauchy seqⁿ of real nos
is convergent.

Every convergent seqⁿ of
real nos is bounded.

⇒ A cauchy seqⁿ of real nos
is bounded. □

Definition 4.3.2 (Cauchy convergence criterion): *A Sequence of real numbers is convergent if and only if it is cauchy sequence.*

Definition 4.3.3 (Contractive Sequence): *We say that the sequence x_n of real numbers is contractive sequence if there exists a constant c , $0 < c < 1$ such that,*

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|, \forall n$$

Theorem 4.3.3. Contractive sequence is cauchy sequence.

Proof. let x_n is contractive sequence

$\therefore \exists c, 0 < c < 1$ such that

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|, \forall n$$

for $\varepsilon > 0$ choose $k(\varepsilon) \in \mathbb{N}$ \exists for $m > n$

$$|x_m - x_n|$$

$$= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$\leq c|x_{m-1} - x_{m-2}| + c|x_{m-2} - x_{m-3}| + \dots + c|x_n - x_{n-1}|$$

$$\leq c^2|x_{m-2} - x_{m-3}| + c^2|x_{m-3} - x_{m-4}| + \dots + c|x_n - x_{n-1}|$$

$$\leq (c^{m-n} + c^{m-n-1} + \dots + c)|x_n - x_{n-1}|$$

$$\leq \frac{c(1 - c^{m-n})}{1 - c} |x_n - x_{n-1}|$$

$$\leq \varepsilon \quad \because \frac{c(1 - c^{m-n})}{(1 - c)} < 1$$

$\therefore x_n$ is cauchy sequence.

□

Divergent Sequence Let x_n be sequence of real numbers

$$|x_n - x_{n+1}|$$

$$|x_{n+2} - x_{n+1}| \leq C \cdot |x_{n+1} - x_n|$$

a) $x_n \rightarrow +\infty$ and $\lim x_n = +\infty$

if every $\alpha \in \mathbb{R}$ there exists a natural number $k(\alpha)$ such that if $n \geq k(\alpha)$, then $x_n > \alpha$.

b) $x_n \rightarrow -\infty$ and $\lim x_n = -\infty$

if every $\beta \in \mathbb{R}$ there exists a natural number $k(\beta)$ such that if $n \geq k(\beta)$, then $x_n < \beta$.

We say that x_n is properly divergent if $\lim x_n = +\infty$ or $-\infty$

4.4 Infinite Series

$$\alpha \underset{n \rightarrow \infty}{\sim} \in \mathbb{R}$$

$$\frac{n^2}{n}$$

Definition 4.4.1 (Infinite Series): If x_n is sequence in \mathbb{R} , then the infinite series generated by x_n is sequence S_n

$$\underline{S_1} = x_1$$

$$S_2 = x_1 + x_2 = S_1 + x_2$$

.

.

.

$$S_n = x_1 + x_2 + \dots + x_n \quad \checkmark$$

Denoted by $\sum x_n$ or $\sum_{n=1}^{\infty} x_n$

Example 20:

$$1. \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots \quad \underline{|r| < 1}$$

~~r ≠ 1~~

$$S_n = \sum_{i=1}^n x_i$$

$$2. \sum_{n=1}^{\infty} (-1)^n = (-1) + 1 + (-1) + \dots$$

-1
0
-1
0

$$3. \sum \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \sum \left(\frac{1}{n} - \frac{1}{n+1} \right) 1 + 1$$

Theorem 4.4.1 (The n^{th} term test). if $\sum x_n$ converges then $\lim x_n = 0$

Proof. By definition $\sum x_n$ converges if S_n converges,

$$\text{Since } = \sum_{i=1}^n x_i$$

$$\therefore x_n = S_n - S_{n-1}$$

$$\therefore \lim x_n = \lim S_n - \lim S_{n-1} = 0$$

□

Definition 4.4.2 (Cauchy Criterion for Series): *The series $\sum x_n$ converges if and only if $\forall \varepsilon > 0$,*

$\exists M(\varepsilon) \in \mathbb{N} \exists$ if $m > n \geq M(\varepsilon)$ then

$$|S_m - S_n| = |x_{n+1} + x_{n+2} + \dots + x_m| < \varepsilon$$

Theorem 4.4.2. let x_n be a sequence of non-negative real numbers then the series $\sum x_n$ converges if and only if the sequence S_k of partial sum is bounded.

$$\sum x_n = \lim S_k = \sup\{S_k : k \in \mathbb{N}\}$$

Theorem 4.4.3. Show that, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, $|r| < 1$

Proof. Suppose,

$$\underline{S_{n+1} = 1 + r + \dots + r^n}$$

$$\underline{S_n = 1 + r + \dots + r^{n-1}}$$

$$\underline{rS_n = (r + r^2 + \dots + r^n)}$$

$$\therefore \underline{S_{n+1} - rS_n = 1}$$

$$\therefore \lim_{n \rightarrow \infty} (S_{n+1} - rS_n) = 1 \lim_{n \rightarrow \infty} 1 = 1$$

$$\therefore (S - rS) = 1 (\dots \text{where } S \sum_{n=0}^{\infty})$$

$$S(1 - r) = 1$$

$$S = \frac{1}{(1 - r)}$$

Theorem 4.4.4. The p Series $\sum \frac{1}{n^p}$ converges when $\underline{p > 1}$

Proof. if $k_1 = 2 - 1 = 1, S_{k_1} = 1$

$$k_1 = 2^2 - 1 = 3, 2^p < 3^p$$

$$2 < 3$$

$$2^p < 3^p$$

$$\underline{\underline{\frac{1}{2^p} > \frac{1}{3^p}}}$$

$$\sum \frac{1}{n^p} = 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{\dots} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} + \dots}_{\dots} \quad \square$$

$$\leq 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \dots + \frac{1}{4^p}$$

$$\leq 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \dots$$

$$\underline{\underline{\frac{2}{2^p} + \frac{2^2}{2^{2p}} + \frac{2^3}{2^{3p}} \dots}}$$

$$\underline{\underline{\frac{1}{2^{p-1}} + \frac{1}{2^{2p-2}} + \frac{1}{2^{3(p-1)}}}}$$

$$S_{k_2} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p} \right) < \frac{1}{1^p} + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

Handwritten proof for the convergence of the p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \sum_{n=1}^{\infty} \frac{1}{2^{p-1}} \cdot \frac{1}{2^{p-1}} \cdot \dots \cdot \frac{1}{2^{p-1}} \cdot \frac{1}{2^p} = \frac{1}{2^{p-1}} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2^{p-1}} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^{p-1}} \cdot 2 = \frac{1}{2^{p-2}}$$

$|r| < 1$

further, if $k_3 = 2^3 - 1$ then

$$S_{k_3} < S_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}} \cdot \frac{1}{4^{p-1}}$$

finally, let $r = \frac{1}{2^{p-1}}$ Since $p > 1$

Using mathematical induction

we can show that if $k_j = 2^j - 1$

$$0 < S_{k_j} < 1 + r + r^2 + \dots + r^{j-1} < \frac{1}{1-r}$$

\Rightarrow The p-series converges if $p > 1$ □

The alternating harmonic series

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots$$

is convergent

$$\uparrow \text{let } S_{2n} = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n} \right) \quad \lim \text{ exist.}$$

$$\downarrow S_{2n+1} = 1 - \left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{1}{4} - \frac{1}{5} \right) - \dots - \left(\frac{1}{2n} - \frac{1}{2n+1} \right) \quad \lim S_{2n+1} = \lim S_{2n} + \frac{1}{2n+1}$$

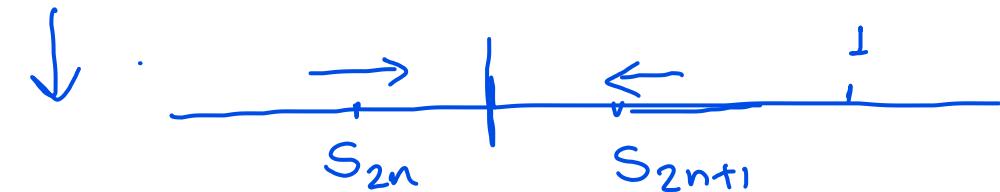
$$= \lim S_{2n} + \lim \frac{1}{2n+1}$$

$$\text{Since } 0 < S_{2n} < S_{2n} + \frac{1}{2n+1} = S_{2n+1} \leq 1$$

S_{2n} and S_{2n+1} both bounded and monotone, so by monotone convergence theorem, must

be convergent and to same point.

$$\sum \frac{(-1)^{n+1}}{n} \text{ must be convergent.}$$



Theorem 4.4.5 (The Comparison Test). Let x_n and y_n be real sequence and for some $k \in \mathbb{N}$

$$0 \leq x_n \leq y_n, \forall n \geq k$$

a) Convergent of $\sum y_n \Rightarrow$ Convergence of $\sum x_n$

b) Divergence of $\sum x_n \Rightarrow$ divergence of $\sum y_n$

$$\sum x_n \leq \sum y_n$$

$$\infty \leq \sum x_n \leq \sum y_n = \infty$$

Proof. a) Suppose $\sum y_n$ is convergent,

i.e for any $\varepsilon > 0, \exists M(\varepsilon) \in \mathbb{N} \ni m > n \geq M(\varepsilon)$

$$|y_{n+1} + \dots + y_m| < \varepsilon$$

if $m > \text{Sup}(k, M(\varepsilon))$

$$0 \leq x_{n+1} + \dots + x_m \leq y_{n+1} + \dots + y_m < \varepsilon$$

$\Rightarrow \sum x_n$ converges.

$$A \rightarrow B$$

$$\neg B \rightarrow \neg A$$

b) This statement is contrapositive to a)

□

Theorem 4.4.6 (Limit Comparison Test). Suppose x_n and y_n are strictly positive sequences and

Suppose following limit exists

$$r = \lim_{n \rightarrow \infty} \left(\frac{x_n}{y_n} \right)$$

a) If $r \neq 0$ then $\sum x_n$ convergent iff $\sum y_n$ convergent.

$$r=0 \quad \sum y_n \Rightarrow \sum x_n$$

$$r = \lim_{n \rightarrow \infty} \frac{x_n}{y_n}$$

$$r \neq 0 \quad \sum y_n \Leftrightarrow \sum x_n$$

b) If $\underline{r = 0}$ then if, $\sum y_n$ convergent then $\sum x_n$ convergent.

Proof. a) Given $r = \lim \frac{x_n}{y_n}$

\therefore by definition, For any $\varepsilon > 0$, $\exists, k(\varepsilon) \in \mathbb{N}$

such that $\left| \frac{x_n}{y_n} - r \right| < \varepsilon, \forall n \geq k(\varepsilon)$

As $r \neq 0, \Rightarrow r > 0 \Rightarrow \varepsilon \frac{r}{2}$

$$r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$$

$$\left(\frac{r}{2} \right) y_n < x_n < \left(\frac{3r}{2} \right) y_n$$

$$\therefore \underline{\left(\frac{r}{2} \right) y_n < x_n}$$

\Rightarrow if x_n converges then $\sum y_n$ also converges. ... (by comparison test)

$$\therefore x_n < \left(\frac{3r}{2}\right) Y_n$$

\Rightarrow If $\sum y_n$ converges then x_n also converges. ... (by comparison test)

$$r = 0 \text{ i.e } \lim \left(\frac{x_n}{y_n} \right) = 0$$

\therefore by definition, For any $\varepsilon > 0$, $\exists, k(\varepsilon) \in \mathbb{N}$

such that

$$\left| \frac{x_n}{y_n} - 0 \right| < \varepsilon$$

$$\left| \frac{x_n}{y_n} \right| < \varepsilon$$

$$\frac{x_n}{y_n} < \varepsilon$$

$$0 < x_n < \varepsilon y_n$$

\therefore By comparison test,

$\sum x_n$ converges if $\sum y_n$ converges.

□

Definition 4.4.3 (Absolute Convergence): *let x_n be sequence in \mathbb{R} . We say that $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent . A series is said to be conditionally convergent if it is convergent but not absolutely convergent.*

Example 21:

$\sum \frac{(-1)^n}{n}$ is convergent but $\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$ is not convergent
 $\therefore \sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 4.4.7. *If a series is absolutely convergent then it is convergent.*

Proof. $\sum |x_n|$ is convergent

\therefore for any $\varepsilon > 0$ $M(\varepsilon) \in \mathbb{N}$

$$|x_{n+1}| + |x_{n+2}| + \dots + |x_m| < \varepsilon \forall m > n > M(\varepsilon)$$

by triangular inequality.

$$|x_{n+1} + x_{n+2} + \dots + x_m| \leq \varepsilon$$

$$|x_{n+1}| + |x_{n+2}| + \dots + |x_m| \leq \varepsilon \forall m > n > M(\varepsilon)$$

$\Rightarrow \sum x_n$ is convergent.

□

Theorem 4.4.8 (Limit Comparison Test- II-), Suppose x_n and y_n are non-zero real sequence and Suppose that following limit exists in \mathbb{R}

$$r = \lim \left(\frac{x_n}{y_n} \right)$$

a) If $r \neq 0$ then $\sum x_n$ absolutely convergent iff $\sum y_n$ is absolutely convergent.

b) If $r = 0$ and $\sum y_n$ is absolutely convergent then $\sum x_n$ absolutely convergent.

Theorem 4.4.9 (Root test). Let x_n be sequence in \mathbb{R} . Suppose that the limit $r = \lim |x_n|^{\frac{1}{n}}$ exists in \mathbb{R} then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

Proof. $r < 1$, $r = \lim |x_n|^{\frac{1}{n}}, \exists r_1, r_1 \in (r, 1)$

$$|x_n|^{\frac{1}{n}} \leq r_1$$

$$\therefore |x_n| \leq \sum r_1^n$$

by comparison test,

$|x_n| < (r_1)^n$ it is convergent

$|x_n| < (r_1)^n$ it is absolutely convergent.

Comparison

$$r = \lim \frac{x_n}{y_n}$$

Root $r = \lim |x_n|^{\frac{1}{n}}$

$$\begin{cases} r \neq 0 \\ r=0 \end{cases}$$

$$\begin{cases} \sum |x_n| \Leftrightarrow \sum |y_n| \\ \sum |y_n| \Rightarrow \sum |x_n| \end{cases}$$

$$\begin{cases} r < 1 & \text{cgt} \\ r > 1 & \text{dvt} \end{cases}$$

$r=1$ Test fails

Theorem 4.4.10 (Ratio Test). Let x_n be non-zero sequence in \mathbb{R} . Suppose $r = \lim \left| \frac{x_{n+1}}{x_n} \right|$ exists

then $\sum x_n$ is absolutely convergent when $r < 1$ and divergent when $r > 1$

Proof. $r < 1, r_1 \in (r, 1)$

$$\left| \frac{x_{n+1}}{x_n} \right| \leq r_1, \forall n > k(\varepsilon)$$

$$|x_{n+1}| \leq r_1 |x_n|$$

$$|x_{n+1}| \leq r_1 |x_n| < r_1 \cdot r_1 |x_{n-1}| < \dots < r_1^n |x_1|$$

$$\therefore |x_{n+1}| < r_1^n \cdot c$$

$$\therefore \sum |x_{n+1}| < \sum r_1^n \cdot c$$

\therefore by comparison test,

$\sum x_n$ is absolutely convergent

□

4.5 Establish the converges/divergence of series

Example 22:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum \left(\frac{1}{n+2} - \frac{1}{n+1} \right)$$

Diagram illustrating the decomposition of the series into two parts, $\sum \frac{1}{n+2}$ and $\sum \frac{1}{n+1}$, which are shown to converge to zero.

$$\underline{S_n} = 1 - \frac{1}{n+2} \rightarrow 0$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots$$

The series is converges to zero

or $(n+1)(n+2) > n.n$

$$\therefore \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

$$\therefore 0 < \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

by comparison test,

$\sum \frac{1}{n^2}$ is convergent.

