

Unit - I - Real Numbers \mathbb{R}

~~Groups Mat~~
~~($\mathbb{R}, +, \cdot$)~~

* Algebraic Properties of \mathbb{R} :-Add[~] Mult[~]On set of \mathbb{R} there are two binary operators $\underline{+}$ & $\underline{\cdot}$

These two operations follows few properties :-

A1) Commutative property of addition

$$a+b = b+a \quad \forall a, b \in \mathbb{R}$$

A2) Associative property of add[~]

$$(a+b)+c = a+(b+c) \quad \forall a, b, c \in \mathbb{R}$$

A3) Existence of zero element (Additive Identity) ✓

$$a+0 = 0+a = a \quad \forall a \in \mathbb{R}$$

A4) Existence of negative element (Additive Inverse)

$$a+(-a) = (-a)+a = 0 \quad \forall a \in \mathbb{R}$$

✓ M1) Commutative property of multiplication

$$a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$$

M2) Associative p of multi.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in \mathbb{R}$$

M3) Existence of unit element (Multiplicative identity)

$$a \cdot 1 = 1 \cdot a = a \quad \forall a \in \mathbb{R}$$

M4) Existence of multiplicative inverse / Reciprocals ✓

$$a \cdot \left(\frac{1}{a}\right) = \frac{1}{a} \cdot a = 1 \quad \forall a \in \mathbb{R} - \{0\}$$

D) Distributive property of multiplication over add[~]

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a$$

$$= a \cdot b + a \cdot c$$

$$\forall a, b, c \in \mathbb{R}$$

* Order Properties of IR

Why do we study Real Analysis?

seq of func
 $\sum f_n(x) = \sum \frac{x_i}{n}$

Convergence of Series of func's

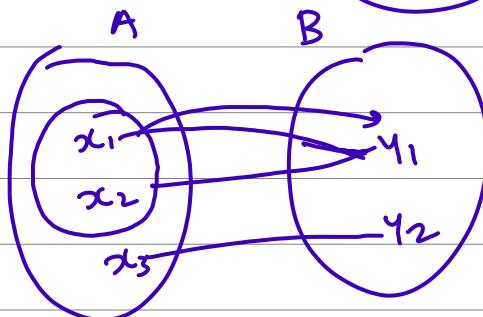
$$\text{CLT} \rightarrow \frac{\sum x_i - \mu}{\sigma} \rightarrow \text{N}(0, 1)$$



$$\left(\frac{\sum x_i}{n} \right) \rightarrow \bar{x}$$

Set Func

func



Prob.

input element \rightarrow Set $n(A) \in \mathbb{N}$
 classi $n(S)$

input Set \rightarrow Real

ST-201

(A)

Borel func

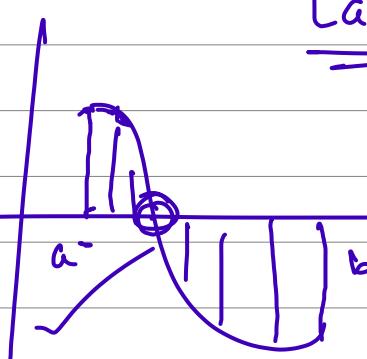
$$x^3 + 2x + 3 = 0 \quad x=? \quad \mathbb{R}$$

[a, b]

closed

ST-705
 Numerical Methods

Cont?



Serier $\rightarrow P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$E(X) = \sum_x \frac{e^{-\lambda} \lambda^x}{x!} x$$

$$= e^{-\lambda} \sum_x \frac{\lambda^x}{x!} x$$

Geometric $P(X=x) = pq^{x-1}$

$$E(X) = \sum x \cdot pq^{x-1} = \frac{1}{p}$$

$$\begin{aligned} &= p \cdot \sum x \cdot q^{x-1} \\ &= p \cdot \frac{1}{(1-q)^2} \\ &= p \cdot \frac{1}{p^2} = \frac{1}{p} \end{aligned}$$

$$\sum x^n = \frac{1}{1-x} \quad |x|<1$$

$$\frac{d}{dx} \sum x^n = \frac{d}{dx} \frac{1}{1-x}$$

$$\text{&} \sum n x^{n-1} = \frac{1}{(1-x)^2}$$

Inverse fun^c $\leftarrow X$ Random Variable

~~ST-201~~
Probability
~~ST-301~~
Asymptotic

- ① Unbiased
- ② Suff
- ③ Eff
- ④ Consistent \rightarrow as $n \rightarrow \infty$

$$\bar{x} \rightarrow \mu \quad \frac{1}{n} \sum x_i \rightarrow \mu$$

$$X_{(n)} \xrightarrow{\text{C.E.}} N(0, \sigma^2)$$

$$2\bar{X} \rightarrow 2\mu$$

* Order Properties of \mathbb{R}

\mathbb{P} set of +ve nos $\in \mathbb{R}$, $\mathbb{P} \subseteq \mathbb{R}$, \mathbb{R}^+

\mathbb{R}^+ satisfies following properties

- ① if $a, b \in \mathbb{R}^+ \Rightarrow a+b \in \mathbb{R}^+$
- ② if $a, b \in \mathbb{R}^+ \Rightarrow a \cdot b \in \mathbb{R}^+$

- ③ if $a \in \mathbb{R}$, then exactly one of the following is true:-

$$\begin{array}{c} \text{---} \\ \text{G.S} \end{array} \quad \begin{array}{c} a \in \mathbb{R}^+, \\ \text{---} \\ \times \end{array} \quad \begin{array}{c} a=0, \\ \text{---} \\ \times \end{array} \quad \begin{array}{c} -a \in \mathbb{R}^+, \\ \text{---} \\ \checkmark \end{array}$$

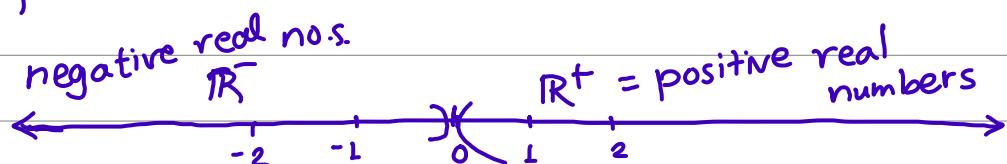
$a \in \mathbb{R}$ $\begin{matrix} > 0 & < 0 & = 0 \end{matrix}$

$\begin{matrix} a \in \mathbb{R}^+ & a \in \mathbb{R}^- & -a \in \mathbb{R}^+ \end{matrix}$

Law of
Trichotomy

Let $a, b \in \mathbb{R}$

- (a) If $a-b \in \mathbb{R}^+ \Rightarrow a>b$ or $b<a$
- (b) If $a-b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a>b$ or $b \leq a$



✓ $\mathbb{R}^+ = \{x / x > 0, x \in \mathbb{R}\}$

$\mathbb{R}^- = \{x / x < 0, x \in \mathbb{R}\}$

Proof :- (a) $a, b \in \mathbb{R}$, $a-b \in \mathbb{R}^+ \Rightarrow a-b > 0 \Rightarrow a > b$

(b) $a, b \in \mathbb{R}$ $a-b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a-b \geq 0 \Rightarrow a \geq b$

Theo:- Let $a, b, c \in \mathbb{R}$

① If $a > b$ & $b > c \Rightarrow a > c$ (Transitivity)

② If $a > b \Rightarrow a+c > b+c$

③ If $a > b$, $c > 0 \Rightarrow a \cdot c > b \cdot c$

If $a \geq b$, $c < 0 \Rightarrow a \cdot c < b \cdot c$

① If $a > b$, & $b > c$

$$a - b > 0 \text{ & } b - c > 0$$

$$\Rightarrow a - b \in \mathbb{R}^+ \text{ & } b - c \in \mathbb{R}^+$$

$$\Rightarrow (a - b) + (b - c) \in \mathbb{R}^+$$

$$\Rightarrow (a - c) \in \mathbb{R}^+$$

$$\Rightarrow a - c > 0$$

$$\Rightarrow a > c$$

(by order properties)

QED

② $a - b \in \mathbb{R}^+$

$$a - b + c - c \in \mathbb{R}^+$$

$$(a + c) - (b + c) \in \mathbb{R}^+$$

$$a + c > b + c$$

③ $a > b$, $c > 0$

to prove $a \cdot c > b \cdot c$

$$\begin{array}{c} a > b, \\ \underline{a - b \in \mathbb{R}^+} \end{array}$$

$$c(a - b) = c \cdot a - c \cdot b \Leftarrow \text{as } c > 0$$

$$\cancel{c a - c b > 0} \quad \Leftarrow$$

$$ca > cb$$

similarly $a > b$, $c < 0 \Rightarrow ac < bc$

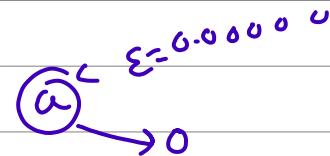
Theo:-

If $a \in \mathbb{R}$, such that $0 \leq a < \varepsilon \Rightarrow \varepsilon \in \mathbb{R}$, $\varepsilon > 0$

$$\Rightarrow a = 0$$

$$\varepsilon = 0.005$$

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By method of contradiction.

