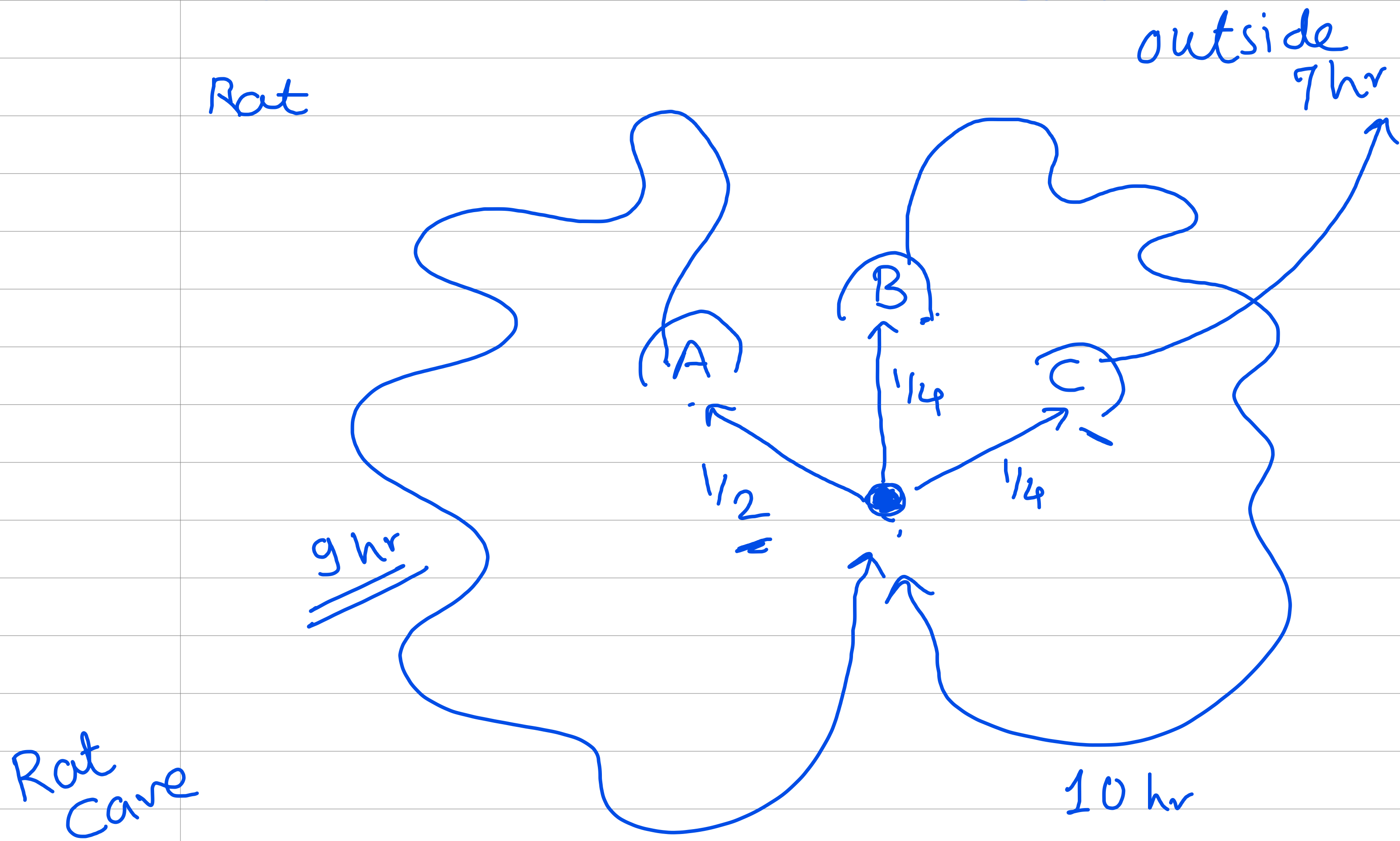
① Distributions Revision

Discrete Continuous



A	1/2	$\to$	9 hr return
B	1/4	$\to$	10 hr return
C	1/4	$\to$	7 hr leave

$\cancel{x}$ μ Assume $E(X) = ?$

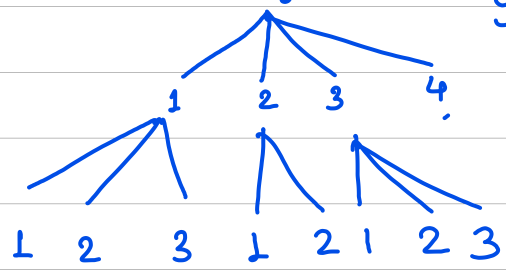
A	1/2	$\longrightarrow$	$9+\mu$ ✓
B	1/4	$\longrightarrow$	$10+\mu$ ✓
C	1/4	$\longrightarrow$	7

$$\mu = \frac{1}{2}(9+\mu) + \frac{1}{4}(10+\mu) + \mu$$

$=$

Offspring Distribution :-

X_0 :- Zeroth generation



$X_1 = 4$ 1st generation size

$X_2 = 8$ 2nd generation size

$X_n = 0$ & $X_m = 0 \ \exists\ m > n$

$X_{n+1} = 0$

Dinosaurs extints $\to X_n = 0$

$X_0 = 1$ $X_1 = 3$, $X_2 = 5$ $\lim_{n\to\infty} X_n = 0$

Sum of expectations $y = \sum_{i=1}^{n} x_i$ ($n \leftarrow$ fixed, $x_i \leftarrow$ Random) $\Rightarrow E(y) = n \cdot E(X) = \sum_{i=1}^{n} E(X_i)$

$V(X) = E(V(X/Y)) + V(E(X/Y))$

$E(X) = E_Y(E_{X/Y}(X/Y))$

Random sum of Random number $V(y) = \sum_{i=1}^{n} V(X_i) + \sum_i \sum_{j, i \neq j} Cov(X_i, X_j)$

$y = \sum_{i=1}^{N} X_i$ ($N \leftarrow$ random, $X_i \leftarrow$ random)

↳ compound distributions

Y ① $N \sim$ Bino $\to y$ Comp. Bino. distribution (K, p) $E(Y)\ Kp \cdot E(X)$

Y ② $\sim$ Pois $\sim$ Comp. Poi. distribution $E(Y)\ \lambda \cdot E(X)$

likewise Comp. Geometric, Negative Binomial

$E(Y) = E\left(\sum_{i=1}^{N} X_i\right) = E_N\left(E_{Y/N}\left(\sum_{i=1}^{N} X_i / N = n\right)\right)$ $\quad E(X_i) = \mu$

$V(X_i) = V(X)$ $\quad E(X_i) = E(X) = \mu$

$= E_N\left(\sum_{i=1}^{N} E(X_i)\right) = E_N(n \cdot E(X)) = E(X) \cdot E(N)$ ✓

$V(Y) = V\left(\sum_{i=1}^{N} X_i\right) = E(V(X/Y)) + V(E(X/Y))$

✓ $= E\left(V\left(\sum_{i=1}^{N} X_i / N\right)\right) + V\left(E\left(\sum_{i=1}^{N} X_i / N\right)\right)$

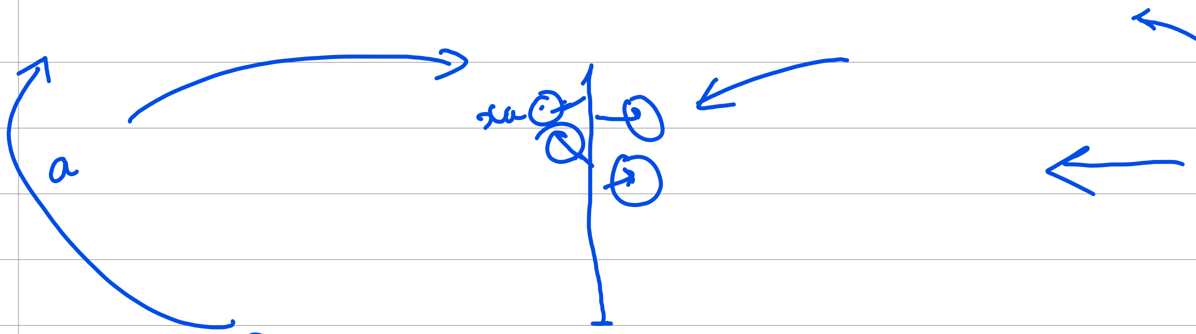
✓ $= E_N(N\,V(X)) + V_N(N\,E(X)) \Rightarrow E(N)\,V(X) + [E(X)]^2\,V(N)$

Comp. Bino. $\Rightarrow V(Y) = kp \cdot V(X) + [E(X)]^2 (kpq)$ | $\Rightarrow$ Comp. Poisson $\Rightarrow V(Y) = \lambda [V(X) + [E(X)]^2]$

$\Rightarrow \lambda \cdot E(X^2)$

$V(X) = E(X^2) - [E(X)]^2$

Matching hat problem



$a \to x_a \Rightarrow$ —

$b \to x_b$

$n \to$ $0, 1, 2, 2, 3, \ldots, n$

$a_1 \quad a_2 \quad \ldots \quad a_n$

$\downarrow \quad \downarrow \quad \quad \downarrow$

$1 \quad 0 \quad \quad 0$

$\Rightarrow \sum_{i=1}^{n} a_i$ matching hats

$\to n$

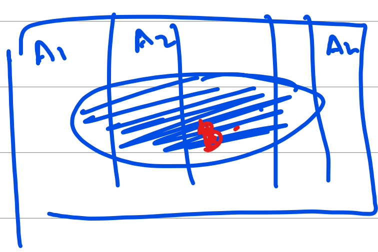
1	n	$1/n$ ✓	1
2	$n-1$	$\frac{1}{(n-1)} \times \frac{1}{n} + \frac{1}{(n-2)}(1 - 1/n)$	

Baye's Rule

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A_i/B) = \frac{P(B/A_i)}{\sum_{i=1}^{n} P(B/A_i)}$$

ΣA_i



$\cup A_i = \Omega$

mutually exclusive & exhaustive

e.g. Family 2 children

$X_i = 0$, if i is female $1/2$

$= 1$ if i is male ✓ $1/2$

$$P\left[\sum_{i=1}^{2} X_i = 2 / \sum_{i=1}^{2} X_i \geq 1\right] = ?$$

$E \quad \Omega = \{BB, BG, GB, GG\}$

$A = \{BB, BG, GB\}$ ✓

$B = \{BB\}$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

e.g. H/T coin

Maths $\to$ A $\to$ 1/2

Stats $\to$ A $\to$ 1/3

$P[\text{Stats}/A] = ?$

$P[A/\text{stats}] = 1/3 \qquad P[A/\text{Maths}] = 1/2$

$P[\text{Stats}] = 1/2 \qquad P[\text{Maths}] = 1/2$

$$P[\text{Stats}/A] = \frac{P[\text{Stats}, A]}{P[\text{Stats}, A] + P[\text{Maths}, A]}$$

$$P[\text{Stats}/A] = \frac{1/3}{1/3 + 1/2} = \frac{2}{2+3} = \frac{2}{5}$$

Monty Hall Problem / Paradox ? Show 80

Monty → Hall

$R \to 2$ ✓

$M \to 3$ open

Paradox

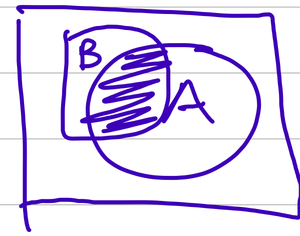
1000	Papers	Letters	Original	Rashmi → 2	M open		
	W	1	1	2	3	2	N
	N	3	2	2	1 / 3	2	W
	W	3	3	2	1	2	N
	2/3						1/3

$E(X/Y)$

$= V(X/Y) \cdot E(X)$

$+ E(X/Y) \cdot V(X)$

$P(B/A) = \dfrac{P(B \cap A)}{P(A)}$



X, Y ind //

x/y	1	2	3
Prob	1/4	1/2	1/4

$S = X + Y \Rightarrow$

y \ x	1	2	3
1	2	$\boxed{3}$	4
2	$\boxed{3}$	4	5
3	4	5	6

$E(X/S)$ $\quad \underline{\underline{S=3}} \to (X=1)$ 1/4

$(X=2)$ 1/2

$(X=3)$ 1/4

	X	1/4	1/2	1/4
	Y	1	2	3
1/4	1	2	3	4
1/2	2	3	4	5
1/4	3	4	5	6

S	2	3	4
P(S)	1/16	$2 \times \frac{1}{2} \times \frac{1}{4}$ = 1/4	$\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{4}$

$\frac{1}{16} + \frac{1}{4} + \frac{1}{16}$

$\frac{\cancel{2}\,1}{\cancel{16}\,8} + \frac{\cancel{1}}{\cancel{4}\,8} = \frac{3}{8}$

5	6
1/8	1/16

	1/16	1/4	3/8	1/8	1/16
S =	2	3	4	5	6
X =	1	1, 2	1, 2, 3.	2, 3	3
X=1	1	1/2	1/6		
X=2		1/2	2/3		
			1/6		

$\dfrac{P(X=1, S=3)}{P(S=3)} = \dfrac{P(X=1, Y=2)}{1/4} = \dfrac{P(X=1)\,(P(Y=2))}{1/4}$

$= \dfrac{\cancel{1/4} \times 1/2}{\cancel{1/4}} = 1/2$

$\dfrac{P(X=2, S=4)}{P(S=4)} = \dfrac{\cancel{1/2 \times 1/2}}{3/8} = \frac{2}{3}$

$\dfrac{P(X=1, S=4)}{P(S=4)} = \dfrac{1/4 \quad 1/4}{3/8} = \frac{3}{8}$ $\quad \dfrac{1/16 \; 2}{3/8}$

$$\frac{P(X=1,S=4)}{P(S=4)} = \frac{P(X=1).P(Y=3)}{3/8} \quad \frac{1/4 \times 1/4}{3/8} = \frac{2}{} \quad \frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$$

$$\frac{P(X=2,S=4)}{P(S=4)} = \frac{P(X=2).P(Y=2)}{P(S=4)} = \frac{\frac{1}{2}\times\frac{1}{2}}{3/8} = \frac{1/4}{3/8_2} = \frac{2}{3}$$

$$\frac{1}{6}+\frac{2}{3} = \frac{1+4}{6} = \frac{5}{6} \rightarrow P(X=3/S=4)\ 1/6$$

$E(X/S=2) = 1$

$E(X/S=3) = 1\times\frac{1}{2} + 2\times\frac{1}{2} = 1.5$

$E(\ \ /S=4) =$ ______

$E(X) = \cancel{E_X[E(X/Y)]} = E_Y[E_{X/Y}(X/Y)]$

MGF of compound distributions $\quad Y=\sum_{i=1}^{N} X_i$

$E(X_i) = E(X)$

$M_Y(t) = E(e^{tY}) = E_N(E_{Y/N}(e^{tY}/N=n))$

$= E_N(E(e^{t\sum_{i=1}^{N}X_i}/N=n))$

$X_i = i.i.d.$

$E(e^{tX})$ $\approx$ t

$E(t^{X})$

$e^{\log t}$

$= E_N(\prod_{i=1}^{N} E(e^{tX_i}))$

$= E_N([E(e^{tX})]^N)$

$M_Y(t) = E_N([M_X(t)]^N) = P_N(M_X(t))$

$= E_N(e^{N.\log M_X(t)}) = M_N(\log M_X(t))$

Ballot Problem

n, m

$\underline{\underline{n > m}}$

A		B
1	>	0
2	>	0, 1
⋮	>	
	>	
n	>	m

$$P_{n,m} = \frac{n}{n+m} \cdot P_{n-1,m} + \frac{m}{n+m} P_{n,m-1} \qquad \boxed{\frac{n-m}{n+m}} \checkmark$$

Ballot Problem

$$= \frac{n}{n+m} \cdot \frac{(n-1)-m}{(n-1)+m} + \frac{m}{n+m} \cdot \frac{(n)-(m-1)}{n+m-1}$$

$$= \frac{n^2 - n - \cancel{mn} + \cancel{mn} - m^2 + m}{(n+m)(m+n-1)} = \frac{n^2 - n - m^2 + m}{(n+m)(m+n-1)}$$

$$= \frac{(n^2-m^2) - (n-m)}{(n+m)(n+m-1)} = \frac{(n-m)(n+m) - (n-m)}{(n+m)(n+m-1)}$$

$$= \frac{(n-m)}{(n+m)} \frac{\cancel{(n+m-1)}}{\cancel{(n+m-1)}}$$

Excercise 1.8> $X_1 \sim Poi(\lambda_1)$ $X_2 \sim Poi(\lambda_2)$ $X_1 + X_2 \sim ?$

$Z = X_1 + X_2$

$$M_Z(t) = M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

$Z \sim Poi(\lambda_1 + \lambda_2)$

Conditional distribution of $X_1 / X_1 + X_2$

$$P(X_1 = x / X_1 + X_2) = \frac{P(X_1 = x,\ X_1 + X_2 = Z)}{P(X_1 + X_2 = Z)} = \frac{P(X_1 = x, X_2 = Z - x)}{P(Z)}$$

$$= \frac{P(X_1 = x) \cdot P(X_2 = Z - x)}{P(Z)}$$

$$= \frac{\cancel{e^{-\lambda_1}} \lambda_1^{x}}{x!} \cdot \frac{\cancel{e^{-\lambda_2}} \lambda_2^{Z-x}}{(Z-x)!} \Big/ \frac{\cancel{e^{-(\lambda_1+\lambda_2)}} (\lambda_1+\lambda_2)^{Z}}{Z!}$$

