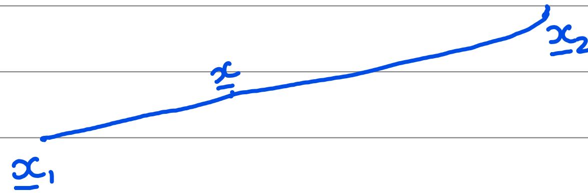
* Line segment : $\underline{x} \in \mathbb{R}^n$



scaler

$\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in (0,1)$

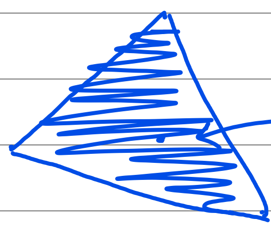
line segment joining $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$ $\{\underline{x} / \underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 ,\ \alpha \in (0,1)\}$

* Line passing $\underline{x}_1$ $\underline{x}_2 \in \mathbb{R}^n$ $\alpha + 1 - \alpha$

line segment ← convex combⁿ $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in (0,1)$

line , linear combⁿ $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$ $\alpha \in \mathbb{R}$

vector space → $\boxed{\underline{x} = \sum \alpha_i \underline{x}_i}$ Linear Combⁿ $\alpha_i \in \mathbb{R}$



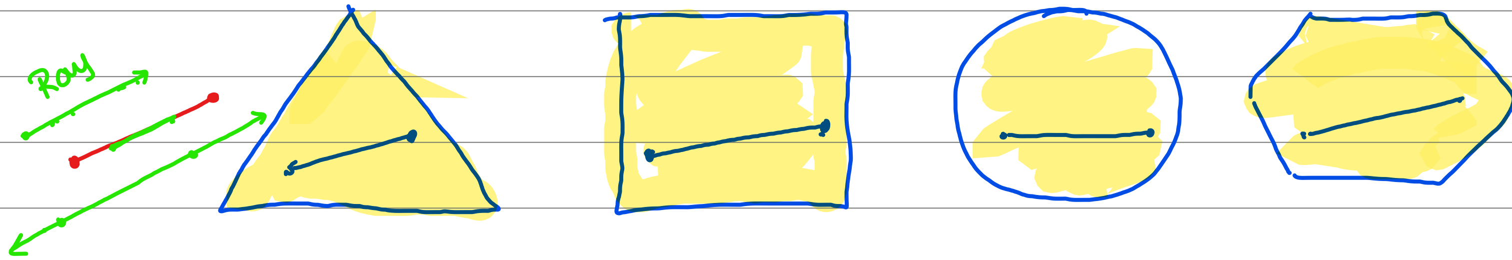
$\underline{x} = \sum \alpha_i \underline{x}_i$ $\sum \alpha_i = 1$, $\alpha_i \geq 0$

↳ Convex Combination

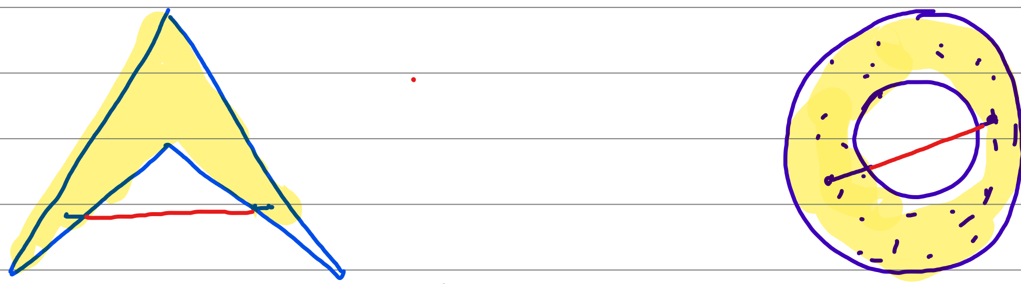
$$\left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \alpha_i x_i \ ,\ \alpha_i \geq 0 \ ,\ \sum \alpha_i = 1 \right\}$$

Convex set if $\forall \underline{x}_1$ & $\underline{x}_2 \in A$, $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \quad \forall \alpha \in (0,1)$

then if $\underline{x} \in S \ \forall \alpha$ A is convex set



Convex sets



Non convex sets

* Ray is convex set

$$\underline{x} = \underline{x}_0 + \underline{d}\,\alpha \qquad \alpha \geq 0$$

$\underline{x}_1$ & $\underline{x}_2 \in A$

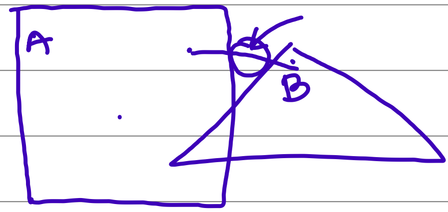
$$\lambda \qquad \underline{x}_1 = \underline{x}_0 + \underline{d}\,\alpha_1$$

$$1-\lambda \qquad \underline{x}_2 = \underline{x}_0 + \underline{d}\,\alpha_2$$

$$\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2 = (\lambda + 1 - \lambda)\underline{x}_0 + \underline{d}(\lambda\alpha_1 + (1-\lambda)\alpha_2)$$

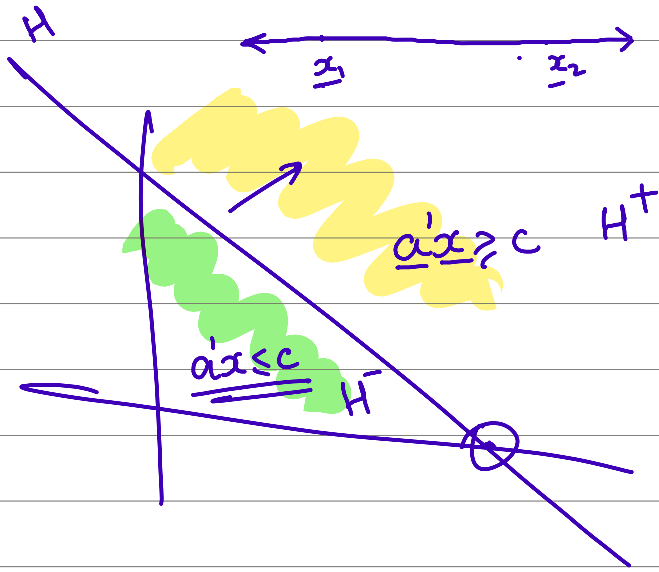
$$= x_0 + d(\qquad) \quad \geq 0$$

$\in A$



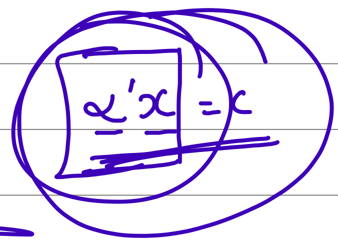
* Union of convex sets may or may not be convex.

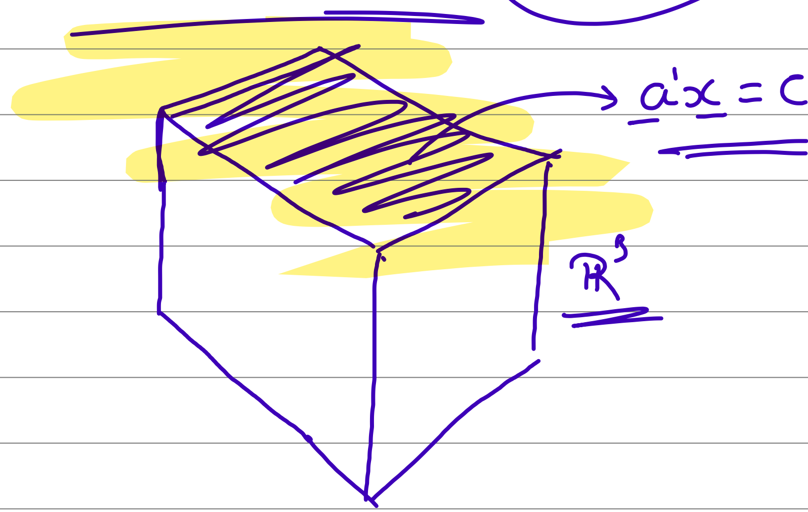
* Intersection of convex sets is also convex.



$$\sum_{i=1}^{2} \alpha_i x_i$$

$$\sum_{i=1}^{n} \alpha_i x_i = c$$





Hyperplane

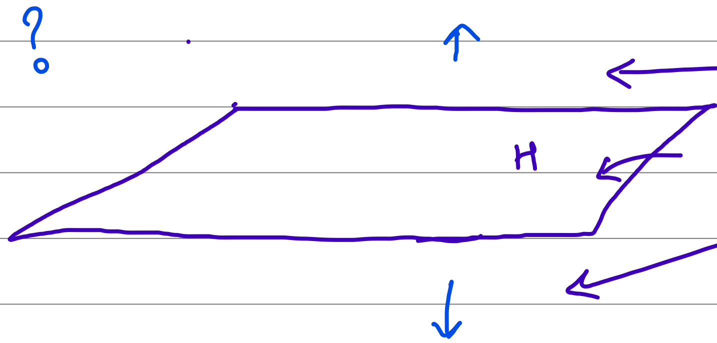
$H_0^+ = \{\underline{x} / a'x > c\}$ +ve open half space

$H^+ = \{\underline{x} / \underline{a}'\underline{x} \geq c\}$ ← positive closed half space

$H = \{\underline{x} / \underline{a}'\underline{x} = c\}$ Hyperplane

$H^- = \{\underline{x} / a'x \leq c\}$ ← Negative closed half space

$H_0^- = \{\underline{x} / \underline{a}'\underline{x} < c\}$ -ve open half space



Hyperplane $H = \{\underline{x} \mid \underline{a}'\underline{x} = c\}$

let $\underline{x}_1, \underline{x}_2 \in H \Rightarrow \underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

$\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$$\left[\begin{aligned} \underline{a}'(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) &= \alpha \underline{a}'\underline{x}_1 + (1-\alpha)\underline{a}'\underline{x}_2 \\ &= \alpha c + (1-\alpha)c \\ &= c \end{aligned}\right]$$

$\Rightarrow \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in H$

$\Rightarrow$ Hyperplane is convex set

H^+, H^-, H_0^+, H_0^- all are convex sets

Open ball $\underline{B = \{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| < r \}}$

To show :- B is convex set.

Let $\underline{x}, \underline{y} \in B \Rightarrow \underline{\alpha \cdot \underline{x}} + (1-\alpha)\, \underline{y} \boxed{\in B}$

$\underline{x} \in B \Rightarrow \|\underline{x} - \underline{x}_0\| < r$

$\underline{y} \in B \Rightarrow \|\underline{y} - \underline{x}_0\| < r$

$\alpha \in (0,1)$

$$\|\alpha \underline{x} + (1-\alpha)\underline{y} - \underline{x}_0\|$$

$$= \|\alpha \underline{x} + (1-\alpha)\underline{y} - (\alpha \underline{x}_0 + (1-\alpha) \cdot x_0)\|$$

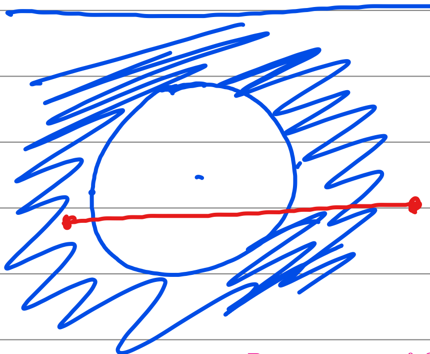
$$= \|\alpha(\underline{x} - \underline{x}_0) + (1-\alpha)(\underline{y} - x_0)\|$$

$$\leq \alpha \underbrace{\|\underline{x} - \underline{x}_0\|}_{<r} + (1-\alpha) \underbrace{\|y - x_0\|}_{<r} \qquad \alpha \in (0,1)$$

$$< \alpha r + (1-\alpha) r$$

$$< r$$

$\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| = r \}$ Convex or Not ✓



$\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| \geq r \}$ convex or not ✓

If C is convex set $\underline{\underline{\lambda C}}$ is also convex set.

$$\lambda C = \{ \underline{y} \mid \underline{y} = \lambda \underline{x} \ , \underline{x} \in C \}$$

to show λC as convex set

$$\underline{y}_1 , \underline{y}_2 \in \lambda C \quad \Rightarrow \quad \underline{y}_1 = \lambda \underline{x}_1 \ , \ \underline{y}_2 = \lambda \underline{x}_2 \quad \underline{x}_1, \underline{x}_2 \in C$$

$$\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 = \alpha \cdot \lambda \underline{x}_1 + (1-\alpha) \lambda \underline{x}_2$$

$$= \lambda (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$= \lambda \underline{x} \qquad \left(\text{as } \underline{x}_1 , x_2 \in C \overset{\text{as C is convex}}{\Rightarrow} x \in C \right)$$

$$\Rightarrow \ \alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 \in \lambda C$$

C, D are convex sets C+D is also convex

$$\rightarrow \qquad C + D = \{ \underline{Z} \mid \underline{z} = \underline{x} + \underline{y} \ , \underline{x} \in C , \underline{y} \in D \}$$

$$\underline{Z}_1 , \underline{Z}_2 \in C + D \ \Rightarrow \ Z_1 = \underline{x}_1 + \underline{y}_1$$

$$Z_2 = \underline{x}_2 + \underline{y}_2 \qquad \underline{x}_1 \underline{x}_2 \in C , \ \underline{y}_1 \underline{y}_2 \in D.$$

$$\alpha Z_1 + (1-\alpha) Z_2 = \alpha \cdot (\underline{x}_1 + \underline{y}_1) + (1-\alpha)(\underline{x}_2 + \underline{y}_2)$$

$$= \underbrace{\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2}_{\in C} + \underbrace{\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2}_{\in D}$$

convex sets

Intersection of any convex sets is convex

Let $\{S_i\}_{i=1}^{\infty}$ be collection of convex sets

$\cap S_i$ is convex

$\underline{x}, \underline{y} \in \cap S_i$

$\Rightarrow \underline{x}, \underline{y} \in S_i \quad \forall i$

$\Rightarrow \alpha \underline{x} + (1-\alpha) \underline{y} \in S_i \quad \forall i$ (S_i is convex)

$\Rightarrow \alpha \underline{x} + (1-\alpha) \underline{y} \in \cap S_i$

$\Rightarrow \cap S_i$ is convex.

- A set S∈Rn is convex if and only if every convex combination of any finite number of points of S is contained in S

∴ Assume that every convex combⁿ of any finite no. of points of S is in S.

② → ①

$\Rightarrow$ it is also true for $n=2$

$\Rightarrow$ if $\forall \underline{x}_1, \underline{x}_2 \in S \Rightarrow \alpha \cdot \underline{x}_1 + (1-\alpha) \underline{x}_2 \in S \quad \forall \alpha (0,1)$

$\Rightarrow$ S is convex set

(II) Assume S is convex and for any finite n $\sum_{i=1}^{n} \alpha_i \underline{x}_i \in S$

$\rightarrow$ let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n \in S$ $\quad \sum_{i=1}^{n} \alpha_i = 1$

we will prove this by mathematical induction

As S is convex, $\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S$ $\quad \alpha + 1 - \alpha = 1$

$\therefore$ So the above statement is true for $n=2$

Assume it is true for $n=K$ $\Rightarrow \sum_{i=1}^{K} \alpha_i \underline{x}_i$ $= 1$

$\sum_{i=1}^{K} \alpha_i = 1$

We have to prove it for $n=K+1$

$$\sum_{i=1}^{K+1} \beta_i \underline{x}_i = \sum_{i=1}^{K} \beta_i x_i + \beta_{K+1} x_{K+1} \qquad \sum_{i=1}^{K+1} \beta_i = 1$$

imp?

$$\sum_{i=1}^{K} \beta_i = 1 - \beta_{K+1}$$

$$\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} = 1$$

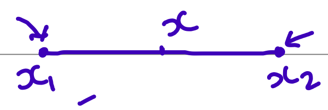
$$= (1-\beta_{K+1}) \boxed{\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} \cdot x_i} + \beta_{K+1} x_{K+1}$$

$\boxed{\ldots} \rightarrow \in S$

$$= (1-\beta_{K+1})\, x^* + \beta_{K+1} x_{K+1}$$

$\in S$ as S is convex set

* Vertices



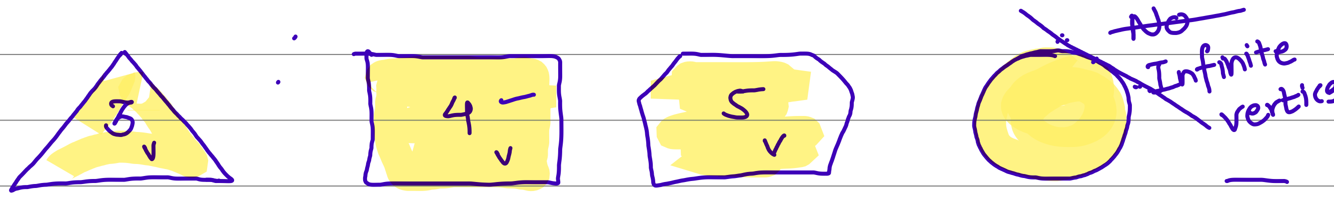
2 vertices

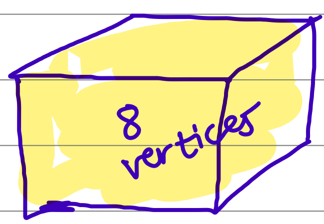
$\Rightarrow \quad x = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$x_2 = \alpha \underline{x}_1 + (1-\alpha)x_2 = \underline{x}_2$

