Open ball $\underline{B = \{ \underline{x} \mid ||\underline{x} - \underline{x}_0|| < r \}}$

To show :- B is convex set.

Let $\underline{x}, \underline{y} \in B \quad \Rightarrow \quad \underline{\alpha \cdot \underline{x}} + (1-\alpha)\, \underline{y} \; \boxed{\in B}$

$\underline{x} \in B \Rightarrow ||\underline{x} - \underline{x}_0|| < r$

$\underline{y} \in B \Rightarrow ||\underline{y} - \underline{x}_0|| < r$

$\alpha \in (0,1)$

$$||\alpha \underline{x} + (1-\alpha)\underline{y} - \underline{x}_0||$$

$$= ||\alpha \underline{x} + (1-\alpha)\underline{y} - (\alpha \underline{x}_0 + (1-\alpha) \cdot x_0)||$$

$$= ||\alpha(\underline{x} - \underline{x}_0) + (1-\alpha)(\underline{y} - x_0)||$$

$$\leq \alpha \underbrace{||\underline{x} - \underline{x}_0||}_{<r} + (1-\alpha) \underbrace{||y - x_0||}_{<r} \qquad \alpha \in (0,1)$$

$$< \alpha r + (1-\alpha) r$$

$$< r$$

---

$\{ \underline{x} \mid ||\underline{x} - \underline{x}_0|| = r \}$ Convex or Not ✓

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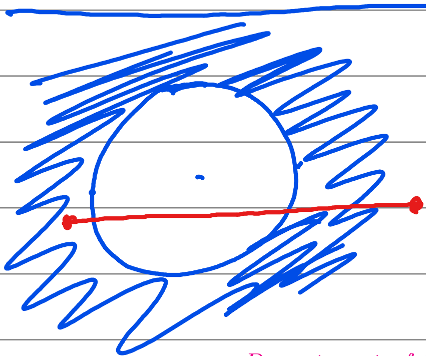


$\{ \underline{x} \mid ||\underline{x} - \underline{x}_0|| \geq r \}$ convex or not ✓

If $C$ is convex set $\underline{\underline{\lambda C}}$ is also convex set.

$$\lambda C = \{ \underline{y} \mid \underline{y} = \lambda \underline{x} \, , \underline{x} \in C \}$$

to show $\lambda C$ as convex set

$$\underline{y}_1, \underline{y}_2 \in \lambda C \quad \Rightarrow \quad \underline{y}_1 = \lambda \underline{x}_1 \, , \underline{y}_2 = \lambda \underline{x}_2 \quad \underline{x}_1, \underline{x}_2 \in C$$

$$\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 = \alpha \cdot \lambda \underline{x}_1 + (1-\alpha) \lambda \underline{x}_2$$

$$= \lambda (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$= \lambda \underline{x} \qquad \left( \text{as } \underline{x}_1, x_2 \in C \Rightarrow x \in C \right)$$

as $C$ is convex

$$\Rightarrow \alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 \in \lambda C$$

$C, D$ are convex sets $C+D$ is also convex

$$\rightarrow \quad C+D = \{ \underline{z} \mid \underline{z} = \underline{x} + \underline{y} \, , \underline{x} \in C, \underline{y} \in D \}$$

$$\underline{z}_1, \underline{z}_2 \in C+D \Rightarrow z_1 = \underline{x}_1 + \underline{y}_1$$

$$z_2 = \underline{x}_2 + \underline{y}_2 \qquad \underline{x}_1, \underline{x}_2 \in C, \ \underline{y}_1, \underline{y}_2 \in D.$$

$$\alpha z_1 + (1-\alpha) z_2 = \alpha \cdot (\underline{x}_1 + \underline{y}_1) + (1-\alpha)(\underline{x}_2 + \underline{y}_2)$$

$$= \underbrace{\alpha \cdot \underline{x}_1 + (1-\alpha) \underline{x}_2}_{\in C} + \underbrace{\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2}_{\in D}$$

convex sets

Intersection of any convex sets is convex

Let $\{S_i\}_{i=1}^{\infty}$ be collection of convex sets

$\cap S_i$ is convex

$\underline{x}, \underline{y} \in \cap S_i$

$\Rightarrow \underline{x}, \underline{y} \in S_i \quad \forall i$

$\Rightarrow \alpha \underline{x} + (1-\alpha)\underline{y} \in S_i \quad \forall i$ ($S_i$ is convex)

$\Rightarrow \alpha \underline{x} + (1-\alpha)\underline{y} \in \cap S_i$

$\Rightarrow \cap S_i$ is convex.

