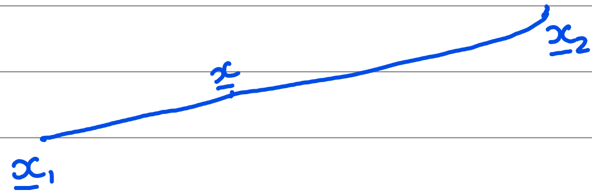
* Line segment : $\underline{x} \in \mathbb{R}^n$




scaler

$\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in (0,1)$

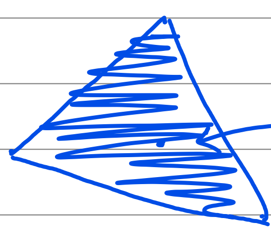
line segment $\{\underline{x} / \underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 ,\ \alpha \in (0,1)\}$
joining $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$

* Line passing $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$ $\alpha + 1 - \alpha$

line segment ← convex combⁿ $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\underline{\alpha \in (0,1)}$

line, linear combⁿ $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in \mathbb{R}$


vector space → $\boxed{\underline{x} = \sum \alpha_i \underline{x}_i}$ Linear Combⁿ $\alpha_i \in \mathbb{R}$



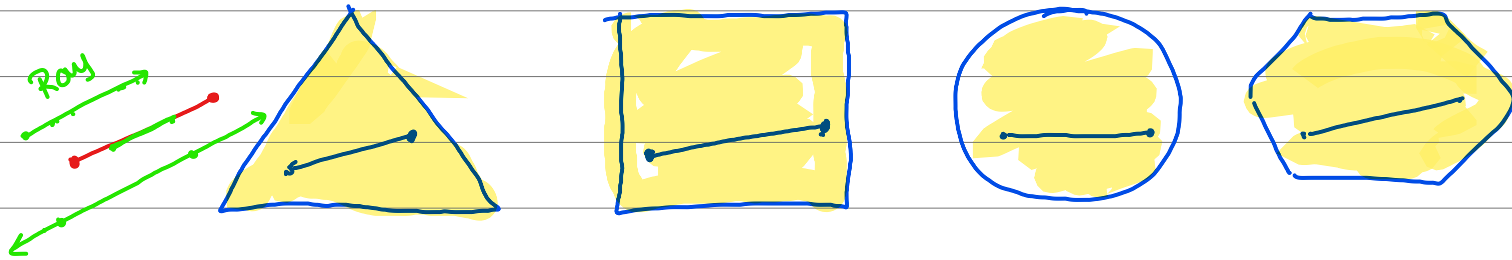
$\underline{x} = \sum \alpha_i \underline{x}_i$ $\underline{\sum \alpha_i = 1}$, $\underline{\alpha_i \geq 0}$

↳ Convex Combination

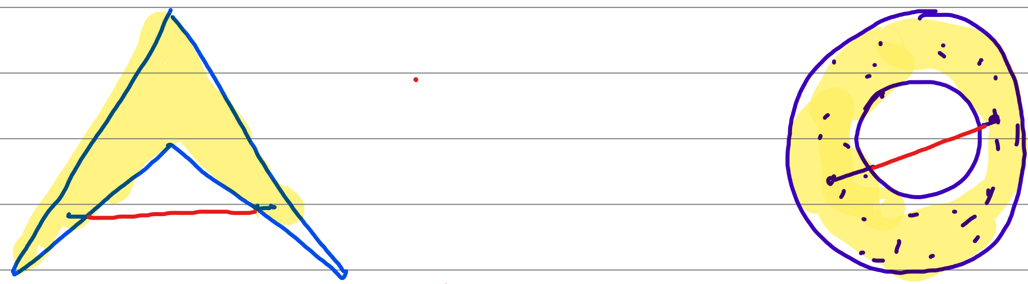
$$\left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \alpha_i x_i ,\ \alpha_i \geq 0 ,\ \sum \alpha_i = 1 \right\}$$

Convex set if $\forall\, \underline{x}_1 \,\&\, \underline{x}_2 \in A$, $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \;\; \forall\, \alpha \in (0,1)$

then if $\underline{x} \in S \;\forall\, \alpha$ A is convex set




Convex sets



Non convex sets

* Ray is convex set

$$\underline{x} = \underline{x}_0 + \underline{d}\,\alpha \qquad \alpha \geq 0$$

$\underline{x}_1 \,\&\, \underline{x}_2 \in A$

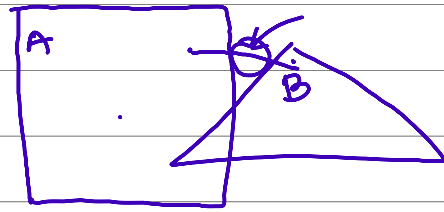
$\lambda \qquad \underline{x}_1 = \underline{x}_0 + \underline{d}\,\alpha_1$

$1-\lambda \qquad \underline{x}_2 = \underline{x}_0 + \underline{d}\,\alpha_2$

$$\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2 = (\lambda + 1 - \lambda)\underline{x}_0 + \underline{d}\left(\lambda\alpha_1 + (1-\lambda)\alpha_2\right)$$