Assume $a > 0 \Rightarrow (as 0 \leq a < \varepsilon) \Rightarrow a$ is positive
 $\Rightarrow \frac{a}{2}$ is positive

we can assume $\varepsilon = a/2 < a$

for any $\varepsilon > 0$, $0 \leq a < \varepsilon$ but for $\varepsilon_0 \Rightarrow 0 < \varepsilon_0 < a$

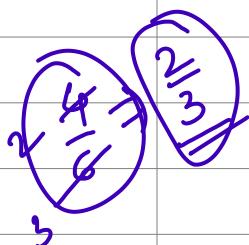
which contradicts to our assumption. \square

Rational Nos. :- $Q = \{x / x = p/q, p, q \in \mathbb{Z}, q \neq 0\}$

* Theo. There doesnot exist any rational no. $\sqrt{2} \in Q$

$$\underline{\underline{r^2 = 2}}$$

\Rightarrow Assume that $\sqrt{2}$ is rational no.



$$\sqrt{2} = \frac{p}{q}$$

common divisor of p, q is 1
 $\hookrightarrow (p, q) = 1$.

$$\Rightarrow 2 = \frac{p^2}{q^2}$$

$$\Rightarrow 2q^2 = p^2 = p \cdot p$$

$$\rightarrow p = 3 \quad p^2 = 9 \quad p = 4 \quad p^2 = 16$$

$\Rightarrow p$ is divisible by 2

$$\Rightarrow p = 2 \cdot m$$

$$\Rightarrow p^2 = 2^2 \cdot m^2 = 4m^2$$

~~$p = \text{even}$~~

$$\Rightarrow 2q^2 = 4m^2$$

$$\Rightarrow q^2 = 2m^2$$

$\Rightarrow q$ is divisible by 2 (****)

$(p,q) = 1$ but here $(p,q) = 2$

which contradicts to our assumption -

$$\begin{aligned}
 & \text{Q} \quad \frac{\text{even}^2}{(2 \cdot n)^2} \quad \frac{\text{odd}^2}{(2n+1)^2} \\
 &= 2^{\text{even}} \cdot (2^n)^2 \quad = 2^{\text{odd}} \cdot \underline{(2n+1)^2} \\
 & \qquad \qquad \qquad = 4n^2 + 2n + 1 \\
 & \qquad \qquad \qquad = 2 \cdot \underline{(2n^2+n)} + 1 \\
 & \qquad \qquad \qquad = \text{odd}
 \end{aligned}$$

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Theo:- If $ab > 0$ then either ① $a > 0, b > 0$
 ② $a < 0, b < 0$

① If $a, b \in \mathbb{R}$ show that $a^2 + b^2 = 0 \Leftrightarrow a = 0, b = 0$

② If $0 < c < 1$ show that $0 < c^2 < c < 1$ ✓

③ If $x, y \in \mathbb{Q}$, $x+y \in \mathbb{Q}$, $\underline{x \cdot y \in \mathbb{Q}}$ ✓

If $x \in \mathbb{Q}, y \in \mathbb{Q}^c$, $x+y \in \mathbb{Q}^c$

$$\begin{aligned}
 & (a+b)^2 = a^2 + b^2 + 2ab = 0 \\
 & a^2 + b^2 = 0 \Rightarrow 2ab = 0 \\
 & \Rightarrow a \cdot b = 0
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{?}{=} a = 0 \quad \text{or} \quad b = 0 \\
 & \text{if } a = 0 \quad \& \quad b \neq 0 \Rightarrow a^2 + b^2 > 0
 \end{aligned}$$

$$a^2 + b^2 = 0 \Rightarrow b = 0$$

$$a, b \in \mathbb{R}^+, c > 0,$$

$$a > b \Rightarrow c \cdot a > c \cdot b$$

$$0 < c < 1$$

, as $c > 0$

$$c < 1$$

$$c \cdot c < 1 \cdot c$$

$$\underline{0 < c^2 < c < 1}$$



$$x, y \in \mathbb{Q} \Rightarrow x+y \in \mathbb{Q}$$

$$x = \frac{p_1}{q_1}, y = \frac{p_2}{q_2} \Rightarrow x+y = \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$$

$$\text{as } p_1 q_1, p_2 q_2 \in \mathbb{Z},$$

$$\underline{p_1 q_2 \in \mathbb{Z}, p_2 q_1 \in \mathbb{Z}}, q_1 q_2 \in \mathbb{Z}$$

$$p_1 q_2 + p_2 q_1 \in \mathbb{Z}$$

$$= \frac{p^*}{q^*} \in \mathbb{Q}$$

Absolute values, $\forall a \in \mathbb{R}$ $|a| = \begin{cases} +a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$

$|a| = \max(a, -a)$

Theo:- ① $|ab| = |a||b| \Rightarrow a, b \in \mathbb{R}$ ✓

② $|a|^2 = a^2 \Rightarrow a \in \mathbb{R}$ $-c \leq a+b \leq c \Rightarrow |a+b| \leq c$

③ If $c \geq 0$ then $|a| \leq c$ iff $-c \leq a \leq c$

④ $-|a| \leq a \leq |a|$ ✓

Proof:- ① If $a, b > 0 \Rightarrow ab > 0 \Rightarrow |a|=a, |b|=b, |a \cdot b|=ab=|a||b|$

$a > 0, b < 0 \Rightarrow ab < 0 \Rightarrow |a|=a, |b|=-b, |ab|=-ab=a \cdot (-b)=|a||b|$

$a < 0, b > 0 \Rightarrow ab < 0 \Rightarrow$ Simillarly

$a \cdot b < 0 \Rightarrow ab > 0 \Rightarrow |a|=-a, |b|=-b, |ab|=ab=(-a) \cdot (-b)$

$= |a||b|$ ✓

② $|a|^2 = a^2$

If $a > 0, |a|=a \Rightarrow |a|^2 = a^2$ ✓

$a < 0 \quad |a| = -a \Rightarrow |a| = (-a)^2 = a^2$ ✓

$a = 0 \quad |a| = a = 0 \Rightarrow |a|^2 = a^2 = 0$ ✓

③ If $c \geq 0$ then

$|a| \leq c$ iff $-c \leq a \leq c$ ✓

i) $|a| \leq c \Leftrightarrow |a| = \max(a, -a) \leq c$

$\Leftrightarrow a \leq c \text{ & } -a \leq c$

$\Leftrightarrow a \leq c \text{ & } a \geq -c$

$\Leftrightarrow -c \leq a \leq c$ ✓

ii) $|a| \leq c$ iff $-c \leq a \leq c$ ✓

now assume $c = |a| \geq 0, |a| \leq |a|$

$\Rightarrow -|a| \leq a \leq |a|$

* Triangular inequality :- If $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$

proof :- If $a, b \in \mathbb{R}$, then

$$\begin{array}{r} -|a| \leq a \leq |a| \\ + \quad -|b| \leq b \leq |b| \\ \hline -|a|-|b| \leq a+b \leq |a|+|b| \end{array}$$

$$\text{put } c = |a| + |b|$$

$$\Rightarrow -(|a| + |b|) \leq a+b \leq |a| + |b|$$

$$\Rightarrow -c \leq a+b \leq c$$

$$\Rightarrow |a+b| \leq c$$

$$\Rightarrow |a+b| \leq |a| + |b|$$

$$|a| = |-a|$$

* Corollary : If $a, b \in \mathbb{R}$, then (a) $||a|-|b|| \leq |a-b|$

$$(b) |a-b| \leq |a| + |b|$$

(a) If $a, b \in \mathbb{R}$

$$\checkmark a = \underline{a-b+b}$$

$$|a| = |\underline{a-b+b}| \leq |a-b| + |b| \quad \text{--- (by triangular inequality)}$$

$$\checkmark b = b-a+a$$

$$|b| = |b-a+a| \leq \underline{|b-a|} + |a| = \underline{|a-b|} + |a| \quad (\text{by tri. inequality})$$

$$\text{From (a)} \quad |a| - |b| \leq |a-b|.$$

$$(**) \quad |b| - |a| \leq |a-b| \Rightarrow |a| - |b| \geq -|a-b|$$

$$\Rightarrow \boxed{-c \leq a \leq c \Rightarrow |a| \leq c}$$

$$\Rightarrow -|a-b| \leq |a| - |b| \leq |a-b|$$

$$\text{put } c = |a-b| \Rightarrow ||a|-|b|| \leq c \Rightarrow ||a|-|b|| \leq |a-b|$$

$$\frac{-|a-b|}{c} \leq \frac{|a|-|b|}{c} \leq \frac{|a-b|}{c}$$

$\Rightarrow |a|-|b| \leq c$

Manoj C Patil

$$\textcircled{b} \quad |a-b| \leq |a| + |b|$$

We have triangular inequality $|a+b| \leq |a| + |b|$

replace b by $-b$

$$|a+(-b)| \leq |a| + |-b|$$

$$\Rightarrow |a-b| \leq |a| + |-b| \quad (|-b| = |b|)$$

$$\Rightarrow |a-b| \leq |a| + |b|$$

$$a-b \leq a+b \quad ?$$

$$a, b \in \mathbb{R}$$

Triangular inequality

$$|\sum_{i=1}^n a_i| \leq \sum_{i=1}^n |a_i|$$

* Real line

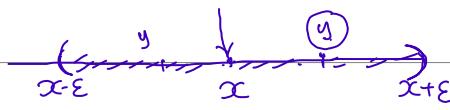
Extended Real Numbers



ε -nbhd.

$$V_\varepsilon(x) = \{y / y \in (x-\varepsilon, x+\varepsilon)\} \subseteq \mathbb{R}$$

$$(x-\varepsilon, x+\varepsilon) = V_\varepsilon(x)$$



interval length $\Rightarrow 2\varepsilon$

$$|x-y| \leq \varepsilon \Rightarrow y \in V_\varepsilon(x)$$

$$V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$$

$$y \in V_\varepsilon(x) \Rightarrow |x-y| \leq \varepsilon$$

$$\Rightarrow x-\varepsilon \leq y \leq x+\varepsilon$$

$$\delta_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) - \{x\}$$

deleted nbhd of x

e.g. we know $|a+b| \leq |a| + |b|$ but if $|a+b| = |a| + |b|$ iff $ab > 0$

i) If $ab > 0$ to prove $|a+b| = |a| + |b|$
 If $a > 0, b > 0, a+b > 0 \Rightarrow |a+b| = a+b = |a| + |b|$
 $a < 0, b < 0, a+b < 0 \Rightarrow |a+b| = -(a+b) = (-a) + (-b) = |a| + |b|$

$a > 0, b < 0$

If $|a+b| = |a| + |b|$ to prove $ab > 0$

$$\begin{aligned} |a+b|^2 &= (|a| + |b|)^2 \\ (a+b)^2 &= |a|^2 + |b|^2 + 2|a||b| \\ \Rightarrow a^2 + b^2 + 2ab &= a^2 + b^2 + 2|a||b| \\ \Rightarrow ab &= |a||b| = |a \cdot b| \quad \text{on} \\ \Rightarrow ab &> 0 \end{aligned}$$

e.g. If $x, y, z \in \mathbb{R}$ & $\underline{x \leq z}$ show that
 $x \leq y \leq z$ iff $|x-y| + |y-z| = |x-z|$

i) Let $\underline{x \leq y \leq z}$. $x-y$ & $y-z$ negative.

