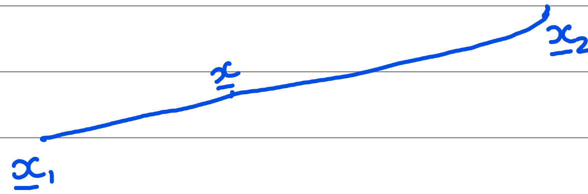
* Line segment : $\underline{x} \in \mathbb{R}^n$



scaler

$\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in (0,1)$

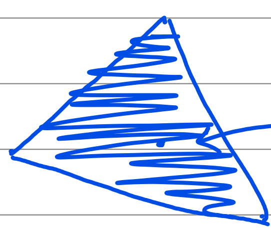
line segment $\{\underline{x} \,/\, \underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2,\ \alpha \in (0,1)\}$
joining $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$

* Line passing $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$ $\alpha + 1 - \alpha$

line segment ← convex comb^n $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in (0,1)$

line , linear comb^n $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in \mathbb{R}$

vector space → $\boxed{\underline{x} = \sum \alpha_i \underline{x}_i}$ Linear Comb^n $\alpha_i \in \mathbb{R}$



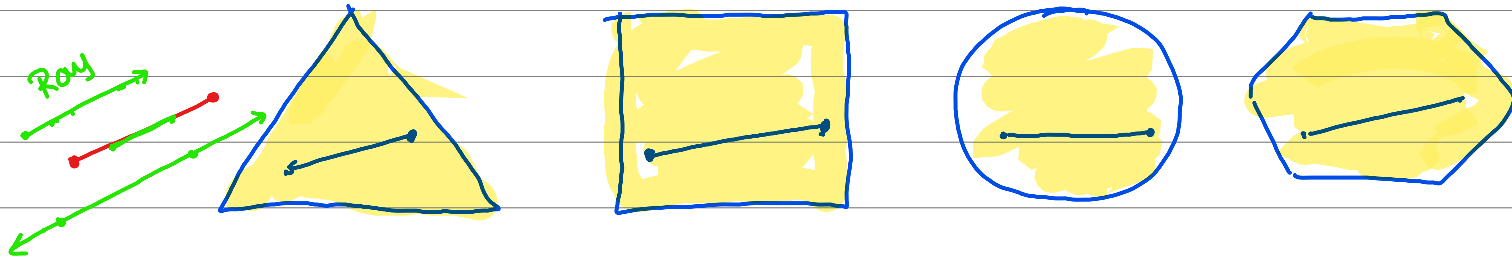
$\underline{x} = \sum \alpha_i \underline{x}_i$ $\sum \alpha_i = 1$, $\alpha_i \geq 0$

↳ Convex Combination

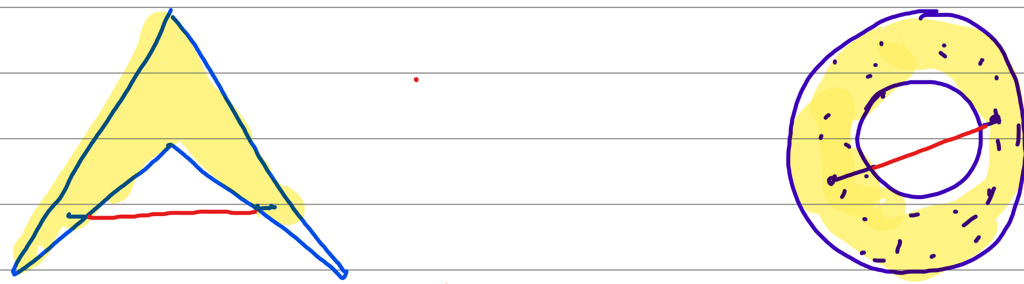
$$\left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \alpha_i x_i,\ \alpha_i \geq 0,\ \sum \alpha_i = 1 \right\}$$