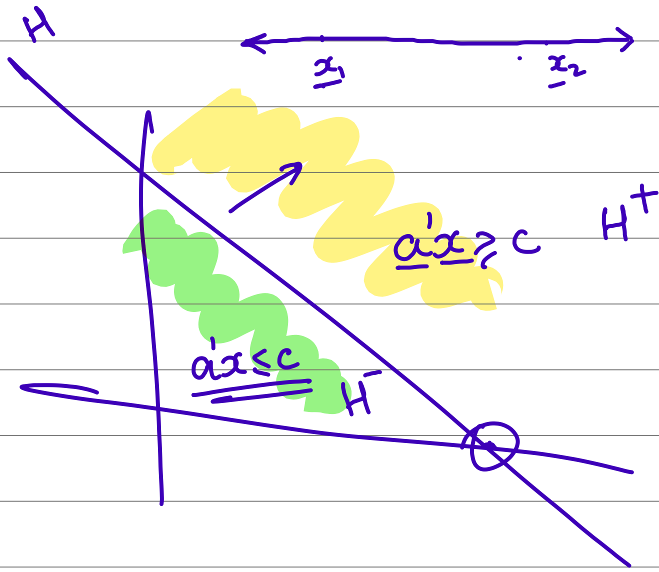
$$= x_0 + d(\qquad) \geq 0$$

$\in A$



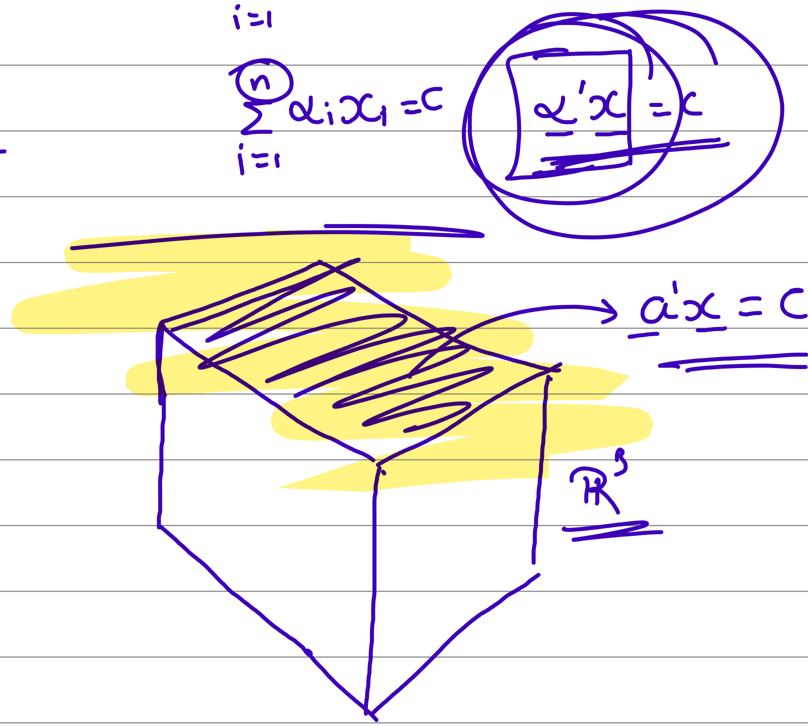

* Union of convex sets
  may or may not be convex.

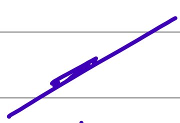
* Intersection of convex sets
  is also convex.




$\sum_{i=1}^{2} \alpha_i x_i$

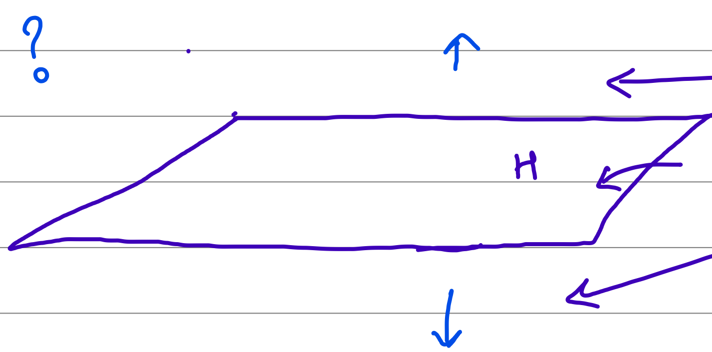
$\sum_{i=1}^{n} \alpha_i x_i = c$



Hyperplane

$H_0^+ = \{\underline{x} / \underline{a}'\underline{x} > c\}$ +ve open half space




$H^+ = \{\underline{x} / \underline{a}'\underline{x} \geq c\}$ ← positive closed half space

$H = \{\underline{x} / \underline{a}'\underline{x} = c\}$ Hyperplane

$H^- = \{\underline{x} / \underline{a}'\underline{x} \leq c\}$ ← Negative closed half space

$H_0^- = \{\underline{x} / \underline{a}'\underline{x} < c\}$ −ve open half space

Hyperplane $H = \{\underline{x} \,/\, \underline{a}'\underline{x} = c\}$

let $\underline{x}_1, \underline{x}_2 \in H \Rightarrow \underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

$\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$$\underline{a}'(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \alpha \underline{a}'\underline{x}_1 + (1-\alpha)\underline{a}'\underline{x}_2$$
$$= \alpha c + (1-\alpha)c$$
$$= c$$

$\Rightarrow \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in H$

$\Rightarrow$ Hyperplane is convex set

$H^+$, $H^-$, $H_0^+$, $H_0^-$ all are convex sets

Open ball $\underline{B = \{ \underline{x} \,/\, \| \underline{x} - \underline{x}_0 \| < r \}}$

To show :- $B$ is convex set.