Example 23:

$$2^{(\frac{-1}{n})}$$

$$\sum 2^{-\frac{1}{n}}$$

$$\sum 2^n$$

$$\lim_{n \rightarrow \infty} 2^{(\frac{-1}{n})} = 1 \neq 0$$

\therefore by n^{th} term test

$2^{(\frac{-1}{n})}$ is divergent

Example 24:

$$\frac{n}{2^n}$$

Applying ratio test

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{(n+1)/2^{(n+1)}}{n/2^n} \right| = \left| \frac{n+1}{n} \right| \cdot \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{2} < 1$$

$\therefore \frac{\sum n}{2^n}$ is convergent.

Cauchy

Definition 4.5.1 (Integral test): Let f be a positive decreasing function on $\{t, t \geq 1\}$ then the

series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral

\int_1^{\infty} f(t) dt = \lim_{b \rightarrow \infty} \int_1^b f(t) dt

exists. In the case of convergence, the partial sum

$$x_n = \frac{1}{n} \downarrow$$

$$f(x) = \frac{1}{x}$$

$S_n = \sum_{k=1}^n f(k)$ and sum $S = \sum_{k=1}^{\infty} f(k)$ satisfy the estimates

$$\int_{n+1}^{\infty} f(t) dt \leq S - S_n \leq \int_1^{\infty} f(t) dt$$

Example 25:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

10-1, 25

$$= \int_1^{\infty} \frac{1}{t^p} dt, \quad x_n \frac{1}{n^p}$$

$$= \left[\frac{t^{-p+1}}{-p+1} \right]_1^{\infty}$$

$$= \frac{1}{1-p} \left[\frac{1}{t^{p-1}} \right]_1^{\infty}$$

$$\frac{1}{p-1}, p > 1$$

$\therefore \sum \frac{1}{n^p}$ is convergent

Definition 4.5.2 (Raabies Test): Let x_n be non-zero sequence in \mathbb{R} and let

$a = \lim n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right)$ whenever this limit exists then $\sum x_n$ absolutely convergent when $a > 1$
 and is not absolutely convergent when $a \leq 1$

$n(1 -$

Example 26:

$$x_n = \frac{1}{n^p}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \left| \frac{n^p}{(n+1)^p} \right| = \left| \frac{1}{1 + \frac{1}{n}} \right|$$

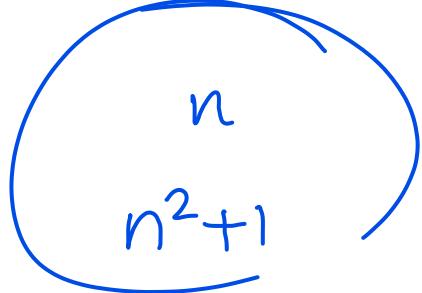
$$\therefore \lim n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right) = \lim n \left(1 - \left| \left(\frac{1}{1 + \frac{1}{n}} \right)^p \right| \right)$$

$$= \lim \left(\frac{\left(1 + \frac{1}{n} \right)^p - 1}{\frac{1}{n} \left(1 + \frac{1}{n} \right)^p} \right)$$

$$= \lim \left(\frac{p \left(1 + \frac{1}{n} \right)^{p+1} \left(-\frac{1}{n^2} \right)}{\frac{1}{n} \left(1 + \frac{1}{n} \right)^p} \right)$$

$$= \lim \left(\frac{-\frac{p}{n} \left(1 + \frac{1}{n} \right)}{\left(1 + \frac{1}{n} \right)^p} \right)$$

?



n

$n^2 + 1$

0

L'Hospital Rule

$$= \lim_{n \rightarrow \infty} \left(\frac{-p \left(1 + \frac{1}{n} \right)}{n} \right)$$

$$= \lim_{n \rightarrow \infty} p \left(-\frac{1}{n} - \frac{1}{n^2} \right)$$

$$= p$$

Example 27:

$$x_n = \frac{1}{n(n+1)}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} \right| = \left| \frac{n}{n+2} \right| = \left| \frac{1}{1 + \frac{2}{n}} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{1}{1 + \frac{2}{n}} \right| = 1$$

\therefore Ratio test fails ($\because r = 1$)

we know $n(n+1) > n.n$

$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$

$$\therefore \frac{1}{n(n+1)} < \frac{1}{n^2} \quad (0 < x_n < y_n)$$

by comparison test

As $\sum \frac{1}{n^2}$ is convergent, $\sum \frac{1}{n(n+1)}$ is also convergent.

Example 28:

$$\frac{n!}{n^n}$$

Using Raabies test, we have,

$$\text{consider, } \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\Rightarrow \left(1 - \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right)$$

$$= \frac{\left(1 + \frac{1}{n}\right)^n - 1}{\left(1 + \frac{1}{n}\right)^n}$$

$$= n \left(1 - \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right)$$

$$\text{as } n \rightarrow \infty, \quad r = \lim n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right) = 0 < 1$$

$\therefore \sum x_n = \sum \frac{n!}{n^n}$ is not absolutely convergent. i.e divergent.

Example 29:

$$\frac{n^2}{\sqrt{n+1}} = \left| \frac{x_{n+1}}{x_n} \right| = \frac{(n+1)^2}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{n^2}$$

$$= \left(1 + \frac{1}{n}\right)^2 \frac{\sqrt{1 + \frac{1}{n}}}{1 - \frac{2}{n}}$$

$$\Rightarrow n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right)$$

$$= \frac{\sqrt{1 + \frac{2}{n}} - \left(1 + \frac{1}{n}\right)^2 \sqrt{1 + \frac{1}{n}}}{\sqrt{\frac{1}{n^2} + \frac{2}{n^3}}}$$

$\therefore \sum \frac{n^2}{\sqrt{n+1}}$ is not absolutely convergent. i.e divergent.

4.6 Test for Non-Absolute Convergence

Definition 4.6.1 (Alternative Series): A sequence of non-zero real numbers is said to be alternating if the terms $(-1)^{(n+1)}x_n$, $n \in \mathbb{N}$ are all positive (or all negative) real numbers. If the sequence x_n is alternating, we say that the series $\sum x_n$ is alternating series.

Theorem 4.6.1 (Alternating Series test). Let z_n be decreasing sequence with strictly positive numbers with $\lim z_n = 0$ then the alternating series $\sum (-1)^{n+1}z_n$ is convergent.

Proof. Given that z_n decreasing sequence and let $S_n = \sum (-1)^{n+1}z_n$

We have

$$S_{2n} = (z_1 - z_2) + (z_3 - z_4) + \dots + (z_{2n-1} - z_{2n}) -$$

and Since $(z_k - z_{k+1}) \geq 0$, it follows that S_{2n} is increasing sequence

$$S_{2n} = z_1 - (z_2 - z_3) + \dots - (z_{n-2} - z_{n-1}) - z_{2n}$$

$$\therefore S_{2n} \leq z_1$$

\therefore bounded by MCT, S_{2n} must be convergent to some number $c \in \mathbb{R}$.

We have to show that entire $S_n \rightarrow c$ if $\varepsilon > 0$, let $k \in \mathbb{N}$. if $n \geq k$

$$|S_{2n} - c| \leq \frac{\varepsilon}{2} \text{ and } z_{2n+1} \leq \frac{\varepsilon}{2}$$

$$|S_{2n+1} - c| = |S_{2n} + z_{n+1} - c|$$

$$\leq |S_{2n} - c| + |z_{n+1}|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$S_n \rightarrow c$$

$$S_n = \sum (-1)^{n+1} z_n \text{ is convergent.}$$

Lemma 4.6.2 (Abels Lemma). $x_n, y_n \in \mathbb{R}$ $S_n = \sum_{i=1}^n y_i$ with $S_0 = 0$ if $m > n$ then,

$$\sum_{k=n+1}^m x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

$$S_n \rightarrow ?$$

$$S_{2n} \rightarrow c \quad \checkmark$$

$$\underline{\underline{S_{n+1}}} \rightarrow c$$

$$\sum (\underline{\underline{(-1)^{n+1}}}) z_n$$

$$z_n \downarrow 0$$

$$\underline{\underline{z_n \leq \varepsilon/2}}$$

$$\exists N \forall n > N$$

$$\underline{\underline{S_{2n+1}}} = \underline{\underline{S_{2n}}} + \underline{\underline{(z_{2n+1})}}$$

□

$$\text{Proof. } y_k = S_k - S_{k-1} \quad \left(\because S_k = \sum_{i=1}^k y_i \& S_{k-1} = \sum_{i=1}^{k-1} y_i \right)$$

$$x_k y_k = x_k S_k - x_k S_{k-1}$$

$$\begin{aligned} & \sum_{k=n+1}^m x_k y_k \\ &= \sum_{k=n+1}^{m-1} (x_k S_k - x_k S_{k-1}) \end{aligned}$$

$$= x_{n+1} S_{n+1} - x_{n+1} S_n + x_{n+2} S_{n+2} - x_{n+2} S_{n+1} + \dots + x_m S_m - x_m S_{m-1}$$

$$= (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k \quad \checkmark$$

□

Theorem 4.6.3 (Diricblet's Test). *If x_n is decreasing $\neq 0$, if $S_n = \sum y_i$ is bounded then $x_n y_n$ is convergent.*

Proof. Let $S_n \leq B$, $\forall n \in \mathbb{N}$. if $m > n$, by abels lemma and $x_k - x_{k+1} > 0$ (as x_n is decreasing)

Consider,

$$\left| \sum_{k=n+1}^m x_k y_k \right| = \left| (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k \right|$$

$$\leq |(x_m S_m - x_{n+1} S_n)| + \sum_{k=n+1}^{m-1} |(x_k - x_{k+1})| |S_k|$$

Suppose,

$$\begin{aligned} & S_m, S_n, S_k = B \\ & \leq |x_m - x_{n+1}| B + B \sum_{k=n+1}^{m-1} |(x_k - x_{k+1})| |S_k| \\ & \leq \frac{\epsilon}{2B} B + B \frac{\epsilon}{2B} \end{aligned}$$

$\leq \varepsilon$

$\therefore \sum x_n y_n$ is convergent.

□

Theorem 4.6.4 (Abel's Test). *If x_n convergent monotone sequence and y_n is convergent then the series is $x_n y_n$ also convergent.*

Proof. Let x_n is decreasing x

$$u_n = x_n - x \text{ decreasing } 0$$

$\sum u_n y_n$ is convergent by diricblets test

$$\begin{aligned} & \sum_n x_n y_n \\ &= \sum_n (x + u_n) y_n \\ &= x \sum_n y_n + \sum_n u_n y_n \end{aligned}$$

$\sum_n x_n y_n$ is convergent sequence.

□

Example 30:

$\sum a_n$ convergent then

1. $\sum b_n = \frac{a_n}{n}$ is convergent sequence.

2. $\sum n^{1/n} a_n$ is divergent sequence.

3. $\sum a_n \sin n$ is divergent sequence.

4. $\sum \frac{\sqrt{a_n}}{n}$ is convergent sequence.

5. $\sum \sqrt{a_n}$ is divergent sequence.

Function and Continuity

Definition 5.0.1 (Cluster Point): Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is cluster point of A if every $\delta > 0 \exists$ atleast one point $x \in A$, $x \neq c \ni |x - c| < \delta$

Theorem 5.0.1. The number $c \in \mathbb{R}$, is cluster point of $A \subseteq \mathbb{R}$ if and only if \exists sequence a_n in A such that $\lim(a_n) = c$ and $a_n \neq c$, $\forall n$

Proof. If c is cluster point of A then for any $n \in \mathbb{N}$ the $\frac{1}{n}$ neighbourhood $v_{1/n}(c)$ contains atleast one point a_n in A distinct from c , then $a_n \in A$, $a_n \neq c$ & $|a_n - c| < \frac{1}{n} \Rightarrow \lim a_n = c$ conversly, if \exists a sequence a_n in $A^{\setminus\{c\}}$ with $\lim(a_n) = c$, then for any $\delta > 0$, $\exists k$ such that

if $n \geq k$, then $a_n \in v_\delta(c)$. Therefore, δ neighbourhood $v_\delta(c)$ contains the point a_n , $\forall n \geq k$ which belong to A and are distinct from c . \square

Definition 5.0.2 (Limit of Function): *Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be the cluster point of A . for a function $f : A \rightarrow \mathbb{R}$ a real number L is said to be limit of f at c if, given any $\varepsilon > 0$, $\exists \delta > 0, \exists x \in A$ and $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$ then we say f converges to L at c .*

Theorem 5.0.2. *If $f : A \rightarrow \mathbb{R}$ and if c is a cluster point of A , then f can have only one limit at c .*

Proof. We will prove this by method of contradiction.

Let L and L' be limits of f at c

For any $\varepsilon > 0$, $\exists \delta \left(\frac{\varepsilon}{2} \right) > 0 \exists x \in A$ and $0 < |x - c| < \delta \left(\frac{\varepsilon}{2} \right)$

$$\Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

Also, $\exists \delta' \left(\frac{\varepsilon}{2} \right) > 0 \quad \exists \quad x \in A \text{ and } |x - c| < \delta' \left(\frac{\varepsilon}{2} \right)$

$$\Rightarrow |f(x) - L'| < \frac{\varepsilon}{2}$$

$$|L - L'|$$

$$= |L - f(x) + f(x) - L'|$$

$$\leq |L - f(x)| + |f(x) - L'|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

Since, $\varepsilon > 0$ is arbitrary, $L = L'$ □

Theorem 5.0.3 (Sequential Criterion). *Let $f : A \rightarrow \mathbb{R}$ and let c be a cluster point of A then the*

following are equivalent.

$$1. \lim_{x \rightarrow c} f(x) = L$$

$$2. \text{for every } x_n \text{ in } A, x_n \rightarrow c, x_n \neq c, \quad \forall n \in \mathbb{N} \Rightarrow f(x_n) \rightarrow L.$$

Definition 5.0.3 (Divergence Criterion): Let $A \subseteq \mathbb{R}$ let $f : A \rightarrow \mathbb{R}$ and let $c \in \mathbb{R}$ be cluster point of A .

a) If $L \in \mathbb{R}$ then f does not have limit L at c iff \exists sequence x_n in A with $x_n \neq c, \forall n \in \mathbb{N}$ such that sequence x_n converges to c . but the sequence $f(x_n)$ does not converges to L

b) the function does not have a limit L at c iff $\exists x_n$ in A with $x_n \neq c, \forall n \in \mathbb{N}$ such that the sequence x_n converges to c but the sequence $f(x_n)$ does not converges in \mathbb{R}

$$f(x) =$$

$$f(x) = \begin{cases} +1 & \text{if } x > 0 \\ -0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Theorem 5.0.4 (Limit Theorem). Let $A \subseteq \mathbb{R}$. and $c \in \mathbb{R}$, be cluster point of A we say that f is

bounded on neighbourhood of c if \exists a δ neighbourhood of $V_\delta(c)$ of c and constant $M > 0 \exists$

✓ $|f(x)| \leq M \quad \forall x \in A \cap V_\delta(c)$

Theorem 5.0.5. If $A \subseteq \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ has limit at $c \in \mathbb{R}$ then f is bounded on some neighbourhood of c

Proof. If $L = \lim_{x \rightarrow c} f$ then for $\varepsilon = 1, \exists \delta_c < 0$

Such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < 1$

$$|f(x)| - |L| \leq |f(x) - L| < 1$$

if $x \in A \cap V_\delta(c)$, $x \neq c$ then,

$$|f(x)| < |L| + 1$$

if $c \notin A$, Take $M = |L| + 1$

while if $c \in A$, Take $M = \text{Sup}\{|f(x)|, |L| + 1\}$

$$\therefore |f(x)| \leq M$$

\therefore by limit theorem

$\therefore f$ is bounded on neighbourhood of c . □

Definition 5.0.4: Let $A \subseteq \mathbb{R}$ and let f & g be function defined on A to \mathbb{R} . We define the sum $f + g$, the difference $f - g$ and the product $f.g$ on $A \rightarrow \mathbb{R}$ to be function from A to \mathbb{R} given by,

$$(f + g)(x) = f(x) + g(x) \quad \checkmark$$

$$(f - g)(x) = f(x) - g(x) \quad \checkmark$$

$$(f.g)(x) = f(x) \cdot g(x) \quad \checkmark$$

\dot{ } \quad \cdot

Further if $b \in \mathbb{R}$

$$(bf)(x) = b \cdot f(x)$$

finally, if $h(x) \neq 0$,

$$\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)}$$

Theorem 5.0.6. Let $A \subseteq \mathbb{R}$ let f & g be function on $A \rightarrow \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of $A \rightarrow \mathbb{R}$ & let