Convex set if $\forall\, \underline{x}_1 \,\&\, \underline{x}_2 \in A$, $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \quad \forall\, \alpha \in (0,1)$

then if $\underline{x} \in S \;\forall\, \alpha$ A is convex set



Convex sets



Non convex sets

* Ray is convex set

$$\underline{x} = \underline{x}_0 + \underline{d}\,\alpha \qquad \alpha \geq 0$$

$\underline{x}_1 \,\&\, \underline{x}_2 \in A$

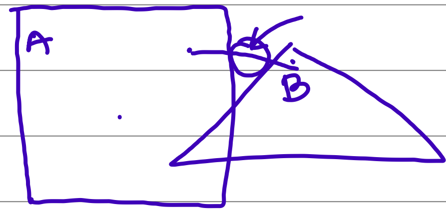
$$\lambda \qquad \underline{x}_1 = \underline{x}_0 + \underline{d}\,\alpha_1$$

$$1-\lambda \qquad \underline{x}_2 = \underline{x}_0 + \underline{d}\,\alpha_2$$

$$\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2 = (\lambda + 1 - \lambda)\underline{x}_0 + \underline{d}\,(\lambda\alpha_1 + (1-\lambda)\alpha_2)$$

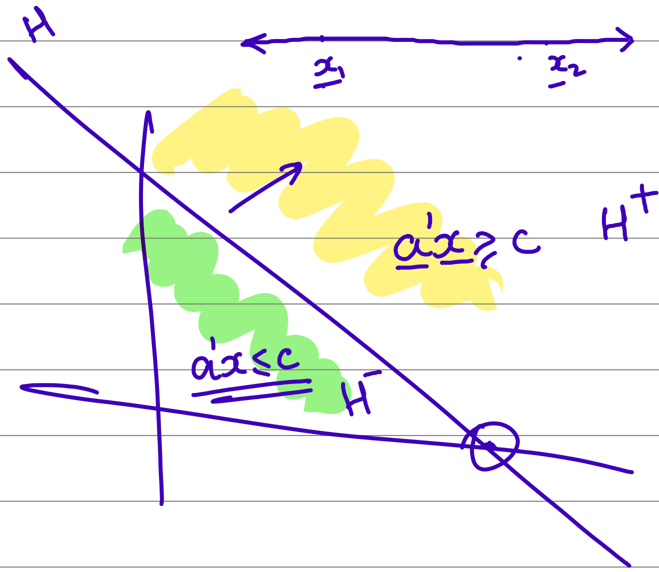
$$= x_0 + d\,(\qquad) \qquad \geq 0$$

$\in A$



* Union of convex sets may or may not be convex.

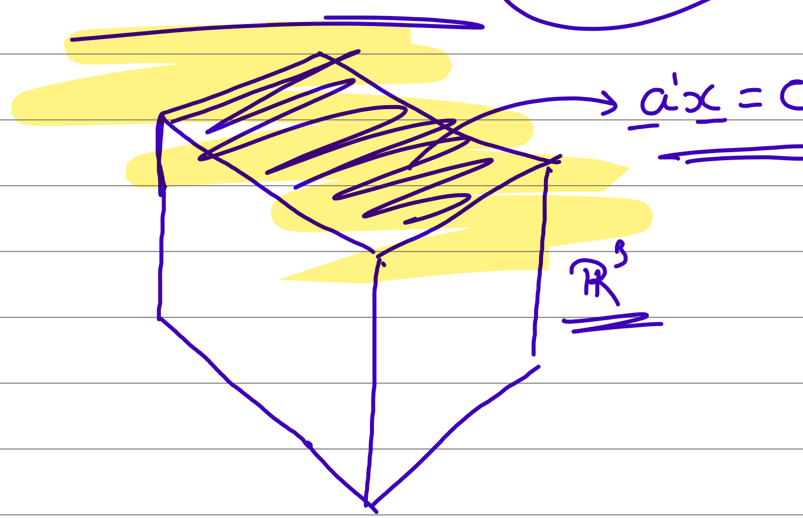
* Intersection of convex sets is also convex.