Let $\underline{x}, \underline{y} \in B \quad \Rightarrow \quad \underline{\alpha \cdot \underline{x}} + (1-\alpha)\, \underline{y} \;\boxed{\in B}$

$\underline{x} \in B \Rightarrow \| \underline{x} - \underline{x}_0 \| < r$

$\underline{y} \in B \Rightarrow \| \underline{y} - \underline{x}_0 \| < r$

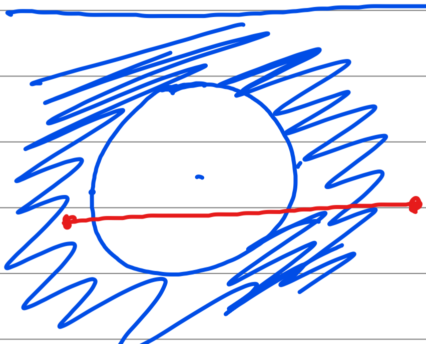
$\alpha \in (0,1)$

$$
\begin{aligned}
& \| \alpha \underline{x} + (1-\alpha) \underline{y} - \underline{x}_0 \| \\
&= \| \alpha \underline{x} + (1-\alpha) \underline{y} - (\alpha \underline{x}_0 + (1-\alpha) \cdot x_0) \| \\
&= \| \alpha (\underline{x} - \underline{x}_0) + (1-\alpha)(\underline{y} - x_0) \| \\
&\leq \alpha \underbrace{\| \underline{x} - \underline{x}_0 \|}_{<r} + (1-\alpha) \underbrace{\| y - x_0 \|}_{<r} \qquad \alpha \in (0,1) \\
&< \alpha r + (1-\alpha) r \\
&< r
\end{aligned}
$$

---

$\{ \underline{x} \,/\, \| \underline{x} - \underline{x}_0 \| = r \}$ Convex or Not

---



$\{ \underline{x} \,/\, \| \underline{x} - \underline{x}_0 \| \geq r \}$ convex or not

If C is convex set $\lambda C$ is also convex set.

$$\lambda C = \{ \underline{y} \mid \underline{y} = \lambda \underline{x} , \underline{x} \in C \}$$

to show $\lambda C$ as convex set

$$\underline{y}_1, \underline{y}_2 \in \lambda C \Rightarrow \underline{y}_1 = \lambda \underline{x}_1 , \underline{y}_2 = \lambda \underline{x}_2 \quad \underline{x}_1, \underline{x}_2 \in C$$

$$\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 = \alpha \cdot \lambda \underline{x}_1 + (1-\alpha) \lambda \underline{x}_2$$

$$= \lambda (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

as C is convex

$$= \lambda \underline{x} \qquad (\text{as } \underline{x}_1, x_2 \in C \Rightarrow x \in C)$$

$$\Rightarrow \alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 \in \lambda C$$

C, D are convex sets C+D is also convex

$$\rightarrow \quad C+D = \{ \underline{Z} \mid \underline{z} = \underline{x} + \underline{y} , \underline{x} \in C, \underline{y} \in D \}$$

$$\underline{z}_1, \underline{z}_2 \in C+D \Rightarrow z_1 = \underline{x}_1 + \underline{y}_1$$
$$z_2 = \underline{x}_2 + \underline{y}_2 \qquad \underline{x}_1, \underline{x}_2 \in C, \ \underline{y}_1, \underline{y}_2 \in D.$$

$$\alpha z_1 + (1-\alpha) z_2 = \alpha \cdot (\underline{x}_1 + \underline{y}_1) + (1-\alpha)(\underline{x}_2 + \underline{y}_2)$$

$$= \underbrace{\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2}_{\in C} + \underbrace{\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2}_{\in D}$$

convex sets

Intersection of any convex sets is convex

Let $\{S_i\}_{i=1}^{\infty}$ be collection of convex sets

$\cap S_i$ is convex

$\underline{x}, \underline{y} \in \cap S_i$

$\Rightarrow \underline{x}, \underline{y} \in S_i \quad \forall\, i$

$\Rightarrow \alpha \underline{x} + (1-\alpha)\underline{y} \in S_i \quad \forall\, i$ ($S_i$ is convex)

$\Rightarrow \alpha \underline{x} + (1-\alpha)\underline{y} \in \cap S_i$

$\Rightarrow \cap S_i$ is convex.

- A set $S \in R^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S

∴ Assume that every convex combⁿ of any finite no. of points of S is in S.

② → ①

$\Rightarrow$ it is also true for $n=2$

$\Rightarrow$ if $\forall\, \underline{x}_1, \underline{x}_2 \in S \Rightarrow \alpha \cdot \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S \quad \forall\, \alpha\,(0,1)$

$\Rightarrow$ S is convex set

② Assume S is convex and for any finite n $\sum_{i=1}^{n} \alpha_i \underline{x}_i \in S$

→ let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in S$ $\sum_{i=1}^{n} \alpha_i = 1$

we will prove this by mathematical induction

As S is convex, $\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S$ $\alpha + 1 - \alpha = 1$

∴ So the above statement is true for $n=2$

Assume it is true for $n=K$ ⇒ $\sum_{i=1}^{K} \alpha_i \underline{x}_i = 1$

$\sum_{i=1}^{K} \alpha_i = 1$

We have to prove it for $n=K+1$

$$\sum_{i=1}^{K+1} \beta_i \underline{x}_i = \sum_{i=1}^{K} \beta_i x_i + \beta_{K+1} x_{K+1} \qquad \sum_{i=1}^{K+1} \beta_i = 1$$

imp?

$$\sum_{i=1}^{K} \beta_i = 1 - \beta_{K+1}$$

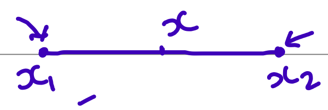
$$\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} = 1$$

$$= (1-\beta_{K+1}) \underbrace{\boxed{\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} \cdot x_i}}_{\in S} + \beta_{K+1} x_{K+1}$$

$$= (1-\beta_{K+1}) x^* + \beta_{K+1} x_{K+1}$$

$\in S$ as $S$ is convex set

* Vertices

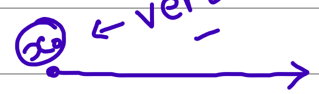



2 vertices

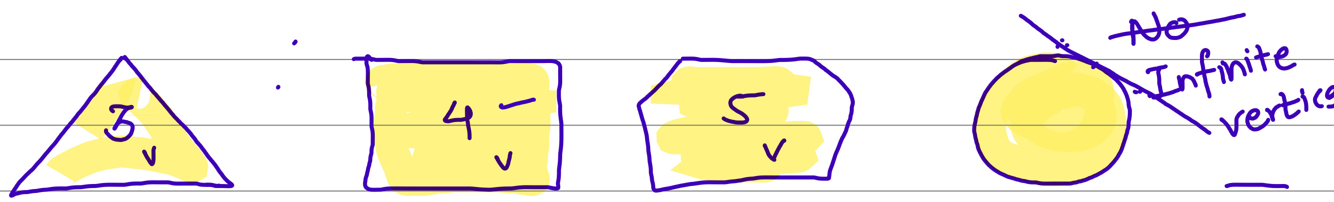
$\Rightarrow \quad x = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

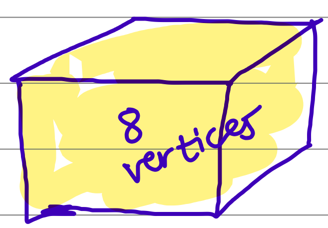
$x_2 = \alpha \underline{x}_1 + (1-\alpha)x_2 = \underline{x}_2$