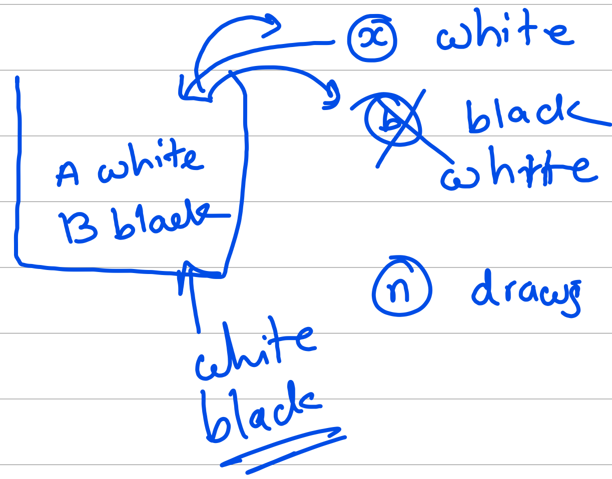
$$= \frac{z!}{x!\,(z-x)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-x}$$

$$= {}^{z}C_{x}$$

$$X_1 / X_1+X_2 \sim \text{Bino}\left(z, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

1.19 excercise

Urn



a b $M_0 = a$

1 W|B W B W $M_1 = a \times \frac{a}{a+b} + (a+1)\frac{b}{a+b} = \frac{a(a+b)}{(a+b)} + \frac{b}{(a+b)}$

M

$M_0 = a$ white ball

$$M_1 = a\left(\frac{a}{a+b}\right) + (a+1)\left(\frac{b}{a+b}\right)$$

3+7

n, M_n, n+1, White

$$M_{n+1} = M_n\left(\frac{?}{a+b}\right) + (M_n+1)\left(\frac{(a+b)-?}{a+b}\right)$$

$$= M_n \cdot \left(\frac{M_n}{a+b}\right) + (M_n+1)\left(\frac{(a+b)-M_n}{a+b}\right)$$

$1 - \frac{M_n}{a+b}$

$$= \cancel{\frac{M_n^2}{(a+b)}} + (M_n+1) - \cancel{\frac{M_n^2}{(a+b)}} - \frac{M_n}{(a+b)}$$

① $M_{n+1} = 1 + M_n\left(1 - \frac{1}{a+b}\right)$ ② $M_0 = a$

$$M_n = a+b - b\left(1 - \frac{1}{a+b}\right)^n$$

$$M_1 = 1 + a\left(1 - \frac{1}{a+b}\right) = 1 + ax$$

$$M_2 = 1 + (1+ax)\cdot x = 1 + x + ax^2$$

$$M_3 = 1 + (1 + x + ax^2)x = 1 + x + x^2 + ax^3$$

$$= 1 + x + x^2 + x^3 + (a-1)x^3$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad = \left(\frac{1-x^3}{1-x}\right) + (a-1)x^3$$

$$\sum_{n=0}^{j} x^n = \frac{1-x^{j+1}}{1-x} \qquad M_n = \left(\frac{1-x^{n+1}}{1-x}\right) + (a-1)x^n$$

$$x = 1 - \frac{1}{a+b} \Rightarrow \frac{1}{1-x} = (a+b)$$

$$M_n = (a+b) - \frac{x^{n+1}}{1-x} + (a-1)x^n$$

$$= (a+b) - (a+b)x^{n+1} + (a-1)x^n$$

$$= (a+b) - x^n\left[(a+b)\cdot x - a + 1\right]$$

$$= (a+b) - \left(1 - \frac{1}{a+b}\right)^n\left[(a+b)\left(1 - \frac{1}{a+b}\right) - a + 1\right]$$

$$\cancel{a} + b - \cancel{1} - \cancel{a} + \cancel{1}$$

Sheldon Ross

$$= (a+b) - b\left(1 - \frac{1}{a+b}\right)^n$$

J. Medhi ↓

Stochastic Process

Seq^n of Random Variable

✓ X_n = fortune of gambler ~~a~~ after n^{th} game

$X_n = X_{n-1} + Z_n$ $Z_n = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$

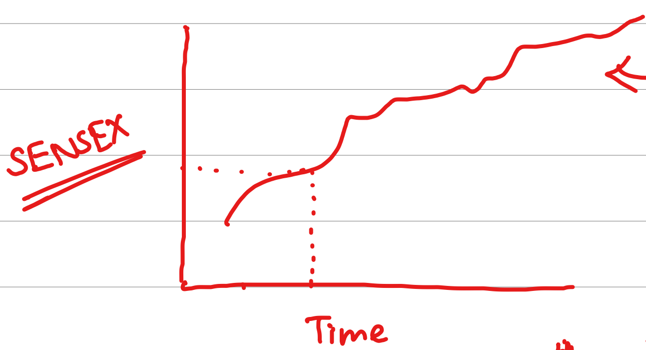
100 → 99, 101

$X_0 \quad X_1 \quad X_2 \ldots \quad X_n \ldots$

Discrete Seq^n of Random

✓ $\{X_i\}_{i=1}^{\infty}$

9.00 → 9.12 9.12 — 9.15 9-30 ——— 3:30

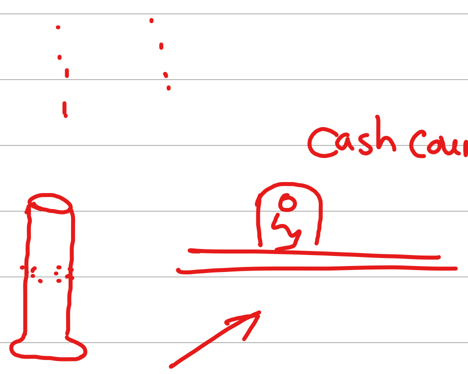


continous ~~X_n~~

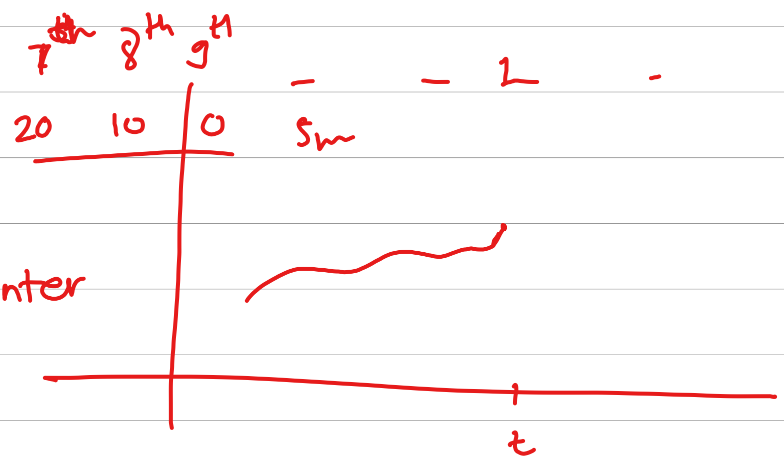
$\{X_t\}_{t \in I}$

I = Time index set

Rain

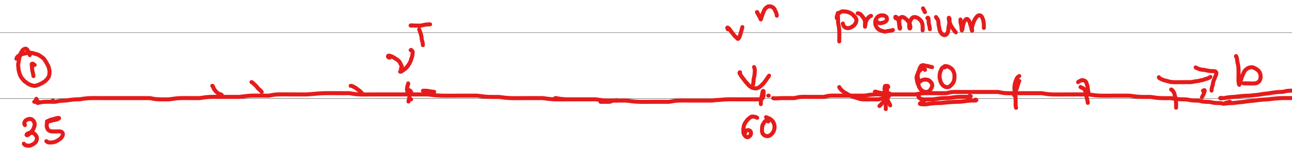


Cash counter



	Statespace	Time domain	Example
1	Discrete	Discrete	* No. of covid patients on n^{th} day * No. of customers on a shop . " -
2	Discrete	Conti	* Pop^n size → conti time discrete pop^n * No. of accidents up to time t
3	Continuous	Discrete	* Milk by cow on n^{th} day * Min^m/Max^m temp on n^{th} day
4	Continuous	Contin.	* Fever - body temp on t time * Blood conc up to time t
5	Mixture	Discrete Cont	* Speed of internet * Rainfall up to time t

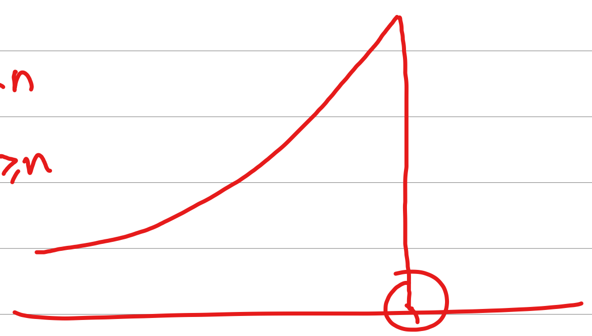
Endowment → $x \rightarrow n$ 35 → 60 ↷ b



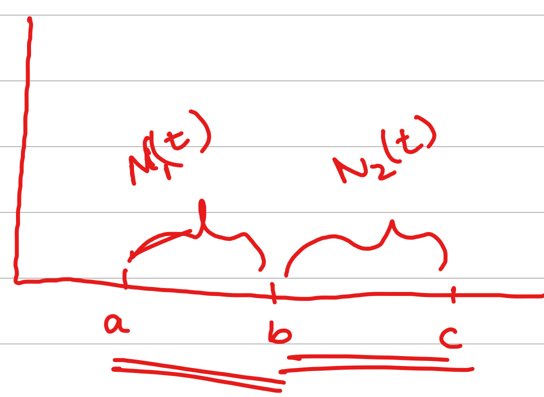
mixture

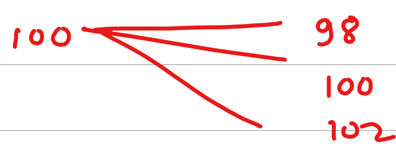
Present = v^T $T < n$

v^n $T \geq n$



Independent Increments





disjoint time interval

$N_1(t)$ & $N_2(t)$

dist^n indep

independent increments

$t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$

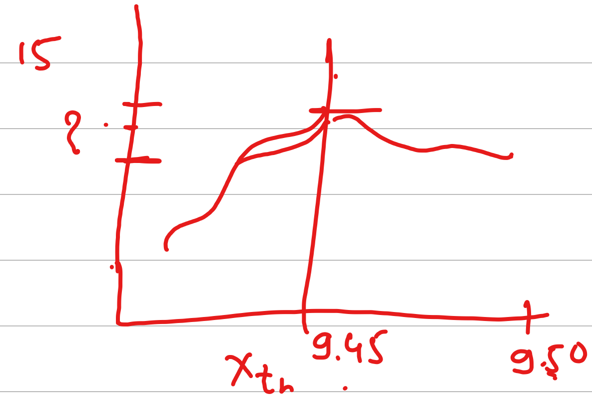
$\rightarrow X_{t_0}, X_{t_1}, X_{t_2}, \ldots < X_{t_{n-1}} < X_{t_n}$

$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1} \ldots, X_{t_n} - X_{t_{n-1}}$ indep. inde-

12 - 2

12 - 1

12.30 - 1.30



Joint same X_{t_0+u} X_{t_1+u} X_{t_2+u} X_{t_n}

Joint t_0 t_1 t_2 t_{n-1} t_n

$(X_{t_0}, X_{t_1}, X_{t_2}, X_{t_{n-1}}, X_{t_n})$

$X_{t_1} - X_{t_0}$ $X_{t_2} - X_{t_1}$ $X_{t_n} - X_{t_{n-1}}$

independent Stationarity strictly

$s < t$ $u > 0$

increment $(s, t]$ $X(t) - X(s)$ 12-2 . No. of students

✓ interval length $(t-s)$

$(s+u, t+u]$ $X(t+u) - X(s+u)$ 7-9.

✓ interval length $(t-s)$

X_{t_0}

weak stationarily μ
$$\begin{cases} E(X(t)) = \mu & \text{constant } \forall t \in I \\ Cov(X(t), X(s)) = fun(t-s) \end{cases}$$

Gambler → Random walk

Renewal

Stationarity

Probabilistic Replica

Find the p.g.f. of the sum Sn = X1 + + Xn of n independent and identical zero-truncated Poisson variates. Find E (Sn) and Pr {Sn = m}, m = n, n + 1, n + 2..

$X_i \sim$ zero-truncated poisson

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!} \Big/ (1-e^{-\lambda}) \rightarrow E(X)$$

$$S_n = \sum_{i=1}^{n} X_i \Rightarrow E(S_n) = n \cdot E(X) = \frac{n\cdot\lambda}{1-e^{-\lambda}}$$

$$P_S(t) = E(t^{S}) = E\left(t^{\sum_{i=1}^{n} x_i}\right) = [P_X(t)]^n$$

$$P_X(t) = \frac{1}{1-e^{-\lambda}}\, e^{-\lambda} \sum_{x=1}^{\infty} \frac{(\lambda t)^x}{x!}$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}}\left[e^{\lambda t} \underbrace{\sum_{x=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^x}{x!}}_{\to 1} - e^{-\lambda t} \right]$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}}\left[e^{\lambda t} - e^{-\lambda t}\right]$$

Markov Chains:-

discrete statespace & discrete time domain stochastic process following Markov property.

Markove Property:- Future depends on present not on past

Notation $\quad P_{ij}^{(n)} = P[X_n = j / X_0 = i]$

$i \qquad\qquad\qquad\qquad j$

$0 \quad 1 \quad 2 \quad 3 \quad \dots \quad n-1 \quad n$

$P_{ij}^{(n)}$ Starting from state i process reaches to state j in n-steps

Stationary $\rightarrow$ Time-homonegenous process

& Non-stationary

Non-stationary $P_{ij}^{(m,n)}$ = Starting from i process reaches to state j from m to n steps (No. of steps - $n-m$)

$P_{ij}^{(m,n+m)} = P[X_{n+m} = j / X_m = i]$

statespace = $S = \{1, 0\} \Rightarrow 1$ if it rains today $\quad 0$ o.w.

0	0		1	1	1	0
1	2	3	8	9	15	30

$P_{10}^{(15)} \qquad\qquad P_{10}^{(15,30)}$

$P[X_{n+1} = j / X_n = i] = P_{ij}^{1} = P_{ij}$

$P[X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0]$

$P[\underbrace{X_{30} = 1}_{\text{future}} / \underbrace{X_{29} = 1}_{\text{present}}, \underbrace{X_{28} = 1, \dots\dots X_0 = 0}_{\text{past}}] = P[X_{30} = 1 / X_{29} =$

$P_{ij}^{(1)} = P[X_{n+1}=j / X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0] = P[X_{n+1}=j / X_n=i]$

$\{X_n, n \geq 0\}$ as Markov chain.

one-step transition prob. matrix

$$\begin{array}{c|cc} & 0 & 1 \leftarrow \text{future state} \\ \hline \text{Current state } 0 & P_{00} & P_{01} \\ 1 & P_{10} & P_{11} \end{array}$$

$P_{00} \Rightarrow P[X_{n+1}=0 / X_n=0]$

$P_{10} \Rightarrow P[X_{n+1}=0 / X_n=1]$

starting ending

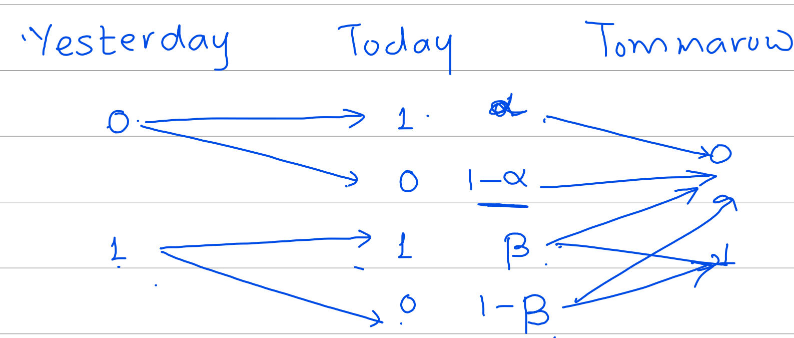
$P^{(2)}$ = Two step transition prob. matrix

$$P^{(2)} = \begin{array}{c|cc} & 0 & 1 \ \text{Future} \\ \hline \text{current state } 0 & P_{00}^{(2)} & P_{01}^{(2)} \\ 1 & P_{10}^{(2)} & P_{11}^{(2)} \end{array}$$

n-step

$$S=\{0,1\} \quad P^{(n)} = \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} \\ P_{10}^{(n)} & P_{11}^{(n)} \end{bmatrix}$$

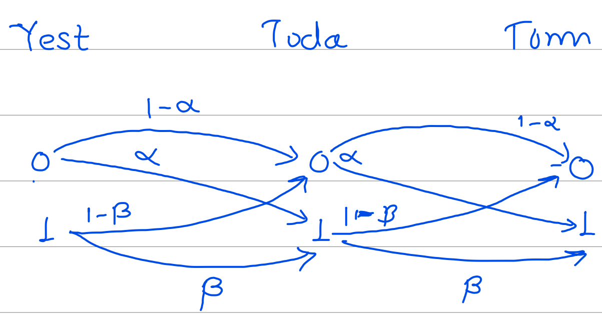
$S = \{1, 2, \ldots, K\} \longrightarrow$

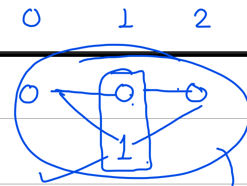


one-step

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & 1-\beta & \beta \end{array}$$

Two-step





$$= P_{00} \cdot P_{00} + P_{01} P_{10}$$
$$= (1-\alpha)^2 + \alpha(1-\beta)$$

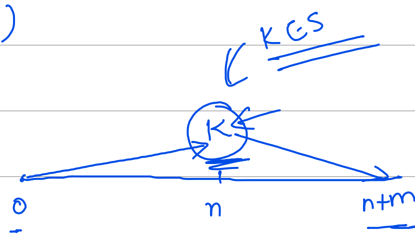
Tommor

$$\text{Yest} \quad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & P_{00}^{(2)} & P_{01}^{(2)} \\ 1 & P_{10}^{(2)} & P_{11}^{(2)} \end{array}$$

n-step transition probabilities

Chapman- Kolmogorov's — (CK-eq^n)

$$P_{ij}^{(n+m)} = \sum_{k \in S} P_{ik}^{(n)} P_{kj}^{(m)}$$



$\{X_n, n \geq 0\}$ is Markov chain?