$$\begin{aligned} \text{LHS} &= |x-y| + |y-z| \\ &= (\cancel{y-x}) + (\cancel{z-y}) \\ &= \cancel{z-x} \\ &= |x-z| \checkmark \\ &= \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} x \leq y &\Rightarrow x-y \leq 0, y \leq z \Rightarrow y-z \leq 0 \\ |x-y| &= -(x-y) \\ &= y-x \\ |y-z| &= -(y-z) \\ &= z-y. \end{aligned}$$

$$\cancel{x \leq z} \Rightarrow x-z \leq 0$$

$$|x-z| = z-x$$

$$|x-y| \neq |x-z|$$

ii) Let $|x-y| + |y-z| = |x-z|$,

$$x \leq z \Rightarrow x-z \leq 0, z-x \geq 0$$

$$x-z = x-y + y-z$$

$$\text{Top} \rightarrow x \leq y \leq z$$

$$x-z = \underline{x-y+y-z}$$

$$|x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z|$$

but $|x-z| = |x-y| + |y-z|$ so. we can use

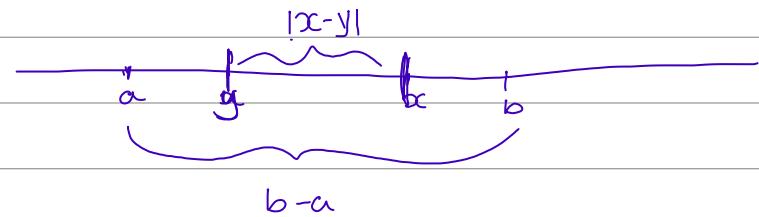
$$\boxed{|a+b| = |a| + |b| \text{ iff } \underline{ab \geq 0}}$$

$x-y \geq 0$
 $x \geq y$

Only two possibilities

$(x-y)(y-z) > 0$	$\Rightarrow x > y, y > z \Rightarrow x > y \geq z$, which is not possible
$(x-y)(y-z) \leq 0$	$\Rightarrow x \leq y, y \leq z \Rightarrow x \leq y \leq z$

if $a < x < b$
 $a < y < b$
 $|x-y| \leq b-a$



?

$$\begin{aligned} -a > -x > -b &\Rightarrow -b < -x < -a \\ a < y < b &\quad \underline{\quad a < y < b} \\ -(b-a) < y-x < b-a & \\ \Rightarrow |x-y| &\leq b-a \end{aligned}$$

e.g. $|x-1| > |x+1|$ $x \in ?$ $\underline{x < 0}$

$\underline{-3} \leq x < 0$ ≤ 0 -0.01

$$|x+1| + |x-2| = 7 \quad x = -3 \quad x = 4$$

$$\text{e.g. } \min\{a,b\} = \frac{1}{2}(a+b - |a-b|) \quad \max\{a,b\} = \frac{1}{2}(a+b + |a-b|)$$

$\rightarrow \text{① Let } \underline{a < b}, \quad \min(a,b) = a \quad \max(a,b) = b$

$$|a-b| = b-a$$

$$\text{RHS} = \frac{1}{2}(a+b + |a-b|)$$

$$= \frac{1}{2}(a+b + b-a)$$

$$= \frac{2b}{2} = \max(a,b)$$

$$\text{RHS} = \frac{1}{2}(a+b - |a-b|)$$

$$= \frac{1}{2}(a+b - (b-a))$$

$$= \frac{2a}{2} = \min(a,b)$$

github.com/manojcpatil/Lecture-scribbles

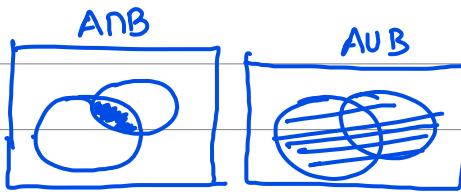
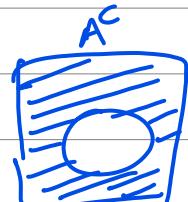
{ Absent Aditya, Dipali, Anushree, Aakash, Lalit, Vishal, Bhavesh, Hikita, Sopan
Nikhil, Pranit, Bhanuja

* Set Operations, $A, B \subseteq \mathbb{R}$. Union : $A \cup B = \{x / x \in A \text{ or } x \in B\}$

Intersection : $A \cap B = \{x / x \in A \text{ & } x \in B\}$

Complement $A^c = \{x / x \notin A, x \in \mathbb{R}\}$

Subtract $A - B = \underline{A \cap B^c} = A \setminus B$
 $= A - (A \cap B)$



① subset $A \subseteq B$: $x \in A \Rightarrow x \in B \not\Rightarrow x \in A$

Theo:-

- ① $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- ② $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$A \subseteq B$



$A=B$ $\Rightarrow A \subseteq B \& B \subseteq A$? Can we say that?
 $(A \cup B)^c = A^c \cap B^c$ Can you prove this?

To prove ① $(A \cup B)^c \subseteq A^c \cap B^c$
 ② $A^c \cap B^c \subseteq (A \cup B)^c$

→ Let $x \in (A \cup B)^c$

$$\Leftrightarrow x \notin A \cup B$$

$$\Leftrightarrow x \notin A \text{ and } x \notin B$$

$$\Leftrightarrow x \in A^c \text{ and } x \in B^c$$

$$\Leftrightarrow x \in A^c \cap B^c.$$

$$\Rightarrow (A \cup B)^c \subseteq A^c \cap B^c$$

$$\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c$$

$$\Rightarrow =$$



$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (\text{Distributive law})$$

$$\rightarrow \text{To prove } A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \checkmark$$

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Let $x \in \underline{A \cap (B \cup C)}$

$\Rightarrow x \in \underline{(A \cap B) \cup (A \cap C)}$

$$\Leftrightarrow x \in A \text{ and } x \in B \cup C$$

$$\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } C)$$

$$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$$

$$\Leftrightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$$

$$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$$

$$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad] \rightarrow$$

$$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

equal

e.g. If A and B are two sets such that

$$A \subseteq B \text{ iff } A \cap B = A$$

$$\text{i) } A \subseteq B \Rightarrow x \in A \Rightarrow x \in B \Rightarrow x \in A$$

\Rightarrow each and every element of A is in B.

$$\Rightarrow A \cap B = A \subseteq B$$

$$\text{ii) } A \cap B = A \Rightarrow \text{To prove } A \subseteq B$$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} \subseteq B$$

$$A \cap B \subseteq B$$

$$\Rightarrow A \subseteq B$$

$$A \setminus (B \cup C) = A \cap (B \cup C)^c.$$

$$\begin{aligned} & A \setminus B \cap A \setminus C \\ &= (A \cap B^c) \cap (A \cap C^c) \end{aligned}$$

$$A \cap (B \cup C)^c \subseteq (A \cap B^c) \cap (A \cap C^c)$$

\subseteq

$$\text{Let } x \in A \cap (B \cup C)^c$$

$$\Leftrightarrow x \in A \text{ and } x \notin (B \cup C)^c$$

\cup - or/and

$$\Leftrightarrow x \in A \text{ and } x \notin B \cup C$$

$$\Leftrightarrow x \in A \text{ and } [x \notin B \text{ and } x \notin C]$$

$$\Leftrightarrow x \in A \text{ and } x \notin B \text{ and } x \in A \setminus B$$

$$\Leftrightarrow x \in A \setminus B \text{ and } x \in A \setminus C$$

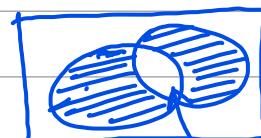
$$\Leftrightarrow x \in A \setminus B \cap A^c$$

Similarly we can prove ② part too

* Symmetric Difference :-

$$\begin{cases} A \Delta B = \{x / x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\} = (A \cup B) \cap (A \cap B)^c \\ = (A \cup B) - (A \cap B) \\ = (A \cup B) \setminus (A \cap B) \end{cases}$$

$$x \in (A \cup B) \cap (A \cap B)^c$$



$$- (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

$x \in A \setminus B$

$$\therefore = (A \cap B^c) \cup (B \cap A^c)$$

$$= \underline{(A \cap B^c)} \cup \underline{B} \cap \underline{(B \cap A^c)} \cup \underline{A^c}$$

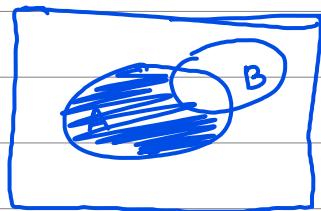
$$= (A \cup B) \cap \underbrace{B^c \cup B}_{B} \cap \underbrace{A \cup A^c}_{B} \cap \underbrace{B^c \cup A^c}_{B}$$

$$= (A \cup B) \cap (B^c \cup A^c)$$

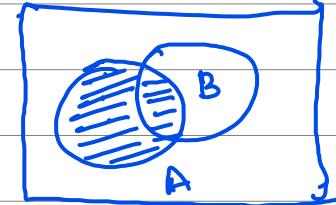
$$= (A \cup B) \cap (A \cap B)^c$$

$$= (A \cup B) \setminus (A \cap B)$$

$$\begin{aligned} A \setminus B \\ = A \cap B^c \end{aligned}$$



$$A \setminus (B \setminus A)$$



$$A \setminus (B \setminus A) = B \cap A^c$$

$$\downarrow \\ A \cap (B \setminus A)^c$$

$$= A \cap (B \cap A^c)^c$$

$$= A \cap (\underline{A \cup B^c})^c$$

$$= \underline{\underline{A}}$$

Lower bound :-

Upper bound

$$S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \Rightarrow \text{low } S = 0 \notin S$$

$$\text{Upper bound (S)} = 1 \in S$$

Bounded below.

$$\mathbb{N}, \mathbb{Z}, \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$$

Bounded Above.

$$\mathbb{I}^+, \mathbb{Z}^+, \mathbb{R}^+$$

Bounded set bounded below & Above

$$\{(-1)^{2n}, n \in \mathbb{N}\} = \{1\} \text{ singleton set}$$

Unique?

$$\{(-1)^n, n \in \mathbb{N}\} = \{-1, 1\}$$

lower bound = -1 $\in S$ Upper bound = +1 $\in S$

$$\underline{S} = (0, 1]$$

$$x > 0 \rightarrow x \in S$$

$$x \leq 1 \rightarrow x \in S$$

$$\begin{cases} 10^{10000} \\ -10 \\ -0.5 \end{cases}$$

Lower bound = 0 $\notin S$ Upper bound = 1 $\in S$

$$\underline{u} \text{ l.b. of } S \Rightarrow \forall x \in S, \underline{x} \geq \underline{u}$$

Sup:- u is \sup of S if ① u is upper bound
 ② v is any other upper bound of S then $v \geq u$.

u must be least upper bound.