$\alpha = 0$




No vertex



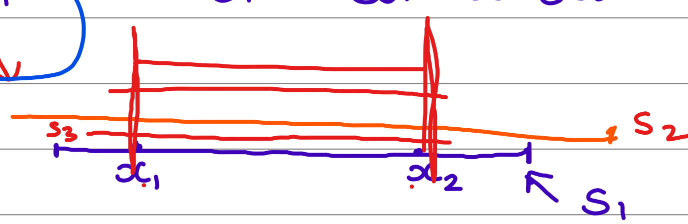



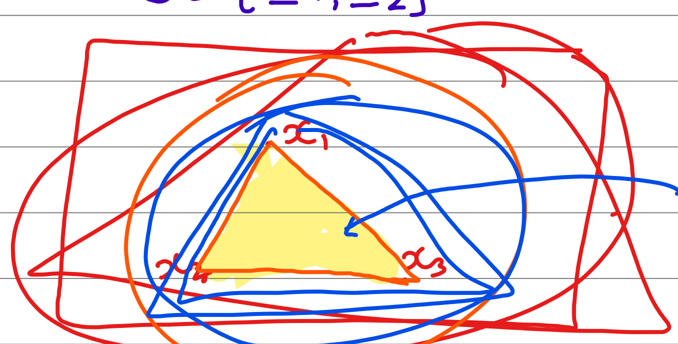

Convex Hull

$$\boxed{Co(S) \Rightarrow \bigcap_{i=1}^{\infty} S_i}$$

$S_i$ is convex set containing $S$

$S = \{\underline{x}_1, \underline{x}_2\}$

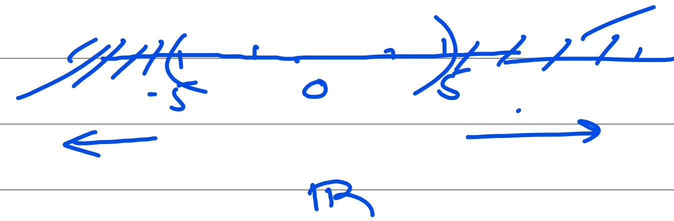



$S = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$

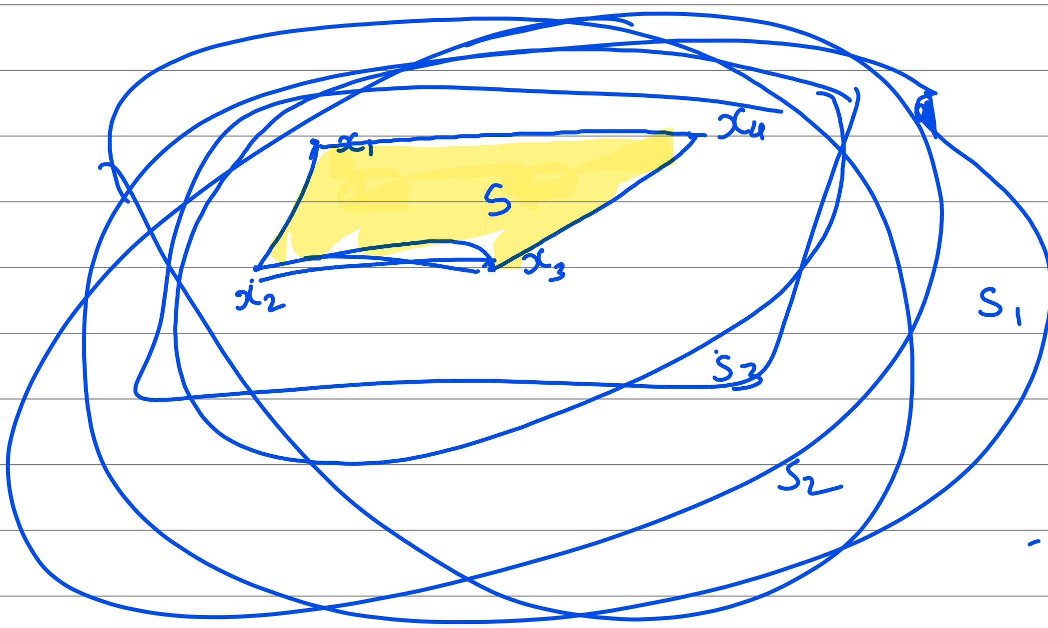
$S = \{ \underset{\sim}{x} \mid \|\underset{\sim}{x}\| \geq 5 \}$

$Co(S) = \mathbb{R}^n$

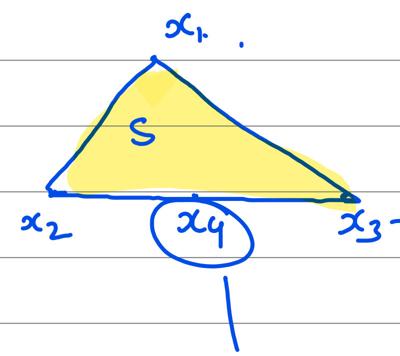


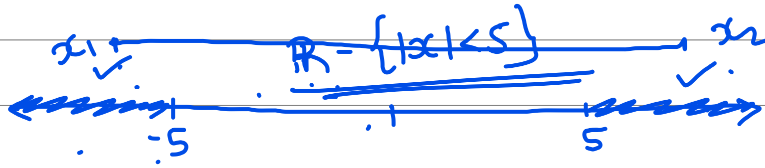

if set is convex $\underline{\underline{Co(S) = S}}$

$\underline{\underline{Co(S)}} = \underline{\underline{\cap S_i}}$ , $S_i$ is convex set containing $S$.



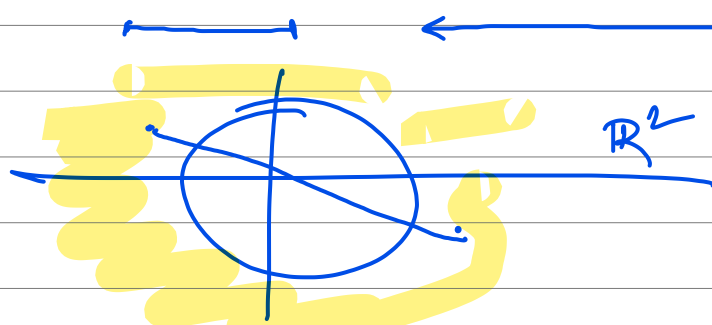

$\underline{\underline{\cap S_i}} = Co(S)$



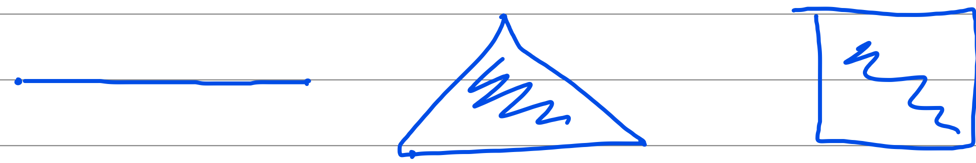




$S = \{ |x| > 5 \}$

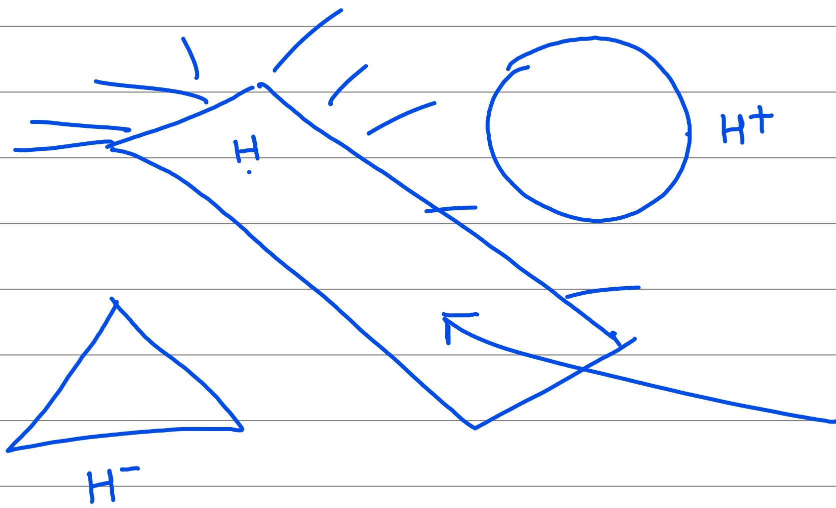
$\underline{\underline{\mathbb{R} = Co(S)}}$



