

# Real Analysis

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# Introduction to Real Analysis

## 1.1 The Algebraic Properties of $\mathbb{R}$

Algebraic Properties of  $\mathbb{R}$  On the set  $\mathbb{R}$  of real numbers there are two binary operations, denoted by  $+$  and  $\cdot$  and called addition and multiplication, respectively. These operations satisfy the following properties :

- (A1)  $a + b = b + a$  for all  $a, b \in \mathbb{R}$  (commutative property of addition);
- (A2)  $(a + b) + c = a + (b + c)$  for all  $a, b, c \in \mathbb{R}$  (associative property of addition) ;

- (A3) there exists an element  $0 \in \mathbb{R}$  such that  $0 + a = a$  and  $a + 0 = a$  for all  $a \in \mathbb{R}$  (existence of a zero element) ;
- (A4) for each  $a \in \mathbb{R}$  there exists an element  $a \in \mathbb{R}$  such that  $a + (-a) = 0$  and  $(-a) + a = 0$  (existence of negative elements) ;
- (M1)  $ab = ba$  for all  $a, b \in \mathbb{R}$  (commutative property of multiplication) ;
- (M2)  $(ab)c = a(bc)$  for all  $a, b, c \in \mathbb{R}$  (associative property of multiplication) ;
- (M3) there exists an element  $1 \in \mathbb{R}$  distinct from 0 such that  $1a = a$  and  $a1 = a$  for all  $a \in \mathbb{R}$  (existence of a unit element) ;
- (M4) for each  $a \in \mathbb{R} - \{0\}$  there exists an element  $1/a \in \mathbb{R}$  such that  $a(1/a) = (1/a)a = a$  and
- (D)  $a(b + c) = (ab) + (ac)$  and  $(b + c)a = (ba) + (ca)$  for all  $a, b, c \in \mathbb{R}$  (distributive property of multiplication over addition).

## 1.2 The Order Properties of $\mathbb{R}$

There is a nonempty subset  $\mathbb{R}^+$  of  $\mathbb{R}$ , called the set of positive real numbers, that satisfies the following properties :

1. If  $a, b \in \mathbb{R}^+$ , then  $a + b \in \mathbb{R}$ .
  2. If  $a, b \in \mathbb{R}^+$ , then  $ab$  belongs to  $\mathbb{R}$ .
  3. If  $a \in \mathbb{R}$ , then exactly one of the following holds :  $a \in \mathbb{R}^+$  OR  $a = 0$  OR  $(-a) \in \mathbb{R}^+$ .
1. Let  $a, b, c \in \mathbb{R}$

if  $a > b$  and  $b > c$  then  $a > c$

Given that,

$$a > b \text{ and } b > c$$

$$\therefore a - b > 0 \text{ and } b - c > 0 \cdots \text{i.e } (a - b), (b - c) \in \mathbb{R}^+$$

$$\therefore (a - b) + (b - c) > 0 \cdots (1^{st} \text{ order prop})$$

$$\therefore a - c > 0 \Rightarrow a > c$$

2. If  $a > b$  then  $a + c > b + c$

Given that,

$$a > b \text{ i.e. } a - b > 0$$

$$\therefore a - b \in \mathbb{R}^+$$

$$\therefore a + c - c - b > 0$$

$$\therefore (a + c) - (b + c) > 0$$

$$\therefore a + c > b + c$$

3. If  $a > b$  and  $c > 0$  then,  $ca > cb$

Given that,  $a > b$  &  $c > 0 \therefore (a - b) > 0 \& c > 0$

$$\text{i.e. } (a - b), c \in \mathbb{R}^+$$

$$\therefore (a - b) \cdot c \in \mathbb{R}^+ \dots (2^{\text{nd}} \text{ order prop})$$

$$\therefore (a - b) \cdot c > 0 \Rightarrow a \cdot c - bc > 0 \Rightarrow ac > bc$$

4. If  $a > b$  and  $c < 0$  then,  $ca < cb$

Given that,  $a > b$  &  $c < 0$

$\therefore (a - b) \in \mathbb{R}^+ \& -c \in \mathbb{R}^+ \dots (3^{\text{rd}} \text{ order prop})$

$\therefore -c(a - b) > 0$

$\therefore -ca + cb > 0$

$\therefore cb > ca$

$\therefore ca < cb$

### 1.3 Absolute Value and Real Line

#### Absolute value and Real line

Absolute value:-

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } +a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

**Theorem 1.3.1.** For  $a, b \in \mathbb{R}$

a)  $|ab| = |a|.|b| \forall a, b \in \mathbb{R}$

b)  $|a|^2 = a^2 \forall a \in \mathbb{R}$

c) if  $c > 0$  then  $|a| \leq c$  iff  $-c \leq a \leq c$

d)  $-|a| \leq a \leq |a| \forall a \in \mathbb{R}$

*Proof.* a)  $|ab| = |a| \cdot |b| \forall a, b \in \mathbb{R}$

- if  $a = 0$  or  $b = 0 \Rightarrow ab = 0 = |ab| = |a| \cdot |b|$

- if  $a > 0$  or  $b > 0 \Rightarrow ab > 0$

$$|ab| = a \cdot b = |a| \cdot |b|$$

- if  $a > 0$  or  $b < 0 \Rightarrow ab < 0$

$$\therefore |ab| = -a \cdot b = (-a) \cdot b = a \cdot (-b) = |a| \cdot |b|$$

- if  $a < 0$  or  $b > 0 \Rightarrow ab < 0$

$$\therefore |ab| = -ab = |a| \cdot |b|$$

- if  $a < 0$  or  $b < 0 \Rightarrow ab > 0$

$$\therefore |ab| = ab = |a| \cdot |b|$$

- Hence proved -

b)  $|a|^2 = a^2 \forall a \in \mathbb{R}$

$$\forall a \in \mathbb{R}, a^2 \in \mathbb{R} \text{ i.e } a^2 \geq 0$$

$$\text{let } |a^2|^2 = |a| \cdot |a| = a \cdot a = a^2, \text{ if } a > 0$$

$$(-a) \cdot (-a) = a^2, \text{ if } a < 0$$

$$\text{Hence, } |a|^2 = a^2$$

c) if  $c > 0$  then  $|a| \leq c$  iff  $-c \leq a \leq c$

Given that,

$$c > 0 \text{ & } |a| \leq c$$

i) To show  $-c \leq a \leq c$

$$\text{Now, } |a| = \max(a, -a) \leq c$$

$$\Rightarrow a \leq c \& -a \leq c$$

$$\Rightarrow a \leq c \& a \geq -c$$

$$-c \leq a \leq c$$

ii) Given that,  $-c \leq a \leq c$  & To show:-  $|a| \leq c$

$$\Rightarrow a \leq c \& -a \leq -c$$

$$\therefore |a| \leq c \dots (|a| = \max(a, -a))$$

d) For  $a \neq 0 \in \mathbb{R}$ ,  $|a| > 0 \dots |a| = \max(a, -a)$

Put  $c = |a| > 0$  in c)

$$\therefore -c \leq a \leq c \Rightarrow -|a| \leq a \leq |a|$$

□

## 1.4 Triangular Inequality

### Triangular Inequality:-

**Theorem 1.4.1.** If  $a, b \in \mathbb{R}$  then  $|a + b| \leq |a| + |b|$

*Proof.* if  $a, b \in \mathbb{R}$  then

$$-|a| \leq a \leq |a|$$

+

$$-|b| \leq b \leq |b|$$


---

$$-(|a| + |b|) \leq a + b \leq (|a| + |b|) \dots (\text{Theorem:-1.1.1-d}))$$

let  $|a| + |b| = c$

$$\therefore -c \leq a + b \leq c$$

$$\Rightarrow |a + b| \leq c \dots \dots (Th^m - 1.1.1 - c)$$

$$\therefore |a + b| \leq |a| + |b|$$

□

**Corollary 1.4.1.1.** If  $a, b \in \mathbb{R}$

a)  $||a| - |b|| \leq |a - b|$

b)  $|a - b| \leq |a| + |b|$

*Proof.* a) We know that,  $a, b \in \mathbb{R}$

$$a = a - b + b$$

$$\therefore |a| = |a - b + b| \leq |a - b| + |b|$$

$$\Rightarrow |a| - |b| \leq |a - b| \quad (1.1)$$

Also,  $b = b - a + a$

$$|b| = |b - a + a| \leq |b - a| + |a|$$

$$\therefore |b| - |a| \leq |a - b| \quad (1.2)$$

from equation (1.1) & (1.2)

$$||a| - |b|| \leq |a - b|$$

b)

if  $a, b, c \in \mathbb{R}$

$$\therefore |a + c| \leq |a| + |c|$$

Put  $c = -b$ ,  $|c| = |-b| = |b|$

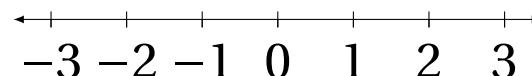
$$\therefore |a + (-b)| \leq |a| + |-b|$$

$$\therefore |a - b| \leq |a| + |b|$$

□

**Corollary 1.4.1.2.** If  $a_1, a_2 \dots a_n$  are any real no then  $|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$

**Definition 1.4.1** (Real line): A convenient and Familiar interpretation of real no system is the real line.



**Definition 1.4.2** ( $\epsilon$ -Neighbourhood:-): let  $a \in \mathbb{R}$  &  $\epsilon > 0$ , then  $\epsilon$ - neighbourhood of  $a$  is the set

$$V_\epsilon(a) = \{x | x \in \mathbb{R}, |x - a| < \epsilon\} \dots 0 \leq |x - a| < \epsilon$$

$$\therefore V_\epsilon(a) = (a - \epsilon, a + \epsilon) \subseteq \mathbb{R}$$

Since  $|x - a| < \epsilon \Rightarrow -\epsilon < x - a < \epsilon \Rightarrow a - \epsilon < x < a + \epsilon$

**Definition 1.4.3** (Deleted- $\epsilon$ -Neighbourhood:-):  $\delta_\epsilon(a) = v_\epsilon(a) - \{a\}$

$$= (a - \epsilon, a + \epsilon) - \{a\}$$

$$i.e 0 < |x - a| < \epsilon$$

### Example 1:

If  $a, b \in \mathbb{R}$ . Show that  $|a + b| = |a| + |b|$  if and only if  $ab \geq 0$

*Proof.* i) Given that  $ab \geq 0$ , To prove-  $|a + b| = |a| + |b|$

if  $ab \geq 0 \Rightarrow a \geq 0, b \geq 0$  or  $a \leq 0, b \leq 0$

$$a + b \geq 0$$

$$\therefore |a| = a, |b| = b$$

$$|a + b| = a + b$$

$$= |a| + |b|$$

$$a + b \leq 0$$

$$\therefore |a| = -a, |b| = -b$$

$$|a + b| = -(a + b)$$

$$= -a - b$$

$$= |a| + |b|$$

ii) Given that  $|a + b| = |a| + |b|$ , To prove  $ab \geq 0$

$$|a + b|^2 = (|a| + |b|)^2$$

$$\therefore a^2 + b^2 + 2ab = |a|^2 + |b|^2 + 2 \cdot |a| \cdot |b|$$

$$\therefore 2ab = 2|a|.|b| \dots (\because |a|^2 = a^2)$$

$$ab = |a| \cdot |b|$$

$$ab = |ab| \dots (\text{Theorem:- 1.1.1-a}))$$

$$\therefore ab \geq 0$$

□

### Example 2:

Show that if  $a, b \in \mathbb{R}$  then

i)  $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|), \min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$

ii)  $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$

*Proof.* i) let  $a > b \Rightarrow |a - b| = a - b$

$$\max(a, b) = a \quad (1.3)$$

Consider, RHS

$$\begin{aligned} &= \frac{1}{2}(a + b - |a - b|) \\ &= \frac{1}{2}(a + b + a - b) \dots \text{from (1.3)} (a - b) \geq 0 \\ &= a \\ &= \text{LHS} \end{aligned}$$

$$\text{let } \min(a, b) = b \quad (1.4)$$

Consider, RHS =  $\frac{1}{2}(a + b - |a - b|)$

$$= \frac{1}{2}(a + b - (a - b))$$

$$= b$$

$$= \text{LHS}$$

ii) Suppose,  $a > b > c$

$$\text{LHS} = \min\{a, b, c\} = c$$

$$\text{RHS} = \min\{\min\{a, b\}, c\} = \min\{b, c\}$$

$$\text{RHS} = C$$

$$= \text{LHS}$$

$$\text{Hence, } \min\{a, b, c\} = \min\{\min\{a, b\}, c\}$$



### Example 3:

If  $x, y, z \in \mathbb{R}$  &  $x \leq z$ , Show that  $x \leq y \leq z$  if and only if  $|x - y| + |y - z| = |x - z|$

$$x \leq z \Rightarrow x - z \leq 0 \therefore |x - z| = z - x$$

*Proof.* i) Given that  $x \leq y \leq z, x, y, z \in \mathbb{R}$

$$\therefore |x - y| = y - x \& |y - z| = z - y$$

To show  $|x - y| + |y - z| = |x - z|$

$$\text{Consider, LHS} = |x - y| + |y - z|$$

$$= y - x + z - y$$

$$= z - x$$

$$= |x - z|$$

$$= \text{RHS}$$

ii) Given that  $|x - y| + |y - z| = |x - z|$

To show,  $x \leq y \leq z$

$$\text{let } a = (x - y), b = (y - z)$$

$$\therefore |(x - y) + (y - z)| = |x - y| + |y - z|$$

$$\Rightarrow (x - y)(y - z) \geq 0 \dots (\because \text{if } |a + b| = |a| + |b| \Leftrightarrow ab \geq 0)$$

$$\therefore a, b \geq 0$$

$$\text{i.e. } (x - y), (y - z) \geq 0$$

$$x \geq y, y \geq z$$

$$\therefore x \geq y \geq z$$

which is not possible Since  $x \leq z$ -(given)

$$a, b \leq 0$$

$$(x - y), (y - z) \leq 0$$

$$\therefore x \leq y, y \leq z$$

$$\therefore x \leq y \leq z$$



**Example 4:**

If  $a < x < b, a < y < b$ . Show that  $|x - y| < b - a$ .

*Proof.* Given that,

$$a < x < b, a < y < b$$

$$0 < x - a < b - a \tag{1.5}$$

$$0 < y - a < b - a \tag{1.6}$$

multiplying by (-1) to (1.6) and add in (1.5)

$$\begin{array}{rcl} 0 & \leqslant & -a \\ + & & -(b-a) \\ \hline & & -(b-a) \leqslant x - y \leqslant b - a \end{array}$$

$$-(b-a) \leqslant x - y \leqslant b - a \Rightarrow |x - y| < b - a$$

□

**Definition 1.4.4** (Upper bound): Let  $S \neq \emptyset \subseteq \mathbb{R}$ , the set  $s$  is said to be bounded above if  $\exists a \in \mathbb{R} \stackrel{s.t}{\Rightarrow} x \leq a \forall x \in S$  Each such ' $a$ ' is called as upper bound of  $S$ .

**Definition 1.4.5** (Lower bound): Let  $S \neq \emptyset \subseteq \mathbb{R}$ . The set  $S$  is said to be bounded below if  $\exists b \in \mathbb{R} \stackrel{s.t}{\Rightarrow} x \geq b \forall x \in S$  Each such  $b$  is called as lower bound of  $S$ .

**Definition 1.4.6** (Bounded Set): If both lower and upper bound exist.

**Definition 1.4.7** (Unbounded set): If set  $S$  is not bounded.

**Definition 1.4.8** (Supremum & Infimum): Let  $S$  be a non-empty subset of  $\mathbb{R}$  if  $S$  is bounded above/below then a no  $u$  is said to be supremum/Infimum (least upper bound or greatest lower bound) of  $S$  if it satisfies the conditions:-

- i)  $u$  is an upper(lower) bound of  $S$ .
- ii) if  $v$  is any upper(lower) bound of  $S$  then  $u \leq v (u \geq v)$ .

## 1.5 Completeness Property

**Statement:-**If set is bounded below then its infimum must be exists and if set is bounded above then its supremum must be exists this property is known as completeness property.

let  $\mathbb{N} = 1, 2, \dots$  bounded below

Unbounded= $\{\infty\}$  = Supremum

Lower bound=  $\{\infty, \dots, -1, 0, 1\}$  = Infimum = 1

### Example 5:

Let  $A \subseteq B$  then Prove that,

I)  $\inf A \geq \inf B$

II)  $\sup A \leq \sup B$

*Proof.* I) Given that,  $A \subseteq B, x \in A \Rightarrow x \in B$

also,  $\inf A = u$  and  $\inf B = v \dots$  (assume)

if  $u$  is  $\inf A$  then, by definition,

i)  $u$  is lower bound,  $x \geq u \forall x \in A$

ii) if  $u_1$  is another lower bound, then  $u_1 < u \forall u_1$ . Assume that,  $\inf B \geq \inf A$

Assume that,  $\inf B \geq \inf A$

i.e  $v \geq u$

i.e  $x \geq v \geq u, \forall x \in B$

$\therefore x \geq v \geq u, \forall x \in A$

$\Rightarrow$  if  $u$  is  $\inf$ , we can not have lower bound greater than  $u$ .

So, our assumption is wrong.

Hence,  $u \geq V$  i.e  $\inf A \geq \inf B$

II) let  $\sup A = u$  and  $\sup B = v$

if  $u$  is supremum of  $A$  then, by definition

- i)  $u$  is upper bound of  $A$  i.e  $x \leq u, \forall x \in A$
- ii) if  $u_1$  is any other upper bound then  $u \leq u_1 \forall u_1$

Assume that,  $\text{Sup } A \geq \text{sup } B$

$$u \geq v$$

i.e  $v \leq u$

$$x \leq v \leq u, \forall x \in B$$

$$x \leq v \leq u, \forall x \in A$$

$\Rightarrow$  if  $u$  is sup of  $A$  then we can not have upper bound less than  $u$ . So assumption is wrong.

Hence,  $u \leq V$  i.e  $\text{sup } A \leq \text{sup } B$



### Example 6:

$S = 1 - \frac{(-1)^n}{n}, n \in \mathbb{N}$ . Find infimum & suptemum

$$S = \{2, 1/2, 1 + 1/3, 1 - 1/4, 1 + 1/5, 1 - 1/6, \dots\}$$

$$\therefore \inf s = 1/2 \text{ of } \sup s = 2$$

### Example 7:

$$S = \frac{(-1)^n}{n}, n \in \mathbb{N}$$

$$S = \{-1, 1/2, -1/3, 1/4, -1/5, \dots\}$$

$$LB = (-\infty, -1] \Rightarrow \text{Inf} = -1 \in S,$$

$$UB = [1/2, \infty) \Rightarrow \sup = 1/2 \in S$$

### Example 8:

$$S = \left\{ \frac{1}{m} - \frac{1}{n}, m, n \in \mathbb{N} \right\}$$

$$S = \{0, 1/2, -1/2, 1 - 1/3, -2/3, 1, -1, \dots\}$$

$$LB = (-\infty, -1] \Rightarrow \inf S = -1 \notin S,$$

$$UB = (1, \infty) \Rightarrow \sup S = 1 \notin S$$

### Example 9:

$$S = \left\{ \frac{n-1}{n}, n \in \mathbb{N} \right\} = \left\{ 1 - \frac{1}{n}, n \in \mathbb{N} \right\}$$

$$S = \{0, 1 - 1/2, 1 - 1/3, 1 - 1/4, \dots\}$$

$$LB = (-\infty, 0] \Rightarrow \inf S = 0 \in S,$$

$$UB = [1, \infty) \Rightarrow \sup S = 1 \notin S$$

Chapter **2**

## Sets Operations

### 2.1 Set Operations

1. Union  $A \cup B = \{x / x \in A \text{ or } x \in B\}$
2. Intersection  $A \cap B = \{x / x \in A \text{ and } x \in B\}$
3. Complement  $A^c = \{x / x \in A, x \in \Omega\}$
4. Substraction  $A - B = A \setminus B = A \cap B^c = \{x / x \in A \text{ but } x \notin B\}$

**Theorem 2.1.1.** if  $A, B, C$  are sets then,

$$a) A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

$$b) A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)^C$$

*Proof.* To Prove:-

$$\text{i) } A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

$$\text{ii) } (A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$$

$$\text{i) } A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

$$\text{i) let } x \in A \setminus (B \cup C) \text{ i.e } x \in A \cap (B \cup C)^C$$

$$\Rightarrow x \in A \text{ and } x \in (B \cup C)^C$$

$$\Rightarrow x \in A \text{ and } x \notin (B \cup C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ and } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \cap B^C \text{ and } x \in A \cap C^C$$

$$\Rightarrow x \in (A \setminus B) \cap (A \setminus C)$$

$$\therefore A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C) \quad (2.1)$$

ii)  $x \in (A \setminus B) \cap (A \setminus C)$

$$\Rightarrow x \in A \setminus B \text{ and } x \in A \setminus C$$

$$\Rightarrow x \in (A \cap B^C) \text{ and } x \in (A \cap C^C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ and } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \in (B \cup C)^C)$$

$$\Rightarrow x \in A \cap (B \cup C)$$

$$A \setminus B \cap A \setminus C \subseteq A \setminus (B \cup C) \quad (2.2)$$

from (2.1) & (2.2)

$$A \setminus (B \cup C) = A \setminus B \cap A \setminus C$$

ii)  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

To Prove:-

i)  $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$

ii)  $A \setminus B \cup A \setminus C \subseteq A \setminus (B \cap C)$

i) let  $x \in A \setminus (B \cap C)$  i.e  $x \in A \cap (B \cap C)^C$

$$\Rightarrow x \in A \text{ and } x \in (B \cap C)^C$$

$$\Rightarrow x \in A \text{ and } x \notin (B \cap C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ or } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \cap B^C \text{ or } x \in A \cap C^C$$

$$\Rightarrow x \in (A \setminus B) \cup (A \setminus C)$$

$$\therefore A \setminus (B \cap C) \subseteq A \setminus B \cup A \setminus C \quad (2.3)$$

ii)  $x \in A \setminus B \cup A \setminus C$

$$\Rightarrow x \in A \setminus B \text{ or } x \in A \setminus C$$

$$\Rightarrow x \in (A \cap B^C) \text{ or } x \in (A \cap C^C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ or } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \in (B \cap C)^C)$$

$$\Rightarrow x \in A \cap (B \cap C)^C$$

$$\Rightarrow x \in A \setminus (B \cap C)$$

$$A \setminus B \cup A \setminus C \subseteq A \setminus (B \cap C) \quad (2.4)$$

from (2.3) & (2.4)

$$A \setminus (B \cap C) = A \setminus B \cup A \setminus C$$

-Hence Proved- □

## 2.2 Distributive Law

### Distributive Law:-

a)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

*Proof.* a) To Prove:-

$$\text{i) } A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

let  $x \in A \cup (B \cap C)$

$$\Rightarrow x \in A \text{ or } x \in B \cap C$$

$$\Rightarrow x \in A \text{ or } x \in B \text{ and } x \in C$$

$$\Rightarrow x \in A \text{ or } x \in B \text{ and } x \in A \text{ or } x \in C$$

$$\Rightarrow x \in A \cup B \text{ and } x \in A \cup C$$

$$\Rightarrow x \in (A \cup B) \cap (A \cup C)$$

$$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \quad (2.5)$$

$$\text{ii) } (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

let  $x \in (A \cup B) \cap (A \cup C)$

$$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$$

$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$

$$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$$

$$\Rightarrow x \in A \cup (B \cap C)$$

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \quad (2.6)$$

from (2.5) & (2.6)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

b) To Prove:-

i)  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

let  $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$  and  $x \in (B \cup C)$

$\Rightarrow x \in A$  and  $(x \in B$  or  $x \in C)$

$\Rightarrow (x \in A$  and  $x \in B)$  or  $(x \in A$  and  $x \in C)$

$\Rightarrow x \in (A \cap B)$  or  $(x \in A \cap C)$

$\Rightarrow x \in (A \cup B) \cup (A \cap C)$

$$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad (2.7)$$

ii)  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

let  $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in (A \cap B)$  or  $x \in (A \cap C)$

$\Rightarrow (x \in A$  and  $x \in B)$  or  $(x \in A$  and  $x \in C)$

$\Rightarrow x \in A$  and  $(x \in B$  or  $x \in C)$

$$\Rightarrow x \in A \text{ and } (x \in B \cup C)$$

$$\Rightarrow x \in A \cap (B \cup C)$$

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \quad (2.8)$$

from (2.7) & (2.8)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

□

**Theorem 2.2.1.** If  $A$  &  $B$  are sets, Show that  $A \subseteq B$  if and only if  $A \cap B = A$

*Proof.* i) Assume that  $A \subseteq B$  to Prove that  $A \cap B = A$

$$\text{let } x \in A \Rightarrow x \in B$$

$$\therefore x \in B \dots (\because A \subseteq B)$$

$$\Rightarrow x \in A \cap B$$

$$\therefore A \subseteq A \cap B \quad (2.9)$$

Also, by definition,

$$A \cap B \subseteq A \quad (2.10)$$

from (2.9) and (2.10)

$$A = A \cap B \quad (2.11)$$

ii) Assume that  $A \cap B = A$ , to prove  $A \subseteq B$

We know that,  $A \cap B \subseteq B$

$$\Rightarrow A \subseteq B \quad (2.12)$$

from (2.11) and (2.12)

$$A \subseteq B \text{ iff } A = A \cap B \quad (2.13)$$

-Hence Proved-

□

### 2.3 Basic Notations Theory

**Definition 2.3.1** (Cartesian Product): let  $A \& B$  be two sets,

$A = \langle 2, 3, 4 \rangle \& \langle 1, 5, 6 \rangle$  then cartesian product is given by

$$A \times B = \{\langle 2, 1 \rangle, \langle 2, 5 \rangle, \langle 2, 6 \rangle, \langle 3, 5 \rangle, \langle 3, 6 \rangle, \langle 4, 1 \rangle, \langle 4, 5 \rangle, \langle 4, 6 \rangle\}$$

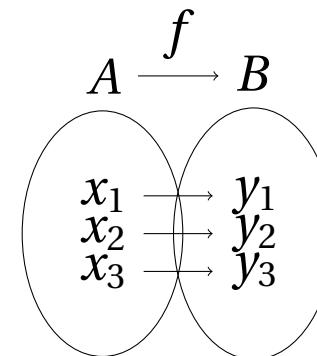
**Definition 2.3.2** (Function): Let  $A \& B$  be sets then a function from  $A$  to  $B$  is a set  $f$  of ordered

pairs in  $A \times B$  such that for each  $a \in A$  then there exists a unique  $b \in B$  with  $(a, b) \in f$ .