Inf:- if v is inf of S if ① v is lower bound
 ② u is any other lower bound of S then $v \leq u$

v must be greatest lower bound.



No upper bound / Supremum :- $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{I}, \mathbb{Z} \dots$

No lower bound / Inf :- $-\mathbb{N}, \bar{\mathbb{Q}}, \bar{\mathbb{R}}, \bar{\mathbb{I}}, \mathbb{Z}^-$

$$S = \{1, 2, 3, 4\}$$

$$\sup(S) = 4$$

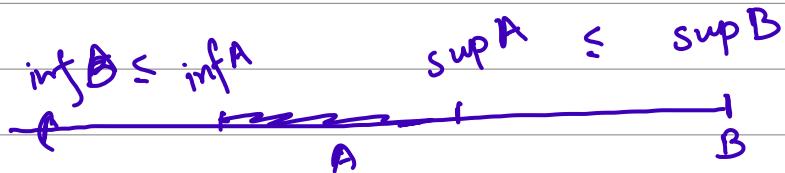
set of Upper bounds = $[4, \infty)$

$$\inf(S) = 1$$

set of lower bounds = $(-\infty, 1]$

* Completeness property:- Lower bound exists \Rightarrow Inf exists
 Upper \Rightarrow Sup

$$A \subseteq B \Rightarrow \inf A \geq \inf B$$



Infr Suppose

Absent:- Adity, Dipali, Anushree, Pradip, Akash Lalit
vishal, komal, Samiya, Groninda, Atul, Chetan,
Yogita .

Completeness Property :-

Lower bound \rightarrow inf exists
Upper \rightarrow sup exists

$\mathbb{N} \Rightarrow$ 0 lower - 1 inf
 $\mathbb{R}^- \Rightarrow$ 1 upper, 0 sup

Supremum	Infimum
least upper	Greatest Lower
bound	bound

{ LUB :- If u is up least upper bound of sets then
 ① u is upper bound ✓
 ② if v is any other ^{upper bound} of S then $v \geq u$.

GLB :-

Theo:- If sup exists, then it is unique.

proof:- We will prove this by method of contradiction.

Assume for set $S \exists u_1 \& u_2$ be two sup.

- | | |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\left\{ \begin{array}{l} \text{by def'n } u_1 \text{ is sup} \\ \text{① } u_1 \text{ is upper bound of } S \\ \text{② If } \exists u_2 \text{ as any other ub.} \\ \text{then } u_2 > u_1 \end{array} \right.$ | $\left\{ \begin{array}{l} u_2 \text{ is sup.} \\ \text{① } u_2 \text{ is upperbound of } S \\ \text{② If } \exists u_1 \text{ as any other} \\ \text{then } u_1 > u_2 \end{array} \right.$ |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

$$\Rightarrow u_1 = u_2$$

\Rightarrow Sup is unique

Theo:- If $A, B \subseteq \mathbb{R}$ and $A \subseteq B$ then if inf & sup of B exists \exists then

- ① $\inf(A) \geq \inf(B)$
- ② $\sup(A) \leq \sup(B)$

Proof:- ① Assume $\inf(A) = u_A$ $\inf(B) = u_B$

$A \subseteq B$ to prove $\inf(A) \geq \inf(B)$

$$u_A \geq u_B -$$

We will prove this by method of contradiction

$$\boxed{u_A \leq u_B}$$

by def of infimum

① u_B is lower bound of B .

② u_x any lower bound of $B \Rightarrow u_x \leq u_B$ (Greatest Lower bound)

u_B Lower bound of $B \Rightarrow \forall x \in B, u_B \leq x$

$A \subseteq B \Rightarrow \forall x \in A \Rightarrow x \in B$

u_B is also lower bound of A as $\forall x \in A \subseteq B, u_B \leq x$

$\left\{ \begin{array}{l} u_A \text{ is inf of } A \Rightarrow u_A \text{ is also lower bound of } A \\ \text{if } \exists \text{ some other lower bound of } A \text{ then } u_A \text{ is greater than that lower bound} \end{array} \right.$
 $\Rightarrow u_B \leq u_A$

which contradicts to our assumption.

$A \subseteq B \quad \inf A \geq \inf B$

Assume $u_A \leq u_B$.

① $u_B \leq \inf B \Rightarrow \begin{cases} u_B \text{ lower bound} \rightarrow x > u_B \Rightarrow x \in B \\ u_x \text{ lower bound of } B, \Rightarrow u_x \leq u_B \end{cases}$

$x > u_B \Rightarrow x \in B$

$\Rightarrow x > u_B \Rightarrow x \in A \subseteq B$

$\Rightarrow u_B$ is lower bound of A . ~~✓~~

$\left\{ \begin{array}{l} u_A \text{ is inf}(A) \\ \text{i) } u_A \text{ l.b.} \end{array} \right.$

ii) u_y l.b. of $A, u_y \leq u_A$

\downarrow

$\underline{u_B \leq u_A}$



$S = \left\{ \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}$, Set of Lower bounds = $(-\infty, -1]$

Upper bounds = $[1/2, \infty)$

$S = \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots \right\}$

$\inf = -1 \in S$

$\sup = 1/2 \in S$

$$S = \left\{ \frac{(-1)^n}{n}, n \in \mathbb{N} \right\} \quad \inf S = -1$$

$$\text{set of LBS} = \sup S = 2$$

$$\text{UBS} =$$

$$S = \left\{ 1 - \frac{(-1)^1}{1}, 1 - \frac{(-1)^2}{2}, \dots \right\} = \left\{ 2, \frac{1}{2}, \frac{4}{3}, \dots \right\}$$

$$S = \left\{ \frac{1}{m} + \frac{1}{n}, m, n \in \mathbb{N} \right\}$$

$$2 \sup \quad \inf 0$$

$$? \quad \left\{ \frac{(-1)^m}{m} + \frac{(-1)^n}{n}, m, n \in \mathbb{N} \right\} ? \quad -2 \quad 1$$

* Cartesian Product

$$A \times B = \{ \langle x, y \rangle / x \in A, y \in B \}$$

$$A = \{2, 3, 4\} \quad B = \{1, 5\}$$

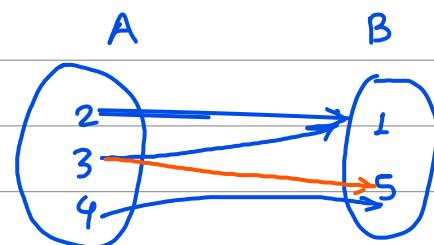
$$A \times B = \{ \langle 2, 1 \rangle, \langle 2, 5 \rangle, \langle 3, 1 \rangle, \langle 3, 5 \rangle, \langle 4, 1 \rangle, \langle 4, 5 \rangle \}$$

$$f = \{ \langle \underset{\uparrow}{2}, \underset{\uparrow}{1} \rangle, \langle 3, 1 \rangle, \langle 4, 5 \rangle \}$$

$$f(2) = 1$$

$$f(3) = 1$$

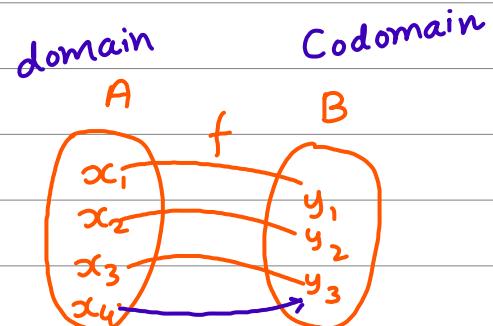
$$f(4) = 5$$



One-one fun^c (Injective)

$x, x_2 \in A$
 $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

not one-one $x_3 \neq x_4 \Rightarrow f(x_3) = f(x_4) = y_3$



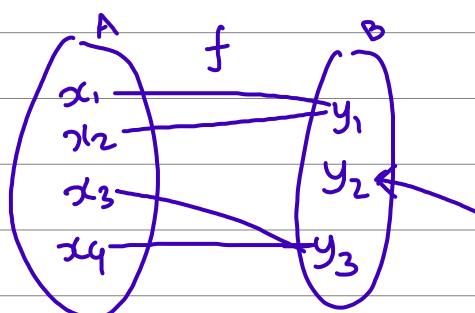
one-one

$$f(x) = x^3 \quad \forall x \in \mathbb{I}$$

$$f(x) = x^2 \quad \forall x \in \mathbb{N} \quad \checkmark$$

*

Into



$$\text{domain} = \{x_1, x_2, x_3, x_4\}$$

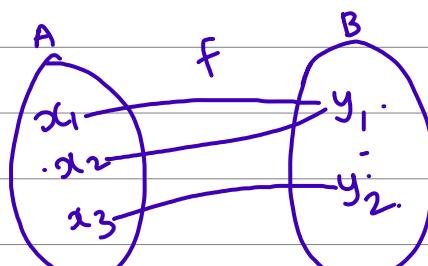
$$\text{Codomain} = \{y_1, y_2, y_3\}$$

$$\text{Range}(f) = \{y_1, y_3\}$$

into

$$\text{Range} \subset \text{Codomain}$$

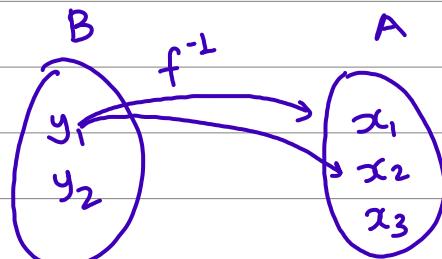
Onto
surjective



$$\text{Codomain} = \text{Range}$$

onto

Set fun^c
(f^{-1})