$\alpha = 0$

No vertex



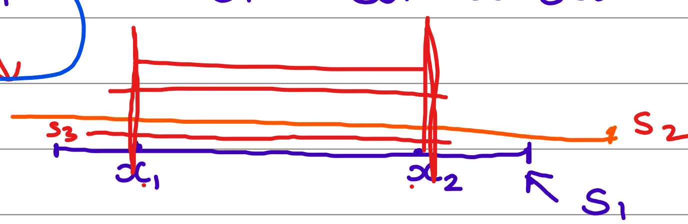


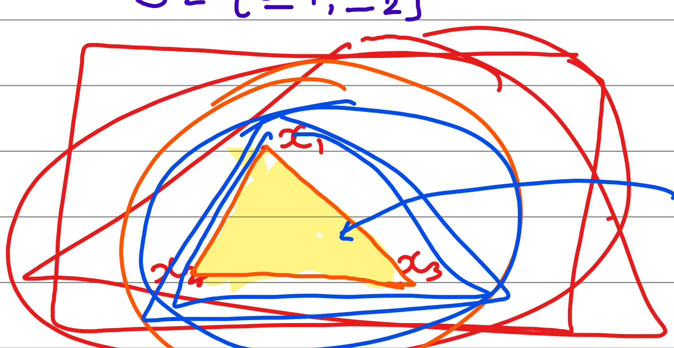
Convex Hull

$$\boxed{Co(S) \Rightarrow \bigcap_{i=1}^{\infty} S_i}$$

S_i is convex set containing S

$S = \{\underline{x}_1, \underline{x}_2\}$

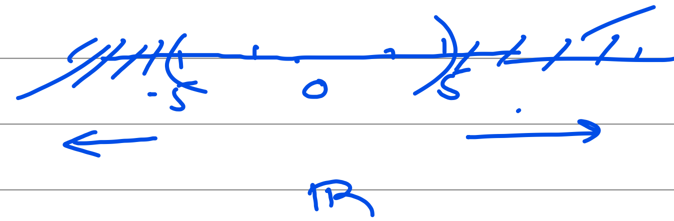




$S = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$

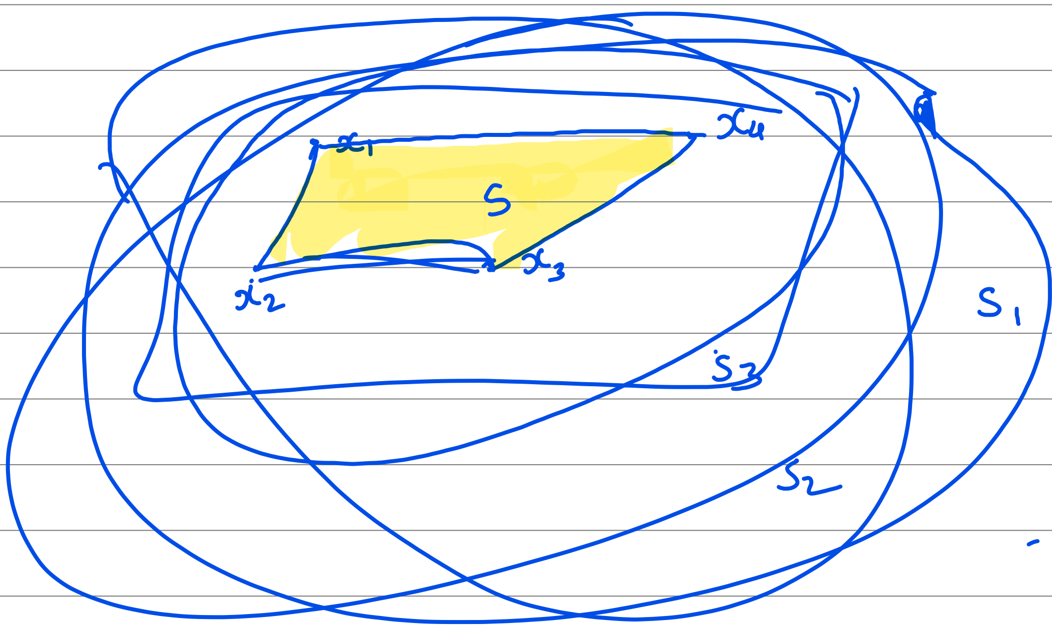
$S = \{ \underline{x} \mid ||\underline{x}|| \geq 5 \}$

$Co(S) = \mathbb{R}^n$

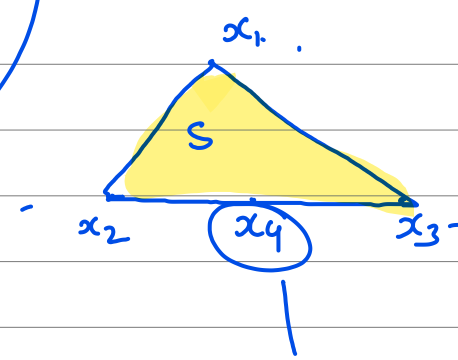


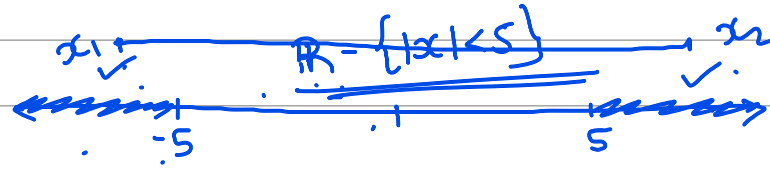
if set is convex $\underline{\underline{Co(S) = S}}$

$Co(S) = \cap S_i$, S_i is convex set containing S.



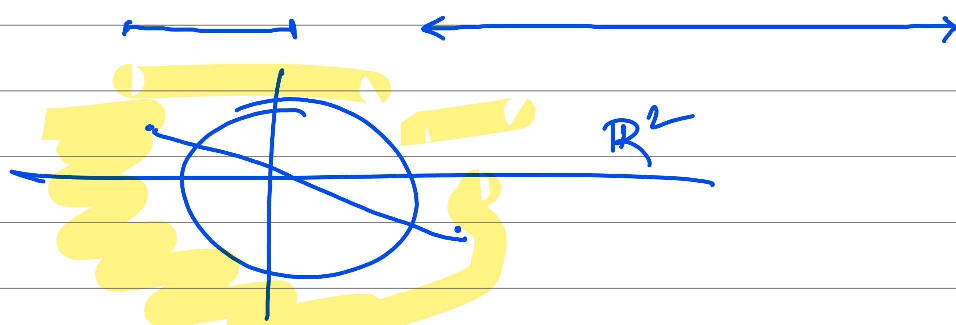
$\cap S_i = Co(S)$



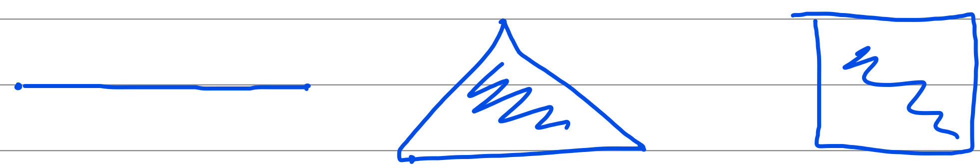


$S = \{ |x| > 5 \}$

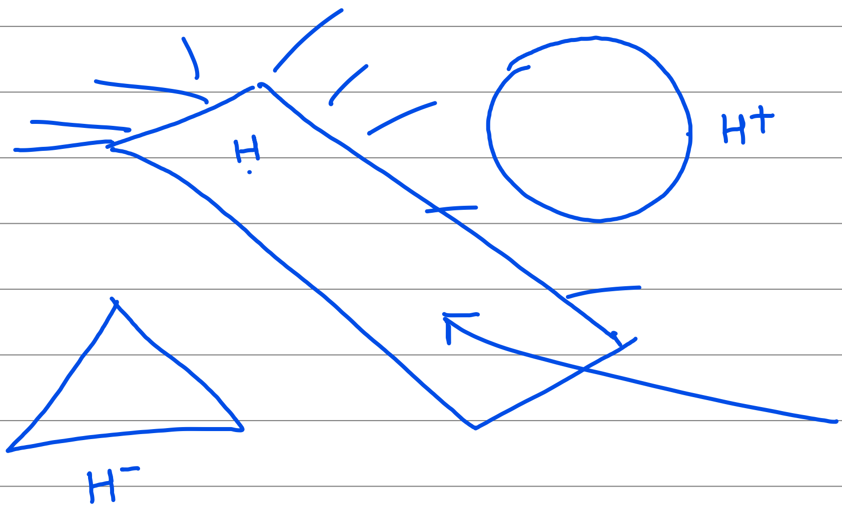
$\underline{\underline{\mathbb{R} = Co(S)}}$



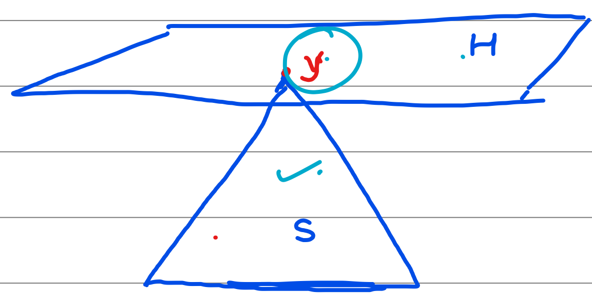
$$Co(S) = \left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \lambda_i x_i, \quad x_i \in S, \quad \sum \lambda_i = 1, \ \lambda_i \geq 0 \right\}$$



Hyperplane:- $H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$



Separating Hyperplane



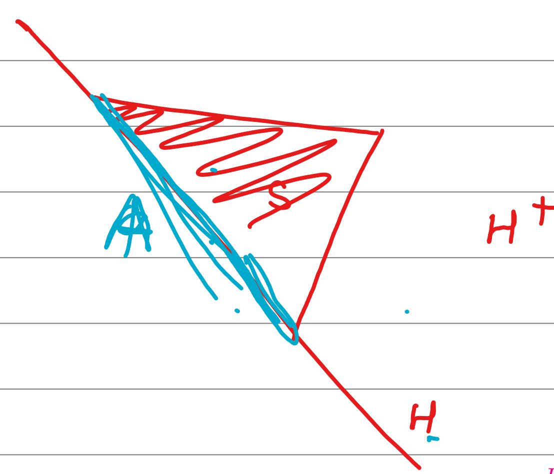
$S \in H^-$ or $S \in H^+$

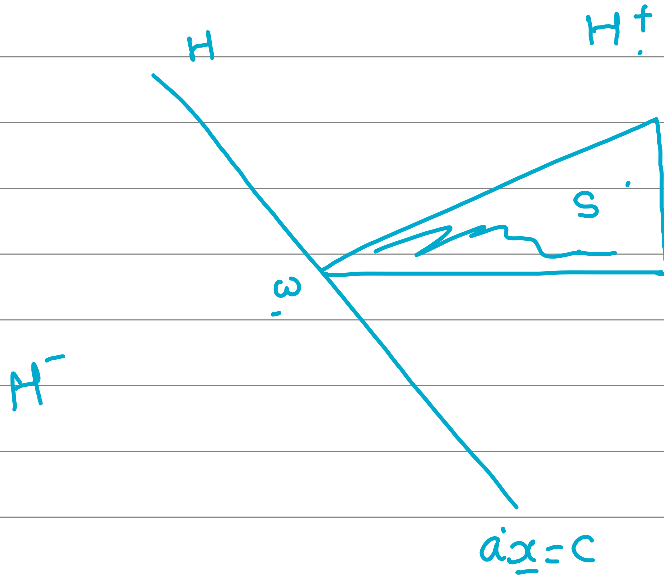
$\underline{a}'\underline{x} \leq c \ \ni x \in S$ or $\underline{a}'\underline{x} \geq c \ \ni x \in S$

$S \in H^-$

$H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$

$\underline{a}'\underline{y} = c \Rightarrow \underline{y} \in S \ni \underline{y} \in H$





$S \in H$

$\forall x \in S$, $\underline{a'x} \geq c$

ω $\quad a'\omega = c$

$\underline{\omega \in S \cap H}$

$\min\limits_{x \in S} \underline{a'x} \Rightarrow \underline{a'\omega}$

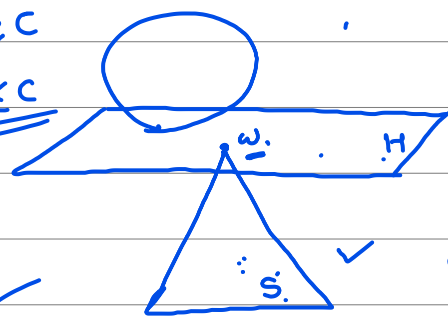
④

---

① Supporting hyperplane :- $a'x \geq c$, $a'x \leq c$

$H : \{ \underline{x} \,/\, \underline{a'x} = c \}$ $\quad \boxed{a'\omega = c}$

$\underline{\omega \in S}$ ← boundary S.



$\underline{\omega \in S \cap H}$

$\underline{a'x} = c$

$S \Rightarrow \underline{a'x \leq c}$

$\underline{a'\omega} = c$

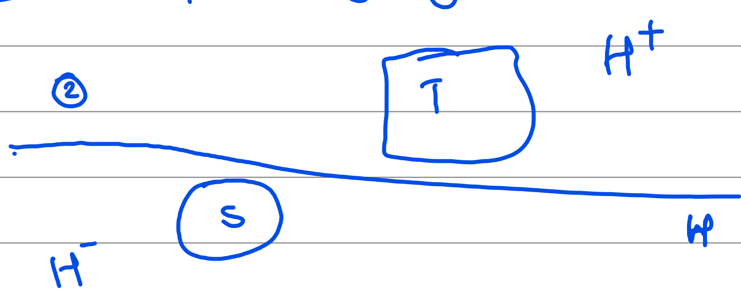
$S \in H^-$

$x \in S$

$\underline{a'x \leq c}$

Separating hyperplane

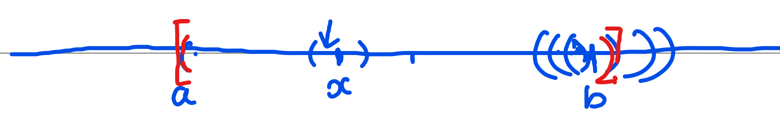
②

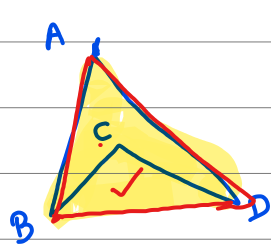


$S \in H^-$ & $T \in H^+$

Closure. $S = (a,b)$

$S' = (a,b) \cup \{a\} \cup \{b\} = [a,b]$



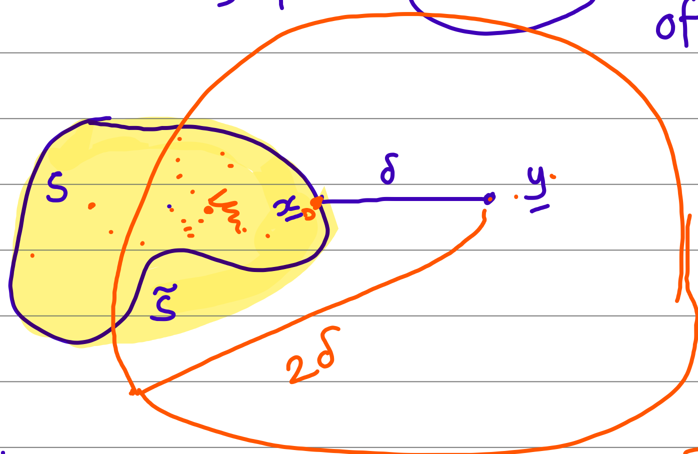


$|x - c| < \varepsilon$

$c - \varepsilon, \; c + \varepsilon$

$S \subseteq \mathbb{R}^n$ convex, y ~~boundary point~~ exterior to closure of S

$\exists\, \underline{a} \in \mathbb{R}^n$, $\underline{a} \neq \underline{0} \in \mathbb{R}^n$, $\underline{a}'\underline{y} < \inf_{\underline{x} \in S} \underline{a}'\underline{x}$



$\tilde{S}$ closure

let $\delta = \inf_{x \in \tilde{S}} ||\underline{x} - \underline{y}||$ $\delta > 0$ as y is exterior to $\tilde{S}$

$y \notin \tilde{S}$

$B_{2\delta} = \{ \underline{x} \,/\, ||\underline{x} - \underline{y}|| < 2\delta \}$