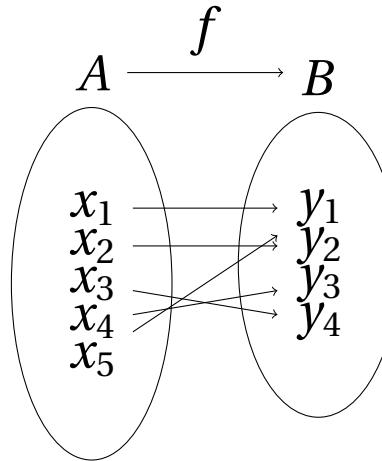
In other words if  $\langle a, b \rangle \in f \& \langle a, b' \rangle \in f \Rightarrow b = b'$

## Types of Function

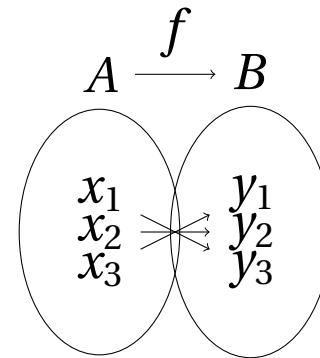
**Definition 2.3.3** (One-One (Injective) Function): *The Function  $f$  is said to be injective (or One-One) if whenever  $x_1 \neq x_2$ , then  $f(x_1) \neq f(x_2)$ .*



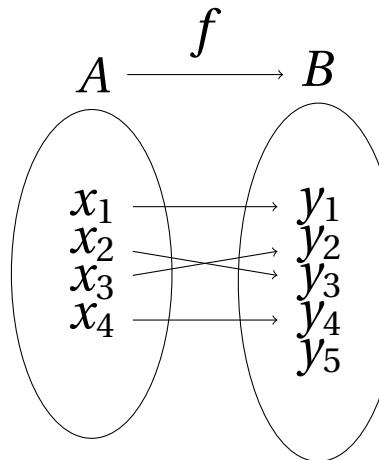
**Definition 2.3.4** (Onto (Surjective) Function): *The function  $f$  is said to be Surjective if  $f(A) = B$  i.e if the range  $R(f) = B$ .*



**Definition 2.3.5** (One-One & Onto (Bijective) Function): *The Function  $f$  is both one-one and onto then it is said to be bijective.*



**Definition 2.3.6** (Into Function): *If  $f$  is not onto then it is called as into function.*



**Definition 2.3.7** (Composite Function): If  $f : A \rightarrow B$  and  $g : A \rightarrow C$  and if  $R(f) \subseteq D(g) = B$  then the composite function  $g \circ f$  is the function from  $A \rightarrow C$   
 $g \circ f : A \rightarrow C$  is composite function if  $g \circ f(x) = g(f(x))$   $x \in A$

### Example 10:

$$f : \mathbb{R} \rightarrow \mathbb{R}, g : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 2x, g(y) = 3y^2 - 1$$

*Proof.* Given that,  $f(x) = 2x, g(y) = 3y^2 - 1$

$$g \circ f(x) = g(f(x))$$

$$= g(2x)$$

$$= 3(2x)^2 - 1$$

$$= 12x^2 - 1$$

$$f \circ g(y) = f(g(y))$$

$$= f(3y^2 - 1)$$

$$= 2(3y^2 - 1)$$

$$= 6y^2 - 2$$

$$\therefore g \circ f \neq f \circ g$$

□

**Example 11:**

Show that if  $f : A \rightarrow B$  then,  $E, F$  are subsets of  $A$  then,

a)  $f(E \cup F) = f(E) \cup f(F)$  and

b)  $f(E \cap F) \subseteq f(E) \cap f(F)$

*Proof.* a)  $f : A \rightarrow B, E, F \subseteq A$

$$f(E) = \{y / y = f(x), x \in E \subseteq A\} \subseteq B$$

$$f(F) = \{y / y = f(x), x \in F \subseteq A\} \subseteq B$$

$$f(E \cup F) = \{y / y = f(x), x \in E \cup F\}$$

To Prove,

i)  $f(E \cup F) \subseteq f(E) \cup f(F)$

ii)  $f(E) \cup f(F) \subseteq f(E \cup F)$

let  $y \in f(E \cup F)$

$$\Leftrightarrow y = f(x), x \in E \cup F$$

$$\Leftrightarrow y = f(x), x \in E \text{ or } x \in F$$

$$\Leftrightarrow y = f(x), x \in E \subseteq A \text{ or } y = f(x), x \in F \subseteq A$$

$$\Leftrightarrow y \in f(E) \text{ or } y \in f(F)$$

$$\Leftrightarrow y \in f(E) \cup f(F)$$

$$\therefore f(E \cup F) \subseteq f(E) \cup f(F) \& f(E) \cup f(F) \subseteq f(E \cup F)$$

$$f(E \cup F) = f(E) \cup f(F)$$

To Prove,

b)  $f(E \cap F) \subseteq f(E) \cap f(F)$

let  $y \in f(E \cap F)$

$$\Rightarrow y = f(x), x \in E \cap F$$

$$\Rightarrow y = f(x), x \in E \text{ and } x \in F$$

$$\Rightarrow y = f(x), x \in E \text{ and } y = f(x), x \in F$$

$$\Rightarrow y \in f(E) \text{ and } y \in f(F)$$

$$\Rightarrow y \in f(E) \cap f(F)$$

$$\therefore f(E \cap F) \subseteq f(E) \cap f(F)$$

□

### **Example 12:**

Example for  $f(E) \cap f(F) \subsetneq f(E \cap F)$

let  $f(x) = x^2$

$$E = \{1, 2\}, f(E) = \{1, 4\}$$

$$F = \{-2, 4\}, f(F) = \{4, 16\}$$

$$E \cap F = \{\phi\}, f(E) \cap f(F) = \{4\}$$

$$f(E \cap F) = \{\phi\}$$

$$f(E) \cap f(F) \subsetneq f(E \cap F)$$

**Example 13:**

Show that if  $f : A \rightarrow B$  and  $G, H$  are subsets of  $B$  then,

a)  $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$  and

b)  $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

*Proof.* a)  $f : A \rightarrow B$

$$f^{-1}(G) = \{x / f(x) \in G\} \subseteq A$$

$$f^{-1}(H) = \{x / f(x) \in H\} \subseteq A$$

$$\text{let } x \in f^{-1}(G \cup H)$$

$$\Leftrightarrow f(x) \in G \cup H$$

$$\Leftrightarrow f(x) \in G \text{ or } f(x) \in H$$

$$\Leftrightarrow x \in f^{-1}(G) \text{ or } x \in f^{-1}(H)$$

$$\Leftrightarrow x \in f^{-1}(G) \cup f^{-1}(H)$$

$$\therefore f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$$

*-Hence Proved-*

b) let  $x \in f^{-1}(G \cap H)$

$$\Leftrightarrow f(x) \in G \cap H$$

$$\Leftrightarrow f(x) \in G \text{ and } f(x) \in H$$

$$\Leftrightarrow x \in f^{-1}(G) \text{ and } x \in f^{-1}(H)$$

$$\Leftrightarrow x \in f^{-1}(G) \cap f^{-1}(H)$$

$$\therefore f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$$

*-Hence Proved-*

**Example 14:**

Show that if  $f : A \rightarrow B$  is injective &  $E \subseteq A$  then  $f^{-1}(f(E)) = E$ . Give an example to show that equality need not hold if  $f$  is not injective.

*Proof.* Given that,  $f : A \rightarrow B$  is injective

i.e if  $x \neq y \Rightarrow f(x) \neq f(y) \forall x, y \in A$

$$E = \{x / x \in E, f(x) \in B\} \subseteq A$$

$$f(E) = \{y / y = f(x) \in f(E), x \in A\} \subseteq B$$

To prove  $f^{-1}(f(E)) = E$

$$\text{let } x \in f^{-1}(f(E))$$

$$\Rightarrow f(x) \in f(E)$$

$\Rightarrow x \in E \dots (\because f \text{ is one-one function})$

$$f^{-1}(f(E)) \subseteq E \quad (2.14)$$

Now, let  $x \in E$

$$\Rightarrow f(x) \in f(E) \quad f^{-1}(H) = \{x / f(x) \in H, x \in A\}$$

$$x \in f^{-1}(f(E))$$

$$E \subseteq f^{-1}(f(E)) \quad (2.15)$$

from (2.14) & (2.15)

$$f^{-1}(f(E)) = E$$

□

**Example 15:**

$$\text{let } f(x) = x^2$$

$$E\{1, 2\} \Rightarrow f(E)\{1, 4\}$$

$$f^{-1}(f(E)) = \{(1, -2, -2)\}$$

$$f^{-1}(f(E)) \neq E$$

**Example 16:**

Show that if  $f : A \rightarrow B$  is surjective and  $E \subseteq A$  then  $f(f^{-1}(H)) = H$ . Give an example to show that equality need not hold if  $f$  is not surjective.

*Proof.*  $f : A \rightarrow B, H \subseteq B$  and  $f$  is surjective i.e every element in  $B$  has inverse image in  $A$

To prove:  $f(f^{-1}(H)) = H$

$$\text{let } y \in f(f^{-1}(H))$$

$$\Rightarrow f(x) \in f(f^{-1}(H))$$

$$\Rightarrow x \in f^{-1}(H)$$

$$\Rightarrow y = f(x) \in H$$

$$\therefore f(f^{-1}(H)) \subseteq H \quad (2.16)$$

let  $y \in H$  then

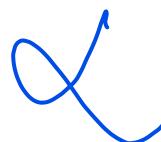
$\exists x \in A$  such that,

$$y = f(x) \in H \dots (\because f \text{ is onto})$$

$$\Rightarrow x \in f^{-1}(H)$$

$$\Rightarrow f(x) \in f(f^{-1}(H)) \dots (x \in E \Rightarrow f(x) \in f(E))$$

$$\Rightarrow y \in f^{-1}(H)$$



$$\therefore H \subseteq f(f^{-1}(H)) \quad (2.17)$$

from (2.16) & (2.17)

$$f(f^{-1}(H)) = H$$

$\phi$

□

- Definition 2.3.8** (Finite & Infinite Sets):
- 1. The empty set  $\phi$  is said to have zero elements.
  - 2. If  $n \in \mathbb{N}$ , a set  $S$  is said to have  $n$  elements if there exists a bijection from set  $\mathbb{N} = \{1, 2, \dots, n\}$  onto  $S$ .
  - 3. A set  $S$  is said to be finite if it is either empty or it has  $n$  elements for some  $n \in \mathbb{N}$ .
  - 4. A set  $S$  is said to be infinite if it is not finite.

**Theorem 2.3.1** (Uniqueness Theorem). If  $S$  is finite set, then the number of elements in  $S$  is

unique number in  $\mathbb{N}$ .

fixed

The set  $\mathbb{N}$  of natural numbers is an infinite set.

**Theorem 2.3.2.** Suppose that  $S$  &  $T$  are sets and  $T \subseteq S$

subsets of  $\mathbb{R}$

-  $S, T \subseteq \mathbb{R}$ , &  $T \subseteq S$ .

a) If  $S$  is finite Set, then  $T$  is a finite Set.

b) If  $T$  is an infinite set then  $S$  is an infinite Set.

*Proof.* a)  $T \subseteq S$  and  $S$  is finite Set

i) Suppose  $S = \phi \Rightarrow T = \phi \Rightarrow T$  is finite

ii) When  $S \neq \phi$  then there are two possibilities.

1)  $T = \phi \Rightarrow T$  is a finite Set **or**

2)  $T \neq \phi$

We will prove this by method of mathematical induction.

- $\#(S) = 1$  and as  $T \neq \phi \Rightarrow S = T$

Hence as  $S$  is finite  $\Rightarrow T$  is finite

- Now assume that this statement is true for  $\#(S) = k$

i.e  $\#(S) = k \& T \subseteq S \Rightarrow T$  is finite set.

- Now, let's prove it for  $\#(S) = k + 1$

As  $S$  is finite, it has bijection with  $N_{k+1}$

$$S = \{f(1), f(2), \dots, f(k+1)\} \quad (2.18)$$

let's define,  $S_1 = S - f(k+1)$

$$\therefore \#(S)_1 = k \text{ and } T_1 = T - f(k+1) \quad \#(S_1)$$

Now, if  $f(k+1) \notin T \Rightarrow T_1 = T \subseteq S_1$

and as  $\#(S)_1 = k \& T \subseteq S_1 \subseteq T$  is finite  $\Rightarrow$

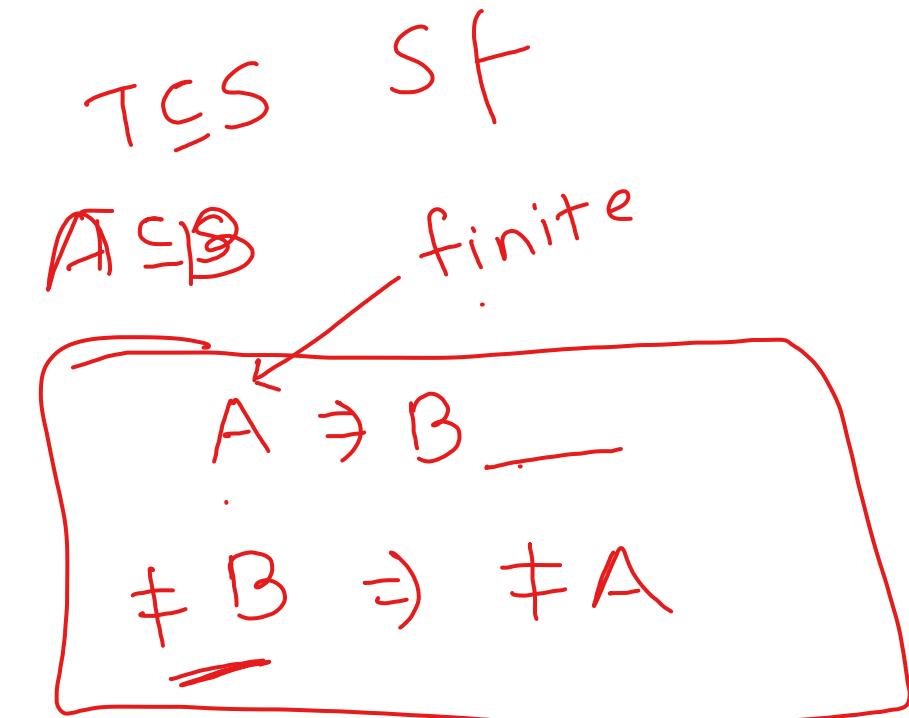
if  $f(k+1) \in T_1 \Rightarrow T_1 = T - f(k+1) \subseteq S_1$

$\therefore T_1 \subseteq S_1, \#(S_1) = k \Rightarrow T_1$  is finite  $\Rightarrow T$  is finite.

-Hence Proved-

b) (b) is a contrapositive statement to (a).

Hence, if  $T$  is infinite  $\Rightarrow S$  is also infinite.





enumerable

**Definition 2.3.9** (Countably Infinite): A set is said to be denumerable or countably infinite if there exists bijection of  $\mathbb{N}$  onto  $S$ .

**Definition 2.3.10** (Countable Set): A set  $S$  is said to be countable if it is either finite or denumerable.

**Definition 2.3.11** (Uncountable Set): A set  $S$  is said to be uncountable if it is not countable.

The following statements are equivalent :—

1.  $S$  is a countable set.

2.  $\exists$  surjection of  $\mathbb{N}$  onto  $\underline{\underline{S}}$ .

3.  $\exists$  injection of  $S$  onto  $\mathbb{N}$

$$\{x_1, x_2, \dots\} \xrightarrow{\text{Bij}} \{x_n, x_{n+1}, \dots\} \xrightarrow{\text{Surj}} \{x_{n+2}, \dots\}$$

**Example**

1. Set of even/odd numbers are denumerable .
2. Set of all integers(denumerable).
3. The union of two disjoint denumerable sets is again denumerable .
4. The sets  $\mathbb{N}$ ,  $\mathbb{N}^2$ ,  $\mathbb{N}^n$  are denumerable .

**Theorem 2.3.3.** Suppose that  $S$ & $T$  are sets and  $T \subseteq S$

a) If  $S$  is countable, then  $T$  is a countable set.

b) If  $T$  is an uncountable then  $S$  is an uncountable Set.

**Theorem 2.3.4.** The Set  $\mathbb{Q}$  of rational numbers is denumerable.

*Proof.* lets prove it for  $\mathbb{Q}^+$  first.

$$\mathbb{Q} = \left\{ \frac{p}{q}, q \neq 0 \right\}, \mathbb{Q}^+ = \left\{ 1, \frac{1}{2}, \dots, \frac{2}{1}, \frac{2}{2}, \frac{2}{3} \right\}$$

We can map  $\mathbb{Q}^+$  with  $\mathbb{N}^2$  however, mapping will not be injection as

$$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} \dots \text{ or } \frac{1}{1} = \frac{2}{4} = \frac{3}{6} \dots$$

To proceed  $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$  is countable.

lets define,  $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}^+$  is mapping of ordered pairs  $< m, n >$  into rational no  $\frac{m}{n}$

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 3 & 4 \end{array} \dots$$

$$\begin{array}{cccc} 2 & 2 & 2 & 2 \\ \hline 1 & 2 & 3 & 4 \end{array} \dots$$

$$\begin{array}{cccc} 3 & 3 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 \end{array} \dots$$

:

:

$\Rightarrow \mathbb{Q}^+$  is countable

Similarly,  $\mathbb{Q}^-$  is also countable

So,  $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$  is countable.... ( $\because$  Union of two disjoint denumerable sets is again denumerable)



- Countable union of countable sets again countable.

## 2.4 Archimedean Property

If  $x \in \mathbb{R}$ , then there exists  $n_x \in \mathbb{N}$  subject to  $x < n_x$ .

*Proof.* By method of contradiction,

$$x \in \mathbb{R}, n_x < x \forall n_x \in \mathbb{N}$$

$\therefore x$  is upper bound for set  $\mathbb{N}$

By completeness property, the set which has upper bound must have supremum (says)

$$n_x < u \quad n_x \in \mathbb{N}$$

$$n_{x+1} \leq u \quad \forall n_x$$

$$n_x \leq u - 1 \quad \forall n_x$$

$\therefore u - 1$  is also upper bound  $< u$  (by definition)

But we know that, Supremum is the least upper bound i.e there exists no other upper bound which is less than  $u$ .

So our assumption is wrong.

Hence,  $x < n_x, x \in \mathbb{R}$

**Corollary 2.4.0.1.** If  $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$  then  $\inf S = 0$

*Proof.*  $S \neq \emptyset$  and 0 is lower bound of  $S$ .

$\therefore$  By completeness Property, set  $S$  has infimum ( $v$ )

Let,  $\varepsilon \in \mathbb{R}, \frac{1}{\varepsilon} > 0 \Rightarrow \frac{1}{\varepsilon} \in \mathbb{R}$

$\therefore$  By archimedean property

$\exists n \in \mathbb{N}, 0 < \frac{1}{\varepsilon} < n \Rightarrow 0 < \frac{1}{n} < \varepsilon \Rightarrow 0 \text{ is inf } (S)$

□

**Corollary 2.4.0.2.** If  $t > 0$ ,  $\exists n_t \in \mathbb{N} \Rightarrow 0 < \frac{1}{n_t} < t$

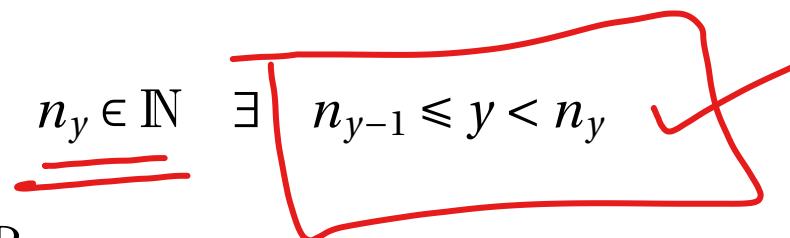
*Proof.*  $t > 0, \frac{1}{t} > 0 \Rightarrow \frac{1}{t} \in \mathbb{R}$

$\therefore$  By archimedean property,  $\exists n \in \mathbb{N}$  subject to  $\frac{1}{t} < n, \exists n_t \in \mathbb{N}$

$$\Rightarrow 0 < \frac{1}{n_t} < t$$

□

**Corollary 2.4.0.3.** If  $y > 0$ ,  $\exists n_y \in \mathbb{N}$



*Proof.* Given that  $y > 0$  i.e  $y \in \mathbb{R}$

$y < n_y, \exists n_y \in \mathbb{N}$  ...By archimedean property

$$E_y = \{n \mid y < n, n \in \mathbb{N}\}$$

$\Rightarrow y$  is lower Bound of  $E_y$

$\Rightarrow$  least element of  $E_y$  is  $\inf(n_y)$

$\Rightarrow \underline{n_{y-1}} \leq y < \underline{n_y}$

□

**Theorem 2.4.1** (Density Theorem). If  $x$  &  $y$  are any real numbers with  $x < y$ , then  $\exists$  a rational numbers  $r \in \mathbb{Q}$  such that  $x < r < y$

*Proof.* assume  $x > 0, x \in \mathbb{R}$

Given,  $x > y \Rightarrow y - x > 0, y - x \in \mathbb{R}$

$\exists n \in \mathbb{N}, \frac{1}{n} < y - x \dots$  (corollary 2.4.0.2)

$$x, y \in \mathbb{R}, \quad \underline{x < y} \Rightarrow y - x > 0$$

$$[\Rightarrow \exists r \in \mathbb{Q} \quad \underline{x < r < y}]$$

$$y - x > 0, y - x \in \mathbb{R}$$

$$\frac{1}{n} < y - x \quad n \in \mathbb{N}$$

$$\underline{\underline{1 < ny - nx}}$$

$$\underline{\underline{1 < ny - n_* x}}$$

(2.19)

~~Assume~~  $\underline{\underline{n_* x}}$

Also,  $x > 0 \Rightarrow \underline{\underline{n_x}} > 0$  then  $\exists m \in \mathbb{N}$  such that  $\underline{\underline{m-1}} \leq \underline{\underline{n_x}} < \underline{\underline{m}}$  ... (corollary 2.4.0.3)

$$\underline{\underline{nx + 1 < ny}}$$

from (2.19)

$$\boxed{n_x < m \leq n_{x+1} < n_y}$$

$$\Rightarrow n_x < m < n_y$$

$$\Rightarrow x < \frac{m}{n} < y$$

$$\Rightarrow x < r < y, \text{ where } r = \frac{m}{n} = \text{rational number}$$

$$m-1 \leq nx < m$$

$$\underline{\underline{nx \leq m}} \leq \underline{\underline{nx+1}} \leq \underline{\underline{ny}}$$

-Hence Proved-

□

**Corollary 2.4.1.1.** If  $x$  and  $y$  are any real numbers with  $x < y$  then  $\exists$  an irrational number

$$\underline{\underline{r \in \mathbb{Q}^c}} \exists x < r < y$$

*Proof.* By density theorem,

If  $x < y$  then  $\exists r_1 \in \mathbb{Q} \exists x < r_1 < y$ . Here  $x < y$

$$x, y \in \mathbb{R} \quad \underline{x < y}. \quad \exists r \in \mathbb{Q} \quad x < r < y$$

$$x, y \in \mathbb{R} \quad x < r <$$

$$\underline{\underline{\sqrt{2}}} \quad \underline{\underline{\sqrt{2}}} \quad \underline{\underline{\frac{x}{\sqrt{2}}}}$$

$$\underline{\underline{\sqrt{2}}} < \underline{\underline{\sqrt{2}}} \in \mathbb{R}$$

$$\underline{\underline{\frac{x}{\sqrt{2}}}} \in \mathbb{R}, \underline{\underline{\frac{y}{\sqrt{2}}}} \in \mathbb{R}$$

$$\therefore \sqrt{2}x < \sqrt{2}y$$

$r/\sqrt{2}$  irr

$$\sqrt{2}x < r_1 < \sqrt{2}y$$

$$x < \frac{r_1}{\sqrt{2}} < y$$

$$x < r < y \quad \text{where} \quad r = \frac{r_1}{\sqrt{2}} = \text{irrational number}$$

-Hence Proved-

□

### Intervals:-

- $\underline{[a, b]} = \{x / a \leq x \leq b\} = \underline{\text{Closed}}$



- $\underline{(a, b)} = \{x / a < x < b\} = \underline{\text{Open}}$



- $\underline{\underline{[a, b)}} = \{x / a \leq x < b\} = \underline{\underline{\text{Half Closed- Half Open}}}$



- $\underline{(a, b]} = \{x / a < x \leq b\} = \underline{\underline{\text{Half Closed- Half Open}}}$

**Intersection:-**

Finite :-  $\bigcap_{i=1}^n \left[ 0, \frac{1}{n} \right] = \left[ 0, \frac{1}{n} \right]$



Arbitrary:-

$$x \in \mathbb{R} \quad \text{such that } \frac{1}{n} < x$$

$$\bigcap_{i=1}^{\infty} \left[ 0, \frac{1}{n_i} \right] = \{0\} = [0, 0.\underline{0000}1]$$

$$= \{0\}$$

$$x < n_x$$

$$\bigcap_{i=1}^{\infty} \left( 0, \frac{1}{n} \right) = \{0\}$$

$$\bigcap_{n=1}^{\infty} (n, \infty) = \emptyset$$

$$\bigcup_{n=1}^{\infty} (-n, n) = \{-\infty, \infty\}$$

$$\bigcap_{n=1}^{\infty} (-n, n) = \{-1, 1\}$$

$$i=1 \quad \left[ 0, \frac{1}{1} \right] = [0, 1]$$

$$i=2 \quad \left[ 0, \frac{1}{2} \right] = [0, \frac{1}{2}]$$

$$[0, 1] \cap [0, \frac{1}{2}] = [0, \frac{1}{2}]$$

$$[0, 1] \cap [0, \frac{1}{2}] \cap [0, \frac{1}{3}] = [0, \frac{1}{3}]$$

$$\bigcap_{i=1}^n \left[ 0, \frac{1}{n} \right] = \left[ 0, \frac{1}{n} \right]$$

$$\bigcap_{n=1}^{\infty} \left[ -1, 1 + \frac{1}{n} \right] = [-1, 1] \quad \checkmark$$

$$\bigcup_{n=1}^{\infty} \left[ -1, 1 - \frac{1}{n} \right] = [-1, 1) \quad \checkmark$$

$$\bigcap_{n=1}^{\infty} [-n, n] = [-1, 1]$$

~~$\bigcap_{n=1}^{\infty}$~~

$$\bigcap_{n=1}^{\infty} [-n, n] = (-\infty, \infty)$$

**Theorem 2.4.2.**  $\mathbb{R}$  is uncountable.

~~we will prove this by method of contradiction~~  
 Proof. Assume that  $\mathbb{R}$  is countable so does  $(0, 1)$  is countable

We can write one-one correspondence with  $\mathbb{N}$  as,

$$b_1 = 0.a_{11}a_{12}a_{13}\dots \neq C$$

$$b_2 = 0.a_{21}a_{22}a_{23}\dots \neq C$$

$$b_3 = 0.a_{31}a_{32}a_{33}\dots \neq C$$

 $\vdots$ 
 $\vdots$ 
 $\vdots$ 
 $\vdots$ 

$$a \dots \underline{\quad} \quad \dots$$

O.C

$$C \neq a$$

$$C_2 \neq a_{22}$$

$$C_i \neq a_{ii}$$

$T \subseteq S$ ,  $S$  is countable  $\Rightarrow T$  is also countable  
 $T$  is uncountable  $\Rightarrow S$  is also uncountable

$(0, 1) \subseteq \mathbb{R}$   
 uncountable

Assume  $\underline{(0, 1)}$  countable ✓

~~Finite~~ or denumerable ✓

onto  
one-one  $\mathbb{N}$

$$S = \underline{(0, 1)} = \{b_1, b_2, b_3, \dots\}$$

$\uparrow$        $\downarrow$        $\downarrow$        $\downarrow$   
 $1, 2, 3, \dots$

$$b_i = 0.a_{i1}a_{i2}a_{i3}\dots a_{ii} \neq C$$

$$b_i = 0.C_1C_2C_3\dots \in (0, 1)$$

$$C_1 \neq a_{11}$$

$$C_2 \neq a_{22}$$

$$C_3 \neq a_{33}$$

:

:

$$C_i \neq a_{ii}$$

As  $C_i \neq a_i$  there does not exists any  $C_i \neq C$

⇒ Our counting Scheme is wrong.