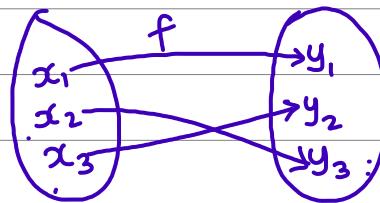
$$f^{-1}(y_1) = \{x_1, x_2\}$$

$$f^{-1}(y_2) = \{x_3\}.$$

set func

$$f^{-1}(y) = \{x / x \in A, f(x) = y \in B\} \quad \checkmark$$

One-One & onto \Rightarrow (Injective + Surjective) \Rightarrow Bijective fun^c

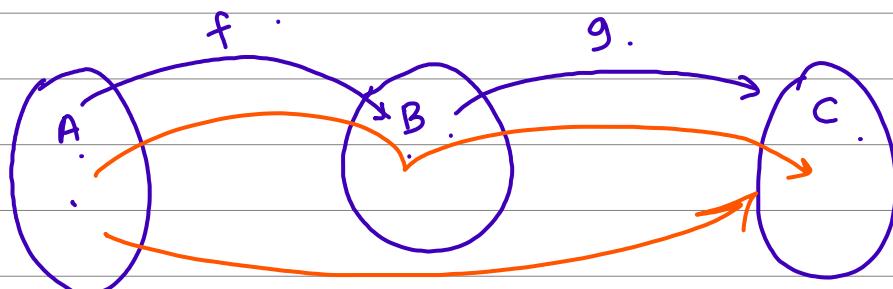


one-one & onto
In Sur
Bijective

$$f(x) = 2x \Rightarrow x \in \mathbb{I}, \mathbb{II}$$

$$f(x) = x^2 \nRightarrow x \in \mathbb{N}$$

Composite
func



$$D(f) = A$$

$$R(f) \subseteq B$$

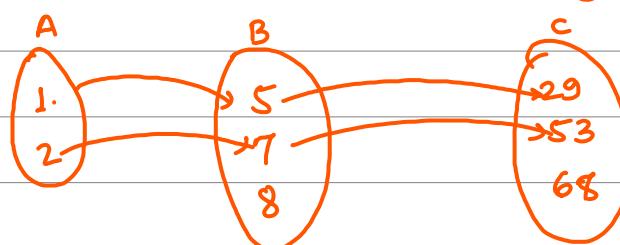
$$\begin{aligned} D(g) &= B \\ R(g) &\subseteq C \end{aligned}$$

$$(2x+3)^2 + 4$$

$$\underline{\underline{g \circ f}}$$

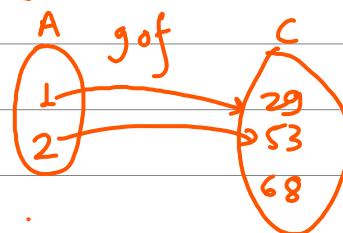
$$f(x) = 2x+3$$

$$g(y) = y^2 + 4$$



$$D(f)=A, \quad \underline{R(f) \subseteq D(g)}$$

$$gof$$



$g \circ f$ may or may not
 $\neq f \circ g$

$$f: \mathbb{R} \rightarrow \mathbb{R} \quad \& \quad g: \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 2x+3$$

$$g(y) = y^2 + 4$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) = g(2x+3) = (2x+3)^2 + 4 \\ &= 4x^2 + 12x + 13 \end{aligned}$$

$$\begin{aligned} f \circ g(x) &= f(g(x)) = f(x^2+4) = 2(x^2+4) + 3 \\ &= 2x^2 + 11 \end{aligned}$$

$g \circ f \neq f \circ g$

$$f: A \rightarrow B, \quad E, F \subseteq A$$

$$f(E \cup F) = \underset{\text{set}}{f(E)} \cup \underset{\text{set}}{f(F)}$$

$$f(E) = \{y / y = f(x), x \in E \subseteq A\} \quad \checkmark$$

$$f(F) = \{y / y = f(x), x \in F \subseteq A\}$$

$$\underline{f(E \cup F)} = \{y / \underline{y = f(x)}, \underline{x \in E \cup F} \subseteq A\} \quad \checkmark$$

$$\text{To prove: } - f(E) \cup f(F) = f(E \cup F)$$

\subseteq

\supseteq

Let $y \in f(E) \cup f(F)$

$\Leftrightarrow \underline{y \in f(E)}$ or $y \in f(F)$

$\Leftrightarrow y = f(x)$ and

$\Leftrightarrow x \in E$ or $x \in F$

$\Leftrightarrow y = f(x)$ and $x \in E \text{ or } F$

$\Leftrightarrow y = f(x)$ and $x \in E \cup F$

$\Leftrightarrow y \in f(E \cup F)$

\Rightarrow As $y \in f(E) \cup f(F) \Rightarrow y \in f(E \cup F)$

So $f(E) \cup f(F) \subseteq f(E \cup F)$

Similarly we can obtain,

$f(E \cap F) \subseteq f(E) \cap f(F)$

$\Rightarrow f(E \cup F) = f(E) \cup f(F)$

$f(E \cap F) \subseteq f(E) \cap f(F)$

Let $y \in f(E \cap F)$

$\Rightarrow y = f(x)$ and $x \in E \cap F$

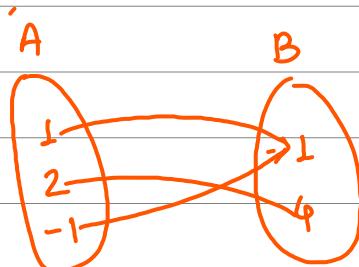
$\Rightarrow \underline{y=f(x)} \text{ & } \underline{x \in E} \text{ and } y=f(x) \text{ & } x \in F$

$\Rightarrow y \in f(E) \text{ and } y \in f(F)$

$\Rightarrow y \in f(E) \cap f(F)$

$\Rightarrow f(E \cap F) \subseteq f(E) \cap f(F)$

✓ $f(E \cap F) \neq f(E) \cap f(F)$



$$\begin{aligned} E &= \{1, 2\} & f(E) &= \{1, 4\} \\ F &= \{-1, 2\} & f(F) &= \{1, 4\} \\ E \cap F &= \{2\} & f(E \cap F) &= \{4\} \end{aligned}$$

$$f(E) \cap f(F) = \{1, 4\}$$

$\Rightarrow f(E \cap F) \neq f(E) \cap f(F)$

$$f : A \rightarrow B \quad G, H \subseteq B$$

$$i) f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$$

$$f^{-1}(G) = \{x / \underline{f(x)=y \in G}, x \in A\}$$

$$f^{-1}(H) = \{x / \underline{f(x)=y \in H}, x \in A\}$$

$$f^{-1}(G \cup H) = \{x / \underline{f(x) \in G \cup H}, x \in A\}$$

To prove

$$f^{-1}(G \cup H) \subseteq f^{-1}(G) \cup f^{-1}(H)$$

\supseteq

Let

$$x \in f^{-1}(G \cup H)$$

$$\Rightarrow f(x) \in G \cup H$$

$$x \in A$$

$$\Rightarrow f(x) \in G \quad \text{or} \quad f(x) \in H$$

$$\cancel{x \in A}$$

$$\Rightarrow x \in A \quad \& \quad f(x) \in G \quad \text{or} \quad x \in A \quad \& \quad f(x) \in H$$

$$\Rightarrow x \in f^{-1}(G) \quad \text{or} \quad x \in f^{-1}(H)$$

$$\Rightarrow x \in f^{-1}(G) \cup f^{-1}(H)$$

$$\Rightarrow f^{-1}(G \cup H) \subseteq f^{-1}(G) \cup f^{-1}(H)$$

Similarly we can obtain $f^{-1}(G \cup H) \supseteq f^{-1}(G) \cup f^{-1}(H)$

Theorem: $f: A \rightarrow B$ is injective $E \subseteq A$

$$f^{-1}(f(E)) = E$$



$\therefore f: A \rightarrow B$ & as injective func
for $x_1, x_2 \in A$, if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$$f^{-1}(D) = \{x / f(x) \in D, x \in A\} \checkmark$$

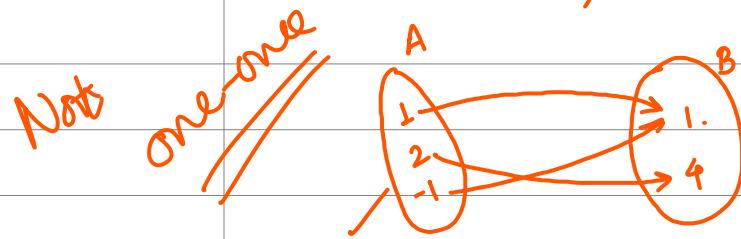
$$f(E) = \{y / y = f(x), x \in E\}$$

Let $x \in f^{-1}(f(E))$

$\Rightarrow y = f(x) \in f(E)$ & $x \in A, y \in B$

\exists some $\alpha \in E$? $(x \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$ one-one ↗

$\Rightarrow f^{-1}(f(E)) \subseteq E$



$$E = \{1, 2\} \quad \checkmark$$

$$f(E) = \{1, 4\}$$

$$f^{-1}(f(E)) = f^{-1}(\{1, 4\})$$

$$= \{-1, 1, 2\}$$

$$E \neq f^{-1}(f(E))$$

Finite Set :

$$A = \{x_1, x_2, x_3, \dots, x_n\}$$

$B = \{1, 2, 3, \dots, n\} \subseteq \mathbb{N}$ n is some fixed $\in \mathbb{N}$

Infinite ✓

$\mathbb{Q}, \mathbb{N} \quad \{1, 2, 3, \dots, \underline{\underline{\dots}}\}$

even odd square

complex Real

$S, T \subseteq \mathbb{R}, T \subseteq S$

To prove If S is finite then T is also finite.

→ S is finite if it is either empty or it has n element

① $S = \emptyset$ & $T \subseteq S \Rightarrow T = \emptyset \Rightarrow T$ is finite.