- A set S∈Rn is convex if and only if every convex combination of any finite number of points of S is contained in S

$\therefore$ Assume that every convex combⁿ of any finite no. of points of S is in S.

② → ①

$\Rightarrow$ it is also true for $n=2$

$\Rightarrow$ if $\forall\, \underline{x}_1, \underline{x}_2 \in S \Rightarrow \alpha \cdot \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S \quad \forall \alpha \in (0,1)$

$\Rightarrow S$ is convex set

(II) Assume S is convex and for any finite n $\sum_{i=1}^{n} \alpha_i \underline{x}_i \in S$

→ let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in S$ $\quad \sum_{i=1}^{n} \alpha_i = 1$

we will prove this by mathematical induction

As S is convex, $\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S$ $\quad \alpha_1 + 1 - \alpha = 1$

∴ So the above statement is true for $n=2$

Assume it is true for $n=K$ $\Rightarrow \sum_{i=1}^{K} \alpha_i \underline{x}_i = 1$

$$\sum_{i=1}^{K} \alpha_i = 1$$

We have to prove it for $n=K+1$

$$\sum_{i=1}^{K+1} \beta_i \underline{x}_i = \sum_{i=1}^{K} \beta_i x_i + \beta_{K+1} x_{K+1} \qquad \sum_{i=1}^{K+1} \beta_i = 1$$

imp ?

$$\sum_{i=1}^{K} \beta_i = 1 - \beta_{K+1}$$

$$\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} = 1$$

$$= (1-\beta_{K+1}) \boxed{\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} \cdot x_i} + \beta_{K+1} x_{K+1}$$

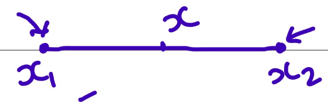
→ $\in S$

$$= (1-\beta_{K+1})\, x^* + \beta_{K+1} x_{K+1}$$

$\in S$ as $S$ is convex set

* Vertices

2 vertices

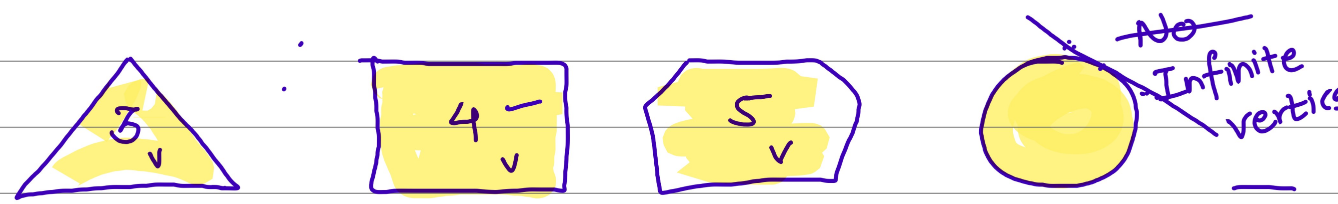


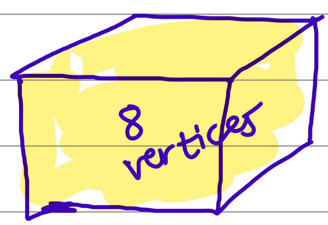

$\Rightarrow \quad x = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$x_2 = \alpha \underline{x}_1 + (1-\alpha)x_2 = \underline{x}_2$

$\uparrow \alpha = 0$


No vertex




Convex Hull

$\boxed{Co(S) \Rightarrow \bigcap_{i=1}^{\infty} S_i}$

$S_i$ is convex set containing S

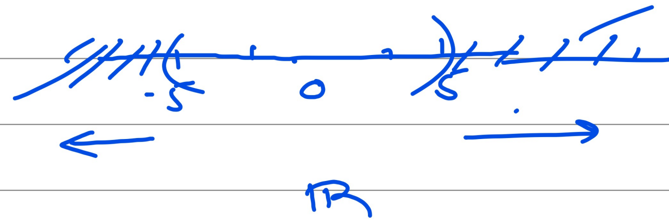
$S = \{\underline{x}_1, \underline{x}_2\}$