$p^{(n+m)}$

$$P_{ij}^{(n+m)} = P[X_{n+m} = j / X_0 = i]$$

$$= \sum_{k \in S} P[\underbrace{X_{n+m} = j}_{\text{future}} / \underbrace{X_m = k}_{\text{present}}, \underbrace{X_0 = i}_{\text{past}}] \cdot P[X_m = k / X_0 = i] \quad k \in S$$

$$= \sum_{k \in S} P[X_{n+m} = j / X_m = k] \cdot P[X_m = k / X_0 = i]$$

stationary processes

$$= \sum_{k \in S} P[X_n = j / X_0 = k] \cdot P[X_m = k / X_0 = i]$$

$$= \sum_{k \in S} P_{kj}^{(n)} \cdot P_{ik}^{(m)}$$

if we have used ~~(ex)~~ 'it' in Matrix form

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

if $n = m = 1$,

$$P^{(1+1)} = P^{(2)} = P^{(1)} \cdot P^{(1)}$$

$$= \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix}$$

$$= \begin{bmatrix} (1-\alpha)^2 + \alpha(1-\beta) & \alpha[(1-\alpha) + \beta] \\ (1-\alpha)(1-\beta) + \beta(1-\beta) & (1-\alpha)(1-\beta) + \beta^2 \end{bmatrix}$$

$\{X_n, n \geq 0\}$ denotes whether it rains on n^{th} day or not

$$X_n = \begin{cases} 0 \\ 1 \end{cases}$$

Monday $\Rightarrow$ 0

Monday $\to$ 1

Saturday $\to$ 1

one step transition

$$\begin{array}{cc} & X_{n+1} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ X_n \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

Five step. $P^{(5)}$

$P[X_5 = 1 / X_0 = 1]$

$$= P^{(5)} = P^{(4)} \cdot P$$

$$= P^{(3)} \cdot P^{(2)}$$

$$= P^{(3)} \cdot \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= P^{(3)} \frac{1}{6}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$$

$$= \frac{1}{6^2} P^{(3)} \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$$

$$= \frac{1}{6^2} P \cdot P^{(2)} \cdot \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$$

$$= \frac{1}{6^4} P \cdot \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix} \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$$

$7 \times 2 \times 7 \times 3$

49×6

$21 \times (15 + 22)$

$21 \times (37)$

71

$$= \frac{1}{6^4} P \begin{bmatrix} 778 & 518 \\ 777 & 519 \end{bmatrix}$$

1296

$= 1$

Row sum 1

$$= \frac{1}{6^4} \begin{bmatrix} 778 & 518 \\ 777 & 519 \end{bmatrix} \times \frac{1}{6} \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$$

$$= \frac{1}{6^5} \begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} 4666 & 3110 \\ 4665 & \boxed{3111} \end{bmatrix}$$

5-step transition prob. matrix

$$P[X_5 = 1 / X_0 = 1] = \frac{3111}{6^5} = \underline{\underline{0.4}}$$

MC $\{X_n, n \geq 0\}$ Markov chain with statespace $S = \{0, 1, 2\}$

One-step TPM

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1/2 & 1/3 & 1/6 \\ 1 & 0 & 1/3 & 2/3 \\ 2 & 1/2 & 0 & 1/2 \end{array}$$

initial prob. ① $P[X_0 = 0] = P[X_0 = 1] = 1/2$?

② $X_1 = ?$

$P[X_3 = 0]$, $P[X_3 = 1]$, $P[X_3 = 2]$; $E(X_3) = ?$

$P[X_1 = 1] = ?$ $= P[X_1 = 1 / X_0 = 0] \cdot P[X_0 = 0] + P[X_1 = 1 / X_0 = 1] \cdot P[X_0 = 1] + P[X_1 = 1 / X_0 = 2] \cdot P[X_0 = 2]$

$= \sum_k P[X_1 = 1 / X_0 = k] \cdot P[X_0 = k]$

$1/3 \times 1/2 + 1/3 \times 1/2 + 1/2 \times 0$

$= 1/3$

$P[X_1 = ?] = \alpha \cdot P$ $\qquad \alpha = (\alpha_0\ \alpha_1\ \alpha_2)'$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

$P[X_1 = \ ] = \frac{1}{12}[3 \quad 4 \quad 5]$

$P(X_1 = 0) = 3/12$ $\quad P(X_1 = 1) = 4/12$ $\quad P(X_1 = 2) = 5/12$

$P[X_3 = ?] \Rightarrow P(X_3 = 0) \quad P(X_3 = 1) \quad P(X_3 = 2) = \alpha \cdot P^3$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} \cdot P^2 \qquad = \frac{1}{12}\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} P^2$$

$1 \times 3 \quad 3 \times 3$

$$= \frac{1}{12}\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} P$$

9, 15, 6, 8, $\cancel{3+16+15}$

$$= \frac{1}{72}\begin{bmatrix} 24 & 14 & 34 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

24 + 56

X_3

$$= \frac{1}{432}\begin{bmatrix} 174 & 76 & 182 \end{bmatrix}$$

$$E(X_3) = \frac{1}{432}\left[0 \times 174 + 1 \times 76 + 2 \times 182\right]$$

$$= \frac{1}{432}\left[76 + 364\right] = \frac{1}{432}\left[440\right] > 1. = 1.0185$$

Solve This :)

Let the TPM of two state markov chain is $P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$

Show that, by mathematical induction

$$P^{(n)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{bmatrix}$$

Put $n=1$,

$$P^{(1)} = P = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^1 & \frac{1}{2} - \frac{1}{2}(2p-1) \\ \frac{1}{2} - \frac{1}{2}(2p-1)^1 & \frac{1}{2} + \frac{1}{2}(2p-1) \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Assume that it is true for $n=k$

$$P^{(k)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^k & \frac{1}{2} - \frac{1}{2}(2p-1)^k \\ \frac{1}{2} - \frac{1}{2}(2p-1)^k & \frac{1}{2} + \frac{1}{2}(2p-1)^k \end{bmatrix}$$

$$P^{(k+1)} = P^{(k)} \cdot P = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$= \begin{bmatrix} p\left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right) + (1-p)\left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right) & (1-p)\left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right) + p\left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right) \\ p\left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right) + (1-p)\left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right) & (1-p)\left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right) + p\left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1} & \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1} & \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1} \end{bmatrix}$$

$S=\{1,2\}$ $\quad P = \begin{array}{c|cc} & 1 & 2 \\ 1 & 2/3 & 1/3 \\ 2 & 1/3 & 2/3 \end{array} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad P_{12}^{(3)} = ?$

$$P^{(3)} = P^{(2)} \cdot P = \frac{1}{3^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} P$$

$$= \frac{1}{3^2}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3^3} \begin{array}{c|cc} & 1 & 2 \\ 1 & 14 & 13 \\ 2 & 13 & 14 \end{array}$$

$$\left(P^{(3)}\right)_{12} = \frac{13}{27} = 0.48$$

$$P = \begin{array}{c|cc} & 0 & 1 \\ 0 & 0.7 & 0.3 \\ 1 & 0.5 & 0.5 \end{array} = \frac{1}{10}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}, \quad P_{\boxed{11}}^{(4)} = ?$$

$$P^{(2)} = \frac{1}{10^2}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix} = \frac{1}{10^2}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}$$

$$P^{(4)} = P^{(2)} \cdot P^{(2)} = \frac{1}{10^4}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}$$

$$= \frac{1}{10^4}\begin{bmatrix} 6256 & 3744 \\ 6240 & \boxed{3760} \end{bmatrix}$$

$$P_{11}^{(4)} = \frac{3760}{10^4}$$

$$P = \begin{array}{c|cc} & 0 & 1 \\ 0 & & \\ 1 & & \boxed{} \end{array}$$

$$\begin{array}{cc} 0 & 1 \\ \end{array}$$
$$\begin{bmatrix} 0 & 1 \end{bmatrix} \quad P^4 \quad \begin{array}{cc} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \end{array}$$

$$\frac{1}{10^2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix} P^2 \begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{10^2}\begin{bmatrix} 5 & 5 \end{bmatrix} P^2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{5}{10^4}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{5}{10^4}\begin{bmatrix}12 & 8\end{bmatrix}\begin{bmatrix}36\\40\end{bmatrix}$$

$$= \frac{5}{10^4}\begin{bmatrix}752\end{bmatrix} = \frac{3760}{10^4}$$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 \end{array} \times \frac{1}{6}$$

$$P^{(3)}_{23}$$

$$= \begin{array}{ccc} 1 & 2 & 3 \\ \hline 0 & 1 & 0 \end{array} \; P^3 \; \begin{array}{c|c} 0 & 1 \\ 0 & 2 \\ 1 & 3 \end{array}$$

$$= \begin{bmatrix}3 & 2 & 2\end{bmatrix} P \begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix}3 & 2 & 2\end{bmatrix}\begin{bmatrix}2 & 3 & 1\\1 & 2 & 3\\3 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix}14 & 17 & 11\end{bmatrix}\begin{bmatrix}1\\3\\1\end{bmatrix}$$

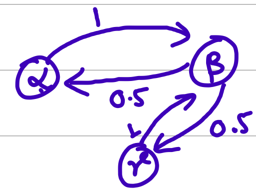
$$= \frac{1}{6^3}\begin{bmatrix}76\end{bmatrix} = \frac{76}{216}$$

$C(\alpha) = \{\alpha, \beta, \gamma\}$ Irreducible

$$\begin{array}{ccc} \alpha & \beta & \gamma \\ \hline 0.25 & 0.25 & 0.5 \end{array}$$

$$X_0 = \frac{1}{4}\begin{bmatrix}1 & 1 & 2\end{bmatrix}$$

$$\begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline \alpha & 0 & 1 & 0 \\ \beta & 1/2 & 0 & 1/2 \\ \gamma & 0 & 1 & 0 \end{array}$$



B $P[X_4 = \gamma] = ?$ $\Rightarrow \frac{1}{4}\begin{bmatrix}1 & 1 & 2\end{bmatrix} P^4 \begin{bmatrix}0\\0\\1\end{bmatrix}$

P.Q.4.

$P[X_4 = \gamma / X_2 = \alpha] \Rightarrow [1\ 0\ 0]\ P^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Classifi

Periodicity is class property ie. $i \leftrightarrow j \Rightarrow d(i) = d(j)$

$\rightarrow \quad i \rightarrow j \quad P_{ij}^{n} > 0 \quad$ for some n

$\quad j \rightarrow i \quad P_{ji}^{m} > 0 \quad$ for some m

$d(i) = \gcd\{t / P_{ii}^{t} > 0\}$ $\qquad d(j) = \gcd\{t / P_{jj}^{t} > 0\}$

$$P_{ii}^{(n+m)} = \sum_{k \in S} P_{ik}^{n} P_{ki}^{m} \geq \underbrace{P_{ij}^{n} P_{ji}^{m}}_{>0}$$

$$P_{jj}^{(n+m)} = \sum_{k \in S} P_{jk}^{m} P_{kj}^{n} \geq \underbrace{P_{ji}^{m} P_{ij}^{n}}_{>0}$$

$\Rightarrow$ $n+m$ is multiple of $d(i)$ $\Rightarrow n+m = d(i) \times$ something

Similarly, $P_{jj}^{(n+m)} > 0 \Rightarrow n+m$ is multiple of $d(j)$

$\Rightarrow n+m = d(j) \times$ something

for some s, $P_{ii}^{(s)} > 0 \Rightarrow s = d(i) \times$ something —— (1)

$$P_{jj}^{(n+m+s)} \geq \underbrace{P_{ji}^{(m)}}_{>0} \underbrace{P_{ii}^{(s)}}_{>0} \underbrace{P_{ij}^{(n)}}_{>0} > 0$$

$\Rightarrow n+m+s = d(j) \times$ something

$\Rightarrow s = d(j) \times$ something —— (2) (as $d(j)$ divides $n+m$)

From (1) & (2)

$\Rightarrow d(i) = d(j)$

Transitivity :- $i \leftrightarrow j$, $j \leftrightarrow k$ $\Rightarrow$ $i \leftrightarrow k$

$i \leftrightarrow j$ $\quad P_{ij}^{n} > 0$, $P_{ji}^{m} > 0$ for some n, m

$j \leftrightarrow k$ $\quad P_{jk}^{s} > 0$, $P_{kj}^{t} > 0$ for some s, t

To show $i \leftrightarrow k$:-> $\quad P_{ik}^{n+s} \geq \underbrace{P_{ij}^{n}}_{>0} \underbrace{P_{jk}^{s}}_{>0}$ $\quad\quad P_{ki}^{t+m} \geq \underbrace{P_{kj}^{t}}_{>0} \cdot \underbrace{P_{ji}^{m}}_{>0}$

> 0 $\quad\quad > 0$

$i \to k$ $\quad\quad k \to i$

$\Rightarrow$ $i \leftrightarrow k$

Recurrent: Starting from state i, Process reenters in the same state with probability 1

① $F_i = 1$

② $\sum_n P_{ii}^{n} = \infty$

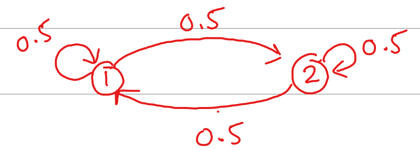
For any state i, F_i denote the probability that the process, starting from state i, will ever reaches state i.

State i is said to recurrent if $F_i = 1$, & transient if $F_i < 1$

$f_i^{(n)} \Rightarrow$ Prob. that starting from state i process reaches to state i in n steps first time.

$\boxed{F_i = \sum_n f_i^{n}}$ ✓

Irreducible



$f_1^{1} = 0.5$ $\quad\quad$ ~~1-1-1~~

$f_1^{2} = 0.5^2$ $\quad\quad$ 1-2-1

$f_1^{3} = 0.5^3$ $\quad\quad$ 1-2-2-1

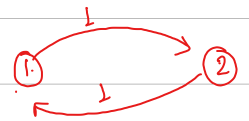
$\Rightarrow f_1^{n} = 0.5^n$

$F_1 = \sum_n f_1^{n} \Rightarrow = 0.5 + 0.5^2 + \dots$

$= 0.5(1 + 0.5 + 0.5^2 + \dots)$

$= 0.5\left(\frac{1}{1-0.5}\right) = \cancel{0.5} \cdot \frac{1}{\cancel{0.5}} = 1$

1 & 2 recurrent



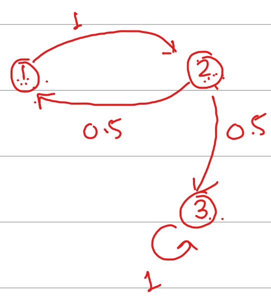
$f_1^{(1)} = 0$ $\quad F_1 = f_1^{(1)} + f_1^{(2)} + 0 + 0$

$f_1^{(2)} = 1$ $\quad = 0 + 1 + 0$

$f_1^{(3)} = 0$ $\quad = 1$

$\Rightarrow$ 1 & 2 recurrent

Reducible



$C(1) = \{1,2\} = C(2)$ $\quad C(3) = \{3\}$

$d(1) = gcd\{2,4,6,\ldots\}$ $\quad d(3) = 1$

$= 2$

$= d(2)$

(Periodicity is class property)

$f_2^1 = 0, \quad f_2^2 = 0.5, \quad f_2^3 = 0, \quad f_2^4 = 0 \; \ldots$

$f_1^1 = 0 \quad f_1^2 = 0.5 \quad f_1^3 = 0 \quad f_1^4 = 0 \; \ldots$

$F_{2i} = 0.5 < 1 \quad \Rightarrow$ 1&2 ~~is~~ are transient states

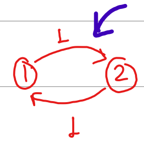
$F_3 = f_3^{(1)} + f_3^{(2)} + f_3^{(3)} + \ldots = 1 + 0 + 0 + \ldots = 1 \Rightarrow$ 3 is ~~recurrent~~. persistent

$P_{ii}^{(n)} \rightarrow$ Starting from i, process reaches to state i, in n steps

To prove

i recurrent iff $\sum_n P_{ii}^n = \infty$ | Transient iff $\sum P_{ii}^n < \infty$

$C(1) = \{1,2\} = C(2)$ Irreducible MC



Recurrent ~~Transient~~

$\sum_n P_{11}^{(n)} = P_{11}^1 + P_{11}^2 + P_{11}^3 + \ldots \qquad = \sum_n P_{11}^{2n}$

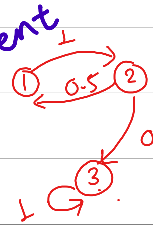
$= 0 + 1 + 0 + 1 + \quad$ infinite - Recurrent

$= \infty$

$\cdot C(1) = \{1,2\}$ ~~recurrent~~ transient

$\cdot C(3) = \{3\}$

Reducible MC



$\sum_n P_{11}^{(n)} = P_{11}^1 + P_{11}^2 + P_{11}^3 + P_{11}^4 + \ldots \qquad = \sum_n P_{11}^{2n}$

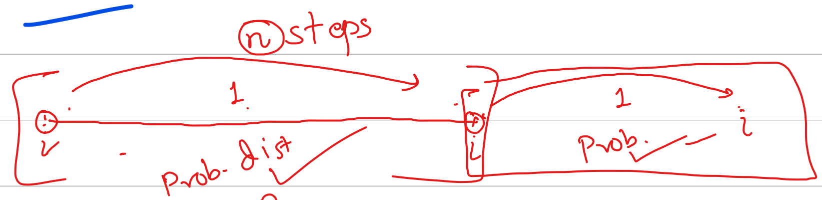
$= 0 + 0.5 + 0 + 0.5^2 + \ldots \qquad = \sum_n 0.5^n$

$= \frac{1}{1-0.5} = \frac{1}{0.5} = 2 < \infty$ $\Rightarrow$ Transient

Finite

(i) Recurrent iff $\sum P_{ii}^{(n)} = \infty$

Markov property



infinite $\longrightarrow$ enters

infinite Process probabilistic replica — $F_{ii} = 1$, 0

i state transient, {starting from i ~~stat~~ process reaches to state i w.p. < 1

$F_{ii} < 1$

$1 - F_{ii} > 0$ it will never return to state i

No. visits Finite no.