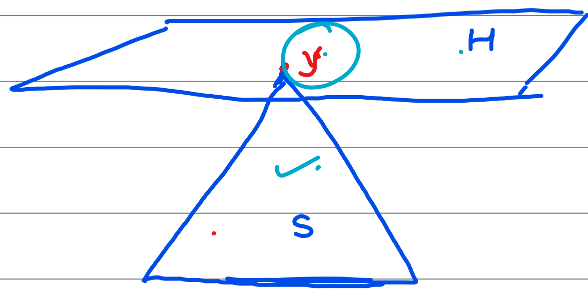
$$Co(S) = \left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \lambda_i x_i \,.\; x_i \in S \quad \sum \lambda_i = 1,\; \lambda_i \geq 0 \right\}$$



Hyperplane:- $H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$




Separating Hyperplane




$S \in H^-$ or $S \in H^+$

$\underline{a}'\underline{x} \leq c$ or $a'x \geq c$

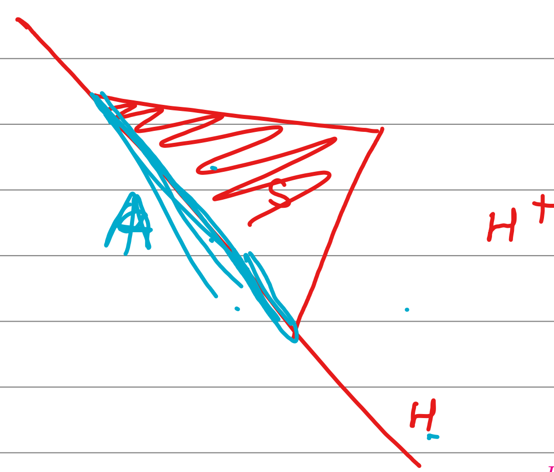
$\ni x \in S$ $\ni x \in S$

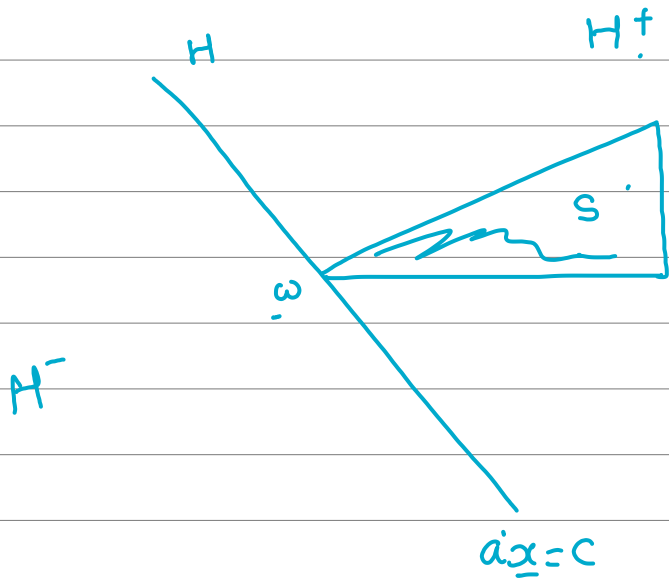
$S \in H^-$

$H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$

$\underline{a}'\underline{y} = c \Rightarrow \underline{y} \in S \ni \underline{y} \in H$


$S \in H^-$






$S \in H$

$\forall x \in S, \quad a'x \geq c$

$\omega \quad a'\omega = c$

$\omega \in S \cap H$

$\min_{x \in S} a'x \Rightarrow a'\omega$

4

① Supporting hyperplane:- $a'x \geq c$, $-a'x \leq c$

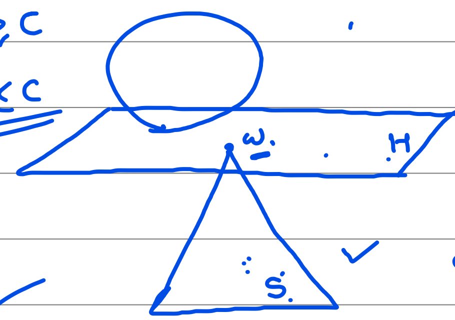
$H : \{ x \,/\, a'x = c \}$ $\boxed{a'\omega = c}$

$\omega \in S$ ← boundary S.