② We will prove this by mathematical induction

✓ S is finite ✓

$$\underline{\#(S)=1} \text{ & } T \subseteq S \quad \text{②.1} \quad \begin{cases} \#(T)=0 \text{ or } \#(T)=1 \\ T=\emptyset \text{ or } \frac{T=S}{\text{S } T \text{ is finite}} \end{cases}$$

✓ $\underline{\#(S)=K}$ & $T \subseteq S \Rightarrow T$ is finite.

$$\#(S)=\underline{K+1} \Rightarrow S = \{x_1, x_2, \dots, \overset{\nwarrow}{x_{K+1}} \underset{\substack{\uparrow \\ L}}{\underset{\substack{\uparrow \\ 2}}{\dots}}, \underset{\substack{\uparrow \\ K+1}}{\underline{x_{K+1}}}\}.$$

$$S_1 = S - \{f(K+1)\} \quad \& \quad \underline{T \subseteq S}$$

$$\begin{array}{l} f(K+1) \in T \quad \text{or} \\ T \notin S, \\ T_1 = T - \{f(K+1)\} \subseteq S, \end{array} \quad \left| \begin{array}{l} f(K+1) \notin T \\ T \subseteq S_1 \subseteq S \\ \Rightarrow T \text{ is finite} \end{array} \right.$$

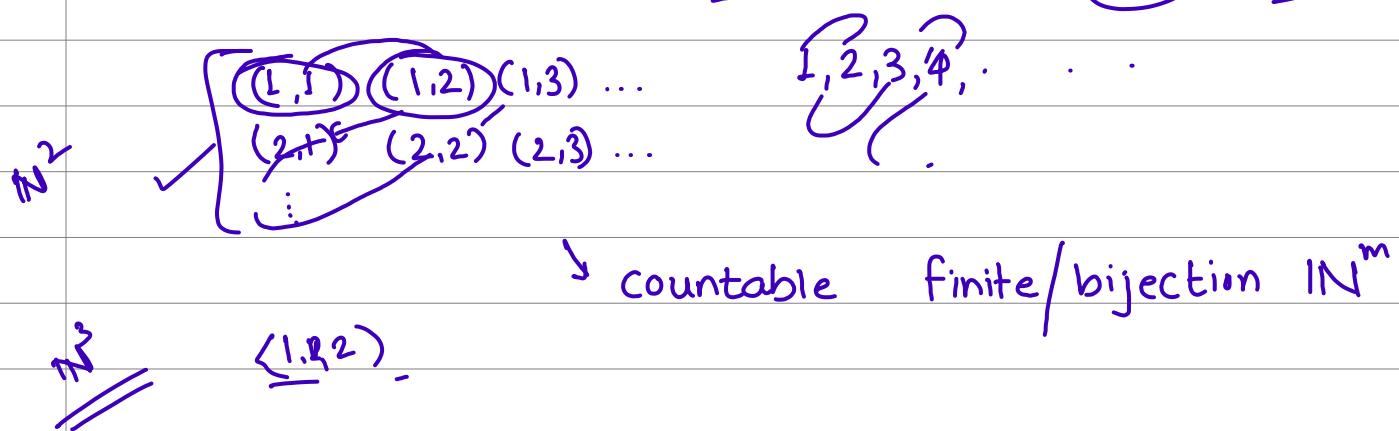
$$\begin{aligned} & T_1 \subseteq S_1 \text{ & } \#(S_1)=K \\ \Rightarrow & T_1 \text{ is finite set} \end{aligned}$$

$$\begin{aligned} & \#(T_1) + 1 = \#(T) \\ \Rightarrow & T \text{ is also finite} \end{aligned}$$

* Countable Set?

even $\{2n, n \in \mathbb{N}\} = \{2, 4, 6, 8, 10, \dots\}$] Bijection with \mathbb{N}
 odd $\{1, 2, 3, 4, 5, \dots\}$ of subset of \mathbb{N}

Countable set :- finite or bijection with \mathbb{N}_-^m , $m \in \mathbb{N}$.



$$S, T \subseteq \mathbb{R}, T \subseteq S$$

① If S is countable $\Rightarrow T$ is also countable

• S is countable $\Rightarrow S$ is either finite or bijection with \mathbb{N} denumerable

① if S is finite $\Rightarrow T$ is finite $\Rightarrow T$ is countable
 (by previous theo.)

② if S has bijection with \mathbb{N}
 - $\mathbb{N} = \{1, 2, 3, \dots\}$ denumerable

We can write $S = \{f(1), f(2), \dots\}$ —

$T \subseteq S \Rightarrow T = \emptyset, T = \text{finite}, \text{if } T \text{ is infinite}$
 $\Rightarrow T = \{f(n_1), f(n_2), f(n_3), \dots\}$ —

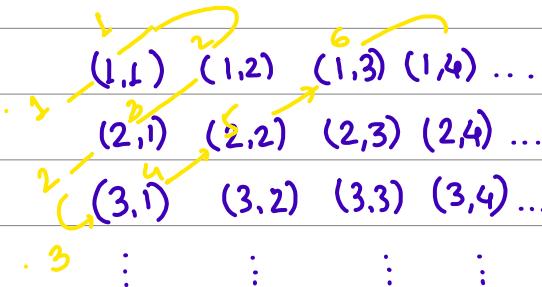
$$B = \{\overset{\uparrow}{n_1}, \overset{\uparrow}{n_2}, \overset{\uparrow}{n_3}, \dots\}$$

\mathbb{N}^2
Countable

1, 2, 3, 4, 5, ...

countably infinite / denumerable

\mathbb{N}^2



countable

Set of Rational Numbers is denumerable.

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

Let's discuss \mathbb{Q}

$$2.5 \quad \frac{5}{2} \cdot \frac{10}{4} \dots$$

$\mathbb{Q} \subset \mathbb{R}^2$

<u>1/1</u>	<u>1/2</u>	<u>1/3</u>	<u>1/4</u>	...
<u>2/1</u>	<u>2/2</u>	<u>2/3</u>	<u>2/4</u>	...
<u>3/1</u>	<u>3/2</u>	<u>3/3</u>	<u>3/4</u>	...
:	:	:	:	

5000

↪ countable

Archimedean Property

$$x \in \mathbb{R} \quad \exists n_x \in \mathbb{N}, \quad x < n_x$$

$$23.56 \in \mathbb{R} \quad \exists 24, 25 \in \mathbb{N}, \quad 23.56 < 24$$

Set of \mathbb{N} is bounded below \Rightarrow It has no upper bound.
 \Rightarrow It doesn't have supremum.

We will prove this by method of contradiction

$x \in \mathbb{R}$ \nexists any $n_x \in \mathbb{N}$, $\Rightarrow x < n_x$

$\Rightarrow n \leq x \Rightarrow \underline{\underline{n \in \mathbb{N}}}$

$\Rightarrow x$ is upper bound for \mathbb{N}

Completeness \curvearrowleft it has some supr let u .

$\Rightarrow n \leq u \Rightarrow \underline{\underline{n \in \mathbb{N}}}$

$\Rightarrow n+1 \leq u \Rightarrow n \in \mathbb{N}$

$\Rightarrow \underline{\underline{n \leq u-1}} \nexists \underline{\underline{n \in \mathbb{N}}} \Rightarrow u-1$ upper bound of \mathbb{N}

by defⁿ of supremum if u is sup $\Rightarrow u$ upper bound

② There exist no upper bound less than u

but $u-1$ is upper bound

u --- but u is sup.

\therefore This contradicts to our assumption that u is sup.

$\therefore \exists n_x \in \mathbb{N}, \Rightarrow x < n_x$ (QED)

* If $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ then $\inf S = 0$

let $\epsilon > 0$, $\frac{1}{\epsilon} \in \mathbb{R}$ $0 < \frac{1}{\epsilon}$

by Archimedean property

$\varepsilon > 0$, $\frac{1}{n_\varepsilon} \in \mathbb{R}$, $\exists n_\varepsilon \in \mathbb{N}$ $\forall n < n_\varepsilon$ $\frac{1}{n} < \frac{1}{n_\varepsilon} < \varepsilon$

Any $\varepsilon > 0$. $\Rightarrow 0 < \frac{1}{n_\varepsilon} < \varepsilon$

\Rightarrow for set S , 0 is lower bound

Assume $\inf S > 0$, $\varepsilon = \inf S > 0$

\Rightarrow of n_ε , $\exists 0 < \frac{1}{n_\varepsilon} < \varepsilon$

$$0 < \frac{1}{n_\varepsilon} < \underline{\inf S}$$

$\inf S$ is infimum of $S \Rightarrow S$ is lower bound

$\Rightarrow \inf S \leq \frac{1}{n_\varepsilon} \quad n \in \mathbb{N}$

\therefore Our assumption is wrong $\inf S > 0$

$\Rightarrow \inf S = 0$

If $t > 0$, $\exists n_t \in \mathbb{N}$, $\Rightarrow 0 < \frac{1}{n_t} < t$

* Let $y \in \mathbb{R}$ $\exists n_y \in \mathbb{N}$,

$$n_y - 1 \leq y \leq n_y$$

2.34 $\in \mathbb{R}$

$\exists z \in \mathbb{N}$

$3 - 1 \leq y \leq 3 \Rightarrow 2 \leq y \leq 3$

Let $y \in \mathbb{R}$, $E_y = \{n \mid n > y, n \in \mathbb{N}\}$

if $y = \underline{2.34}$ $E_y = \{\underline{3}, 4, 5, 6, \dots\}$

$y \leq n \Rightarrow n \in E_y$
 $\Rightarrow y$ is Lower bound of E_y

by completeness property, \exists inf of $E_y = u$

$\therefore u \leq n \forall n \in \mathbb{N}$

if u is inf, $u+1$ cannot be lower bound of S

$$\begin{array}{c} \exists \text{ some } n \in \mathbb{N}, \\ \underline{=} \quad \underline{n} < \underline{u+1} \\ \Rightarrow \quad \underline{n-1} < \underline{u}. \end{array}$$

Density Theo:- $x, y \in \mathbb{R}$, $x < y$, $\exists r \in \mathbb{Q}$, $x < r < y$

$$x < y, y - x > 0 \quad \exists n \in \mathbb{N}, \quad 0 < \frac{1}{n} < y - x$$

$$\Rightarrow 1 < ny - nx$$

$$\Rightarrow \underline{nx+1} \leq \underline{ny}.$$

Assume
 $x \geq 0, nc > 0 \quad m \in \mathbb{N}$

$$nx \leq \underbrace{m-1 \leq nc \leq m}_{?} \leq ny$$

$$r = \frac{m}{n}$$

$$\Rightarrow nx \leq m \leq ny \Rightarrow x \leq \frac{m}{n} \leq y$$

by density theo.