$\sim$ Geo

$E(\text{No of visits}) \Rightarrow \dfrac{1}{1 - F_{ii}}$

- State i is recurrent if, with probability 1, a process starting from state i, will eventually return.
- However, by Markovian Property, the process probabilistically restarts itself upon returning to state i. Hence with prob. 1 it will return to i.
- Repeating this argument, with probability 1, the no. of visits to state i will be infinite & will thus have infinite expectation.

On the other hand, if state i is transient, there is positive prob., $1 - F_{ii} > 0$, that it will never return again. Hence the no. of visits is Geometric with $E(n) = \dfrac{1}{1 - F_{ii}}$

$\therefore$ State i is recurrent iff $E(\text{no. of visits to } i / X_0 = i) = \infty$

Let $I_n = \begin{cases} 1 & \text{if } x_n = i \\ 0 & \text{if } x_n \neq i \end{cases}$

$E(I_n) = P(I_n = 1 / X_0 = i)$
$= P(X_n = i / X_0 = i)$
$= P_{ii}^n$

$\therefore \sum_n I_n$ denotes no. of visits to i. ✓

$E\left(\sum_n I_n / X_0 = i\right) = \sum_n E(I_n / X_0 = i)$

$= \sum_n P_{ii}^n$

$\Rightarrow$ i is recurrent iff $\sum_n P_{ii}^n = \infty$ ✓

Theo $\boxed{\text{Recurrence is class property so does transience}}$

Recurrence

$\hookrightarrow$ $F_{ii} = 1$ or/and $\sum P_{ii}^n = \infty$

Transience

$F_{ii} < 1$ $\sum P_{ii}^n < \infty$

$i \leftrightarrow j \Rightarrow$ i is recurrent $\Rightarrow$ j is also recurrent

$\rightarrow$ $i \leftrightarrow j$, ⊙ $i \rightarrow j$ for some s, $P_{ij}^s > 0$
$j \rightarrow i$ for some t, $P_{ji}^t > 0$

i is recurrent $\Rightarrow \sum_n P_{ii}^n = \infty$

$P_{jj}^{s+t+n} = \sum_{k \in S} P_{jk}^t P_{kk}^n P_{kj}^s$ $< \sum_{n=0}^{\infty} P_{jj}^n = \infty$

$$\geq P_{ji}^{t} P_{ii}^{n} P_{ij}^{s}$$

$$\sum_n P_{jj}^{s+t+n} \geq \sum_n P_{ji}^{t} P_{ii}^{n} P_{ij}^{s}$$

$$\geq \underset{>0}{P_{ji}^{t}} \underset{\infty}{\left(\sum_n P_{ii}^{n}\right)} \underset{>0}{P_{ij}^{s}}$$

$$\Rightarrow \sum_n P_{jj}^{n} \geq \sum_n P_{jj}^{s+t+n} \geq \infty$$

$\leq \infty$

$$\Rightarrow \sum_n P_{jj}^{n} = \infty$$

Transience is class property.

$i \leftrightarrow j$, if i is transient $\Rightarrow$ j is also transient.

$i \leftrightarrow j$ $\quad \cdot i \to j, \quad 1 \geq P_{ij}^{s} > 0$

$\quad\quad\quad j \to i, \quad 1 \geq P_{ji}^{t} > 0$

i is transient $\Rightarrow \sum_n P_{ii}^{n} < \infty$

$$P_{jj}^{s+t+n} \geq P_{ji}^{t} P_{ii}^{n} P_{ij}^{s}$$

$$\sum P_{jj}^{s+t+n} \geq \boxed{\underset{\leq 1}{P_{ji}^{t}} \underset{<\infty}{\left(\sum P_{ii}^{n}\right)} \underset{\leq 1}{P_{ij}^{s}}} \leq \infty$$

$$\sum P_{jj}^{s+t+n} \leq \infty$$

Recurrence

$i \leftrightarrow j$ $\quad i$ transient $\Rightarrow j$ transient

By method of contradiction

i transient but j is recurrent

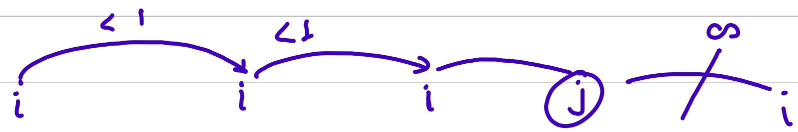
j recurrent $\Rightarrow$ i recurrent

which is contradiction

Irreducible MC :- Only one class, All states are communicating with each other.

Ergodic

Let μ_{jj} denotes the expected no. of transitions need to return to state j



Recurrent

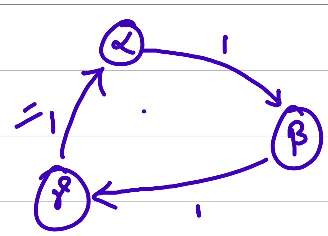
o $F_i = 1$

o $\sum P_{ii} = \infty$

$$\mu_{jj} = \begin{cases} \infty, & \text{if } j \text{ is transient.} \\ \sum n f_i^n & \text{if } j \text{ is recurrent.} \end{cases}$$

$\mu_{jj} = \infty$ null recurrent $\qquad \mu_{jj} < \infty$ +ve recurrent

$$\begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline \alpha & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \\ \gamma & 1 & 0 & 0 \end{array}$$



First time

$\mu_{\alpha\alpha} = \sum_n n \cdot f_\alpha^n$

$= 1\cdot0 + 2\cdot0 + 3\cdot1 + 4\cdot0 + 0 \ldots$

$= 3$

$\mu_{\alpha\alpha} = 3 < \infty$

positive recurrent aperiodic $\Rightarrow$ ergodic

$C(\alpha) = \{\alpha, \beta, \gamma\} \Rightarrow$ Irreducible

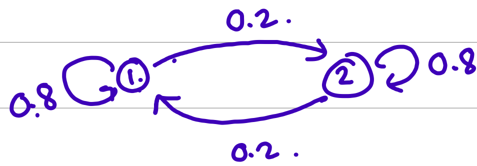
$F_\alpha = f_\alpha^1 + f_\alpha^2 + f_\alpha^3 + f_\alpha^4 \ldots$

$= 0 + 0 + 1 + 0 \ldots$

$= 1 \Rightarrow \alpha, \beta, \gamma$ all are recurrent

$\mu_{\alpha\alpha} = 3 < \infty \Rightarrow \alpha$ is +ve recurrent

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.8 & 0.2 \\ 2 & 0.2 & 0.8 \end{array} \qquad \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$



$C(1) = \{1,2\}$ $\Rightarrow$ Irreducible $\quad 1 \to 2 \to 2 \to 1$

First time (4) (2)

$F_1 = f_1^1 + f_1^2 + f_1^3 + f_1^4 + \dots$

$= 0.8 + 0.2^2 + 0.2^2 \cdot 0.8^1 + 0.2^2 \cdot 0.8^2 + \dots + 0.2^2 (0.8)^{n-2} + \dots$

$= 0.8 + 0.2^2 [1 + 0.8 + 0.8^2 + \dots] \quad < \infty \qquad 0.8 < 1$

$= 0.8 + 0.2^2 \left[\frac{1}{1-0.8}\right]$

$= 0.8 + 0.2$

$= 1$ $\Rightarrow$ 1, 2 are recurrent states

$\mu_{11} = \sum_n n \cdot f_1^n = 1 \cdot 0.8 + 2 \cdot 0.2^2 + 3 \cdot 0.2^2 \cdot 0.8 + \dots + n \cdot 0.2^2 \cdot 0.8^{n-2} + \dots$

$= 0.8 + 0.2^2 [2 + 3 \cdot 0.8 + \dots + n \cdot 0.8^{n-2} + \dots]$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{d}{dx} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2}$$

$= 0.8 + 0.2^2 \left[\sum_n (n+1) x^{n-1}\right] = 0.8 + 0.2^2 \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1} + 0.2^2 \sum_{n=1}^{\infty} x^{n-1}$

$= 0.8 + \cancel{(0.2)^2} \cdot \frac{1}{\cancel{(1-0.8)^2}} + 0.2^2 \cdot \sum_{n=0}^{\infty} x^n = 0.8 + 1 + 0.2 \quad < \infty$

1, 2 +ve recurrent.

* Ergodic state :- Aperiodic, positive recurrent

i ergodic state $\Rightarrow$ $d(i) = 1$, $\bar{F}_i = 1$ or $\sum P_{ii}^n = \infty$

$\mu_{ii} < \infty$

* Ergodic MC $\Rightarrow$ All states are ergodic.

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$C(1) = \{1,2\}$ $\quad$ 1, 2, positive recurrent

$d(1) = \gcd(1,2,\dots) = 1 = d(2)$

1, 2 positive recurrent, aperiodic $\Rightarrow$ ergodic

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0.5 & 0.5 \\ 3 & 1 & 0 & 0 \end{array}$$

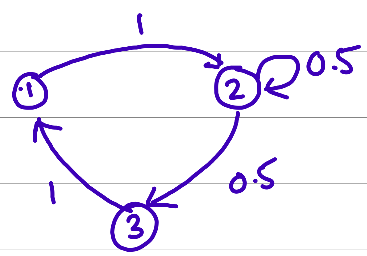
states :- Aperiodic $\quad$ Recurrent $\quad$ Transient

+ve $\quad$ null

Ergodic

$\Rightarrow$ MC Ergodic ?

$\Rightarrow$



$C(1) = \{1,2,3\}$ $\Rightarrow$ Irreducible

$d(2) = \gcd\{1,2,\dots\} = 1$ $\Rightarrow$ Aperiodic

Periodicity is class property $\Rightarrow$ All states are aperiodic.

$$F_2 = f_2^1 + f_2^2 + \dots = 0.5 + 0 + 0.5 \cdot 1 \cdot 1 + 0 + \dots$$
$$= 0.5 + 0.5$$
$$= 1 \qquad \Rightarrow \text{Recurrent}$$

$\because$ Recurrence is class property. $\Rightarrow$ All states are recurrent.

$$\mu_2 = 1 \cdot f_2^1 + 2 \cdot f_2^2 + 3 \cdot f^3 + 4 \cdot f_2^4 + \dots$$
$$= 1 \cdot 0.5 + 0 + 3 \cdot 0.5 + 0 + \dots$$
$$= 0.5 + 1.5$$
$$= 2 < \infty \qquad \Rightarrow \text{Positive Recurrent}$$

All states are positive recurrent & aperiodic, i.e. all states are ergodic

$\therefore$ MC is ergodic MC.

Limit Theo. :-

If j is transient, then $\sum_n P_{jj}^n < \infty$

$P_{jj}^n \to 0$ as $n \to \infty$

We have only finite no. of transitions returning to j.

$\underline{\pi_j} = \lim_{n\to\infty} P_{jj}^n$ $\quad\quad$ $\pi_{ij} = \lim_{n\to\infty} P_{ij}^n$

+ve recurrent state $\pi_j > 0$

null recurrent $\pi_j = 0$? —

Stationary

CK eq^n

$$P^{n+1} = P^n \cdot P$$

$$\lim_{n\to\infty} P_{ij}^{(n+1)} = \lim_{n\to\infty} \sum_{k\in S} P_{ik}^{(n)} \cdot P_{kj}^{(1)}$$

$$\underline{\underline{\pi_{ij}}} = \sum_{k\in S} \pi_{ik} P_{kj}$$

In Matrix form

$$P^{n+1} = P^n \cdot P$$

$$\left(\lim_{n\to\infty} P^{n+1}\right) = \left(\lim_{n\to\infty} P^n\right) \cdot P$$

$$\underline{\pi} = \underline{\pi} \cdot P$$

Recurrent limiting $\longrightarrow$ π = Stationary dist^n

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0.5 & 0.5 \\ 3 & 1 & 0 & 0 \end{array} \qquad \underline{\underline{\Pi}} = [\Pi_1, \Pi_2, \Pi_3]'$$

$$\underline{\Pi}' = \underline{\Pi}' P$$

$$[\Pi_1 \ \Pi_2 \ \Pi_3] = [\Pi_1 \ \Pi_2 \ \Pi_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Pi_1 = \Pi_3$$

$$\Pi_2 = \Pi_1 + \Pi_2/2 \Rightarrow \Pi_1 = \Pi_2/2$$

$$\Pi_3 = \Pi_2/2$$

$$\underline{\Pi} = \underline{\underline{\left[\frac{\Pi_2}{2}, \Pi_2, \frac{\Pi_2}{2}\right]}} \Rightarrow \sum_{i=1}^{3} \Pi_i = 1 \Rightarrow$$

$$\Rightarrow \frac{\Pi_2}{2} + \Pi_2 + \frac{\Pi_2}{2} = 2\Pi_2 = 1$$

$$\Pi_2 = 1/2$$

✓ $\underline{\Pi} = [1/4, 1/2, 1/4]$

$$P = \frac{1}{10}\begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix} \qquad \Pi = ?$$

$$\underline{\Pi} = \underline{\Pi} P$$

$$[\Pi_1 \ \Pi_2 \ \Pi_3] = [\Pi_1 \ \Pi_2 \ \Pi_3] \begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix} \frac{1}{10}$$

$$10\Pi_1 = 3\Pi_1 + 4\Pi_2 + 3\Pi_3 \Rightarrow 7\Pi_1 = 4\Pi_2 + 3\Pi_3 \Rightarrow 7\Pi_1 = 4\Pi_2 + \frac{3}{2}(\Pi_1 + \Pi_2)$$

$$10\Pi_2 = 4\Pi_1 + 3\Pi_2 + 3\Pi_3 \Rightarrow 7\Pi_2 = 4\Pi_1 + 3\Pi_3$$

$$10\Pi_3 = 3\Pi_1 + 3\Pi_2 + 4\Pi_3 \Rightarrow 6\Pi_3 = 3\Pi_1 + 3\Pi_2 \Rightarrow 2\Pi_3 = \Pi_1 + \Pi_2$$

$$11\Pi_1 \ \cancel{14\Pi_1} = 8\Pi_2 + \cancel{3\Pi_1} + 3\Pi_2 \quad \Rightarrow \Pi_1 = \Pi_2$$

$$\underline{\pi} = [\pi_1\ \ \pi_2\ \ \pi_3] = [\pi_1\ \ \pi_1\ \ \pi_1]$$

$$\sum \pi_i = 1\ , \quad \pi_1 + \pi_1 + \pi_1 = 1 \quad \Rightarrow \pi_1 = 1/3$$

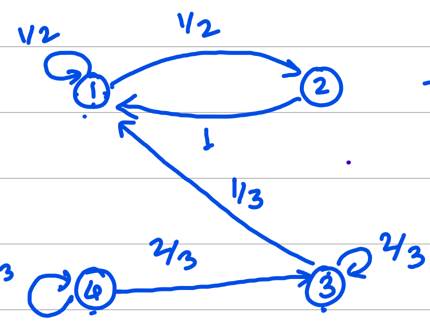
$$\pi = [1/3\ \ 1/3\ \ 1/3]$$

①
$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 6 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix} \Rightarrow ? \quad \underline{[x, x, y, y]} \quad \underline{\underline{x, y}} \in \underline{\underline{(0,1)}}$$

$\underline{x+y = 1/2}$ ✓

Stationary distribution may or may not be unique.