$S = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$

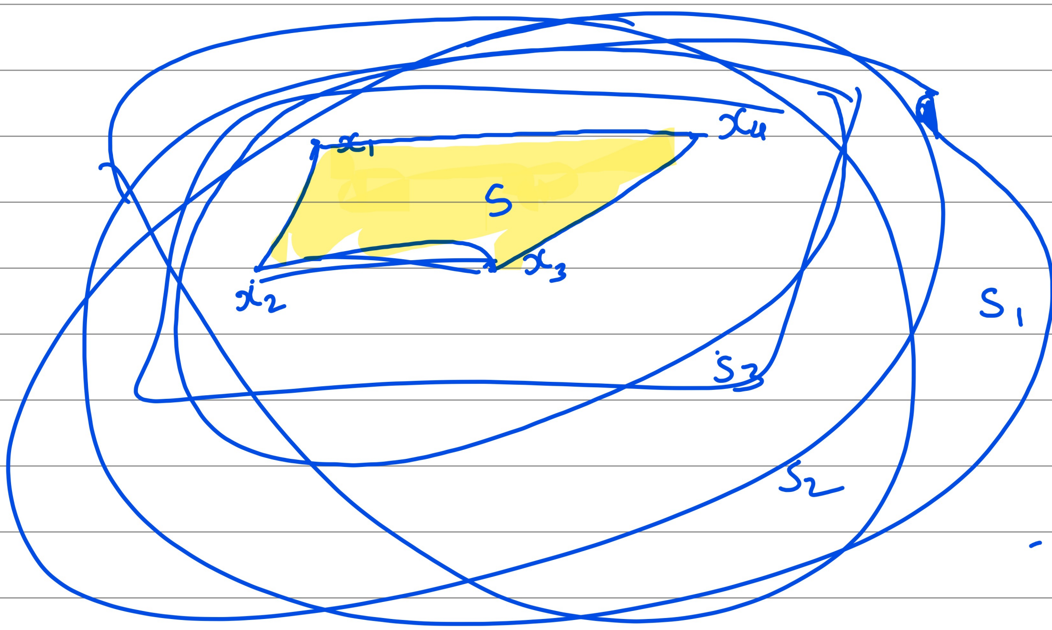
$S = \{ \underline{x} \mid \|\underline{x}\| \geq 5 \}$

$Co(S) = \mathbb{R}^n$

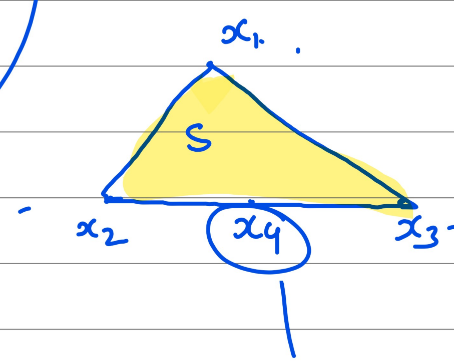


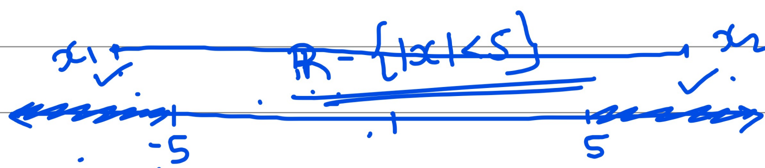

if set is convex
$\underline{\underline{Co(S) = S}}$

$Co(S) = \cap S_i$ , $S_i$ is convex set containing $S$.