⇒ Our assumption is wrong.

⇒  $(0, 1)$  must be uncountable .

⇒  $\mathbb{R}$  is uncountable. □

## 2.5 Cauchy Schwartz Inequality

Let  $a_i, b_i \in \mathbb{R} \forall i$  then

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \stackrel{=} {\downarrow} \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \quad \checkmark$$

*Proof.* let  $x \in \mathbb{R}$  then,

$$a_i x + b_i \in \mathbb{R} \dots \because a_i, b_i \in \mathbb{R}$$

$$\therefore (a_i x + b_i)^2 \geq 0 \quad \checkmark$$

$$a_i^2 x^2 + 2a_i x b_i + b_i^2 \geq 0$$

$$\begin{aligned} & (a_i x + b_i)^2 \geq 0 & = 0 \\ & \sum (a_i x + b_i)^2 = 0 & \\ & x = \frac{-B \pm \sqrt{B^2 - 2A}}{2} \end{aligned}$$

$$\Rightarrow \left( \sum_{i=1}^n a_i^2 \right) x^2 + 2 \left( \sum_{i=1}^n a_i b_i \right) x + \sum_{i=1}^n b_i^2 \geq 0$$

$$Ax^2 + 2Bx + C \geq 0 \quad : \quad (2.20)$$

where,

$$A = \sum_{i=1}^n a_i^2, B = \sum_{i=1}^n a_i b_i, C = \sum_{i=1}^n b_i^2$$

let  $x = \frac{-B}{A}$

$\therefore$  from (2.20)

$$A\left(\frac{B}{A}\right)^2 + 2B\left(\frac{-B}{A}\right) + C \geq 0 \Rightarrow \frac{B^2}{A} - \frac{2B^2}{A} + C \geq 0$$

$$\Rightarrow \frac{-B^2}{A} + C \geq 0$$

$$\Rightarrow C \geq \frac{B^2}{A}$$

$$\Rightarrow A \cdot C \geq B^2$$

$$\Rightarrow B^2 \geq A \cdot C$$

$$\therefore \left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right)$$

-Hence Proved-

□

**Note:-**

Equality hold if  $a_i$  and  $b_i$  is equal to zero.

If  $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$  then  $Ax^2 + 2Bx + C = 0$

$$a_i = b_i \quad \checkmark$$

how to  
(prove it?)

?

A

$$Ax^2 + 2Bx + C = 0$$

$$\frac{A(-B + \sqrt{B^2 - 4AC})^2}{2A^2} + 2B \left( \frac{-B + \sqrt{B^2 - 4AC}}{2A} \right) + C = 0$$

$$\frac{B^2 - 2B\sqrt{B^2-4AC}}{4A} + \frac{B^2-4AC}{A} + \frac{-B^2 + B\sqrt{B^2-4AC}}{A} + C = 0$$

$$\frac{2B^2 - 2By - 4AC - 4B^2 + 4By}{4A} + C = 0$$

$$\frac{2By - 2B^2 - 4AC}{4A} + C = 0$$

✓

$$\frac{B\sqrt{B^2-4AC} - B^2 - 2AC}{2A} + C = 0 \Rightarrow ?$$

=

## Elements of Point Set Topology

$$A \subseteq B, A \neq B \Rightarrow A \subset B$$

↑  
proper

### 3.1 Terminology and Notations

**Definition 3.1.1** (Member of a set): *If an element  $x$  is in a set  $A$ , we write  $x \in A$  and say that  $x$  is a member of  $A$ , or that  $x$  belongs to  $A$ . If  $x$  is not in  $A$ , we write  $x \notin A$*

**Definition 3.1.2** (Subset): *If every element of a set  $A$  also belongs to a set  $B$ , we say that  $A$  is a subset of  $B$  and write  $A \subseteq B$  or  $B \supseteq A$ .*

**Definition 3.1.3** (Proper Subset): *We say that a set  $A$  is a proper subset of a set  $B$  if  $A \subset B$ , but there is at least one element of  $B$  that is not in  $A$ .*

$$A \subset B \quad \text{if } A \subseteq B \text{ but } A \neq B$$

↑  
A  $\subseteq$  B

**Definition 3.1.4 (Equal Sets):** Two sets  $A$  and  $B$  are said to be equal, and we write  $\underline{A = B}$ , if they contain the same elements. i.e.  $A \subseteq B$  and  $B \supseteq A$ .

$$A = B \text{ iff } \begin{matrix} A \subseteq B, B \subseteq A \\ \wedge \end{matrix}$$

A set is normally defined by either listing its elements explicitly, or by specifying a property that determines the elements of the set.

- The set of natural numbers  $\mathbb{N} := \{1, 2, 3, \dots\}$  lists
- The set of integers  $\mathbb{Z} := \{0, 1, -1, 2, -2, 3, -3, \dots\}$  Rule
- The set of rational numbers  $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$ ,
- The set of real numbers  $\mathbb{R}$ .

$$\frac{m}{n}$$

**Definition 3.1.5 (Open Set):** A subset  $G$  of  $\mathbb{R}$  is open in  $\mathbb{R}$  if for each  $x \in G$  there exists a neighbourhood  $\forall v$  of  $x$  such that  $v \subseteq G$ .

**Definition 3.1.6 (Closed set):** A subset  $f$  of  $\mathbb{R}$  is closed in  $\mathbb{R}$  if the complement  $f^C$  is open in  $\mathbb{R}$ ,

$G$  is open iff for  $x \in G \exists \epsilon > 0$

$$x \in (x - \epsilon_x, x + \epsilon_x) \subseteq G$$

e.g.  $(-\infty, \infty) = \mathbb{R}$  - open as well as closed

$(0, 1)$  - open

$(a, b)$  - open

$[a, \infty)$  - not open but closed ?

$[a, b]$  - not open but closed

$\emptyset$  - open and closed

$[a, b)$  - neither open nor closed

$(a, b]$  - neither open nor closed

$\mathbb{Q}$  - not closed not open

$\mathbb{N}$  - closed but not open

$\mathbb{I}$  - closed but not open

**Definition 3.1.7** (Interior point): For some  $x \in s$  if  $\exists$  open interval  $I_x \ni x \in I_x \subseteq S$  then  $x$  is called interior point of set  $S$ .

**Definition 3.1.8** (Interior of Set): Collection of all interior point is called interior of set  $(S_i)$ .

example  $S = \{[0, 1], [0, 1), (0, 1]\}, S_i(0, 1)$

$S^i$

**Theorem 3.1.1.** Finite union of open sets is open.

*Proof.* let  $A$  and  $B$  be two finite open sets.

Claim-  $A \cup B$  is open set.

$\therefore A$  &  $B$  be two open set.

$\Rightarrow \forall x \in A, \exists I_x \subseteq A$  and  $\forall x \in B \exists I_x \subseteq B$

let  $x \in A \cup B$

$x \in A$  or  $x \in B$

$\therefore x \in I_x \subseteq A$  or  $x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A \cup B$

$\Rightarrow A \cup B$  is open set.



**Theorem 3.1.2.** *Finite intersection of open set is open.*

*Proof.* let  $A$  &  $B$  be two open sets.

claim-  $A \cap B$  is open.

let  $x \in A \cap B$

$\therefore x \in A$  or  $x \in B$

$\Rightarrow \exists I_x \ni x \in I_x \subseteq A$  and  $x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A \cap B$

$\Rightarrow A \cap B$  is open set.



**Theorem 3.1.3.** *Arbitrary union of open sets is open.*

*Proof.* let  $\{A_i\}_{i=1}^{\infty}$  be collection of open sets.

claim-  $\bigcup_{i=1}^{\infty} A_i$  is open set



$$\text{let } x \in \bigcup_{i=1}^{\infty} A_i$$

$\Rightarrow x \in A_j, \text{ for some } j \in I$

$\Rightarrow x \in I_x \subseteq A_j, \text{ for some } j \in I$

$\Rightarrow x \in I_x \subseteq A_j \subseteq \bigcup_{i=1}^{\infty} A_i$

$\therefore \bigcup_{i=1}^{\infty} A_i$  is open set. □

**Theorem 3.1.4.** *Arbitrary intersection of open sets may or may not be open set.*

*Proof.* Set  $S_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

$\bigcap_{n=1}^{\infty} S_n = \{1\}$  which is not open set. □

**Theorem 3.1.5.** *Finite union of two closed set is closed.*

*Proof.* let  $A$  &  $B$  closed set.

Claim-  $A \cup B$  is closed set.

Since,  $A^C$  &  $B^C$  are open sets.

$\Rightarrow A^C \cap B^C$  is open set.

$\Rightarrow (A \cup B)^C$  is open set

$\Rightarrow A \cup B$  is closed set

□

**Theorem 3.1.6.** *Finte intersection of two closed set is closed.*

*Proof.* let  $A$  &  $B$  two closed set.

$\Rightarrow A^C$  &  $B^C$  are two open sets.

$\Rightarrow A^C \cup B^C$  is again open set.

$\Rightarrow (A \cap B)^C$  is open set

$\Rightarrow A \cap B$  is closed set

□

**Theorem 3.1.7.** *Arbitrary union of closed sets may not be closed.*

**Example 17:**

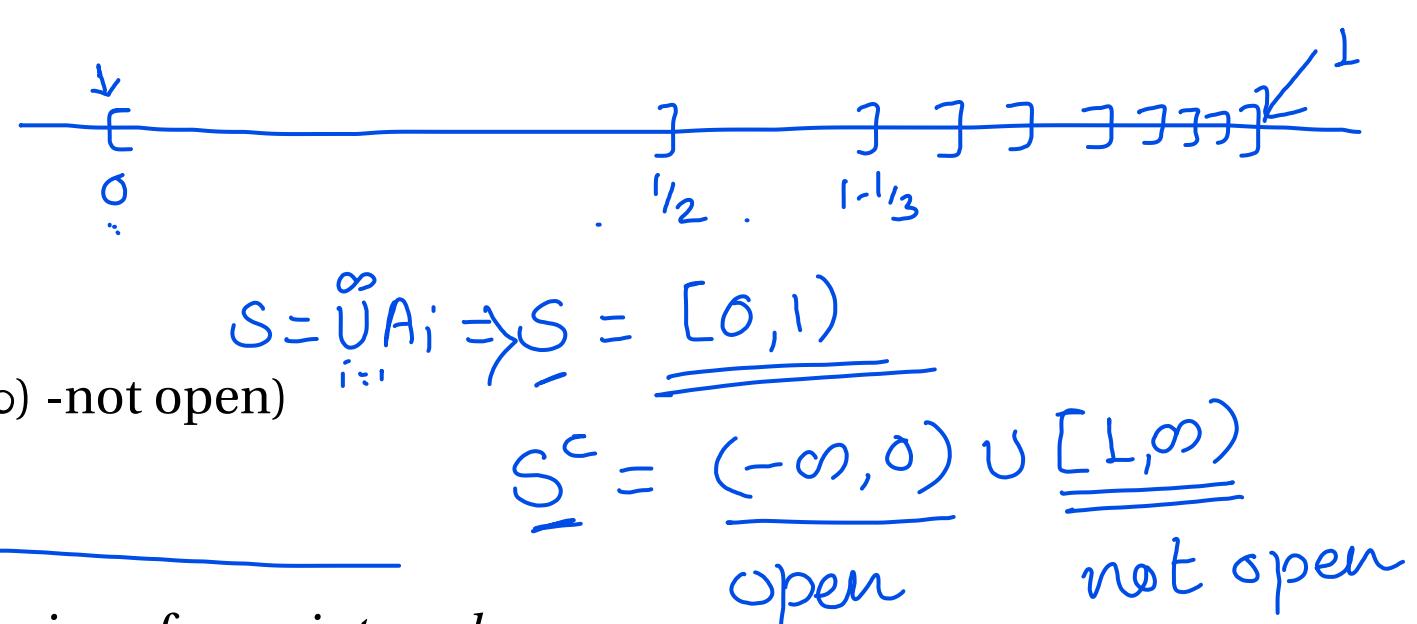
[Counter example]  $A_n = [0, \underline{n}]$ ,  $\cup A_n$   $[0, \infty)$ -closed

$$A_n = \left[ 0, 1 - \frac{1}{n} \right]$$

$$A_1 = \overline{\{0\}}$$

$$A_2 = \left[0, \frac{1}{2}\right] \cup A_n[0, 1) \text{ -not closed}$$

$$A_3 = \left[ 0, 1 - \frac{1}{3} \right] \dots (\because (-\infty, 0) \cup [1, \infty) \text{-not open})$$



**Theorem 3.1.8.** *Every open set is union of open intervals*

*Proof.* Suppose  $S = \{x_1, x_2, x_3\}$

let  $S$  be an open set,  $S = \{x_1, x_2, x_3 \dots\} = \{x_i\}$

for each  $x_i \in I_{x_i} \subseteq S$

$$\{x_i\} \subseteq I_{x_i} \subseteq S$$

$$S = \cup\{x_i\} \subseteq \cup_{i \in I} \subseteq I_{x_i} \subseteq S$$

Hence, Every open set is union of open intervals

**Theorem 3.1.9.** *Interior of set is open set.*

*Proof.* Given that, Let  $S^i$  is interior.

$S$  is open set.

Claim-  $x \in S^i, \exists I_x \in S^i \ni x \in I_x \subseteq S^i$

let  $x \in S^i$

$\Rightarrow x$  is interior point of  $S$

$\Rightarrow x \in I_x \subseteq S^i$

let  $y \in I_x \Rightarrow y \in S \Rightarrow y \in I_x \subseteq S$

$\Rightarrow y \in S^i, y \in I_x$

$\therefore y$  is also interior point of  $S$

this is true for all  $y \in I_x$

$\therefore I_x \subseteq S^i \Rightarrow x \in I_x \subseteq S^i$

$\Rightarrow S^i$  is open set. □

**Theorem 3.1.10.** *Interior of set is largest open subset of set.*

*Proof.* let  $S \subseteq \mathbb{R}$ ,  $S^i$  is interior set of  $S$ .

Claim:-  $S^i \subseteq S$  is largest open set.

We prove this by method of contradiction

Assume that,  $T$  is largest open subset of set  $S$ .

( $S^i$  is not largest) i.e  $S^i \subseteq T \subseteq S$

$S^i$  is proper subset of  $T$

Since,  $S^i \in T$

$\exists$  some  $x \in T, x \notin S^i$

Now,  $x \in T \subseteq S \Rightarrow x$  is interior point of  $S$

This contradicts to our assumption that  $x \notin S^i$

$\therefore$  Our assumption is wrong.

Hence, Interior of set is largest open subset. □

**Definition 3.1.9** (Limit point of set): Let  $c$  be the limit point of set  $S$  iff for any  $\varepsilon > 0$ ,  $\exists x \in S \exists$

$$0 < |x - c| < \varepsilon$$

$$\text{i.e } -\varepsilon < x - c < \varepsilon$$

$$\text{i.e } c - \varepsilon < x < c + \varepsilon$$

$$\text{i.e } x \in \delta_\varepsilon(c)$$

$$\Rightarrow \#(\delta_\varepsilon \cap A) \neq 0$$

example-  $S = \left\{ \frac{1}{n}, n \in \mathbb{R} \right\}$ , 0 is limit point of  $S$ .

Correct it

**Definition 3.1.10** (Derived Set): The set of all limit points of Set  $S$  is called the derived set of  $S$  and denoted by  $S'$ .

$$S' = \{c \mid c \text{ is limit point of } S\}$$

**Definition 3.1.11** (Closed Set): The set  $S$  is said to be closed set if it contains all of its limit points (i.e  $S' \subseteq S$ )

**Definition 3.1.12** (Closure Set):  $S = S \cup S'$

**Example 18:**

$$1. S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}, S = \{0\} \notin S \text{ [Neither open nor closed]}$$

$$\bar{S} = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \cup \{0\}$$

$$2. S = \mathbb{Q}, S' = \mathbb{R} \bar{S} = \mathbb{R}$$

$$3. S = \mathbb{I}, S' = \phi \bar{S} = \mathbb{I}$$

$$4. S = \mathbb{N}, S' = \phi \bar{S} = \mathbb{N}$$

Note:- If  $S$  is closed then  $S = \bar{S}$

**Theorem 3.1.11.** Let  $S \subseteq T$  then  $S' \subseteq T'$

*Proof.* Let  $c \in S'$

$\Rightarrow c$  is limit point of  $S$

for any  $\varepsilon > 0, \delta_\varepsilon(c) \cap S \neq \emptyset$

$\Rightarrow \delta_\varepsilon(c) \cap T \neq \emptyset$  as  $S \subseteq T$

$\Rightarrow c$  is limit point of  $T$

$\Rightarrow c \in T'$

$\therefore S' \subseteq T'$

□

**Theorem 3.1.12.** Show that  $(S \cup T)' = S' \cup T'$

*Proof.* To prove,  $(S \cup T)' = S' \cup T'$

i.e

a)  $(S \cup T)' \subseteq S' \cup T'$

b)  $S' \cup T' \subseteq (S \cup T)'$

- first we prove part b)

$$S \subseteq S \cup T \Rightarrow S' \subseteq (S \cup T)'$$

$$T \subseteq S \cup T \Rightarrow T' \subseteq (S \cup T)'$$

$$\Rightarrow S' \cup T' \subseteq (S \cup T)' \tag{3.1}$$

- a) let  $c \in (S \cup T)'$

$\Rightarrow c$  is limit points of  $S \cup T$

$\Rightarrow \exists S \cup T \exists x \in \delta_\varepsilon(c)$

$\Rightarrow x \in S \exists x \in \delta_\varepsilon(c)$  or  $x \in T \exists x \in \delta_\varepsilon(c)$

$\Rightarrow c$  is limit point of  $S$  or  $c$  is limit point of  $T$

$\Rightarrow c \in S'$  or  $c \in T'$

$\Rightarrow c \in S' \cup T'$

$$(S \cup T)' \subseteq S' \cup T' \quad (3.2)$$

from (3.1) and (3.2)

$$(S \cup T)' = S' \cup T'$$

□

**Theorem 3.1.13.** *Finite intersection of two closed set is closed.*

*Proof.* let  $S$  &  $T$  be two closed sets.

$$\therefore S' \subseteq S \text{ and } T' \subseteq T$$

Claim:  $S \cap T$  is closed

i.e  $(S \cap T)' \subseteq (S \cap T)$

We know,

$$S \cap T \subseteq S \Rightarrow (S \cap T)' \subseteq S' \subseteq S$$

$$S \cap T \subseteq T \Rightarrow (S \cap T)' \subseteq T' \subseteq T$$

$$(S \cap T)' \subseteq (S \cap T)$$

$\therefore S \cap T$  is closed set. □

**Theorem 3.1.14.** let  $S$  &  $T$  be subsets of  $\mathbb{R}$ ,  $S' \cap T'$  may or may not be subset of  $S \cap T'$

*Proof.*  $\therefore S' = [1, 2], T' = [2, 3]$

$$(S \cap T) = \emptyset \text{ & } S' \cap T' = \{2\}$$

$$\Rightarrow (S' \cap T')' = \emptyset$$

$$\therefore S' \cap T' \not\subseteq (S' \cap T')'$$

**Definition 3.1.13** (Dense Set): A Subset  $A \subseteq \mathbb{R}$  is said to be dense set in  $\mathbb{R}$  if every point of  $\mathbb{R}$  is point of  $A$  or limit point of  $\mathbb{R}$  or equivalently if closure of  $A$  is  $\mathbb{R}$

$$\overline{A} = A' \cup A = \mathbb{R}$$

- A set  $A$  is said to be dense in itself if  $\overline{A} = A$
- A set  $A$  is said to be nowhere dense relative to  $\mathbb{R}$  if no neighborhood of  $\mathbb{R}$  is contained in the closure of  $A$
- A set is said to be perfect if it is identical with its derived set or equivalently a set which is closed and dense in itself.

**Theorem 3.1.15.** Set is closed if and only if its complement is open.

*Proof.* a) let  $S$  be closed set

To prove-  $S^c$  is open.

let  $x \in S^c$

$\Rightarrow x$  is not limit point of  $S(\bar{S} = S)$

for some  $\varepsilon > 0, V_\varepsilon(x) \cap S = \emptyset$

$(x - \varepsilon, x + \varepsilon) \subseteq S^c$

$\therefore S^c$  is open.

b) let  $S^c$  is open set

To prove-  $S$  is closed set

By method of contradiction,

Assume that  $S$  is not closed.

$\therefore \exists$  some limit point of  $x$  of  $S \ni x \notin S$

$\Rightarrow x \in S^c$

for some  $\varepsilon > 0 \exists (x - \varepsilon, x + \varepsilon) \subseteq S^c \dots (\because S^c$  is open set)

$\therefore V_\varepsilon(x) \cap S = \emptyset$

which is not possible as  $x$  is limit point

⇒ Our Assumption is that  $x \in S$  is wrong

⇒ All limit point of  $S$  are in  $S$

⇒ is closed set.

□

**Theorem 3.1.16.** *Derived set of set is closed.*

*Proof.* let  $S \subseteq \mathbb{R}$ ,  $S'$  is derived set of  $S$ .

To prove-  $S'$  is closed i.e  $(S')' \subseteq S' = S''$

let  $c \in S'' \Rightarrow c$  is limit point of  $S'$

i.e every  $\varepsilon - neighborhood$   $v$  of  $c$  contains atleast one point  $x$  of  $S' \ni x \neq c$

i.e  $x \in S' \Rightarrow x$  is limit point of set  $S$ .

$\therefore$  Every  $\varepsilon$  neighborhood  $v$  of  $x$  contains atleast one point of  $S$ .

As  $x \in v$ ,  $v$  is also a  $\varepsilon$  neighborhood of  $x$

$\therefore v$  contains atleast one point of  $S$ .

In this way, we can prove that, every  $\varepsilon$  neighborhood  $v$  of  $c$  contains infinitely many points of  $S$ .

$\therefore C$  is limit point of set  $S$ .

Also  $c \in S'$

As  $c \in S'' \Rightarrow c \in S'$ ,  $S'' \subseteq S' \Rightarrow S'$  is closed set when  $S'' = \phi$

then  $S'' \subseteq S' \Rightarrow S'$  is closed set. □

### 3.2 Compact Set

**Definition 3.2.1** (Open Cover): *Let  $A$  be a subset of  $\mathbb{R}$ . An open cover of  $A$  is a collection*

*$G = \{G_\alpha\}$  of open sets in  $\mathbb{R}$  whose union contains  $A$  i.e*

$$A \subseteq \bigcup_\alpha G_\alpha$$

**Definition 3.2.2** (Subcover): if  $G'$  is subcollection of sets from  $G$  such that the union of sets in  $G'$  also contains  $A$  then  $G'$  is called a subcover of  $G$

**Definition 3.2.3** (Finite Subcover): A subset  $k$  of  $\mathbb{R}$  is said to be compact if every open cover of  $\mathbb{R}$  has finite subcover.

### Example 19:

$$1. S = (0, 1), G_i = \left(0, 1 - \frac{1}{i}\right)$$

$$\cap G_i = (0, 1) \supseteq (0, 1)$$

$$\cap G_i = \left(0, 1 - \frac{1}{n}\right) \not\subseteq (0, 1)$$

$\therefore (0, 1)$  is not compact

2.  $\mathbb{N}$  is not compact

### 3.3 Heine Borel theorem

**Theorem 3.3.1** (Heine Borel theorem). *The set  $k$  is compact set if and only if it is closed & bounded.*

*Proof.* Given that,  $k$  is compact set.

i.e Every open cover exists finite subcover.

claim-  $k$  is bounded & closed.

1.  $k$  is bounded

$$G_i = (-i, i), G = \mathbb{R}$$

$$\cup_{i=1}^n G_i = (-n, n), k \subseteq (-n, n)$$

$\therefore k$  is bounded

2.  $k$  is closed i.e  $k^c$  is open

$$\text{let } x \in k^c$$

$$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$

$$G_1 = (-\infty, x - 1) \cup (x + 1, \infty)$$

$$G_2 = \left(-\infty, x - \frac{1}{2}\right) \cup \left(x + \frac{1}{2}, \infty\right)$$

...

...

...

$$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$

$\therefore k$  is closed.

Hence, from a) and b),

$k$  is compact if and only if it is closed and bounded.

□



## Sequence and Series

***Definition 4.0.1*** (Sequence and Series): *A sequence of real numbers is function defined on the set  $\mathbb{N}$  whose range is contained in the set  $\mathbb{R}(x : \mathbb{N} \rightarrow \mathbb{R})$*

*Denoted by  $x, (x_n), (x_n, n \in \mathbb{N})$*

*example  $\frac{1}{n}, \frac{1}{n^2}, 2n, n^2 + 1, n^2 - n$*

- *Constant Sequence-  $x_n = x, \forall n \in \mathbb{N}$*
- *Increasing Sequence-  $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$*

- *Strictly increasing sequence*-  $x_n < x_{n+1} \forall n \in \mathbb{N}$
- *Decreasing Sequence*-  $x_n \geq x_{n+1}, \forall n \in \mathbb{N}$
- *Strictly Decreasing Sequence*-  $x_n > x_{n+1}, \forall n \in \mathbb{N}$

**Definition 4.0.2** (Fibonacci Sequence):  $x_1, x_2, x_{n+2} = x_{n+1} + x_n$

- Limit of Sequence- A Sequence  $(x_n) \in \mathbb{R}$  is said to be converge to  $x \in \mathbb{R}$  or  $x$  is said to be limit of  $(x_n)$  if for every  $\varepsilon > 0 \exists > 0 k(\varepsilon) \in \mathbb{N}$  such that

$$|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$$

If sequence has limit, we say that sequence is convergent. IF it has no limit has no limit, we say that is divergent.

$$\lim(x_n) = x \text{ or } x_n \rightarrow x$$

- *Ocillating Sequence*:  $(x_n) = (-1)^n, n \in \mathbb{N}$ - (non convergent)

$$(x_n) = \frac{(-1)^n}{n}, n \in \mathbb{N}$$

**Definition 4.0.3** (Uniqueness of limit point): A sequence in  $\mathbb{R}$  have atmost limit point one.

let  $x_1$  &  $x_2$  be two limit points of  $x_n$

$\therefore$  for any  $\varepsilon > 0 \ \forall, n \geq k_1(\varepsilon)$  &  $|x_n - x_1| < \varepsilon$

$\exists k_1(\varepsilon) \in \mathbb{N} \ \exists |x_n - x_1| < \varepsilon, \forall n \geq k_1(\varepsilon)$

$\exists k_2(\varepsilon) \in \mathbb{N} \ \exists |x_n - x_2| < \varepsilon, \forall n \geq k_2(\varepsilon)$

$k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\forall n \in \mathbb{N} \ \exists n \geq k(\varepsilon)$

$$|x_1 - x_2|$$

$$= |x_1 - x_n + x_n - x_2|$$

$$\leq |x_n - x_1| + |x_n - x_2|$$

$$\leq \varepsilon + \varepsilon$$

$$\leq 2\varepsilon$$

As this statement is true for any  $\varepsilon > 0$ ,  $x_1 = x_2$

Hence, Sequence have atmost one limit point.