$S \in H^-$

$x \in S$

$a'x \leq c$




$\omega \in S \cap H$

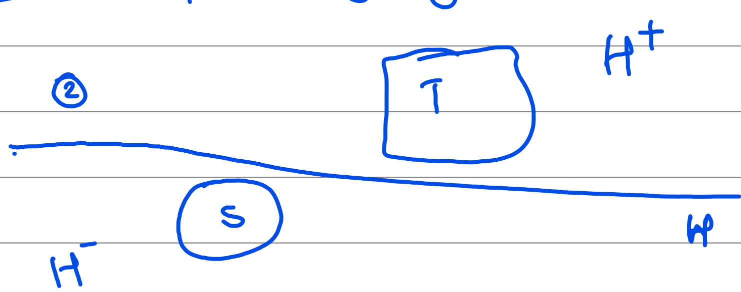
$a'x = c$

$S \Rightarrow a'x \leq c$

$a'\omega = c$

Separating hyperplane

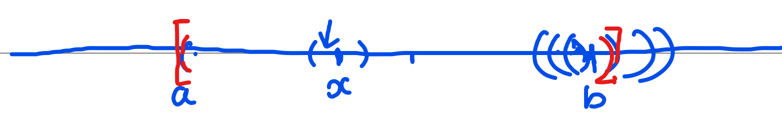
②

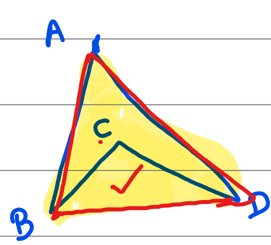



$S \in H^-$ & $T \in H^+$

Closure. $S = (a,b)$

$S' = (a,b) \cup \{a\} \cup \{b\} = [a,b]$



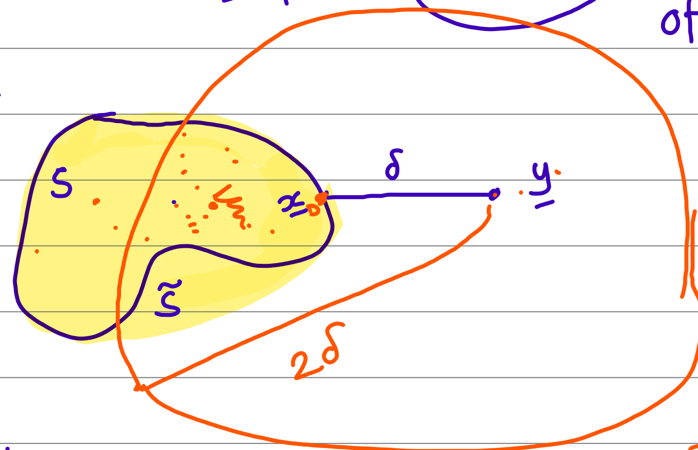

$|x - c| < \varepsilon$

$c - \varepsilon, \; c + \varepsilon$

$S \subseteq \mathbb{R}^n$ convex, $y$ ~~boundary point~~ exterior to closure of $S$

$\underline{a} \neq \underline{0} \in \mathbb{R}^n$

$\exists$ $\exists$ $\underline{a} \in \mathbb{R}^n$, $\underline{a}'\underline{y} < \inf\limits_{\underline{x} \in S} \underline{a}'\underline{x}$




$\tilde{S}$ closure

let $\underline{\delta} = \inf\limits_{x \in \tilde{S}} ||\underline{x} - \underline{y}||$ $\quad$ $\delta > 0$ as $y$ is exterior to $\tilde{S}$

$y \notin \tilde{S}$

$B_{2\delta} = \{ \underline{x} \mid ||\underline{x} - \underline{y}|| < 2\delta \}$

$\delta = \inf\limits_{x \in \tilde{S}} ||\underline{x} - \underline{y}|| = \inf\limits_{x \in \tilde{S} \cap B_{2\delta}} ||\underline{x} - \underline{y}||$

$\tilde{S} \cap B_{2\delta}$ is closed & bounded.

lets define $f : \tilde{S} \cap B_{2\delta} \to \mathbb{R}$, $\quad f(\underline{x}) = ||\underline{x} - \underline{y}||$

$f$ contin____,

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by max$^m$ min$^m$ theo., $f$ attains its extremum in that set

$\exists$ some $\underline{x}_0 \in \tilde{S} \cap B_{2\delta} \ni$

$\delta = \min\limits_{\underline{x} \in \tilde{S} \cap B_{2\delta}} ||\underline{x} - \underline{y}|| = ||\underline{x}_0 - \underline{y}||$

$\Rightarrow \underline{x}_0$ is boundary pt. of $\tilde{S}$

To show $\underline{a} = \underline{x}_0 - \underline{y}$

$\underline{a}'\underline{y} \le \inf_{\underline{x}} \underline{a}'\underline{x}$

$\underline{x}, \underline{x}_0 \in \tilde{S}$, $\alpha \underline{x} + (1-\alpha)\underline{x}_0 \in \tilde{S}$

$$||\alpha \underline{x} + (1-\alpha)\underline{x}_0 - y|| \geq ||\underline{x}_0 - \underline{y}||$$

$||\underline{x}|| = \sqrt{\underline{x}'\underline{x}}$

$$||(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)|| \geq ||\underline{x}_0 - \underline{y}||$$

$$||(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)||^2 \geq ||\underline{x}_0 - \underline{y}||$$

$$\Rightarrow \left((\underline{x}_0 - \underline{y}) + \underline{\alpha}(\underline{x} - \underline{x}_0)\right)^T \left((x_0 - y) + \alpha(\underline{x} - \underline{x}_0)\right) \geq (\underline{x}_0 - \underline{y})^T(\underline{x}_0 - y)$$

$$\Rightarrow \left[(\underline{x}_0 - \underline{y})' + \alpha(\underline{x} - \underline{x}_0)'\right]\left[(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\right] \geq (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})$$

$$\Rightarrow (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y}) + \alpha(\underline{x}_0 - \underline{y})'(\underline{x} - x_0) + \alpha(\underline{x} - \underline{x}_0)'(\underline{x}_0 - \underline{y}) + \alpha^2(\underline{x} - \underline{x}_0)'(\underline{x} - \underline{x}_0)$$

$$\geq (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})$$

$$2\alpha(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) + \alpha^2|x - x_0|^2 \geq 0$$

$$2(\underline{x}_0 - \underline{y})'(x - x_0) + \alpha|x - x_0|^2 \geq 0$$

let $\alpha \to 0$

$$(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) \geq 0$$

$$\Rightarrow (\underline{x}_0 - \underline{y})'\underline{x} \geq (\underline{x}_0 - \underline{y})'\underline{x}_0$$