1. If $\lim_{x \rightarrow c} f = L$ & $\lim_{x \rightarrow c} g = M$ then

$$\lim_{x \rightarrow c} (f \pm g) = L \pm M$$

$$\lim_{x \rightarrow c} (f \cdot g) = L \cdot M$$

$$\lim_{x \rightarrow c} (b \cdot f) = b \cdot L$$

$$2. \lim_{x \rightarrow c} \left(\frac{f}{c} \right) = \frac{L}{H}$$



where, $h(x) \neq 0$ and $\lim_{x \rightarrow c} h(x) = H \neq 0$

Theorem 5.0.7. Let $A \subseteq \mathbb{R}$ let $f : A \rightarrow \mathbb{R}$ and let $c \in \mathbb{R}$ be the cluster point of A .

if $a \leq f(x) \leq b \forall x \in A, x \neq c$ and if $\lim_{x \rightarrow c} f$ exists then $a \leq \lim_{x \rightarrow c} f \leq b$

Proof. Given, $f : A \rightarrow \mathbb{R}$ and c is cluster point of A .

let $x_n \in A$ such that $x_n \rightarrow c$

$$\therefore f(x_n) \rightarrow L = \lim_{x \rightarrow c} f(x_n) = \lim_{x \rightarrow c} f(x)$$

Also,

$$a \leq f(x) \leq b$$

$$a \leq f(x_n) \leq b$$

$$a \leq \lim_{x \rightarrow c} f(x_n) \leq b$$

$$a \leq \lim_{x \rightarrow c} f(x) \leq b$$

$$a \leq L \leq b$$

□

Theorem 5.0.8 (Squeeze Theorem). *Let $A \subseteq \mathbb{R}$ let $f, g, h : \rightarrow \mathbb{R}$ & $c \in \mathbb{R}$ be a cluster point of A . If*

$$\underbrace{f(x) \leq g(x) \leq h(x)}_{\text{for all } x \in A, x \neq c} \quad \& \quad \lim_{x \rightarrow c} f = \lim_{x \rightarrow c} h \text{ then, } \lim_{x \rightarrow c} g = L.$$

Proof. Given, $f, g, h : \rightarrow \mathbb{R}$ & c is cluster point of A $x_n \in A \Rightarrow x_n \rightarrow c$

$$f(x_n) \rightarrow L = \lim_{x \rightarrow c} f(x_n) = \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} h(x_n)$$

i.e $f(x_n) \rightarrow L$ & $h(x_n) \rightarrow L$

Also,

$$f(x) \leq g(x) \leq h(x)$$

$$f(x_n) \leq g(x_n) \leq h(x_n)$$

$$\lim_{x_n \rightarrow c} f(x_n) \leq \lim_{x_n \rightarrow c} g(x_n) \leq \lim_{x_n \rightarrow c} h(x_n)$$

$$\therefore L \leq \lim_{x \rightarrow c} g(x_n) \leq L$$

$$\lim_{x \rightarrow c} g(x_n) = L$$

i.e $g(x_n) \rightarrow L$

i.e $\lim_{x \rightarrow c} g = L$

-Hence Proved-

□

Definition 5.0.5: Let $A \in \mathbb{R}$ & let $f : A \rightarrow \mathbb{R}$

1. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A, x > c\}$ then we say that $L \in \mathbb{R}$ is right hand limit of f at c

$\lim_{x \rightarrow c^+} f(x) = L$ If given any $\varepsilon > 0$ $\exists \delta(\varepsilon) > 0 \quad \exists \forall x \in A$ with $0 < x - c < \delta$ then $|f(x) - L| < \varepsilon$

2. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, 0) = \{x \in A, x < c\}$ then we say that $L \in \mathbb{R}$ is left hand limit of f at c

$$\lim_{x \rightarrow c^-} f(x) = L \text{ If given any } \varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \exists \quad \forall x \in A \text{ with } 0 < -x + c < \delta \text{ then } |f(x) - L| < \varepsilon$$

5.1 Continuous Function

Definition 5.1.1 (Continuous Function): Let $A \subseteq \mathbb{R}$ let $f : A \rightarrow \mathbb{R}$ & let $c \in A$ we say that f is continuous at c if given any $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0 \quad \exists$ if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$ iff f fails to be continuous at c then we say that f is discontinuous at c .

Theorem 5.1.1. A function $f : A \rightarrow \mathbb{R}$ is continuous at point $c \in A$ if and only if given any $\varepsilon > 0$, $v_\varepsilon(f(c))$ of $f(c)$ \exists of c such that if x is any point of $A \cap v_\delta(c)$ then $f(x) \in v_\varepsilon(f(c))$ i.e $f(A \cap v_\delta(c)) \subseteq v_\varepsilon(f(c))$

Proof. $\therefore \lim_{x \rightarrow c} = L$

i.e any $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0 \quad \exists$

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

and $\lim f(x) = f(c)$

for any $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0 \quad \exists$

$$|x - c| < \delta, \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\therefore x \in A \cap v_\delta(c) \Rightarrow f(x) \in v_\varepsilon(f(c)), \quad \forall x$$

$$\therefore f(A \cap v_\delta(c)) \subseteq v_\varepsilon(f(c)) \dots (\because \text{if } A \subset B \Rightarrow x \in A \Rightarrow x \in B \text{ then } A \subseteq B)$$

□

Definition 5.1.2 (Combination of Continuous function): Let $A \subseteq \mathbb{R}$. Let f & g be function on A to \mathbb{R} , let $b \in \mathbb{R}$, Suppose that $c \in A$ & that f & g are continuous at c .

a) then $f + g$, $f - g$, $f \cdot g$ and $b \cdot f$ are continuous at c ✓

b) if $h : A \rightarrow \mathbb{R}$ is continuous at $c \in A$ & if $h(x) \neq 0$, $x \in A$, then $\left(\frac{f}{h}\right)$ is also continuous at c

Definition 5.1.3 (Continuous Point): Let $A \subseteq \mathbb{R}$ & $f : A \rightarrow \mathbb{R}$. if $B \subseteq A$ we say that f is continuous at c

uous on set B iff is continuous at every point of B

Example 31:

Continuous

- $f(x) = x, \quad x \in \mathbb{R}$ ✓

- $f(x) = x^2, \quad x \in \mathbb{R}$ ✓

- $f(x) = \frac{1}{x}, \quad x \in \mathbb{R}^+, \{0\} \quad x \in \underline{\mathbb{R} - \{0\}}$

\mathbb{R}^+

\mathbb{R}^-

- $f(x) = \underline{\text{Polynomial function}} \quad x \in \mathbb{R}$ ✓

- $f(x) = \underline{\text{Rational function}}$

$$\frac{f(x)}{g(x)}$$

- $f(x) = \text{Trignometric function}$ ✓

- ~~$f(x) = \sqrt{f}, \quad x \in \mathbb{R}$~~

Example 32:

Discontinuous

- $\psi(x) = \frac{1}{x}, \quad x = 0 \quad \checkmark$

~~Dirichlet's~~

- $\psi(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases} \quad \checkmark$

discont everywhere

$\varepsilon > 0$

$0 < |x - c| < \delta_\varepsilon$

$c \in \mathbb{Q}^c$

$(c - \delta, c + \delta)$

\Rightarrow

$|f(x) - f(c)| < \varepsilon$

\Rightarrow

$|f(c) - 0| < \varepsilon$

$|0 - 1| < \varepsilon$

discontinuous at every pt.

- $\sin(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$

- $\sin(x) = \begin{cases} 0, & x = 0 \end{cases}$

discont at $x = 0 \quad \checkmark$

$$\text{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$$

✓ $\psi(x) = [x]$ = greatest integer function discount at integer

Theorem 5.1.2. Let $A \subseteq \mathbb{R}$. Let $f : A \rightarrow \mathbb{R}$ & let $|f|$ be defined by $|f|(x) = |f(x)| \quad \forall x \in A$

1. If f is continuous at point $c \in A$ then $|f|$ is continuous at c .

2. If f is continuous on A then $|f|$ is continuous on A .

Theorem 5.1.3. Let $A, B \subseteq \mathbb{R}$ & let $f : A \rightarrow \mathbb{R}$ & $g : B \rightarrow \mathbb{R}$ be function such that $f(A) \subseteq B$ if

✓ f is continuous at point $c \in A$ and g is continuous at $b = f(c) \in B$ then the composition $g \circ f : A \rightarrow \mathbb{R}$ is continuous at c .

Proof. Let W be ε -neighbourhood of $g(b)$. since g is continuous at b there is a δ -neighbourhood of v of $b = f(c)$ such that if $y \in B \cap v$ then $g(y) \in W$. Since f is also continuous at c , ther is a v -neighbourhood v of $c \ni x \in A \cap U$ then $f(x) \in v$

Since $f(A) \subseteq B$, it follows that if $x \in A \cap U$ then $f(x) \in B \cap v$ so that $g(f(x)) = g(f(x)) \in W$ But,

Since W is an arbitrary ε -neighbourhood of $g(b)$ this implies $g \circ f$ is continuous at c . \square

5.2 Continuous function on Interval

$$\underline{|f(x)| \leq M}$$

Definition 5.2.1 (Bounded Function): A function $f : A \rightarrow \mathbb{R}$ is said to be bounded on A if \exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$

Theorem 5.2.1 (Boundedness Theorem-). Let $I = [a, b]$ be a closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then f is bounded on I .

Proof. Suppose f is bounded on I .

then, for any $n \in \mathbb{N}$, $\exists x_n \in I \quad \exists |f(x_n)| > k$.

Since, I is bounded, sequence x_n is bounded.

\therefore By Bolzano weistress theorem,

\exists subsequence x_{n_k} that converges to some x

Since, I is closed, elements of sequence $x_{n_k} \in I \Rightarrow x \in I$.

then, f is continuous at x so that $f(x_{n_k})$ converges to $f(x)$

$$\underline{|f(x_n)| > n_k > k}$$

$$\Rightarrow |f(x_{nk})| > n_k > k \quad \forall k \in \mathbb{N}$$

\therefore Our assumption is wrong.

Hence, f must be bounded. □

Definition 5.2.2 (Absolute Extremum): Let $A \subseteq \mathbb{R}$ and let $f : A \rightarrow \mathbb{R}$. We say that f has an absolute maximum on A if there is $x^* \in A$ such that

$$f(x^*) \geq f(x), \quad \forall x \in A$$

Relative

We say that f has absolute minimum on A if there is $x^* \in A$ such that

$$f(x^*) \leq f(x), \quad \forall x \in A$$

Theorem 5.2.2 (Maximum-Minimum Theorem). Let $I = [a, b]$ be closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then f has an absolute maximum and absolute minimum on I .

Proof. $f(I) = \{f(x); \quad x \in I\}$

I is a closed bounded and f is continuous on I then $f(x)$ is also bounded $\subseteq \mathbb{R}$

\therefore By completeness property, it has supremum and infimum

$$\therefore S^* = \text{Sup}\{f(I)\}, \quad S_* = \text{Inf}\{f(I)\}$$

claim- To show , $\exists x^*, x_* \in I$

$\exists S^* = f(x^*) = \text{absolute maximum}$

$S_* = f(x_*) = \text{absolute minimum}$

$$S_* = \text{Inf}\{f(I)\}$$

if $n \in \mathbb{N}$ then $S^* - \frac{1}{n}$ is not upper bound

$$\therefore S^* - \frac{1}{n} < f(x_n) < S^*, \quad \forall n \in \mathbb{N}$$

Since, I is bounded x_n is bounded By Bolzano weistress theorem,

$\exists x_{n_k}$ subsequence of x_n and $x_{n_k} \rightarrow \text{some } x^*$

Also, As I is closed and $x_{n_k} \in I \Rightarrow x^*$ must be in I

$\Rightarrow f$ is continuous at x^* , $\lim f(x_{n_k}) = f(x^*)$

$$S^* - \frac{1}{n} < f(x_{n_r}) \leq S^*, \quad \forall r \in \mathbb{N}$$

\therefore by squeeze theorem

$$\lim f(x_{n_r}) = S^*$$

$$\therefore S^* = f(x^*) \text{ i.e } f(x^*) \geq f(x), \quad \forall x$$

$\therefore x^*$ is absolute maximum

Similarly, we show x_* is absolute minimum □

Theorem 5.2.3 (Location of Root). Let $I = [a, b]$ & let $f : I \rightarrow \mathbb{R}$ be continuous on I . If $f(a) < 0$ $f(b)$ or $f(b) < 0 < f(a)$, then $\exists c \in (a, b) \ni f(c) = 0$. Bisection Method

Proof. Assume that $f(a) < 0$ $f(b)$

Let $I_1 = [a_1, b_1]$ where, $a_1 = a, b_1 = b$

let $P_1 = \frac{a+b}{2}$ if $f(P_1) = 0$ then $c = P_1$

if $P_1 \neq 0$, then either $f(P_1) > 0$ or $f(P_1) < 0$

if $f(P_1) > 0$ then $a_2 = a_1, b_2 = P_1$ and if $f(P_1) < 0$

$a_2 = P_1, b_2 = b_1$ thus, we get $I = [a_2, b_2] \in I_1$

continuing this bisection, we obtain intervals I_1, I_2, \dots, I_k

In this process, we terminate by locating a point $P_n \in I_n \ni f(P_n) = 0$

if process does not terminate, we obtain nested sequence of bounded interval

$$I_n = [a_n, b_n]$$

$$\exists f(a_n) < 0 \text{ & } f(b_n) > 0$$

$$\text{& length of interval } b_n - a_n = \frac{(b-a)}{2^{n-1}}$$

$$\Rightarrow \exists \text{ a point } c \in I_n \quad \forall n \in \mathbb{N}$$

$$a_n \leq c \leq b_n, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 0 \leq c - a_n \leq b_n - a_n$$

$$\Rightarrow 0 \leq c - a_n \leq \frac{(b-a)}{2^{n-1}}$$

$$\Rightarrow \lim f(a_n) = \lim f(b_n) = f(c)$$

$$\Rightarrow 0 \leq b_n - c \leq b_n - a_n$$

$$a_0, b_0$$

$$\frac{b-a}{2^{n-1}}$$

$$a_n \leq c \leq b_n$$

$$0 \leq c - a_n \leq b_n - a_n$$

$$a_n \leq c$$

$$-c \leq -a_n$$

$$0 \leq c - a_n \leq \frac{b-a}{2^{n-1}}$$

$$c = \lim a_n \leq 0$$

$$b_n - c \leq b_n - a_n$$

$$b_n - c \leq \frac{b-a}{2^{n-1}}$$

$$\Rightarrow 0 \leq b_n - c \leq \frac{(b-a)}{2^{n-1}}$$

?



Theorem 5.2.4 (Bolzano's Intermediate Theorem). Let I be an interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I if $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$ then a point $c \in I$ between a & $b \exists f(c) = k$.