**Definition 4.0.4** (Tail Sequence): *If  $\{x_1, x_2, \dots\}$  is sequence of real numbers and if  $m$  is given natural number then  $m$ -tail of  $x_n$  is sequence*

$$x_m = \{x_{m+n} / x_{m+1}, x_{m+2}, \dots\}$$

**Theorem 4.0.1.** *Let  $x_n$  be sequence of real numbers and let  $m \in \mathbb{N}$  then  $m$ -tail  $x_m$  of  $x_n$  converges if & only if  $x_n$  converges.*

*Proof.* Let  $x_n \rightarrow x$  i.e  $\lim_{n \rightarrow \infty} x_n = x$

$\Rightarrow$  for any  $\varepsilon > 0$ ,  $\exists k(\varepsilon) \in \mathbb{N}$

such that  $|x_n - x| < \varepsilon$ ,  $\forall n \geq k(\varepsilon)$

$\Rightarrow x - \varepsilon < x_n < x + \varepsilon$ ,  $\forall n \geq k(\varepsilon)$

$$\Rightarrow x - \varepsilon < x_k, x_{k+1}, \dots < x + \varepsilon$$

$$\text{let } y_n = x_{m+n}, n$$

$$\Rightarrow x - \varepsilon < y_{k-m}, y_{k+1-m}, \dots < x + \varepsilon$$

$$\Rightarrow x - \varepsilon < y_n < x + \varepsilon \forall n \geq k(\varepsilon) - m = k_1(\varepsilon)$$

$$\Rightarrow |y_n - x| < \varepsilon \forall n \geq k_1(\varepsilon)$$

$$y_n \rightarrow x$$

-Hence proved-

□

**Theorem 4.0.2.** Let  $x_n$  be a sequence of real numbers and  $x \in \mathbb{R}$  if  $a_n$  is sequence of positive real numbers with  $\lim a_n = 0$  and iff for some constant  $c > 0$  and some  $m \in \mathbb{N}$ , we have  $|x_n - x| \leq ca_n, \forall n \geq m$  then it follows that  $\lim x_n = x$

*Proof.* Given that  $\lim a_n = 0$

$$\text{i.e } a_n \rightarrow 0$$

$\therefore$  by definition, for any  $\varepsilon > 0, \frac{\varepsilon}{c} (\because c > 0)$

$\exists k(\varepsilon) \in \mathbb{N}$  such that

$$|a_n - 0| < \frac{\varepsilon}{c}$$

$$a_n < \frac{\varepsilon}{c} \dots (\because a_n > 0)$$

let  $k_1(\varepsilon) = \max(m_1 k_1(\varepsilon))$

$\forall n \geq k_1(\varepsilon)$

$$|x_n - x|$$

$$\leq c a_n$$

$$\leq c(\varepsilon/c)$$

$$\leq \varepsilon, \forall n \geq k_1(\varepsilon)$$

$$\therefore x_n \rightarrow x$$

□

**Definition 4.0.5** (Bounded Sequence): A Sequence of real numbers  $x_n$  is said to be bounded if  $\exists m > 0$  such that  $|x_n| \leq m, \forall n \in \mathbb{N}$

**Theorem 4.0.3.** The Convergent sequence of real numbers is bounded.

*Proof.* let  $x_n \rightarrow x$

$\therefore$  by definition, for any  $\varepsilon > 0$ ,  $\exists k(\varepsilon) \in \mathbb{N}$

such that  $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geq k(\varepsilon)$

let

$$M = \max\{|x_1|, |x_2|, \dots, |x_k|, x + \varepsilon\}$$

$\therefore |x_n| \leq M, \forall n$

$\Rightarrow x_n$  is bounded.

-Hence Proved-

□

**Theorem 4.0.4.** a) Let  $x_n$  and  $y_n$  be sequence of real numbers that converges to  $x$  and  $y$  respectively and let  $c \in \mathbb{R}$  then, the sequence  $X + Y, X - Y, XY$  and  $CX$  converges to  $x + y, x - y, xy$  and  $cx$

b) If  $x_n \rightarrow x$  and  $z_n$  is sequence of non-zero real numbers that converges to  $z$  and if  $z \neq 0$  then

$$\frac{X}{Z} \rightarrow \frac{x}{z}$$

*Proof.* a) given that  $x_n \rightarrow x$

$\therefore$  by definition, for any  $\varepsilon > 0, \frac{\varepsilon}{2} > 0$

$\exists k_1(\varepsilon) \in \mathbb{N}$  such that

$$|x_n - x| < \varepsilon/2, \forall n \geq k_1(\varepsilon)$$

also,  $y_n \rightarrow y$

$\therefore$  by definition, for any  $\varepsilon > 0, \frac{\varepsilon}{2} > 0$

$\exists k_2(\varepsilon) \in \mathbb{N}$  such that

$$|y_n - y| < \varepsilon/2, \forall n \geq k_2(\varepsilon)$$

$$\text{let } k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$$

$\therefore \forall n \geq k(\varepsilon)$

i)  $|x_n + y_n - (x + y)| = |x_n - x + y_n - y|$

$\leq |x_n - x| + |y_n - y| \dots$  (triangular inequality)

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$\therefore (x_n + y_n) \rightarrow (x + y)$

ii)  $|(x_n - y_n) - (x - y)| = |x_n - x - y_n + y|$

$\leq |x_n - x| + |y_n - y| \dots$  (triangular inequality)

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$\therefore (x_n - y_n) \rightarrow (x - y)$

iii)  $x_n \rightarrow x$

$\therefore$  by definition, for any  $\varepsilon > 0, \frac{\varepsilon}{2M} > 0, \dots$  ( $\because M > 0$ )

$\exists k_1(\varepsilon) \in \mathbb{N}$  such that

$$|x_n - x| < \varepsilon/2M, \forall n \geq k_1(\varepsilon)$$

also,  $y_n \rightarrow y$

$\therefore$  by definition, for any  $\varepsilon > 0, \frac{\varepsilon}{2|x|} > 0, \dots (\because |x| > 0)$

$\exists k_2(\varepsilon) \in \mathbb{N}$  such that

$$|y_n - y| < \varepsilon/2|x|, \forall n \geq k_2(\varepsilon)$$

let  $k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\therefore \forall n \geq k(\varepsilon)$

$$\begin{aligned} |(x_n y_n) - (xy)| &= |x_n y_n - xy_n + xy_n - xy| \\ &\leq |y_n||x_n - x| + |x_n||y_n - y| \dots \text{(triangular inequality)} \\ &\leq M \frac{\varepsilon}{2M} + |x|_2 \frac{\varepsilon}{2} \end{aligned}$$

$$\leq \varepsilon$$

$$\therefore x_n y_n \rightarrow xy$$

iv)  $x_n \rightarrow x$

$\therefore$  by definition, for any  $\varepsilon > 0, \frac{\varepsilon}{|c|} > 0, \dots (\because |c| > 0)$

$\exists k(\varepsilon) \in \mathbb{N}$  such that

$$|x_n - x| < \frac{\varepsilon}{|c|}, \forall n \geq k(\varepsilon)$$

$$|(cx_n - cx)| = |c|. |x_n - x|$$

$$\leq |c| \cdot \frac{\varepsilon}{|c|}$$

$$\leq \varepsilon$$

$$\therefore cx_n \rightarrow cx$$

b)  $x_n \rightarrow x$  and  $z_n \rightarrow z$

$\therefore$  by definition, for any  $\varepsilon > 0, \exists |z|.m > 0$

$\exists k(\varepsilon) \in \mathbb{N}$  such that

$$|z_n - z| < \varepsilon. |z|.m, \forall n \geq k(\varepsilon)$$

$$\text{let } y_n = \frac{1}{z_n}$$

$$\begin{aligned} \text{consider, } |(y_n - x)| &= \left| \frac{1}{z_n} - \frac{1}{z} \right| \\ &= \frac{|z - z_n|}{|z_n.z|} \end{aligned}$$

$$\leq \frac{\varepsilon. |z|. m}{|z_n|. |z|}$$

$$\leq \frac{\varepsilon. m}{|z_n|}$$

$\leq \varepsilon \dots (\because z_n \text{ is bounded } m < z_n < m)$

$$\therefore \frac{1}{x_n} \rightarrow \frac{1}{z}$$

$$\therefore y_n \rightarrow y$$

we know that,  $x_n y_n \rightarrow xy \dots (\because \text{if } x_n \rightarrow x \text{ & } y_n \rightarrow y \text{ then } x_n y_n \rightarrow xy)$

$$\therefore \frac{x_n}{z_n} \rightarrow \frac{x}{y}$$

-Hence Proved-



**Theorem 4.0.5.** If  $x_n \rightarrow x$  and if  $x_n \geq 0, \forall n \in \mathbb{N}$  then  $x = \lim x_n \geq 0$

*Proof.* Given that,  $x_n \rightarrow x$

$\therefore$  by definition, for any  $\varepsilon > 0$

$\exists k(\varepsilon) \in \mathbb{N}$

such that  $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

we will prove this by method of contradiction.

let if possible  $x < 0$

$$\therefore -x > 0$$

Assume,  $0 < \varepsilon < -x$

$$\therefore x - \varepsilon < 0 \text{ and } x + \varepsilon < 0 \text{ &}$$

$$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geq k(\varepsilon)$$

$$\therefore x_n < 0$$

which contradicts to given statement that  $x_n \geq 0$

$\therefore$  Our assumption is wrong.

$$\therefore x = \lim x_n \geq 0$$

-Hence Proved-



**Theorem 4.0.6.** If  $x_n \rightarrow x, y_n \rightarrow y$  are convergent sequence of real numbers and if  $x_n \leq y_n, \forall n \in \mathbb{N}$  then  $\lim x_n \leq \lim y_n$

*Proof.* Given that,  $x_n \rightarrow x$ , and  $y_n \rightarrow y$  also,  $x_n \leq y_n, \forall n$

$$\Rightarrow y_n - x_n \geq 0$$

$$\Rightarrow z_n \geq 0$$

Now,  $y_n - x_n \rightarrow y - x$  (say  $z$ )

As,  $z_n \geq 0, z_n \rightarrow z$

$\therefore z \geq 0 \dots$  (by above theorem)

$$\therefore y - x \geq 0$$

$$\therefore y \geq x$$

$$\therefore x \leq y$$

-Hence Proved □

**Theorem 4.0.7.** If  $x_n$  is convergent to some  $x \in \mathbb{R}$  and  $a \leq x_n \leq b, \forall n$  then  $a \leq x \leq b$

*Proof.* Given that,  $x_n \rightarrow x$  and  $a \leq x_n \leq b$

$$\text{let } a_n = a \& b_n = b$$

$$\text{i.e } a_n \rightarrow a \& b_n \rightarrow b$$

$$\therefore a_n \leq x_n \leq b_n$$

i.e  $a_n \leq x_n \& x_n \leq b_n$

$\lim a_n \leq \lim x_n \& \lim x_n \leq b_n \dots$  (by above theorem)

$a \leq x$  and  $x \leq b \therefore a \leq x \leq b$

-Hence Proved-



#### 4.1 Squeeze Theorem

**Theorem 4.1.1.** Suppose  $x_n$ ,  $y_n$  and  $z_n$  are sequence of real numbers  $\exists x_n \leq y_n \leq z_n, \forall n \in \mathbb{N}$  and  $\lim x_n \leq \lim y_n$  then  $y_n$  is convergent and  $\lim x_n = \lim y_n = \lim z_n$ .

*Proof.* Given that,  $x_n \leq y_n \leq z_n, \forall n$

let,  $\lim x_n = \lim z_n = w$

i.e  $x_n \rightarrow w$  and  $z_n \rightarrow w$

$\therefore$  by definition, for any  $\varepsilon > 0 \ \exists$

$k_1(\varepsilon) \in \mathbb{N}$  and  $k_2(\varepsilon) \in \mathbb{N}$  such that

$$|x_n - w| < \varepsilon, \forall n \geq k_1(\varepsilon) \text{ and } |z_n - w| < \varepsilon, \forall n \geq k_2(\varepsilon)$$

$$\therefore w - \varepsilon \leq x_n \leq w + \varepsilon \text{ and } w - \varepsilon \leq z_n \leq w + \varepsilon$$

$$\therefore w - \varepsilon \leq x_n \leq y_n \text{ and } y_n \leq z_n \leq w + \varepsilon$$

$$\therefore w - \varepsilon \leq x_n \leq y_n \leq z_n \leq w + \varepsilon$$

i.e  $w - \varepsilon \leq y_n \leq w + \varepsilon$

i.e  $|y_n - w| < \varepsilon, \forall n \in k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$$\therefore y_n \rightarrow w$$

$$\therefore \lim x_n = \lim y_n = \lim z_n = w$$

□

**Theorem 4.1.2.** Given that,  $x_n \rightarrow x$  then Show that,

a)  $|x_n| \rightarrow |x|$

b)  $\sqrt{x_n} \rightarrow \sqrt{x}$

*Proof.* Given that,  $x_n \rightarrow x$

$\therefore$  by definition, for any  $\varepsilon > 0$ ,  $\exists k(\varepsilon) \in \mathbb{N}$

such that  $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

consider,

$$||x_n| - |x||$$

$\leq |x_n - x| \dots$  (by corollary of triangular inequality)

$$\leq \varepsilon$$

$$|x_n| \rightarrow |x|$$

Given that,  $x_n \rightarrow x$

$\therefore$  by definition, for any  $\varepsilon > 0$ ,  $\sqrt{x} > 0$ ,  $\frac{\varepsilon}{\sqrt{x}} > 0$ ,  $\varepsilon\sqrt{x} > 0$

$\exists k(\varepsilon) \in \mathbb{N}$  such that

$$|x_n - x| \leq \varepsilon\sqrt{x}, \forall k(\varepsilon) \in \mathbb{N}$$

As,  $\sqrt{x} > 0$

$$\therefore 0 < \sqrt{x} < \sqrt{x_n} + \sqrt{x}$$

$$\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{x_n} + \sqrt{x}} \quad (4.1)$$

$$|\sqrt{x_n} - \sqrt{x}|$$

$$= \frac{|\sqrt{x_n} - \sqrt{x}| \cdot |\sqrt{x_n} + \sqrt{x}|}{|\sqrt{x_n} - \sqrt{x}|}$$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$$

$$\leq \frac{|x_n - x|}{\sqrt{x}} \dots \text{(from 4.1)}$$

$$\leq \frac{\varepsilon \cdot \sqrt{x}}{\sqrt{x}}$$

$$\leq \varepsilon$$

$$\therefore \sqrt{x_n} \rightarrow \sqrt{x}$$

□

## 4.2 Monotone Sequence

- *Monotone decreasing:*  $x_n \geq x_{n+1}, \forall n$
- *Monotone increasing:*  $x_n \leq x_{n+1}, \forall n$

$x_n$  is called as monotone if it is increasing or decreasing.

**Theorem 4.2.1** (Monotone Convergence theorem). *A monotone sequence of real numbers is convergent if and only if*

a) *If  $x_n$  is bounded increasing sequence*

$$\lim(x_n) = \text{Sup}\{x_n, n \in \mathbb{N}\}$$

b) If  $x_n$  is bounded decreasing sequence

$$\lim(x_n) = \inf\{x_n, n \in \mathbb{N}\}$$

*Proof.* We know that, Convergent sequence must be bounded.

Conversely, let  $x_n$  be monotone bounded sequence.

a) Assume  $x_n$  is increasing and bounded.

As  $x_n$  is bounded  $M \in \mathbb{R}, |x_n| \leq M, \forall n$

let,  $S = \{x_n, \forall n \in \mathbb{N}\}$

$M$  upper bound of  $S$

$\therefore$  By completeness property,  $\exists x^* \in \mathbb{R}$

$\exists x^* = \sup\{x_n, n \in \mathbb{N}\}$

$\therefore x_n \leq x^* \forall \mathbb{N}$

for any  $\varepsilon > 0$   $x^* - \varepsilon$  is not supremum of  $S$

$\therefore x^* - \varepsilon < x_k \leq x^*$ , for some  $k$

$$\Rightarrow x^* - \varepsilon < x_k \leq x_{k+1} \leq x_{k+2} \leq \dots \leq x^*$$

$$\therefore x^* - \varepsilon < x_n < x^*, \forall n \geq k(\varepsilon)$$

$$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$$

$$\therefore x^* = \lim x_n$$

i.e  $x_n$  is convergent sequence.

b) Assume  $x_n$  is decreasing and bounded.

As  $x_n$  is bounded  $M \in \mathbb{R}, |x_n| \leq M, \forall n$

let,  $S = \{x_n, \forall n \in \mathbb{N}\}$

$-M$  lower bound of  $S$

$\therefore$  By completeness property,  $\exists x^* \in \mathbb{R}$

$\exists x^* = \inf\{x_n, n \in \mathbb{N}\}$

$\therefore x_n \geq x^* \forall \mathbb{N}$

for any  $\varepsilon > 0$   $x^* + \varepsilon$  is not lower bound of  $S$

$\therefore x^* < x_k < x^* + \varepsilon$ , for some  $k$

$\Rightarrow x^* < \dots \leq x_{k+2} \leq x_{k+1} \leq x_k < x^* + \varepsilon$

$\therefore x^* < x_n < x^* + \varepsilon$

$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$

$\therefore x^* = \lim x_n$

i.e  $x_n$  is convergent sequence.

□

**Theorem 4.2.2.** If  $x_n$  converges to  $x$  then any subsequences  $x_{n_k}$  of  $x_n$  also converges to  $x$ .

*Proof.* for any  $\varepsilon > 0 \exists k(\varepsilon) \in \mathbb{N}$  such that,

$$|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$$

let subsequence  $x_{n_k} = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$

As  $x_n \rightarrow x \Rightarrow x - \varepsilon < x_n < x + \varepsilon$

Also,  $n_k \geq n \geq k(\varepsilon)$

$$\rightarrow x - \varepsilon < x_{n_k} < x + \varepsilon, \forall n_k \geq k(\varepsilon)$$

$$\therefore x_{n_k} \rightarrow x$$

□

**Theorem 4.2.3** (Monotone Subsequence theorem). *If  $x_n$  is sequence of real numbers then there is subsequence of  $x_n$  that is monotone.*

*Proof.* We will say that  $m^{th}$  term  $x_m$  is a peak if  $x_m \geq x_n \forall n \geq m$ .

Note that, In a decreasing sequence, every term is peak while in increasing sequence, no term is peak.

Case-1:-

$x_n$  has infinitely many peaks. In this case, we list the peaks by,

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \dots \geq x_{m_k}, \dots$$

$\therefore$  subsequence  $x_{m_k}$  is decreasing subsequence of  $x_n$ .

Case-2:-

$x_n$  has finitely number of peaks.

let these peaks be denoted by,

$$x_{m_1}, x_{m_2}, x_{m_3} \dots x_{m_r}$$

let  $S_1 = m_r + 1$  be the first index beyond the last peak since  $x_{S_1}$  is not peak  $\exists S_2 > S_1$

$\exists x_{S_1} < x_{S_2}$  since  $x_{S_2}$  is not peak  $\exists S_3 > S_2$

$\exists x_{S_2} < x_{S_3}$  continuing this way, we obtain an increasing sequence.  $\square$

**Theorem 4.2.4** (Bozano- Weistress theorem). *A bounded sequence of real numbers has convergent subsequence.*

*Proof.* Let  $x_n$  be bounded sequence.

$\therefore$  by monotone subsequence theorem,

$\exists x_{n_k}$  subsequence of  $x_n$  that is monotone.

As  $x_n$  is bounded  $x_{n_k}$  is also bounded

$\therefore$  by monotone convergence theorem,

$x_{n_k}$  is monotone and bounded so convergent.  $\square$

### 4.3 Cauchy Sequence

**Definition 4.3.1** (Cauchy Sequence): *A sequence of real numbers is said to be cauchy if for every  $\varepsilon > 0$ ,  $\exists H(\varepsilon) \in \mathbb{N}$  such that  $|X_n - X_m| < \varepsilon$ ,  $\forall n, m \geq H(\varepsilon)$*

**Theorem 4.3.1.** *Every convergent sequence is cauchy.*

*Proof.* let  $x_n \rightarrow x$

for any  $\frac{\varepsilon}{2} > 0$ ,  $\exists k(\varepsilon) \in \mathbb{N}$

$\exists |X_n - x| < \frac{\varepsilon}{2}$ ,  $\forall n \geq k(\varepsilon)$  let,  $k_1, k_2 \in \mathbb{N}$  such that  $\forall k_1, k_2 \geq k(\varepsilon)$

$$|X_{k_1} - x_{k_2}|$$

$$\leq |X_{k_1} - x| + |X_{k_2} - x|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

Hence, every convergent sequence is cauchy. □

**Theorem 4.3.2.** *A cauchy sequence of real numbers is bounded.*

*Proof.* let  $x_n$  be cauchy sequence and

let  $\varepsilon = 1$  if  $H = H(1)$  and  $n \geq H$  then  $n \geq H$ .

$$M = \sup\{|x_1|, |x_2|, |x_3|, \dots, |x_{H-1}|, |x_H| + 1\}$$

then it follows that  $|x_n| \leq M \forall n$

$\therefore$  cauchy sequence of real numbers is bounded. □

**Definition 4.3.2** (Cauchy convergence criterion): *A Sequence of real numbers is convergent if and only if it is cauchy sequence.*

**Definition 4.3.3** (Contractive Sequence): *We say that the sequence  $x_n$  of real numbers is contractive sequence if there exists a constant  $c$ ,  $0 < c < 1$  such that,*

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|, \forall n$$

**Theorem 4.3.3.** *Contractive sequence is cauchy sequence.*

*Proof.* let  $x_n$  is contractive sequence

$\therefore \exists c, 0 < c < 1$  such that

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|, \forall n$$

for  $\varepsilon > 0$  choose  $k(\varepsilon) \in \mathbb{N}$   $\exists$  for  $m > n$

$$\begin{aligned} & |x_m - x_n| \\ &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n| \\ &\leq c|x_{m-1} - x_{m-2}| + c|x_{m-2} - x_{m-3}| + \dots + c|x_n - x_{n-1}| \\ &\leq c^2|x_{m-2} - x_{m-3}| + c^2|x_{m-3} - x_{m-4}| + \dots + c|x_n - x_{n-1}| \\ &\leq (c^{m-n} + c^{m-n-1} + \dots + c)|x_n - x_{n-1}| \\ &\leq \frac{c(1 - c^{m-n})}{1 - c} |x_n - x_{n-1}| \\ &\leq \varepsilon \quad \because \frac{c(1 - c^{m-n})}{1 - c} < 1 \end{aligned}$$

$\therefore x_n$  is cauchy sequence.



**Divergent Sequence** Let  $x_n$  be sequence of real numbers

a)  $x_n \rightarrow +\infty$  and  $\lim x_n = +\infty$

if every  $\alpha \in \mathbb{R}$  there exists a natural number  $k(\alpha)$  such that if  $n \geq k(\alpha)$ , then  $x_n > \alpha$ .

b)  $x_n \rightarrow -\infty$  and  $\lim x_n = -\infty$

if every  $\beta \in \mathbb{R}$  there exists a natural number  $k(\beta)$  such that if  $n \geq k(\beta)$ , then  $x_n < \beta$ .

We say that  $x_n$  is properly divergent if  $\lim x_n = +\infty$  or  $-\infty$

#### 4.4 Infinite Series

**Definition 4.4.1** (Infinite Series): *If  $x_n$  is sequence in  $\mathbb{R}$ , then the infinite series generated by  $x_n$  is sequence  $S_n$*

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

.

.

.

$$S_n = x_1 + x_2 + \dots + x_n$$

Denoted by  $\sum x_n$  or  $\sum_{n=1}^{\infty} x_n$

### Example 20:

$$1. \sum_{n=0}^{\infty} r_n = 1 + r + r^2 + \dots$$

$$2. \sum_{n=1}^{\infty} (-1^n) = (-1) + 1 + (-1) + \dots$$

$$3. \sum \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$$

**Theorem 4.4.1** (The  $n^{th}$  term test). if  $\sum x_n$  converges then  $\lim x_n = 0$

*Proof.* By definition  $\sum x_n$  converges if  $S_n$  converges,

$$\text{Since } = \sum_{i=1}^n x_i$$

$$\therefore x_n = S_n - S_{n-1}$$

$$\therefore \lim x_n = \lim S_n - \lim S_{n-1} = 0$$

□

**Definition 4.4.2** (Cauchy Criterion for Series): *The series  $\sum x_n$  converges if and only if  $\forall \varepsilon > 0$ ,*

*$\exists M(\varepsilon) \in \mathbb{N}$  such that if  $m > n \geq M(\varepsilon)$  then*

$$|S_m - S_n| = |x_{n+1} + x_{n+2} + \dots + x_m| < \varepsilon$$

**Theorem 4.4.2.** *let  $x_n$  be a sequence of non-negative real numbers then the series  $\sum x_n$  converges if and only if the sequence  $S_k$  of partial sum is bounded.*

$$\sum x_n = \lim S_k = \sup\{S_k : k \in \mathbb{N}\}$$

**Theorem 4.4.3.** *Show that,  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$*

*Proof.* Suppose,

$$S_{n+1} = 1 + r + \dots + r^n$$

$$S_n = 1 + r + \dots + r^{n-1}$$

$$rS_n = (r + r^2 + \dots + r^n)$$

$$\therefore S_{n+1} - rS_n = 1$$

$$\therefore \lim_{n \rightarrow \infty} (S_{n+1} - rS_n) = \lim_{n \rightarrow \infty} 1 = 1$$

$$\therefore (S - rS) = 1 (\dots \text{where } S \sum_{n=0}^{\infty})$$

$$S(1 - r) = 1$$

$$S = \frac{1}{(1 - r)}$$

□

**Theorem 4.4.4.** *The p Series  $\sum \frac{1}{n^p}$  converges when  $p > 1$*

*Proof.* if  $k_1 = 2 - 1 = 1, S_{k_1} = 1$

$$k_1 = 2^2 - 1 = 3, 2^p < 3^p$$

$$S_{k_2} = \frac{1}{1^p} + \left( \frac{1}{2^p} + \frac{1}{3^p} \right) < \frac{1}{1^p} + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

further, if  $k_3 = 2^3 - 1$  then

$$S_{k_3} < S_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}} \frac{1}{4^{p-1}}$$

finally, let  $r = \frac{1}{2^{p-1}}$  Since  $p > 1$

Using mathematical induction

we can show that if  $k_j = 2^j - 1$

$$0 < S_{k_j} < 1 + r + r^2 + \dots + r^{j-1} < \frac{1}{1-r}$$

$\Rightarrow$  The p-series converges if  $p > 1$

□

### The alternating harmonic series

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots$$

is convergent

$$\text{let } S_{2n} = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots + \left( \frac{1}{2n-1} - \frac{1}{2n} \right)$$

$$S_{2n+1} = 1 - \left( \frac{1}{2} - \frac{1}{3} \right) - \left( \frac{1}{4} - \frac{1}{5} \right) - \dots - \left( \frac{1}{2n} - \frac{1}{2n+1} \right)$$

Since  $0 < S_{2n} < S_{2n} + \frac{1}{2n+1} = S_{2n+1} \leq 1$

$S_{2n}$  and  $S_{2n+1}$  both bounded and monotone, so by monotone convergence theorem, must be convergent and to same point.