$\delta = \inf_{x \in \tilde{S}} ||\underline{x} - \underline{y}|| = \inf_{x \in \tilde{S} \cap B_{2\delta}} ||\underline{x} - \underline{y}||$

$\tilde{S} \cap B_{2\delta}$ is closed & bounded.

lets define $f: \tilde{S} \cap B_{2\delta} \to \mathbb{R}$, $f(\underline{x}) = ||\underline{x} - \underline{y}||$

f contin____,

ST-101

by max^m min^m theo., f attains its extremum in that set

$\exists$ some $\underline{x}_0 \in \tilde{S} \cap B_{2\delta}$ $\ni$

$$\delta = \min_{\underline{x} \in \tilde{S} \cap B_{2\delta}} ||\underline{x} - \underline{y}|| = ||\underline{x}_0 - \underline{y}||$$

$\Rightarrow$ $\underline{x}_0$ is boundary pt. of $\tilde{S}$

To show $\underline{a} = \underline{x}_0 - \underline{y}$

$\underline{a}'\underline{y} \le \inf_{\underline{x}} \underline{a}'\underline{x}$

$\underline{x}, \underline{x}_0 \in \tilde{S}$, $\alpha \underline{x} + (1-\alpha)\underline{x}_0 \in \tilde{S}$

$$\|\alpha \underline{x} + (1-\alpha)\underline{x}_0 - y\| \geq \|\underline{x}_0 - \underline{y}\|$$

$$\|(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\| \geq \|\underline{x}_0 - \underline{y}\|$$

$\|\underline{x}\| = \sqrt{\underline{x}'\underline{x}}$

$$\|(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\|^2 \geq \|\underline{x}_0 - \underline{y}\|$$

$$\Rightarrow \left((\underline{x}_0 - \underline{y}) + \underline{\alpha}(\underline{x} - \underline{x}_0)\right)^T\left((x_0 - y) + \alpha(\underline{x} - \underline{x}_0)\right) \geq (\underline{x}_0 - \underline{y})^T(\underline{x}_0 - y)$$

$$\Rightarrow \left[(\underline{x}_0 - \underline{y})' + \alpha(\underline{x} - \underline{x}_0)'\right]\left[(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\right] \geq (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})$$

$$\Rightarrow (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y}) + \alpha(\underline{x}_0 - \underline{y})'(\underline{x} - x_0) + \alpha(\underline{x} - \underline{x}_0)'(\underline{x}_0 - y) + \alpha^2(\underline{x} - \underline{x}_0)'(\underline{x} - \underline{x}_0)$$

$$\geq (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})$$

$$2\alpha(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) + \alpha^2|x - x_0|^2 \geq 0$$

$$2(\underline{x}_0 - \underline{y})'(x - x_0) + \alpha|x - x_0|^2 \geq 0$$

let $\alpha \to 0$

$$(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) \geq 0$$

$$\Rightarrow (\underline{x}_0 - \underline{y})'\underline{x} \geq (\underline{x}_0 - \underline{y})'\underline{x}_0$$

$$\Rightarrow \underline{a}'\underline{x} \geq \underline{a}'(\underline{x}_0 - \underline{y} + \underline{y})$$

let $a = \underline{x}_0 - \underline{y}$

$$\geq a'(x_0-y) + a'y$$

$\|z\| = \sqrt{z'z}$

$$a'x \geq (x_0-y)'(x_0-y) + a'y \qquad \forall\, x \in S$$

$$\geq \delta^2 + a'y \qquad \delta > 0$$

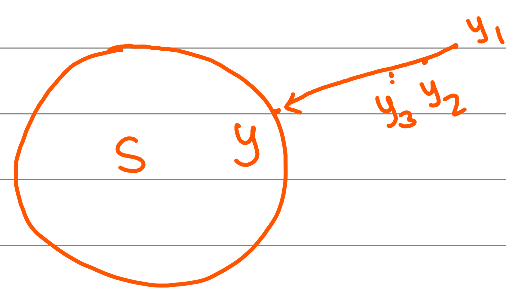
$$\inf_{x\in S} a'x \geq a'y$$

$$a'y < \inf_{x \in S} a'x$$

Theo. S convex, y boundary of S

To show $\exists\, H, \ni S \in H^+/H^-$, $y \in S \cap H$

→ let y_n be seq^n of points exterior to closure of S
Assume $y_n \to y$



$$\exists\, a_n \in \mathbb{R}^n, \qquad a_n'y_n \leq \inf_{x\in S} a_n'x \qquad \|a_n\| = 1$$

$$a_n'y_n - a_n'y + a_n'y \leq a_n'x \qquad \forall\, x \in S$$

for large n, $y_n \to y \Rightarrow a_n'y_n \to a'y$

$\underline{a}'y \nless \underline{a}'\underline{x}$

$$\underline{a}_n'\underline{y} < \underline{a}_n'\underline{x}$$

$\|a_n\|$ ∋ seq^n of a_n bounded.
Bolzano Weierstrauss The$^\circ$ for seq^{n}s.

∃ convergent subseq^n $\{a_{n_k}\}$
Suppose it converges to $a_{n_k} \to a$

$$\underline{a}_{n_k}'\underline{y} < \underline{a}_{n_k}'\underline{x}$$

letting $k \to \infty$, $\underline{a}_{n_k} \to \underline{a}$

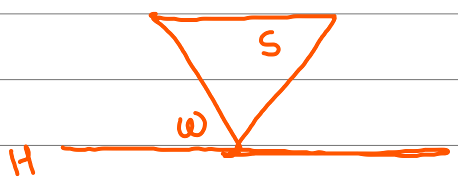
$$\underline{a}'\underline{y} = \underline{a}_{n_k}'\underline{y} < \underline{a}_{n_k}'\underline{x} = \underline{a}'\underline{x}$$

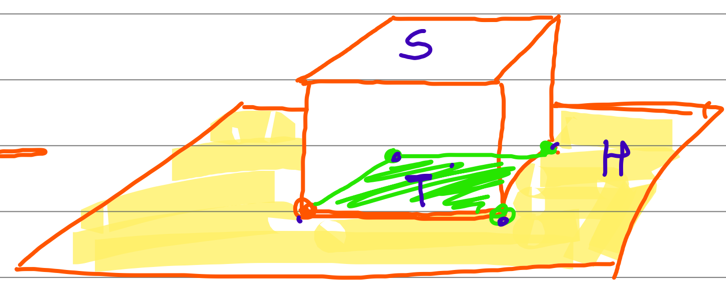
$$\ni \underline{a}'\underline{y} \le \underline{a}'\underline{x} \qquad \forall x \in S$$

$H = \{\underline{x} / \underline{a}'\underline{x} = \underline{a}'\underline{y}\}$ is supporting hyperplane. at y

---

$$T = S \cap H = \{\omega\}$$





H Supportive Hyperplane, $T = S \cap H$
by method of contradiction

let $\underline{x}_0$ be extreme pt. of T but not of S.

let $\exists$ some $x_1, x_2 \in S \ni \underline{x}_0 = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$H = \{ \underline{x} \mid \underline{a}'\underline{x} = c \}$ supportive hyperplane of S

let $S \in H^+$, $\underline{a}'\underline{x} \geq c \ \forall \ \underline{x} \in S$.

$\Rightarrow \underline{a}'\underline{x}_1 \geq c, \ \underline{a}'\underline{x}_2 \geq c$

$\underline{x}_0 \in T = S \cap H \quad \Rightarrow \underline{a}'\underline{x}_0 = c$

$\Rightarrow \underline{a}'(\alpha x_1 + (1-\alpha)\underline{x}_2) = c$

$\Rightarrow \alpha \underline{a}'\underline{x}_1 + (1-\alpha)\underline{a}'\underline{x}_2 = c$

$\alpha \in (0,1), (1-\alpha) \in (0,1)$

$\Rightarrow \alpha \cdot \underset{\geq c}{\underline{\qquad}} + (1-\alpha)\underset{\geq c}{(\qquad)} = c$

it is only possible if $\underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

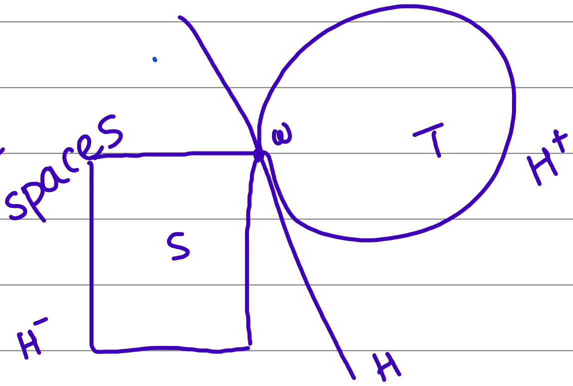
$\Rightarrow \underline{x}_1, \underline{x}_2 \in H$ & $\Rightarrow x_1, x_2 \in S \cap H$

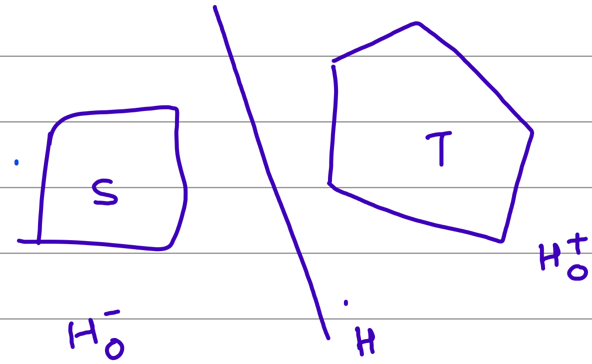
$\Rightarrow x_1, x_2 \in T$

$\therefore$ Which contradicts to our assumption that x_0 is extreme pt. of T.

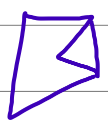
Separating hyperplane

closed half spaces





open half spaces



convex set

S convex

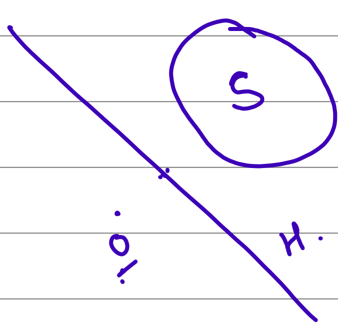
$\underline{0} \notin S$

$\tilde{S} = S$

$\underline{0} \notin \tilde{S}$

exterior pt.

$$\left[\underline{0} = \underline{a}'\underline{0} < \inf_{\underline{x} \in S} \underline{a}'\underline{x} \right] \Rightarrow \underline{a}'\underline{x} > 0 \quad \forall\ \underline{x} \in S \qquad \text{—(1)}$$

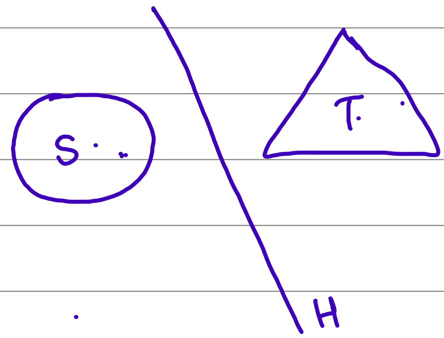


if $H = \{ \underline{x} \mid \underline{a}'\underline{x} = c \}$ —(2)

$0 < c < \inf_{x \in S} \underline{a}'\underline{x}$

$S \in H_0^+$

$\underline{0} \in H$

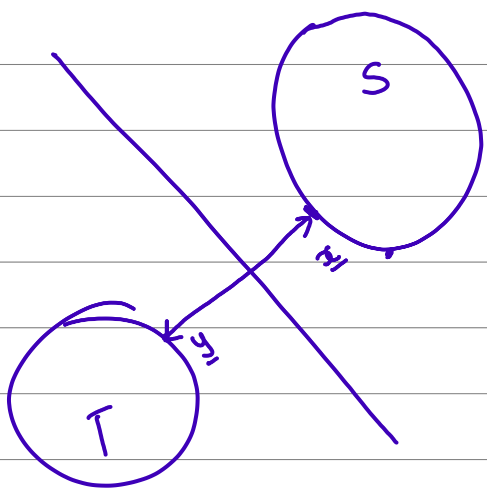


$\underline{0} \notin S-T$

$S-T = \{ \underline{z} \mid \underline{z} = \underline{x} - \underline{y},\ \underline{x} \in S,\ \underline{y} \in T \}$

$\underline{a}'\underline{z} \geq 0 \quad \forall\ \underline{z} \in S-T$

$\underline{a}'\underline{x} - a'\underline{y} > 0$



$\inf\limits_{z \in S-T} a'\underline{z} > 0$

$\inf\limits_{\substack{x \in S \\ y \in T}} \underline{a}'\underline{x} - \underline{a}'\underline{y} > 0$

$\inf\limits_{\underline{x} \in S} \underline{a}'\underline{x} > \sup\limits_{y \in T} \underline{a}'\underline{y}.$

S.T

$0 \notin S-T$

$0 = \underline{a}'\underline{0} < \inf\limits_{z \in S-T} a'z$

$\inf\limits_{z \in S-T} a'z > 0$

$\inf\limits_{x \in S} a'\underline{x} - \sup\limits_{y \in T} a'y > 0$

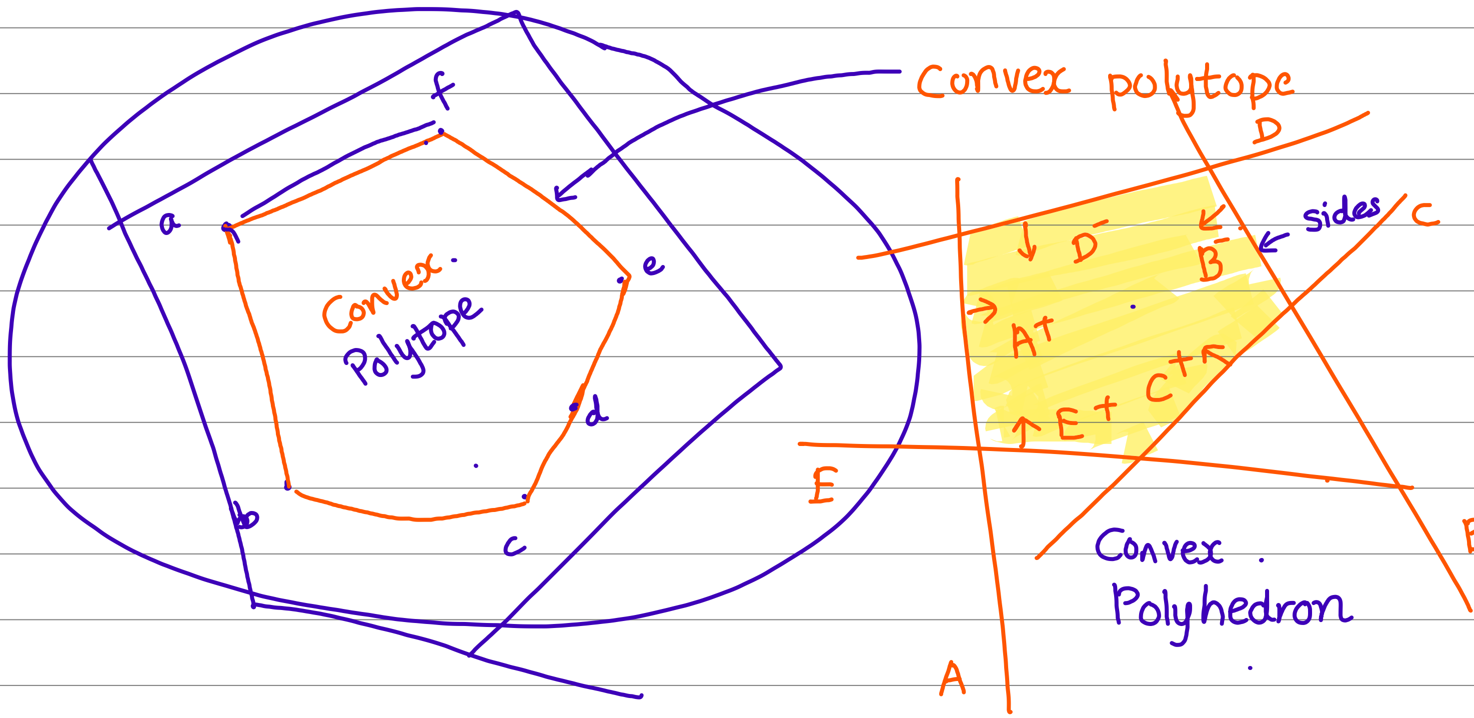
$\inf a'x > c > \sup a'y.$

$\inf\limits_{\underline{x} \in S} \underline{a}'\underline{x} > c > \sup\limits_{\underline{y} \in T} \underline{a}'\underline{y}$

$H = \{ \underline{x} \mid \underline{a}'\underline{x} = c \}$ ✓

$S \in H_0^{+} \qquad T \in H_0^{-}$

strictly separating.



Convex Polyhedron

$\underline{S} = \{\underline{a}, b, c, d, e, f\}$ $\qquad \underline{\underline{Co(S)}}$ = Convex hull / $\underline{\text{Polytope}}$

$Co(S) = \{\underline{z} \mid z = \sum \alpha_i x_i,\ x_i \in S\}$ $\qquad$ Vertices $(Co(S)) = \{a, b, c, e, f\} = V$

$\underline{x}$

$u \in V \Rightarrow u \notin S$

$S = \{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m\}$ $\qquad Co(S) = \{\underline{x} \mid \underline{x} = \sum_{i=1}^{m} \alpha_i x_i\}$

$V =$ vertices $(Co(S))$

$\underline{u} \in V$ but $u \notin S$

$\underline{u} \in Co(S)$

$\Rightarrow \quad \underline{u} = \sum_{i=1}^{m} \alpha_i x_i \qquad \alpha_i \in (0,1) \qquad \sum \alpha_i = 1 \qquad \alpha_i < 1$

$\underline{u} = \alpha_1 x_1 + \sum_{i=2}^{m} \alpha_i x_i = \underline{\alpha_1 x_1} + (1-\alpha_1) \underbrace{\sum_{i=2}^{m} \frac{\alpha_i x_i}{(1-\alpha_1)}}_{x^*} = \alpha_1 x_1 + (1-\alpha_1) x^*$

$A\underline{x} \le \underline{b}$

$a_1'\underline{x} = a_{11}x_1 + a_{12}x_2 \le b_1$ H H^+ H^-

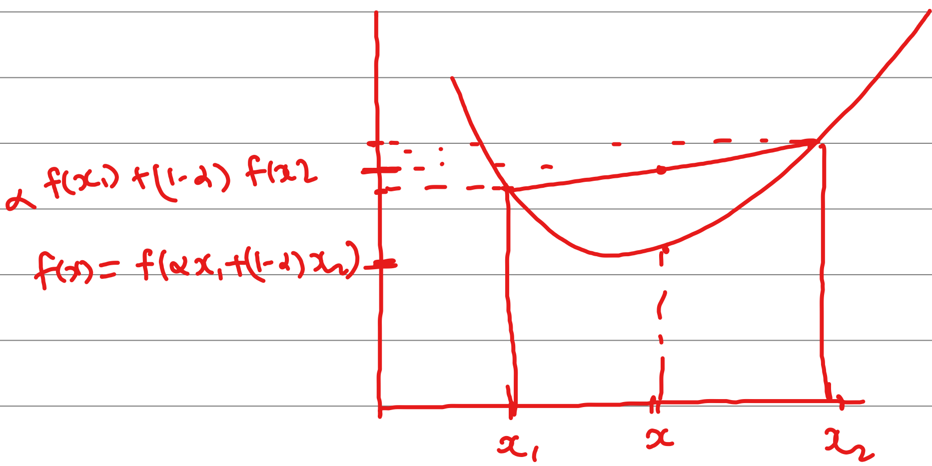
$a_2'\underline{x} = a_{21}x_1 + a_{22}x_2 \le b_2$ H H^+ H^-

$\underline{x} \ge \underline{0}$ H^+

$H = \{ \underline{x} \,/\, \underline{a}_1'\underline{x} = b_1 \}$

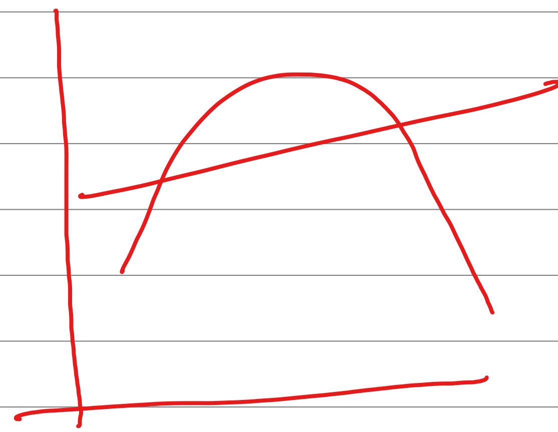
$\{ x : / \, Ax \le b \}$

Polyhedron / Polyhedral



convex

e.g x^2



concave

eg $-x^2$

$f(\underline{x}) = \underline{c}'\underline{x} + \underline{d}$

$\underline{x}_1, \underline{x}_2 \in S$

$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \underline{c}'(\alpha \underline{x}_1 + (1-\alpha)\cdot \underline{x}_2) + \underline{d}$

$= \underline{\alpha}\, \underline{c}'\underline{x}_1 + (1-\alpha)\, \underline{c}'\underline{x}_2 + \alpha \underline{d} + (1-\alpha)\underline{d}$

$= \alpha(\underline{c}'\underline{x}_1 + \underline{d}) + (1-\alpha)(\underline{c}'\underline{x}_2 + \underline{d})$

$= \alpha \cdot f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$

$\le$ $\Rightarrow$ convex

$\ge$ $\Rightarrow$ concave

f convex fun^c on T, $\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in T$.

$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

$T_K = \{ \underline{x} / \underline{x} \in T, f(\underline{x}) \leq K \}$

let $\underline{x}_1, \underline{x}_2 \in T_K$

$\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in T$ and $f(\underline{x}_1) \leq K$, $f(x_2) \leq K$

$\Rightarrow$ as T is convex, & $\underline{x}_1, x_2 \in T$ $\Rightarrow$ $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in T$

$$f(x) = f(\alpha x_1 + (1-\alpha) x_2)$$

$$\leq \alpha \underbrace{f(x_1)}_{\leq K} + (1-\alpha) \underbrace{f(x_2)}_{K} \qquad \text{(as } f \text{ is convex)} \quad \alpha \in (0,1)$$

$$\leq K$$

$\underline{x} \in T$ & $f(\underline{x}) \leq K$ $\Rightarrow$ $\underline{x} \in T_K$

$\Rightarrow$ T_K is convex set.

---

f, g are convex functions

f is convex fun^c $\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in S$

$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$

$\mu \geq 0$

$(\mu \cdot f)(\underline{x}) = \mu \cdot f(\underline{x})$

$$(\mu f)(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \mu \cdot f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2)$$

$$\leq \mu \cdot [\alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)]$$

$$\leq \alpha \cdot \mu \cdot f(x_1) + (1-\alpha)\, \mu \cdot f(x_2)$$

$$\leq \alpha \cdot (\mu f)(x_1) + (1-\alpha)(\mu f)(x_2)$$

μf is also convex.

$(f+g)(\underline{x}) = f(\underline{x}) + g(\underline{x})$

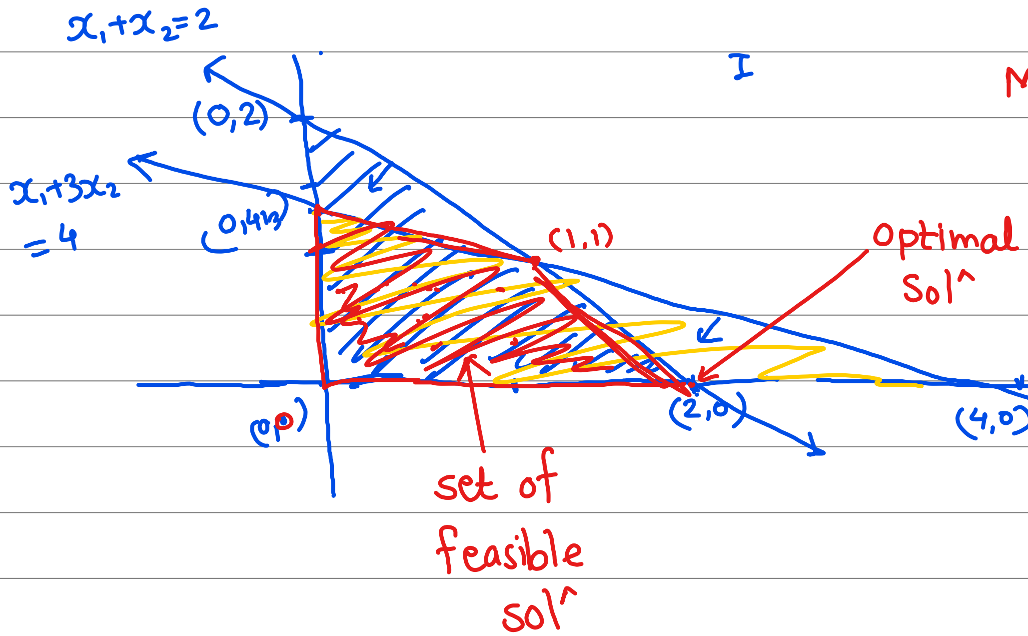
$$(f+g)(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) + g(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2)$$

$$\leq \alpha f(x_1) + (1-\alpha) f(x_2) + \alpha g(x_1) + (1-\alpha) g(x_2)$$ f & g conv

$$\leq \alpha (f(x_1) + g(x_1)) + (1-\alpha)(f(x_2) + g(x_2))$$

$$\leq \alpha \cdot (f+g)(\underline{x}_1) + (1-\alpha)(f+g)(\underline{x}_2)$$

I



Max $2x_1+x_2$

$x_1+x_2 \le 2$

$x_1+3x_2 \le 4$

$x_1, x_2 \geq 0$

$x_1+x_2=2$

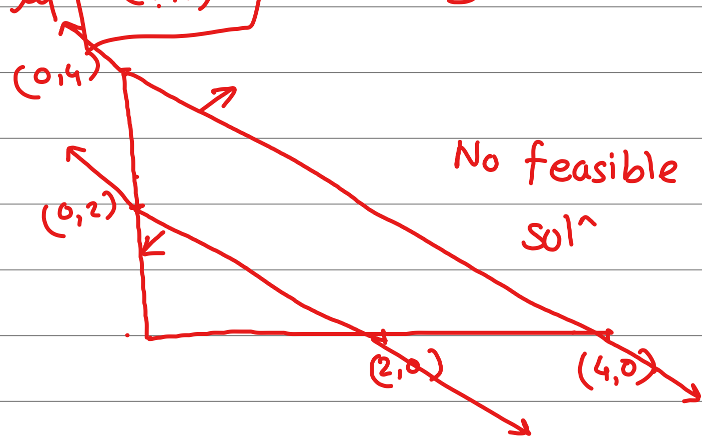
$(0,2)$ $(2,0)$

$x_1+3x_2 = 4$

$(0,4/3)$ $(4,0)$

		$Z=2x_1+x_2$
x_1	$(0,0)$	0
x_2	$(0,4/3)$	$4/3$
x_3	$(2,0)$	4
x_4	$(1,1)$	3

$$Z_0 = \min\{\underline{c}'\underline{x}_i,\ i=1:k\}$$



$x_1+x_2 \le 2$

$x_1+x_2 \geq 4$

Let Max $\underline{Z} = \underline{c}'\underline{x}$

$A\underline{x} \le b$

$T = \{\underline{x} \mid A\underline{x} \le b\}$ constraint set.

let $\underline{x}^1$ & $\underline{x}^2$ are to optimum sol^n to LP

$$\text{Max } Z = \underset{\underline{x} \in T}{\text{Max}}\ \underline{c}'\underline{x} = \underline{c}'\underline{x}^1 = \underline{c}'\underline{x}^2 = Z_0 \text{ (say) optimum value}$$

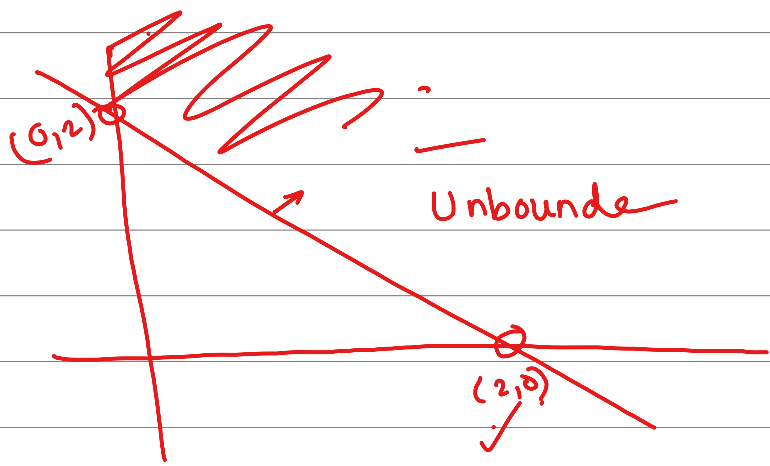
$\underline{x}^0 = \alpha \underline{x}^1 + (1-\alpha)\underline{x}^2$

To show $\underline{x}^0$ is also optimum sol^n

$$\underline{c}'\underline{x}^0 = c'(\alpha \underline{x}^1 + (1-\alpha)\underline{x}^2)$$
$$= \alpha \underline{c}'\underline{x}^1 + (1-\alpha)\underline{c}'\underline{x}^2$$
$$= \alpha Z_0 + (1-\alpha) Z_0$$
$$= Z_0$$

$\Rightarrow$ $\underline{x}^0$ is also optimum sol^n

$\Rightarrow$ Set of optimum sol^n is also convex set.



$$\text{Min } Z = 2x_1 + 3x_2$$
$$x_1 + x_2 \geq 2$$

$\boxed{Z_0 =}$ $\min\{\underline{c}'\underline{x}_i,\ i = 1:K\}$

$\underline{x} \in T \Rightarrow \quad \underline{x} = \sum_{i=1}^{K} \alpha_i x_i \quad . \quad \alpha_i \geq 0 \quad \sum \alpha_i = 1$

$$\underline{c}'\underline{x} = \sum \alpha_i \underline{c}'\underline{x}_i$$
$$\geq \sum \alpha_i Z_0 = Z_0 \quad \Rightarrow$$

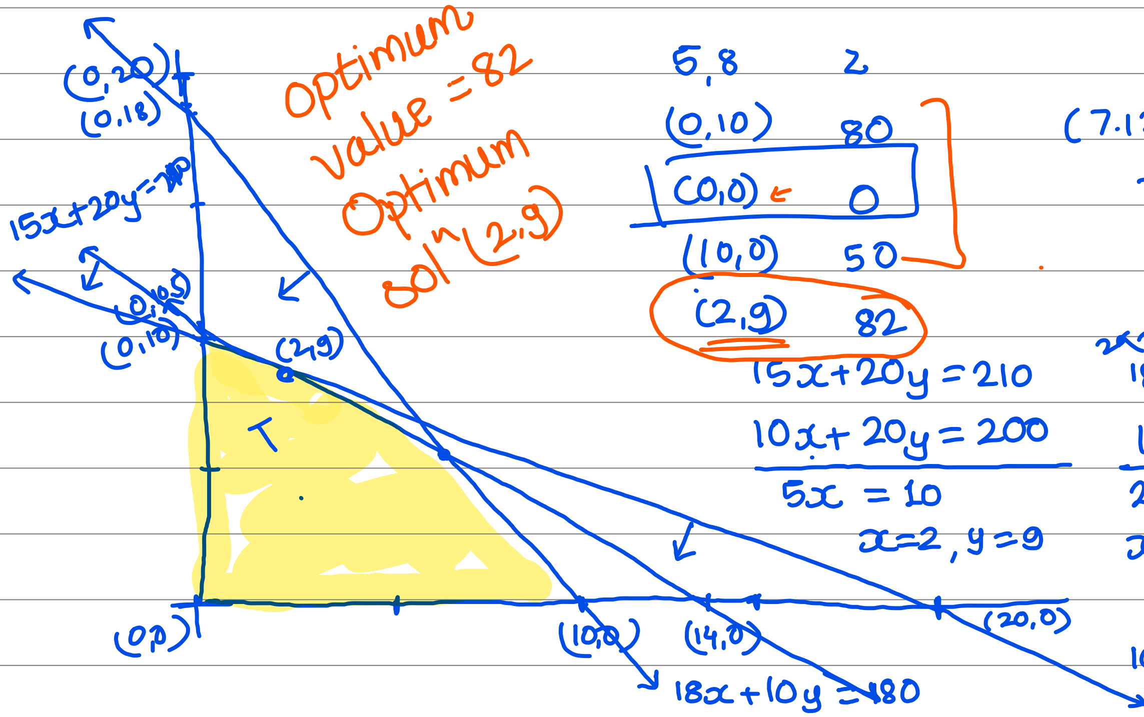
$$\underline{c}'\underline{x} \geq z_0 \qquad \forall\ \underline{x} \in T$$

$$\min_{\underline{x}\in T} \underline{c}'\underline{x} \geq z_0 = \min\{\underline{c}'\underline{x}_i, \quad i = 1:k\}$$

max $5x + 8y$

subject to

$18x + 10y \leq 180$	$18x + 10y = 180$	$(0,18)$	$(10,0)$
$10x + 20y \leq 200$	$10x + 20y = 200$	$(0,10)$	$(20,0)$
$15x + 20y \leq 210$	$15x + 20y = 210$	$(0,10.5)$	$(14,0)$
$x, y \geq 0$	$x, y \geq 0$		



Optimum value = 82

Optimum sol^n (2,9)

5, 8	z
(0,10)	80
(0,0)	0
(10,0)	50
(2,9)	82

(7.12, 5.14)