Proof. 1. Assume that, $a < b, a, b \in I, f$ continuous on I

Define $g(x) = f(x) - k$

As $f(x)$ is continuous, $g(x)$ is also continuous on I

Also, $f(a) < k < f(b)$

$$f(a) - k < 0 < f(b) - k$$

$$g(a) < 0 < g(b)$$

$$f(b) < k < f(a)$$

$$g(x) = k - f(x)$$

\therefore by location of root theorem

$$\exists c \ni g(c) = 0$$

i.e $f(c) - k = 0$

$\therefore f(c) = k$

2. Assume that, $a > b, a, b \in I, f$ continuous on I

Define $h(x) = k - f(x)$

As $f(x)$ is continuous, $h(x)$ is also continuous on I

Also, $f(a) < k < f(b)$

$$k - f(a) < 0 < k - f(b)$$

$$h(a) < 0 < h(b)$$

\therefore by location of root theorem

$$\exists c \ni h(c) = 0$$

i.e $k - f(c) = 0$

$$\therefore f(c) = k$$

□

Corollary 5.2.4.1. Let $I = [a, b]$ be a closed bounded interval. Let $f : I \rightarrow \mathbb{R}$ be continuous on I if $k \in \mathbb{R}$ is any number satisfying $\underline{\text{Inf}} f(I) \leq k \leq \underline{\text{Sup}} f(I)$ then \exists a number $c \in I \ni f(c) = k$

Proof. Given that, I is a closed bounded interval and $f : I \rightarrow \mathbb{R}$ is continuous on I

\therefore By maximum- minimum theorem,

$\exists \underline{x^*}, \underline{x_*} \in I$ such that $f(x^*) = \underline{\text{Sup}}\{f(I)\}$ ✓

$$f(x_*) = \underline{\text{Inf}}\{f(I)\}$$

✓ Also, Given that, $\underline{\text{Inf}} f(I) \leq k \leq \underline{\text{Sup}} f(I)$

i.e $f(x^*) \leq k \leq f(x_*)$

\therefore by Bolzano intermediate theorem,

$\exists c \in I \ni f(c) = k$ ✓

$$\begin{array}{c} \underline{m} \quad \underline{M} \\ f(x_*) \leq k \leq f(x^*) \\ \hline \exists c \in [x_*, x^*] \subseteq [a, b] \\ f(c) = k \end{array}$$

~~-Hence Proved-~~

Theorem 5.2.5. Let I be closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then,
the set $f(I) = \{f(x) : x \in I\}$ be closed bounded interval.

Proof. let ,

$$\checkmark m = \inf\{f(I)\}$$

$$f(I) = \{f(x), x \in I\}$$

$$\checkmark M = \sup\{f(I)\}$$

$$f(I) \subseteq [m, M]$$

by maximum - minimum theorem, $m, M \in f(I)$

$$\underline{\underline{f(I) \subseteq [m, M]}}$$

if $k \in [m, M]$



\therefore by bolzano-itermediate theorem

$$\exists c \in I, \underline{f(c) = k} \quad \leftarrow$$

Hence, $k \in f(I)$

$$\Rightarrow [m, M] \subseteq f(I)$$

$\therefore f(I)$ is the interval $[m, M]$ ✓

□

5.3 Continuity

Definition 5.3.1 (Uniform Continuous): Let $A \subseteq \mathbb{R}$ & let $f : A \rightarrow \mathbb{R}$. We say that f is uniformly continuous on A if for each $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ \exists if $x, y \in A$ are any numbers satisfying

$$|x - y| < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$$

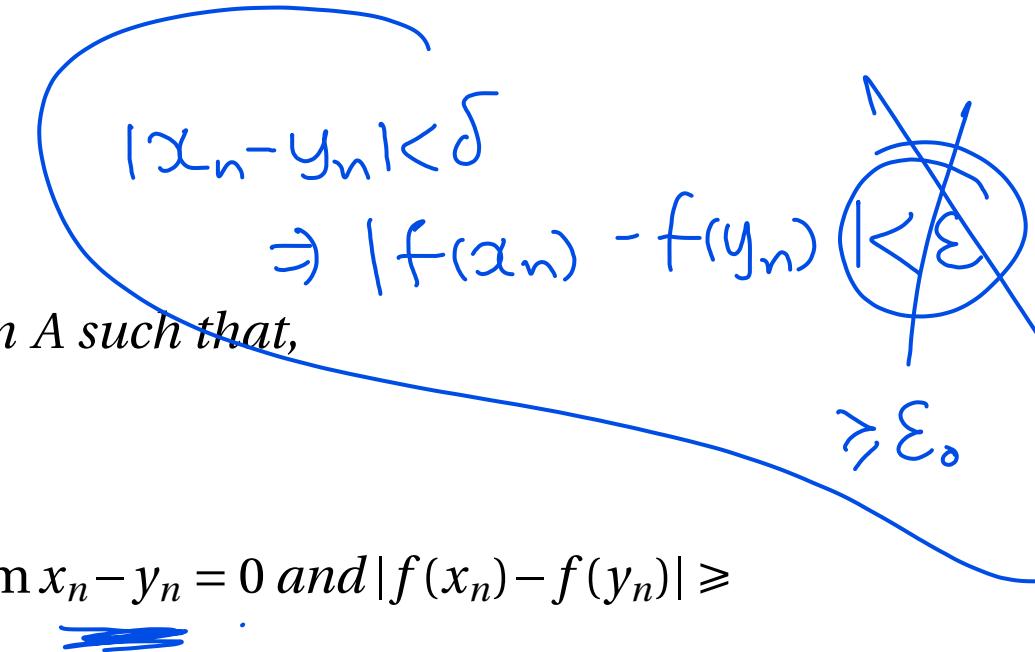
Definition 5.3.2 (Non- Uniform Continuity): Let $A \subseteq \mathbb{R}$ & let $f : A \rightarrow \mathbb{R}$ then following statements are equivalent.

i) f is not uniformly continuous on A . ✓

ii) $\exists a_n \quad \varepsilon_0 > 0 \quad \exists$ for every $\delta > 0$ there are points x_δ, y_δ in A such that,

$$|x_\delta - y_\delta| < \delta \text{ and } |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$$

iii) $\exists a_n \quad \varepsilon_0 > 0$ and two sequences x_n & y_n in A such that $\lim x_n - y_n = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}$



Theorem 5.3.1 (Uniform Continuity Theorem). Let I be closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then f is uniform continuous on I .

$\checkmark I \rightarrow \mathbb{R}$ be continuous on I then f is uniform continuous on I .

Contradiction

Proof. If f is not uniform continuous on I then,

$\exists \varepsilon_0 > 0$ and two sequences $x_n, y_n \in I$

$$|x_n - y_n| < \frac{1}{n} \quad \& \quad |f(x_n) - f(y_n)| \geq \varepsilon_0$$

Since I is bounded x_n, y_n are bounded.

\exists subsequence x_{n_k} of x_n that converges to some elements $z \in I$ (as I closed) as

$$|x_n - y_n| < \frac{1}{n} \quad \forall n$$

Subsequence y_{n_k} of y_n also converges to z

$$|y_{n_k} - z|$$

$$= |y_{n_k} - x_{n_k} + x_{n_k} - z|$$

$$\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

$\therefore y_{n_k}$ is also converges to z

+ Cont \rightarrow uniformly cont
closed bounded + cont \Rightarrow uniformly cont
Interval

$$\lim x_n - y_n = 0$$

$$\begin{aligned} \lim x_n &= \lim y_n \\ \downarrow & \\ x_n &\rightarrow y_{n_k} \end{aligned}$$

$$|y_{n_k} - z| < \varepsilon$$

$y_{n_k} \rightarrow z$
 $f(y_{n_k}) \rightarrow f(z)$
 $f(y_{n_k})$ converges

Now if f is continuous at z both $f(x_{n_k})$ and $f(y_{n_k})$ must converges $f(z)$

But this not possible as $|f(x_n) - f(y_n)| \geq \varepsilon_0$

\therefore Our assumption is wrong. □

Definition 5.3.3 (Lipschitz Function): Let $A \subseteq \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ if there exists a constant $k > 0$ such that

$|f(x) - f(u)| < k|x - u| \quad \forall x, u \in A$ then f is said to be a Lipschitz function on A

Theorem 5.3.2. Lipschitz function is an uniformly continuous function always.

Proof. for Lipschitz function

$$|f(x) - f(u)| < k|x - u|$$

$$\text{Now, } |x - u| < \frac{\varepsilon}{k} = \delta, \quad \frac{\varepsilon}{k} > 0 \text{ ask } k > 0$$

$$|f(x) - f(u)| < k \cdot \frac{\varepsilon}{k}$$

$$< \varepsilon$$

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

Lipschitz function is always uniformly continuous function. □

Theorem 5.3.3. *If $f : A \rightarrow \mathbb{R}$ is uniformly continuous on subset A of \mathbb{R} and if x_n is a cauchy sequence in A , then $f(x_n)$ is cauchy sequence in \mathbb{R} .*

Proof. let x_n is a cauchy sequence in A and let $\varepsilon > 0$ choose $x, y \in A$, $\delta > 0$

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

Since, x_n is a cauchy sequence $\exists H(\delta)$

$$|x_n - x_m| < \delta, \quad \forall n, m \geq H(\delta)$$

(as f is uniformly continuous)

$$|f(x_n) - f(x_m)| < \varepsilon$$

Therefore, the sequence $f(x_n)$ is cauchy sequence. □

Theorem 5.3.4 (Continuous Extension Theorem). *A function f is uniformly continuous on (a, b) iff it can be defined at the end points a & b such that the extended function is continuous on $[a, b]$.*

Proof. Assume that function f is continuous on $[a, b]$

\therefore by definition,

for any $\varepsilon > 0$, $\frac{\varepsilon}{2} > 0$, $\exists \delta(\varepsilon) > 0$ such that,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

let x_1 & $x_2 \in [a, b]$

by definition,

for any $\varepsilon > 0$, $\frac{\varepsilon}{2} > 0$, $\exists \delta(\varepsilon) > 0$ such that,

$$|x_1 - c| < \frac{\delta}{2} \Rightarrow |f(x_1) - f(c)| < \frac{\varepsilon}{2}$$

and,

for any $\varepsilon > 0$, $\frac{\varepsilon}{2} > 0$, $\exists \delta(\varepsilon) > 0$ such that,

$$|x_2 - c| < \frac{\delta}{2} \Rightarrow |f(x_2) - f(c)| < \frac{\varepsilon}{2}$$

consider,

$$|x_1 - x_2|$$

$$= |x_1 - c + c - x_2|$$

$$\leq |x_1 - c| + |x_2 - c|$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2}$$

$$\leq \delta$$

and,

$$|f(x_1) - f(x_2)|$$

$$= |f(x_1) - f(c) + f(c) - f(x_2)|$$

$$\leq |f(x_1) - f(c)| + |f(x_2) - f(c)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\leq \varepsilon$

$$\therefore |x_1 - x_2| \leq \delta \Rightarrow |f(x_1) - f(x_2)| \leq \varepsilon$$

$\therefore f$ is uniformly continuous on (a, b)

Conversely, Suppose f is uniformly continuous on (a, b) . Lets define $f(a)$ & $f(b)$

Lets x_n be sequence in (a, b) $\exists \lim x_n = a$

$\Rightarrow x_n$ is cauchy sequence and as f is uniformly continuous on (a, b) and $x_n \in (a, b)$

\therefore by sequential criteria , $\lim f(x_n) = L$ exists if y_n is any other sequence in (a, b) that converges to a then

$$\lim x_n - y_n = a - a = 0$$

$$\lim f(y_n) = \lim(f(y_n) - f(x_n) + f(x_n)) = L$$

So we define, $L = f(a)$

then f is continuous at a

Similarly, we can find some $M = f(b)$ and we can say that f is continuous on extended

$[a, b]$



Definition 5.3.4 (Step Function): *$I \subseteq \mathbb{R}$ be an interval and let $S : I \rightarrow \mathbb{R}$ then S is called a step function if it has only a finite number of distinct values.*

5.4 Continuity And Gauges

Definition 5.4.1 (Partition): *A partition of an interval $I = [a, b]$ is collection $P = \{I_1, I_2, \dots, I_n\}$ of non-overlapping closed intervals whose union is $[a, b]$. We generally denote $I_i = [x_{i-1}, x_i]$ where $a = x_0 < \dots < x_{i-1} < x_i < \dots < x_n = b$*

The points x_i ($i = 0, 1, 2, \dots, n$) are called the partition points of p . If a point t_i has been chosen from each interval I_i , for ($i = 0, 1, 2, \dots, n$) then the points t_i are called tags and set of ordered pairs $\dot{p} = \{(I_1, t_1), (I_2, t_2), \dots, (I_n, t_n)\}$ is called as tagged partition of I

Definition 5.4.2: *A gauge on I is a strictly positive function defined on I . If δ is a gauge on I , then a tagged partition \dot{p} is said to be δ -fine if*

$$t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$$

If a partition p of $I = [a, b]$ is a δ -fine & $x \in I$, then \exists a tag t_i in p such that $|x - t_i| \leq \delta(t_i)$

Alternative proof of Boundedness Theorem

Proof. Since f is continuous on I , then for each $t \in I$ $\exists \delta(t) > 0 \ni$ if $x \in I$ and

$$|x - t| < \delta(t) \text{ then } |f(x) - f(t)| < 1$$

Thus, δ -gauge on I let $\{(I_i, t_i)\}_{i=1}^n$ be δ -fine partition on I and let

$$k = \max\{|f(t_i)| \mid i = 1, 2, \dots, n\}$$

Given any $x \in I \ni i$ with $|x - t_i| \leq \delta(t_i)$

$$\begin{aligned} |f(x)| &= |f(x) - f(t_i) + f(t_i)| \\ &\leq 1 + k \end{aligned}$$

Since $x \in I$ is arbitrary, f is bounded. □

Definition 5.4.3 (Monotone and Inverse Function): If $A \subseteq \mathbb{R}$, then a function $f : A \rightarrow \mathbb{R}$ is

said to be increasing on A if whenever $x_1, x_2 \in A$ and $x_1 < x_2$ then $f(x_1) \leq f(x_2)$

if $x_1, x_2 \in A$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ then f is called strictly increasing function.

Similarly, for decreasing function,

$x_1 < x_2$ then $f(x_1) \geq f(x_2)$ and strictly decreasing function

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Theorem 5.4.1. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be increasing on I . Suppose $c \in I$ is not endpoint of I then ,

$$1. \lim_{x \rightarrow c^-} = \text{Sup}\{f(x) / x \in I, x < c\}$$

$$2. \lim_{x \rightarrow c^+} = \text{Sup}\{f(x) / x \in I, x > c\}$$

Proof. 1. Let $x \in I$ & $x < c \Rightarrow f(x) < f(c)$

So, for set $\{f(x) / x \in I, x > c\}$, $f(c)$ is upper bound, So by completeness property,

\exists Supremum, say L .

if $\varepsilon > 0$, then $L - \varepsilon$ is not upper bound

Hence, $\exists \quad y_\varepsilon \in I, \quad y_\varepsilon < c$

$\exists \quad L - \varepsilon < f(y_\varepsilon) \leq L$

Since, f is increasing, if $\delta_\varepsilon = c - y_\varepsilon$ and if

$0 < c - y < \delta_c$ then

$$y_\varepsilon < y < c$$

So that, $t - \varepsilon < f(y_\varepsilon) \leq f(y) \leq L$

$\Rightarrow |f(y) - L| < \varepsilon$ when $0 < c - y < \delta_c$

Simillarly we can prove (ii)

□

Theorem 5.4.2 (Continuous Inverse Function). *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be strictly monotone and continuous on I then the function g —inverse to f is strictly monotone and continuous on $I = f(I)$*

Proof. Let f is strictly increasing

Since f is continuous on I

By preservation of interval theorem,

$J = f(I)$ is also an interval. Also,

$f : I \rightarrow \mathbb{R}$ is strictly monotone and injective on I , therefore, inverse function $g : J \rightarrow \mathbb{R}$ exists if

$y_1, y_2 \in J$ with $y_1 < y_2$ then

$y_1 = f(x_1), \quad y_2 = f(x_2)$ for some $x_1, x_2 \in I$

$\Rightarrow x_1 < x_2$ as function is increasing

$\Rightarrow x_1 = g(y_1) < g(y_2) = x_2$

Since, y_1, y_2 arbitrary elements of J with

$y_1 < y_2$, we conclude that g is strictly increasing on J .

Now, we have to show that g is continuous on J .

As $g(J) = I$ is an interval.

Indeed, if g is discontinuous at a point $c \in J$, then the jump at c is non-zero so that $\lim_{y \rightarrow c^-} g < \lim_{y \rightarrow c^+} g$

$$\lim_{y \rightarrow c^+} g$$

if we choose any number $x \neq g(c)$ satisfying $\lim_{y \rightarrow c^-} g < x < \lim_{y \rightarrow c^+} g$

then, $x \neq g(y)$, for any $y \in J$

Hence, $x \notin I$ which contradicts to our given condition that I is interval.

\therefore The inverse function g is continuous on J . □

Chapter 6

Differentiation

6.1 Derivative

$$\lim_{x \rightarrow c^-}$$

$$= \lim_{h \rightarrow 0^-} \frac{f(c-h) - f(c)}{-h}$$

Definition 6.1.1 (Derivative): Let $I \subseteq \mathbb{R}$ be an interval. let $f : I \rightarrow \mathbb{R}$ and let $c \in I$. We say that a real number L is derivative of f at c if given any $\varepsilon > 0$ $\exists \delta(\varepsilon) > 0$ \exists if $x \in I$ satisfies

$$0 < |x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$$

.. ..

We say, f is differentiable at c .

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

Theorem 6.1.1. If $f : I \rightarrow \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c .

Proof. $\forall x \in I, x \neq c$

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c)$$

$$\lim_{x \rightarrow c} f(x) - f(c)$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}(x - c)$$

$$= f'(c).0$$

$$= 0$$

$$\lim_{x \rightarrow c} f(x) = f(c)$$

$\therefore f$ is continuous at point c

$f : I \rightarrow \mathbb{R}$ is derivable at c

for any $\epsilon > 0 \exists \delta_\epsilon > 0, |x - c| < \delta_\epsilon \Rightarrow$

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon$$

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = L$$

$$\frac{x \neq c}{f(x) - f(c)} = \frac{f(x) - f(c)}{x - c} \times x - c$$

$$\lim_{x \rightarrow c} f(x) - f(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \times \lim_{x \rightarrow c} (x - c)$$

$$\lim_{x \rightarrow c} f(x) - f(c) = f'(c) \times 0$$

$$\Rightarrow \boxed{\lim_{x \rightarrow c} f(x) = f(c)}$$

if $f : I \rightarrow \mathbb{R}$ is continuous at point c then f may or may not be derivable at c . □

Example 33:

$f(x) = |x|$ is continuous at 0 but not differentiable at 0.

Theorem 6.1.2. Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$ & let $f : I \rightarrow \mathbb{R}$ & $g : I \rightarrow \mathbb{R}$ be functions that are differentiable at c then

a) $(\alpha f)'(c) = \alpha f'(c)$, $\alpha \in \mathbb{R}$

b) $(f + g)'(c) = f'(c) + g'(c)$

c) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

d) $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$ ($g(c) \neq 0$)

Proof. a) $(\alpha f)'(c)$

$$= \lim_{x \rightarrow c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x - c}$$

~~$$(f+g)'(c) =$$~~

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

$$\lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} = g'(c)$$

$$\lim_{x \rightarrow c} \frac{(f+g)(x) - (f+g)(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{\alpha f(x) - \alpha f(c)}{x - c}$$

$$= \alpha \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$$\underline{(\alpha f)'(c) = \alpha f'(c)}$$

b) $\lim_{x \rightarrow c} \frac{(f + g)(x) - (f + g)(c)}{x - c} = (f + g)'(c)$

$$\therefore (f + g)'(c)$$

$$= \lim_{x \rightarrow c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}$$

$$f'(c) + g'(c)$$

$$= \lim_{x \rightarrow c} \frac{(f(x) + g(x)) - (f(c) + g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{(f(x) - f(c)) - (g(x) - g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \left\{ \frac{f(x) - f(c)}{x - c} \right\} + \lim_{x \rightarrow c} \left\{ \frac{g(x) - g(c)}{x - c} \right\}$$

$$\underline{(f+g)'(c) = f'(c) + g'(c)}$$

c) Let $h(x) = fg(x)$

$$h'(c) = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c}$$

$$(fg)'(c)$$

$$= \lim_{x \rightarrow c} \frac{fg(x) - fg(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(x) - f(c).g(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(x) - f(c).g(x) + f(c).g(x) - f(c).g(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{(f(x) - f(c))g(x) + f(c).(g(x) - g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \rightarrow c} g(x) + \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \cdot f(c)$$

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

d) let $h = \frac{f}{g}$

$$\therefore h'(c) = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c}$$

$$\therefore \left(\frac{f}{g} \right)'(c)$$

$$= \lim_{x \rightarrow c} \frac{\left(\frac{f}{g} \right)(x) - \left(\frac{f}{g} \right)(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(c) - f(c).g(c) + f(c).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow c} \frac{(f(x) - f(c)).g(c) + f(c).(g(x) - g(c))}{g(x).g(c)(x - c)} \\
 &= \lim_{x \rightarrow c} \left(\frac{1}{g(x).g(c)} \right) \left[\lim_{x \rightarrow c} \left(\frac{f(x) - f(c)}{x - c} \right) . g(c) - \lim_{x \rightarrow c} \left(\frac{g(x) - g(c)}{x - c} \right) f(c) \right] \\
 &= \frac{1}{(g(c))^2} [f'(c).g(c) - g'(c)f(c)] \\
 \left(\frac{f}{c} \right)'(c) &= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad (g(c) \neq 0)
 \end{aligned}$$

□

Theorem 6.1.3. Let f be defined on an interval I containing point c . Then f is differential at c iff \exists a function ψ on I that is continuous at c and satisfies $f(x) - f(c) = \psi(x)(x - c)$ $x \in I$ In this case, $\psi(c) = f'(c)$

Proof. If $f'(c)$ exists we can define,

$$\psi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c}, & \text{for } x \neq c, x \in I \\ f'(c), & \text{for } x = c \end{cases}$$

$\lim_{x \rightarrow c} \psi(x) = f'(c)$

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c} \psi(x)$$

$(x - c) \psi(x) = f(x) - f(c)$

Now, assume that ψ function is continuous at c and satisfies

$f(c)$

$$\frac{f(x) - f(c)}{x - c} = \psi(x) \cdot (x - c)$$

~~$\psi(c)$~~

$f'(c)$

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c} \psi(x) = \psi(c)$$

$\psi(c)$ exists

$\therefore f$ is differentiable at c and $\psi(c) = f'(c)$

□

6.2 Chain Rule

Theorem 6.2.1 (Chain Rule). *Let I, J be intervals in \mathbb{R} . Let $g : I \rightarrow \mathbb{R}$ & $f : J \rightarrow \mathbb{R}$ be functions such that $f(J) \subseteq I$ and let $c \in J$*

If f is differentiable at c & g is differentiable at $f(c)$ then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$

Proof. Given that f is differentiable at c

$\therefore \exists$ function ψ on $J \ni$

$$f(x) - f(c) = \psi(x) \cdot (x - c) \text{ & } f'(c) = \psi(c)$$

Also, g is differentiable at $f(c)$

\exists function ψ on $I \ni$

$$g(f(x)) - g(f(c)) = \psi(f(x)) \cdot (f(x) - f(c)) \text{ & } g'(f(c)) = \psi(f(c))$$

Consider,

$$\begin{aligned} & g \circ f(x) - g \circ f(c) \\ &= g(f(x)) - g(f(c)) \\ &= \psi(f(x)) \cdot (f(x) - f(c)) \\ &= \psi(f(x)) \cdot (\psi(x) \cdot (x - c)) \end{aligned}$$

$$= [\psi(f(x)).\psi(x)].(x - c)$$

$\therefore g \circ f$ is differentiable at c

Also, $\lim_{x \rightarrow c} \frac{g \circ f(x) - g \circ f(c)}{(x - c)}$

$$= \lim_{x \rightarrow c} [\psi(f(x)).\psi(x)]$$

$$= \psi(f(c)).\psi(c)$$

$$(g \circ f)'(c) = g'(f(c)).f'(c)$$

□

Definition 6.2.1 (Inverse Function): Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ strictly monotone and continuous on I . let $J = f(I)$ and let $g : J \rightarrow \mathbb{R}$ be strictly monotone and continuous function inverse to f .

Theorem 6.2.2. If f is differentiable at c , $c \in I$ & $f'(c) \neq 0$ then g is differentiable at $d = \underline{f(c)}$ &

$$\boxed{g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}}$$

Proof. Given that, f is differentiable at $c \in I$

$$(g \circ f)'(c) = \underline{\underline{g'(f(c))}} \cdot \underline{\underline{f'(c)}}$$

$\therefore \exists \psi$ on I continuous at c \exists

$$f(x) - f(c) = \psi(x).(x - c) \text{ & } \psi(c) = f'(c)$$

Since $f'(c) \neq 0 \Rightarrow \psi(c) \neq 0$

\exists neighbourhood of c , $v = (c - \delta, c + \delta)$

$$\exists \quad \psi(x) \neq 0 \quad \forall x \in v \cap I$$

If $U = f(v \cap I)$ then inverse function g satisfies

$$f(g(y)) = y, \quad \forall y \in U$$

$$y - d = f(g(y)) - f(c) = \psi(g(y)).(g(y) - g(d))$$

since, $\psi(g(y)) \neq 0, \quad \forall y \in U$

$$g'(y) - g(d) = \frac{1}{\psi(g(y))}(y - d)$$

Since, $\psi(g(y))$ is continuous at d

$\therefore g'(d)$ exists and

$$g'(d) = \frac{1}{\psi(g(d))} = \frac{1}{\psi(c)} = \frac{1}{f'(c)}$$
□

Theorem 6.2.3. Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is an even function [i.e $f(-x) = f(x) \forall x$] and has

derivative at every point, then the derivative f' is an odd function. Also, prove that if $g : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable odd function, then g' is even function.

Proof. a) Given that f is even function

$$f(x) = f(-x) \quad \forall x$$



Also, f is differentiable at c

$$\therefore \underline{f'(c)} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ exists}$$



To prove, f' is odd function

$$\text{i.e } \underline{f'(-c) = -f'(c)}$$

consider,

$$\underline{\underline{f'(-c)}} = \lim_{x \rightarrow -c} \frac{f(x) - f(-c)}{x + c}$$

$$= \lim_{-x \rightarrow c} \frac{f(x) - f(c)}{-x + c} \quad \begin{matrix} f(x) \\ \downarrow \\ f(-x) - f(-c) \\ \downarrow \\ -x + c \end{matrix}$$

$$= \lim_{-x \rightarrow c} \frac{f(x) - f(c)}{-x + c} \quad ?$$

$$= -f'(c)$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{-(x - c)}$$

$$= -f'(c)$$

$\therefore f'$ is odd function.

b) Given that g is odd function

$$g(x) = g(-x) \forall x$$

Also, g is differentiable at c

$$\therefore g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c}$$

exists

To prove, g' is even function

$$\text{i.e } g'(-c) = -g'(c)$$

consider,

$$\underline{\underline{g'(-c)}}$$

$$= \lim_{x \rightarrow -c} \frac{g(x) - g(-c)}{x + c}$$

$$\frac{g(-x) - g(-c)}{-x + c}$$

$$= \lim_{-x \rightarrow c} \frac{-g(x) + g(c)}{-x + c}$$

$$\frac{-g(x) + g(c)}{-x + c}$$

$$= \lim_{-x \rightarrow c} \frac{g(x) - g(c)}{(x - c)}$$

$$\frac{g(x) - g(c)}{x - c}$$

$$= g'(c)$$

$\therefore g'$ is even function.

□

Theorem 6.2.4 (Interior Extremum). *Let c be an interior point of the interval I at which $f : I \rightarrow \mathbb{R}$ has a relative extremum. If derivative of f at c exists, then $f'(c) = 0$*

Proof. Let f has relative maximum at c

if $f'(c) > 0$, $\exists V_\varepsilon(c) \subseteq I$

$$\frac{f(x) - f'(c)}{x - c} > 0, \quad \forall x \in V_\varepsilon(c), x \neq c$$

if $x \in V, x > c$

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f'(c)}{x - c} > 0$$

$$\therefore f(x) > f(c) \quad \forall x > 0, \quad x \in V_\varepsilon(c)$$

but, f has relative maximum at c .

So, our assumption is wrong that $f'(c) > 0$

Similarly, we can show that $f'(c) < 0$

$$\therefore f'(c) = 0$$

□

Corollary 6.2.4.1. Let $f : I \rightarrow \mathbb{R}$ be continuous on an interval I and suppose that f has relative extremum at an interior point c of I then either the derivative of f at c does not exist or it is equal to 0.

Theorem 6.2.5 (Rolle's theorem). If a function f defined on $[a, b]$ is

1. Continuous on $[a, b]$ ✓

2. derivable on (a, b) ✓

3. $f(a) = f(b)$ ✓

then $\exists c \in \mathbb{R}, c \in (a, b) \ni f'(c) = 0$

Proof. Since, f is continuous $[a, b] \Rightarrow f$ is bounded

by maximum- minimum theorem,

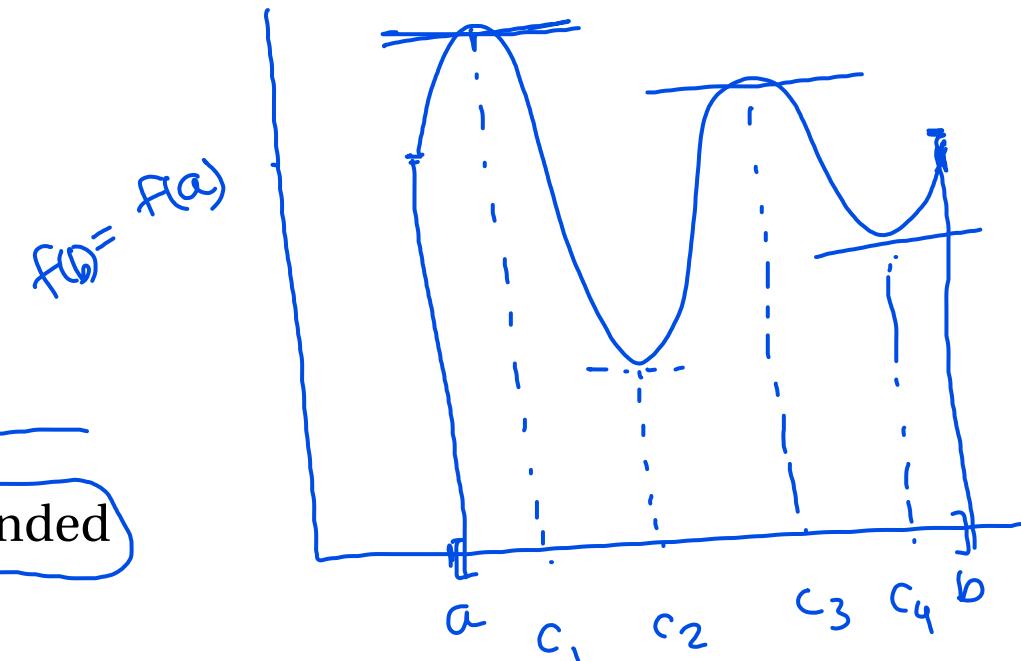
If $m = \inf\{f(I)\}$ and $M = \sup\{f(I)\}$ then $\exists c, d \in (a, b)$

$f(c) = m$ & $f(d) = M$

there are two possibilities $m = M$ or $m \neq M$

If $m = M$

$\Rightarrow \inf\{f(I)\} = \sup\{f(I)\} \Rightarrow f$ is continuous ✓



$$f(I) = \{ f(x), x \in I \}$$

bounded.

$$\frac{F(x) = C = M - M}{\frac{d}{dx} f(x)} = 0 \quad \nexists x \in [a, b]$$

$$\Rightarrow f'(c) = 0, \quad \forall c \in (a, b)$$

If $m \neq M$

$$\Rightarrow f(c) = m \neq f(a) \Rightarrow c \neq a$$

$$\Rightarrow f(c) = m \neq f(b) \Rightarrow c \neq b$$

$\Rightarrow c$ lies in (a, b)

Now, we have to show $f'(c) = 0$

IF $f'(c) < 0$, $\exists (c, c + \delta), \delta_1 > 0$ for every x of which $f(x) < f(c) = m$ which contradicts to

our assumption that infimum attains at c .

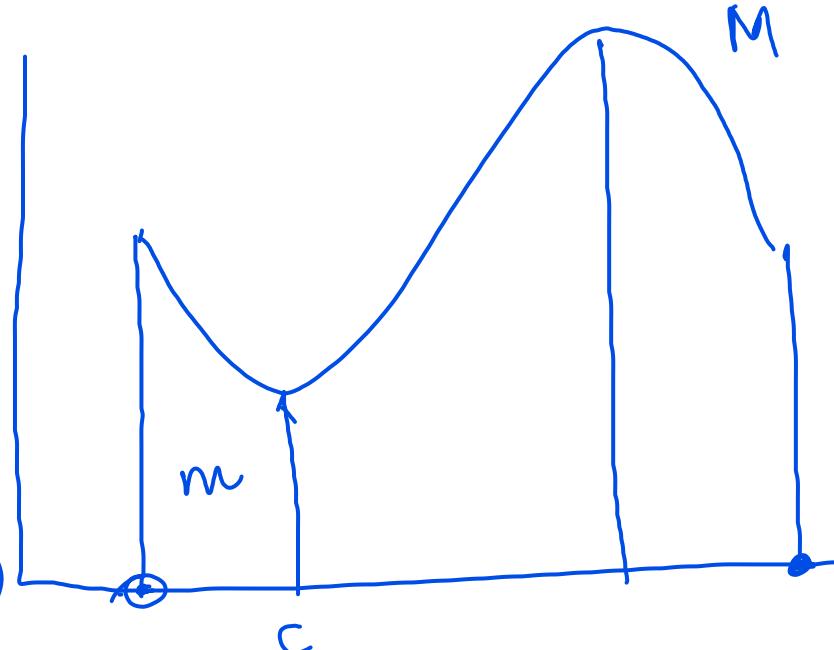
Similarly, $f'(c) > 0$ is not possible

$$\therefore f'(c) = 0$$

Theorem 6.2.6 (Langrange's Mean Value theorem). If a function f defined on $[a, b]$

i) Continuous on $[a, b]$

ii) differentiable on (a, b)



$$m = f(c)$$

$$\overline{c \in [a, b]}$$

$$\underline{c \in (a, b)}$$

$$f(x) > f(c) \Rightarrow x \in I$$

c abs. min

$$f'(c) > 0$$

$$x \in (c - \delta, c)$$

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} > 0$$

$$f(x) - f(c) < 0 \quad \square$$

$$f(x) < f(c)$$

then $\exists c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

Proof. Let us define function ψ on $[a, b]$ such that

$\psi(x) = f(x) - Ax$, where A is constant.

As $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) ,

$\psi(x)$ is also continuous on $[a, b]$ and differentiable on (a, b)

Assume, $\psi(a) = \psi(b)$

$$f(a) - A.a = f(b) - A.b$$

$$f(b) - f(a) = A(b - a)$$

$$\therefore A = \frac{f(b) - f(a)}{b - a}$$

$$\therefore \psi(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} \right) x$$

- i) $\psi(x)$ is continuous on $[a, b]$

ii) $\psi(x)$ is derivable on (a, b)

iii) $\psi(a) = \psi(b)$

\therefore by rolle's theorem

$$\psi'(c) = f'(c) - \left(\frac{f(b) - f(a)}{b - a} \right)$$

$$f'(c) = \left(\frac{f(b) - f(a)}{b - a} \right)$$

□

Theorem 6.2.7 (Cauchy Mean Value theorem). *If f, g defined on $[a, b]$*

i) continuous on $[a, b]$

ii) derivable on (a, b)

iii) $g'(x) \neq 0, \quad \forall x \in (a, b) \exists \quad c \in (a, b) \quad \exists$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Let us define function $\psi(x)$ on $[a, b] \ni$

$\psi(x) = f(x) - Ag(x)$

i) $\psi(x)$ is continuous on $[a, b]$

ii) $\psi(x)$ is derivable on (a, b)

iii) $\psi(a) = \psi(b)$

$$\Rightarrow f(a) + A.g(a) = f(b) - A.g(b)$$

$$\therefore f(b) - f(a) = A(g(b) - g(a))$$

$$A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

\therefore by rolles theorem,

$$\psi'(c) = 0$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

□

6.3 Taylor's Theorem

Theorem 6.3.1 (Taylor's Theorem). *If a function f defined on $[a, a + h]$ is such that*

- i) $(n-i)^th$ derivative f^{n-1} is continuous on $[a, a + h]$ and
- ii) n^th derivative f^n exists on $(a, a + h)$ then \exists atleast one real number θ between 0 & 1 ($0 < \theta < 1$) that,

$$f(a+h) = f(a) + hf'(a) + \left(\frac{h^2}{2!}\right)f''(a) + \left(\frac{h^3}{3!}\right)f'''(a) + \dots + \left(\frac{h^{n-1}}{(n-1)!}\right)f^{n-1}(a) + \left(\frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]}\right)f^n(a+\theta h)$$

f(x) = f(x₀) +

where p is given positive integer R_n

forms of remainder form-

- ✓
- i) $R_n = \left(\frac{h^n (1-\theta)^{n-p}}{p[(n-1)!]} \right) f^n(a + \theta h)$ $h = (x - x_0)$
 ii) $R_n = \left(\frac{h^n (1-\theta)^{n-1}}{(n-1)!} \right) f^n(a + \theta h) \Rightarrow \text{Cauchy}$
 iii) $R_n = \left(\frac{h^n}{n!} \right) f^n(a + \theta h) \Rightarrow \text{Called as Langranges Forms of remainder}$

Theorem 6.3.2 (Maclaurins Theorem). $f(x) = \underbrace{f(0)}_{=} + xf'(0) + \left(\frac{x^2}{2!} \right) f''(0) + \left(\frac{x^3}{3!} \right) f'''(0) + \dots + \left(\frac{x^{n-1}}{(n-1)!} \right) f^{n-1}(0) + \left(\frac{x^n (1-\theta)^{n-p}}{p[(n-1)!]} \right) f^n(\theta x)$

Example 34:

$$f(x) = e^x$$

\therefore By Maclaurins theorem,

$$f(x) = \underbrace{f(0)}_{=} + \underbrace{xf'(0)}_{=} + \left(\frac{x^2}{2!} \right) f''(0) + \dots$$

$$1 + x + \frac{x^2}{2!} + \dots$$

$$\sum n \frac{x^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

Example 35:

$$f(x) = \sin(x)$$
✓

∴ By Maclaurins theorem,

$$f(x) = \sin 0 + x \cos 0 + \frac{x^2}{2!}(-\sin 0) + \frac{x^3}{3!}(-\cos 0) + \dots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$
✓

Example 36:

$$f(x) = \log(1 + x)$$
✓

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}$$

∴ By Maclaurins theorem,

$$f(x) = f(0) + xf'(0) + \left(\frac{x^2}{2!}\right)f''(0) + \left(\frac{x^3}{3!}\right)f'''(0) + \dots$$

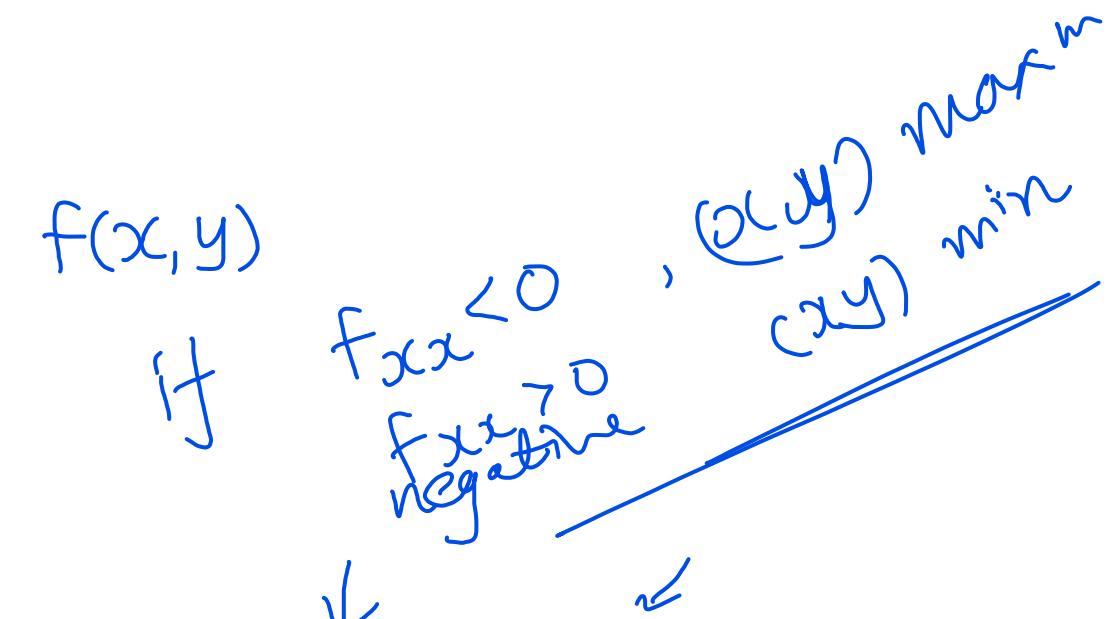
$$f(x) = 0 + x(1) + \left(\frac{x^2}{2!}\right)(-1) + \left(\frac{x^3}{3!}\right)(2) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

6.4 Maximum or Minimum for function of two variables

- $f(a, b)$ is extreme value of $f(x, y)$. if
- i) $f_x(a, b) = 0 \leq f_y(a, b)$
 - ii) $f_{xx}(a, b) = f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$

and this extreme value is maximum or minimum according as $f_{xx}(a, b)$ or $f_{yy}(a, b)$ is negative or positive.



Further investigation needed if,

$$f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 = 0$$

$$\begin{aligned} \checkmark f_x &= 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \\ f_y &= 3y^2 - 12 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2 \end{aligned}$$

Example 37:

find maximum and minimum of

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20 = 0 \quad \checkmark$$

Proof. $f_x(x, y) = 0$

i.e $3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

$f_y(x, y) = 0$

i.e $3y^2 + 12 = 0$

$y^2 = 4$

$x = \pm 2$

$$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = 0$$

x	y	f_{xx}	f_{yy}	f_{xy}	
1	2	6	12	0	$72 > 0$
-1	2	-6	12	0	$-72 < 0$
-1	-2	-6	-12	0	$72 > 0$
1	-2	6	-12	0	$-72 < 0$

(1, 2) (-1, -2) extremum

$$f(x, y) \equiv \frac{1+8-3-24+20}{=} = 2$$

$$f(-1, -2) = \frac{-1-8+3+24+20}{=} = \frac{38}{=}$$

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 6y, \quad f_{xy}(x, y) = 0$$

for $x = 1, y = 2$

$$f_{xx}(x, y) = 6x = 6, \quad f_{yy}(x, y) = 6y = 12, \quad f_{xy}(x, y) = 0$$

for $x = -1, y = -2$

$$f_{xx}(x, y) = 6x = -6, \quad f_{yy}(x, y) = 6y = -12, \quad f_{xy}(x, y) = 0$$

for $x = -1, y = 2$

$$f_{xx}(x, y) = 6x = -6, \quad f_{yy}(x, y) = 6y = 12, \quad f_{xy}(x, y) = 0$$

for $x = 1, y = -2$

$$f_{xx}(x, y) = 6x = 6, \quad f_{yy}(x, y) = 6y = -12, \quad f_{xy}(x, y) = 0$$

minimum=(1, 2)

maximum=(-1, -2)

□

Sequence and Series of Function

7.1 Sequence of Function

Definition 7.1.1 (Sequence of Function): Let $A \subseteq \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$

$\exists f_n : A \rightarrow \mathbb{R}$ we shall say that (f_n) is a sequence of function on A to \mathbb{R}

Definition 7.1.2 (Pointwise Convergent): Let f_n be a sequence of function on $A \subseteq \mathbb{R}$ to \mathbb{R} . let $A_0 \subseteq A$ & let $f_n : A_0 \rightarrow \mathbb{R}$ we say that the sequence f_n converges on A_0 to f iff for each $x \in A_0$ the sequence $f_n(x)$ converges to f (x)

The sequence $f_n : A \rightarrow \mathbb{R}$ converges to function $f_n : A_0 \rightarrow \mathbb{R}$ on A_0 iff for each $\varepsilon > 0$ & $x \in A_0$ \exists

$$\underline{k(\varepsilon_1 x)} \in \mathbb{N} \quad \exists \quad |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq k(\varepsilon_1 x)$$

Example 38:

$$f_n(x) = \frac{x}{n}, \quad f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\text{i.e. } |\frac{x}{n} - 0| < \varepsilon \Rightarrow \left| \frac{x}{n} \right|$$

$$\therefore \frac{|x|}{n} < \varepsilon$$

$$\therefore n > \frac{|x|}{\varepsilon} = K(\varepsilon, x)$$

Example 39:

$$f_n(x) = x^n \quad \checkmark \quad x \in (-1, 1]$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$f(x) \begin{cases} 0 \\ 1 \end{cases}$$

$-1 < x < 1$
 $x=1$ ✓

$$|x^n - 0| < \varepsilon, \quad -1 < x < 1$$

$$|x^n| < \varepsilon$$

$$n \log x < \log \varepsilon$$

$$n < \log\left(\frac{\varepsilon}{x}\right)$$

$$\therefore n > \log\left(\frac{x}{\varepsilon}\right)$$

Example 40:

$$f_n(x) = \frac{x^2 + nx}{n}, \quad x \in \mathbb{R}, \quad f(x) = x$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\left| \frac{x^2}{n} + x - x \right| < \varepsilon \left[\left| \frac{x^2}{n} \right| < \varepsilon \right]$$

$$\therefore \frac{x^2}{\varepsilon} < n$$

$f(\varepsilon)$

$$\therefore n > \frac{x^2}{\varepsilon}$$

Definition 7.1.3 (Uniform Convergence): A sequence of function on $A \subseteq \mathbb{R}$ to \mathbb{R} converges uniformly on $A_0 \subseteq A$ to a function $f : A_0 \rightarrow \mathbb{R}$ iff for each $\varepsilon > 0$ there is a natural number $k(\varepsilon)$ (depending on ε but not on $x \in A_0$) \exists

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq k(\varepsilon)$$

denoted by, $f_n(x) \rightharpoonup f(x)$ on A_0

Lemma 7.1.1. A sequence f_n of function on $A \subseteq \mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \rightarrow \mathbb{R}$ iff for some $\varepsilon_0 > 0 \exists$ subsequence f_{n_k} of f_n and a sequence x_k in A_0 such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}$$

Example 41:

$$f_n(x) = \frac{x_k}{n_k}, \quad f(x) = 0, x_k = k, n_k = k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\left| \frac{x_k}{n_k} - 0 \right| \Rightarrow \left| \frac{k}{k} - 0 \right|$$

$$\Rightarrow |1 - 0|$$

$$\Rightarrow |1| \geq \varepsilon$$

Example 42:

$$f_n(x) = \frac{x^2 + nx}{n}, \quad f(x) = x, x_k = k, n_k = -k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\left| \frac{x_k^2}{n_k} + x_k - x_k \right| \geq \varepsilon_0 \Rightarrow$$

$$\left| \frac{k^2}{k} \right| \geq \varepsilon_0$$

$$\therefore |k| > \varepsilon$$

\therefore not uniformly convergent

Example 43:

$$f_n(x) = x^n$$

$$f(x) = \begin{cases} 0 & ; 0 \leq x < 1 \\ 1 & ; x = 1 \end{cases}$$

$$x_k = \left(\frac{1}{2}\right)^{\left(\frac{1}{k}\right)}, \quad n_k = k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\therefore |x_k^{n_k} - 0| \geq \varepsilon_0$$

$$\left| \left(\frac{1}{2} \right)^{\left(\frac{1}{k} \right)} - 0 \right| \geq \varepsilon_0$$

$$\therefore \left| \frac{1}{2} \right| > \varepsilon$$

\therefore Not uniformly convergent

Definition 7.1.4 (Uniform Norm): If $A \subseteq \mathbb{R}$ & $\psi : A \rightarrow \mathbb{R}$ is a function we say that ψ is bounded on A . If the set $\psi(A)$ is bounded subset of \mathbb{R} if ψ is bounded we define the uniform norm of ψ on A by, $\|\psi\|_A = \text{Sup}\{|\psi(x)| : x \in A\}$

Note that, it follows that if $\varepsilon > 0$,

$$\|\psi\|_A \leq \varepsilon \Leftrightarrow |\psi(x)| \leq \varepsilon, \quad \forall x \in A$$



Lemma 7.1.2. A sequence f_n of bounded function on $A \subseteq \mathbb{R}$ uniformly on A to f if and only if

$$\|f_n - f\|_A \rightarrow 0$$

Example 44:

$$f(x) = x \quad [0, 1]$$

$$\text{Sup}\{|\psi(x)| : x \in A\} = 1$$

$$\|f\|_A = 1$$

Example 45:

$$f_n(x) = \frac{x}{n}, \quad f(x) = 0, \quad [0, 1]$$

$$|f_n(x) - f(x)| = |x|$$

$$\therefore \|f_n - f\|_A = \frac{1}{n}^n \rightarrow 0$$

Example 46:

$$f_n(x) = x^n \quad [0, k], \quad f(x) = 0$$

$$|f_n(x) - f(x)| = |x^n|$$

$$\therefore ||f_n - f||_A = |k^n|$$

Example 47:

$$f_n(x) = x^n(1-x) \quad x \in [0, 1], f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$|x^n(1-x) - 0| = |x^n(1-x)|$$

$$f_n(x) = x^n - x^{n+1}$$

$$\therefore f'_n(x) = nx^{n-1} - (n+1)x^n = 0$$

$$\Rightarrow nx^{n-1} = (n+1)x^n$$

$$\Rightarrow \frac{n}{n+1} = x$$

$$x = \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$$\therefore \|f_n - f\|_A = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \left(1 - \frac{1}{\left(1 + \frac{1}{n}\right)}\right)$$

$$= \left(1 + \frac{1}{n}\right)^{-n} \left(\frac{\frac{1}{n}}{1 + \frac{1}{n}}\right)$$

$$= \left(1 + \frac{1}{n}\right)^{-n} \left(\frac{1}{1 + \frac{1}{n}}\right) \rightarrow 0$$

7.2 Cauchy Criteria for Uniform Convergence

Theorem 7.2.1. Let f_n be a sequence of bounded function on $A \subseteq \mathbb{R}$ then this seqence converges uniformly on A to a bounded function f iff for each $\varepsilon > 0$ $\exists H(\varepsilon) \in \mathbb{N} \exists$

$$\|f_m - f_n\|_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$$

Proof. If $f_n(x) \rightharpoonup f(x)$ then for $\varepsilon > 0 \quad \exists k \left(\frac{\varepsilon}{2} \right) \ni$

$$\|f_n - f\|_A \leq \left(\frac{\varepsilon}{2} \right) \quad \forall n \geq k \left(\frac{\varepsilon}{2} \right)$$

Hence, if both $m, n \geq k \left(\frac{\varepsilon}{2} \right)$

$$|f_m(x)' - f_n(x)|$$

$$= |f_m(x) - f(x) + f(x)f_n(x)|$$

$$\leq |f_m(x) - f(x)| + |f(x)f_n(x)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon \quad \forall m, n \geq k \left(\frac{\varepsilon}{2} \right)$$

Conversely,

Suppose, $\varepsilon > 0$, $\exists H(\varepsilon) \in \mathbb{N}$

$$\exists ||f_m - f_n||_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$$

\therefore for each $x \in A$

$$|f_m(x) - f_n(x)| \leq ||f_m(x) - f_n(x)||_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$$

$\Rightarrow f_m(x)$ is cauchy sequence and hence convergent.

$$\therefore \exists \quad f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in A$$

We have $|f_m(x) - f_n(x)| \leq \varepsilon, \quad \forall m \geq H(\varepsilon)$

$\therefore f_n(x) \xrightarrow{\sim} f(x)$ on A

□

7.3 Series of Function



If f_n is sequence of function defined on subset D of \mathbb{R} with values in \mathbb{R} , the sequence of partial sums S_n of infinite series $\sum f_n$ is defined for x in D by,



$$S_1(x) = f_1(x) \quad -$$

$$S_2(x) = f_2(x) + S_1(x) \quad -$$

:

:

$$S_{n+1}(x) = S_n(x) + f_{n+1}(x)$$

:

:

- In the case sequence S_n of functions f_n converges to function f on D we say that $\sum f_n$ converges on D to f

- If the series $\sum |f_n(x)|$ converges for each $x \in D$, we say that $\sum f_n$ converges absolutely on D .

abs. convergence

- if (S_n) sequence of partial sums is uniformly convergent on D to f , we say that $\sum f_n$ is uniformly converges on D .
- If f_n is continuous on $D \subseteq \mathbb{R}$ to \mathbb{R} for each $n \in \mathbb{N}$ and if $\sum f_n$ converges f on D uniformly, then f is continuous on D .

Definition 7.3.1 (Cauchy Criterion): f_n be a sequence of f_n on $D \subseteq \mathbb{R}$ to \mathbb{R} , the series $\sum f_n$ is uniformly convergent on D iff for every uniformly $\varepsilon > 0$, $\exists \tilde{M}(\varepsilon)$

$$\exists |f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \varepsilon, \quad \forall m > n \leq M(\varepsilon) \quad m > n > \tilde{M}(\varepsilon)$$

Theorem 7.3.1 (Weistress M-test). Let M_n be a sequence of positive real numbers such that $|f_n(x)| \leq M_n \quad \forall x \in D \quad \forall n \in \mathbb{N}$. If the series $\sum M_n$ is convergent then $\sum f_n$ is uniformly convergent on D .

Proof. M_n is convergent,

By cauchy criterion for series,

for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

$$\exists M_{n+1} + M_{n+2} + \dots + M_m < \varepsilon, \quad \forall m > n \leq k(\varepsilon)$$

$$\exists |f_{n+1}(x)| + |f_{n+2}(x)| + \dots + |f_m(x)| < M_{n+1} + M_{n+2} + \dots + M_m < \varepsilon$$

Also,

$$\underbrace{|f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)|}_{\text{By Cauchy criterion}} \leq |f_{n+1}| + |f_{n+2}| + \dots + |f_m| < \quad \forall m > n \geq k(\varepsilon)$$

\therefore By Cauchy criterion,

$\sum f_n$ is uniformly convergent on D .

□

Definition 7.3.2 (Power Series): A series of real function $\sum f_n$ is said to be power series around $x = c$ if the function has the form $f_n(x) = a_n(x - c)^n$ where a_n and $c \in \mathbb{R}$ and where $n = 0, 1, 2, \dots$

Example 48:

Power Series

$$\sum a_n x^n = a_0 x^0 + a_1 x^1 + \dots + a_n x^n + \dots$$

$$\frac{x^n}{n}$$

$$c=0, a_n = 1/n$$

$$\underline{\underline{\sum x^n}}$$

$$c=0, a_n = 1$$

$$\sum_{n=0}^{\infty} n!x^n, \quad \sum_{n=0}^{\infty} x^n, \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$a_n = n!, \quad 1, \quad \frac{1}{n!}$$

$$\underline{c=0}$$

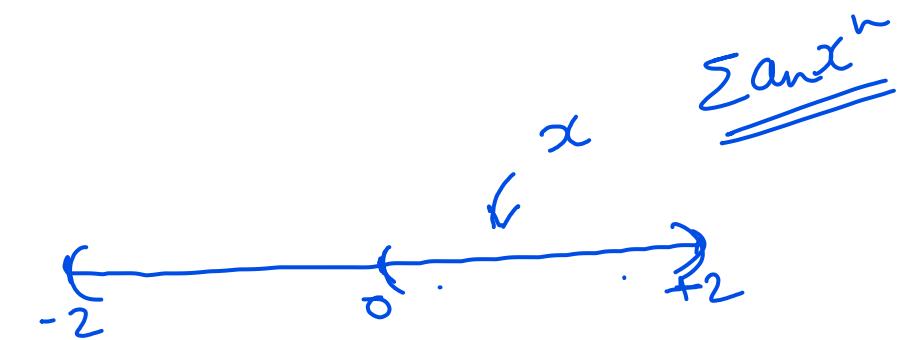
$\frac{1}{n}$

Definition 7.3.3 (Radius of Convergence): $\sum a_n x^n$ be a power series if a sequence $|a_n|^{\frac{1}{n}}$ is bounded, we set $\rho = \lim \text{Sup} |a_n|^{\frac{1}{n}}$ if this sequence is not bounded, we set $\rho = +\infty$

We define radius of convergece of $\sum a_n x^n$ to be given by,

$$R = \begin{cases} 0 & ; \text{if } \rho = +\infty \\ \frac{1}{\rho} & ; \text{if } 0 < \rho < +\infty \\ \infty & ; \rho = 0 \end{cases}$$

The interval of convergence is the open interval $(-R, R)$



Example 49:

$$\sum \frac{x^n}{2^n} \Rightarrow \left| \frac{1}{2^n} x^n \right|$$

$$\Rightarrow a_n \cdot x^n$$

$$\rho = \lim Sup |a_n|^{\frac{1}{n}}$$

$$\Rightarrow \lim Sup \left| \frac{1}{2^n} \right|^{\frac{1}{n}}$$

$$\Rightarrow \frac{1}{2}$$

$$\Rightarrow R = \frac{1}{\rho} = 2$$

Example 50:

$$\sum n x^n \Rightarrow a_n x^n$$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

$$R = \lim \left| \frac{n}{n+1} \right|$$

$$R = \lim \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$R = 1$$

Differentiation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < R$$

$$y = \sum a_n x^n \quad |x| \leq R$$

$$\frac{dy}{dx} = \sum n \cdot a_n x^{n-1}$$

$$\sum n \cdot x^{n-1} = \frac{+1}{(1-x)^2}$$

$$\sum n(n-1) x^{n-2} = \frac{+2}{(1-x)^3}$$

Chapter **8**

Riemann Integral

8.1 Introduction

Riemann Integral

If $I = [a, b]$ be closed bounded interval in \mathbb{R} then partition of I is a finite ordered set $\mathbb{P} = (x_0, x_1, \dots, x_{n-1}, x_n)$ of points in I such that $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$.

The points of P are used to divide $I = [a, b]$ into non-overlapping sub-intervals.

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n]$$

Norm of $P = ||p|| = \max\{|x_i - x_{i-1}|, i = 1, 2, \dots, n\}$

The norm of partition is merely the length of largest sub-interval into which the partition divide if point t_i has been chosen from each sub-interval $I_i = [x_{i-1}, x_i] = \forall i = 1 : n$ then the points are called as tags of sub-intervals $I - i$.

A set of ordered pairs

$\dot{p} = \{(x_{i-1}, x_i), t_i\}_{i=1}^n$ is tagged partition of $[a, b]$

Definition 8.1.1 (Riemann Sum): *If \dot{p} is the tagged partition, we define Riemann sum of function. $f : [a, b] \rightarrow \mathbb{R}$ corresponding to \dot{p} to be the number,*

$$S(f, \dot{p}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

Definition 8.1.2 (Riemann Integral): *A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be Riemann integrable on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for $\varepsilon > 0 \exists \delta_\varepsilon > 0 \exists$ if \dot{p} is any tagged partition of $[a, b]$ with $||\dot{p}|| < \delta_\varepsilon$ then*

$$|S(f, \dot{p}) - L| < \varepsilon$$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $R[a, b]$

i.e $\|\dot{p}\| \rightarrow 0 \Rightarrow S(f, \dot{p}) \rightarrow L$

Definition 8.1.3: If $f \in R[a, b]$ then the number L is uniquely determined and called as Riemann Integral of f over $[a, b]$

$$L = \int_a^b f(x) dx$$

Theorem 8.1.1. If $f \in R[a, b]$ then the value of the integral is uniquely determined.

Proof. Assume that L' & L'' both satisfy the definition and

let $\varepsilon > 0 \quad \exists \quad \delta'_{\frac{\varepsilon}{2}} > 0 \quad \exists$ if \dot{p}_1 is tagged partition with $\|\dot{p}_1\| < \delta'_{\frac{\varepsilon}{2}}$ then

$$|S(f, \dot{p}_1) - L'| < \frac{\varepsilon}{2}$$

Similarly, $\exists \quad \delta''_{\frac{\varepsilon}{2}} > 0 \quad \exists$ if \dot{p}_2 is tagged partition with $\|\dot{p}_2\| < \delta''_{\frac{\varepsilon}{2}}$ then

$$|S(f, \dot{p}_2) - L''| < \frac{\varepsilon}{2}$$

Now, let $\delta_\varepsilon = \min\left(\delta'_{\frac{\varepsilon}{2}}, \delta''_{\frac{\varepsilon}{2}}\right)$

let \dot{p} be tagged partition with $||\dot{p}|| < \delta_\varepsilon$

$\Rightarrow |S(f, \dot{p}) - L'| < \frac{\varepsilon}{2}$ and

$|S(f, \dot{p}_2) - L''| < \frac{\varepsilon}{2}$

So, $|L' - L''| = |L' - S(f, \dot{p}) + s(f, \dot{p}) - L''|$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

As ε is arbitrary, $L' = L''$ □

Theorem 8.1.2. Every constant function on $[a, b]$ is in $R[a, b]$.

Proof. Let $f(x) = k \quad \forall x \in [a, b]$ be the constant function, if $\dot{p} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is any tagged partition on $[a, b]$

$$S(f, \dot{P}) = \sum_{i=1}^n k(x_i - x_{i-1}) = k(b - a)$$

Hence, for any $\varepsilon > 0$, we can choose $\delta_\varepsilon > 0 \quad \exists \quad ||\dot{P}|| < \delta_\varepsilon \text{ & } |S(f, \dot{P}) - k(b - a)| = 0 < \varepsilon$

$$\int_a^b f(x) dx = k(b - a)$$

$\therefore f(x)$ is an Riemann integrable $f \in R[a, b]$

□

8.2 Some Properties of Integral

Theorem 8.2.1. Suppose that f & g are in $R[a, b]$ then

a) If $k \in \mathbb{R}$, the function $k.f$ is in $R[a, b]$ and $\int_a^b kf = k \int_a^b f$

b) the function f & g is in $R[a, b]$ and $\int_a^b f + g = \int_a^b f + \int_a^b g$

c) $f(x) \leq g(x) \quad \forall \quad x \in [a, b]$ then $\int_a^b f \leq \int_a^b g$

Theorem 8.2.2. If $f \in R[a, b]$ then f is bounded on $[a, b]$.

Proof. Assume that f is unbounded on $[a, b]$

As $f \in \mathcal{R}[a, b]$, then for any $\varepsilon > 0$ $\exists \delta_\varepsilon > 0$
such that $\|\dot{p}\| < \delta_\varepsilon$ then $|S(f, \dot{p}) - L| < \varepsilon$

Now, let $\underline{Q} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be partition on $[a, b]$ with $\|\underline{Q}\| < \delta$. Since $|f|$ is not bounded on $[a, b]$, \exists atleast one sub-interval $[x_{k-1}, x_k]$ on $[a, b]$ which $|f|$ is not bounded.