$$\Rightarrow \underline{a}'\underline{x} \geq \underline{a}'(\underline{x}_0 - \underline{y} + \underline{y})$$

let $a = \underline{x}_0 - \underline{y}$

$$\geq a'(x_0 - y) + a'y$$

$\|z\| = \sqrt{z'z}$

$$a'x \geq \underline{(x_0-y)'(x_0-y)} + a'y \quad \forall x \in S$$

$$\geq \delta^2 + a'y \qquad \delta > 0$$

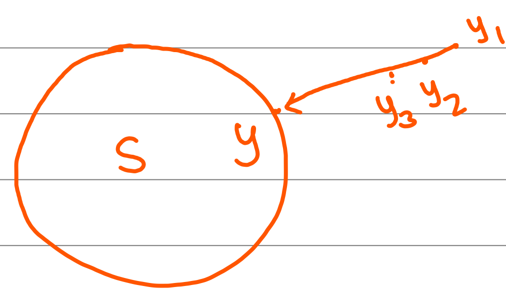
$$\inf_{x \in S} a'x \geq a'y \quad \checkmark$$

$$a'y < \inf_{x \in S} a'x$$

Theo. $S$ convex, $y$ boundary of $S$

To show $\exists\, H, \ni S \in H^+/H^-$, $y \in S \cap H$

$\rightarrow$ [ let $y_n$ be seq$^n$ of points exterior to closure of $S$
Assume $y_n \to y$ ]




$$\exists\, a_n \in \mathbb{R}^n, \qquad a_n' y_n \leq \inf_{x \in S} a_n' x \qquad \|a_n\| = 1$$

$$a_n' y_n - a_n' y + a_n' y \leq a_n' x \qquad \forall x \in S$$

for large $n$, $y_n \to y \Rightarrow a_n' y_n \to a' y$

$\underline{a}'\underline{y} \nless \underline{a}'\underline{x}$

$$\underline{a}_n'\underline{y} < \underline{a}_n'\underline{x}$$

$\|a_n\| \ni$ seq$^n$ of an bounded.
Bolzano Weierstrauss Theo for seq$^n$s.

$\exists$ convergent subseq$^n$ $\{a_{n_k}\}$
Suppose it converges to $a_{n_k} \to a$

$$\underline{a}_{n_k}'\underline{y} < \underline{a}_{n_k}'\underline{x}$$

letting $k \to \infty$, $\underline{a}_{n_k} \to \underline{a}$

$$\underline{a}'\underline{y} = \underline{a}_{n_k}'\underline{y} < \underline{a}_{n_k}'\underline{x} = \underline{a}'\underline{x}$$

$\ni$ $\underline{a}'\underline{y} \leq \underline{a}'\underline{x}$ $\forall x \in S$

$H = \{\underline{x} / \underline{a}'\underline{x} = \underline{a}'\underline{y}\}$ is supporting hyperplane. at $y$

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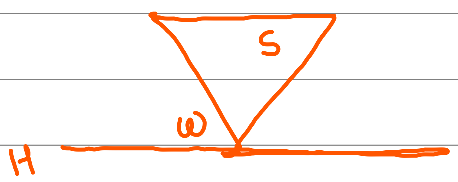
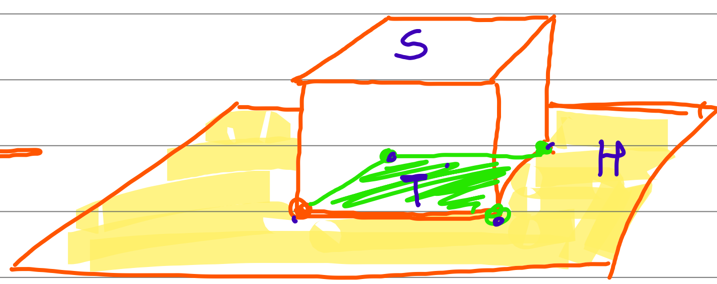
$$T = S \cap H = \{\omega\}$$



H Supportive Hyperplane, $T = S \cap H$
by method of contradiction

let $\underline{x}_0$ be extreme pt. of $T$ but not of $S$.

let $\exists$ some $x_1, x_2 \in S \ni \underline{x}_0 = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$H = \{\underline{x} \mid \underline{a}'\underline{x} = c\}$ supportive hyperplane of S

let $S \in H^+$, $\underline{a}'\underline{x} \geq c \ \forall \underline{x} \in S$.

$\Rightarrow \underline{a}'\underline{x}_1 \geq c, \ \underline{a}'\underline{x}_2 \geq c$

$\underline{x}_0 \in T = S \cap H \qquad \Rightarrow \underline{a}'\underline{x}_0 = c$

$\Rightarrow \underline{a}'(\alpha x_1 + (1-\alpha)\underline{x}_2) = c$

$\Rightarrow \alpha \underline{a}'\underline{x}_1 + (1-\alpha)\underline{a}'\underline{x}_2 = c$

$\alpha \in (0,1), (1-\alpha) \in (0,1)$

$$\Rightarrow \alpha \cdot \underbrace{\qquad}_{\geq c} + (1-\alpha)\underbrace{(\qquad)}_{\geq c} = c$$

it is only possible if $\underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

$\Rightarrow \underline{x}_1, \underline{x}_2 \in H$ & $\Rightarrow x_1, x_2 \in S \cap H$

$\Rightarrow \underline{x_1, x_2 \in T}$

$\therefore$ Which contradicts to our assumption that $x_0$ is extreme pt. of T.