76.85

$$15x + 20y = 210$$
$$10x + 20y = 200$$
$$5x = 10$$
$$x = 2,\ y = 9$$

$\times 2\ \ 36 \quad 20 \quad 360$

$$18x + 10y = 180$$
$$15x + 20y = 210$$
$$21x = 150$$
$$x = 7.12$$
$$y = 5.14$$

$10x + 20y = 200$

$$\boxed{\begin{array}{l} 2x_1+3x_2 \le 10 \\ 3x_1+2x_2 \ge 5 \\ 3x_1+5x_2 \le 15 \end{array}}$$
$$x_1, x_2 \ge 0$$

$$\begin{array}{l} 2x_1+3x_2+x_3 = 10 \\ 3x_1+2x_2-x_4 = 5 \\ 3x_1+5x_2+x_5 = 15 \end{array}$$

$$\begin{array}{l} 2x_1+3x_2 \le 10 \\ x_1+5x_2 \le 15 \\ x_1 \ge 0, \ \boxed{x_2 \le 0} \end{array} \Rightarrow \begin{array}{l} 2x_1-3x_2' \le 10 \\ x_1-5x_2' \le 15 \\ x_1 \ge 0 \quad x_2' \ge 0 \end{array}$$

$$x_2' = -x_2$$

$$x_B = [x_1 \ x_2 \ x_3]$$
Basic

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{bmatrix} \underbrace{\begin{matrix} 2 & 3 & 1 \\ 3 & 2 & 0 \\ 3 & 5 & 0 \end{matrix}}_{B} & \underbrace{\begin{matrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{matrix}}_{R} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 15 \end{bmatrix}$$

Basis

$$\underset{\substack{m\times n \\ 3\times 5}}{A}\underline{x} = \underline{b} \qquad m=3, \ n=5$$

$$\rho(A) \le \min(m,n)$$

$$A = [B \ \ R] \qquad \rho(A) = \rho(B)$$

$$A\underline{x} = \underline{b}$$

$$\underset{m\times m}{[B} \ \ \underset{m\times(n-m)}{R]} \begin{bmatrix} \underline{x}_B \\ \underline{x}_{NB} \end{bmatrix} = \underline{b}$$

$$B\underline{x}_B + R\underline{x}_{NB} = \underline{b}$$
$$B^{-1}B\underline{x}_B + B^{-1}R\underline{x}_{NB} = B^{-1}\underline{b}$$

$$\underline{x}_B = B^{-1}\underline{b} - B^{-1}R\underline{x}_{NB}$$

Basic Soln $\rightarrow \underline{x}_{NB} = \underline{0}, \quad \underline{x}_B = B^{-1}\underline{b}$

Max $3x_1+5x_2$

$x_1+2x_2 \le 6$ $\quad$ $x_1+2x_2+x_3 = 6$

$3x_1+x_2 \le 5$ $\quad$ $3x_1+x_2+x_4 = 5$

$x_1,x_2 \geq 0$ $\quad$ $x_1,x_2,x_3,x_4 \geq 0$

$$A=\begin{bmatrix}1&2&1&0\\3&1&0&1\end{bmatrix} \quad \underline{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \quad \underline{b}=\begin{bmatrix}6\\5\end{bmatrix}$$

$m=2, \quad n=4$

$\rho(A)=\underline{m}=2$

$x_1+x_2 \le 2$

$\rightarrow \boxed{\cancel{2x_1+2x_2 \le 4}}$

(1) $B=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\quad$ $R=\begin{bmatrix}1&2\\3&1\end{bmatrix}$ $\quad$ $\underline{x}_B=\begin{bmatrix}x_3\\x_4\end{bmatrix}$ $\quad$ $x_{NB}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$B^{-1}\underline{b}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}6\\5\end{bmatrix}=\begin{bmatrix}6\\5\end{bmatrix}$$

$x_B=[6,5]$

$x_{NB}=[0,0]$

$$\underline{x}=\begin{pmatrix}x_1 & x_2 & x_3 & x_4\\ 0, & 0, & 6, & 5\end{pmatrix}$$ Basic Sol^n

(2) $B=\begin{bmatrix}1&2\\3&1\end{bmatrix}$ $\quad$ $\underline{x}_B=[x_1,x_2]$ $\quad$ $\underline{x}_{NB}=[x_3\ x_4]$

$$B^{-1}=\frac{-1}{5}\begin{bmatrix}1&-2\\-3&1\end{bmatrix} \quad B^{-1}\underline{b}=\frac{-1}{5}\begin{bmatrix}1&-2\\-3&1\end{bmatrix}\begin{bmatrix}6\\5\end{bmatrix}=\frac{-1}{5}\begin{bmatrix}-4\\-13\end{bmatrix}=\begin{bmatrix}4/5\\13/5\end{bmatrix}$$

Basic Sol^n $\quad$ $\underline{x}=[4/5, 13/5, 0, 0]$

$a_1 \; a_2 \ldots a_p$ $\quad \sum_{i=1}^{p} \alpha_i a_i = 0 \Rightarrow \quad \Rightarrow \alpha_i = 0$ l.i.

for some i $\alpha_i \neq 0$

↳ linearly dependent

$$\underline{x} = (\underbrace{x_1 x_2 \ldots x_p}_{>0}, \underbrace{x_{p+1} \ldots x_n}_{=\underline{0}})$$

$$\underset{m\times n}{A}\underline{x} = \underline{b}_{m\times 1}$$

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \ldots + x_n \underline{a}_n = \underline{b}_{m\times 1}$$

$$\Rightarrow \quad x_1 \underline{a}_1 + \underline{x}_2 \underline{a}_2 + \ldots + x_p \underline{a}_p = \underline{b}_{m\times 1}$$

① If p vectors are linearly independent

$p \leq m$ ✓

↳ i) $p = m$, $\underline{x}$ is f.s. is also b.f.s.

ii) $p < m$

we can add $m-p$ linearly independent columns

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \ldots + x_p \underline{a}_p + 0 \; \underline{a}_{p+1} + 0 \cdot \underline{a}_{p+2} + \ldots + 0 \cdot \underline{a}_m = \underline{b}$$

$\Rightarrow$ we get degenerate b.f.s.

② If p. ~~vectors~~ columns corresponding to feasible sol^n are linearly dependent.

$$\sum_{i=1}^{p} \alpha_i a_i = 0 \Rightarrow \text{for some } \alpha_j \neq 0$$

Assume $\alpha_j \neq 0$

$$\frac{x_j}{\alpha_j} \sum_{i=1}^{p} \alpha_i a_i = 0 .$$

$$\sum_{i=1}^{p} x_i \underline{a}_i = \underline{b}. \qquad (A\underline{x} = \underline{b}$$

$$\sum_{i=1}^{p} x_i a_i - \frac{x_j}{\alpha_j} \sum_{i=1}^{p} \alpha_i a_i = \underline{b} - \underline{0}$$

$$\underbrace{\sum_{\substack{i=1 \\ i \neq j}}^{p} x_i a_i}\ + \underline{x_j a_j - x_j a_j} = \underline{b}$$

$\underline{x} = [1, 1, 1]$

$$\sum \alpha_i \underline{a}_i = 0 \Rightarrow \alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

choosing $\alpha_1 = 1, \ \alpha_2 = 1, \ \alpha_3 = -1$

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$A\underline{x} = \underline{b}$

$\underline{x} = [1, 1, 1]$

$[2, 2, 0]$

$\underline{x}$ extreme

~~dependent~~

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_K \underline{a}_K = \underline{b}. \qquad 0 < z < \min \frac{x_i}{|\alpha_i|}$$

$$\pm z \left[\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \dots + \alpha_K \underline{a}_K = \underline{0}\right] \qquad \text{for some } \alpha_j \neq 0$$

$z > 0$

$x_i - z\alpha_i \geq 0$

$$\frac{1}{2}(\underline{x} + z\underline{\alpha}) + \frac{1}{2}(\underline{x} - z\underline{\alpha}) = \underline{x} \qquad \min\left(\frac{x_i}{\alpha_i}\right) \geq z \quad \forall i$$

$$(x_1 + z\alpha_1)\underline{a}_1 + \dots + (x_K + z\alpha_K)\underline{a}_K = \underline{b}$$

$$\text{Max } z = 6x_1 + 5x_2$$
$$\text{S.t. } x_1 + x_2 \leq 5$$
$$3x_1 + 2x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

$$x_1 + x_2 + x_3 = 5$$
$$3x_1 + 2x_2 + x_4 = 12$$
$$\Rightarrow x_i \geq 0$$

$$x_3 = 5 - x_1 - x_2 \quad \text{—(1)}$$
$$x_4 = 12 - 3x_1 - 2x_2 \quad \text{—(2)}$$

0, 0 , 6, 5

$(5, 12, 0, 0)$

(x_3, x_4, x_1, x_2)

Basic Nonbasic

$z = 0$

$\min(5, 4)$

Basic Nonbasic

(x_1, x_3, x_2, x_4)

6 0 5 0

$(4, 1, 0, 0)$

$z = 24$

$$3x_1 = 12 - 2x_2 - x_4$$
$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 \quad \text{—(3)}$$
$$x_3 = 5 - 4 + \frac{2}{3}x_2 + \frac{1}{3}x_4 - x_2$$
$$\rightarrow x_3 = 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 \quad \text{—(4)}$$

$$z = 6x_1 + 5x_2$$
$$= 6\cdot\left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) + 5x_2$$
$$z = 24 + x_2 - 2x_4$$

$\min(6, 3)$

$$\frac{1}{3}x_2 = 1 - x_3 + \frac{1}{3}x_4 \Rightarrow x_2 = 3 - 3x_3 + x_4$$

$$x_1 = 4 - \frac{2}{3}(3 - 3x_3 + x_4) - \frac{1}{3}x_4$$

$$x_1 = 2 + 2x_3 - x_4$$

Basic		NonB	
x_1	x_2	x_3	x_4
2	3	0	0
6	5	0	0

$z = 27$

$z = 6x$

$$z = 24 + x_2 - 2x_4$$
$$= 24 + 3 - 3x_3 + x_4 - 2x_4$$
$$z = 27 - 3x_3 - x_4$$

$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ (2 & 3 & 0 & 0) \end{array}$ $z = 27$

Simplex Method.

	C_j	6	5	0	0		
C_j	Basic	x_1	x_2	x_3	x_4	b	θ
0	x_3	1	1	1	0	5	5/1
0	x_4	$\boxed{3}$	2	0	1	12	12/3 = 4 → leaving
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	6	5	0	0		
		most +ve entering					
0	x_3	0	$\boxed{1/3}$	1	-1/3	1	3
6	x_1	1	2/3	0	1/3	4	6
	Z_j	6	4	0	2	24	
	$C_j - Z_j$	0	1 most +ve	0	-2		

min (beside 5/1)

min → $R_1 - R_2'$

$\frac{1}{3} R_2$

			x_1	x_2	x_3	x_4	b
$3R_1$	5	x_2	0	1	3	-1	3
$R_2 - \frac{2}{3}R_1'$	6	x_1	1	0	-2	1	2
		Z_j	6	5	3	1	27
		$C_j - Z_j$	0	0	-3	-1	

3

Min $z = -x_1 - x_2$

s.t. $x_1 + x_2 \le 2$

$x_1 - x_2 \le 1$

$x_2 \le 1$

$x_1, x_2 \geq 0$

Max $w = x_1 + x_2$

s.t. $x_1 + x_2 + x_3 = 2$

$x_1 - x_2 + x_4 = 1$

$x_2 + x_5 = 1$

$x_i \geq 0 \ \forall \ i = 1:5$

	Cost	Basic	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	x_5 (0)	b	θ
	0	x_3	1	1	1	0	0	2	2
	0	x_4	$\boxed{1}$	−1	0	1	0	1	1 → leaving variable (min θ)
	0	x_5	0	1	0	0	1	1	
		z_j	0	0	0	0	0	0	
		$c_j - z_j$	1 ↑ max +ve (entering variable)	1	0	0	0		

	Cost	Basic	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	x_5 (0)	b	θ
$R_1 - R_2$	0	x_3	0	$\boxed{2}$	1	−1	0	1	1/2 → leaving
	1	x_1	1	−1	0	1	0	1	–
	0	x_5	0	1	0	0	1	1	1
		z_j	1	−1	0	1	0	1	
		$c_j - z_j$	0	2 ↑ entering	0	−1	0		

	Cost	Basic	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	x_5 (0)	b	θ
$\frac{1}{2}R_1$	1	x_2	0	1	1/2	−1/2	0	1/2	–
$R_2 + R_1$	1	x_1	1	0	1/2	1/2	0	3/2	3
$R_3 - R_1$	0	x_5	0	0	−1/2	$\boxed{1/2}$	1	1/2	1 →
		z_j	1	1	1	0	0	2	
		$c_j - z_j$	0	0	−1	0 ↑	0		

Optimal Sol^n $(3/2, 1/2, 0, 0, 1/2)$ ✓

Optimal Value $z = 2$

			1	1	0	0	0		θ
$R_1+\frac{1}{2}R_3$	1	x_2	0	1	0	0	1	1	
$R_2-\frac{1}{2}R_3$	1	x_1	1	0	1	0	-1	1	
$2R_3$	0	x_4	0	0	-1	1	2	1	
		z_j	1	1	1	0	0	2	
		c_j-z_j	0	0	-1	0	0		

Alternate $\rightarrow$ Optimal Sol^n :- $(1,1,0,1,0)$ ✓

Optimal Value :- 2

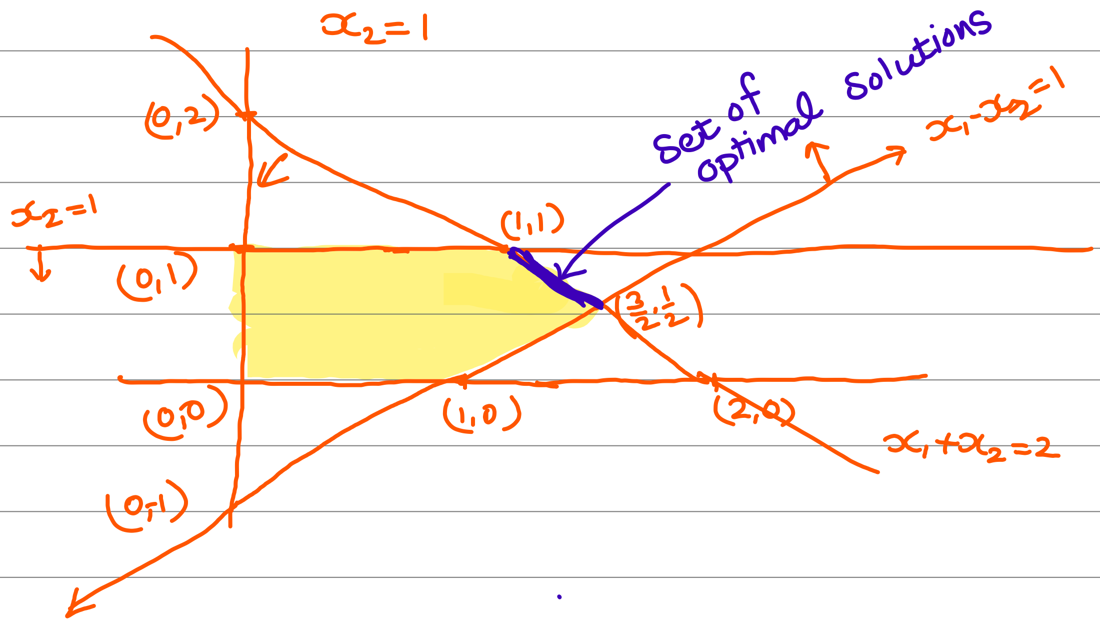
Min $z = -x_1 - x_2$

$x_1 + x_2 \leq 2$ $\quad\quad x_1 + x_2 = 2 \quad\quad (0,2),\ (2,0)$

$x_1 - x_2 \leq 1$ $\quad\quad x_1 - x_2 = 1 \quad\quad (0,-1),\ (1,0)$

$x_2 \leq 1$ $\quad\quad x_2 = 1$

$x_1, x_2 \geq 0$



Extreme pt.s.	Z value. min $z=-x_1-x_2$
(0,0)	0
(0,1)	-1
(1,1)	-2 min
$(\frac{3}{2},\frac{1}{2})$	-2

Two phase method :-

$$\text{Min } z = -2x_1 - x_2 \qquad \text{Max } -z = 2x_1 + x_2$$

s.t.