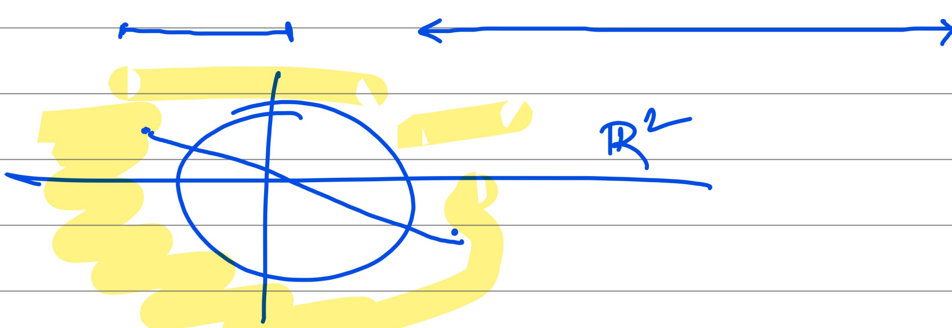

$\cap S_i = Co(S)$



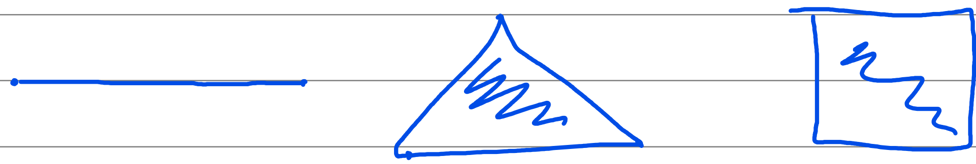




$S = \{ |x| > 5 \}$

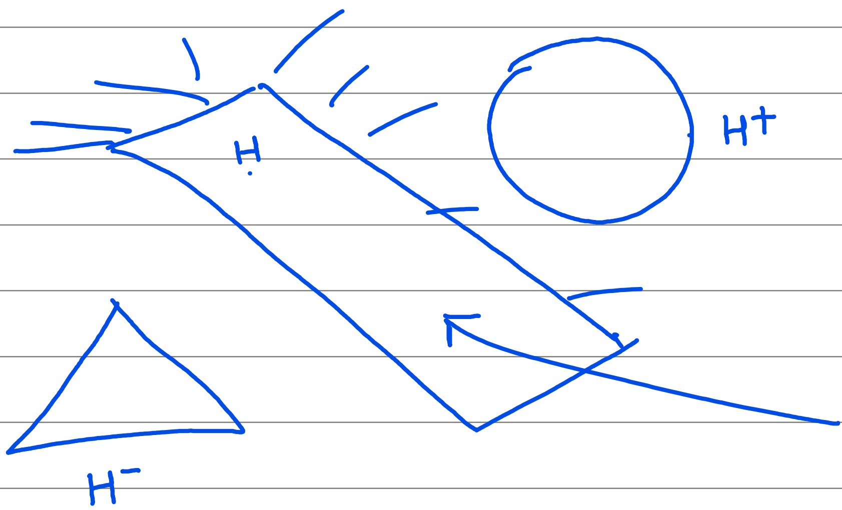
$\underline{\underline{\mathbb{R} = Co(S)}}$



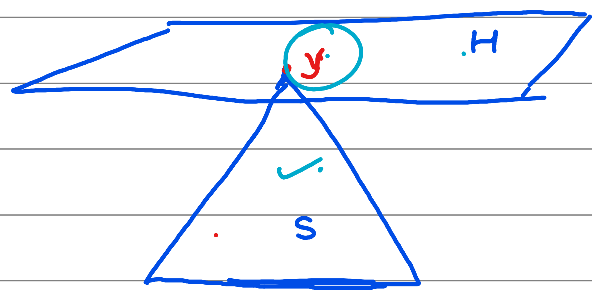
$$Co(S) = \left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \lambda_i x_i, \quad x_i \in S, \quad \sum \lambda_i = 1, \ \lambda_i \geq 0 \right\}$$



Hyperplane:- $H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$




Separating Hyperplane




$S \in H^-$ or $S \in H^+$

$\underline{a}'\underline{x} \leq c$ or $\underline{a}'\underline{x} \geq c$

$\forall x \in S$ $\qquad \forall x \in S$

$S \in H^-$

$H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$

$\underline{a}'\underline{y} = c \Rightarrow \underline{y} \in S \ni \underline{y} \in H$


$S \in H^-$