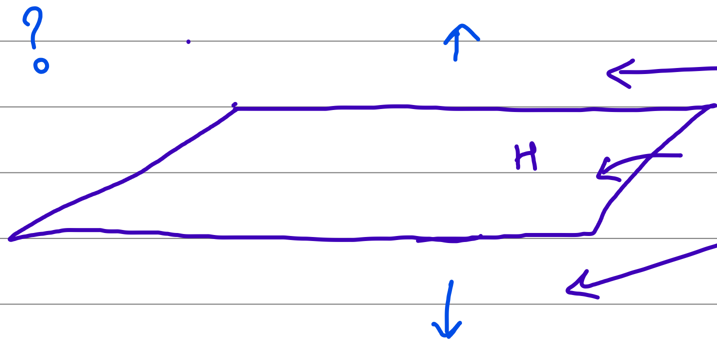


$$\sum_{i=1}^{2} \alpha_i x_i$$

$$\sum_{i=1}^{n} \alpha_i x_i = c$$

$\underline{\alpha}'\underline{x} = c$





Hyperplane

$H_0^+ = \{\underline{x} \,/\, a'x > c\}$ +ve open half space

$H^+ = \{\underline{x} \,/\, \underline{a}'\underline{x} \geq c\}$ ← positive closed half space

$H = \{\underline{x} \,/\, \underline{a}'\underline{x} = c\}$ Hyperplane

$H^- = \{\underline{x} \,/\, a'x \leq c\}$ ← Negative closed half space

$H_0^- = \{\underline{x} \,/\, \underline{a}'\underline{x} < c\}$ -ve open half space

Hyperplane $H = \{\underline{x} / \underline{a}'\underline{x} = c\}$

let $\underline{x}_1, \underline{x}_2 \in H \Rightarrow \underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

$\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$$\begin{aligned} \underline{a}'(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) &= \alpha \underline{a}'\underline{x}_1 + (1-\alpha)\underline{a}'\underline{x}_2 \\ &= \alpha c + (1-\alpha)c \\ &= c \end{aligned}$$

$\Rightarrow \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in H$

$\Rightarrow$ Hyperplane is convex set

H^+, H^-, H_0^+, H_0^- all are convex sets

Open ball $\underline{B = \{ \underline{x} \mid \|\underline{x} - \underline{x}_0\|} < r \}$

To show :- B is convex set.

Let $\underline{x}, \underline{y} \in B \Rightarrow \underline{\alpha \cdot \underline{x}} + (1-\alpha)\,\underline{y} \boxed{\in B}$

$\underline{x} \in B \Rightarrow \|\underline{x} - \underline{x}_0\| < r$

$\underline{y} \in B \Rightarrow \|\underline{y} - \underline{x}_0\| < r$

$\alpha \in (0,1)$

$$\|\alpha \underline{x} + (1-\alpha)\underline{y} - \underline{x}_0\|$$

$$= \|\alpha \underline{x} + (1-\alpha)\underline{y} - (\alpha \underline{x}_0 + (1-\alpha)\cdot x_0)\|$$

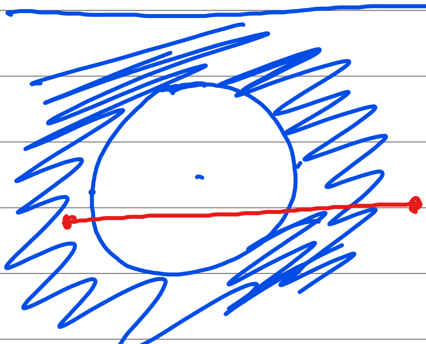
$$= \|\alpha(\underline{x} - \underline{x}_0) + (1-\alpha)(\underline{y} - x_0)\|$$

$$\leq \alpha \underbrace{\|\underline{x} - \underline{x}_0\|}_{<r} + (1-\alpha)\underbrace{\|y - x_0\|}_{<r} \qquad \alpha \in (0,1)$$

$$< \alpha r + (1-\alpha) r$$

$$< r$$

$\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| = r \}$ Convex or Not ✓



$\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| \geq r \}$ convex or not ✓

If C is convex set λC is also convex set.

$$\lambda C = \{ \underline{y} \mid \underline{y} = \lambda \underline{x} \, , \underline{x} \in C \}$$

to show λC as convex set

$\underline{y}_1, \underline{y}_2 \in \lambda C \Rightarrow \underline{y}_1 = \lambda \underline{x}_1 \, , \underline{y}_2 = \lambda \underline{x}_2 \quad \underline{x}_1, \underline{x}_2 \in C$

$$\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 = \alpha \cdot \lambda \underline{x}_1 + (1-\alpha) \lambda \underline{x}_2$$

$$= \lambda (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$= \lambda \underline{x} \qquad \left(\text{as } \underline{x}_1, x_2 \in C \Rightarrow x \in C \right)$$

as C is convex

$$\Rightarrow \alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 \in \lambda C$$

C, D are convex sets C+D is also convex

$\rightarrow \quad C+D = \{ \underline{z} \mid \underline{z} = \underline{x} + \underline{y} \, , \underline{x} \in C, \underline{y} \in D \}$

$\underline{z}_1, \underline{z}_2 \in C+D \Rightarrow z_1 = \underline{x}_1 + \underline{y}_1$

$z_2 = \underline{x}_2 + \underline{y}_2 \qquad \underline{x}_1, \underline{x}_2 \in C, \ \underline{y}_1, \underline{y}_2 \in D.$

$$\alpha z_1 + (1-\alpha) z_2 = \alpha \cdot (\underline{x}_1 + \underline{y}_1) + (1-\alpha)(\underline{x}_2 + \underline{y}_2)$$

$$= \underbrace{\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2}_{\in C} + \underbrace{\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2}_{\in D}$$

convex sets

Intersection of any convex sets is convex

Let $\{S_i\}_{i=1}^{\infty}$ be collection of convex sets

$\cap S_i$ is convex

$\underline{x}, \underline{y} \in \cap S_i$

$\Rightarrow \underline{x}, \underline{y} \in S_i \quad \forall\, i$

$\Rightarrow \alpha \underline{x} + (1-\alpha)\underline{y} \in S_i \quad \forall\, i$ (S_i is convex)

$\Rightarrow \alpha \underline{x} + (1-\alpha)\underline{y} \in \cap S_i$

$\Rightarrow \cap S_i$ is convex.

· A set $S \in R^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S

∴ Assume that every convex combⁿ of any finite no. of points of S is in S.

2 → 1

$\Rightarrow$ it is also true for $n=2$

$\Rightarrow$ if $\forall\, \underline{x}_1, \underline{x}_2 \in S \Rightarrow \alpha \cdot \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S \quad \forall\, \alpha\,(0,1)$

$\Rightarrow$ S is convex set

(II) Assume S is convex and for any finite n $\sum_{i=1}^{n} \alpha_i \underline{x}_i \in S$

→ let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in S$ $\qquad \sum_{i=1}^{n} \alpha_i = 1$

we will prove this by mathematical induction

As S is convex, $\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S$ $\qquad \alpha + 1 - \alpha = 1$

∴ So the above statement is true for $n=2$

Assume it is true for $n=K$ $\Rightarrow \sum_{i=1}^{K} \alpha_i \underline{x}_i$ =1

$\sum_{i=1}^{K} \alpha_i = 1$

We have to prove it for $n=K+1$

$$\sum_{i=1}^{K+1} \beta_i \underline{x}_i = \sum_{i=1}^{K} \beta_i x_i + \beta_{K+1} x_{K+1} \qquad \sum_{i=1}^{K+1} \beta_i = 1$$

imp ?

$$\sum_{i=1}^{K} \beta_i = 1 - \beta_{K+1}$$

$$\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} = 1$$

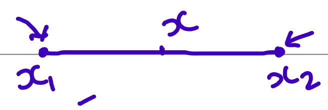
$$= (1-\beta_{K+1}) \boxed{\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} \cdot x_i} + \beta_{K+1} x_{K+1}$$

$\in S$

$$= (1-\beta_{K+1})\, x^* + \beta_{K+1} x_{K+1}$$

$\in S$ as S is convex set

* Vertices

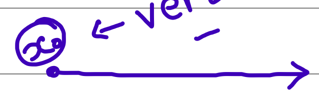


2 vertices

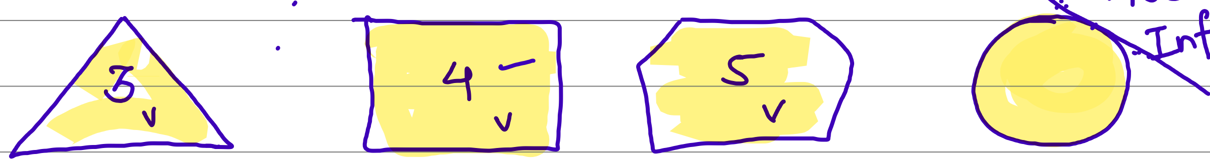
$\Rightarrow \quad x = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