$x_1 + x_2 \geq 2$	$x_1 + x_2 - x_3 = 2$	$x_3 = -2$
$x_1 + x_2 \leq 4$	$x_1 + x_2 + x_4 = 4$	$x_4 = 4$
$x_1, x_2 \geq 0$	$x_i \geq 0 \ \forall i$	

After introducing artificial variable

First phase

$$\text{Min } \omega = a \rightarrow \text{Max } -\omega = -a$$

$x_1 + x_2 - x_3 + a = 2 \Rightarrow a = 2$ ✓

$x_1 + x_2 + x_4 = 4 \Rightarrow x_4 = 4$ ✓

$x_i \geq 0 \quad a \geq 0$

	c_j	0	0	0	0	-1		
	j	x_1	x_2	x_3	x_4	a	b	θ
-1	a	1	1	-1	0	1	2	2 →
0	x_4	1	1	0	1	0	4	4
	z_j	-1	-1	1	0	-1		
	$c_j - z_j$	1 ↑	1	-1	0	0		

Second phase

		Coeff c_j	2	1	0	0		
	2	x_1	1	1	-1	0	2	–
$R_2 - R_1$	0	x_4	0	0	1	1	2	2 →
		z_j	2	2	-2	0	4	
		$c_j - z_j$	0	-1	2 ↑	0		
$R_1 + R_2$	2	x_1	1	1	0	1	4	
	0	x_3	0	0	1	1	2	
		z_j	2	2	0	2	8	
		$c_j - z_j$	0	-1	0	-2		

$(x_1, x_2) = (4, 0)$

$z = -8$

Min $z = x_1 + x_2$

S.t. $x_1 + 2x_2 \le 2$ | $x_1 + 2x_2 + x_3 = 2$

$3x_1 + 5x_2 \ge 15$ | $3x_1 + 5x_2 - x_4 + a = 15$

$x_1, x_2 \ge 0$ | $x_i \ge 0 \ \forall i, \ a \ge 0$

Phase-I Min a

		0	0	0	0	-1		
		x_1	x_2	x_3	x_4	a	b	θ
0	x_3	1	2	1	0	0	2	1 →
-1	a	3	5	0	-1	1	15	3
	Z_j	-3	-5	0	1	-1	-15	
	$C_j - Z_j$	3	5 ↑	0	-1	0		

			x_1	x_2	x_3	x_4	a	b	θ
$\frac{1}{2}R_1$	0	x_2	1/2	1	1/2	0	0	1	2 →
$R_2 - 5R_1$	-1	a	1/2	0	-5/2	-1	1	10	20
		Z_j	-1/2	0	5/2	1	-1	-10	
		$C_j - Z_j$	1/2 ↑	0	-5/2	-1	0		

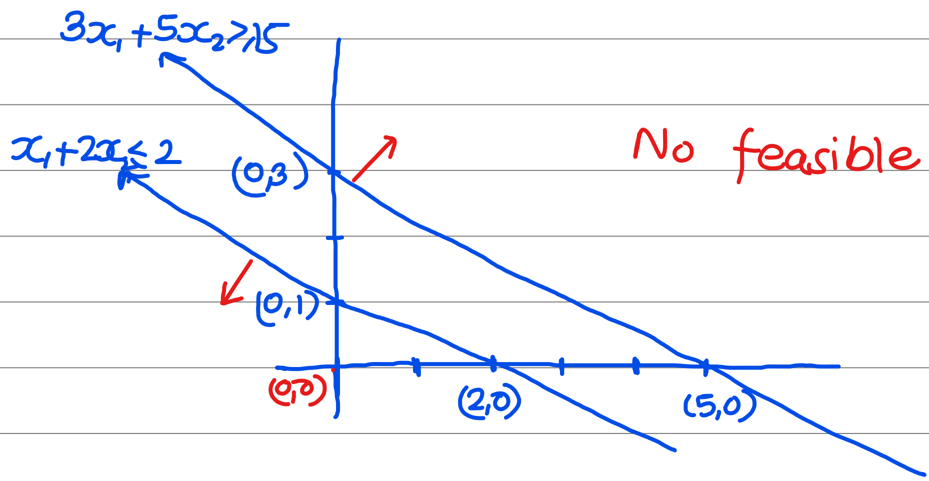
			x_1	x_2	x_3	x_4	a	b
$2R_1$	0	x_1	1	2	1	0	0	2
$R_2 - \frac{1}{2}R_1$	-1	a	0	-1	-3	-1	1	9
		Z_j	0	1	3	1	-1	
		$C_j - Z_j$	0	-1	-3	-1	0	

As in the final sol^n of Phase-I artificial variable is still positive $(a = 9)$

$\Rightarrow$ The LPP is infeasible.

Min $z = x_1 + x_2$

$x_1 + 2x_2 \leq 2$	$x_1 + 2x_2 = 2$	$(0,1)\ (2,0)$
$3x_1 + 5x_2 \geq 15$	$3x_1 + 5x_2 = 15$	$(0,3)\ (5,0)$
$x_1, x_2 \geq 0$		



No feasible region/ sol^n

Imp

Min $z = 2 - x_2$

$x_1 - x_2 = 4$	$x_1 - x_2 + a_1 = 4$	$x_1 = 4$
$-x_2 - x_3 = 0$	$x_2 + x_3 + a_2 = 0$	$a_2 = 0$

Min $a_1 + a_2$

Direct second phase.

		0	1	0	0	0		
		x_1	x_2	x_3	a_1	a_2	b	
0	x_1	1	-1	0	1	0	4	–
0	x_3	0	1	1	0	1	0	0 →
	z_j	0	0	0				
	$c_j - z_j$	0	1 ↑	0				

$R_1 + R_2$

		0	1	0	
		x_1	x_2	x_3	b
0	x_1	1	0	1	4
1	x_2	0	1	1	0
	z_j	0	1	1	
	$c_j - z_j$	0	0	-1	

$x_2 = 0$

$z = 2$

?

Big-M   Charne's M Technique

$$\text{Min } Z = 4x_1 + 8x_2 + 3x_3 + Ma_1 + Ma_2$$

$x_1 + x_2 \geq 2$ | $x_1 + x_2 - x_4 + a_1 = 2$
$2x_2 + x_3 \geq 5$ | $2x_2 + x_3 - x_5 + a_2 = 5$
$x_i \geq 0 \ \forall i$ | $x_i \geq 0 \ \forall i \quad a_i \geq 0$

			-4	-8	-3	0	0	$-M$	$-M$		
			x_1	x_2	x_3	x_4	x_5	a_1	a_2	b	θ
	$-M$	a_1	1	1	0	-1	0	1	0	2	2 →
	$-M$	a_2	0	2	1	0	-1	0	1	5	$5/2$
		z_j	$-M$	$-3M$	$-M$	M	M	$-M$	$-M$		
		$c_j - z_j$	$M-4$	$3M-8$ ↑	$M-3$	$-M$	$-M$	0	0		
	-8	x_2	1	1	0	-1	0		0	2	$-$
$R_2 - 2R_1$	$-M$	a_2	-2	0	1	2	-1		1	1	1
		z_j	$2M-8$	-8	$-M$	$8-2M$	M		$-M$		
		$c_j - z_j$	$4-2M$	0	$M-3$	$2M-8$ ↑	$-M$		0		

			-4	-8	-3	0	0		
$R_1 + R_2$	-8	x_2	0	1	$1/2$	0	$-1/2$	$5/2$	5
$\frac{1}{2}R_2$	0	x_4	-1	0	$1/2$	1	$-1/2$	$1/2$	1 →
		z_j	0	-8	-4	0	4	-20	
		$c_j - z_j$	-4	0	1 ↑	0	-4		
$R_1 - \frac{1}{2}R_2$	-8	x_2	1	1	0	-1	0	2	
$2R_2$	-3	x_3	-2	0	1	2	-1	1	
		$c_j - z_j$	-6	0	0	-2	-3		

$$\text{Min } Z = -x_1 + x_2 + Ma_1 + Ma_2 \qquad \text{Max } z = x_1 - x_2 - Ma_1 - Ma_2$$

s.t. $x_1 - 2x_2 - x_3 = 1$ $\quad x_1 - 2x_2 - x_3 + a_1 = 1$

$-x_1 + 2x_2 - x_4 = 1$ $\quad -x_1 + 2x_2 - x_4 + a_2 = 1$

$x_i \geq 0 \ \forall i$ $\quad x_i \geq 0 \quad a_i \geq 0 \ \forall i$

	Basic	1	-1	0	0	-M	-M		
		x_1	x_2	x_3	x_4	a_1	a_2	b	θ
-M	a_1	1	-2	-1	0	1	0	1	1
-M	a_2	-1	2	0	-1	0	1	1	–
	z_j	0	0	M	M	-M	-M		
	$c_j - z_j$	1	-1	-M	-M	0	0		
1	x_1	1	-2	-1	0		0	1.	–
R_2+R_1 -M	a_2	0	0	-1	-1		1	2	–
	z_j	1.	-2	-1+M	M		-M		
	$c_j - z_j$	0	1	1-M	-M		0		

Unbounded solⁿ

$$\min \ Z = -x_1 + x_2$$

$x_1 - 2x_2 - x_3 = 1$ $\quad x_1 - 2x_2 \geq 1$

$-x_1 + 2x_2 - x_4 = 1$ $\quad -x_1 + 2x_2 \geq 1$

$x_1 - 2x_2 \leq -1$

Dual

$$3y_1 + 5y_2 \geq 6$$
$$2y_1 + 6y_2 \geq 5$$
$$\text{Min } 18y_1 + 25y_2$$

$$\text{Max } 6x_1 + 5x_2$$

y_1	$3x_1 + 2x_2 \leq 18$	$\times 3$	$\times 1$	0
y_2	$5x_1 + 6x_2 \leq 25$	$\times 0$	$\times 1$	$\times 6/5$
	$x_1, x_2 \geq 0$			

$$6x_1 + 5x_2 \leq 9x_1 + 6x_2 \leq 54 \qquad z \leq 54$$

$$\leq 8x_1 + 8x_2 \leq 43 \qquad z \leq 43$$

$$\frac{6}{5}x_1 \cancel{5} + \frac{36}{5}x_2 \leq 30 \qquad z \leq 30$$

$$\left[\begin{array}{ll} \text{Max } z = \underline{c}'\underline{x} & \text{Min } \omega = \underline{b}'\underline{y} \\ A\underline{x} \leq \underline{b} & A'\underline{y} \geq \underline{c} \\ x \geq 0 & \underline{y} \geq 0 \end{array}\right]$$

SLP DSLP

		SLP	DSLP
①	Constraint	$A_{m \times n}$	$A'_{n \times m}$
②	Constant	$\underline{b}$	$\underline{c}$
③	No. of variables	n	m
	No. of constraints	m	n

④ Constraints $\leq$ Variables ≥ 0

$\geq$ ≤ 0

$=$ unrestricted

$y_1 \leftarrow x_1 + x_2 = 2$

$\mapsto x_1 + x_2 \leq 2 \rightarrow y_1' \rightarrow y_1$

$\mapsto x_1 + x_2 \geq 2 \rightarrow -y_1''$

$$y_1 = \underset{\geq 0}{y_1'} - \underset{\geq 0}{y_1''}.$$

---

⑤ Variable $\underline{\underline{x_i \geq 0}}$ Constraint $\geq$

$\underline{\underline{x_i \leq 0}}$ $\leq$

unrestricted x_i $=$

Dual of dual is primal.

SLP Max $z = \underline{c}'\underline{x}$

$A\underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

DSLP Min $w = \underline{b}'\underline{y}$

$A'\underline{y} \geq \underline{c}$ $\underline{y} \geq 0$

D(DSLP) Max $v = c'\underline{u}$

$(A')'\underline{u} \leq (b')' \Rightarrow A\underline{u} \leq \underline{b}$

$\underline{u} \geq \underline{0}$

Hence Proved.

SLP Max $z = \underline{c}'\underline{x} = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$A\underline{x} \leq \underline{b}$

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \longrightarrow y_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \longrightarrow y_2$

$\vdots$

$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n = b_k$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \longrightarrow y_m$

convert this equality in less than type.

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \rightarrow y_1$

$\vdots$

$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \leq b_k \longrightarrow y_k^1$

$\vdots$

$-a_{k1}x_1 - a_{k2}x_2 - \ \ - a_{kn}x_n \leq -b_k \rightarrow y_k^2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \rightarrow y_m$

Dual :-

Min $\omega = b_1y_1 + b_2y_2 + \ldots + b_k(y_k^1 - y_k^2) + \ldots + b_my_m$

$a_{11}y_1 + a_{21}y_2 + \ldots + a_{k1}(y_k^1 - y_k^2) + \ldots + a_{m1}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \ldots + a_{k2}(y_k^1 - y_k^2) + \ldots + a_{m2}y_m \geq c_2$

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{kn}(y_k^1 - y_k^2) + \ldots + a_{mn}y_m \geq c_m$

$y_i \geq 0 \ \forall i \qquad y_k^1, y_k^2 \geq 0$

lets define $y_K = y_K^1 - y_K^2$
as $y_K^1 \geq 0$ & $y_K^2 \geq 0 \Rightarrow y_K$ is unrestricted in sign.

$x_p \to$ unrestricted. $\to$ p^{th} constraint

Max $c_1x_1 + c_2x_2 + \dots + c_px_p + \dots + c_nx_n$

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1p}x_p + \dots + a_{1n}x_n \leq b_1$

$\vdots$

$a_{m1}x_1 + \quad + \quad + a_{mp}x_p + \quad a_{mn}x_n \leq b_m$

$\underline{\underline{x_i \geq 0}}$ $\forall i$ except p

x_p unrestricted

let $x_p = x_p^1 - x_p^2$

Max $c_1x_1 + \dots + c_p(x_p^1 - x_p^2) + \dots + c_nx_n$

$y_1 -$ $a_{11}x_1 + a_{12}x_2 + \dots \quad + \underline{a_{1p}(x_p^1 - x_p^2)} + \dots \quad \leq b_1$

$\vdots$

$y_m -$ $a_{m1}x_1 + \dots \quad + \dots \quad + a_{mp}(x_p^1 - x_p^2) \quad + \dots \quad \leq b_m$

$\underline{\underline{x_i \geq 0}}$ $\forall i$ $\quad x_p^1, x_p^2 \geq 0$

Dual
Min

$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geq c_1$

$\vdots$

$\left.\begin{array}{l} a_{1p}y_1 + a_{2p}y_2 + \dots + a_{mp}y_m \geq c_p \\ -a_{1p}y_1 - a_{2p}y_2 - \dots - a_{mp}y_m \geq -c_p \end{array}\right]$ —— *

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n$

$\therefore$ from $\underline{\underline{*}}$

- Any feasible solution to primal (SLP) has value z greater than or at least equal to the value v for any feasible solution to dual (DSLP).

SLP Min $z = \underline{c}'\underline{x}$
$A\underline{x} \geq \underline{b}$
$\underline{x} \geq \underline{0}$

DSLP Max $v = \underline{b}'\underline{y}$
$A'\underline{y} \leq \underline{c}$
$\underline{y} \geq \underline{0}$

Let x_0 be any feasible sol^n to primal $\Rightarrow \underline{x}_0 \geq \underline{0}$
let y_0 be any feasible sol^n to dual

$$z_0 = \underline{c}'\underline{x}_0 \geq (A'\underline{y}_0)'\underline{x}_0 = \underline{y}_0'A\underline{x}_0 \geq \underline{y}_0'\underline{b} = \underline{b}'\underline{y}_0 = v_0$$

Alternative.

SLP Max $z = \underline{c}'\underline{x}$
$A\underline{x} \leq \underline{b}$
$\underline{x} \geq 0$

DSLP Min $v = \underline{b}'\underline{y}$
$A'\underline{y} \leq \underline{c}$
$\underline{y} \geq \underline{0}$

Let x_0 & y_0 be any feasible sol^ns to SLP & DSLP resp.

$$z_0 = \underline{c}'\underline{x}_0 \leq (A'\underline{y}_0)'\underline{x}_0 = \underline{y}_0'A\underline{x}_0 \leq \underline{y}_0'\underline{b} = \underline{b}'\underline{y}_0 = v_0$$

→ If SLP min $\Rightarrow$ min $z \geq$ max v

if SLP max $\Rightarrow$ min $v \geq$ max z

$\underline{x}_0$ & $\underline{y}_0$ are feasible sol^n to SLP & DSLP resp.
(min) (max)

with $\underline{c}'\underline{x}_0 = \underline{b}'\underline{y}_0$

weak duality then $\Rightarrow$ $\underline{c}'\underline{x} \geq \underline{b}'\underline{y}$ $\forall \underline{x}$ $\forall \underline{y}$.

$\underline{b}'\underline{y} \leq \underline{c}'\underline{x}$

as $\underline{x}_0$ is feasible sol^n

$\Rightarrow \quad \underline{b}'\underline{y} \leq \underline{c}'\underline{x}_0 \quad \forall \underline{y}$

$\Rightarrow \max_{\underline{y}} \underline{b}'\underline{y} \leq \underline{c}'\underline{x}_0 = \underline{b}'\underline{y}_0$ $\Rightarrow$ y_0 is optimal sol^n to DSLP

Similarly we can obtain that x_0 is O.S. to SLP

$\underline{b}'\underline{y} \leq \underline{c}'\underline{x} \quad \forall \underline{y} \ \forall \underline{x}$

as y_0 is feasible sol^n to DSLP

$\underline{b}'\underline{y}_0 \leq \underline{c}'\underline{x} \quad \forall \underline{x}$

$\underline{c}'\underline{x}_0 = \underline{b}'\underline{y}_0 \leq \min_{\underline{x}} \underline{c}'\underline{x}$ $\Rightarrow$ x_0 is optimal.

min $\quad x_1 - x_2 \rightarrow -\infty$

∞

$c'x \geq b'y \leftarrow \infty.$

$\min c'x \geq \max b'y \rightarrow$ infeasible.

