Infinite Series :-

$\sum x_n$ $\quad \sum_{i=1}^{\infty} x_n = ?$

$S_1 = x_1$

$S_2 = x_1 + x_2$

$S_n = \sum_{i=1}^{n} x_i$

$S_n \to S$

$S_n \to S$

$S_{n+1} \to S$

$\lim_n S_n - S_{n-1} = 0$

$\lim_n \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i = 0$

$\therefore \lim x_n = 0$

$\sum \frac{1}{n}$ $\quad \sum \frac{(-1)^n}{n}$ $\quad \sum x^n = \frac{1}{1-x}$ $\quad |x|<1$

$= \sum \frac{1}{n}$

$= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{✓} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{✓} + \dots$

$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}$

$\geq 1 + \frac{1}{2}\left[1 + 1 + 1 + \dots\right.$

$\geq \infty$

$\sum \frac{(-1)^{n+1}}{n}$

$= \underbrace{1 - \frac{1}{2}} + \underbrace{\frac{1}{3} - \frac{1}{4}} + \underbrace{\frac{1}{5} - \frac{1}{6}}$

$S_n = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \dots$

$= 1 - \underbrace{\left(\frac{1}{2 \cdot 3}\right)}_{>0} - \underbrace{\left(\frac{1}{4 \cdot 5}\right)}_{>0}$

$S_n \downarrow$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \ldots\ldots + \cancel{\frac{1}{n}} - \frac{1}{n+1} \longrightarrow 0$$

$$\Rightarrow 1$$

$\sum \frac{(-1)^n}{n}$ cgt $\quad \frac{(-1)^n}{n} \to 0$

$\sum r^n \quad |r|<1$ cgt $\quad r^n \to 0 \quad 0.5^2$

$\sum \frac{1}{n^2}$ cgt $\quad \frac{1}{n^2} \to 0$

$\sum \frac{1}{n}$ dvge $\quad \frac{1}{n} \to 0$

Cauchy Criterion for convergence of series

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for any $\varepsilon>0 \ \exists \ K(\varepsilon) \in \mathbb{N} \ \ni \ |S_n - S_m| < \varepsilon \quad \forall \ n \geq m \geq K(\varepsilon)$

$$\Rightarrow \left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} x_i \right| < \varepsilon \quad \forall \ n \geq m \geq K(\varepsilon)$$

$$\Rightarrow \left| \sum_{i=m+1}^{n} x_i \right| < \varepsilon$$

Let xn be a sequence of non-negative real numbers then the series P \sum xn converges if and only if the sequence Sk of partial sum is bounded.

① Let $\sum_{n=1}^{\infty} x_n$ converges, $\sum_{n=1}^{\infty} x_n = s$

$\Rightarrow$ $S_n = \sum_{i=1}^{n} x_i$ = seq^n of partial sums

$\Rightarrow$ S_n is convergent to s.

$\Rightarrow$ Every convergent seq^n is bounded

$\Rightarrow$ S_n is bounded.

② S_n is bounded $\Rightarrow$ To prove S_n is cgt.

$S_{n+1} = \sum_{i=1}^{n+1} x_i = S_n + x_{n+1}$ as $x_{n+1} \geq 0$

$S_{n+1} \geq S_n$

S_n is monotonically $\uparrow$ & bounded

by MCT it is cgt.

Show that $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots = \frac{1}{1-r}$ if $|r| < 1$

$S_{n+1} = \sum_{i=1}^{n+1} r^i = 1 + r + r^2 + \quad + r^{n-1} + r^n$

$S_n = \sum_{i=1}^{n} r^i = 1 + r + r^2 + \ldots + r^{n-1}$

$r S_n = r + r^2 + \ldots + r^n = S_{n+1} - 1$

$1 + r S_n = S_{n+1}$?

$\lim (1 + r \cdot S_n) = \lim S_{n+1}$

$1 + r \cdot S = S$ $\Rightarrow$ $S = 1/(1-r)$?

$p>1$

$$\sum_n \frac{1}{n^p} = 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}} + \underbrace{\frac{1}{8^p}} + \cdots$$

$$\leq 1 + \underbrace{\frac{1}{2^p} + \frac{1}{2^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}} + \underbrace{\frac{1}{8^p}} + \cdots$$

$$\leq 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots$$

$$\leq 1 + \frac{2}{2^p} + \frac{2^2}{2^{2p}} + \frac{2^3}{2^{3p}} + \cdots$$

$$\leq 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$$

as $p>1$, $\frac{1}{2^{p-1}} < 1 \Rightarrow \sum r^n \to \frac{1}{1-r}$ cgt.

$\Rightarrow \sum \frac{1}{n^p}$ is also cgt.

Comparison Test :- $K \in \mathbb{N} \ni \underline{0 \leq x_n \leq y_n} \ \forall\ n \geq K$

$\sum y_n$ cgt. $\Rightarrow \sum x_n$ cgt

by Cauchy criterion for convergence of series
for any $\varepsilon > 0$ $\exists M(\varepsilon) \in \mathbb{N}$. $|y_{m+1} + y_{m+2} + \ldots + y_n| < \varepsilon$ $\forall n > m > M(\varepsilon)$

$K'(\varepsilon) = \max(M(\varepsilon), K)$

for $n, m \geq K'(\varepsilon)$

$$|y_{m+1} + \dots + y_n| < \varepsilon$$

by Δ inequality

$$|x_{m+1} + x_{m+2} + \dots + x_n| < |y_{m+1} + \dots + y_n| < \varepsilon$$

by Cauchy criterion Σx converges.

$\Sigma \frac{1}{n^2+n}$ $\qquad \Sigma x_n \qquad \Sigma y_n = \Sigma \frac{1}{n^2} \to$ cgt

$$r = \lim \frac{x_n}{y_n} = \lim \frac{1}{n^2+n} \times n^2 = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

$$r \neq 0$$

$0 < x_n, \quad 0 < y_n \quad \forall n$

$$r = \lim \frac{x_n}{y_n}$$

for any $\varepsilon > 0 \ \exists \ K(\varepsilon) \in \mathbb{N} \ni \left|\frac{x_n}{y_n} - r\right| < \varepsilon$

$$r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$$

$$y_n(r-\varepsilon) \leq x_n \leq y_n(r+\varepsilon)$$

$x_n \leq y_n, \quad \Sigma y_n$ cgt $\Rightarrow \Sigma x_n$ cgt if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt

if $\sum x_n$ cgt $\Rightarrow$ $\sum y_n$ cgt

if $r=0$, $r-\varepsilon < \frac{x_n}{y_n} < r+\varepsilon$

$-\varepsilon < \frac{x_n}{y_n} < \varepsilon$

$-\varepsilon < 0 < \frac{x_n}{y_n} < \varepsilon$

$x_n \leq y_n \cdot \varepsilon$

$\Rightarrow$ if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt.