

Unit - I - Real Numbers \mathbb{R}

~~Groups Mat~~
~~($\mathbb{R}, +, \cdot$)~~

* Algebraic Properties of \mathbb{R} :-Add[~] Mult[~]On set of \mathbb{R} there are two binary operators $\underline{+}$ & $\underline{\cdot}$

These two operations follows few properties :-

A1) Commutative property of addition

$$a+b = b+a \quad \forall a, b \in \mathbb{R}$$

A2) Associative property of add[~]

$$(a+b)+c = a+(b+c) \quad \forall a, b, c \in \mathbb{R}$$

A3) Existence of zero element (Additive Identity) ✓

$$a+0 = 0+a = a \quad \forall a \in \mathbb{R}$$

A4) Existence of negative element (Additive Inverse)

$$a+(-a) = (-a)+a = 0 \quad \forall a \in \mathbb{R}$$

✓ M1) Commutative property of multiplication

$$a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$$

M2) Associative p of multi.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in \mathbb{R}$$

M3) Existence of unit element (Multiplicative identity)

$$a \cdot 1 = 1 \cdot a = a \quad \forall a \in \mathbb{R}$$

M4) Existence of multiplicative inverse / Reciprocals ✓

$$a \cdot \left(\frac{1}{a}\right) = \frac{1}{a} \cdot a = 1 \quad \forall a \in \mathbb{R} - \{0\}$$

D) Distributive property of multiplication over add[~]

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a$$

$$= a \cdot b + a \cdot c$$

$$\forall a, b, c \in \mathbb{R}$$

* Order Properties of IR

Why do we study Real Analysis?

seq of func
 $\sum f_n(x) = \sum \frac{x_i}{n}$

Convergence of Series of func's

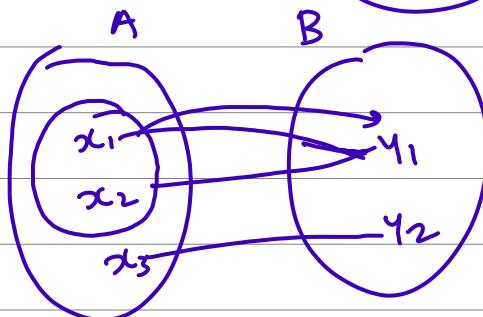
$$\text{CLT} \rightarrow \frac{\sum x_i - \mu}{\sigma} \rightarrow \text{N}(0, 1)$$



$$\left(\frac{\sum x_i}{n} \right) \rightarrow \bar{x}$$

Set Func

func



Prob.

input element \rightarrow Set $n(A) \in \mathbb{N}$
 classi $n(S)$

input Set \rightarrow Real

ST-201

(A)

Borel func

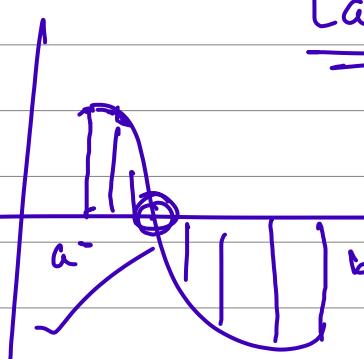
$$x^3 + 2x + 3 = 0 \quad x=? \quad \mathbb{R}$$

[a, b]

closed

ST-705
Numerical Methods

Cont?



Serier $\rightarrow P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$E(X) = \sum_x \frac{e^{-\lambda} \lambda^x}{x!} x$$

$$= e^{-\lambda} \sum_x \frac{\lambda^x}{x!} x$$

Geometric $P(X=x) = pq^{x-1}$

$$E(X) = \sum x \cdot pq^{x-1} = \frac{1}{p}$$

$$\begin{aligned} &= p \cdot \sum x \cdot q^{x-1} \\ &= p \cdot \frac{1}{(1-q)^2} \\ &= p \cdot \frac{1}{p^2} = \frac{1}{p} \end{aligned}$$

$$\sum x^n = \frac{1}{1-x} \quad |x| < 1$$

$$\frac{d}{dx} \sum x^n = \frac{d}{dx} \frac{1}{1-x}$$

$$\text{& } \sum n x^{n-1} = \frac{1}{(1-x)^2}$$

Inverse fun^c $\leftarrow X$ Random Variable

ST-201
Probability
ST-301
Asymptotic

- ① Unbiased
- ② Suff
- ③ Eff
- ④ Consistent \rightarrow as $n \rightarrow \infty$

$$\bar{x} \rightarrow \mu \quad \frac{1}{n} \sum x_i \rightarrow \mu$$

$$X_{(n)} \xrightarrow{\text{C.E.}} N(0, \sigma^2)$$

$$2\bar{X} \rightarrow 2\mu$$

* Order Properties of R

P set of tve nos $\in \mathbb{R}$, $P \subseteq \mathbb{R}$, \mathbb{R}^+

\mathbb{R}^+ satisfies following properties

- ① if $a, b \in \mathbb{R}^+$ $\Rightarrow a+b \in \mathbb{R}^+$
② if $a, b \in \mathbb{R}^+$ $\Rightarrow a \cdot b \in \mathbb{R}^+$

③ if $a \in \mathbb{R}$ then exactly one of the following is true:-

$$\frac{a \in \mathbb{R}^+}{X}, \text{ OR } \frac{a=0}{X}, \text{ OR } \frac{-a \in \mathbb{R}^+}{\checkmark}$$

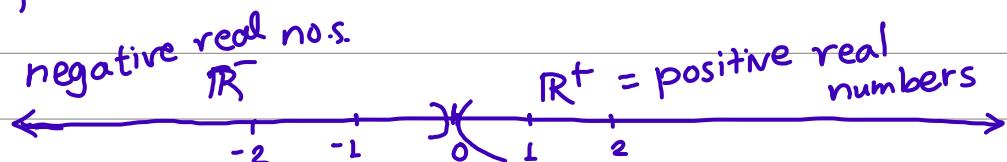
$$\begin{array}{c} \underline{a \in \mathbb{R}} \\ \begin{matrix} > 0 & < 0 & = 0 \\ a \in \mathbb{R}^+ & a \in \mathbb{R}^- \\ -a \in \mathbb{R}^+ \end{matrix} \end{array}$$

Law of Trichotomy

Let $a, b \in \mathbb{R}$

- a) If $\underline{a} - b \in \mathbb{R}^+$ $\Rightarrow a > b$ or $b < a$

b) If $\underline{a} - b \in \mathbb{R}^+ \cup \{0\}$ $\Rightarrow a \geq b$ or $b \leq a$



$$\checkmark \quad \mathbb{R}^+ = \{x \mid x > 0, x \in \mathbb{R}\}$$

$$R^- = \{x / x < 0, x \in R\}$$

Proof :- @ $a, b \in \mathbb{R}$, $a - b \in \mathbb{R}^+$ $\Rightarrow a - b > 0 \Rightarrow a > b$

$$\textcircled{b} \quad a, b \in \mathbb{R} \quad a - b \in \mathbb{R}^+ \cup \underline{\underline{\{0\}}} \Rightarrow a - b \geq 0 \Rightarrow a \geq b$$

Theo:- Let $a, b, c \in \mathbb{R}$

- ① If $a > b$ & $b > c \Rightarrow a > c$ (Transitivity)
 - ② If $a > b \Rightarrow a+c > b+c$
 - ③ If $a > b, c > 0 \Rightarrow a \cdot c > b \cdot c$
If $a > b, c < 0 \Rightarrow a \cdot c < b \cdot c$

① If $a > b$, & $b > c$

$$a - b > 0 \text{ & } b - c > 0$$

$$\Rightarrow a - b \in \mathbb{R}^+ \text{ & } b - c \in \mathbb{R}^+$$

$$\Rightarrow (a - b) + (b - c) \in \mathbb{R}^+$$

$$\Rightarrow (a - c) \in \mathbb{R}^+$$

$$\Rightarrow a - c > 0$$

$$\Rightarrow a > c$$

(by order properties)

QED

② $a - b \in \mathbb{R}^+$

$$a - b + c - c \in \mathbb{R}^+$$

$$(a + c) - (b + c) \in \mathbb{R}^+$$

$$a + c > b + c$$

③ $a > b$, $c > 0$

to prove $a \cdot c > b \cdot c$

$$\begin{array}{c} a > b, \\ \underline{a - b \in \mathbb{R}^+} \end{array}$$

$$c(a - b) = c \cdot a - c \cdot b \Leftarrow \text{as } c > 0$$

$$\cancel{c a - c b > 0} \quad \Leftarrow$$

$$ca > cb$$

similarly $a > b$, $c < 0 \Rightarrow ac < bc$

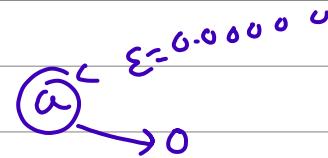
Theo:-

If $a \in \mathbb{R}$, such that $0 \leq a < \varepsilon \Rightarrow \varepsilon \in \mathbb{R}$, $\varepsilon > 0$

$$\Rightarrow a = 0$$

$$\varepsilon = 0.005$$

08



By method of contradiction.

Assume $a > 0 \Rightarrow (as 0 \leq a < \varepsilon) \Rightarrow a$ is positive
 $\Rightarrow \frac{a}{2}$ is positive

we can assume $\varepsilon = a/2 < a$

for any $\varepsilon > 0, 0 \leq a < \varepsilon$ but for $\varepsilon_0 \Rightarrow 0 < \varepsilon_0 < a$

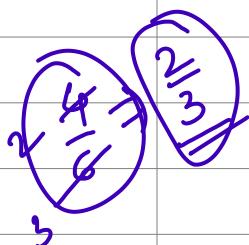
which contradicts to our assumption. \square

Rational Nos. :- $Q = \{x / x = p/q, p, q \in \mathbb{Z}, q \neq 0\}$

* Theo. There doesnot exist any rational no. $\sqrt{2} \in Q$

$$\underline{\underline{r^2 = 2}}$$

\Rightarrow Assume that $\sqrt{2}$ is rational no.



$$\sqrt{2} = \frac{p}{q}$$

common divisor of p, q is 1
 $\hookrightarrow (p, q) = 1$

$$\Rightarrow 2 = \frac{p^2}{q^2}$$

$$\Rightarrow 2q^2 = p^2 = p \cdot p$$

$$\rightarrow p = 3 \quad p^2 = 9 \quad p^2 = 16 =$$

$\Rightarrow p$ is divisible by 2

$$\Rightarrow p = 2 \cdot m$$

$$\Rightarrow p^2 = 2^2 \cdot m^2 = 4m^2$$

~~$p = \text{even}$~~

$$\Rightarrow 2q^2 = 4m^2$$

$$\Rightarrow q^2 = 2m^2$$

$\Rightarrow q$ is divisible by 2 (****)

$(p,q) = 1$ but here $(p,q) = 2$

which contradicts to our assumption -

$$\begin{aligned}
 & \text{Q} \quad \frac{\text{even}^2}{(2 \cdot n)^2} \quad \frac{\text{odd}^2}{(2n+1)^2} \\
 &= 2^{\text{even}} \cdot (2^n)^2 \quad = 2^{\text{odd}} \cdot \underline{(2n+1)^2} \\
 & \qquad \qquad \qquad = 4n^2 + 2n + 1 \\
 & \qquad \qquad \qquad = 2 \cdot \underline{(2n^2+n)} + 1 \\
 & \qquad \qquad \qquad = \text{odd}
 \end{aligned}$$

1

Theo:- If $ab > 0$ then either ① $a > 0, b > 0$
 ② $a < 0, b < 0$

① If $a, b \in \mathbb{R}$ show that $a^2 + b^2 = 0 \Leftrightarrow a = 0, b = 0$

② If $0 < c < 1$ show that $0 < c^2 < c < 1$ ✓

③ If $x, y \in \mathbb{Q}$, $x+y \in \mathbb{Q}$, $\underline{x \cdot y \in \mathbb{Q}}$ ✓

If $x \in \mathbb{Q}, y \in \mathbb{Q}^c$, $x+y \in \mathbb{Q}^c$

$$\begin{aligned}
 & (a+b)^2 = a^2 + b^2 + 2ab = 0 \\
 & a^2 + b^2 = 0 \Rightarrow 2ab = 0 \\
 & \Rightarrow a \cdot b = 0
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{?}{=} a = 0 \quad \text{or} \quad b = 0 \\
 & \text{if } a = 0 \quad \& \quad b \neq 0 \Rightarrow a^2 + b^2 > 0
 \end{aligned}$$

$$a^2 + b^2 = 0 \Rightarrow b = 0$$

$$a, b \in \mathbb{R}^+, c > 0,$$

$$a > b \Rightarrow c \cdot a > c \cdot b$$

$$0 < c < 1$$

$$\text{as } c > 0$$

$$c < 1$$

$$c \cdot c < 1 \cdot c$$

$$0 < c^2 < c < 1$$



$$x, y \in \mathbb{Q} \Rightarrow x+y \in \mathbb{Q}$$

$$x = \frac{p_1}{q_1}, y = \frac{p_2}{q_2} \Rightarrow x+y = \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$$

$$\text{as } p_1 q_1, p_2 q_2 \in \mathbb{Z},$$

$$\underline{p_1 q_2 \in \mathbb{Z}, p_2 q_1 \in \mathbb{Z}}, q_1 q_2 \in \mathbb{Z}$$

$$p_1 q_2 + p_2 q_1 \in \mathbb{Z}$$

$$= \frac{p^*}{q^*} \in \mathbb{Q}$$

Absolute values, $\forall a \in \mathbb{R}$ $|a| = \begin{cases} +a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$

$|a| = \max(a, -a)$

Theo:- ① $|ab| = |a||b| \Rightarrow a, b \in \mathbb{R}$ ✓

② $|a|^2 = a^2 \Rightarrow a \in \mathbb{R}$ $-c \leq a+b \leq c \Rightarrow |a+b| \leq c$

③ If $c \geq 0$ then $|a| \leq c$ iff $-c \leq a \leq c$

④ $-|a| \leq a \leq |a|$

Proof:- ① If $a, b > 0 \Rightarrow ab > 0 \Rightarrow |a|=a, |b|=b, |a||b|=ab=|a||b|$

$a > 0, b < 0 \Rightarrow ab < 0 \Rightarrow |a|=a, |b|=-b, |ab|=-ab=a \cdot (-b)=|a||b|$

$a < 0, b > 0 \Rightarrow ab < 0 \Rightarrow$ Simillarly

$a, b < 0 \Rightarrow ab > 0 \Rightarrow |a|=-a, |b|=-b, |ab|=ab=(-a) \cdot (-b)$

$= |a||b|$ ✓

② $|a|^2 = a^2$

If $a > 0, |a|=a \Rightarrow |a|^2 = a^2$ ✓

$a < 0 \quad |a| = -a \Rightarrow |a| = (-a)^2 = a^2$ ✓

$a = 0 \quad |a| = a = 0 \Rightarrow |a|^2 = a^2 = 0$ ✓

③ If $c \geq 0$ then

$|a| \leq c$ iff $-c \leq a \leq c$

i) $|a| \leq c \Leftrightarrow |a| = \max(a, -a) \leq c$

$\Leftrightarrow a \leq c \text{ & } -a \leq c$

$\Leftrightarrow a \leq c \text{ & } a \geq -c$

$\Leftrightarrow -c \leq a \leq c$ ✓

ii) $|a| \leq c$ iff $-c \leq a \leq c$ ✓

now assume $c = |a| \geq 0, |a| \leq |a|$

$\Rightarrow -|a| \leq a \leq |a|$

* Triangular inequality :- If $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$

proof :- If $a, b \in \mathbb{R}$, then

$$\begin{array}{r} -|a| \leq a \leq |a| \\ + \quad -|b| \leq b \leq |b| \\ \hline -|a|-|b| \leq a+b \leq |a|+|b| \end{array}$$

$$\text{put } c = |a| + |b|$$

$$\Rightarrow -(|a| + |b|) \leq a+b \leq |a| + |b|$$

$$\Rightarrow -c \leq a+b \leq c$$

$$\Rightarrow |a+b| \leq c$$

$$\Rightarrow |a+b| \leq |a| + |b|$$

$$|a| = |-a|$$

* Corollary : If $a, b \in \mathbb{R}$, then (a) $||a|-|b|| \leq |a-b|$

$$(b) |a-b| \leq |a| + |b|$$

(a) If $a, b \in \mathbb{R}$

$$\checkmark a = \underline{a-b+b}$$

$$|a| = |\underline{a-b+b}| \leq |a-b| + |b| \quad \text{--- (by triangular inequality)}$$

$$\checkmark b = b-a+a$$

$$|b| = |b-a+a| \leq |\underline{b-a}| + |a| = |\underline{a-b}| + |a| \quad (\text{by tri. inequality})$$

$$\text{From (a)} \quad |a| - |b| \leq |a-b|.$$

$$(**) \quad |b| - |a| \leq |a-b| \Rightarrow |a| - |b| \geq -|a-b|$$

$$\Rightarrow \boxed{\begin{array}{l} -c \leq a \leq c \\ \Rightarrow |a| \leq c \end{array}} \quad \boxed{-|a-b| \leq |a| - |b| \leq |a-b|}$$

$$\text{put } c = |a-b| \Rightarrow ||a|-|b|| \leq c \Rightarrow ||a|-|b|| \leq |a-b|$$

$$\frac{-|a-b|}{c} \leq \frac{|a|-|b|}{c} \leq \frac{|a-b|}{c}$$

$\Rightarrow |a|-|b| \leq c$

Manoj C Patil

$$\textcircled{b} \quad |a-b| \leq |a| + |b|$$

We have triangular inequality $|a+b| \leq |a| + |b|$

replace b by $-b$

$$|a+(-b)| \leq |a| + |-b|$$

$$\Rightarrow |a-b| \leq |a| + |-b| \quad (|-b| = |b|)$$

$$\Rightarrow |a-b| \leq |a| + |b|$$

$$a-b \leq a+b \quad ?$$

$$a, b \in \mathbb{R}$$

Triangular inequality

$$|\sum_{i=1}^n a_i| \leq \sum_{i=1}^n |a_i|$$

* Real line

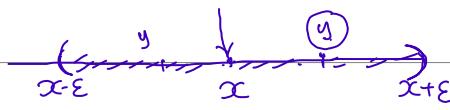
Extended Real Numbers



ε -nbhd.

$$V_\varepsilon(x) = \{y / y \in (x-\varepsilon, x+\varepsilon)\} \subseteq \mathbb{R}$$

$$(x-\varepsilon, x+\varepsilon) = V_\varepsilon(x)$$



interval length $\Rightarrow 2\varepsilon$

$$|x-y| \leq \varepsilon \Rightarrow y \in V_\varepsilon(x)$$

$$V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$$

$$y \in V_\varepsilon(x) \Rightarrow |x-y| \leq \varepsilon \Rightarrow \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow x-\varepsilon \leq y \leq x+\varepsilon$$

$$\delta_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) - \{x\} \quad \text{deleted nbhd of } x$$

e.g. we know $|a+b| \leq |a| + |b|$ but if $|a+b| = |a| + |b|$ iff $ab > 0$

i) If $ab > 0$ to prove $|a+b| = |a| + |b|$
 If $a > 0, b > 0, a+b > 0 \Rightarrow |a+b| = a+b = |a| + |b|$
 $a < 0, b < 0, a+b < 0 \Rightarrow |a+b| = -(a+b) = (-a) + (-b) = |a| + |b|$

$a > 0, b < 0$

If $|a+b| = |a| + |b|$ to prove $ab > 0$

$$\begin{aligned} |a+b|^2 &= (|a| + |b|)^2 \\ (a+b)^2 &= |a|^2 + |b|^2 + 2|a||b| \\ \Rightarrow a^2 + b^2 + 2ab &= a^2 + b^2 + 2|a||b| \\ \Rightarrow ab &= |a||b| = |a \cdot b| \quad \text{on} \\ \Rightarrow ab &> 0 \end{aligned}$$

e.g. If $x, y, z \in \mathbb{R}$ & $\underline{x \leq z}$ show that
 $x \leq y \leq z$ iff $|x-y| + |y-z| = |x-z|$

i) Let $\underline{x \leq y \leq z}$. $x-y$ & $y-z$ negative.

$$\begin{aligned} \text{LHS} &= |x-y| + |y-z| \\ &= (\cancel{y-x}) + (\cancel{z-y}) \\ &= \cancel{z-x} \\ &= |x-z| \checkmark \\ &= \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} x \leq y &\Rightarrow x-y \leq 0, y \leq z \Rightarrow y-z \leq 0 \\ |x-y| &= -(x-y) \\ &= y-x \\ &= z-y. \end{aligned}$$

$$\cancel{x \leq z} \Rightarrow x-z \leq 0$$

$$|x-z| = z-x$$

$$|x-y| \neq (x-y)$$

ii) Let $|x-y| + |y-z| = |x-z|$,

$$x \leq z \Rightarrow x-z \leq 0, z-x \geq 0$$

$$x-z = x-y + y-z$$

$$\text{Top} \rightarrow x \leq y \leq z$$

$$x-z = \underline{x-y+y-z}$$

$$|x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z|$$

but $|x-z| = |x-y| + |y-z|$ so. we can use

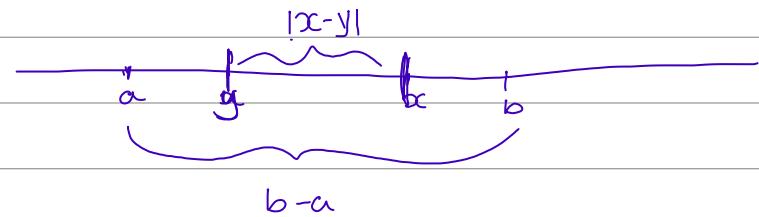
$$\boxed{|a+b| = |a| + |b| \text{ iff } \underline{ab \geq 0}}$$

$x-y \geq 0$
 $x \geq y$

Only two possibilities

$(x-y)(y-z) > 0$	$\Rightarrow x > y, y > z \Rightarrow x > y \geq z$, which is not possible
$(x-y)(y-z) \leq 0$	$\Rightarrow x \leq y, y \leq z \Rightarrow x \leq y \leq z$

if $a < x < b$
 $a < y < b$
 $|x-y| \leq b-a$



?

$$\begin{aligned} -a > -x > -b &\Rightarrow -b < -x < -a \\ a < y < b &\quad \underline{\quad a < y < b} \\ -(b-a) < y-x < b-a & \\ \Rightarrow |x-y| &\leq b-a \end{aligned}$$

e.g. $|x-1| > |x+1|$ $x \in ?$ $\underline{x < 0}$

$\underline{-3} \leq x < 0$ ≤ 0 -0.01

$$|x+1| + |x-2| = 7 \quad x = -3 \quad x = 4$$

$$\text{e.g. } \min\{a,b\} = \frac{1}{2}(a+b - |a-b|) \quad \max\{a,b\} = \frac{1}{2}(a+b + |a-b|)$$

$\rightarrow \textcircled{1} \cdot \text{ Let } \underline{a < b}, \quad \min(a,b) = a \quad \max(a,b) = b$

$$|a-b| = b-a$$

$$\text{RHS} = \frac{1}{2}(a+b + |a-b|)$$

$$= \frac{1}{2}(a+b + b-a)$$

$$= \frac{2b}{2} = \max(a,b)$$

$$\text{RHS} = \frac{1}{2}(a+b - |a-b|)$$

$$= \frac{1}{2}(a+b - (b-a))$$

$$= \frac{2a}{2} = \min(a,b)$$

github.com/manojcpatil/Lecture-scribbles

{ Absent Aditya, Dipali, Anushree, Aakash, Lalit, Vishal, Bhavesh, Hikita, Sopan
 Nikhil, Pranit, Bhanuja