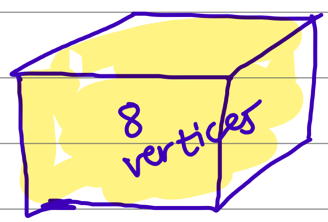
$x_2 = \alpha \underline{x}_1 + (1-\alpha)x_2 = \underline{x}_2$

$\alpha = 0$



No vertex

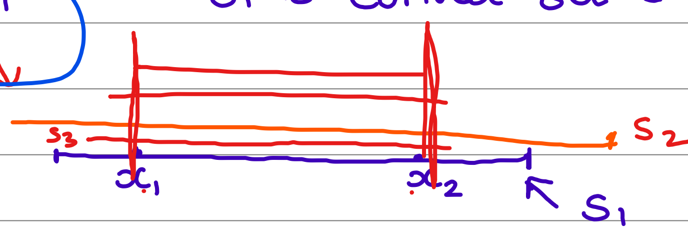


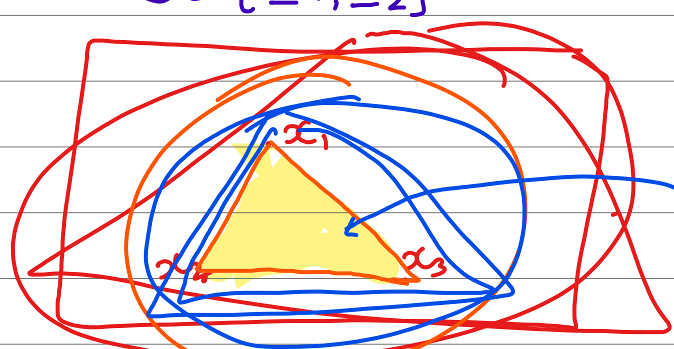


Convex Hull

$\boxed{Co(S) \Rightarrow \bigcap_{i=1}^{\infty} S_i}$ S_i is convex set containing S

$S = \{\underline{x}_1, \underline{x}_2\}$

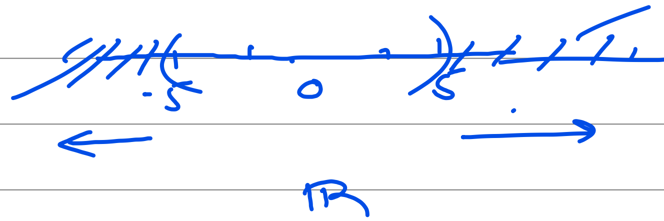




$S = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$

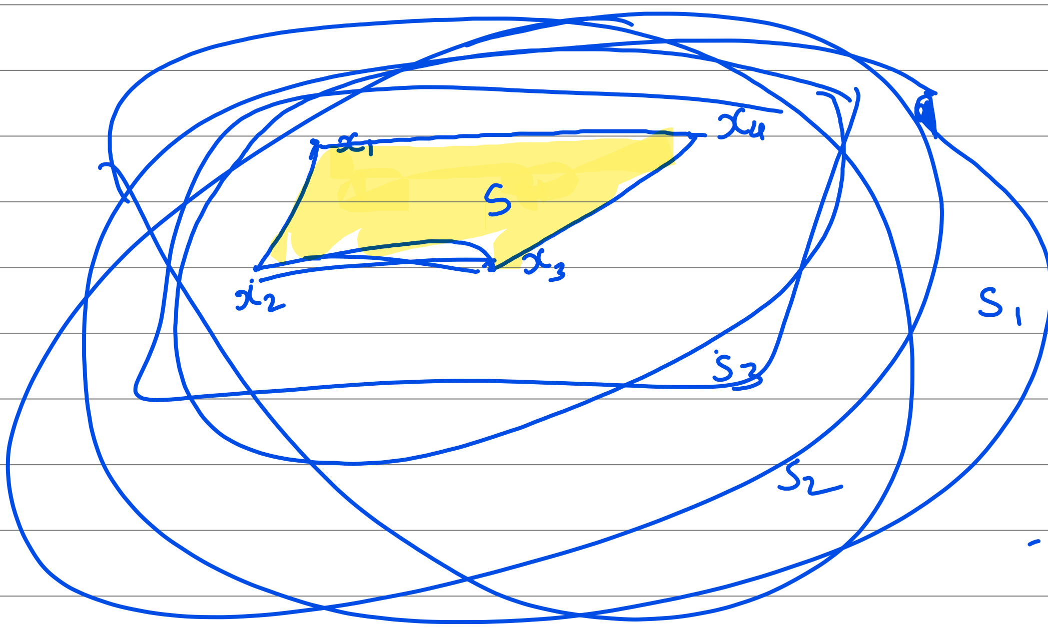
$$S = \{ \underline{x} \mid \|\underline{x}\| \geq 5 \}$$