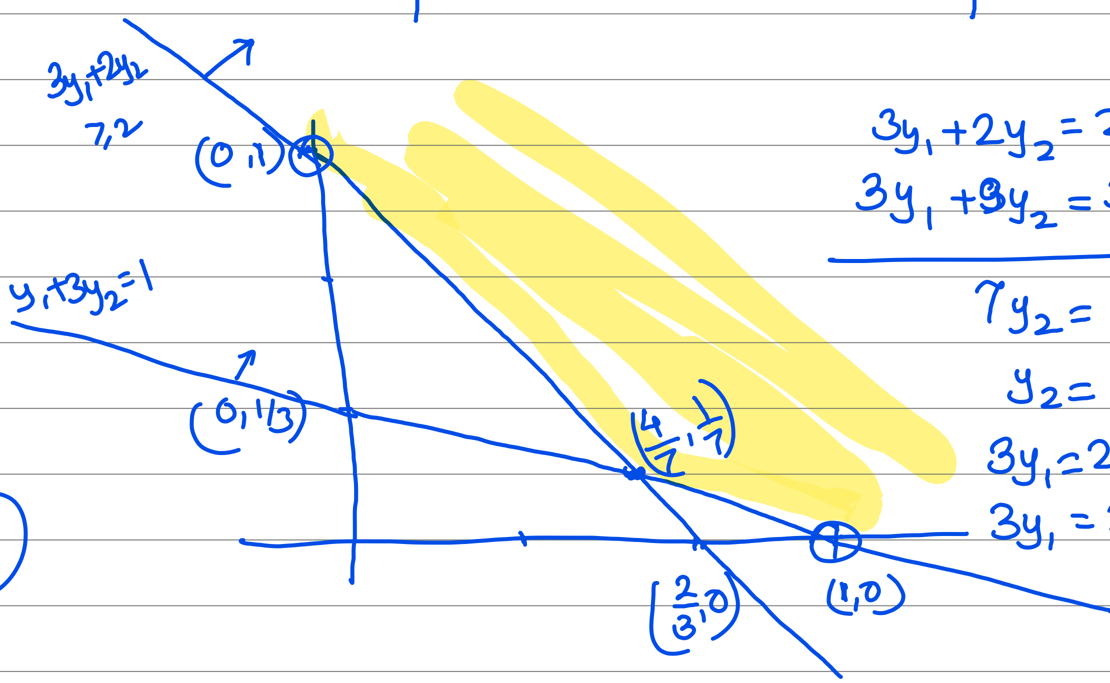
$-\infty \rightarrow$

Solve the following problem ~~by~~ simplex

Max $z = 2x_1 + x_2$	Min $v = 32y_1 + 30y_2$	$(0,1)$ $(2/3, 0)$
$3x_1 + x_2 \leq 32$	$3y_1 + 2y_2 \geq 2$	$(0, 1/3)$ $(1,0)$
$2x_1 + 3x_2 \leq 30$	$y_1 + 3y_2 \geq 1$	
$x_1, x_2 \geq 0$	$y_1, y_2 \geq 0$	

Last table

		2	1	0	0	
	Basic	x_1	x_2	x_3	x_4	b
2	x_1	1	0	3/7	−1/7	66/7
1	x_2	0	1	−2/7	3/7	26/7
	$c_j - z_j$	0	0	−4/7	−1/7	158/7



$$3y_1 + 2y_2 = 2$$
$$3y_1 + 9y_2 = 3$$

$$7y_2 = 1$$
$$y_2 = 1/7$$
$$3y_1 = 2 - 2y_2$$
$$3y_1 = 2 - \frac{2}{7} = \frac{12}{7}$$

	32	30
$(0,1)$		30
$(1,0)$		32
$(\frac{4}{7}, \frac{1}{7})$		$\frac{158}{7}$

Dual Sim

Min $2x_1 + x_2$ / Max $z = -2x_1 - x_2$

$x_1 - x_2 \geq 2$ $\qquad x_1 - x_2 - x_3 = 2 \quad (-1)$

$x_1 + x_2 \leq 4$ $\qquad x_1 + x_2 + x_4 = 4$

$x_1, x_2 \geq 0$ $\qquad x_i \geq 0$

		-2	-1	0	0		
	Basic	x_1	x_2	x_3	x_4	b	
0	x_3	$\boxed{-1}$	+1	+1	0	-2	leaving $\rightarrow$
0	x_4	1	1	0	1	4	
	$C_j - Z_j$	-2 $\uparrow$	-1	0	0		
	θ	2	–	–	–		

$-R_1$	-2	x_1	1	-1	-1	0	2
$R_2 - R_1$	0	x_4	0	2	1	1	2
		$C_j - Z_j$	0	-3	-2	0	

Min $Z = -2x_1 - x_2 - x_3$

$4x_1 + 6x_2 + 3x_3 \leq 8$

$x_1 - 9x_2 + x_3 \leq -3$

$-2x_1 - 3x_2 + 5x_3 \leq -4$

Max $Z = 2x_1 + x_2 + x_3$

$4x_1 + 6x_2 + 3x_3 + x_4 = 8$

$x_1 - 9x_2 + x_3 + x_5 = -3$

$-2x_1 - 3x_2 + 5x_3 + x_6 = -4$

$\ni i \quad x_i \geq 0$

		2	1	1	0	0	0		
		x_1	x_2	x_3	x_4	x_5	x_6	b	
0	x_4	$\boxed{4}$	6	3	1	0	0	8	2 →
0	x_5	1	-9	1	0	1	0	-3	-
0	x_6	-2	-3	5	0	0	1	-4	-
	$C_j - Z_j$	2 ↑	1	1	0	0	0		

$\frac{1}{4}R_1$	2	x_1	1	3/2	3/4	1/4	0	0	2	leaving
$R_2 - R_1$	0	x_5	0	$\boxed{-21/2}$	1/4	-1/4	1	0	-5	→
$R_3 + 2R_1$	0	x_6	0	0	13/2	1/2	0	1	0	
		Z_j	2	3	3/2	1/2	0	0	4	
		$C_j - Z_j$	0	-2 ↑	-1/2	-1/2	0	0		
		θ	-	4/21	-	2				

$R_1 - \frac{3}{2}R_2$	2	x_1	1	0	11/14	3/14	1/7	0	9/7	-
$-\frac{2}{21}R_2$	1	x_2	0	1	-1/42	1/42	-2/21	0	10/21	-
	0	x_6	0	0	13/2	1/2	0	1	0	
		Z_j	2	1	65/42	19/42	4/21	0		
		$C_j - Z_j$	0	0	-23/42	-19/42	-4/21	0		

Branch & Bound Method

LP1

Max $z = 3x_1 + 2x_2$

S.t. $2x_1 - x_2 \le 6$

$x_1 + 3x_2 \le 15$

$x_1, x_2 \geq 0$ $\boxed{\text{\& integers}}$

$6x_1 - 3x_2 = 18$

$x_1 + 3x_2 = 15$

$7x_1 \qquad 33$

$x_1 = \frac{33}{7}$ $x_2 = \frac{24}{7}$

LP2 Max $z = 3x_1 + 2x_2$

$2x_1 - x_2 \le 6$

$x_1 + 3x_2 \le 15$

$x_1 \le 4$

$x_i \geq 0$

LP3 Max $z = 3x_1 + 2x_2$

$2x_1 - x_2 \le 6$

$x_1 + 3x_2 \le 15$

$x_1 \geq 5$

$x_i \geq 0$

LP4 Max $z = 3x_1 + 2x_2$

$2x_1 - x_2 \le 6$

$x_1 + 3x_2 \le 15$

$x_1 \le 4$

$x_2 \le 3$

LP5 Max $z = 3x_1 + 2x_2$

$2x_1 - x_2 \le 6$

$x_1 + 3x_2 \le 15$

$x_1 \le 4$

$x_2 \geq 4$

LP1

$x_1 = \frac{33}{7}$ $\quad x_2 = \frac{24}{7}$ $\quad z = \frac{123}{7}$

$x_1 \le 4$ $\qquad\qquad$ $x_1 \ge 5$

LP2 $\qquad\qquad$ LP3

$x_1 = 4$ $\quad x_2 = \frac{11}{3}$ $\quad z = \frac{47}{3}$

Infeasible

~~$x_1 =$ $x_2 =$ $z =$~~

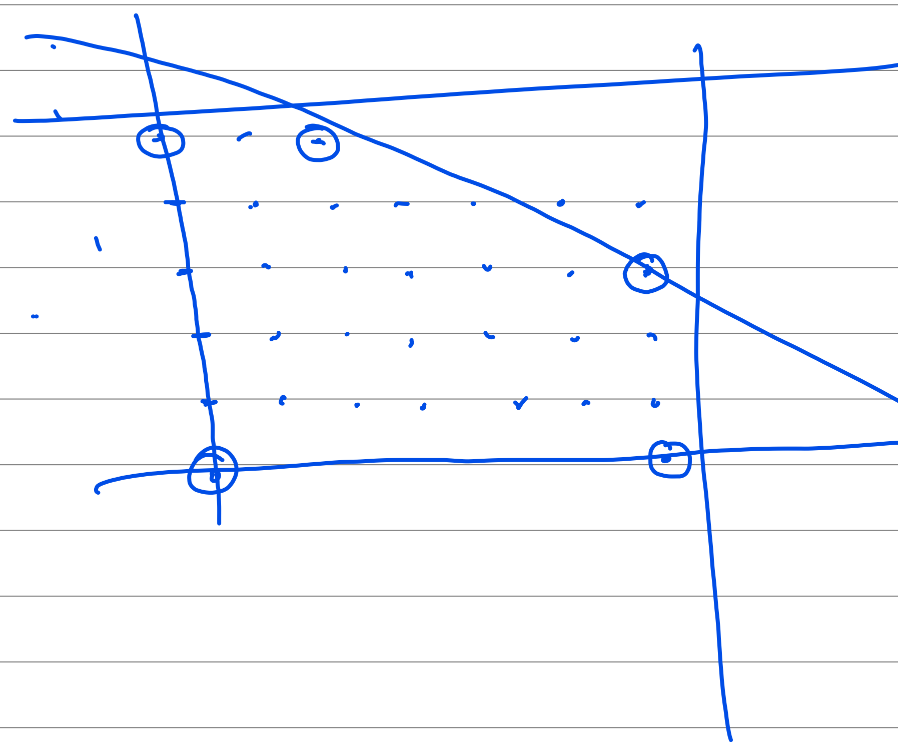
$x_2 \le 3$ $\qquad\qquad$ $x_2 \ge 4$

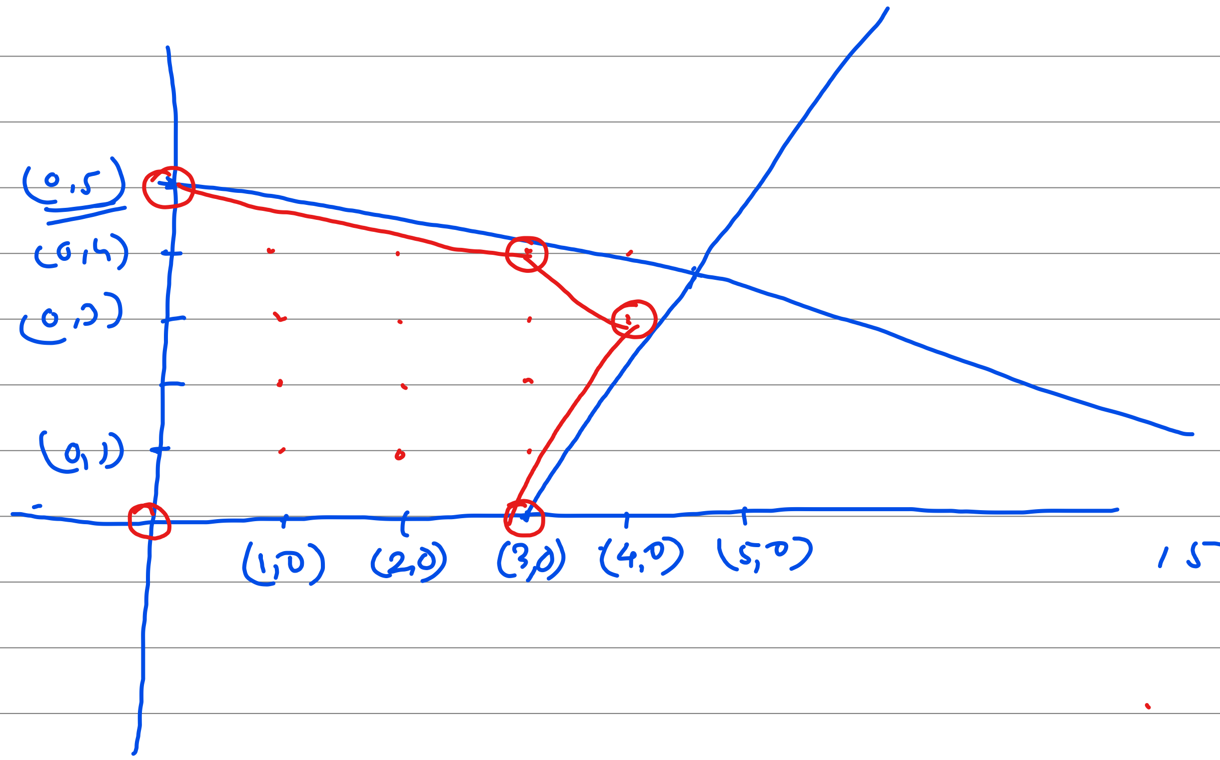
LP4 $\qquad\qquad$ LP5

$x_1 = 4$ $\qquad\qquad$ $x_1 =$

$x_2 = 3$ $\qquad\qquad$ $x_2 =$

$z = 18$ $\qquad\qquad$ $z =$




(0,5)
(0,4)
(0,3)
(0,1)
(1,0)
(2,0)
(3,0)
(4,0)
(5,0)

Non-linear programming

Quadratic Programming

$$\text{Min} \quad z = \underbrace{\underline{c}'\underline{x}} + \frac{1}{2}\underbrace{\underline{x}'D\underline{x}}$$

linear [$g(\underline{x}) \geq 0$

Max $z = \underline{c}'\underline{x} + \underline{x}'G\underline{x}$

$g(\underline{x}) \leq \underline{0}$]

20% 30% 50% =100

Max R_p $\omega_1 R_1 + \omega_2 R_2 + \omega_3 R_3$ $-$

Max $V(\underline{\omega}'\underline{R})$ $-\omega'\Sigma\omega$

$$\text{Max} \quad (R_p) - V(R_p)$$

4

$\omega_1 + \omega_2 + \omega_3 = 1$

$\omega_1 \geq 0 \quad \omega_2 \geq 0 \quad \omega_3 \geq 0$

$$\text{Max} \quad z = \boxed{\underline{c}'\underline{x} + \frac{1}{2}\underline{x}'D\underline{x}}$$ ← n- no. of unknown variables

$$\boxed{\underline{g}(\underline{x}) \leq \underline{0}} \Rightarrow \underline{g}(\underline{x}) + \underline{s}^2 = 0$$ ← m constraints

$$L = \underline{c}'\underline{x} + \frac{1}{2}\underline{x}'D\underline{x} - \underline{\lambda}\left(\underline{g}(\underline{x}) + \underline{s}^2\right)$$

w.r.t. $\underline{x} \Rightarrow$ $\quad c' + \frac{1}{\cancel{2}}\cancel{2}D\underline{x} - \lambda g'(\underline{x}) = 0$

w.r.t. $\underline{\lambda} \Rightarrow$ $\quad g(x) + s^2 = 0$

w.r.t. $s \Rightarrow$ $\quad 2\lambda s = 0 \quad \Rightarrow \quad \lambda s = \underline{0}$

} Kuhn-Tucker Conditions.

diag

$$\text{Max} \quad z = \underline{c}'\underline{x} + \frac{1}{2}\underline{x}'D\underline{x}$$

$g(x) \leq 0$

$\lambda_{m\times 1}$, $\underline{u}_{n\times 1}$

$m \rightarrow A\underline{x} \leq \underline{b}$

$n \rightarrow \underline{x} \geq \underline{0}$

$A\underline{x} - \underline{b} + s^2 = 0$

$-\underline{x} + \underline{y}^2 = 0$

$$\begin{bmatrix} A \\ -I \end{bmatrix}\underline{x} - \begin{bmatrix} \underline{b} \\ 0 \end{bmatrix}$$

$$L = \underline{c}'\underline{x} + \frac{1}{2}x'Dx - \lambda(Ax - b + s^2) - u(-x + y^2)$$

w.r.t. $\underline{x} \Rightarrow$ $\boxed{c + D\underline{x} - A'\lambda + \underline{u} = 0}$

w.r.t. $\lambda \Rightarrow$ $Ax - b + s^2 = 0$ $\Rightarrow$ $A\underline{x} \leq b$

w.r.t. $\underline{u} \Rightarrow$ $-x + y^2 = 0$ $\Rightarrow$ $\underline{x} \geq \underline{0}$

w.r.t. $s \Rightarrow$ $\lambda s = \underline{0}$ $\Rightarrow$ $\lambda_i s_i = 0 \quad \forall i$

w.r.t. $y \Rightarrow$ $\underline{u}y = \underline{0}$ $\Rightarrow$ $u_i y_i = 0 \quad \forall i$