②
$$\frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$



Transition Graph

$C(1) = \{1,2\}$, $C(3) = \{3\}$, $C(4) = \{4\}$ $\Rightarrow$ Reducible MC

$d(1) = 1$ $\quad d(3) = 1$ $\quad d(4) = 1$ $\Rightarrow$ aperiodic

$$F_1 = f_1^1 + f_1^2 + f_1^n + \dots$$
$$= \frac{1}{2} + \frac{1}{2}\cdot 1 + 0 + \dots$$
$$= 1$$

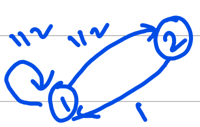
$\Rightarrow F_1 = 1 \Rightarrow$ 1,2 are recurrent

$$F_3 = f_3^1 + f_3^2 + f_3^3 + \dots$$
$$= \frac{2}{3} + 0 + 0 + \dots$$
$$= \frac{2}{3} < 1$$

$\Rightarrow$ State 3 is transient

$$F_4 = f_4^1 + f_4^2 + f_4^3 + \dots$$
$$= \frac{1}{3} + 0 + \dots$$
$$= \frac{1}{3} < 1$$

$\Rightarrow$ 4 is transient



$$\mu_{11} = \sum_n n f_1^n = 1\cdot f_1^1 + 2\cdot f_1^2 + 0 = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5 < \infty$$

Recurrence is class property

$$F_2 = f_2^1 + f_2^2 + f_2^3 + f_2^4 + \dots + f_2^n + \dots$$
$$= 0 + \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} \qquad + \frac{1}{2}\cdot 1\cdot\left(\frac{1}{2}\right)^{n-2} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right)$$
$$= \frac{1}{2}\cdot\left(\frac{1}{1-1/2}\right) = 1 \quad \Rightarrow \quad 2 \text{ is recurrent.}$$

$$\mu_2 = \sum n f_2^n \Rightarrow \quad 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2^2} + \dots + n\cdot\frac{1}{2^{n-1}} + (n+1)\frac{1}{2^n}$$
$$\Rightarrow \sum (n+1)\frac{1}{2^n} = \underline{\sum n\cdot\frac{1}{2^n}} + \sum\frac{1}{2^n} < \infty$$

$\Rightarrow$ 2 is positive recurrent

1,2, aperiodic, positive recurrent $\Rightarrow$ Ergodic States

Lets find out limiting distribution / Stationary

$$[\pi_1 \; \pi_2 \; \pi_3 \; \pi_4] = [\pi_1 \; \pi_2 \; \pi_3 \; \pi_4]\,\frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$6\pi_1 = 3\pi_1 + 6\pi_2 + 2\pi_3 \qquad \cancel{6\pi_1 = 3\pi_1 + 3\pi_1}$$
$$6\pi_2 = 3\pi_1 \qquad \Rightarrow \pi_1 = 2\pi_2$$
$$6\pi_3 = 4\pi_3 + 4\pi_4 \quad \Rightarrow \pi_3 = 0$$
$$6\pi_4 = 2\pi_4 \qquad \Rightarrow \pi_4 = 0 \quad \Rightarrow \quad 4\pi_4 = 0 \quad \Rightarrow \pi_4 = 0$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \Rightarrow \quad 2\pi_2 + \pi_2 + 0 + 0 = 1 \Rightarrow \pi_2 = 1/3$$
$$\Rightarrow \pi_1 = 2/3$$

Stationary Distribution $[1/3 \;\; 2/3 \;\; 0 \;\; 0]$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 1 & 1 \\ & & 1 & 1 \end{bmatrix}$$

$\underline{\underline{\pi}} = [\pi_1 \ \pi_1 \ \pi_3 \ \pi_3]$

$\pi_1 + \pi_1 + \pi_3 + \pi_3 = 1$

$\Rightarrow \underline{\underline{\pi_1 + \pi_3 = 1}}$

$\underline{\underline{\pi_1 = x}}$

unique

$\underline{[x \ x \ 1-x \ 1-x]}$

$\lim_{n\to\infty} P_{ij}^{n}$ $i \to j$ in n steps

$\to \underline{\underline{\pi_j}}$ ✓ $i \to$ doesn't matter

$\pi' = \pi' P$ eqn $\to$ ✓

$\pi' = \pi' P$

$I\underline{\pi} = P'\underline{\pi}$

$\Rightarrow (P' - I)\underline{\pi} = \underline{0}$

$A\underline{x} = \underline{b}$,

$\pi_1 + \pi_2 + \pi_3 = 1$

$$[1 \ 1 \ 1]\begin{bmatrix}\pi_1 \\ \pi_2 \\ \pi_3\end{bmatrix} = 1$$

$$A = \begin{bmatrix} P' - I \\ \text{ones} \end{bmatrix} \qquad b = \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$$

No limiting distribution exists

$$\begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$$

$P^n \Rightarrow$ $P^{2n} = I$

$P^{2n+1} = P$

$\pi' = \pi' P$

initial $= \pi$ $\qquad \pi$

~~$P_2 = \pi P P^2$~~

$P(X_0 = x) = \alpha \qquad \lim_{n\to\infty} P(x_n = x) \to \pi$

$x_1 \qquad \alpha P \qquad\qquad x_2 \qquad \alpha . P^2$

$\alpha = \pi \qquad x \qquad \pi P = P \qquad\qquad \pi . P^2$

$\pi . P . P^2$

$P P^2$

Theo.:- An irreducible aperiodic MC belongs to one of the following classes :-

j transient $\forall\, i \in S$

$P_{ij}^n \to 0$

① Either the states are all transient or all null recurrent in this case, $P_{ij}^n \to 0 \ \forall\, i, j$ and $\nexists$ no stationary distⁿ.

$F_j = 1,\ \mu_{jj} < \infty$

② Or else, all states are positive recurrent, that is

$$\pi_j = \lim_{n\to\infty} P_{ij}^n > 0$$

unique stationary distⁿ $\qquad$ limiting distⁿ

In this case, $\{\pi_j,\ j = 0, 1, 2 \ldots\}$ is stationary distⁿ & $\nexists$ no other stationary distribution.

Proof: We will prove (ii) $\qquad S = \{0, 1, 2, \ldots\} = \{0, 1, 2, \ldots, M, \ldots\}$ for some $M \in S$

$\sum_{j=0}^{\infty} P_{ij}^n = 1 \quad j \in S$

$$\sum_{j=0}^{M} P_{ij}^n \leq \sum_{j=0}^{\infty} P_{ij}^n = 1. \qquad \forall \text{ all } M \in S$$

$\sum_{j=0}^{M} P_{ij}^n \leq 1$

letting $n \to \infty$ yields $\qquad \sum_{j=0}^{M} P_{ij}^n \leq 1$

$$\sum_{j=0}^{\infty} \lim_{n\to\infty} P_{ij}^n \le 1$$

Π_j

$$\sum_{j=0}^{M} \Pi_j \le 1 \qquad \forall\, M \in S$$

$M \to \infty \qquad S = \{0,1,\ldots\}$

implying that

$$\sum_{j=0}^{\infty} \Pi_j \le 1$$

$\geq \quad S = \{0,1,\ldots,M,\ldots\}.$

Now

$$P_{ij}^{n+1} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^1 \geq \sum_{k=0}^{M} P_{ik}^n P_{kj} \qquad \forall\, M \in S \qquad \text{C-K eq}^n$$

CK $P_{ij}^{m+n} = \sum_{k \in S} P_{ik}^m P_{kj}^n$

letting $n \to \infty$ yields

$$\lim_{n\to\infty} P_{ij}^{n+1} \geq \lim_{n\to\infty} \sum_{k=0}^{M} P_{ik}^n P_{kj} \qquad \forall\, M \in S$$

Π_j

$\lim_{n\to\infty} P_{ik}^n \Rightarrow \Pi_k$

$\lim_{n\to\infty} P_{ij}^n = \lim_{n\to\infty} P_{ij}^{n+1} = \Pi_j$

implying that

$$\Pi_j \geq \sum_{k=0}^{M} \left(\lim_{n\to\infty} P_{ik}^n\right) P_{kj} \qquad \forall\, M \in S$$

$$\Pi_j \geq \sum_{k=0}^{M} \Pi_k \cdot P_{kj} \qquad \forall\, M \in S$$

$$\Rightarrow \quad \Pi_j \geq \sum_{k=0}^{\infty} \Pi_k P_{kj} \qquad \forall\, j \geq 0$$

$>$? Q.

To prove equality, assume strict inequality for some j

$$\Rightarrow \quad \Pi_j > \sum_{k=0}^{\infty} \Pi_k P_{kj} \qquad k \to j \qquad j = 0,1,\ldots\infty$$

Summation

$$\Rightarrow \quad \sum_{j=0}^{\infty} \Pi_j > \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Pi_k P_{kj} = \sum_{k=0}^{\infty} \Pi_k \sum_{j=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \Pi_k \quad 1$$

$$\Rightarrow \quad \sum_{j=0}^{\infty} \Pi_j > \sum_{k=0}^{\infty} \Pi_k \qquad \text{[Which is contradiction]}$$

$P_j = P(X_n = j)$

$p^{(2)} = p.p$

$p^{(1)} = \alpha.p$

$\pi' = \pi' p$

$$\Rightarrow \boxed{\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}} \quad \forall\ j \in S = \{0,1,2,\ldots\}$$

Putting $P_j = \dfrac{\pi_j}{\sum_k \pi_k} \Rightarrow P_j$ is stationary dist^n

$x_1 \to \alpha.P$, $x_1 = x_0$, $\alpha = \pi$

$\therefore$ At least one stationary dist^n exists.

Now let $\{P_j, j=0,1,2,\ldots\}$ be any stationary distribution.

Then if $\{P_j, j=0,1,2,\ldots\}$ is the prob. dist^n of X_0 (Initial Prob.dist^n)

$x_1 \sim \alpha P$

$x_1 \sim \pi.P$

$\pi.P = \pi'$ x_2

$$\underline{P_j = P\{X_n = j\}}$$

$$= \sum_{i=0}^{\infty} P\{X_n = j / X_0 = i\} \ P\{X_0 = i\}$$

$$P_j = \sum_{i=0}^{\infty} P_{ij}^n P_i \qquad \text{——— (1)}$$

$$\Rightarrow P_j \geq \sum_{i=0}^{M} P_{ij}^n P_i \qquad \forall\ M \in S$$

Let $n \to \infty$ and $M \to \infty$ $\quad P_j \geq \sum_{i=0}^{M} \left(\lim_{n\to\infty} P_{ij}^n\right) . P_i$, $\lim_{n\to\infty} P_{ij}^n \to \pi_j$

stationary

$$\underline{P_j} \geq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j \underbrace{\left(\sum_{i=0}^{\infty} P_i\right)}_{1} = \underline{\pi_j} \qquad \text{——— (2)}$$

limiting

$$\Rightarrow \underline{P_j \geq \pi_j}$$

Lets prove other side.

$$P_j = \sum_{i=0}^{\infty} P_{ij}^n P_i = \sum_{i=0}^{M} P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_{ij}^n . P_i$$

CK

$$\leq \sum_{i=0}^{M} P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_i \qquad (\text{as } P_{ij}^n \leq 1)$$

$$\lim_{n\to\infty} P_j \leq \sum_{i=0}^{M} \left(\lim_{n\to\infty} P_{ij}^n\right) . P_i + \sum_{i=M+1}^{\infty} P_i$$

$$P_j \le \sum_{i=0}^{M} \pi_j \cdot P_i + \sum_{i=M+1}^{\infty} P_i$$

$$\le \pi_j \left(\sum_{i=0}^{M} P_i\right) + \sum_{i=M+1}^{\infty} P_i$$

letting $M \to \infty$

$$P_j \le \pi_j \underbrace{\left(\sum_{i=0}^{\infty} P_i\right)}_{1} + 0$$

$$P_j \le \pi_j \qquad \text{--- (3)}$$

from (2) & (3) $\Rightarrow$ $P_j = \pi_j \quad j = 0, 1, 2, \ldots,$ is only stationary distⁿ.

If all states are transient or null recurrent and $\{P_j, j = 0, 1, 2, \ldots\}$ is stationary distⁿ, then (1) eqⁿ holds i.e.

$$P_j = \sum_{i=0}^{\infty} \underbrace{P_{ij}^{n}}_{\to 0} \underbrace{P_i}_{\ge 1} \to 0$$

and $P_{ij}^{n} \to 0$, which is impossible.
Thus for case (1), no stationary distⁿ exists.

Note:-

* In the irreducible, positive recurrent, we have that $\pi_j, j \ge 0$ are unique non-negative solution of

$$\pi_j = \sum_i \pi_i P_{ij} \quad \& \quad \sum_j \pi_j = 1$$

i.e. In long run, π_j is the proportion of time that the MC is in state j.

$$\Rightarrow \quad \pi_j = \frac{1}{\mu_{jj}}$$

$$\Rightarrow \quad \lim_{n\to\infty} P_{jj}^{n\cdot d(j)} = \frac{d}{\mu_{jj}} = d\cdot\pi_j \qquad (d = d(j)\ \forall j)$$

Prob. Dist^n $\{P_j, j\geq 0\}$ is said to be stationary for MC if

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij} \qquad j\geq 0$$

* Random Walk

X_0 = initial position $\qquad X_0 = 100$

$$Z_i = \begin{cases} +1 & \text{with } p \\ -1 & \text{with } 1-p \end{cases}$$

$X_1 = X_0 + Z_1 \leftarrow$ outcome of 1^{st} trial/game

$X_2 = X_1 + Z_2$ o

$X_n = X_{n-1} + Z_n \leftarrow$ outcom of n^{th} game/trial

$X_1 = X_0 + Z_1$

$X_2 = X_0 + \sum_{i=1}^{2} Z_i$

$X_n = X_0 + \sum_{i=1}^{n} Z_i$

$\{X_i, i\in I\} \Rightarrow \overset{\text{Statespace}}{S} = \{-\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, \infty\}$

$\uparrow$ Random Walk

R.W. $X_n = X_0 + \sum_{i=1}^{n} Z_i$

if Z_i takes ± 1 values

$$Z_i = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$$

↳ Simple ~~Rand~~ RW

If

$$Z_i = \begin{cases} +\omega & \text{w.p. } 1/2 \\ -\omega & 1/2 \end{cases}$$

↳ symmetric r.w.

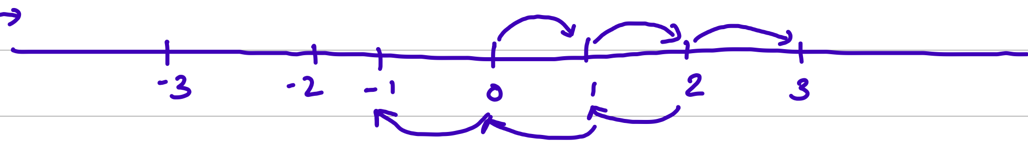
$$Z_i = \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & 1/2 \end{cases}$$

Simple Symmetric RW.

Ireducible MC

$C(0) = \{0, 1, -1, 2, -2, \dots\}$ Single communicating class

S.R.W →



Does random walk follows markov property?

Random Walk $X_n = X_0 + \sum_{i=1}^{n} Z_i$ $Z_i \sim$ i.i.d.

$P[X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0]$

$= P[X_{n+1} = j / X_n = i]$

$\to P[X_0 + \sum_{i=1}^{n} Z_i = j / X_0 + \sum_{i=1}^{n-1} Z_i = i, \dots]$

$= P[\underline{X_0 + \sum_{i=1}^{n-1} Z_i} + Z_i = j / \underline{X_0 + \sum_{i=1}^{n-1} Z_i = i}, \dots]$

$= P[Z_i = j - i / X_0 + \sum Z_i = i] \dots]$

as Z_i are i.i.d

X_n is MC

$$X_n = X_0 + \sum_{i=1}^{n} Z_i \quad \text{or} \quad X_n = X_{n-1} + Z_i$$

* TPM for random walk

$$\begin{array}{c|ccccccc}
 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
-3 & 0 & p & & & & & \\
-2 & q & 0 & p & & & & \\
-1 & & q & 0 & p & & & \\
0 & & & q & 0 & p & & \\
1 & & & & q & 0 & p & \\
2 & & & & & q & 0 & p \\
3 & & & & & & q & \\
\end{array}$$

* Gambler's Ruin Problem

X_0 = initial amount $\qquad X_n$ = Gambler's Fortune at n^{th} game

$$Z_i = \begin{cases} +1 & p \\ -1 & q \end{cases} \quad \text{i.i.d games}$$

Interest $\underline{X_n \to 0}$ Gambler $\underline{\text{Ruined}}$

A B

X_0 $N-X_0$

Markov chain $S = \{0, 1, 2, \ldots, N\}$

0

⋮

N

$$\begin{array}{c|c}
 & \begin{array}{cccccc} 0 & 1 & 2 & & N-1 & N \end{array} \\
\hline
\begin{array}{c} 0 \\ 1 \\ 2 \\ \\ N-1 \\ N \end{array} &
\begin{bmatrix}
1 & 0 & & & & \\
q & 0 & p & & & \\
 & q & 0 & p & & \\
 & & & & & \\
 & & & q & 0 & p \\
 & & & & & 1
\end{bmatrix}
\end{array}$$

Absorbing State → 0

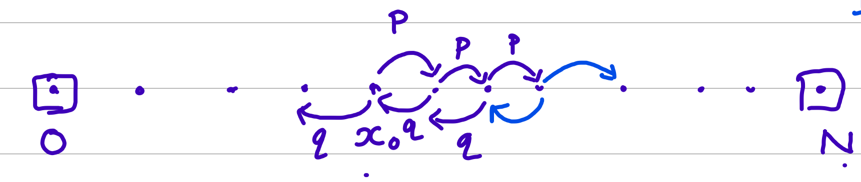
Absorbing State → N

1 +1

-1

$d(i) = \gcd\{n \,/\, P_{ii}^{n} > 0\}$

Transition graph



Irreducible

period i

$d_i = 1$? ✗

$= 2$? ✓

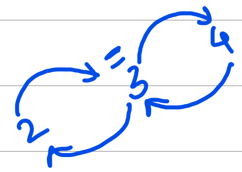
let $N = 4$

✓

	A	B
	x_0	$N - x_0$
+1	$x_0 + 1$	$N - x_0 - 1$
−1	x_0	$N - x_0$

2n

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \multicolumn{5}{c}{\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & & \\ & q & 0 & p & \\ & & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]} \\ 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \\ 4 & & & & & \end{array}$$



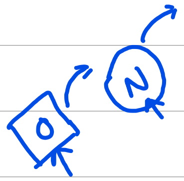
Classes $\{0\}, \{N\}, \underline{\{1, 2, \ldots, N-1\}}$

$\{0\}, \{N\}$ → Recurrent; $\{1, 2, \ldots, N-1\}$ → Transient

Long run probabilities

P_{ij}^{1} $\quad i \to j$ $\qquad j = i-1, \; i+1$

$\qquad\qquad\qquad\qquad\qquad\qquad q \quad\; p$



✓ $P_{00} = 1 = P_{NN}$

✓ $P_{i,i+1} = p = 1 - P_{i,i-1} \qquad i = 1, 2, \ldots, N-1$

Let f_i denote the prob. that starting from i gambler's fortune reaches to state N before reaching 0.

f_i $i \to N$

$i \to N$ $\quad$ $i+1 \to N$ $\quad$ $i-1 \to N$

i → $i+1 \cdot p$; $i-1 \cdot q$

$$f_i = p \cdot f_{i+1} + q \cdot f_{i-1} \qquad i = 1, 2, \ldots, N-1$$

$p + q = 1$ $\qquad$ $p + q + r = 1$, r tie, $r > 0$

$$(p+q) \cdot f_i = p \cdot f_{i+1} + q f_{i-1}$$

$$q(f_i - f_{i-1}) = p(f_{i+1} - f_i)$$

$$\left[(f_{i+1} - f_i) = \frac{q}{p}(f_i - f_{i-1}) \right] \qquad i = 1, 2, \ldots, N-1$$

$f_0 = ?$ $\quad$ $f_0 = 0$

$i = 1$, $\quad f_2 - f_1 = \frac{q}{p}(f_1 - f_0) = \frac{q}{p}(f_1)$.