$$Co(S) = \mathbb{R}^n$$

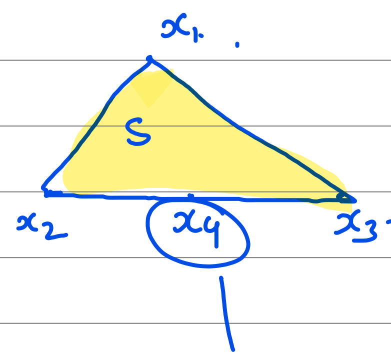


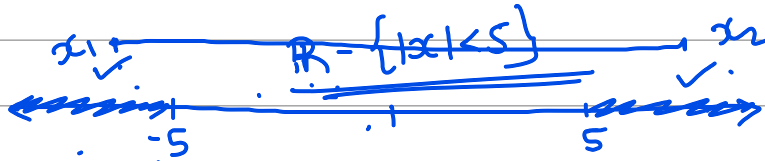
if set is convex $\underline{\underline{Co(S) = S}}$

$\underline{\underline{Co(S)}} = \underline{\underline{\cap S_i}}$, S_i is convex set containing S.



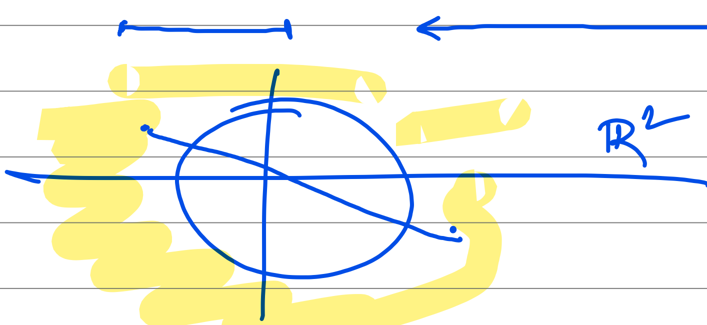
$\underline{\underline{\cap S_i = Co(S)}}$



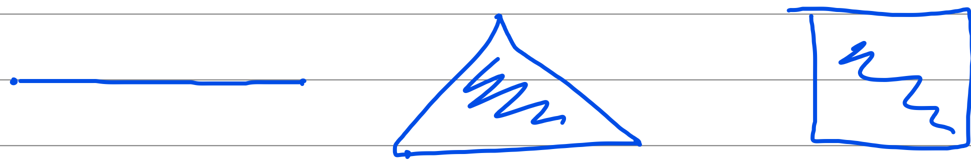


$S = \{ |x| > 5 \}$

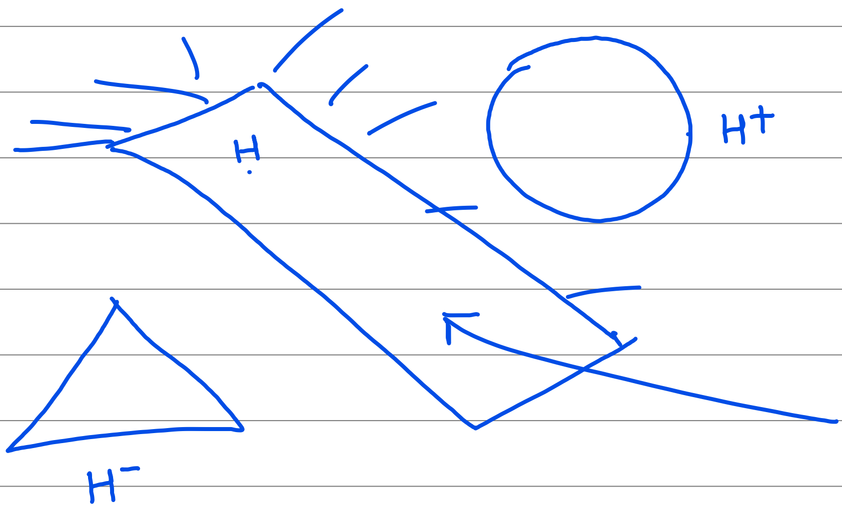
$\underline{\underline{\mathbb{R} = Co(S)}}$ ✓