Let tag \underline{Q} by $t_i = x_i$ for $i \neq k$ and $t_k \in [x_{k-1}, x_k]$ such that,

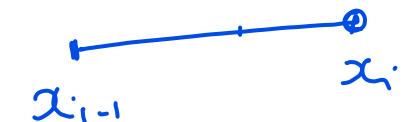
$$|f(t_k)(x_k - x_{k-1})| > |L| + \varepsilon + \left| \sum_{i \neq k}^n f(t_i)(x_i - x_{i-1}) \right|$$

By triangular inequality $|a + b| > |a| - |b|$

$$|S(f, \underline{Q})| \geq |f(t_k)(x_k - x_{k-1})| - \left| \sum_{i \neq k}^n f(t_i)(x_i - x_{i-1}) \right| > |L| + \varepsilon$$

\therefore which is contradict to our assumption.

$\therefore f$ is bounded on $[a, b]$ □



Definition 8.2.1 (Cauchy Criterion for Riemann Integrable function): A function $f : [a, b] \rightarrow \mathbb{R} \in R[a, b]$ if and only if for every $\varepsilon > 0$, $\exists n_\varepsilon > 0$ if \dot{p} & \underline{Q} are any tagged partitions of $[a, b]$ with $\|\dot{p}\| < n_\varepsilon$ & $\|\underline{Q}\| < n_\varepsilon$ then,

$$|S(f, \dot{P}) - S(f, \dot{Q})| < \varepsilon$$

Theorem 8.2.3 (Squeeze theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ then $f \in R[a, b]$ if and only if for every $\varepsilon > 0 \exists$ function α_ε & w_ε in $R[a, b]$ with

$$\alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon \quad \forall x \in R[a, b] \text{ & such that } \int_a^b w_\varepsilon - \alpha_\varepsilon < \varepsilon$$

Proof. \iff Take $\alpha_\varepsilon = w_\varepsilon = f \quad \forall \varepsilon > 0$

\iff Let $\varepsilon > 0$, Since $\alpha_\varepsilon, w_\varepsilon \in R[a, b]$

$\exists \delta_\varepsilon > 0 \quad \exists ||\dot{P}|| < \delta_\varepsilon$ then

$$\left| S(\alpha_\varepsilon, \dot{P}) - \int_a^b \alpha_\varepsilon \right| < \varepsilon \quad \& \quad \left| S(w_\varepsilon, \dot{P}) - \int_a^b w_\varepsilon \right| < \varepsilon$$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon, \dot{P}) \quad \& \quad S(w_\varepsilon, \dot{P}) < \int_a^b w_\varepsilon + \varepsilon$$

As $\alpha_\varepsilon \leq f \leq w_\varepsilon$

$$S(\alpha_\varepsilon, \dot{P}) \leq S(f, \dot{P}) \leq S(w_\varepsilon, \dot{P})$$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leq S(f, \dot{P}) \leq \int_a^b w_\varepsilon + \varepsilon$$

Consider another partition $\|\dot{Q}\| < \delta_\varepsilon$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leq S(f, \dot{Q}) \leq \int_a^b w_\varepsilon + \varepsilon$$

$$\Rightarrow |S(f, \dot{Q}) - S(f, \dot{P})| < \int_a^b (w_\varepsilon - \alpha_\varepsilon) + 2\varepsilon \leq 3\varepsilon$$

Since, $\varepsilon > 0$, is arbitrary, $f \in R[a, b]$ ✓

□

Theorem 8.2.4. If $f : R[a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then $f \in R[a, b]$

Proof. As f is continuous on closed bounded interval $[a, b]$, f is uniformly continuous on $[a, b]$

\therefore for any $\varepsilon > 0$, $\exists \delta_\varepsilon > 0$ \exists if $u, v \in [a, b]$

$$|u - v| < \delta_\varepsilon$$

$$\Rightarrow |f(u) - f(v)| < \frac{\varepsilon}{b-a}$$

Let $p = \{I_i\}_{i=1}^n$ be a partition such that $\|p\| < \delta_\varepsilon$, let $u_i \in I_i$ be a point where f attains minimum value on I_i & $v_i \in I_i$ be a point where f attains maximum value on I_i

Let α_ε be the step function

$$\alpha_\varepsilon(x) = f(u_i) \quad \forall \quad x \in [x_{i-1}, x_i] \quad (i = 1 : n-1)$$

Let w_ε be the step function

$$w_\varepsilon(x) = f(v_i) \quad \forall \quad x \in [x_{i-1}, x_i] \quad (i = 1 : n-1)$$

so, $\alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon(x) \quad \forall x \in [a, b]$

$$0 \leq \int_a^b (w_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^n (f(v_i) - f(u_i))(x_i - x_{i-1})$$

$$< \sum_{i=1}^n \left(\frac{\varepsilon}{b-a} \right) (x_i - x_{i-1}) = \varepsilon$$

\therefore by squeeze theorem,

$f \in R[a, b]$

□

Theorem 8.2.5. If $f : R[a, b] \rightarrow \mathbb{R}$ is monotone on $[a, b]$ then $f \in R[a, b]$

Proof. Suppose f is I on $[a, b]$

Assume $a < b, \varepsilon > 0$

$$h = \frac{f(b) - f(a)}{q} < \frac{\varepsilon}{(b-a)}$$

let $y_k = f(a) + k.h \quad \forall \quad k = 0, 1, \dots q$

let $A_k = f^{-1}[y_{k-1}, y_k] \quad \forall \quad k = 0, 1, \dots q-1$

The sets A_k are pairwise disjoint and have union $[a, b]$

so A_k is either

- a) empty
- b) single point set
- c) non degenerate interval in $[a, b]$

We discard the sets for which a) holds and relabel remaining ones if we adjoin the end points of the remaining intervals A_k , we obtain closed intervals I_k

So we have step functions α_ε & w_ε

$$\alpha_\varepsilon(x) = y_{k-1}, \quad w_\varepsilon(x) = y_k \quad \forall x \in A_k$$

$$\alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon(x) \quad \forall x \in [a, b]$$

$$\begin{aligned} & \int_a^b (w_\varepsilon - \alpha_\varepsilon) \\ &= \sum_{k=1}^q (y_k - y_{k-1})(x_k - x_{k-1}) \\ &= \sum_{k=1}^q h(x_k - x_{k-1}) \\ &= h.(b - a) \end{aligned}$$

so, by squeeze theorem,

$$f \in R[a, b]$$

□

8.3 Fundamental theorem of Integral calculus

Theorem 8.3.1. Suppose, there is finite set E in $[a, b]$ and function $f: F: [a, b] \rightarrow \mathbb{R}$ such that

1. F is continuous on $[a, b]$

2. $F'(x) = f(x) \quad \forall \quad x \in [a, b] \setminus E$

3. $f \in R[a, b]$ then $\int_a^b f = f(b) - f(a)$

Proof. Let $\varepsilon > 0$, since $f \in R[a, b]$ $\exists \delta_\varepsilon > 0$

\exists if p is any tagged partition $\|p\| < \delta_\varepsilon$

$$\left| S(f, p) - \int_a^b f \right| < \varepsilon$$

If the sub-intervals in p are $[x_{i-1}, x_i]$ then

by MVT, $\exists u_i \in (x_{i-1}, x_i)$

$$F'(u_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad \forall i = 1 : n$$

adding $i = 1 : n$

$$\sum_{i=1}^n f(x_i) - f(x_{i-1}) = \sum_{i=1}^n F'(u_i)(x_i - x_{i-1})$$

$$F(a) - F(b) = \sum_{i=1}^n f'(u_i)(x_i - x_{i-1}) = S(f, \dot{p})$$

Assuming $\dot{p}_u = \{[x_i - x_{i-1}], u_i\}_{i=1}^n$

$$\Rightarrow \left| F(a) - F(b) - \int_a^b f \right| < \varepsilon$$

$$\Rightarrow \int_a^b f = F(a) - F(b)$$

□

8.4 Indefinite Integral

Definition 8.4.1 (Indefinite Integral): If $f \in R[a, b]$ then $f(z) = \int_a^z f \quad \forall z \in [a, b]$

Theorem 8.4.1. The indefinite integral F is continuous on $[a, b]$. In fact, if $|f(x)| \leq M \quad \forall x \in [a, b]$ then $|F(z) - F(w)| \leq M|z - w| \quad \forall z, w \in [a, b]$

Proof. If $z, w \in [a, b]$, $w \leq z$

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = f(w) + \int_w^z f$$

$$\Rightarrow \int_w^z f = F(z) - F(w)$$

if $-M \leq f(x) \leq M \quad \forall x \in [a, b]$

$$-M(z-w) \leq \int_w^z f \leq M(z-w)$$

$$\Rightarrow |F(z) - F(w)| \leq \left| \int_w^z f \right| \leq M|z-w|$$

□

8.5 Examples

Example 51:

$$f(x) = x$$

$$g(x) = \frac{1}{x}$$

$$f \circ g = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$g \circ f = g(f(x)) = g(x) = \frac{1}{x}$$

$$f \circ g = g \circ f$$

Example 52:

$$A_n = \{(n+1)k, \quad k \in \mathbb{N}\}$$

$$A_1 = \{2k, \quad k \in \mathbb{N}\}$$

$$A_2 = \{3k, \quad k \in \mathbb{N}\}$$

$$A_1 \cap A_2 = \{6k, \quad k \in \mathbb{N}\}$$

$$\cap A_i = \{\phi\}$$

$$\cup A_i = \mathbb{N} - \{1\}$$

Example 53:

$$\lim \frac{n^2}{n!}$$

$$\lim \frac{n \cdot n}{n \cdot (n-1)!}$$

$$\lim \frac{n}{(n-2)(n-1)!}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{\left(1 - \frac{1}{n}\right)} \lim_{n \rightarrow \infty} \frac{1}{(n-2)!} \\ &= (1)(0) \end{aligned}$$

0

Example 54:

$$\text{Result:- } \lim_{x \rightarrow \infty} (1 + a^x)^{\frac{1}{x}} = e^a$$

$$x_n = (a^n + b^n)^{\frac{1}{n}}, \quad a < b$$

$$= \lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} b \left(\frac{a^n}{b^n} + 1 \right)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} b \left(\left(\frac{a}{b} \right)^n + 1 \right)^{\frac{1}{n}}$$

$$= b \cdot e^{\frac{a}{b}}$$

$\therefore (a^n + b^n)^{\frac{1}{n}}$ is convergent, bounded and cauchy.

Example 55:

$$\sum x_n = \frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{13} + \frac{1}{17}$$

$$\sum |x_n| = \frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} + \frac{1}{17}$$

$$\sum x_n = \frac{(-1)^{n-1}}{4n - (-1)^n}$$

Example 56:

$$f_n(x) = \frac{1}{nx+1}, \quad x \in (0, 1), f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\left| \frac{1}{nx+1} - 0 \right| < \varepsilon$$

$$\therefore \left| \frac{1}{nx+1} \right| < \varepsilon$$

$$|nx+1| > \frac{1}{\varepsilon}$$

Example 57:

Examine convergent of $\sum \left(\frac{1}{2^n} + \frac{1}{3^n} \right)$

$$\sum \left(\frac{1}{2^n} + \frac{1}{3^n} \right) = \sum \frac{1}{2^n} + \sum \frac{1}{3^n}$$

$$\sum \left(\frac{1}{2^n} + \frac{1}{3^n} \right) = \sum \left(\frac{1}{2} \right)^n + \sum \left(\frac{1}{3} \right)^n$$

$$\sum r_1^n + \sum r_2^n \quad r_1 = \frac{1}{2} < 1, r_2 = \frac{1}{3} < 1$$

which is convergent

Example 58:

$$f_n(x) = \frac{1}{x^n} \quad x \in (0, 1)$$

$$f(x) = \begin{cases} \text{not defined} & x = -1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$$

Example 59:

$$\lim_{n \rightarrow \infty} |x_n - x_{n+1}| = 0$$

x_n does not converges for given example j

Example 60:

$$\sum \frac{1}{\sqrt{n^3 + 4}} \text{ Use comparision test}$$

$$n < n^{\frac{3}{2}}, \quad n > 1$$

$$\frac{1}{n} > \frac{1}{n^{\frac{3}{2}}}$$

As $\frac{1}{n}$ is divergent $\Rightarrow \frac{1}{n^{\frac{3}{2}}}$ is also divergent.

Definition 8.5.1 (Taylors expansion for two variables): $f(x, y) =$

$$f(a, b) + \left[(x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right] f(a, b) + \frac{1}{2!} \left[(x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right]^2 f(a, b) + \dots$$

$$\dots \dots \frac{1}{(n-1)!} \left[(x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right]^{n-1} f(a, b) + R_n$$

Example 61:

a_n is bounded, decreasing sequence.

b_n is bounded, increasing sequence

$$x_n = a_n + b_n$$

$$\sum |x_n - x_{n+1}|$$

$$= \sum |a_n + b_n - a_{n+1} - b_{n+1}|$$

$$= \sum |a_n - a_{n+1} + b_n - b_{n+1}|$$

$$\leq |a_n - a_{n+1}| + |b_n - b_{n+1}|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$\sum |x_n - x_{n+1}| \rightarrow 0$$

Example 62:

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \quad p > 0$$

$$\log n < n$$

$$\frac{1}{\log n} > \frac{1}{n}$$

$$\left(\frac{1}{\log n}\right)^p > \frac{1}{n^p}$$

$$\frac{1}{n(\log n)^p} > \frac{1}{n^{p+1}}, \quad p+1 > 1$$

\therefore by comparison test,

As $\sum \frac{1}{n^{p+1}}$ convergent $\Rightarrow \sum \frac{1}{n(\log n)^p}$ is convergent.

Example 63:

$$\sum x_n = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{n \cdot 2^n}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)^{2^{n+1}}}}{n2^n} \right|$$

$$= \left| \frac{n \cdot 2^n}{(n+1)2^{n+1}} \right|$$

$$= \left| \left(\frac{n}{n+1} \right) \frac{1}{2} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{1}{1 + \frac{1}{n}} \right) \frac{1}{2} \right|$$

by ratio test

$$= \frac{1}{2} < 1$$

$\sum x_n = \frac{1}{n2^n}$ is convergent.

Example 64:

$$S = \left\{ 1 + \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}$$

limit point of $S = 1$

Example 65:

$$\sum x_n = \sum \frac{1}{\sqrt{n} + \sqrt{n-1}}$$

$$n > \sqrt{n}$$

$$\frac{1}{n} < \frac{1}{\sqrt{n}}$$

by Ratio test,

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{\sqrt{(n+1)} + \sqrt{n}}}{\frac{1}{\sqrt{n} + \sqrt{n-1}}} \right|$$

$$= \left| \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$= \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right| = 1$$

\therefore Ratio test fails here

$$\sum \frac{1}{\sqrt{n} + \sqrt{n-1}} \times \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}} = \sum \sqrt{n} - \sqrt{n-1}$$

$\therefore S_n = \sqrt{n}$ which divergent

$\therefore \sum \frac{1}{\sqrt{n} + \sqrt{n-1}}$ is divergent.

Example 66:

$$\sum \frac{(2n-1)}{n(n+1)(n+2)} = \frac{1}{1.2.3} + \frac{3}{2.3.4} + \dots$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{(2n-1)}{n(n+1)(n+2)(n+3)}}{\left(\frac{2n-1}{n(n+1)(n+2)} \right)} \right|$$

$$= \left| \frac{(2n+1)n}{(2n-)(n+3)} \right|$$

$$= \left| \frac{\left(2 + \frac{1}{n}\right)}{\left(2 - \frac{1}{n}\right)\left(1 + \frac{3}{n}\right)} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = 1$$

\therefore Ratio test fails here.

$$\begin{aligned}\sum \left(\frac{2n-1}{n(n+1)(n+2)} \right) &= \sum \frac{2n}{n(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)} \\ &= \sum \frac{2}{(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)}\end{aligned}$$

$\therefore x_n$ is convergent.