$i = 2$, $\quad f_3 - f_2 = \frac{q}{p}(f_2 - f_1) = \frac{q}{p}\left(\frac{q}{p} f_1\right) = \left(\frac{q}{p}\right)^2 f_1$

$i = j$ $\quad f_{j+1} - f_j = \left(\frac{q}{p}\right)^j f_1$

$i = N-1$ $\quad f_N - f_{N-1} = \left(\frac{q}{p}\right)^{N-1} f_1$

sum of first 2 terms $\quad \cancel{f_3 - f_2} + \cancel{f_2 - f_1} = \frac{q}{p} f_1 + \left(\frac{q}{p}\right)^2 f_1$

j terms $\quad f_{j+1} - f_1 = f_1 \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^j \right]$

$$f_{j+1} = f_1 \left[1 + \frac{q}{p}' + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^j \right]$$

if $q/p = 1$ $\Rightarrow q = p = 1/2$ $\quad f_{j+1} = f_1[1 + 1 + 1 + \ldots + 1] = (j+1) f_1 \Rightarrow f_j = j f_1$

geometric series

$$\frac{1}{1-x} = 1+x+x^2+\dots+x^n+x^{n+1}+\dots+\dots \qquad x<1$$

$$\frac{1}{1-x} = sum + x^{n+1}[1+x+\dots]$$

$$\frac{1}{1-x} = sum? + \frac{x^{n+1}}{1-x}$$

if $\frac{q}{p} \neq 1 < 1$

$$sum? = \frac{1-x^{n+1}}{1-x}$$

$f_0 = 0 \to N$

$f_i \quad i \to N$

$f_N \quad N \to N$

$= 1$

$$f_j = f_1\left[1+\frac{q}{p}+\dots+\left(\frac{q}{p}\right)^{j-1}\right] = \begin{cases} f_1 \dfrac{1-(q/p)^j}{1-q/p} & q/p \neq 1 \\ j \cdot f_1 & q/p = 1 \end{cases}$$

$$f_N = 1, \quad j = N \Rightarrow \quad f_N = \begin{cases} f_1 \dfrac{1-(q/p)^N}{1-q/p} & q/p<1 \\ N \cdot f_1 & q/p = 1 \end{cases}$$

But we are not going to discuss

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & & & \\ 1 & q & r & p & \\ 2 & & q & r & p \\ 3 & & & & 1 \end{array}$$

$r>0 \qquad p+q+r=1$

$$1 = \begin{cases} f_1 \dfrac{1-(q/p)^N}{1-q/p} \\ f_1 N \end{cases}$$

$$\Rightarrow \quad f_1 = \begin{cases} \dfrac{1-q/p}{1-(q/p)^N} & q/p<1 \\ 1/N & q/p=1 \end{cases}$$

$$f_j = \begin{cases} \dfrac{\cancel{1-q/p}}{1-(q/p)^N} \cdot \dfrac{1-(q/p)^j}{\cancel{1-q/p}} = \dfrac{1-(q/p)^j}{1-(q/p)^N} & q/p < 1 \\ j \cdot \dfrac{1}{N} = \dfrac{j}{N} & \dfrac{q}{p} = 1 \end{cases}$$

Gambler's Ruin prob.

$$1-f_j = \begin{cases} 1 - \dfrac{1-(q/p)^j}{1-(q/p)^N} & q/p < 1 \\ 1 - \dfrac{j}{N} & q/p = 1 \end{cases}$$

limitting $N \to \infty$

$$\lim_{N\to\infty} f_j = \begin{cases} \lim\limits_{N\to\infty} \dfrac{1-(q/p)^j}{1-(q/p)^N} = 1-(q/p)^j & q/p < 1, \ (q/p)^N < 1 \to 0 \\ \lim\limits_{N\to\infty} \dfrac{j}{N} = 0 \end{cases}$$

Thaughtful.

Determine the expected number of steps/games that gambler makes, starting from state i, before reaching 0 or N.

$i \longrightarrow 0 \longrightarrow -i$

$i \longrightarrow N \longrightarrow N-i$

$$B = \min\left\{ m \,\middle|\, \sum_{i=1}^{m} X_i = -i \text{ or } N-i \right\}$$

$$X_i = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } q \end{cases}$$

$$E(X_i) = +1(p) + (-1)(1-p)$$
$$= p - 1 + p$$
$$= 2p - 1$$

by Wald's eq^ns

$$E\left(\sum_{i=1}^{B} X_i\right) = E(B)\cdot E(X_i) = (2p-1)\cdot E(B)$$

started from i, to reach N, $f_i = \dfrac{1-(q/p)^i}{1-(q/p)^N}$ ✓

i, to reach 0, $1-f_i$.

$$\sum_{j=1}^{B} x_j = \begin{cases} -i & 1-f_i \\ n-i & f_i \end{cases}$$

$$E\left(\sum_{j=1}^{B} x_j\right) = -i\cdot(1-f_i) + (N-i)\cdot f_i = -i + \cancel{if_i} + N f_i - \cancel{if_i}$$

$$(2p-1)\cdot E(B) = -i + N\left(\frac{1-(q/p)^i}{1-(q/p)^N}\right)$$

$$E(B) = \frac{1}{(2p-1)}\left[-i + N\left(\frac{1-(q/p)^i}{1-(q/p)^N}\right)\right]$$

$N=4$, $\quad i=2 \quad p=0.6 \quad q=0.4 \quad \frac{q}{p} = \frac{0.4}{0.6} = \frac{2}{3}$

$$f_2 = \frac{1-(q/p)^2}{1-(q/p)^N} = \frac{1-(2/3)^2}{1-(2/3)^4} = \frac{3^2-2^2}{3^4-2^4} \times \frac{\cancel{3^4}\, 3^2}{\cancel{3^2}}$$

winning

$$= \frac{9-4}{81-16} \times 9 = \frac{45}{65} = \frac{9}{13}$$

Ruin $\quad 1-f_2 \Rightarrow 4/13$

~~Irreducible~~ MC

$$H = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.4 & 0.1 \\ 2 & 0.5 & 0.5 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

Reduces

3- absorbing state

stationary / limiting dist^n $[0\ 0\ 1]$

$C(1) = \{1,2\} = C(2)$, $C(3) = \{3\}$ ← absorbing state

After sufficiently large no. of steps process is in → 3

$$P = \begin{array}{c|cc} & S-A & A \\ \hline S-A & N & B \\ A & \underline{O} & I \end{array} \qquad N = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

$A = \{3\}$

$\underline{0} = [0\ 0]$

$$\boxed{j \in A \qquad P[X_n = j, X_{n-1} \notin A, X_{n-2} \notin A, \ldots x_0] = \underline{\alpha}' N^{n-1} B_j}$$

$\uparrow$ j^{th} column of B

$x_0 = 1$

$$\alpha = \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ [1\ \ 0\ \ 0] \end{array}$$

$$P[X_n = 3, x_{n-1} \neq 3, x_{n-2} \neq 3, \ldots x_0 = 1] = [1\ 0\ 0] \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}^{n-1} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

* Simulation of RW

$$z_i = \begin{cases} +1 & p \\ -1 & q \end{cases} \qquad \text{simulate?}$$

$$X_n = X_0 + \sum_{i=1}^{n} z_i = X_{n-1} + Z_n$$

$x_0 = 100$

$n = 7$ simulate?

$X_n = X_{n-1} + Z_n$

$n = 7$

$X_n = X_0 + \sum Z_i$

$Z_i = \begin{cases} +1 & p = \\ -1 & q \end{cases}$

$Z_1 = \text{rbinom}(1, p)$ 1, 1, p.

$Z_2 =$

$=$

$=$

$=$

$=$

?

$X_0 = 100$

$X_1 = X_0 + Z_1$

$X_2 = X_1 + Z_2$

7 R.S.

sample(

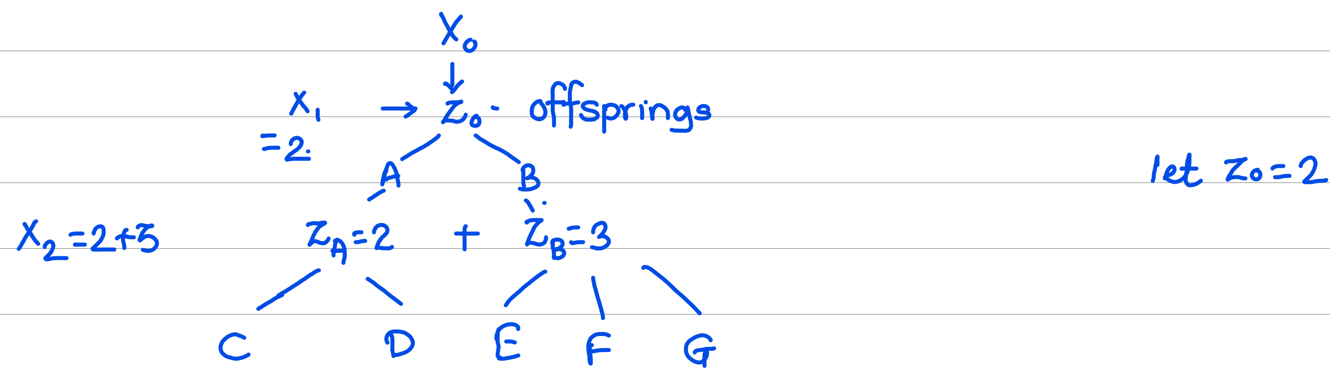
$Z_i = \begin{cases} +2 & 0.25 \\ -1 & 0.35 \\ 0 & 0.2 \\ 1 & 0.2 \end{cases}$

General R.W

Absent: 2001, 3, 6, 10, 16, 17, 22, 23, 33, 35, 38, 39, 43, 44, 45, 47, 50, 51, 55

Branching Process

(Galton-Watson Branching process)



$X_3 \Rightarrow Z_C + Z_D + Z_E + Z_F + Z_G$

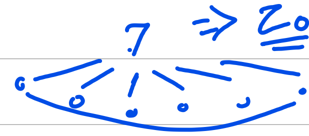
Consider the pop^n consisting of individuals able to produce offspring of the same kind.

Suppose that each individual have produced j offsprings with prob. P_j, $j \geq 0$, independent of each other.

② $Z_i \sim$ i.i.d discrete

X_0 = size of $zero^{th}$ generation

X_n = size of n^{th} generation



let $X_n = 10$

$$X_{n+1} = \sum_{i=1}^{X_n} Z_i = \text{size of } (n+1)^{st} \text{ generation}$$

✓ $X_n = 3$

$$\underline{X_{n+1}} \quad \begin{array}{c} A \\ \downarrow \\ Z_A \end{array} + \begin{array}{c} B \\ \\ Z_B \end{array} + \begin{array}{c} C \\ \\ Z_C \end{array}$$

$X_{n+1} = \sum_{i=1}^{3=X_n} Z_i$ ✓

$Z_0 = 0, 1, 2, \ldots$

$X_n = 0$

$X_{n+1} = ? \Rightarrow X_{n+1} = 0$

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

$$E(X_n) = E_{X_{n-1}}\left(E_Z\left(\sum_{i=1}^{X_{n-1}} Z_i / X_{n-1}=k\right)\right)$$

$$= E_{X_{n-1}}\left(\sum_{i=1}^{k} E(Z_i) / X_{n-1}=k\right)$$

as Z_i are i.i.d. $E(Z)=\mu$. $V(Z)=\sigma^2$

$$= E_{X_{n-1}}\left(k\cdot E(Z) / X_{n-1}=k\right)$$

$$= \mu \cdot E_{X_{n-1}}(X_{n-1})$$

$$E(X_n) = \mu\cdot E(X_{n-1}) = \mu\cdot\mu\cdot E(X_{n-2})$$

$$E(X_{n-1}) = \mu\cdot E(X_{n-2})$$

$$E(X_n) = \mu\cdot\mu\cdot \dots\ E(X_0) = \mu^n E(X_0)$$

Assume $E(X_0)=1$

$$\underline{E(X_n)=\mu^n}$$

$$V(X_n) = V\left(\sum_{i=1}^{X_{n-1}} Z_i\right)$$

$$= E_{X_{n-1}}\left(V_Z\left(\sum_{i=1}^{k} Z_i / X_{n-1}=k\right)\right) + V_{X_{n-1}}\left(E_Z\left(\sum_{i=1}^{k} Z_i / X_{n-1}=k\right)\right)$$

$$= E_{X_{n-1}}(k\cdot\sigma^2) + V_{X_{n-1}}(k\mu)$$

$$V(X_n) = \sigma^2\cdot E(X_{n-1}) + \mu^2 V(X_{n-1})$$

$X_0=1$ $$\underline{\underline{V(X_1)}} = \sigma^2 \underbrace{E(X_0)}_{=1} + \mu^2\cdot \underbrace{V(X_0)}_{=0} = \sigma^2$$

$$V(X_2) = \sigma^2 \cdot E(X_1) + \mu^2 \cdot V(X_1)$$
$$= \sigma^2 \mu + \mu^2 \sigma^2$$
$$= \mu\sigma^2 (1 + \mu)$$

$$V(X_3) = \sigma^2 (E(X_2)) + \mu^2 \cdot V(X_2)$$
$$= \sigma^2 \mu^2 + \mu^2 \cdot \sigma^2 (\mu + \mu^2)$$
$$= \sigma^2 (\mu^2 + \mu^3 + \mu^4)$$
$$= \mu^2 \sigma^2 (1 + \mu + \mu^2)$$

$$V(X_n) = \mu^{n-1} \sigma^2 (1 + \mu + \mu^2 + \dots + \mu^{n-1})$$
↳ Geometric Series.