$\sum \frac{(-1)^{n+1}}{n}$  must be convergent.

**Theorem 4.4.5** (The Comparision Test). *Let  $x_n$  and  $y_n$  be real sequence and for some  $k \in \mathbb{N}$   $0 \leq x_n \leq y_n, \forall n \geq k$*

a) *Convergent of  $\sum y_n \Rightarrow$  Convergence of  $\sum x_n$*

b) *Divergence of  $\sum x_n \Rightarrow$  divergence of  $\sum y_n$*

*Proof.* a) Suppose  $\sum y_n$  is convergent,

i.e for any  $\varepsilon > 0, \exists M(\varepsilon) \in \mathbb{N} \ni m > n \geq M(\varepsilon)$

$$|y_{n+1} + \dots + y_m| < \varepsilon$$

if  $m > Sup(k, M(\varepsilon))$

$$0 \leq x_{n+1} + \dots + x_m \leq y_{n+1} + \dots + y_m < \varepsilon$$

$\Rightarrow \sum x_n$  converges.

b) This statement is contrapositive to a)

□

**Theorem 4.4.6** (Limit Comparison Test). *Suppose  $x_n$  and  $y_n$  are strictly positive sequences and Suppose following limit exists*

$$r = \lim \left( \frac{x_n}{y_n} \right)$$

a) If  $r \neq 0$  then  $\sum x_n$  converges iff  $\sum y_n$  converges.

b) If  $r = 0$  then if,  $\sum y_n$  convergent then  $\sum x_n$  convergent.

*Proof.* a) Given  $r = \lim \frac{x_n}{y_n}$

$\therefore$  by definition, For any  $\varepsilon > 0$ ,  $\exists, k(\varepsilon) \in \mathbb{N}$

such that  $\left| \frac{x_n}{y_n} - r \right| < \varepsilon, \forall n \geq k(\varepsilon)$

As  $r \neq 0, \Rightarrow r > 0 \Rightarrow \varepsilon \frac{r}{2}$

$$r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$$

$$\left( \frac{r}{2} \right) y_n < x_n < \left( \frac{3r}{2} \right) y_n$$

$$\therefore \left( \frac{r}{2} \right) y_n < x_n$$

$\Rightarrow$  if  $x_n$  converges then  $\sum y_n$  also converges. ... ( by comparison test)

$$\therefore x_n < \left(\frac{3r}{2}\right) Y_n$$

$\Rightarrow$  If  $\sum y_n$  converges then  $x_n$  also converges. ... (by comparison test)

$$r = 0 \text{ i.e } \lim \left( \frac{x_n}{y_n} \right) = 0$$

$\therefore$  by definition, For any  $\varepsilon > 0$ ,  $\exists, k(\varepsilon) \in \mathbb{N}$

such that

$$\left| \frac{x_n}{y_n} - 0 \right| < \varepsilon$$

$$\left| \frac{x_n}{y_n} \right| < \varepsilon$$

$$\frac{x_n}{y_n} < \varepsilon$$

$$0 < x_n < \varepsilon y_n$$

$\therefore$  By comparison test,

$\sum x_n$  converges if  $\sum y_n$  converges. □

**Definition 4.4.3** (Absolute Convergence): *let  $x_n$  be sequence in  $\mathbb{R}$ . We say that  $\sum x_n$  is absolutely convergent if the series  $\sum |x_n|$  is convergent . A series is said to be conditionally convergent if it is convergent but not absolutely convergent.*

**Example 21:**

$\sum \frac{(-1)^n}{n}$  is convergent but  $\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$  is not convergent  
 $\therefore \sum \frac{(-1)^n}{n}$  is conditionally convergent.

**Theorem 4.4.7.** *If a series is absolutely convergent then it is convergent.*

*Proof.*  $\sum |x_n|$  is convergent

$\therefore$  for any  $\varepsilon > 0$   $M(\varepsilon) \in \mathbb{N}$

$$|x_{n+1}| + |x_{n+2}| + \dots + |x_m| < \varepsilon \forall m > n > M(\varepsilon)$$

$$|x_{n+1} + x_{n+2} + \dots + x_m| \leq \varepsilon$$

$$|x_{n+1}| + |x_{n+2}| + \dots + |x_m| \leq \varepsilon \forall m > n > M(\varepsilon)$$

$\Rightarrow \sum x_n$  is convergent. □

**Theorem 4.4.8** (Limit Comparison Test- II-). *Suppose  $x_n$  and  $y_n$  are non-zero real sequence and Suppose that following limit exists in  $\mathbb{R}$*

$$r = \lim \left( \frac{x_n}{y_n} \right)$$

a) If  $r \neq 0$  then  $\sum x_n$  absolutely convergent iff  $\sum y_n$  is absolutely convergent.

b) If  $r = 0$  and  $\sum y_n$  is absolutely convergent then  $\sum x_n$  absolutely convergent.

**Theorem 4.4.9** (Root test). Let  $x_n$  be sequence in  $\mathbb{R}$ . Suppose that the limit  $r = \lim |x_n|^{\frac{1}{n}}$  exists in  $\mathbb{R}$  then  $\sum x_n$  is absolutely convergent when  $r < 1$  and is divergent when  $r > 1$ .

*Proof.*  $r < 1$ ,  $r = \lim |x_n|^{\frac{1}{n}}, \exists r_1, r_1 \in (r, 1)$

$$|x_n|^{\frac{1}{n}} \leq r_1$$

$$\therefore |x_n| \leq r_1^n$$

by comparison test,

$|x_n| < (r_1)^n$  it is convergent

$|x_n| < (r_1)^n$  it is absolutely convergent. □

**Theorem 4.4.10** (Ratio Test). Let  $x_n$  be non-zero sequence in  $\mathbb{R}$ . Suppose  $r = \lim \left| \frac{x_{n+1}}{x_n} \right|$  exists then  $\sum x_n$  is absolutely convergent when  $r < 1$  and divergent when  $r > 1$

*Proof.*  $r < 1, r_1 \in (r, 1)$

$$\left| \frac{x_{n+1}}{x_n} \right| \leq r_1, \forall n > k(\varepsilon)$$

$$|x_{n+1}| \leq r_1 |x_n|$$

$$|x_{n+1}| \leq r_1 |x_n| < r_1 \cdot r_1 |x_{n-1}| < \dots < r_1^n |x_1|$$

$$\therefore |x_{n+1}| < r_1^n \cdot c$$

$$\therefore \sum |x_{n+1}| < \sum r_1^n \cdot c$$

$\therefore$  by comparison test,

$\sum x_n$  is absolutely convergent

□

#### 4.5 Establish the converges/divergence of series

##### Example 22:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$\sum_{n=1}^{\infty} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots$$

The series is converges to zero

or  $(n+1)(n+2) > n.n$

$$\therefore \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

$$\therefore 0 < \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

by comparison test,

$\sum \frac{1}{(n+1)(n+2)}$  is convergent.

### Example 23:

$$2^{(\frac{-1}{n})}$$

$$\lim_{n \rightarrow \infty} 2^{(\frac{-1}{n})} = 1 \neq 0$$

$\therefore$  by  $n^{th}$  term test

$2^{(\frac{-1}{n})}$  is divergent

**Example 24:**

$$\frac{n}{2^n}$$

Applying ratio test

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{(n+1)/2^{(n+1)}}{n/2^n} \right| = \left| \frac{n+1}{n} \right| \cdot \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{2} < 1$$

$\therefore \frac{\sum n}{2^n}$  is convergent.

**Definition 4.5.1** (Integral test): *Let  $f$  be a positive decreasing function on  $\{t, t > 1\}$  then the series  $\sum_{k=1}^{\infty} f(k)$  converges if and only if the improper integral*

$$\int_1^{\infty} f(t) dt = \lim_{b \rightarrow \infty} \int_1^b f(t) dt$$

*exists. In the case of convergence, the partial sum*

$S_n = \sum_{k=1}^n f(k)$  and sum  $S = \sum_{k=1}^{\infty} f(k)$  satisfy the estimates

$$\int_{n+1}^{\infty} f(t) dt \leq S - S_n \leq \int_1^{\infty} f(t) dt$$

**Example 25:**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$= \int_1^{\infty} \frac{1}{t^p} dt, \quad x_n \frac{1}{n^p}$$

$$= \left[ \frac{t^{-p+1}}{-p+1} \right]_1^{\infty}$$

$$= \frac{1}{1-p} \left[ \frac{1}{t^{p-1}} \right]_1^{\infty}$$

$$\frac{1}{p-1}, p > 1$$

$\therefore \sum \frac{1}{n^p}$  is convergent

**Definition 4.5.2** (Raabies Test): Let  $x_n$  be non-zero sequence in  $\mathbb{R}$  and let

$a = \lim n \left( 1 - \left| \frac{x_{n+1}}{x_n} \right| \right)$  whenever this limit exists then  $\sum x_n$  absoultey convergent when  $a > 1$  and is not absoultey convergent when  $a < 1$

**Example 26:**

$$x_n = \frac{1}{n^p}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \left| \frac{n^p}{(n+1)^p} \right| = \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$\therefore \lim n \left( 1 - \left| \frac{x_{n+1}}{x_n} \right| \right) = \lim n \left( 1 - \left| \left( \frac{1}{1 + \frac{1}{n}} \right)^p \right| \right)$$

$$= \lim \left( \frac{\left( 1 + \frac{1}{n} \right)^p - 1}{\frac{1}{n} \left( 1 + \frac{1}{n} \right)^p} \right)$$

$$= \lim \left( \frac{p \left( 1 + \frac{1}{n} \right)^{p+1} \left( -\frac{1}{n^2} \right)}{\frac{1}{n} \left( 1 + \frac{1}{n} \right)^p} \right)$$

$$= \lim \left( \frac{-\frac{p}{n} \left( 1 + \frac{1}{n} \right)}{\left( 1 + \frac{1}{n} \right)^p} \right)$$

$$= \lim_{n \rightarrow \infty} \left( \frac{-p \left( 1 + \frac{1}{n} \right)}{n} \right)$$

$$= \lim_{n \rightarrow \infty} p \left( -\frac{1}{n} - \frac{1}{n^2} \right)$$

$$= p$$

### Example 27:

$$x_n = \frac{1}{n(n+1)}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} \right| = \left| \frac{n}{n+2} \right| = \left| \frac{1}{1 + \frac{2}{n}} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{1}{1 + \frac{2}{n}} \right| = 1$$

$\therefore$  Ratio test fails ( $\because r = 1$ )

we know  $n(n+1) > n.n$

$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$

$$\therefore \frac{1}{n(n+1)} < \frac{1}{n^2} \quad (0 < x_n < y_n)$$

by comparison test

As  $\sum \frac{1}{n^2}$  is convergent ,  $\sum \frac{1}{n(n+1)}$  is also convergent.

### **Example 28:**

$$\frac{n!}{n^n}$$

Using Raabies test, we have,

$$\text{consider, } \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\Rightarrow \left( 1 - \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right)$$

$$= \frac{\left(1 + \frac{1}{n}\right)^n - 1}{\left(1 + \frac{1}{n}\right)^n}$$

$$= n \left( 1 - \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right)$$

$$\text{as } n \rightarrow \infty, \quad r = \lim n \left( 1 - \left| \frac{x_{n+1}}{x_n} \right| \right) = 0 < 1$$

$\therefore \sum x_n = \sum \frac{n!}{n^n}$  is not absolutely convergent. i.e divergent.

**Example 29:**

$$\frac{n^2}{\sqrt{n+1}} = \left| \frac{x_{n+1}}{x_n} \right| = \frac{(n+1)^2}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{n^2}$$

$$= \left(1 + \frac{1}{n}\right)^2 \frac{\sqrt{1 + \frac{1}{n}}}{1 - \frac{2}{n}}$$

$$\Rightarrow n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right)$$

$$= \frac{\sqrt{1 + \frac{2}{n}} - \left(1 + \frac{1}{n}\right)^2 \sqrt{1 + \frac{1}{n}}}{\sqrt{\frac{1}{n^2} + \frac{2}{n^3}}}$$

$\therefore \sum \frac{n^2}{\sqrt{n+1}}$  is not absolutely convergent. i.e divergent.

## 4.6 Test for Non-Absolute Convergence

**Definition 4.6.1** (Alternative Series): *A sequence of non-zero real numbers is said to be alternating if the terms  $(-1)^{(n+1)}x_n$ ,  $n \in \mathbb{N}$  are all positive (or all negative) real numbers. If the sequence  $x_n$  is alternating, we say that the series  $\sum x_n$  is alternating series.*

**Theorem 4.6.1** (Alternating Series test). *Let  $z_n$  be decreasing sequence with strictly positive numbers with  $\lim z_n = 0$  then the alternating series  $\sum (-1)^{n+1}z_n$  is convergent.*

*Proof.* Given that  $z_n$  decreasing sequence and let  $S_n = \sum (-1)^{n+1}z_n$

We have

$$S_{2n} = (z_1 - z_2) + (z_3 - z_4) + \dots + (z_{2n-1} - z_{2n})$$

and Since  $(z_k - z_{k+1}) \geq 0$ , it follows that  $S_{2n}$  is increasing sequence

$$S_{2n} = z_1 - (z_2 - z_3) + \dots - (z_{n-2} - z_{n-1}) - z_{2n}$$

$$\therefore S_{2n} \leq z_1$$

$\therefore$  bounded by MCT,  $S_{2n}$  must be convergent to some number  $c \in \mathbb{R}$ .

We have to show that entire  $S_n \rightarrow c$  if  $\varepsilon > 0$ , let  $k \in \mathbb{N}$ . if  $n \geq k$

$$|S_{2n} - c| \leq \frac{\varepsilon}{2} \text{ and } z_{2n+1} \leq \frac{\varepsilon}{2}$$

$$|S_{2n+1} - c| = |S_{2n} + z_{n+1} - c|$$

$$\leq |S_{2n} - c| + |z_{n+1}|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$S_n \rightarrow c$$

$S_n = \sum (-1)^{n+1} z_n$  is convergent.

□

**Lemma 4.6.2** (Abels Lemma).  $x_n, y_n \in \mathbb{R}$   $S_n = \sum_{i=1}^n$  with  $S_0 = 0$  if  $m > n$  then,

$$\sum_{k=n+1}^m x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

$$\text{Proof. } y_k = S_k - S_{k-1} \quad \left( \because S_k = \sum_{i=1}^k y_i \& S_{k-1} = \sum_{i=1}^{k-1} y_i \right)$$

$$x_k y_k = x_k S_k - x_k S_{k-1}$$

$$\begin{aligned} & \sum_{k=n+1}^m x_k y_k \\ &= \sum_{k=n+1}^{m-1} (x_k S_k - x_k S_{k-1}) \end{aligned}$$

$$= x_{n+1} S_{n+1} - x_{n+1} S_n + x_{n+2} S_{n+2} - x_{n+2} S_{n+1} + \dots + x_m S_m - x_m S_{m-1}$$

$$= (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

□

**Theorem 4.6.3** (Diricblet's Test). *If  $x_n$  is decreasing  $\neq 0$ , if  $S_n = \sum y_i$  is bounded then  $x_n y_n$  is convergent.*

*Proof.* Let  $S_n \leq B$ ,  $\forall n \in \mathbb{N}$ . if  $m > n$ , by abels lemma and  $x_k - x_{k+1} > 0$  (as  $x_n$  is decreasing)

Consider,

$$\left| \sum_{k=n+1}^m x_k y_k \right| = \left| (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k \right|$$

$$\leq |(x_m S_m - x_{n+1} S_n)| + \sum_{k=n+1}^{m-1} |(x_k - x_{k+1})| |S_k|$$

Suppose,

$$\begin{aligned} & S_m, S_n, S_k = B \\ & \leq |x_m - x_{n+1}| B + B \sum_{k=n+1}^{m-1} |(x_k - x_{k+1})| |S_k| \\ & \leq \frac{\epsilon}{2B} B + B \frac{\epsilon}{2B} \end{aligned}$$

$\leq \varepsilon$ 

$\therefore \sum x_n y_n$  is convergent.

□

**Theorem 4.6.4** (Abel's Test). *If  $x_n$  convergent monotone sequence and  $y_n$  is convergent then the series is  $x_n y_n$  also convergent.*

*Proof.* Let  $x_n$  is decreasing  $x$

$$u_n = x_n - x \text{ decreasing } 0$$

$\sum u_n y_n$  is convergent by diricblets test

$$\begin{aligned} & \sum_n x_n y_n \\ &= \sum_n (x + u_n) y_n \\ &= x \sum_n y_n + \sum_n u_n y_n \end{aligned}$$

$\sum_n x_n y_n$  is convergent sequence.

□

### Example 30:

$\sum a_n$  convergent then

1.  $\sum b_n = \frac{a_n}{n}$  is convergent sequence.

2.  $\sum n^{1/n} a_n$  is divergent sequence.

3.  $\sum a_n \sin n$  is divergent sequence.

4.  $\sum \frac{\sqrt{a_n}}{n}$  is convergent sequence.

5.  $\sum \sqrt{a_n}$  is divergent sequence.



## Function and Continuity

**Definition 5.0.1** (Cluster Point): *Let  $A \subseteq \mathbb{R}$ . A point  $c \in \mathbb{R}$  is cluster point of  $A$  if every  $\delta > 0 \exists$  atleast one point  $x \in A$ ,  $x \neq c \ni |x - c| < \delta$*

**Theorem 5.0.1.** *The number  $c \in \mathbb{R}$ , is cluster point of  $A \subseteq \mathbb{R}$  if and only if  $\exists$  sequence  $a_n$  in  $A$  such that  $\lim(a_n) = c$  and  $a_n \neq c$ ,  $\forall n$*

*Proof.* If  $c$  is cluster point of  $A$  then for any  $n \in \mathbb{N}$  the  $\frac{1}{n}$  neighbourhood  $v_{1/n}(c)$  contains atleast one point  $a_n$  in  $A$  distinct from  $c$ , then  $a_n \in A$ ,  $a_n \neq c$  &  $|a_n - c| < \frac{1}{n} \Rightarrow \lim a_n = c$  conversly, if  $\exists$  a sequence  $a_n$  in  $A^{\setminus\{c\}}$  with  $\lim(a_n) = c$ , then for any  $\delta > 0$ ,  $\exists k$  such that

if  $n \geq k$ , then  $a_n \in v_\delta(c)$ . Therefore,  $\delta$  neighbourhood  $v_\delta(c)$  contains the point  $a_n$ ,  $\forall n \geq k$  which belong to  $A$  and are distinct from  $c$ .  $\square$

**Definition 5.0.2** (Limit of Function): *Let  $A \subseteq \mathbb{R}$  and let  $c \in \mathbb{R}$  be the cluster point of  $A$ . for a function  $f : A \rightarrow \mathbb{R}$  a real number  $L$  is said to be limit of  $f$  at  $c$  if, given any  $\varepsilon > 0$ ,  $\exists \delta > 0, \exists x \in A$  and  $0 < |x - c| < \delta$  then  $|f(x) - L| < \varepsilon$  then we say  $f$  converges to  $L$  at  $c$ .*

**Theorem 5.0.2.** *If  $f : A \rightarrow \mathbb{R}$  and if  $c$  is a cluster point of  $A$ , then  $f$  can have only one limit at  $c$ .*

*Proof.* We will prove this by method of contradiction.

Let  $L$  and  $L'$  be limits of  $f$  at  $c$

For any  $\varepsilon > 0$ ,  $\exists \delta \left( \frac{\varepsilon}{2} \right) > 0 \exists x \in A$  and  $0 < |x - c| < \delta \left( \frac{\varepsilon}{2} \right)$

$$\Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

Also,  $\exists \delta' \left( \frac{\varepsilon}{2} \right) > 0 \quad \exists \quad x \in A \text{ and } |x - c| < \delta' \left( \frac{\varepsilon}{2} \right)$

$$\Rightarrow |f(x) - L'| < \frac{\varepsilon}{2}$$

$$|L - L'|$$

$$= |L - f(x) + f(x) - L'|$$

$$\leq |L - f(x)| + |f(x) - L'|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

Since,  $\varepsilon > 0$  is arbitrary,  $L = L'$  □

**Theorem 5.0.3** (Sequential Criterion). *Let  $f : A \rightarrow \mathbb{R}$  and let  $c$  be a cluster point of  $A$  then the*

following are equivalent.

$$1. \lim_{x \rightarrow c} f(x) = L$$

$$2. \text{for every } x_n \text{ in } A, x_n \rightarrow c, x_n \neq c, \quad \forall n \in \mathbb{N} \Rightarrow f(x_n) \rightarrow L.$$

**Definition 5.0.3** (Divergence Criterion): *Let  $A \subseteq \mathbb{R}$  let  $f : A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be cluster point of  $A$ .*

a) *If  $L \in \mathbb{R}$  then  $f$  does not have limit  $L$  at  $c$  iff  $\exists$  sequence  $x_n$  in  $A$  with  $x_n \neq c, \forall n \in \mathbb{N}$  such that sequence  $x_n$  converges to  $c$ . but the sequence  $f(x_n)$  does not converges to  $L$*

b) *the function does not have a limit  $L$  at  $c$  iff  $\exists x_n$  in  $A$  with  $x_n \neq c, \forall n \in \mathbb{N}$  such that the sequence  $x_n$  converges to  $c$  but the sequence  $f(x_n)$  does not converges in  $\mathbb{R}$*

$$f(x) =$$

$$f(x) = \begin{cases} +1 & \text{if } x > 0 \\ -0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

**Theorem 5.0.4** (Limit Theorem). *Let  $A \subseteq \mathbb{R}$ . and  $c \in \mathbb{R}$ , be cluster point of  $A$  we say that  $f$  is bounded on neighbourhood of  $c$  if  $\exists$  a  $\delta$  neighbourhood of  $V_\delta(c)$  of  $c$  and constant  $M > 0 \ \exists |f(x)| \leq M \quad \forall x \in A \cap V_\delta(c)$*

**Theorem 5.0.5.** *If  $A \subseteq \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$  has limit at  $c \in \mathbb{R}$  then  $f$  is bounded on some neighbourhood of  $c$*

*Proof.* If  $L = \lim_{x \rightarrow c} f$  then for  $\varepsilon = 1, \exists \delta_c < 0$

Such that  $0 < |x - c| < \delta \Rightarrow |f(x) - L| < 1$

$$|f(x)| - |L| \leq |f(x) - L| < 1$$

if  $x \in A \cap V_\delta(c)$ ,  $x \neq c$  then,

$$|f(x)| < |L| + 1$$

if  $c \notin A$ , Take  $M = |L| + 1$

while if  $c \in A$ , Take  $M = \text{Sup}\{|f(x)|, |L| + 1\}$

$$\therefore |f(x)| \leq M$$

$\therefore$  by limit theorem

$\therefore f$  is bounded on neighbourhood of  $c$ . □

**Definition 5.0.4:** Let  $A \subseteq \mathbb{R}$  and let  $f$  &  $g$  be function defined on  $A$  to  $\mathbb{R}$ . We define the sum  $f + g$ , the difference  $f - g$  and the product  $f.g$  on  $A \rightarrow \mathbb{R}$  to be function from  $A$  to  $\mathbb{R}$  given by,

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(f.g)(x) = f(x).g(x)$$

Further if  $b \in \mathbb{R}$

$$(bf)(x) = b \cdot f(x)$$

finally, if  $h(x) \neq 0$ ,

$$\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)}$$

**Theorem 5.0.6.** Let  $A \subseteq \mathbb{R}$  let  $f$  &  $g$  be function on  $A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be a cluster point of  $A \rightarrow \mathbb{R}$  & let

1. If  $\lim_{x \rightarrow c} f = L$  &  $\lim_{x \rightarrow c} g = M$  then

$$\lim_{x \rightarrow c} (f \pm g) = L \pm M$$

$$\lim_{x \rightarrow c} (f \cdot g) = L \cdot M$$

$$\lim_{x \rightarrow c} (b \cdot f) = b \cdot L$$

$$2. \lim_{x \rightarrow c} \left( \frac{f}{c} \right) = \frac{L}{H}$$

where,  $h(x) \neq 0$  and  $\lim_{x \rightarrow c} h(x) = H \neq 0$

**Theorem 5.0.7.** Let  $A \subseteq \mathbb{R}$  let  $f : A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be the cluster point of  $A$ .

if  $a \leq f(x) \leq b \quad \forall x \in A, x \neq c$  and if  $\lim_{x \rightarrow c} f$  exists then  $a \leq \lim_{x \rightarrow c} f \leq b$

*Proof.* Given,  $f : A \rightarrow \mathbb{R}$  and  $c$  is cluster point of  $A$ .

let  $x_n \in A$  such that  $x_n \rightarrow c$

$$\therefore f(x_n) \rightarrow L = \lim_{x \rightarrow c} f(x_n) = \lim_{x \rightarrow c} f(x)$$

Also,

$$a \leq f(x) \leq b$$

$$a \leq f(x_n) \leq b$$

$$a \leq \lim_{x \rightarrow c} f(x_n) \leq b$$

$$a \leq \lim_{x \rightarrow c} f(x) \leq b$$

$$a \leq L \leq b$$

□

**Theorem 5.0.8** (Squeeze Theorem). *Let  $A \subseteq \mathbb{R}$  let  $f, g, h : \rightarrow \mathbb{R}$  &  $c \in \mathbb{R}$  be a cluster point of  $A$ . If*

$$f(x) \leq g(x) \leq h(x), \quad \forall x \in A, x \neq c \text{ & } \lim_{x \rightarrow c} f = \lim_{x \rightarrow c} h \text{ then, } \lim_{x \rightarrow c} g = L.$$

*Proof.* Given,  $f, g, h : \rightarrow \mathbb{R}$  &  $c$  is cluster point of  $A$   $x_n \in A \Rightarrow x_n \rightarrow c$

$$f(x_n) \rightarrow L = \lim_{x \rightarrow c} f(x_n) = \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} h(x_n)$$

i.e  $f(x_n) \rightarrow L$  &  $h(x_n) \rightarrow L$

Also,

$$f(x) \leq g(x) \leq h(x)$$

$$f(x_n) \leq g(x_n) \leq h(x_n)$$

$$\lim_{x_n \rightarrow c} f(x_n) \leq \lim_{x_n \rightarrow c} g(x_n) \leq \lim_{x_n \rightarrow c} h(x_n)$$

$$\therefore L \leq \lim_{x \rightarrow c} g(x_n) \leq L$$

$$\lim_{x \rightarrow c} g(x_n) = L$$

i.e  $g(x_n) \rightarrow L$

i.e  $\lim_{x \rightarrow c} g = L$

-Hence Proved-

□

**Definition 5.0.5:** Let  $A \in \mathbb{R}$  & let  $f : A \rightarrow \mathbb{R}$

1. If  $c \in \mathbb{R}$  is a cluster point of the set  $A \cap (c, \infty) = \{x \in A, x > c\}$  then we say that  $L \in \mathbb{R}$  is right hand limit of  $f$  at  $c$

$\lim_{x \rightarrow c^+} f(x) = L$  If given any  $\varepsilon > 0$   $\exists \delta(\varepsilon) > 0 \quad \exists \forall x \in A$  with  $0 < x - c < \delta$  then  $|f(x) - L| < \varepsilon$

2. If  $c \in \mathbb{R}$  is a cluster point of the set  $A \cap (-\infty, 0) = \{x \in A, x < c\}$  then we say that  $L \in \mathbb{R}$  is left hand limit of  $f$  at  $c$

$$\lim_{x \rightarrow c^-} f(x) = L \text{ If given any } \varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \exists \quad \forall x \in A \text{ with } 0 < -x + c < \delta \text{ then } |f(x) - L| < \varepsilon$$

## 5.1 Continuous Function

**Definition 5.1.1** (Continuous Function): Let  $A \subseteq \mathbb{R}$  let  $f : A \rightarrow \mathbb{R}$  & let  $c \in A$  we say that  $f$  is continuous at  $c$  if given any  $\varepsilon > 0$ ,  $\exists \delta(\varepsilon) > 0 \quad \exists$  if  $x$  is any point of  $A$  satisfying  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \varepsilon$  iff fails to be continuous at  $c$  then we say that  $f$  is discontinuous at  $c$

**Theorem 5.1.1.** A function  $f : A \rightarrow \mathbb{R}$  is continuous at point  $c \in A$  if and only if given any  $\varepsilon > 0$ ,  $v_\varepsilon(f(c))$  of  $f(c)$   $\exists$  of  $c$  such that if  $x$  is any point of  $A \cap v_\delta(c)$  then  $f(x) \in v_\varepsilon(f(c))$  i.e  $A \cap v_\delta(c) \subseteq v_\varepsilon(f(c))$

*Proof.*  $\therefore \lim_{x \rightarrow c} = L$

i.e any  $\varepsilon > 0$ ,  $\exists \delta(\varepsilon) > 0 \quad \exists$

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

and  $\lim f(x) = f(c)$

for any  $\varepsilon > 0$ ,  $\exists \delta(\varepsilon) > 0 \quad \exists$

$$|x - c| < \delta, \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\therefore x \in A \cap v_\delta(c) \Rightarrow f(x) \in v_\varepsilon(f(c)), \quad \forall x$$

$$\therefore f(A \cap v_\delta(c)) \subseteq v_\varepsilon(f(c)) \dots (\because \text{if } A \subset B \Rightarrow x \in A \Rightarrow x \in B \text{ then } A \subseteq B) \quad \square$$

**Definition 5.1.2** (Combination of Continuous function): Let  $A \subseteq \mathbb{R}$ . Let  $f$  &  $g$  be function on  $A$  to  $\mathbb{R}$ , let  $b \in \mathbb{R}$ , Suppose that  $c \in A$  & that  $f$  &  $g$  are continuous at  $c$

a) then  $f + g, f - g, f \cdot g$  and  $b \cdot f$  are continuous at  $c$

b) if  $h : A \rightarrow \mathbb{R}$  is continuous at  $c \in A$  & if  $h(x) \neq 0, \quad x \in A$ , then  $\left(\frac{f}{h}\right)$  is also continuous at  $c$

**Definition 5.1.3** (Continuous Point): Let  $A \subseteq \mathbb{R}$  &  $f : A \rightarrow \mathbb{R}$ . if  $B \subseteq A$  we say that  $f$  is contin-

uous on set  $B$  iff is continuous at every point of  $B$

### Example 31:

Continuous

- $f(x) = x, \quad x \in \mathbb{R}$
- $f(x) = x^2, \quad x \in \mathbb{R}$
- $f(x) = \frac{1}{x}, \quad x \in \mathbb{R}^+, \{0\}$
- $f(x) = \text{Polynomial function} \quad x \in \mathbb{R}$
- $f(x) = \text{Rational function}$
- $f(x) = \text{Trigonometric function}$
- $f(x) = \sqrt{f}, \quad x \in \mathbb{R}$

**Example 32:**

Discontinuous

- $\psi(x) = \frac{1}{x}, \quad x = 0$

- $\psi(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$   
discount everywhere

- $\sin(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ 0, & x = 0 \end{cases}$   
discount at  $x = 0$

- $\psi(x) = [x]$  = greatest integer function discount at integer

**Theorem 5.1.2.** Let  $A \subseteq \mathbb{R}$ . Let  $f : A \rightarrow \mathbb{R}$  & let  $|f|$  be defined by  $|f|(x) = |f(x)| \quad \forall x \in A$

1. If  $f$  is continuous at point  $c \in A$  then  $|f|$  is countinuous at  $c$
2. If  $f$  is continuous on  $A$  then  $|f|$  is continous on  $A$ .

**Theorem 5.1.3.** Let  $A, B \in \mathbb{R}$  & let  $f : A \rightarrow \mathbb{R}$  &  $g : B \rightarrow \mathbb{R}$  be function such that  $f(A) \subseteq B$  if  $f$  is countinuous at point  $c \in A$  and  $g$  is continuous at  $b = f(c) \in B$  then the composition  $g \circ f : A \rightarrow \mathbb{R}$  is continuous at  $c$ .

*Proof.* Let  $W$  be  $\varepsilon$ -neighbourhood of  $g(b)$ . since  $g$  is continuous at  $b$  there is a  $\delta$ -neighbourhood of  $v$  of  $b = f(c)$  such that if  $y \in B \cap v$  then  $g(y) \in W$ . Since  $f$  is also continuous at  $c$ , ther is a  $v$ -neighbourhood  $v$  of  $c \ni x \in A \cap U$  then  $f(x) \in v$

Since  $f(A) \subseteq B$ , it follows that if  $x \in A \cap U$  then  $f(x) \in B \cap v$  so that  $g \circ f(x) = g(f(x)) \in W$  But, Since  $W$  is an arbitrary  $\varepsilon$ -neighbourhood of  $g(b)$  this implies  $g \circ f$  is continuous at  $c$ .  $\square$

## 5.2 Continuous function on Interval

**Definition 5.2.1** (Bounded Function): *A function  $f : A \rightarrow \mathbb{R}$  is said to be bounded on  $A$  if  $\exists$  a constant  $M > 0$  such that  $|f(x)| \leq M$  for all  $x \in A$*

**Theorem 5.2.1** (Boundedness Theorem-). *Let  $I = [a, b]$  be a closed bounded interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$  then  $f$  is bounded on  $I$ .*

*Proof.* Suppose  $f$  is bounded on  $I$ .

then, for any  $n \in \mathbb{N}$ ,  $\exists x_n \in I \quad \exists |f(x_n)| > k$ .

Since,  $I$  is bounded, sequence  $x_n$  is bounded.

$\therefore$  By Bolzano weistress theorem,

$\exists$  subsequence  $x_{nk}$  that converges to some  $x$

Since,  $I$  is closed, elements of sequence  $x_{nk} \in I \Rightarrow x \in I$ .

then,  $f$  is continuous at  $x$  so that  $f(x_{nk})$  converges to  $f(x)$ .

$$\Rightarrow |f(x_{nk})| > n_k > k \quad \forall k \in \mathbb{N}$$

$\therefore$  Our assumption is wrong.

Hence,  $f$  must be bounded. □

**Definition 5.2.2** (Absolute Extremum): *Let  $A \subseteq \mathbb{R}$  and let  $f : A \rightarrow \mathbb{R}$ . We say that  $f$  has an absolute maximum on  $A$  if there is  $x^* \in A$  such that*

$$f(x^*) \geq f(x), \quad \forall x \in A$$

*We say that  $f$  has absolute minimum on  $A$  if there is  $x^* \in A$  such that*

$$f(x^*) \leq f(x), \quad \forall x \in A$$

**Theorem 5.2.2** (Maximum-Minimum Theorem). *Let  $I = [a, b]$  be closed bounded interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$  then  $f$  has an absolute maximum and absolute minimum on  $I$ .*

*Proof.*  $f(I) = \{f(x); \quad x \in I\}$

$I$  is a closed bounded and  $f$  is continuous on  $I$  then  $f(x)$  is also bounded  $\subseteq \mathbb{R}$

$\therefore$  By completeness property, it has supremum and infimum

$$\therefore S^* = \text{Sup}\{f(I)\}, \quad S_* = \text{Inf}\{f(I)\}$$

claim- To show ,  $\exists x^*, x_* \in I$

$\exists S^* = f(x^*) = \text{absolute maximum}$

$S_* = f(x_*) = \text{absolute minimum}$

$$S_* = \text{Inf}\{f(I)\}$$

if  $n \in \mathbb{N}$  then  $S^* - \frac{1}{n}$  is not upper bound

$$\therefore S^* - \frac{1}{n} < f(x_n) < S^*, \quad \forall n \in \mathbb{N}$$

Since,  $I$  is bounded  $x_n$  is bounded By Bolzano weistress theorem,

$\exists x_{n_k}$  subsequence of  $x_n$  and  $x_{n_k} \rightarrow \text{some } x^*$

Also, As  $I$  is closed and  $x_{n_k} \in I \Rightarrow x^*$  must be in  $I$

$\Rightarrow f$  is continuous at  $x^*$ ,  $\lim f(x_{n_k}) = f(x^*)$

$$S^* - \frac{1}{n} < f(x_{n_r}) \leq S^*, \quad \forall r \in \mathbb{N}$$

$\therefore$  by squeeze theorem

$$\lim f(x_{n_r}) = S^*$$

$$\therefore S^* = f(x^*) \text{ i.e } f(x^*) \geq f(x), \quad \forall x$$

$\therefore x^*$  is absolute maximum

Similarly, we show  $x_*$  is absolute minimum □

**Theorem 5.2.3** (Location of Root). *Let  $I = [a, b]$  & let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . If  $f(a) < 0 f(b)$  or  $f(b) < 0 < f(a)$ , then  $\exists c \in (a, b) \ni f(c) = 0$ .*

*Proof.* Assume that  $f(a) < 0 f(b)$

Let  $I_1 = [a_1, b_1]$  where,  $a_1 = a, b_1 = b$

let  $P_1 = \frac{a+b}{2}$  if  $f(P_1) = 0$  then  $c = P_1$

if  $P_1 \neq 0$ , then either  $f(P_1) > 0$  or  $f(P_1) < 0$

if  $f(P_1) > 0$  then  $a_2 = a_1, b_2 = P_1$  and if  $f(P_1) < 0$

$a_2 = P_1, b_2 = b_1$  thus, we get  $I = [a_2, b_2] \in I_1$

continuing this bisectins, we obtain intervals  $I_1, I_2, \dots, I_k$

In this process, we terminate by locating a point  $P_n \in \exists f(P_n) = 0$

if process does not terminate, we obtain nested sequence of bounded interval

$$I_n = [a_n, b_n]$$

$$\exists f(a_n) < 0 \text{ & } f(b_n) > 0$$

$$\text{& length of interval } b_n - a_n = \frac{(b-a)}{2^{n-1}}$$

$$\Rightarrow \exists \text{ a point } c \in I_n \quad \forall n \in \mathbb{N}$$

$$a_n \leq c \leq b_n, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 0 \leq c - a_n \leq b_n - a_n$$

$$\Rightarrow 0 \leq c - a_n \leq \frac{(b-a)}{2^{n-1}}$$

$$\Rightarrow \lim f(a_n) = \lim f(b_n) = f(c)$$

$$\Rightarrow 0 \leq b_n - c \leq b_n - a_n$$

$$\Rightarrow 0 \leq b_n - c \leq \frac{(b-a)}{2^{n-1}}$$

□

**Theorem 5.2.4** (Bolzano's Intermediate Theorem). *Let  $I$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$  if  $a, b \in I$  and if  $k \in \mathbb{R}$  satisfies  $f(a) < k < f(b)$  then a point  $c \in I$  between  $a$  &  $b \ni f(c) = k$ .*

*Proof.* 1. Assume that,  $a < b, a, b \in I, f$  continuous on  $I$

Define  $g(x) = f(x) - k$

As  $f(x)$  is continuous,  $g(x)$  is also continuous on  $I$

Also,  $f(a) < k < f(b)$

$$f(a) - k < 0 < f(b) - k$$

$$g(a) < 0 < g(b)$$

$\therefore$  by location of root theorem

$$\exists c \ni g(c) = 0$$

i.e  $f(c) - k = 0$

$\therefore f(c) = k$

2. Assume that,  $a > b, a, b \in I, f$  continuous on  $I$

Define  $h(x) = k - f(x)$

As  $f(x)$  is continuous,  $h(x)$  is also continuous on  $I$

Also,  $f(a) < k < f(b)$

$$k - f(a) < 0 < k - f(b)$$

$$h(a) < 0 < h(b)$$

$\therefore$  by location of root theorem

$$\exists c \ni h(c) = 0$$

i.e  $k - f(c) = 0$

$$\therefore f(c) = k$$

□

**Corollary 5.2.4.1.** Let  $I - [a, b]$  be a closed bounded interval. Let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$  if  $k \in \mathbb{R}$  is any number satisfying  $\text{Inf } f(I) \leq k \leq \text{Sup } f(I)$  then  $\exists$  a number  $c \in I \ni f(c) = k$

*Proof.* Given that,  $I$  is a closed bounded interval and  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$

$\therefore$  By maximum- minimum theorem,

$$\exists \quad x^*, x_* \in I \text{ such that } f(x^*) = \text{Sup}\{f(I)\}$$

$$f(x_*) = \text{Inf}\{f(I)\}$$

Also, Given that,  $\text{Inf } f(I) \leq k \leq \text{Sup } f(I)$

$$\text{i.e } f(x^*) \leq k \leq f(x_*)$$

$\therefore$  by Bolzano intermediate theorem,

$$\exists \quad c \in I \quad \exists \quad f(c) = k$$

-Hence Proved-



**Theorem 5.2.5.** Let  $I$  be closed bounded interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$  then,  
the set  $f(I) = \{f(x) : x \in I\}$  be closed bounded interval.

*Proof.* let ,

$$m = \inf\{f(I)\}$$

$$M = \sup\{f(I)\}$$

by maximum - minimum theorem,  $m, M \in f(I)$

$$f(I) \subseteq [m, M]$$

if  $k \in [m, M]$

$\therefore$  by bolzano-itermediate theorem

$$\exists \quad c \in I, \quad f(c) = k$$

Hence,  $k \in f(I)$

$$\Rightarrow [m, M] \subseteq f(I)$$

$\therefore f(I)$  is the interval  $m, M]$

□

### 5.3 Continuity

**Definition 5.3.1** (Uniform Continuous): Let  $A \subseteq \mathbb{R}$  & let  $f : A \rightarrow \mathbb{R}$ . We say that  $f$  is uniformly continuous on  $A$  if for each  $\varepsilon > 0$ ,  $\exists \delta(\varepsilon) > 0 \quad \exists \quad \text{if } x, y \in A \text{ are any numbers satisfying}$   
 $|x - y| < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$

**Definition 5.3.2** (Non- Uniform Continuity): Let  $A \subseteq \mathbb{R}$  & let  $f : A \rightarrow \mathbb{R}$  then following statements are equivalent.

i)  $f$  is not uniformly continuous on  $A$ .

ii)  $\exists a_n \quad \varepsilon_0 > 0 \quad \exists$  for every  $\delta > 0$  there are points  $x_\delta, y_\delta$  in  $A$  such that,

$$|x_\delta - y_\delta| < \delta \text{ and } |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$$

iii)  $\exists a_n \quad \varepsilon_0 > 0$  and two sequences  $x_n$  &  $y_n$  in  $A$  such that  $\lim_{n \rightarrow \infty} x_n - y_n = 0$  and  $|f(x_n) - f(y_n)| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}$

**Theorem 5.3.1** (Uniform Continuity Theorem). *Let  $I$  be closed bounded interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$  then  $f$  is uniform continuous on  $I$ .*

*Proof.* If  $f$  is not uniform continuous on  $I$  then,

$\exists \varepsilon_0 > 0$  and two sequence  $x_n, y_n \in I$

$$|x_n - y_n| < \frac{1}{n} \text{ & } |f(x_n) - f(y_n)| \geq \varepsilon_0$$

Since  $I$  is bounded  $x_n, y_n$  are bounded.

$\exists$  subsequence  $x_{n_k}$  of  $x_n$  that converges to some elements  $z \in I$  (as  $I$  closed) as

$$|x_n - y_n| < \frac{1}{n} \quad \forall n$$

Subsequence  $y_{n_k}$  of  $y_n$  also converges to  $z$

$$|y_{n_k} - z|$$

$$= |y_{n_k} - x_{n_k} + x_{n_k} - z|$$

$$\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

$\therefore y_{n_k}$  is also converges to  $z$

Now if  $f$  is continuous at  $z$  both  $f(x_{n_k})$  and  $f(y_{n_k})$  must converges  $f(z)$

But this not possible as  $|f(x_n) - f(y_n)| \geq \varepsilon_0$

$\therefore$  Our assumption is wrong. □

**Definition 5.3.3** (Lipschitz Function): *Let  $A \subseteq \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$  if there exists a constant  $k > 0$  such that*

$|f(x) - f(u)| < k|x - u| \quad \forall x, u \in A$  then  $f$  is said to be a Lipschitz function on  $A$

**Theorem 5.3.2.** *Lipschitz function is an uniformly continuous function always.*

*Proof.* for Lipschitz function

$$|f(x) - f(u)| < k|x - u|$$

$$\text{Now, } |x - u| < \frac{\varepsilon}{k} = \delta, \quad | < \frac{\varepsilon}{k} > 0 \text{ ask } > 0$$

$$|f(x) - f(u)| < k \cdot \frac{\varepsilon}{k}$$

$$< \varepsilon$$

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

Lipschitz function is always uniformly continuous function. □

**Theorem 5.3.3.** *If  $f : A \rightarrow \mathbb{R}$  is uniformly continuous on subset  $A$  of  $\mathbb{R}$  and if  $x_n$  is a cauchy sequence in  $A$ , then  $f(x_n)$  is cauchy sequence in  $\mathbb{R}$ .*

*Proof.* let  $x_n$  is a cauchy sequence in  $A$  and let  $\varepsilon > 0$  choose  $x, y \in A$ ,  $\delta > 0$

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

Since,  $x_n$  is a cauchy sequence  $\exists H(\delta)$

$$|x_n - x_m| < \delta, \quad \forall n, m \geq H(\delta)$$

(as  $f$  is uniformly continuous)

$$|f(x_n) - f(x_m)| < \varepsilon$$

Therefore, the sequence  $f(x_n)$  is cauchy sequence. □

**Theorem 5.3.4** (Continuous Extension Theorem). *A function  $f$  is uniformly continuous on  $(a, b)$  iff it can be defined at the end points  $a$  &  $b$  such that the extended function is continuous on  $[a, b]$ .*

*Proof.* Assume that function  $f$  is continuous on  $[a, b]$

$\therefore$  by definition,

for any  $\varepsilon > 0$ ,  $\frac{\varepsilon}{2} > 0$ ,  $\exists \delta(\varepsilon) > 0$  such that,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

let  $x_1$  &  $x_2 \in [a, b]$

by definition,

for any  $\varepsilon > 0$ ,  $\frac{\varepsilon}{2} > 0$ ,  $\exists \delta(\varepsilon) > 0$  such that,

$$|x_1 - c| < \frac{\delta}{2} \Rightarrow |f(x_1) - f(c)| < \frac{\varepsilon}{2}$$

and,

for any  $\varepsilon > 0$ ,  $\frac{\varepsilon}{2} > 0$ ,  $\exists \delta(\varepsilon) > 0$  such that,

$$|x_2 - c| < \frac{\delta}{2} \Rightarrow |f(x_2) - f(c)| < \frac{\varepsilon}{2}$$

consider,

$$|x_1 - x_2|$$

$$= |x_1 - c + c - x_2|$$

$$\leq |x_1 - c| + |x_2 - c|$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2}$$

$$\leq \delta$$

and,

$$|f(x_1) - f(x_2)|$$

$$= |f(x_1) - f(c) + f(c) - f(x_2)|$$

$$\leq |f(x_1) - f(c)| + |f(x_2) - f(c)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\leq \varepsilon$

$$\therefore |x_1 - x_2| \leq \delta \Rightarrow |f(x_1) - f(x_2)| \leq \varepsilon$$

$\therefore f$  is uniformly continuous on  $(a, b)$

Conversely, Suppose  $f$  is uniformly continuous on  $(a, b)$ . Lets define  $f(a)$  &  $f(b)$

Lets  $x_n$  be sequence in  $(a, b)$   $\exists \lim x_n = a$

$\Rightarrow x_n$  is cauchy sequence and as  $f$  is uniformly continuous on  $(a, b)$  and  $x_n \in (a, b)$

$\therefore$  by sequential criteria ,  $\lim f(x_n) = L$  exists if  $y_n$  is any other sequence in  $(a, b)$  that converges to  $a$  then

$$\lim x_n - y_n = a - a = 0$$

$$\lim f(y_n) = \lim(f(y_n) - f(x_n) + f(x_n)) = L$$

So we define,  $L = f(a)$

then  $f$  is continuous at  $a$

Similarly, we can find some  $M = f(b)$  and we can say that  $f$  is continuous on extended

$[a, b]$



**Definition 5.3.4** (Step Function):  *$I \subseteq \mathbb{R}$  be an interval and let  $S : I \rightarrow \mathbb{R}$  then  $S$  is called a step function if it has only a finite number of distinct values.*

#### 5.4 Continuity And Gauges

**Definition 5.4.1** (Partition): *A partition of an interval  $I = [a, b]$  is collection  $P = \{I_1, I_2, \dots, I_n\}$  of non-overlapping closed intervals whose union is  $[a, b]$ . We generally denote  $I_i = [x_{i-1}, x_i]$  where  $a = x_0 < \dots < x_{i-1} < x_i < \dots < x_n = b$*

*The points  $x_i$  ( $i = 0, 1, 2, \dots, n$ ) are called the partition points of  $p$ . If a point  $t_i$  has been chosen from each interval  $I_i$ , for ( $i = 0, 1, 2, \dots, n$ ) then the points  $t_i$  are called tags and set of ordered pairs  $\dot{p} = \{(I_1, t_1), (I_2, t_2), \dots, (I_n, t_n)\}$  is called as tagged partition of  $I$*

**Definition 5.4.2:** *A gauge on  $I$  is a strictly positive function defined on  $I$ . If  $\delta$  is a gauge on  $I$ , then a tagged partition  $\dot{p}$  is said to be  $\delta$ -fine if*

$$t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$$

If a partition  $p$  of  $I = [a, b]$  is a  $\delta$ -fine &  $x \in I$ , then  $\exists$  a tag  $t_i$  in  $p$  such that  $|x - t_i| \leq \delta(t_i)$

### Alternative proof of Boundedness Theorem

*Proof.* Since  $f$  is continuous on  $I$ , then for each  $t \in I$   $\exists \delta(t) > 0 \ni$  if  $x \in I$  and

$$|x - t| < \delta(t) \text{ then } |f(x) - f(t)| < 1$$

Thus,  $\delta$ -gauge on  $I$  let  $\{(I_i, t_i)\}_{i=1}^n$  be  $\delta$ -fine partition on  $I$  and let

$$k = \max\{|f(t_i)| \mid i = 1, 2, \dots, n\}$$

Given any  $x \in I \ni i$  with  $|x - t_i| \leq \delta(t_i)$

$$\begin{aligned} |f(x)| &= |f(x) - f(t_i) + f(t_i)| \\ &\leq 1 + k \end{aligned}$$

Since  $x \in I$  is arbitrary,  $f$  is bounded. □

**Definition 5.4.3** (Monotone and Inverse Function): If  $A \subseteq \mathbb{R}$ , then a function  $f : A \rightarrow \mathbb{R}$  is

said to be increasing on  $A$  if whenever  $x_1, x_2 \in A$  and  $x_1 < x_2$  then  $f(x_1) \leq f(x_2)$

if  $x_1, x_2 \in A$  and  $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$  then  $f$  is called strictly increasing function.

Similarly, for decreasing function,

$x_1 < x_2$  then  $f(x_1) \geq f(x_2)$  and strictly decreasing function

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

**Theorem 5.4.1.** Let  $I \subseteq \mathbb{R}$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be increasing on  $I$ . Suppose  $c \in I$  is not endpoint of  $I$  then ,

$$1. \lim_{x \rightarrow c^-} = \text{Sup}\{f(x) / x \in I, x < c\}$$

$$2. \lim_{x \rightarrow c^+} = \text{Sup}\{f(x) / x \in I, x > c\}$$

*Proof.* 1. Let  $x \in I$  &  $x < c \Rightarrow f(x) < f(c)$

So, for set  $\{f(x) / x \in I, x > c\}$ ,  $f(c)$  is upper bound, So by completeness property,

$\exists$  Supremum, say  $L$ .

if  $\varepsilon > 0$ , then  $L - \varepsilon$  is not upper bound

Hence,  $\exists \quad y_\varepsilon \in I, \quad y_\varepsilon < c$

$\exists \quad L - \varepsilon < f(y_\varepsilon) \leq L$

Since,  $f$  is increasing, if  $\delta_\varepsilon = c - y_\varepsilon$  and if

$0 < c - y < \delta_c$  then

$$y_\varepsilon < y < c$$

So that,  $t - \varepsilon < f(y_\varepsilon) \leq f(y) \leq L$

$\Rightarrow |f(y) - L| < \varepsilon$  when  $0 < c - y < \delta_c$

Simillarly we can prove (ii)

□

**Theorem 5.4.2** (Continuous Inverse Function). *Let  $I \subseteq \mathbb{R}$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be strictly monotone and continuous on  $I$  then the function  $g$ —inverse to  $f$  is strictly monotone and continuous on  $I = f(I)$*

*Proof.* Let  $f$  is strictly increasing

Since  $f$  is continuous on  $I$

By preservation of interval theorem,

$J = f(I)$  is also an interval. Also,

$f : I \rightarrow \mathbb{R}$  is strictly monotone and injective on  $I$ , therefore, inverse function  $g : J \rightarrow \mathbb{R}$  exists if

$y_1, y_2 \in J$  with  $y_1 < y_2$  then

$y_1 = f(x_1), \quad y_2 = f(x_2)$  for some  $x_1, x_2 \in I$

$\Rightarrow x_1 < x_2$  as function is increasing

$\Rightarrow x_1 = g(y_1) < g(y_2) = x_2$

Since,  $y_1, y_2$  arbitrary elements of  $J$  with

$y_1 < y_2$ , we conclude that  $g$  is strictly increasing on  $J$ .

Now, we have to show that  $g$  is continuous on  $J$ .

As  $g(J) = I$  is an interval.

Indeed, if  $g$  is discontinuous at a point  $c \in J$ , then the jump at  $c$  is non-zero so that  $\lim_{y \rightarrow c^-} g < \lim_{y \rightarrow c^+} g$

$$\lim_{y \rightarrow c^+} g$$

if we choose any number  $x \neq g(c)$  satisfying  $\lim_{y \rightarrow c^-} g < x < \lim_{y \rightarrow c^+} g$

then,  $x \neq g(y)$ , for any  $y \in J$

Hence,  $x \notin I$  which contradicts to our given condition that  $I$  is interval.

$\therefore$  The inverse function  $g$  is continuous on  $J$ . □



# Differentiation

## 6.1 Derivative

**Definition 6.1.1** (Derivative): *Let  $I \subseteq \mathbb{R}$  be an interval. let  $f : I \rightarrow \mathbb{R}$  and let  $c \in I$ . We say that a real number  $L$  is derivative of  $f$  at  $c$  if given any  $\varepsilon > 0$   $\exists \delta(\varepsilon) > 0$   $\exists$  if  $x \in I$  satisfies*

$$0 < |x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$$

*We say,  $f$  is differentiable at  $c$ .*

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

**Theorem 6.1.1.** If  $f : I \rightarrow \mathbb{R}$  has a derivative at  $c \in I$ , then  $f$  is continuous at  $c$ .