if $x, y \in \mathbb{R}$, $x < y$ then $\exists r \in \mathbb{Q}$, $x < r < y$

$\sqrt{2}$ is irrational no. So does $x\sqrt{2} \& y\sqrt{2}$

\therefore if $x < y$

$$\sqrt{2}x < \sqrt{2}y$$

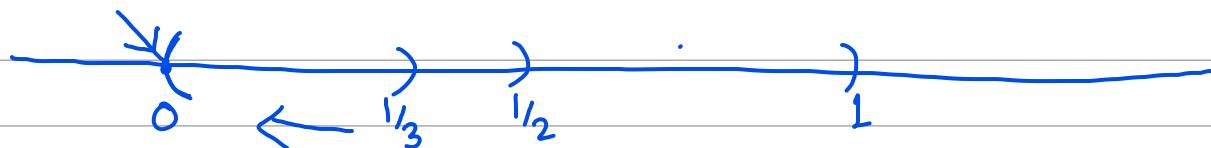
\because by density theo.

$$\sqrt{2}x < r < \sqrt{2}y$$

$$x < \frac{r}{\sqrt{2}} < y$$

Arbitrary intersection

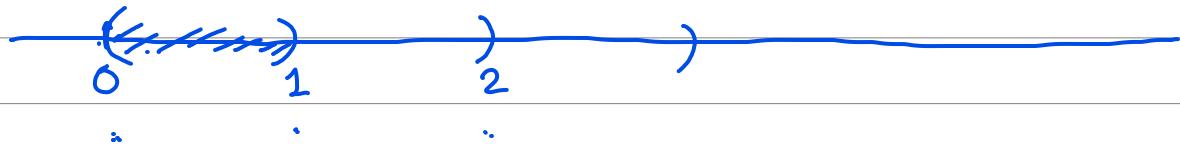
$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) \text{ finite intersection} \Rightarrow \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset$$



$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset$$

$$\bigcap_{n=1}^{\infty} (0, n) = (0, \infty)$$

$$\bigcup_{n=1}^{\infty} (0, n) = (0, \infty)$$



$$(0, 1) \cup (0, 2) = (0, 2)$$

$$(0, 1) \cup (0, 2) \cup (0, \dots) \cup (0, n) = (0, n)$$

$$\left[\frac{1}{k}, 1 \right] \subset \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 \right]$$

$n=1 \Rightarrow \{1\}$ $n=2 \Rightarrow \left[\frac{1}{2}, 1 \right]$
 $n=3 \Rightarrow \left[\frac{1}{3}, 1 \right]$



$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 \right] = (0, 1]$$

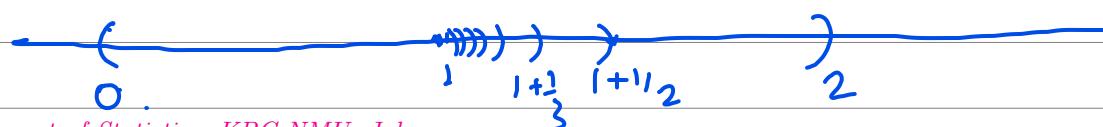
closed

Arbitrary union of closed intervals ~~is again closed.~~ may or may not be an interval

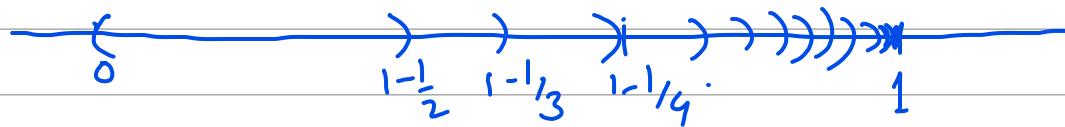
$$\underline{(0, 1]} = \bigcap_{n=1}^{\infty} (0, 1 + \frac{1}{n})$$

\uparrow

$$(0, 2) \cap (0, 1 + \frac{1}{2}) = (0, 1 + \frac{1}{2})$$



$$\bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n}) = (0, 1 - 1) = \emptyset \quad (0, 1 - \frac{1}{1})$$

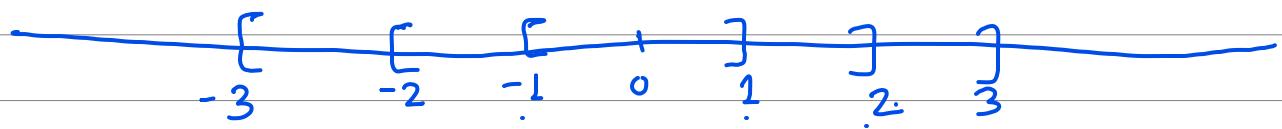


$$= (0, 1)$$

$$\bigcup_{n=1}^{\infty} [0, 1 - \frac{1}{n}] = [0, 1]$$

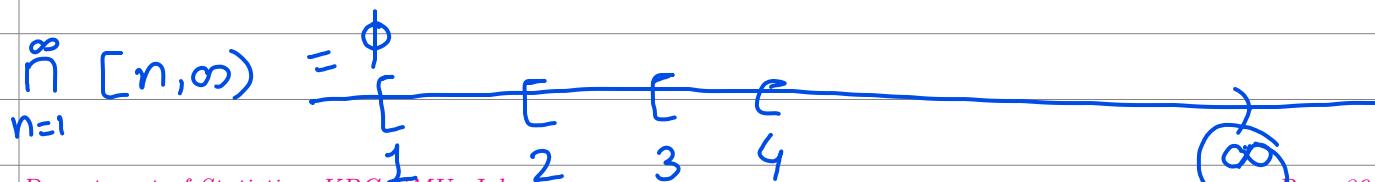
half closed
half open interval

$$\bigcup_{n=1}^{\infty} [-n, n] \checkmark = (-\infty, \infty)$$



$$\bigcup_{n=1}^{\infty} [-n, n] = [-k, k]$$

$$\bigcap_{n=1}^{\infty} [-n, n] = [-1, 1] \checkmark$$



\mathbb{R} is uncountable
 $\Rightarrow (0,1)$ is uncountable

Assume $(0,1)$ is countable.

$\Rightarrow (0,1)$ is denumerable

$\Rightarrow (0,1)$ has one-one & onto ~~cor~~ relation with \mathbb{N}

$\Rightarrow \exists S = (0,1) = \{b_1, b_2, b_3, \dots\}$

$b_i \in (0,1)$ it can be written in the form of $0.a_1 a_2 a_3 \dots$

$$b_1 = 0.\underline{a_{11}} a_{12} a_{13} \dots \quad c_1 \neq a_{11} \quad 0.121234$$

$$b_2 = 0.\underline{a_{21}} a_{22} a_{23} \dots \quad c_2 \neq a_{21}$$

$$b_3 = 0.\underline{a_{31}} a_{32} a_{33} \dots \quad c_3 \neq a_{31}$$

:

$$\checkmark c = 0.c_1 c_2 c_3 \dots$$

$c \in (0,1)$ and as $c_i \neq a_{ii} \forall i$

$\Rightarrow c \neq b_i \forall i$

but as $c \in (0,1) \& c \neq b_i \forall i$

\therefore So our assumption that $(0,1)$ is countable is wrong

$\Rightarrow (0,1) \subseteq \mathbb{R}$, $(0,1)$ is uncountable $\Rightarrow \mathbb{R}$ is uncountable.

* Cauchy Schwartzs inequality

If $a_i, b_i \in \mathbb{R}$ $i=1:n$

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

$a_i, b_i \in \mathbb{R}$, some $x \in \mathbb{R}$

$$a_i x + b_i \in \mathbb{R}$$

$\forall i$

$$(a_i x + b_i)^2 \geq 0$$

$\forall i$

$$a_i^2 x^2 + 2a_i b_i x + b_i^2 \geq 0$$

$\forall i$

$$x^2 \sum_{i=1}^n a_i^2 + 2x \sum_{i=1}^n a_i b_i + \sum_{i=1}^n b_i^2 \geq 0$$

This statement is true for any $x \in \mathbb{R}$

$$A = \sum a_i^2$$

$$B = \sum a_i b_i$$

$$C = \sum b_i^2$$

$$Ax^2 + 2Bx + C \geq 0$$

Assume

$$x = \frac{-B}{A}$$



$$A \cdot \frac{B^2}{A^2} + \frac{2B(-B)}{A} + C \geq 0$$

$$\frac{B^2}{A} - \frac{2B^2}{A} + C \geq 0$$

$$-\frac{B^2}{A} + C \geq 0$$

$$C \geq \frac{B^2}{A}$$

$$AC \geq B^2$$

$$B^2 \leq AC$$

$$(\sum a_i b_i)^2 \leq (\sum a_i^2) (\sum b_i^2)$$

(QED)

$$\left(\sum a_i b_i \right)^2 = \left(\sum a_i^2 \right) \left(\sum b_i^2 \right)$$

?

a_i, b_i

$a_i = b_i$

$$\left(\sum a_i^2 \right)^2 = \left(\sum a_i^2 \right) \left(\sum a_i^2 \right) \checkmark$$

* Set Topology

Open Set

(a, b)

$[a, b]$

$[a, \infty)$

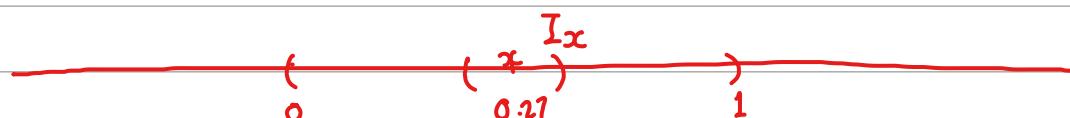
Open | Closed set

If it is actually closed set

$\{1, 2, 3, 4, 5\}$

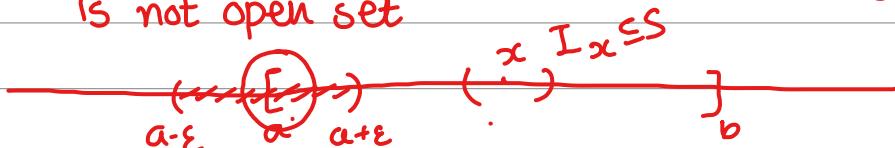
\mathbb{R} $(-\infty, \infty)$ open
Closed both (?)