$\mu < 1$

$$= \mu^{n-1} \sigma^2 \left(\frac{1 - \mu^n}{1 - \mu} \right) \checkmark$$

* Probability of extinction →

$X_n = 0$ for some $n \in \mathbb{N}$

starting from single individual $(X_0 = 1)$ but $X_n = 0$ ____

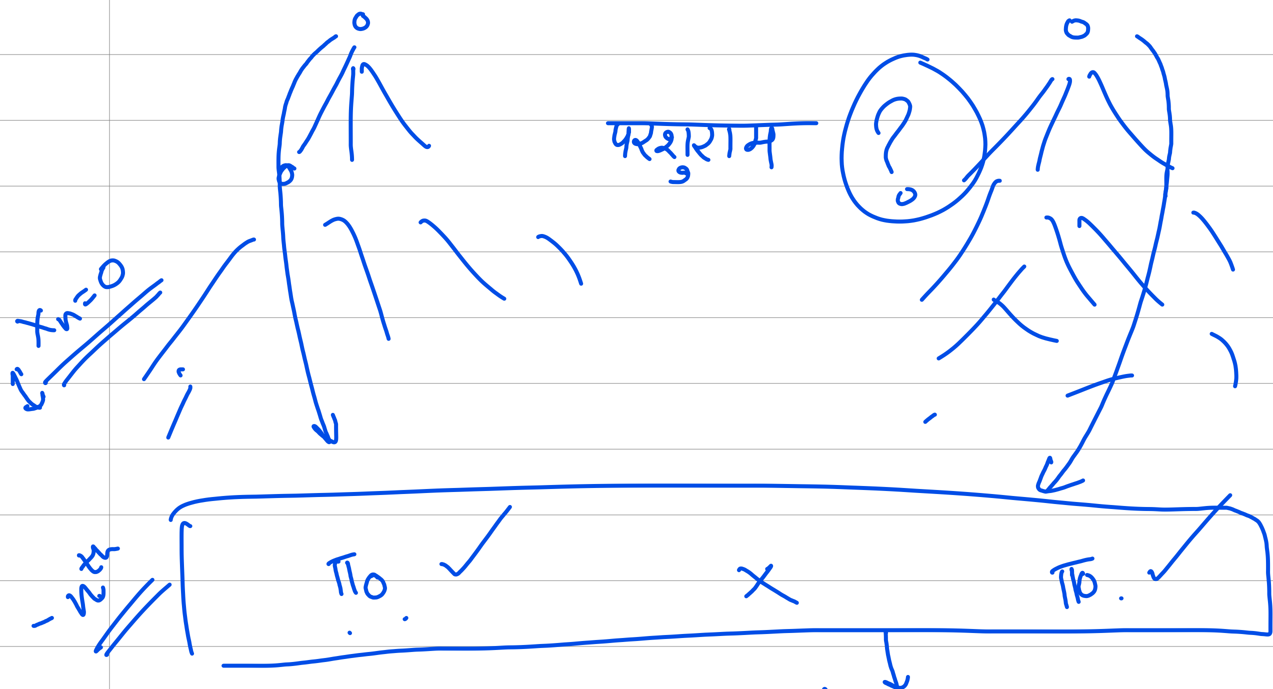
$$\pi_0 = \lim_{n \to \infty} P(X_n = 0) \quad ?$$

$x_0 = 1$

$$\pi_0 = P(\text{Pop}^n \text{ ever dies out}) \quad \leftarrow \text{single individual}$$

$$\pi_0 = \sum_{j=0}^{\infty} (\text{Pop}^n \text{ ever dies out} / X_1 = j) \cdot P_j$$

$$= \sum_{j=0}^{\infty} \pi_0^j \cdot P_j$$



$X_n = 0$

$X_{n+m} = 0 \quad \forall\, m \in \mathbb{N}$

PGF ?

$$P_X(t) = E(t^X) = \sum_x t^x P_x \quad ✓$$

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j \cdot P_j = P_Z(\pi_0)$$

PGF of Z

$$\pi_0 = P_Z(\pi_0)$$

by solving this eq^n we can get π_0 -

e.g. $Z_i \sim Bino(n=2, p=1/3)$

$\mu \Rightarrow n.p. = 2 \times 1/3 < 1$

$\pi_0 = P_Z(\pi_0)$

$t = P_Z(t) \Rightarrow t = \left(\frac{2}{3} + \frac{1}{3}.t\right)^2$

π_0 = Prob. of

$0 \le t \le 1$

$\cancel{4t=0}$

$$9t = 4 + 4t + t^2$$
$$0 = 4 - 5t + t^2$$
$$0 \Rightarrow 4 - 4t - t + t^2$$
$$0 = 4(1-t) - t(1-t)$$
$$0 \Rightarrow (4-t)(1-t)$$
$$\cancel{t=4} \text{ or } t = 1$$

min^n value of π_0 obtained by solving this

$$Z = \begin{cases} 0 & 0.3 \\ 1 & 0.3 \\ 2 & 0.4 \end{cases}$$

$t = \sum t^j \cdot p_j = t^0 \cdot 0.3 + t^1 \cdot 0.3 + t^2 \cdot 0.4$

$$t = 0.3 + 0.3t + 0.4t^2$$
$$10t = 3 + 3t + 4t^2$$
$$4t^2 - 7t + 3 = 0$$
$$\Rightarrow 4t^2 - 4t - 3t + 3 = 0$$
$$\Rightarrow 4t(t-1) - 3(t-1) = 0$$
$$\Rightarrow t = \frac{3}{4} \text{ or } t = 1 \quad \Rightarrow \pi_0 = 3/4$$

Absu 2001, 4, 6, 10, 15, 16, 22, 25, 30, 35, 38, 39, 43, 44, 45, 47, 50, 55

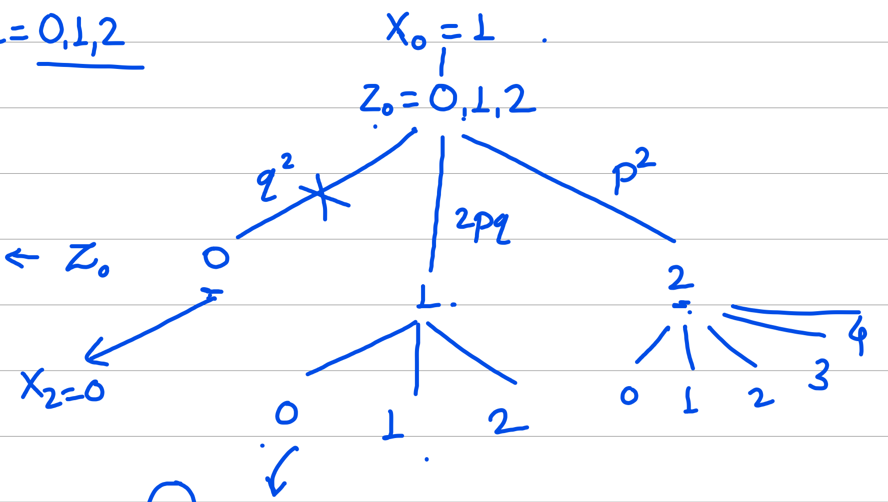
* Calculate the probability that the pop^n become extint for the first time in 3^{rd} generation, consider $X_0 = 1$, $\underline{Z \sim B(2,p)}$

→ So we have to find out $P(\underline{X_3 = 0} / X_2 \neq 0, \underline{X_1 \neq 0}, \underline{X_0 = 1})$

$Z \sim Bino(2, p)$ $\quad Z = \underline{0, 1, 2}$

$$\boxed{\begin{aligned} P(Z=0) &= q^2 \\ P(Z=1) &= 2pq \\ P(Z=2) &= p^2 \end{aligned}}$$

$\underline{X_1} \leftarrow Z_0$



Hint?

$X_1 = 2 \Rightarrow X_2 = \sum_{i=1}^{X_1} Z_i = \sum_{i=1}^{2} Z_i$ $\qquad Z_i \sim B(2,p)$

✓ $\underline{X_2 \sim B(4,p)}$

Obtain it?

$$\boxed{\begin{aligned} &✓\, P(X_3 = 0 / X_2 \neq 0, X_1 \neq 0, X_0 = 1) \\ &= ? \end{aligned}}$$

$X_0 = 1$

$X_1 \sim Bin(2, p)$

$X_2 / X_1 \neq 0 \sim Bin(2 \times X_1, p)$ ✓

✓ $X_3 / X_2 \neq 0, X_1 \neq 0 \sim Bin(2 \times X_2, p)$

Random Walk

→ $X_n = X_0 + \sum_{i=1}^{n} Z_i$ $\qquad Z_i$ - $\underline{i.i.d.}$ $\qquad \underset{r.w.}{X_n}$

$X_n = X_{n-1} + Z_n$ i.i.d. $\qquad Z_n = \underline{X_n - X_{n-1}}$ $\qquad$ indep. increments

* Counting Process

Statespace Time domain
discrete cont

$X_n \rightarrow \{X(t), t \in I\}$

$I = [a,b)$, $[0,\infty)$ etc.

$\{N(t), t \in I\}$ — state discrete, time cont.

e.g. no. of events occured during $(0,t]$
no. of customers arrived in bank / barber / Ticket coun
no. of deaths occurred due to Covid-19

① $N(t) \geq 0$
② $N(t)$ integer valued
③ $s \leq t \Rightarrow N(s) \leq N(t)$ 10-3 $\leq$ 10-5
④ for $s<t$, $N(t)-N(s)$ increment $(s,t]$ 20 25

* Counting Process with independent increments
for $s<t$, $N(t)-N(s)$ is increment,
independent increment if $N(s)$ & $N(t)-N(s)$ are indepen

Do you remember any stoc. process?

$\underbrace{0 \quad\quad s}_{N(s)} \underbrace{\quad\quad t}_{N(t)-N(s)}$

* Counting Process with stationary increments

$s \leq t$ $N(t)-N(s)$ & $N(t+u)-N(s+u)$ if both have same dist^n

if dist^n of no. of events occured in an interval depends on length of interval only
$2 \leq 3$ $N(3)-N(2)$ & $N(13)-N(12)$ if both have same dist^n
$(2,3] \rightarrow 1$ $(12,13] \rightarrow 1$.

* Poisson Process :-

Def^n ① The counting process $\{N(t), t \geq 0\}$ is said to be a poisson process having rate λ, $\lambda > 0$ if
① $N(0) = 0$
② The process has indep. & stationary increments

0,t S,S+t $\sim Poi(\lambda t)$

N(t) $N(S+t)-N(S)$

③ The no of events in any interval of length t . is poisson distributed with parameter λt

$(0,t)$ $N(t) \sim Poi(\lambda t)$, $(S,S+t]$, $N(t+S)-N(S) \sim Poi(\lambda t)$

Def^n ②

Equivalence bet^n two def^{n}s

counting ✓

① $N(t) \geq 0$

② Integer value

③ $s \leq t$ $N(s) \leq N(t)$

④ $(s,t]$, $N(t)-N(s)$ incre

stoch ✓ poiss. process ✓

① $N(0)=0$

② incr. ind. & stationary

③ ____ $\sim Poi(\lambda(\text{length of interval}))$

Poisson process example

$\{N(t), t \geq 0\} \longrightarrow \lambda = 2/\text{day}$ ⑤ ?

poiss

$N(0)=0$

✓ $(0,5]$ $N(5)=?$

$\sim Poi(\lambda \cdot 5) = Poi(10)$

$P(N(5)) = N(5) \sim Poi(10)$

$0,1,2,\ldots$

$E(N(5)) = 10$

Def^{n}② Let $\{N(t), t \geq 0\}$ be a counting process is said to be Poisson process with rate λ, $\lambda > 0$ if

(i) $N(0)=0$

② indep. & stationary increments.

③ $P(N(h)=1) = P_1(h) = \lambda h + o(h)$

$o(h) \to 0$ as $h \to 0$

h small

④ $P(N(h) \geq 2) = o(h) = \sum_{k=2}^{\infty} P_k(h)$

equivalence

Def ① → Def ②

for first two conditions no need to prove

To prove Def ② → ③ cond^ $\boxed{P_1(h) = \lambda h + o(h)}$

$P_n(t) = P[N(t) = n]$

$P_1(h) = P(N(h) = 1)$ $(0, h]$ $N(h) \sim Poi(\lambda h)$

$= e^{-\lambda h} \cdot \lambda h$

$= \lambda h \left[1 - \lambda h + \frac{(\lambda h)^2}{2!} - \frac{(\lambda h)^3}{3!} + \cdots\right]$ $o(h) = o(h) + o(h)$

$= \lambda h + \underbrace{o(h)}_{\to 0}$

$\cancel{P_2(h)}$ $\sum_{k=2}^{\infty} P_k(h) = \sum_{k=2}^{\infty} \frac{e^{-\lambda h} (\lambda h)^k}{k!}$

$P[N(h) \geq 2]$

$= 1 - P_1(h) - P_0(h)$

$= 1 - e^{-\lambda h} \lambda h - e^{-\lambda h}$

$= 1 - (\lambda h + 1) \cdot e^{-\lambda h}$

$= 1 - (\lambda h + 1) \left[1 - \lambda h + \frac{(\lambda h)^2}{2} + \cdots\right]$

$= \cancel{1} - \lambda h \left[1 - \lambda h + \frac{(\lambda h)^2}{2!} + \cdots\right] - \left[\cancel{1} - \lambda h + \frac{(\lambda h)^2}{2!} + \cdots\right]$

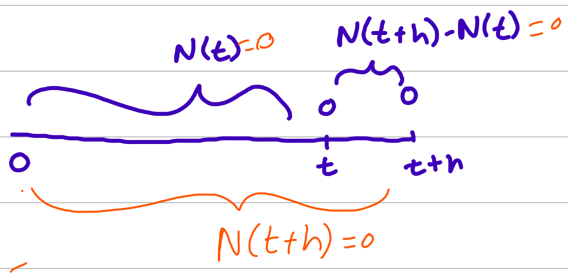
$= o(h)$

Def^n ② $\to$ Def^n ①

Def^n ② ① $N(0)=0$ ② Stationary & indep. ③ $P_1(h) = \lambda h + o(h)$ ④ $\sum_{k=2}^{\infty} P_k(h) = o(h)$ $P(N(h)\geq 2)$

def ① ① —— ② —— ③ increment $\sim$ Poi(λ. length)

$$P_n(t) = P[N(t)=n]$$



$$P_0(t+h) = P[N(t+h)=0]$$

$$= P[N(t)=0, N(t+h)=0]$$

$$= P[\underbrace{N(t)=0}_{(0,t]}, \underbrace{N(t+h)-N(t)=0}_{(t,t+h]}]$$ ind. increment

$$= P[N(t)=0] \cdot P[N(t+h)-N(t)=0]$$

$$= P[N(t)=0] \cdot P_0(h)$$

$$= P[N(t)=0] \cdot [1 - P_1(h) - \sum_{k=2}^{\infty} P_k(h)]$$ ③ ④

$$P[N(t+h)=0] = P(N(t)=0] \cdot [1 - \lambda h + o(h)]$$ $\cancel{+o(h)}$

$$\to \lim_{h\to 0} \frac{P(N(t+h)=0) - P(N(t)=0)}{h} = \lim_{h\to 0} \frac{-\lambda \cancel{h} P(N(t)=0)}{\cancel{h}} + \frac{o(h)}{h}$$

$$P_0'(t) = -\lambda P_0(t)$$

$$\frac{1}{P_0(t)} \cdot P_0'(t) = -\lambda$$

$$\frac{d}{dt} \log P_0(t) = -\lambda$$

$$\int \frac{d}{dt} \log P_0(t) = \int_0^t -\lambda \, dt$$

$$\log P_0(t) = -\lambda t + C$$

$$P_0(t) = e^{-\lambda t} \cdot e^{C}$$

① $N(0)=0$, $\Rightarrow$ $P_0(0) = 1$ $\Rightarrow$ $P_0(0) = e^{-\lambda \cdot 0} e^{C} = e^{C} = 1$

$\Rightarrow C = 0$

$$P_0(t) = e^{-\lambda t}$$ ✓

$P_0(t) = e^{-\lambda t}$

$N(t) \sim Poi(\lambda t)$

Poisson $x = 0, 1, 2, \ldots$ 0, 1,

0 ——— t ——— $t+h$ (n)

$$P_n(t+h) = P[N(t+h)=n]$$

$$= P[N(t+h)=n / N(t)=n] \; P[N(t)=n] + P[N(t+h)=n / N(t)=n-1] \; P[N(t)=n-1] + P[N(t+h)=n / N(t)\le n-2] \; P[N(t)\le n-2]$$

0, 1

0 ——— t ——— $t+h$

P_n

$$= P_0(h) \cdot P_n(t) + P_1(h) \cdot P_{n-1}(t) + \sum_{k\geq 2}^{\infty} P_k(h) \cdot P_{n-k}(t)$$

$$P_n(t+h) = [1-\lambda h] \cdot P_n(t) + \lambda h \, P_{n-1}(t) + o(h) \qquad + o(h)$$

$$\lim_{h\to 0} \frac{P_n(t+h) - P_n(t)}{h} = \lim_{h\to 0} \frac{-\lambda \cancel{h} \cdot P_n(t)}{\cancel{h}} + \frac{\lambda \cancel{h} P_{n-1}(t)}{\cancel{h}} + \frac{o(h)}{h}$$

$\frac{o(h)}{h} \to 0$

$$P_n'(t) = -\lambda [P_n(t) - P_{n-1}(t)]$$

$$P_n'(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

$$e^{\lambda t} P_n'(t) + \lambda e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

$$\frac{d}{dt} e^{\lambda t} P_n(t)$$

$$\boxed{\frac{d}{dt}\left(e^{\lambda t} P_n(t)\right) = \lambda e^{\lambda t} P_{n-1}(t)} \quad \text{——— } (*)$$

$n=1$

$$\frac{d}{dt}\left(e^{\lambda t} P_1(t)\right) = \lambda e^{\lambda t} P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\int_0^t \qquad\qquad \int_0^t \lambda$$

$N(0)=0$ w.p. 1

$N(0)=1$ w.p. 0

$$e^{\lambda t} P_1(t) = \lambda t + C$$

put $t=0$ $\Rightarrow$ $e^{\lambda \cdot 0} P_1(0) = 0 + C$

$\Rightarrow$ $0 = C$

$$e^{\lambda t} P_1(t) = \lambda t$$

$$\Rightarrow \quad P_1(t) = e^{-\lambda t} \lambda t$$

Abs. 2001, 5, 6, 9, 10, 14, 15, 16, 27, 33, 35, 39, 41, 43, 44, 45, 51, 54, 55

$P_0(t) = e^{-\lambda t}$ $\qquad$ $P_1(t) = e^{-\lambda t} \lambda t$

Assume it for $(n-1)$

$$P_{n-1}(t) = \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \quad \text{——— } (**)$$

from $(*)$

$$\frac{d}{dt} e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

from $(**)$

$$\frac{d}{dt} e^{\lambda t} P_n(t) = \lambda \cancel{e^{\lambda t}} \frac{\cancel{e^{-\lambda t}} (\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

$$\int_0^t \frac{d}{ds} e^{\lambda s} P_n(s)\, ds = \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!}\, ds$$

$$e^{\lambda t} P_n(t) = \left[\frac{\lambda^n}{(n-1)!}\,\frac{s^n}{n}\right]_0^t = \frac{(\lambda t)^n}{n!}$$

$$\boxed{P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

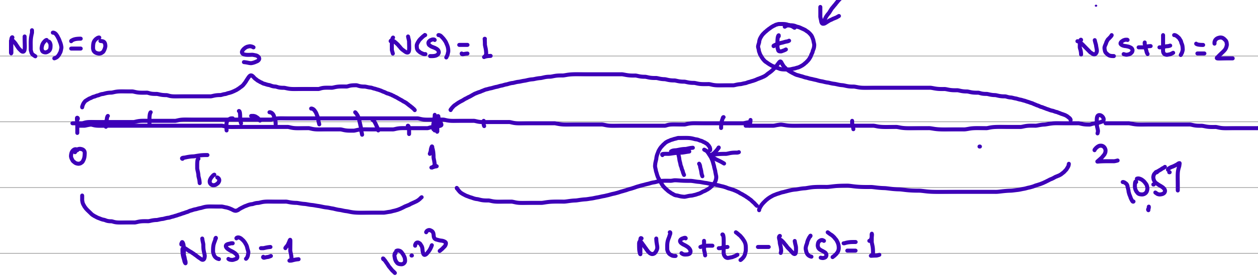
$$N(t) \sim Poi(\lambda t)$$

Inter event

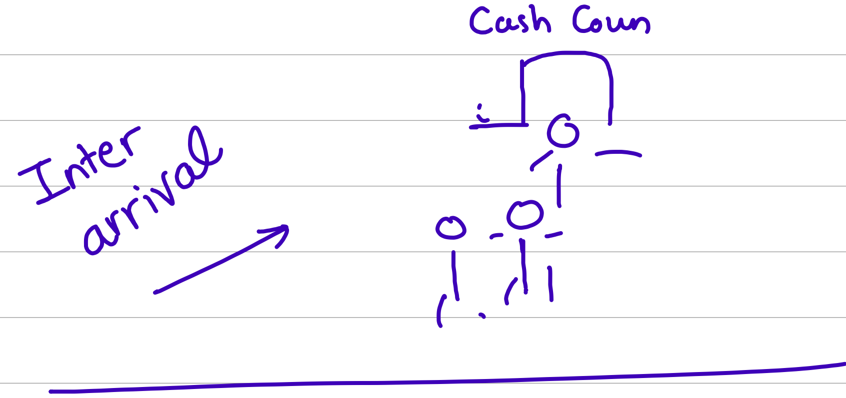
* <u>Interarrival</u> time between poisson events :-

$N(v)=1$, $s<v<s+t$

$N(u)=0$, $u<s$



Bank



$T_0 \rightarrow$
$T_1 \rightarrow 2^{nd}$

T_0, T_1 interarrival

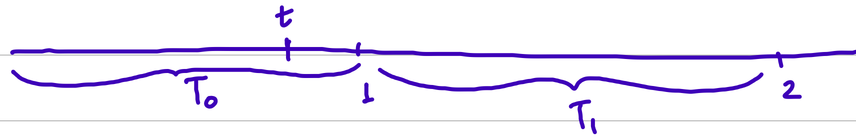
$N(T_0) = 1$

$N(T_0+T_1) = 2$

....