*Proof.*  $\forall x \in I, x \neq c$

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c)$$

$$\lim_{x \rightarrow c} f(x) - f(c)$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}(x - c)$$

$$= f'(c).0$$

$$= 0$$

$$\lim_{x \rightarrow c} f(x) = f(c)$$

$\therefore f$  is continuous at point  $c$

if  $f : I \rightarrow \mathbb{R}$  is continuous at point  $c$  then  $f$  may or may not be derivable at  $c$ . □

### Example 33:

$f(x) = |x|$  is continuous at 0 but not differentiable at 0.

**Theorem 6.1.2.** Let  $I \subseteq \mathbb{R}$  be an interval, let  $c \in I$  & let  $f : I \rightarrow \mathbb{R}$  &  $g : I \rightarrow \mathbb{R}$  be function that are differentiable at  $c$  then

$$a) (\alpha f)'(c) = \alpha f'(c), \quad \alpha \in \mathbb{R}$$

$$b) (f + g)'(c) = f'(c) + g'(c)$$

$$c) (fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

$$d) \left(\frac{f}{c}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad (g(c) \neq 0)$$

*Proof.* a)  $(\alpha f)'(c)$

$$= \lim_{x \rightarrow c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{\alpha f(x) - \alpha f(c)}{x - c}$$

$$= \alpha \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$$\underline{(\alpha f)'(c) = \alpha f'(c)}$$

b)  $\lim_{x \rightarrow c} \frac{(f + g)(x) - (f + g)(c)}{x - c} = (f + g)'(c)$

$$\therefore (f + g)'(c)$$

$$= \lim_{x \rightarrow c} \frac{(f(x) + g(x)) - (f(c) + g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{(f(x) - f(c)) - (g(x) - g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \left\{ \frac{f(x) - f(c)}{x - c} \right\} + \lim_{x \rightarrow c} \left\{ \frac{g(x) - g(c)}{x - c} \right\}$$

$$\underline{(f+g)'(c) = f'(c) + g'(c)}$$

c) Let  $h(x) = fg(x)$

$$h'(c) = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c}$$

$$(fg)'(c)$$

$$= \lim_{x \rightarrow c} \frac{fg(x) - fg(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(x) - f(c).g(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(x) - f(c).g(x) + f(c).g(x) - f(c).g(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{(f(x) - f(c))g(x) + f(c).(g(x) - g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \rightarrow c} g(x) + \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \cdot f(c)$$

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

d) let  $h = \frac{f}{g}$

$$\therefore h'(c) = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c}$$

$$\therefore \left( \frac{f}{g} \right)'(c)$$

$$= \lim_{x \rightarrow c} \frac{\left( \frac{f}{g} \right)(x) - \left( \frac{f}{g} \right)(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(c) - f(c).g(c) + f(c).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$= \lim_{x \rightarrow c} \frac{(f(x) - f(c)).g(c) + f(c).(g(x) - g(c))}{g(x).g(c)(x - c)}$$

$$= \lim_{x \rightarrow c} \left( \frac{1}{g(x).g(c)} \right) \left[ \lim_{x \rightarrow c} \left( \frac{f(x) - f(c)}{x - c} \right) . g(c) - \lim_{x \rightarrow c} \left( \frac{g(x) - g(c)}{x - c} \right) f(c) \right]$$

$$= \frac{1}{(g(c))^2} [f'(c).g(c) - g'(c)f(c)]$$

$$\left( \frac{f}{c} \right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad (g(c) \neq 0)$$

□

**Theorem 6.1.3.** Let  $f$  be defined on an interval  $I$  containing point  $c$ . Then  $f$  is differential at  $c$  iff  $\exists$  a function  $\psi$  on  $I$  that is continuous at  $c$  and satisfies  $f(x) - f(c) = \psi(x)(x - c)$   $x \in I$  In this case,  $\psi(c) = f'(c)$

*Proof.* If  $f'(c)$  exists we can define,

$$\psi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & \text{for } x \neq c, x \in I \\ f'(c) & \text{for } x = c \end{cases}$$

$$\lim_{x \rightarrow c} \psi(x) = f'(c)$$

Now, assume that  $\psi$  function is continuous at  $c$  and satisfies

$$f(x) - f(c) = \psi(x).(x - c)$$

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c} \psi(x) = \psi(c) \text{ exists}$$

$\therefore f$  is differentiable at  $c$  and  $\psi(c) = f'(c)$

□

## 6.2 Chain Rule

**Theorem 6.2.1** (Chain Rule). *Let  $I, J$  be intervals in  $\mathbb{R}$ . Let  $g : I \rightarrow \mathbb{R}$  &  $f : J \rightarrow \mathbb{R}$  be functions such that  $f(J) \subseteq I$  and let  $c \in J$*

If  $f$  is differentiable at  $c$  &  $g$  is differentiable at  $f(c)$  then the composite function  $g \circ f$  is differentiable at  $c$  and  $(g \circ f)'(c) = g'(f(c)).f'(c)$

*Proof.* Given that  $f$  is differentiable at  $c$

$\therefore \exists$  function  $\psi$  on  $J \ni$

$$f(x) - f(c) = \psi(x).(x - c) \text{ & } f'(c) = \psi(c)$$

Also,  $g$  is differentiable at  $f(c)$

$\exists$  function  $\psi$  on  $I \ni$

$$g(f(x)) - g(f(c)) = \psi(f(x)).(f(x) - f(c)) \text{ & } g'(f(c)) = \psi(f(c))$$

Consider,

$$\begin{aligned} & g \circ f(x) - g \circ f(c) \\ &= g(f(x)) - g(f(c)) \\ &= \psi(f(x)).(f(x) - f(c)) \\ &= \psi(f(x)).(\psi(x).(x - c)) \end{aligned}$$

$$= [\psi(f(x)).\psi(x)].(x - c)$$

$\therefore g \circ f$  is differentiable at  $c$

Also,  $\lim_{x \rightarrow c} \frac{g \circ f(x) - g \circ f(c)}{(x - c)}$

$$= \lim_{x \rightarrow c} [\psi(f(x)).\psi(x)]$$

$$= \psi(f(c)).\psi(c)$$

$$(g \circ f)'(c) = g'(f(c)).f'(c)$$

□

**Definition 6.2.1** (Inverse Function): *Let  $I$  be an interval in  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  strictly monotone and continuous on  $I$ . let  $J = f(I)$  and let  $g : J \rightarrow \mathbb{R}$  be strictly monotone and continuous function inverse to  $f$ .*

**Theorem 6.2.2.** *If  $f$  is differentiable at  $c$ ,  $c \in I$  &  $f'(c) \neq 0$  then  $g$  is differentiable at  $d = f(c)$  &*

$$g'(d) = \frac{1}{f(c)} = \frac{1}{f'(g(d))}$$

*Proof.* Given that,  $f$  is differentiable at  $c \in I$

$\therefore \exists \psi$  on  $I$  continuous at  $c \ni$

$$f(x) - f(c) = \psi(x).(x - c) \text{ & } \psi(c) = f'(c)$$

Since  $f'(c) \neq 0 \Rightarrow \psi(c) \neq 0$

$\exists$  neighbourhood of  $c$ ,  $v = (c - \delta, c + \delta)$

$$\exists \quad \psi(x) \neq 0 \quad \forall x \in v \cap I$$

If  $U = f(v \cap I)$  then inverse function  $g$  satisfies

$$f(g(y)) = y, \quad \forall y \in U$$

$$y - d = f(g(y)) - f(c) = \psi(g(y)).(g(y) - g(d))$$

since,  $\psi(g(y)) \neq 0, \quad \forall y \in U$

$$g'(y) - g(d) = \frac{1}{\psi(g(y))}(y - d)$$

Since,  $\psi(g(y))$  is continuous at  $d$

$\therefore g'(d)$  exists and

$$g'(d) = \frac{1}{\psi(g(d))} = \frac{1}{\psi(c)} = \frac{1}{f'(c)}$$
□

**Theorem 6.2.3.** Prove that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an even function [i.e  $f(-x) = f(x) \forall x$ ] and has

*derivative at every point, then the derivative  $f'$  is an odd function. Also, prove that if  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable odd function, then  $g'$  is even function.*

*Proof.* a) Given that  $f$  is even function

$$f(x) = f(-x) \forall x$$

Also,  $f$  is differentiable at  $c$

$$\therefore f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ exists}$$

To prove,  $f'$  is odd function

$$\text{i.e } f'(-c) = -f'(c)$$

consider,

$$\begin{aligned} & f'(-c) \\ &= \lim_{x \rightarrow -c} \frac{f(x) - f(-c)}{x + c} \end{aligned}$$

$$= \lim_{-x \rightarrow c} \frac{f(x) - f(c)}{-x + c}$$

$$= \lim_{-x \rightarrow c} \frac{f(x) - f(c)}{-(x - c)}$$

$$= -f'(c)$$

$\therefore f'$  is odd function.

b) Given that  $g$  is odd function

$$g(x) = g(-x) \forall x$$

Also,  $g$  is differentiable at  $c$

$$\therefore g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \text{ exists}$$

To prove,  $g'$  is even function

$$\text{i.e } g'(-c) = -g'(c)$$

consider,

$$g'(-c)$$

$$= \lim_{x \rightarrow -c} \frac{g(x) - g(-c)}{x + c}$$

$$= \lim_{-x \rightarrow c} \frac{-g(x) + g(c)}{-x + c}$$

$$= \lim_{-x \rightarrow c} \frac{g(x) - g(c)}{(x - c)}$$

$$= g'(c)$$

$\therefore g'$  is even function.

□

**Theorem 6.2.4** (Interior Extremum). *Let  $c$  be an interior point of the interval  $I$  at which  $f : I \rightarrow \mathbb{R}$  has a relative extremum. If derivative  $f'$  at  $c$  exists, then  $f'(c) = 0$*

*Proof.* Let  $f$  has relative maximum at  $c$

$$\begin{aligned} \text{if } f'(c) > 0, \quad \exists \quad V_\varepsilon(c) \subseteq I \\ \frac{f(x) - f'(c)}{x - c} > 0, \quad \forall x \in V_\varepsilon(c), x \neq c \end{aligned}$$

if  $x \in V, x > c$

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f'(c)}{x - c} > 0$$

$$\therefore f(x) > f(c) \quad \forall x > 0, \quad x \in V_\varepsilon(c)$$

but,  $f$  has relative maximum at  $c$ .

So, our assumption is wrong that  $f'(c) > 0$

Similarly, we can show that  $f'(c) < 0$

$$\therefore f'(c) = 0$$

□

**Corollary 6.2.4.1.** Let  $f : I \rightarrow \mathbb{R}$  be continuous on an interval  $I$  and suppose that  $f$  has relative extremum at an interior point  $c$  of  $I$  then either the derivative of  $f$  at  $c$  does not exist or it is equal to 0

**Theorem 6.2.5** (Rolle's theorem). *If a function  $f$  defined on  $[a, b]$  is*

*1. Continuous on  $[a, b]$*

*2. derivable on  $(a, b)$*

*3.  $f(a) = f(b)$*

*then  $\exists c \in \mathbb{R}, c \in (a, b) \ni f'(c) = 0$*

*Proof.* Since,  $f$  is continuous  $[a, b] \Rightarrow f$  is bounded

$\therefore$  by maximum- minimum theorem,

If  $m = \inf\{f(I)\}$  and  $M = \sup\{f(I)\}$  then  $\exists c, d \in (a, b)$

$f(c) = m$  &  $f(d) = M$

there are two possibilities  $m = M$  or  $m \neq M$

If  $m = M$

$\Rightarrow \inf\{f(I)\} = \sup\{f(I)\} \Rightarrow f$  is continuous

$$\Rightarrow f'(c) = 0, \quad \forall c \in (a, b)$$

If  $m \neq M$

$$\Rightarrow f(c) = m \neq f(a) \Rightarrow c \neq a$$

$$\Rightarrow f(c) = m \neq f(b) \Rightarrow c \neq b$$

$\Rightarrow c$  lies in  $(a, b)$

Now, we have to show  $f'(c) = 0$

IF  $f'(c) < 0$ ,  $\exists (c, c + \delta), \delta_1 > 0$  for every  $x$  of which  $f(x) < f(c) = m$  which contradicts to our assumption that infimum attains at  $c$ .

Simillarly,  $f'(c) > 0$  is not possible

$$\therefore f'(c) = 0$$

□

**Theorem 6.2.6** (Langrange's Mean Value theorem). *If a function  $f$  defined on  $[a, b]$*

*i) Continuous on  $[a, b]$*

*ii) differentiable on  $(a, b)$*

then  $\exists c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

*Proof.* Let us define function  $\psi$  on  $[a, b]$  such that

$\psi(x) = f(x) - Ax$ , where  $A$  is constant.

As  $f(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ ,

$\psi(x)$  is also continuous on  $[a, b]$  and differentiable on  $(a, b)$

Assume,  $\psi(a) = \psi(b)$

$$f(a) - A.a = f(b) - A.b$$

$$f(b) - f(b) = A(b - a)$$

$$\therefore A = \frac{f(b) - f(a)}{b - a}$$

$$\therefore \psi(x) = f(x) - \left( \frac{f(b) - f(a)}{b - a} \right) x$$

i)  $\psi(x)$  is continuous on  $[a, b]$

ii)  $\psi(x)$  is derivable on  $(a, b)$

iii)  $\psi(a) = \psi(b)$

$\therefore$  by rolle's theorem

$$\psi'(c) = f'(c) - \left( \frac{f(b) - f(a)}{b - a} \right)$$

$$f'(c) = \left( \frac{f(b) - f(a)}{b - a} \right)$$

□

**Theorem 6.2.7** (Cauchy Mean Value theorem). *If  $f, g$  defined on  $[a, b]$*

i) continuous on  $[a, b]$

ii) derivable on  $(a, b)$

iii)  $g'(x) \neq 0, \quad \forall x \in (a, b) \exists \quad c \in (a, b) \quad \exists$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

*Proof.* Let us define function  $\psi(x)$  on  $[a, b] \ni$

$$\psi(x) = f(x) - Ag(x)$$

i)  $\psi(x)$  is continuous on  $[a, b]$

ii)  $\psi(x)$  is derivable on  $(a, b)$

iii)  $\psi(a) = \psi(b)$

$$\Rightarrow f(a) + A.g(a) = f(b) - A.g(b)$$

$$\therefore f(b) - f(a) = A(g(b) - g(a))$$

$$A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$\therefore$  by rolles theorem,

$$\psi'(c) = 0$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

□

### 6.3 Taylor's Theorem

**Theorem 6.3.1** (Taylor's Theorem). *If a function  $f$  defined on  $[a, a + h]$  is such that*

i)  $(n - i)^t h$  derivative  $f^{n-1}$  is continuous on  $[a, a + h]$  and

ii)  $n^t h$  derivative  $f^n$  exists on  $(a, a + h)$  then  $\exists$  atleast one real number  $\theta$  between 0 & 1

$(0 < \theta < 1)$  that,

$$f(a+h) = f(a) + hf'(a) + \left(\frac{h^2}{2!}\right)f''(a) + \left(\frac{h^3}{3!}\right)f'''(a) + \dots + \left(\frac{h^{n-1}}{(n-1)!}\right)f^{n-1}(a) + \left(\frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]}\right)f^n(a+\theta h)$$

where  $p$  is given positive integer  $\mathbb{R}_n$

forms of remainder form-

$$i) R_n = \left( \frac{h^n (1-\theta)^{n-p}}{p[(n-1)!]} \right) f^n(a + \theta h)$$

$$ii) R_n = \left( \frac{h^n (1-\theta)^{n-1}}{(n-1)!} \right) f^n(a + \theta h) \Rightarrow \text{Cauchy}$$

iii)  $R_n = \left( \frac{h^n}{n!} \right) f^n(a + \theta h) \Rightarrow \text{Called as Langranges Forms of remainder}$

**Theorem 6.3.2** (Maclaurins Theorem).  $f(x) = f(0) + xf'(0) + \left( \frac{x^2}{2!} \right) f''(0) + \left( \frac{x^3}{3!} \right) f'''(0) + \dots + \left( \frac{x^{n-1}}{(n-1)!} \right) f^{n-1}(0) + \left( \frac{x^n (1-\theta)^{n-p}}{p[(n-1)!]} \right) f^n(\theta x)$

### Example 34:

$$f(x) = e^x$$

$\therefore$  By Maclaurins theorem,

$$f(x) = f(0) + xf'(0) + \left( \frac{x^2}{2!} \right) f''(0) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

**Example 35:**

$$f(x) = \sin(x)$$

$\therefore$  By Maclaurins theorem,

$$f(x) = \sin 0 + x \cos 0 + \frac{x^2}{2!}(-\sin 0) + \frac{x^3}{3!}(-\cos 0) + \dots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

**Example 36:**

$$f(x) = \log(1 + x)$$

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}$$

$\therefore$  By Maclaurins theorem,

$$f(x) = f(0) + xf'(0) + \left(\frac{x^2}{2!}\right)f''(0) + \left(\frac{x^3}{3!}\right)f'''(0) + \dots$$

$$f(x) = 0 + x(1) + \left(\frac{x^2}{2!}\right)(-1) + \left(\frac{x^3}{3!}\right)(2) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

#### 6.4 Maximum or Minimum for function of two variables

$f(a, b)$  is extreme value of  $f(x, y)$ . if

i)  $f_x(a, b) = 0 = f_y(a, b)$

ii)  $f_{xx}(a, b) = f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$

and this extreme value is maximum or minimum according as  $f_{xx}(a, b)$  or  $f_{yy}(a, b)$  is negative or positive.

Further investigation needed if,

$$f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 < 0$$

**Example 37:**

find maximum and minimum of

$$f(x, y) = x^3 + y^3 - 3x + 12y + 20 = 0$$

*Proof.*  $f_x(x, y) = 0$

i.e  $3x^2 - 3 = 0$

$$x^2 = 1$$

$$x = \pm 1$$

$$f_y(x, y) = 0$$

i.e  $3y^2 + 12 = 0$

$$y^2 = 4$$

$$y = \pm 2$$

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 6y, \quad f_{xy}(x, y) = 0$$

for  $x = 1, y = 2$

$$f_{xx}(x, y) = 6x = 6, \quad f_{yy}(x, y) = 6y = 12, \quad f_{xy}(x, y) = 0$$

for  $x = -1, y = -2$

$$f_{xx}(x, y) = 6x = -6, \quad f_{yy}(x, y) = 6y = -12, \quad f_{xy}(x, y) = 0$$

for  $x = -1, y = 2$

$$f_{xx}(x, y) = 6x = -6, \quad f_{yy}(x, y) = 6y = 12, \quad f_{xy}(x, y) = 0$$

for  $x = 1, y = -2$

$$f_{xx}(x, y) = 6x = 6, \quad f_{yy}(x, y) = 6y = -12, \quad f_{xy}(x, y) = 0$$

minimum=(1, 2)

maximum=(-1, -2)

□



## Sequence and Series of Function

### 7.1 Sequence of Function

**Definition 7.1.1** (Sequence of Function): *Let  $A \subseteq \mathbb{R}$  be given and suppose that for each  $n \in \mathbb{N}$*

*$\exists f_n : A \rightarrow \mathbb{R}$  we shall say that  $(f_n)$  is a sequence of function on  $A$  to  $\mathbb{R}$*

**Definition 7.1.2** (Pointwise Convergent): *Let  $f_n$  be a sequence of function on  $A \subseteq \mathbb{R}$  to  $\mathbb{R}$ . let  $A_0 \subseteq A$  & let  $f_n : A_0 \rightarrow \mathbb{R}$  we say that the sequence  $f_n$  converges on  $A_0$  to  $f$  iff for each  $x \in A_0$  the sequence  $f_n(x)$  converges to  $f$*

*The sequence  $f_n : A \rightarrow \mathbb{R}$  converges to function  $f_n : A_0 \rightarrow \mathbb{R}$  on  $A_0$  iff for each  $\varepsilon > 0$  &  $x \in A_0 \exists$*

$$k(\varepsilon_1 x) \in \mathbb{N} \quad \exists \quad |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq k(\varepsilon_1 x)$$

**Example 38:**

$$f_n(x) = \frac{x}{n}, \quad f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\text{i.e } |\frac{x}{n} - 0| < \varepsilon \Rightarrow \left| \frac{x}{n} \right|$$

$$\therefore \frac{|x|}{n} < \varepsilon$$

$$\therefore n > \frac{|x|}{\varepsilon}$$

**Example 39:**

$$f_n(x) = x^n$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$|x^n - 0| < \varepsilon, \quad -1 < x < 1$$

$$|x^n| < \varepsilon$$

$$n \log x < \log \varepsilon$$

$$n < \log\left(\frac{\varepsilon}{x}\right)$$

$$\therefore n > \log\left(\frac{x}{\varepsilon}\right)$$

**Example 40:**

$$f_n(x) = \frac{x^2 + nx}{n}, \quad x \in \mathbb{R}, \quad f(x) = x$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\left| \frac{x^2}{n} + x - x \right| < \varepsilon \quad \left| \frac{x^2}{n} \right| < \varepsilon$$

$$\therefore \frac{x^2}{\varepsilon} < n$$

$$\therefore n > \frac{x^2}{\varepsilon}$$

**Definition 7.1.3** (Uniform Convergence): *A sequence of function on  $A \subseteq \mathbb{R}$  to  $\mathbb{R}$  converges uniformly on  $A_0 \subseteq A$  to a function  $f : A_0 \rightarrow \mathbb{R}$  iff for each  $\varepsilon > 0$  there is a natural number  $k(\varepsilon)$  (depending on  $\varepsilon$  but not on  $x \in A_0$ )  $\exists$*

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq k(\varepsilon)$$

*denoted by,  $f_n(x) \rightharpoonup f(x)$  on  $A_0$*

**Lemma 7.1.1.** *A sequence  $f_n$  of function on  $A \subseteq \mathbb{R}$  does not converges uniformly on  $A_0 \subseteq A$  to a function  $f : A_0 \rightarrow \mathbb{R}$  iff for some  $\varepsilon_0 > 0 \exists$  subsequence  $f_{n_k}$  of  $f_n$  and a sequence  $x_k$  in  $A_0$  such that*

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}$$

### Example 41:

$$f_n(x) = \frac{x_k}{n_k}, \quad f(x) = 0, x_k = k, n_k = k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\left| \frac{x_k}{n_k} - 0 \right| \Rightarrow \left| \frac{k}{k} - 0 \right|$$

$$\Rightarrow |1 - 0|$$

$$\Rightarrow |1| \geq \varepsilon$$

**Example 42:**

$$f_n(x) = \frac{x^2 + nx}{n}, \quad f(x) = x, x_k = k, n_k = -k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\left| \frac{x_k^2}{n_k} + x_k - x_k \right| \geq \varepsilon_0 \Rightarrow$$

$$\left| \frac{k^2}{k} \right| \geq \varepsilon_0$$

$$\therefore |k| > \varepsilon$$

$\therefore$  not uniformly convergent

### Example 43:

$$f_n(x) = x^n$$

$$f(x) = \begin{cases} 0 & ; 0 \leq x < 1 \\ 1 & ; x = 1 \end{cases}$$

$$x_k = \left(\frac{1}{2}\right)^{\left(\frac{1}{k}\right)}, \quad n_k = k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\therefore |x_k^{n_k} - 0| \geq \varepsilon_0$$

$$\left| \left( \frac{1}{2} \right)^{\left(\frac{1}{k}\right)} - 0 \right| \geq \varepsilon_0$$

$$\therefore \left| \frac{1}{2} \right| > \varepsilon$$

$\therefore$  Not uniformly convergent

**Definition 7.1.4** (Uniform Norm): If  $A \subseteq \mathbb{R}$  &  $\psi : A \rightarrow \mathbb{R}$  is a function we say that  $\psi$  is bounded on  $A$ . If the set  $\psi(A)$  is bounded subset of  $\mathbb{R}$  if  $\psi$  is bounded we define the uniform norm of  $\psi$  on  $A$  by,  $\|\psi\|_A = \text{Sup}\{|\psi(x)| : x \in A\}$

Note that, it follows that if  $\varepsilon > 0$ ,

$$\|\psi\|_A \leq \varepsilon \Leftrightarrow |\psi(x)| \leq \varepsilon, \quad \forall x \in A$$

**Lemma 7.1.2.** A sequence  $f_n$  of bounded function on  $A \subseteq \mathbb{R}$  uniformly on  $A$  to  $f$  if and only if

$$\|f_n - f\|_A \rightarrow 0$$

**Example 44:**

$$f(x) = x \quad [0, 1]$$

$$\text{Sup}\{|\psi(x)| : x \in A\} = 1$$

$$\|f\|_A = 1$$

**Example 45:**

$$f_n(x) = \frac{x}{n}, \quad f(x) = 0, \quad [0, 1]$$

$$|f_n(x) - f(x)| = |x|$$

$$\therefore \|f_n - f\|_A = \frac{1}{n}^n \rightarrow 0$$

**Example 46:**

$$f_n(x) = x^n \quad [0, k], \quad f(x) = 0$$

$$|f_n(x) - f(x)| = |x^n|$$

$$\therefore ||f_n - f||_A = |k^n|$$

**Example 47:**

$$f_n(x) = x^n(1-x) \quad x \in [0, 1], f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$|x^n(1-x) - 0| = |x^n(1-x)|$$

$$f_n(x) = x^n - x^{n+1}$$

$$\therefore f'_n(x) = nx^{n-1} - (n+1)x^n = 0$$

$$\Rightarrow nx^{n-1} = (n+1)x^n$$

$$\Rightarrow \frac{n}{n+1} = x$$

$$x = \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$$\therefore \|f_n - f\|_A = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \left(1 - \frac{1}{\left(1 + \frac{1}{n}\right)}\right)$$

$$= \left(1 + \frac{1}{n}\right)^{-n} \left(\frac{\frac{1}{n}}{1 + \frac{1}{n}}\right)$$

$$= \left(1 + \frac{1}{n}\right)^{-n} \left(\frac{1}{1 + \frac{1}{n}}\right) \rightarrow 0$$

## 7.2 Cauchy Criteria for Uniform Convergence

**Theorem 7.2.1.** Let  $f_n$  be a sequence of bounded function on  $A \subseteq \mathbb{R}$  then this seqence converges uniformly on  $A$  to a bounded function  $f$  iff for each  $\varepsilon > 0$   $\exists H(\varepsilon) \in \mathbb{N} \exists$

$$\|f_m - f_n\|_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$$

*Proof.* If  $f_n(x) \rightharpoonup f(x)$  then for  $\varepsilon > 0 \quad \exists k \left( \frac{\varepsilon}{2} \right) \ni$

$$\|f_n - f\|_A \leq \left( \frac{\varepsilon}{2} \right) \quad \forall n \geq k \left( \frac{\varepsilon}{2} \right)$$

Hence, if both  $m, n \geq k \left( \frac{\varepsilon}{2} \right)$

$$|f_m(x)' - f_n(x)|$$

$$= |f_m(x) - f(x) + f(x)f_n(x)|$$

$$\leq |f_m(x) - f(x)| + |f(x)f_n(x)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon \quad \forall m, n \geq k \left( \frac{\varepsilon}{2} \right)$$

Conversely,

Suppose,  $\varepsilon > 0$ ,  $\exists H(\varepsilon) \in \mathbb{N}$

$\exists ||f_m - f_n||_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$

$\therefore$  for each  $x \in A$

$|f_m(x) - f_n(x)| \leq ||f_m(x) - f_n(x)||_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$

$\Rightarrow f_m(x)$  is cauchy sequence and hence convergent.