① Open S , $\forall x \in S \exists I_x \subseteq S \ni x \in I_x \subseteq S$
 if $x = \underline{\underline{0.27}}$, $I_x = (\underline{\underline{0.25}}, \underline{\underline{0.30}}) \quad x \in I_x \subseteq S$



ϵ -nbhd of x , $I_x \in (x - \epsilon, x + \epsilon)$ $\epsilon > 0$

$\equiv S = [a, b]$ is not open set



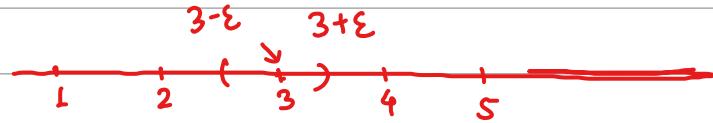
$$(a - \epsilon, a + \epsilon) \subseteq [a, b]$$

$$\underline{(a - \epsilon, a)}$$

open set

$$\begin{bmatrix} \mathbb{R} \checkmark \\ \emptyset \end{bmatrix} \rightarrow x \in \mathbb{R}, I_x \subseteq \mathbb{R} \rightarrow x \in I_x \subseteq \mathbb{R}$$

Set of ① \mathbb{N}



$\exists \delta$

$(3-\varepsilon, 3+\varepsilon) \notin \mathbb{N}$
 \mathbb{N} is not open, II,

Closed set

② \mathbb{Q} is not open

$$\text{let } x = \underline{0.25}. \quad \varepsilon = \underline{0.1} > 0, \quad I_x = (x-\varepsilon, x+\varepsilon) = (\underline{0.15}, \underline{0.35})$$

$\stackrel{\text{in}}{\downarrow} (0.15, 0.35) \notin \mathbb{Q} \leftarrow \text{Set of Rational Nos.}$

[Density Theo. :-? $\forall x, y \in \mathbb{R}, x < y, \exists r \in \mathbb{Q} \rightarrow x < r < y$]
simillarly $\exists r \in \mathbb{Q}^c \rightarrow x < r < y$]

$$\underline{x=0.15}, \quad \underline{y=0.25} \in \mathbb{R} \quad \exists \underline{r \in \mathbb{Q}^c} \Rightarrow \cancel{x < r < y} \quad \underline{0.15 < r < 0.25}$$

$\underline{(0.15, 0.35)} \notin \mathbb{Q}^c$

\mathbb{R}

$$\underline{(0.15, 0.35)} = \{x / \underline{0.15} < x < \underline{0.35}\} \notin \mathbb{Q}$$

$\uparrow \quad \downarrow$
 $\exists \text{ irrational}$

$[a, \infty)$, $\cdot \in \mathbb{R}$ if any nbhd of $a \in S$

Interior point

(a, b) for every pt. in (a, b)
 $\rightarrow x \in \underline{(a, b)}, \exists x \in I_x \subseteq (a, b)$

every pt. of (a, b) is interior pt.

$S = \underline{[a,b]}$, \therefore for $a \in S$, any $\epsilon > 0$, $(a-\epsilon, a+\epsilon) \not\subseteq S$.
 $\therefore a$ is not interior pt. of S .

$$S_i = S - \{\alpha\} = \underline{\underline{(a,b)}}$$

\therefore every pt. of S_i is an interior pt.

Interior Set : Collection of Interior points of sets.

$$S \supseteq S^i$$

$$\begin{array}{lll} [a,b) \ni (a,b) & \leftarrow \text{Set of} \\ & \text{interior} \\ & \text{pts.} \\ \mathbb{R} \ni \mathbb{R} & & [a,\infty) \ni (a,\infty) \\ \emptyset \ni \emptyset & & (-\infty, b] \ni (-\infty, b) \\ [a,b] \ni (a,b) & & \\ [a,b) \ni (a,b) & & \\ [a,b] \ni (a,b) & & \end{array}$$

Finite union of open sets is again open set.

Let A, B as open sets

① If A is open set, then $\forall x \in A \exists I_x \subseteq A. \exists x \in I_x \subseteq A$

if $x \in A \cup B$ and $x \in I_x \subseteq A$
 $A \subseteq \overline{A \cup B}$

$$\Rightarrow x \in I_x \subseteq A \subseteq A \cup B$$

$\Rightarrow A \cup B$ is again open set

$$S = [a, b] \quad S^i = \text{interior set of } S = (a, b)$$

* Finite intersection of open sets is again open

Let $A \& B$ are two open sets.

$$\begin{aligned} \text{then } \forall x \in A &\exists \text{ some } I_x \subseteq A \Rightarrow x \in I_x \subseteq A \\ \forall x \in B &\exists \text{ some } I_x \subseteq B \Rightarrow x \in I_x \subseteq B \end{aligned}$$

$$\text{if } x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

$$\begin{aligned} \text{as } A, B \text{ are open set } \Rightarrow x \in I_x \subseteq A \text{ and } x \in I_x \subseteq B \\ \Rightarrow x \in I_x \subseteq A \& B \\ \Rightarrow x \in I_x \subseteq \overline{A \cap B} \end{aligned}$$

✓ $\Rightarrow A \cap B$ is also open

* Arbitrary Union of open sets is open

Prove: $\{A_i\}_i$ is collection of open sets, $\bigcup_{i=1}^{\infty} A_i$ is open

Let $x \in \bigcup A_i$

$$\Rightarrow x \in A_i \text{ for some } i$$

$(0, 1) \text{ (or, 2)}$

$$\Rightarrow x \in I_x \subseteq A_i \text{ as } A_i \text{ are all open sets}$$

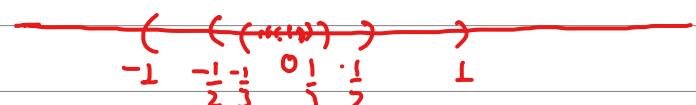
$(0, 1)$ open

$$\Rightarrow x \in I_x \subseteq A_i \subseteq \bigcup_{i=1}^{\infty} A_i \Rightarrow \bigcup A_i \text{ is also open set.}$$

* Arbitrary intersection of open sets may or may not be open

counter example /

$$A_i = \left(-\frac{1}{n}, \frac{1}{n}\right)$$



$$A_1 = (-1, 1)$$

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$\bigcap_{i=1}^{\infty} A_i = \underline{\underline{\{0\}}}$$

$0 \in I_x \notin \{0\}$ \Rightarrow $\bigcap A_i$ is not open here.

$$A_n = (0, \underbrace{1+\frac{1}{n}}_{\text{---}})$$

$$A_1 = (0, 2)$$

$$A_2 = (0, 1 + \frac{1}{2})$$

$$A_3 = (0, 1 + \frac{1}{3})$$

$$\cap A_i = \underline{(0, 1]} \quad \text{is not open set}$$

Closed Set := S is closed if S^c is open.

$$S = [a, \infty) \Rightarrow S^c = (-\infty, a) \text{ open} \Rightarrow S \text{ is closed}$$

$$S = (-\infty, b] \Rightarrow S^c = (b, \infty) \text{ open} \Rightarrow S \text{ is closed}$$

$$S = \underline{\{1, 2, 3\}} \Rightarrow S^c = \underline{(-\infty, 1)} \cup \underline{(1, 2)} \cup \underline{(2, 3)} \cup \underline{(3, \infty)} \therefore \text{finite union of open set is again open}$$

$$\underline{\underline{\mathbb{N}}} = \{1, 2, 3, \dots\} \Rightarrow S^c = \underline{(-\infty, 1)} \cup \bigcup_{n=1}^{\infty} \underline{[n, n+1]} \therefore \text{Arbitrary union of open is open}$$

\mathbb{N} is closed set $\Rightarrow \mathbb{I}, \mathbb{Z}, \dots$ closed sets.

\mathbb{Q} is not closed
 \mathbb{Q}^c is not open
 \mathbb{Q}, \mathbb{Q}^c both are not closed sets.

\mathbb{R} open $\overline{\text{closed}}$

$\mathbb{R}^c = \emptyset \Rightarrow \text{open set}$
 $\Rightarrow \mathbb{R}$ is closed.

$\because \emptyset, \mathbb{R}$ are open and closed

$\emptyset \quad \emptyset^c = \mathbb{R}$ is open $\Rightarrow \emptyset$ is closed.

* Finite union of closed set is closed

intersecting

To prove $A \cup B$ is closed
 $\rightarrow (A \cup B)^c$ is open

limit pt. of set $\{0\} \cup \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$
 not open $\xrightarrow{\text{I is s}}$
 not closed $\xleftarrow{\text{not closed}}$

$$(A \cup B)^c = A^c \cap B^c$$

A^c, B^c are open sets & Finite intersection of opensets is open

$(A \cup B)^c$ is open
 $\Rightarrow A \cup B$ is closed

$$(A \cap B)^c = A^c \cup B^c$$

A^c, B^c open & Finite union of open sets is open
 $\Rightarrow (A \cap B)^c$ is open.

$\frac{1}{n} \rightarrow 0$

Theo:- Arbitrary intersection of closed sets is closed.

(We have discussed,

→ Arbitrary union of open sets is open. ✓

$\{A_i\}$ arbitrary collection of closed sets
 $\Rightarrow \underline{\underline{A_i^c}}$ is open $\Rightarrow i$

$\underline{\underline{\bigcup_{i=1}^{\infty} A_i^c}}$ is open set.

$$\underline{\underline{\bigcup_i A_i^c}} = (\underline{\underline{\bigcap_{i=1}^{\infty} A_i}})^c \text{ open}$$

$\therefore \underline{\underline{\bigcap_{i=1}^{\infty} A_i}}$ is closed set

Arbitrary union of closed sets may or may not be closed.

counter examples $\{[0, 1 - 1/n]\}$ is collection of closed sets



$$A_1 = [0, 1 - 1/2] = [0, 1/2]$$

$$A_2 = [0, 1 - 1/3] = [0, 2/3]$$

$$A_3 = [0, 1 - 1/4] = [0, 3/4]$$

$$S = \bigcup [0, 1 - 1/n] = \underline{\underline{[0, 1)}}$$

$$S^c = \underline{\underline{(-\infty, 0)}} \cup \underline{\underline{[1, \infty)}}$$

\nwarrow not open

$\Rightarrow S^c$ is not open $\Rightarrow S$ is not closed.