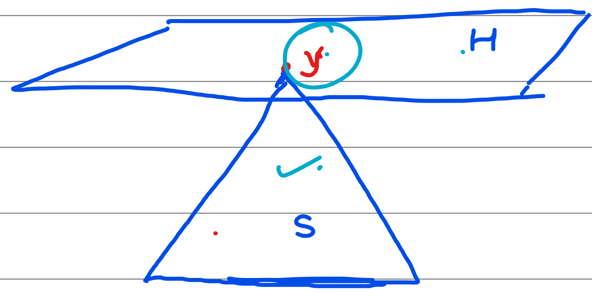
$$Co(S) = \left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \lambda_i x_i, \quad x_i \in S, \quad \sum \lambda_i = 1, \ \lambda_i \geq 0 \right\}$$



Hyperplane :- $H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$



Separating Hyperplane



$S \in H^-$ or $S \in H^+$

$\underline{a}'\underline{x} \leq c$ or $\underline{a}'\underline{x} \geq c$

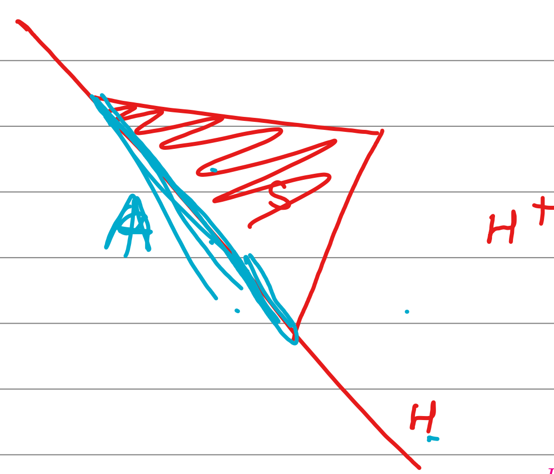
$\ni x \in S$ $\quad$ $\ni x \in S$

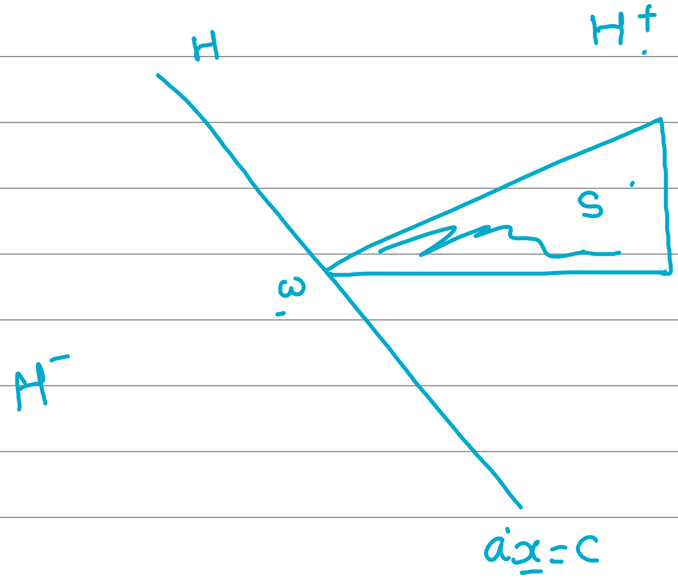
$S \in H^-$

$H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$

$\underline{a}'\underline{y} = c \Rightarrow \underline{y} \in S \ni \underline{y} \in H$

$S \in H^-$





$S \in H$

$\forall\ x \in S,\quad \underline{a}'\underline{x} \geq c$

$\omega \quad \underline{a}'\omega = c$

$\underline{\omega \in S \cap H}$

$\min_{\underline{x} \in S} \underline{a}'\underline{x} \ni \underline{a}'\underline{\omega}$