K-T conditions

$u_i x_i = 0$

$$\text{Min} \quad z = -10x_1 - 25x_2 + 10x_1^2 + x_2^2 + 4x_1x_2$$

s.t. $x_1 + 2x_2 \leq 10$

$x_1 + x_2 \leq 9$

$x_1, x_2 \geq 0$

$$\text{Max} \quad 10x_1 + 25x_2 \underbrace{-10x_1^2 - x_2^2 - 4x_1x_2}$$

$$\text{Max} \quad \underline{c}'\underline{x} + \frac{1}{2}\underline{x}'\underline{D}\underline{x} \quad \Rightarrow c' = [10 \;\; 25]$$

$$\frac{1}{2}D = \begin{bmatrix} -10 & -2 \\ -2 & -1 \end{bmatrix} \Rightarrow D = \begin{bmatrix} -20 & -4 \\ -4 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 25 \end{bmatrix} + \begin{bmatrix} -20 & -4 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Min $z = a_1 + a_2$

$$20x_1 + 4x_2 + \lambda_1 + \lambda_2 - \mu_1 + a_1 = 10$$
$$4x_1 + 2x_2 + 2\lambda_1 + \lambda_2 - \mu_2 + a_2 = 25$$
$$x_1 + 2x_2 + s_1 = 10$$
$$x_1 + x_2 + s_2 = 9$$
$$x_i \geq 0, \quad \lambda_i \geq 0, \quad \mu_i \geq 0, \quad s_i \geq 0$$

$$\lambda_i s_i = 0 \qquad \mu_i x_i = 0 \qquad \forall i$$

Coeff.	Basic	x_1 (0)	x_2 (0)	λ_1 (0)	λ_2 (0)	μ_1 (0)	μ_2 (0)	S_1 (0)	S_2 (0)	a_1 (-1)	a_2 (-1)	b	θ
-1	a_1	$\boxed{20}$	4	1	1	-1	0	0	0	1	0	10	1/2
-1	a_2	4	2	2	1	0	-1	0	0	0	1	25	25/4
0	s_1	1	2	0	0	0	0	1	0	0	0	10	10
0	s_2	1	1	0	0	0	0	0	1	0	0	9	9
	Z_j	-24	-6	-3	-2	1	1	0	0	-1	-1		
	$C_j - Z_j$	24 ↑	6	3	2	-1	-1	0	0	0	0		

Coeff.	Basic	x_1	x_2	λ_1	λ_2	μ_1	μ_2	S_1	S_2	a_2	b
$\frac{R_1}{20}$ 0	x_1	1	1/5	1/20	1/20	-1/20	0	0	0	0	1/2
R_2-4R_1 -1	a_2	0	6/5	9/5	4/5	1/5	-1	0	0	1	23
R_3-R_1 0	s_1 ←	0	9/5	-1/20	-1/20	1/20	0	1	0	0	19/2
R_4-R_1 0	s_2	0	4/5	-1/20	-1/20	1/20	0	0	1	0	17/2
	Z_j	0	-6/5	-9/5	-4/5	-1/5	1	0	0	-1	-23
	$C_j - Z_j$	0	6/5 ↑	9/5	4/5	1/5	-1	0	0	0	

Coeff.	Basic	x_1	x_2	λ_1	λ_2	μ_1	μ_2	S_1	S_2	a_2	b
0	x_2	5	1	1/4	1/4	-1/4	0	0	0	0	5/2
-1	a_2	-6	0	3/2	1/2	1/2	-1	0	0	1	20
0	S_1	-9	0	-1/2	-1/2	1/2	0	1	0	0	5
0	S_2	-4	0	-1/4	-1/4	1/4	0	0	1	0	13/2
	Z_j	6	0	-3/2	-1/2	-1/2	1	0	0	0	-20
	$C_j - Z_j$	-6	0	3/2 ↑	1/2 ↑	1/2 ↑	-1	0	0		

Coeff.	Basic	x_1	x_2	λ_1	λ_2	μ_1	μ_2	S_1	S_2	a_2	b	θ
	x_2	1/2	1	0	0	0	0	1/2	0	0	5	-
-1	a_2	3	0	2	1	0	-1	-1	0	1	15	15/2 →
	μ_1	-18	0	-1	-1	1	0	2	0	0	10	-
	S_2	1/2	0	0	0	0	0	-1/2	1	0	4	-
	Z_j	-3	0	-2	-1	0	1	1	0		-15	
	$C_j - Z_j$	3 ↑	0	2 ↑	1	0	-1	-1	0			

	x_1	x_2	λ_1	λ_2	μ_1	μ_2	S_1	S_2	
x_2	$1/2$	1	0	0	0	0	$1/2$	0	5
λ_1	$3/2$	0	1	$1/2$	0	$-1/2$	$-1/2$	0	7.5
μ_1	$33/2$	0	0	$-1/2$	1	$-1/2$	$3/2$	0	17.5
S_2	$1/2$	0	0	0	0	0	$-1/2$	1	4

$$x_1 = 0, \quad x_2 = 5, \quad Z \Rightarrow -25x_2 + x_2^2$$
$$= -125 + 25$$
$$= -100$$

HW

$$\text{Min } Z = x_1^2 - x_1x_2 + 2x_2^2 - x_1 - x_2$$

s.t.

$$2x_1 + x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Wolfe's Algorithm

Beale's Algorithm

$$\text{Min } f(\underline{x}) = \underline{c}'\underline{x} + \frac{1}{2}\underline{x}'G\underline{x}$$

S.t. $A\underline{x} = \underline{b}$

$\underline{x} \geq \underline{0}$ $\quad \underline{x}_{n\times1}, \underline{b}_{m\times1}, \underline{c}_{n\times1}, A_{m\times n}$

$G_{n\times n}$ symmetric

$$\text{Min } Z = -10x_1 - 25x_2 + 10x_1^2 + 4x_1x_2 + x_2^2$$

S.t. $x_1 + 2x_2 \leq 10 \Rightarrow x_1 + 2x_2 + x_3 = 10$

$x_1 + x_2 \leq 9 \qquad x_1 + x_2 + x_4 = 9$

$x_1, x_2 \geq 0 \qquad x_i \geq 0$

basic	x_1	x_2	x_3	x_4	b	θ
x_3	1	2	1	0	10	10/1
x_4	1	1	0	1	9	9/1

$$f(\underline{x}) = -10x_1 - 25x_2 + 10x_1^2 + 4x_1x_2 + x_2^2$$

$\underline{c'x} + x'Dx$

$$f(\underline{x}) = [-10 \ -25]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [x_1 \ x_2]\begin{bmatrix} 10 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$c = \begin{array}{cc} x_1 & x_2 \\ (-10, & -25) \end{array} \qquad D = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & 10 & 2 \\ x_2 & 2 & 1 \end{array}$$

$$x_1^{(1)} = \min\left(\frac{10}{1}, \frac{9}{1}\right) = 9 \qquad\qquad x_1^{(2)} = \frac{-\alpha_1}{2d_{11}} = \frac{-(-10)}{2\times 10} = \frac{1}{2}$$

Criterion 1 $\qquad$ ✓Criterion 2 as $x_1^{(2)}$ is min^m

$$u_1 = \frac{df}{dx_1} = -10 + \underbrace{20x_1 + 4x_2}$$

$$-20x_1 - 4x_2 + u_1 = -10$$

	x_1	x_2	x_3	x_4	u_1	b
x_3	1	2	1	0	0	10
x_4	1	1	0	1	0	9
u_1	$\boxed{-20}$	-4	0	0	1	-10 →

x_1 enters

		x_1	x_2	x_3	x_4	u_1	b	
$R_1 - R_3$	x_3	0	$9/5$	1	0	$1/20$	$19/2$	$\frac{19\times 5}{2\times 9} =$
$R_2 - R_3$	x_4	0	$4/5$	0	1	$1/20$	$17/2$	
$-\frac{1}{20}R_3$	x_1	1	$1/5$	0	0	$-1/20$	$1/2$	→

$$\overset{\text{Basic}}{(x_3, x_4, x_1} ; \overset{\text{NB}}{x_2\, u_1)} = \overset{\text{Basic}}{(19/2, 17/2, 1/2} ; \overset{\text{NB}}{0, 0)}$$

$$f(\underline{x}) = -10x_1 - 25x_2 + 10x_1^2 + 4x_1x_2 + x_2^2$$

$$x_1 + \frac{x_2}{5} - \frac{u_1}{20} = \frac{1}{2} \Rightarrow x_1 = \frac{1}{2} - \frac{x_2}{5} + \frac{u_1}{20}$$

$$f(\underline{x}_{NB}) = -10\left(\frac{1}{2} - \frac{x_2}{5} + \frac{u_1}{20}\right) - 25x_2 + 10\left(\frac{1}{2} - \frac{x_2}{5} + \frac{u_1}{20}\right)^2 + 4\left(\frac{1}{2} - \frac{x_2}{5} + \frac{u_1}{20}\right)x_2 + x_2^2$$

$$= -5 + 2x_2 - \frac{u_1}{2} - 25x_2 + x_2^2 + 2x_2 - \frac{4}{5}x_2^2 + \frac{4}{\cancel{20}\,5}x_2u_1$$

$$+10\left(\frac{1}{4} + \frac{x_2^2}{25} + \frac{u_1^2}{2\cancel{0}} - \frac{2x_2u_1}{\cancel{100}\,50} - \frac{x_2}{5} + \frac{u_1}{20}\right)$$

$$=$$

$$= -\frac{5}{2} - 23x_2 + \underbrace{\frac{3}{5}x_2^2 + \frac{u_1^2}{40}}$$

$$\alpha = \begin{array}{cc} x_2 & u_1 \\ \end{array}\begin{bmatrix} -23 & 0 \end{bmatrix} \qquad D = \begin{array}{c} \\ \end{array}\begin{array}{cc} x_2 & u_1 \end{array}\begin{bmatrix} 3/5 & 0 \\ 0 & 1/40 \end{bmatrix}$$

entering

$$x_2^{(1)} = \min\left(\frac{19}{2}\times\frac{5}{9}, \frac{17}{2}\times\frac{5}{4}, \frac{1}{2}\times 5\right) \qquad x_2^{(2)} = \frac{-\alpha_2}{2d_{22}} = \frac{-(-23)}{2\times 3/5}$$

$$= \frac{5}{2} \qquad\qquad = \frac{5}{2}\times\frac{23}{3}$$

as $x_2^{(1)}$ is $\min^m$ we are going to use criterion 1

		x_1	x_2	x_3	x_4	u_1	b	
$R_1 - \frac{9}{5}R_3$	x_3	-9	0	1	0	$\boxed{1/2}$	5	$\longrightarrow$
$R_2 - \frac{4}{5}R_3$	x_4	-4	0	0	1	$1/4$	$13/2$	
$5R_3$	x_2	5	1	0	0	$-1/4$	$5/2$	

$$(x_3, x_4, x_2 : x_1, u_1) = (5, 13/2, 5/2; 0, 0)$$

$$f(\cdot) = -\frac{5}{2} - 23x_2 + \frac{3}{5}x_2^2 + \frac{u_1^2}{40}$$

$$5x_1 + x_2 - \frac{1}{4}u_1 = 5/2 \Rightarrow \underline{x_2 = \frac{5}{2} - 5x_1 + \frac{1}{4}u_1}$$

$$f(x, u_1) = -\frac{5}{2} - 23\left(\frac{5}{2} - 5x_1 + \frac{1}{4}u_1\right) + \frac{3}{5}\left(\frac{5}{2} - 5x_1 + \frac{u_1}{4}\right)^2 + \frac{u_1^2}{40}$$

$$= -60 + 115x_1 - \frac{23}{4}u_1 + \frac{u_1^2}{40} + \frac{3}{5}\left(\left(\frac{5}{2} - 5x_1\right)^2 + \frac{u_1^2}{16} + 2\left(\frac{5}{2} - 5x_1\right)\frac{u_1}{4}\right)$$

$$= -60 + 115x_1 - \frac{23}{4}u_1 + \frac{u_1^2}{40} + \frac{3}{5}\left(\frac{25}{4} - 25x_1 + 25x_1^2 + \frac{5u_1}{4} - \frac{5x_1u_1}{2}\right)$$

$$\left[= \frac{-225}{4} + \underbrace{100x_1 - 5u_1} + 15x_1^2 - \frac{3}{2}u_1x_1 + \frac{u_1^2}{16} \right]$$

$$\alpha = \begin{array}{cc} x_1 & u_1 \\ \end{array} \quad D = \begin{array}{cc} x_1 & u_1 \end{array}$$

$$\alpha = \begin{bmatrix} 100 & -5 \end{bmatrix} \qquad D = \begin{bmatrix} 15 & -3/4 \\ -3/4 & 1/16 \end{bmatrix}$$

$\uparrow$ u_1 entering

$$u_1^{(1)} = \min\left(\frac{5}{1/2}, \frac{13}{\cancel{2}} \times \cancel{4}\right) = 10 \qquad u_1^{(2)} = \frac{-\alpha_1}{2d_{11}} = \frac{-(-5)}{2 \times 1/16} = 40$$

	x_1	x_2	x_3	x_4	u_1	b
u_1	-18	0	2	0	1	10
x_4	$1/2$	0	$-1/2$	1	0	4
x_2	$1/2$	1	$1/2$	0	0	5

$$f(\underline{x}) = -\frac{225}{4} + 100x_1 - 5u_1 + 15x_1^2 - \frac{3}{2}u_1x_1 + \frac{u_1^2}{16}$$

$$-18x_1 + 2x_3 + u_1 = 10 \Rightarrow u_1 = 10 + 18x_1 - 2x_3$$

$$f(\underline{x}_{NB}) = f(x_1, x_3)$$
$$= -\frac{225}{4} + 100x_1 + 15x_1^2 - 5(10 + 18x_1 - 2x_3)$$
$$-\frac{3}{2}x_1(10 + 18x_1 - 2x_3) \quad + \frac{1}{16}(10 + 18x_1 - 2x_3)^2$$

$$= \left(-\frac{225}{4} - 50\right) + x_1(100 - 90 - 15) + 10x_3 + 3x_1x_3$$
$$+ \frac{1}{16}\left((10 + 18x_1)^2 + 4x_3^2 - 2x_3(10 + 18x_1)\right)$$
$$+ \frac{1}{16}\left(100 + 360x_1 + 324x_1^2 + 4x_3^2 - 20x_3 - 36x_1\right)$$
$$= \left(-\frac{425}{4} + \frac{25}{4}\right) + x_1\left(-5 + \frac{81}{4}\right) + x_3\left(10 - \frac{\cancel{20}5}{\cancel{16}4}\right) + \cdots$$
$$\qquad\qquad\qquad \geq 0 \qquad\qquad\qquad \geq 0$$

$$= -100 + \text{some fun}^c \text{ of Nonbasic Variables.}$$

HW →

Gomory's Cutting Plane Method.

Min $Z = -9x_1 - 10x_2$ → Max $9x_1 + 10x_2$

Sub to $x_1 \leq 3$ $x_1 + x_3 = 3$

$2x_1 + 5x_2 \leq 15$ $2x_1 + 5x_2 + x_4 = 15$

$x_i \geq 0 \quad \forall i$

		9	10	0	0	
	Basic	x_1	x_2	x_3	x_4	b
0	x_3	1	0	1	0	3
0	x_4	2	5	0	1	15
	Z_j					
	$C_j - Z_j$					

		9	10	0	0	
	Basic	x_1	x_2	x_3	x_4	
9	x_1	1	0	1	0	3
→ 10	x_2	0	1	$-2/5$	$1/5$	$9/5$
	Z_j	9	10	5	2	
	$C_j - Z_j$	0	0	-5	-2	

Non-integer

$$x_2 - \frac{2}{5}x_3 + \frac{1}{5}x_4 = 9/5$$

$$x_2 = \frac{9}{5} + \frac{2}{5}x_3 - \frac{1}{5}x_4$$

$$-\frac{1}{5} - (-1)$$

$$u_1 = -1 + \frac{4}{5} + \frac{2}{5}x_3 + \frac{4}{5}x_4$$

$$-\frac{2}{5}x_3 - \frac{4}{5}x_4 + u_1 = -\frac{1}{5}$$

pure

		9	10	0	0	0	
		x_1	x_2	x_3	x_4	u_1	b
9	x_1	1	0	1	0	0	3
10	x_2	0	1	-2/5	1/5	0	9/5
0	u_1	0	0	-2/5	$\boxed{-4/5}$	1	-1/5
	Z_j	9	10	5	2	0	
	$C_j - Z_j$	0	0	-5	-2	0	
	θ_j	-	-	25/2	5/2	-	