$\therefore \exists f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in A$

We have  $|f_m(x) - f_n(x)| \leq \varepsilon, \quad \forall m \geq H(\varepsilon)$

$\therefore f_n(x) \xrightarrow{\sim} f(x)$  on  $A$

□

### 7.3 Series of Function

If  $f_n$  is sequence of function defined on subset  $D$  of  $\mathbb{R}$  with values in  $\mathbb{R}$ , the sequence of partial sums  $S_n$  of infinite series  $\sum f_n$  is defined for  $x$  in  $D$  by,

$$S_1(x) = f_1(x)$$

$$S_2(x) = f_2(x) + S_1(x)$$

:

:

$$S_{n+1}(x) = S_n(x) + f_{n+1}(x)$$

:

:

- In the case sequence  $S_n$  of functions  $f_n$  converges to function  $f$  on  $D$  we say that  $\sum f_n$  converges on  $D$  to  $f$
- If the series  $\sum |f_n(x)|$  converges for each  $x \in D$ , we say that  $\sum f_n$  converges absolutely on  $D$ .

- if  $(S_n)$  sequence of partial sums is uniformly convergent on  $D$  to  $f$ , we say that  $\sum f_n$  is uniformly converges on  $D$
- If  $f_n$  is continuous on  $D \subseteq \mathbb{R}$  to  $\mathbb{R}$  for each  $n \in \mathbb{N}$  and if  $\sum f_n$  converges  $f$  on  $D$  uniformly, then  $f$  is continuous on  $D$

**Definition 7.3.1** (Cauchy Criterion):  *$f_n$  be a sequence of  $f_n$  on  $D \subseteq \mathbb{R}$  to  $\mathbb{R}$ , the series  $\sum f_n$  is uniformly convergent on  $D$  iff for every uniformly  $\varepsilon > 0$ ,  $\exists M(c)$*

$$\exists |f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \varepsilon, \quad \forall m > n \leq M(\varepsilon)$$

**Theorem 7.3.1** (Weistress M-test). *Let  $M_n$  be a sequence of positive real numbers such that  $|f_n(x)| \leq M_n \quad \forall x \in D \quad \forall n \in \mathbb{N}$ . If the series  $M_n$  is convergent then  $\sum f_n$  is uniformly convergent on  $D$*

*Proof.*  $M_n$  is convergent,

By cauchy criterion for series,

for any  $\varepsilon > 0$ ,  $\exists k(\varepsilon) \in \mathbb{N}$

$$\exists M_{n+1} + M_{n+2} + \dots + M_m < \varepsilon, \quad \forall m > n \leq k(\varepsilon)$$

$$\exists |f_{n+1}(x)| + |f_{n+2}(x)| + \dots + |f_m(x)| < M_{n+1} + M_{n+2} + \dots + M_m < \varepsilon$$

Also,

$$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| \leq |f_{n+1}| + |f_{n+2}| + \dots + |f_m| < \quad \forall m > n \geq k(\varepsilon)$$

$\therefore$  By Cauchy criterion,

$\sum f_n$  is uniformly convergent on  $D$ .

□

**Definition 7.3.2** (Power Series): A series of real function  $\sum f_n$  is said to be power series around  $x = c$  if the function has the form  $f_n(x) = a_n(x - c)^n$  where  $a_n$  and  $c \in \mathbb{R}$  and where  $n = 0, 1, 2, \dots$

### Example 48:

Power Series

$$\sum a_n x^n = a_0 x^0 + a_1 x + \dots + a_n x^n + \dots$$

$$\sum_{n=0}^{\infty} n!x^n \quad \sum_{n=0}^{\infty} x^n \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\frac{1}{n}$

**Definition 7.3.3** (Radius of Convergence):  $\sum a_n x^n$  be a power series if a sequence  $|a_n|^{\frac{1}{n}}$  is bounded, we set  $\rho = \lim Sup |a_n|^{\frac{1}{n}}$  if this sequence is not bounded, we set  $\rho = +\infty$

We define radius of convergece of  $\sum a_n x^n$  to be given by,

$$R = \begin{cases} 0 & ; \text{if } \rho = +\infty \\ \frac{1}{\rho} & ; \text{if } 0 < \rho < +\infty \\ \infty & ; \rho = 0 \end{cases}$$

The interval of convergence is the open interval  $(-R, R)$

### Example 49:

$$\sum \frac{x^n}{2^n} \Rightarrow \left| \frac{1}{2^n} x^n \right|$$

$$\Rightarrow a_n \cdot x^n$$

$$\rho = \lim Sup |a_n|^{\frac{1}{n}}$$

$$\Rightarrow \lim Sup \left| \frac{1}{2^n} \right|^{\frac{1}{n}}$$

$$\Rightarrow \frac{1}{2}$$

$$\Rightarrow R = \frac{1}{\rho} = 2$$

**Example 50:**

$$\sum n x^n \Rightarrow a_n x^n$$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

$$R = \lim \left| \frac{n}{n+1} \right|$$

$$R = \lim \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$R = 1$$

Chapter **8**

## Riemann Integral

### 8.1 Introduction

#### Riemann Integral

If  $I = [a, b]$  be closed bounded interval in  $\mathbb{R}$  then partition of  $I$  is a finite ordered set  $\mathbb{P} = (x_0, x_1, \dots, x_{n-1}, x_n)$  of points in  $I$  such that  $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ .

The points of  $P$  are used to divide  $I = [a, b]$  into non-overlapping sub-intervals.

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n]$$

Norm of  $P = ||p|| = \max\{|x_i - x_{i-1}|, i = 1, 2, \dots, n\}$

The norm of partition is merely the length of largest sub-interval into which the partition divide if point  $t_i$  has been chosen from each sub-interval  $I_i = [x_{i-1}, x_i] = \forall i = 1 : n$  then the points are called as tags of sub-intervals  $I - i$ .

A set of ordered pairs

$\dot{p} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$  is tagged partition of  $[a, b]$

**Definition 8.1.1** (Riemann Sum): *If  $\dot{p}$  is the tagged partition, we define Riemann sum of function.  $f : [a, b] \rightarrow \mathbb{R}$  corresponding to  $\dot{p}$  to be the number,*

$$S(f, \dot{p}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

**Definition 8.1.2** (Riemann Integral): *A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be Riemann integrable on  $[a, b]$  if there exists a number  $L \in \mathbb{R}$  such that for  $\varepsilon > 0 \exists \delta_\varepsilon > 0 \exists$  if  $\dot{p}$  is any tagged partition of  $[a, b]$  with  $||\dot{p}|| < \delta_\varepsilon$  then*

$$|S(f, \dot{p}) - L| < \varepsilon$$

The set of all Riemann integrable functions on  $[a, b]$  will be denoted by  $R[a, b]$

i.e  $\|\dot{p}\| \rightarrow 0 \Rightarrow S(f, \dot{p}) \rightarrow L$

**Definition 8.1.3:** If  $f \in R[a, b]$  then the number  $L$  is uniquely determined and called as Riemann Integral of  $f$  over  $[a, b]$

$$L = \int_a^b f(x) dx$$

**Theorem 8.1.1.** If  $f \in R[a, b]$  then the value of the integral is uniquely determined.

*Proof.* Assume that  $L'$  &  $L''$  both satisfy the definition and

let  $\varepsilon > 0 \quad \exists \quad \delta'_{\frac{\varepsilon}{2}} > 0 \quad \exists$  if  $\dot{p}_1$  is tagged partition with  $\|\dot{p}_1\| < \delta'_{\frac{\varepsilon}{2}}$  then

$$|S(f, \dot{p}_1) - L'| < \frac{\varepsilon}{2}$$

Similarly,  $\exists \quad \delta''_{\frac{\varepsilon}{2}} > 0 \quad \exists$  if  $\dot{p}_2$  is tagged partition with  $\|\dot{p}_2\| < \delta''_{\frac{\varepsilon}{2}}$  then

$$|S(f, \dot{p}_2) - L''| < \frac{\varepsilon}{2}$$

Now, let  $\delta_\varepsilon = \min\left(\delta'_{\frac{\varepsilon}{2}}, \delta''_{\frac{\varepsilon}{2}}\right)$

let  $\dot{p}$  be tagged partition with  $||\dot{p}|| < \delta_\varepsilon$

$\Rightarrow |S(f, \dot{p}) - L'| < \frac{\varepsilon}{2}$  and

$|S(f, \dot{p}_2) - L''| < \frac{\varepsilon}{2}$

So,  $|L' - L''| = |L' - S(f, \dot{p}) + s(f, \dot{p}) - L''|$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

As  $\varepsilon$  is arbitrary,  $L' = L''$  □

**Theorem 8.1.2.** Every constant function on  $[a, b]$  is in  $R[a, b]$ .

*Proof.* Let  $f(x) = k \quad \forall x \in [a, b]$  be the constant function, if  $\dot{p} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$  is any tagged partition on  $[a, b]$

$$S(f, \dot{P}) = \sum_{i=1}^n k(x_i - x_{i-1}) = k(b - a)$$

Hence, for any  $\varepsilon > 0$ , we can choose  $\delta_\varepsilon > 0 \quad \exists \quad ||\dot{P}|| < \delta_\varepsilon \text{ & } |S(f, \dot{P}) - k(b - a)| = 0 < \varepsilon$

$$\int_a^b f(x) dx = k(b - a)$$

$\therefore f(x)$  is an Riemann integrable  $f \in R[a, b]$

□

## 8.2 Some Properties of Integral

**Theorem 8.2.1.** Suppose that  $f$  &  $g$  are in  $R[a, b]$  then

a) If  $k \in \mathbb{R}$ , the function  $k.f$  is in  $R[a, b]$  and  $\int_a^b kf = k \int_a^b f$

b) the function  $f$  &  $g$  is in  $R[a, b]$  and  $\int_a^b f + g = \int_a^b f + \int_a^b g$

c)  $f(x) \leq g(x) \quad \forall \quad x \in [a, b]$  then  $\int_a^b f \leq \int_a^b g$

**Theorem 8.2.2.** If  $f \in [a, b]$  then  $f$  is bounded on  $[a, b]$

*Proof.* Assume that  $f$  is unbounded on  $[a, b]$

As  $f \in [a, b]$ , then for any  $\varepsilon > 0 \quad \exists \delta_\varepsilon > 0$

such that  $\|\dot{p}\| < \delta_\varepsilon$  then  $|S(f, \dot{p}) - L| < \varepsilon$

Now, let  $Q = \{[x_{i-1}, x_i]\}_{i=1}^n$  be partition on  $[a, b]$  with  $\|Q\| < \delta$ . Since  $|f|$  is not bounded on  $[a, b]$ ,  $\exists$  atleast one sub-interval  $[x_{k-1}, x_k]$  on  $[a, b]$  which  $|f|$  is not bounded.

Let tag  $Q$  by  $t_i = x_i$  for  $i \neq k$  and  $k \in [x_{k-1}, x_k]$  such that,

$$|f(t_k)(x_k - x_{k-1})| > |L| + \varepsilon + \left| \sum_{i \neq k}^n f(t_i)(x_i - x_{i-1}) \right|$$

By triangular inequality  $|a + b| > |a| - |b|$

$$|S(f, Q)| \geq |f(t_k)(x_k - x_{k-1})| - + \left| \sum_{i \neq k}^n f(t_i)(x_i - x_{i-1}) \right| > |L| + \varepsilon$$

$\therefore$  which is contradict to our assumption.

$\therefore f$  is bounded on  $[a, b]$  □

**Definition 8.2.1** (Cauchy Criterion for Riemann Integrable function): *A function  $f : [a, b] \rightarrow \mathbb{R} \in R[a, b]$  if and only if for every  $\varepsilon > 0, \exists n_\varepsilon > 0$  if  $\dot{p}$  &  $Q$  are any tagged partitions of  $[a, b]$  with  $\|\dot{p}\| < n_\varepsilon$  &  $\|\dot{Q}\| < n_\varepsilon$  then,*

$$|S(f, \dot{P}) - S(f, \dot{Q})| < \varepsilon$$

**Theorem 8.2.3** (Squeez theorem). Let  $f : [a, b] \rightarrow \mathbb{R}$  then  $f \in R[a, b]$  if and only if for every  $\varepsilon > 0$   $\exists$  function  $\alpha_\varepsilon$  &  $w_\varepsilon$  in  $R[a, b]$  with

$$\alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon \quad \forall x \in R[a, b] \text{ & such that } \int_a^b w_\varepsilon - \alpha_\varepsilon < \varepsilon$$

*Proof.*  $\iff$  Take  $\alpha_\varepsilon = w_\varepsilon = f \quad \forall \varepsilon > 0$

$\iff$  Let  $\varepsilon > 0$ , Since  $\alpha_\varepsilon, w_\varepsilon \in R[a, b]$

$\exists \delta_\varepsilon > 0 \quad \exists ||\dot{P}|| < \delta_\varepsilon$  then

$$\left| S(\alpha_\varepsilon, \dot{P}) - \int_a^b \alpha_\varepsilon \right| < \varepsilon \quad \& \quad \left| S(w_\varepsilon, \dot{P}) - \int_a^b w_\varepsilon \right| < \varepsilon$$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon, \dot{P}) \quad \& \quad S(w_\varepsilon, \dot{P}) < \int_a^b w_\varepsilon + \varepsilon$$

As  $\alpha_\varepsilon \leq f \leq w_\varepsilon$

$$S(\alpha_\varepsilon, \dot{p}) \leq S(f, \dot{p}) \leq S(w_\varepsilon, \dot{p})$$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leq S(f, \dot{p}) \leq \int_a^b w_\varepsilon + \varepsilon$$

Consider another partition  $\|\dot{Q}\| < \delta_\varepsilon$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leq S(f, \dot{Q}) \leq \int_a^b w_\varepsilon + \varepsilon$$

$$\Rightarrow |S(f, \dot{Q}) - S(f, \dot{p})| < \int_a^b (w_\varepsilon - \alpha_\varepsilon) + 2\varepsilon \leq 3\varepsilon$$

Since,  $\varepsilon > 0$ , is arbitrary,  $f \in R[a, b]$

□

**Theorem 8.2.4.** If  $f : R[a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  then  $f \in R[a, b]$

*Proof.* As  $f$  is continuous on closed bounded interval  $[a, b]$ ,  $f$  is uniformly continuous on  $[a, b]$

$\therefore$  for any  $\varepsilon > 0$ ,  $\exists \delta_\varepsilon > 0$   $\exists$  if  $u, v \in [a, b]$

$$|u - v| < \delta_\varepsilon$$

$$\Rightarrow |f(u) - f(v)| < \frac{\varepsilon}{b-a}$$

Let  $p = \{I_i\}_{i=1}^n$  be a partition such that  $\|p\| < \delta_\varepsilon$ , let  $u_i \in I_i$  be a point where  $f$  attains minimum value on  $I_i$  &  $v_i \in I_i$  be a point where  $f$  attains maximum value on  $I_i$

Let  $\alpha_\varepsilon$  be the step function

$$\alpha_\varepsilon(x) = f(u_i) \quad \forall \quad x \in [x_{i-1}, x_i] \quad (i = 1 : n-1)$$

Let  $w_\varepsilon$  be the step function

$$w_\varepsilon(x) = f(v_i) \quad \forall \quad x \in [x_{i-1}, x_i] \quad (i = 1 : n-1)$$

$$\text{so, } \alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon(x) \quad \forall x \in [a, b]$$

$$0 \leq \int_a^b (w_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^n (f(v_i) - f(u_i))(x_i - x_{i-1})$$

$$< \sum_{i=1}^n \left( \frac{\varepsilon}{b-a} \right) (x_i - x_{i-1}) = \varepsilon$$

$\therefore$  by squeeze theorem,

$f \in R[a, b]$

□

**Theorem 8.2.5.** If  $f : R[a, b] \rightarrow \mathbb{R}$  is monotone on  $[a, b]$  then  $f \in R[a, b]$

*Proof.* Suppose  $f$  is I on  $[a, b]$

Assume  $a < b, \varepsilon > 0$

$$h = \frac{f(b) - f(a)}{q} < \frac{\varepsilon}{(b-a)}$$

let  $y_k = f(a) + k.h \quad \forall \quad k = 0, 1, \dots q$

let  $A_k = f^{-1}[y_{k-1}, y_k] \quad \forall \quad k = 0, 1, \dots q-1$

The sets  $A_k$  are pairwise disjoint and have union  $[a, b]$

so  $A_k$  is either

- a) empty
- b) single point set
- c) non degenerate interval in  $[a, b]$

We discard the sets for which a) holds and relabel remaining ones if we adjoin the end points of the remaining intervals  $A_k$ , we obtain closed intervals  $I_k$

So we have step functions  $\alpha_\varepsilon$  &  $w_\varepsilon$

$$\alpha_\varepsilon(x) = y_{k-1}, \quad w_\varepsilon(x) = y_k \quad \forall x \in A_k$$

$$\alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon(x) \quad \forall x \in [a, b]$$

$$\begin{aligned} & \int_a^b (w_\varepsilon - \alpha_\varepsilon) \\ &= \sum_{k=1}^q (y_k - y_{k-1})(x_k - x_{k-1}) \\ &= \sum_{k=1}^q h(x_k - x_{k-1}) \\ &= h.(b - a) \end{aligned}$$

so, by squeeze theorem,

$$f \in R[a, b]$$

□

### 8.3 Fundamental theorem of Integral calculus

**Theorem 8.3.1.** Suppose, there is finite set  $E$  in  $[a, b]$  and function  $f : F : [a, b] \rightarrow \mathbb{R}$  such that

1.  $F$  is continuous on  $[a, b]$

2.  $F'(x) = f(x) \quad \forall \quad x \in [a, b] \setminus E$

3.  $f \in R[a, b]$  then  $\int_a^b f = f(b) - f(a)$

*Proof.* Let  $\varepsilon > 0$ , since  $f \in R[a, b]$   $\exists \delta_\varepsilon > 0$

$\exists$  if  $p$  is any tagged partition  $\|p\| < \delta_\varepsilon$

$$\left| S(f, p) - \int_a^b f \right| < \varepsilon$$

If the sub-intervals in  $p$  are  $[x_{i-1}, x_i]$  then

by MVT,  $\exists u_i \in (x_{i-1}, x_i)$

$$F'(u_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad \forall \quad i = 1 : n$$

adding  $i = 1 : n$

$$\sum_{i=1}^n f(x_i) - f(x_{i-1}) = \sum_{i=1}^n F'(u_i)(x_i - x_{i-1})$$

$$F(a) - F(b) = \sum_{i=1}^n f'(u_i)(x_i - x_{i-1}) = S(f, \dot{p})$$

Assuming  $\dot{p}_u = \{[x_i - x_{i-1}], u_i\}_{i=1}^n$

$$\Rightarrow \left| F(a) - F(b) - \int_a^b f \right| < \varepsilon$$

$$\Rightarrow \int_a^b f = F(a) - F(b)$$

□

## 8.4 Indefinite Integral

**Definition 8.4.1** (Indefinite Integral): If  $f \in R[a, b]$  then  $f(z) = \int_a^z f \quad \forall z \in [a, b]$

**Theorem 8.4.1.** The indefinite integral  $F$  is continuous on  $[a, b]$ . In fact, if  $|f(x)| \leq M \quad \forall x \in [a, b]$  then  $|F(z) - F(w)| \leq M|z - w| \quad \forall z, w \in [a, b]$

*Proof.* If  $z, w \in [a, b]$ ,  $w \leq z$

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = f(w) + \int_w^z f$$

$$\Rightarrow \int_w^z f = F(z) - F(w)$$

if  $-M \leq f(x) \leq M \quad \forall x \in [a, b]$

$$-M(z-w) \leq \int_w^z f \leq M(z-w)$$

$$\Rightarrow |F(z) - F(w)| \leq \left| \int_w^z f \right| \leq M|z-w|$$

□

## 8.5 Examples

### Example 51:

$$f(x) = x$$

$$g(x) = \frac{1}{x}$$

$$f \circ g = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$g \circ f = g(f(x)) = g(x) = \frac{1}{x}$$

$$f \circ g = g \circ f$$

**Example 52:**

$$A_n = \{(n+1)k, \quad k \in \mathbb{N}\}$$

$$A_1 = \{2k, \quad k \in \mathbb{N}\}$$

$$A_2 = \{3k, \quad k \in \mathbb{N}\}$$

$$A_1 \cap A_2 = \{6k, \quad k \in \mathbb{N}\}$$

$$\cap A_i = \{\phi\}$$

$$\cup A_i = \mathbb{N} - \{1\}$$

**Example 53:**

$$\lim \frac{n^2}{n!}$$

$$\lim \frac{n \cdot n}{n \cdot (n-1)!}$$

$$\lim \frac{n}{(n-2)(n-1)!}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{\left(1 - \frac{1}{n}\right)} \lim_{n \rightarrow \infty} \frac{1}{(n-2)!} \\ &= (1)(0) \end{aligned}$$

0

### **Example 54:**

$$\text{Result:- } \lim_{x \rightarrow \infty} (1 + a^x)^{\frac{1}{x}} = e^a$$

$$x_n = (a^n + b^n)^{\frac{1}{n}}, \quad a < b$$

$$= \lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} b \left( \frac{a^n}{b^n} + 1 \right)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} b \left( \left( \frac{a}{b} \right)^n + 1 \right)^{\frac{1}{n}}$$

$$= b \cdot e^{\frac{a}{b}}$$

$\therefore (a^n + b^n)^{\frac{1}{n}}$  is convergent, bounded and cauchy.

### Example 55:

$$\sum x_n = \frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{13} + \frac{1}{17}$$

$$\sum |x_n| = \frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} + \frac{1}{17}$$

$$\sum x_n = \frac{(-1)^{n-1}}{4n - (-1)^n}$$

**Example 56:**

$$f_n(x) = \frac{1}{nx+1}, \quad x \in (0, 1), f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\left| \frac{1}{nx+1} - 0 \right| < \varepsilon$$

$$\therefore \left| \frac{1}{nx+1} \right| < \varepsilon$$

$$|nx+1| > \frac{1}{\varepsilon}$$

**Example 57:**

Examine convergent of  $\sum \left( \frac{1}{2^n} + \frac{1}{3^n} \right)$

$$\sum \left( \frac{1}{2^n} + \frac{1}{3^n} \right) = \sum \frac{1}{2^n} + \sum \frac{1}{3^n}$$

$$\sum \left( \frac{1}{2^n} + \frac{1}{3^n} \right) = \sum \left( \frac{1}{2} \right)^n + \sum \left( \frac{1}{3} \right)^n$$

$$\sum r_1^n + \sum r_2^n \quad r_1 = \frac{1}{2} < 1, r_2 = \frac{1}{3} < 1$$

which is convergent

### **Example 58:**

$$f_n(x) = \frac{1}{x^n} \quad x \in (0, 1)$$

$$f(x) = \begin{cases} \text{not defined} & x = -1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$$

**Example 59:**

$$\lim_{n \rightarrow \infty} |x_n - x_{n+1}| = 0$$

$x_n$  does not converges for given example j

**Example 60:**

$$\sum \frac{1}{\sqrt{n^3 + 4}} \text{ Use comparision test}$$

$$n < n^{\frac{3}{2}}, \quad n > 1$$

$$\frac{1}{n} > \frac{1}{n^{\frac{3}{2}}}$$

As  $\frac{1}{n}$  is divergent  $\Rightarrow \frac{1}{n^{\frac{3}{2}}}$  is also divergent.

**Definition 8.5.1** (Taylors expansion for two variables):  $f(x, y) =$

$$f(a, b) + \left[ (x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right] f(a, b) + \frac{1}{2!} \left[ (x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right]^2 f(a, b) + \dots$$

$$\dots \dots \frac{1}{(n-1)!} \left[ (x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right]^{n-1} f(a, b) + R_n$$

**Example 61:**

$a_n$  is bounded, decreasing sequence.

$b_n$  is bounded, increasing sequence

$$x_n = a_n + b_n$$

$$\sum |x_n - x_{n+1}|$$

$$= \sum |a_n + b_n - a_{n+1} - b_{n+1}|$$

$$= \sum |a_n - a_{n+1} + b_n - b_{n+1}|$$

$$\leq |a_n - a_{n+1}| + |b_n - b_{n+1}|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$\sum |x_n - x_{n+1}| \rightarrow 0$$

### Example 62:

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \quad p > 0$$

$$\log n < n$$

$$\frac{1}{\log n} > \frac{1}{n}$$

$$\left(\frac{1}{\log n}\right)^p > \frac{1}{n^p}$$

$$\frac{1}{n(\log n)^p} > \frac{1}{n^{p+1}}, \quad p+1 > 1$$

$\therefore$  by comparison test,

As  $\sum \frac{1}{n^{p+1}}$  convergent  $\Rightarrow \sum \frac{1}{n(\log n)^p}$  is convergent.

### Example 63:

$$\sum x_n = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{n \cdot 2^n}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)^{2^{n+1}}}}{n2^n} \right|$$

$$= \left| \frac{n \cdot 2^n}{(n+1)2^{n+1}} \right|$$

$$= \left| \left( \frac{n}{n+1} \right) \frac{1}{2} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \left| \left( \frac{1}{1 + \frac{1}{n}} \right) \frac{1}{2} \right|$$

by ratio test

$$= \frac{1}{2} < 1$$

$\sum x_n = \frac{1}{n2^n}$  is convergent.

**Example 64:**

$$S = \left\{ 1 + \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}$$

limit point of  $S = 1$

### Example 65:

$$\sum x_n = \sum \frac{1}{\sqrt{n} + \sqrt{n-1}}$$

$$n > \sqrt{n}$$

$$\frac{1}{n} < \frac{1}{\sqrt{n}}$$

by Ratio test,

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{\sqrt{(n+1)} + \sqrt{n}}}{\frac{1}{\sqrt{n} + \sqrt{n-1}}} \right|$$

$$= \left| \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$= \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right| = 1$$

$\therefore$  Ratio test fails here

$$\sum \frac{1}{\sqrt{n} + \sqrt{n-1}} \times \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}} = \sum \sqrt{n} - \sqrt{n-1}$$

$\therefore S_n = \sqrt{n}$  which divergent

$\therefore \sum \frac{1}{\sqrt{n} + \sqrt{n-1}}$  is divergent.

**Example 66:**

$$\sum \frac{(2n-1)}{n(n+1)(n+2)} = \frac{1}{1.2.3} + \frac{3}{2.3.4} + \dots$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{(2n-1)}{n(n+1)(n+2)(n+3)}}{\left( \frac{2n-1}{n(n+1)(n+2)} \right)} \right|$$

$$= \left| \frac{(2n+1)n}{(2n-)(n+3)} \right|$$

$$= \left| \frac{\left(2 + \frac{1}{n}\right)}{\left(2 - \frac{1}{n}\right)\left(1 + \frac{3}{n}\right)} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = 1$$

$\therefore$  Ratio test fails here.

$$\begin{aligned}\sum \left( \frac{2n-1}{n(n+1)(n+2)} \right) &= \sum \frac{2n}{n(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)} \\ &= \sum \frac{2}{(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)}\end{aligned}$$

$\therefore x_n$  is convergent.