T_0+T_1 arrival

- $N(t) \sim$ follows poisson process



$$F_{T_0}(t) = P[T_0 \le t] = 1 - P[T_0 > t]$$
$$= 1 - P[N(t) = 0]$$
$$= 1 - e^{-\lambda t}$$

$$T_0 \sim exp(\lambda)$$

$$F_{T_1}(t) = P[T_1 \le t / T_0 = s] = 1 - P[T_1 > t / T_0 = s]$$

? no. of events =

$$= 1 - P[N(t) = 0]$$
$$\cancel{P[N(t+s) - N(s) = 0]} \quad \text{Same}$$
$$= 1 - e^{-\lambda t}$$

$$T_1 \sim exp(\lambda)$$

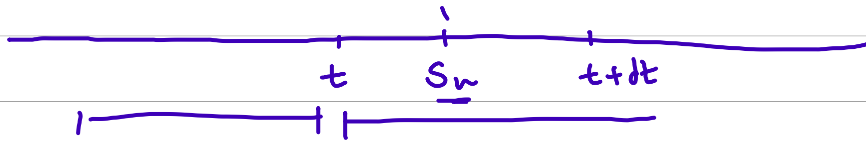
$\Rightarrow$ $T_0, T_1, \ldots$ $\underline{\underline{exp(\lambda)}}$ $\qquad$ $\underline{\underline{E(T_i) = 1/\lambda}}$

* Arrival Time

$S_n \Rightarrow$ Arrival time of n^{th} customer/event

$$S_n = \sum_{i=0}^{n-1} T_i$$

T_i's are i.i.d $exp(\lambda)$, $N(t) \sim Poi(\lambda t)$



$$\underline{P(t < S_n < t+\delta t)} = P\big(\underbrace{N(t)=n-1}_{(0,t]},\ \underbrace{N(t+\delta t)-N(t)=1}_{(t,\,t+\delta t]}\big) \qquad \text{ind. incre.}$$

$$= \underline{P(N(t)=n-1)} \cdot P(N(t+\delta t)-N(t)=1)$$

$$\delta t$$

$$F(t+\delta t) - F(t) = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \cdot e^{-\lambda \delta t}\, \lambda\, \delta t$$

$$\underline{f(t)} = \lim_{\delta t \to 0} \frac{F(t+\delta t)-F(t)}{\delta t} = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \lim_{\delta t \to 0} \frac{e^{-\lambda \delta t}\lambda \cancel{\delta t}}{\cancel{\delta t}}$$

$$= \frac{e^{-\lambda t}}{(n-1)!} \lambda^n t^{n-1}$$

$$= \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}$$

$$S_n \sim \text{Gamma}(\underline{n, \lambda})$$

* Conditional distribution of arrival time

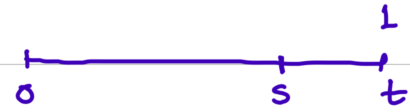
inter arrival
$\Rightarrow$ exp
arrival
$\Rightarrow$ gamma
cond' arrival
$\Rightarrow$ uniform

$$x_i \sim U(0,t) \qquad f(x) = \frac{1}{t} \quad 0<x<t$$

$$f(x_{(1)}, x_{(2)}, \ldots, x_{(n)}) = \frac{n!}{t^n}$$

$\cdot\ N(t) = 1$

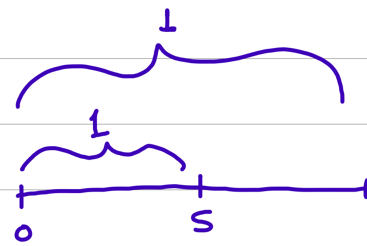
S_n = arrival of n^{th} = $\sum_{i=0}^{n-1} T_i \Rightarrow S_1 = T_0$



For $s \leq t$

$$F(s) = P(S_1 < s \,/\, N(t) = 1)$$

$$= \frac{P[T_0 < s,\ N(t) = 1]}{P[N(t) = 1]}$$



$$= \frac{P[N(s) = 1,\ N(t) - N(s) = 0]}{P[N(t) = 1]}$$

ind. incr

$$= \frac{P[N(s) = 1] \cdot P[N(t) - N(s) = 0]}{P[N(t) = 1]}$$

$$= \frac{e^{-\lambda s}\,\lambda s \quad e^{-\lambda(t-s)}}{e^{-\lambda t}\,\lambda t}$$

$$= \frac{s}{t}$$

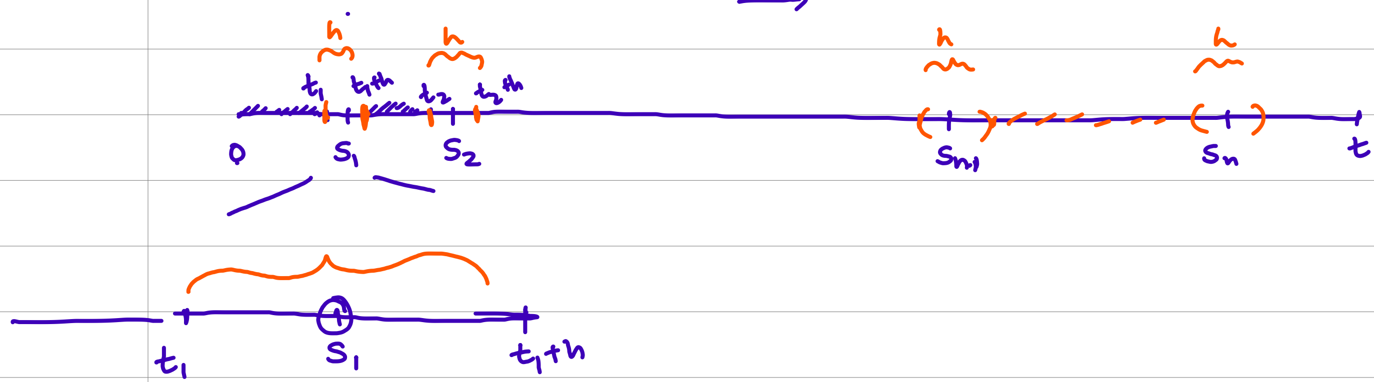
$$F(s) = \frac{s}{t}.$$

$U(0,t) \Rightarrow F(x) = \frac{x}{t}$

$S_1 / N(t)=1 \sim U(0,t)$ ✓

* Given that $N(t)=n$, n arrival times $S_1, S_2, \ldots, S_n$ have same distⁿ as of the order statistics of n indep. random variables from $U(0,t)$

$0\text{-}t \Longrightarrow$ length $\underline{t-nh}$ 0



let t_i such that $0<t_1<S_1<t_1+h<t_2<\ldots<t_n<S_n<t_n+h<t$

$$P[t_i \le S_i \le t_i+h,\ i=1{:}n \,/\, N(t)=n]$$

$$= \frac{P[N(t_i+h)-N(t_i)=1 \ \ni\ i=1{:}n,\ \text{no event elsewhere in } (0,t)]}{P[N(t)=n]} \qquad \overset{t-nh}{\Rightarrow}$$

$$= \frac{[\lambda h\, e^{-\lambda h}]^n \ e^{-\lambda(t-nh)}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}} \qquad P[N(t-nh)=0]$$

$$= \frac{n!\ \cancel{\lambda^n}\ \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}\ (\cancel{\lambda} t)^n} \cdot h^n\ \cancel{e^{-\lambda nh}}\ \cancel{e^{+\lambda nh}}$$

$$= \frac{n!}{t^n}\ h^n$$

$$\lim_{h\to 0} \frac{1}{h^n} P[t_i \le S_i \le t_i+h,\ i=1{:}n \,/\, N(t)=n] = \frac{n!}{t^n}$$

$$\underline{\underline{f(S_1, S_2, \ldots, S_n / N(t)=n)}} = \frac{n!}{t^n}$$

Conditional joint pdf of arrival times is equi to joint pdf of order stats $U(0,t)$

Absentee: 2001, 6, 9, 10, 12, 15, 16, 22, 23, 31, 33, 35, 39, 43, 44, 45, 47,

Poisson Process — Branching Process

① ①

②

③ $\sim Poi(\lambda t)$

$N(t+s) - N(s) \sim Poi(\lambda t)$

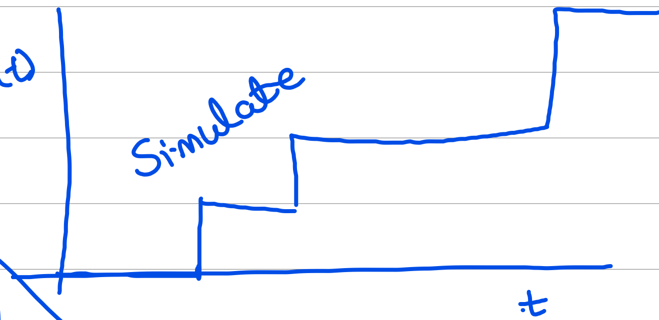
Simulate Poisson?

$N(t) = ?$ $t > 0$

② Interarrival → $expo(1/\lambda)$?

Arrival times ~ $Gamma(n, 1/\lambda)$

$N(t)$



③ Conditional distⁿ of arrival times

$S_1, S_2, \dots, S_n / N(t) = n$

$\sim X_{(1)}, X_{(2)}, \dots, X_{(n)} \sim U(0,t)$

Random Sample from

10 · U(0,7)

order

$S_1, S_2, \dots, S_n / N(7) = 10$ (0, 7)

$\{N(t), t \geq 0\}$ $\lambda = 2/hr$ $t = 15\ hr$

(0 — 15) $N(t) = ?$ 30

$N(22) = 0$

$N(37) = 1$

	Arrival	Interarrival
①	23 min	23 min
②	55 min	32 min
③	75 min	20 min

$\lambda = 2/hr$ $t = 15\ hr$

$T_0 \sim exp(mean\ 1/2)$

$T_1 \sim ex$

$T_2 \sim$

⋮

$T_{100} \sim$

$N(t)$

$(0, t]$

0 – 23	0
23 – 55	1
55 – 75	2

$N(t) = 0$ $0 \leq t < 23$

$\quad\quad 1$ $23 \leq t \leq 55$

Abs. 1, 6 to 10, 13, 15, 22, 23, 25, 27, 33 to 35, 37, 39, 42 to 45, 47, 50, 54 to 56

Poisson Process simulation ($\lambda =$)

① Expo ② Uni - Cond^n ③ Poi ~~X~~

$\lambda = 2$, $0-t$, $t = 15$

Interarrival $\sim exp(1/\lambda)$

$T_i \sim exp(1/\lambda)$

simulate T_i, $n \geq \lambda t$

arr = Arrival time - cumsum

15 ~~15.23~~

14.90

$$N(t) = \begin{cases} 0 & 0 \leq t < S_1 \\ 1 & S_1 \leq t < S_2 \\ 2 & \\ 3 & S_n \\ & 14.90 \leq t \leq t = 15 \end{cases}$$

no. of arrivals 0-t

dataframe(c(0,arr), c(arr,t), 0:n)

$S_1, S_2 \ldots S_n / N(t) = n \sim X_{(1)}, X_{(2)}, \ldots, X_{(n)}$
where $X_{(i)}$ is order statistics of $U(0,t)$
$i = 1:n$

S = sort (runif(n, 0, t))

$$N(t) = \begin{cases} 0 \leq t \leq S & 0 \\ \vdots & \vdots \\ S \leq t \leq t = & n \end{cases}$$

$0, t$

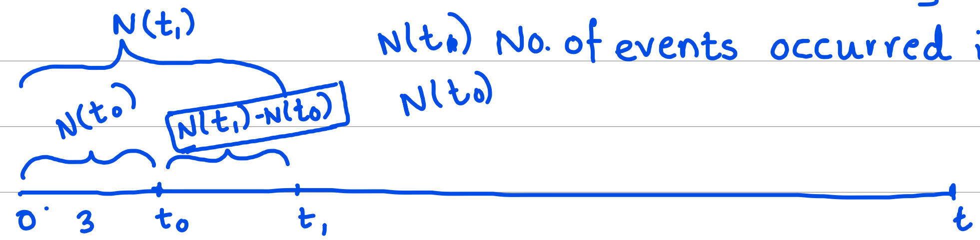
③ sample path $\quad t_0 \le t_1 \le t_2 \le \cdots \le t_n$

simulate $\quad N(t_0)\ N(t_1)\ N(t_2) \ldots N(t_n)$

$N(t_1)$ No. of events occurred in $(0, t_1]$

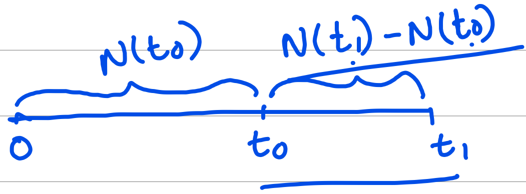
$N(t_0)$ $\quad (0, t_0]$

① $N(0)=0$
② P
③



$N(t_0) \sim Poi(\lambda \cdot t_0)$ $\qquad N(t_0) = 2$

$N(t_1) \sim Poi(\lambda t_1)$ $\qquad N(t_1) / N(t_0) = 2 \ \sim Poi($



increment $\sim Poi(\lambda(t_1 - t_0))$

$t_0 \quad N(t_0) \sim Poi(\lambda t_0)$

$t_1 \quad \underline{N(t_0)} + \underline{N(t_1) - N(t_0)} \sim Poi(\lambda(t_1 - t_0))$

$t_2 \quad N(t_1) + \underline{N(t_2) - N(t_1)} \sim Poi(\lambda(t_2 - t_1))$

$\vdots$

Simulate the Poisson process at time points t with rate without simulating its interarrival time.

i) t = 1.5,2.2,3.8,7.5,8.8 & λ= 1 ii) t = 1.23,2.21,2.83,6.05,7.08,17.8 & λ= 1.5

time pts.	inte	λ.int.			
1.5	1.5	$\to$ Poi( )	N(1.5)		
2.2	2.2-1.5	Poi( )		+	= N(2.2)
3.8	3.8-2.2				
7.5	7.5-3.8				
8.8	8.8-7.5				

Let {Xn,n N} be a Bienayme-Galton-Watson Branching Process with offspring distribution given by P[= 0] = 0.2,P[= 1] = 0.3,P[= 2] = 0.2,P[= 3] = 0.3. If X0 = 2,then realize X1,X2,··· ,X5 where Xn denote size of n

th generation.

$X1 =$ X_2 $X6$

Branching $X_n = \sum_{i=1}^{X_{n-1}} Z_i$

$P(Z_i = Z) = 0.2$ $Z = 0$.

0.3 $Z = 1$.

0.2 $Z = 2$.

0.3 $Z = 3$.

$X_0 = 2$ 2

$X_1 = Z_1 + Z_2$

{ x_1 =sum sample(x_0, 0:3, prob=c(0.2 0.3 0.2 0)

g (s) = 0.25 + 0.35s + 0.3s2 + 0.1s3

$P[Z = 0] = 0.25$

1	0.35
2	0.3
3	0.1

Absent :- 2006 9, 10, 33-35, 39, 43, 44, 45, 47, 55