Markov Chains :-

discrete statespace & discrete time domain stochastic process following Markov property.

Markove Property :- Future depends on present not on past

Notation $P_{ij}^{(n)} = P[X_n = j / X_0 = i]$

$P_{ij}^{(n)}$ Starting from state i process reaches to state j in n-steps

i ... j
0 1 2 3 ... n-1 n

Stationary → Time-homogenous process
& Non-stationary

Non-stationary

$P_{ij}^{(m,n)}$ = Starting from i process reaches to state j from m to n steps (No. of steps - n-m)

✓ $P_{ij}^{(m,n+m)} = P[X_{n+m} = j / X_m = i]$

statespace = $S = \{1, 0\}$ ⇒ 1 if it rains today 0 o.ω.

0 0 ... 1 1 ... 1 ... 0
1 2 3 ... 8 9 ... 15 ... 30

$P_{10}^{(15)}$ $P_{10}^{(15,30)}$ ✓
$\quad ij$

$P[X_{n+1} = j / X_n = i] = P_{ij}^{1} = P_{ij}$

$\{X_n, n \geq 0\}$

$P[X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0]$

$P[\underbrace{X_{30} = 1}_{\text{Future}} / \underbrace{X_{29} = 1}_{\text{present}}, \underbrace{X_{28} = 1, \ldots\ldots X_0 = 0}_{\text{past}}] = P[X_{30} = 1 / X_{29} = 1]$

$p_{ij}^{(1)} = P[X_{n+1}=j / X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0] = P[X_{n+1}=j / X_n=i]$

$\{X_n, n \geq 0\}$ as Markov chain.

one-step transition prob matrix

$$\begin{array}{cc} & \text{future state} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ \text{Current state} \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \end{array}$$

$P_{00} \Rightarrow P[X_{n+1}=0 / X_n=0]$

$P_{10} \Rightarrow P[X_{n+1}=\underline{0} / X_n=\underline{1}]$

starting ending

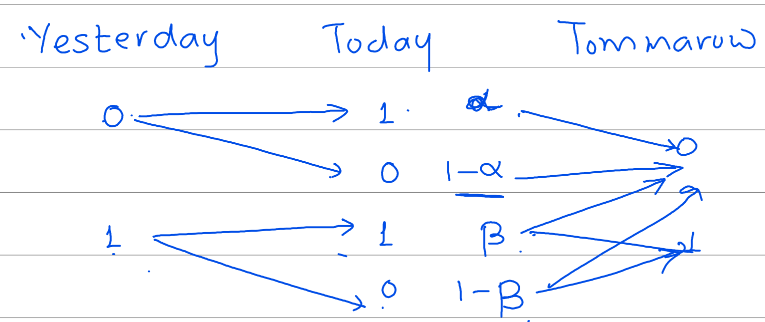
$P^{(2)}$ = Two step transition prob. matrix

$$P^{(2)} = \begin{array}{cc} & \text{Future} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ \text{current state} \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} P_{00}^{(2)} & P_{01}^{(2)} \\ P_{10}^{(2)} & P_{11}^{(2)} \end{bmatrix} \end{array}$$

n-step

$$S=\{0,1\} \quad P^{(n)} = \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} \\ P_{10}^{(n)} & P_{11}^{(n)} \end{bmatrix}$$

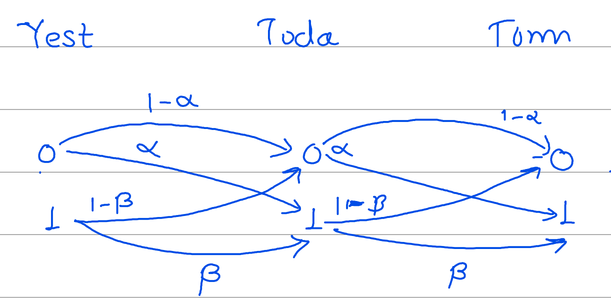
$S = \{1, 2, \ldots, K\}$

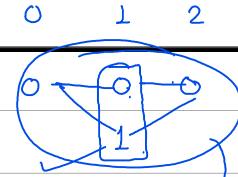


one-step

$$P = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \end{array}$$

Two-step





$= P_{00} \cdot P_{00} + P_{01} P_{10}$

$= (1-\alpha)^2 + \alpha(1-\beta)$

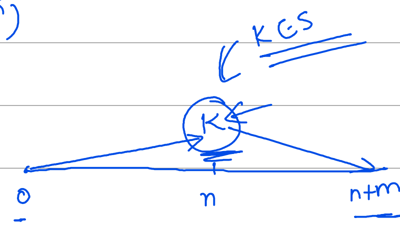
Tommor

$$\text{Yest} \quad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & P_{00}^{(2)} & P_{01}^{(2)} \\ 1 & P_{10}^{(2)} & P_{11}^{(2)} \end{array}$$

n-step transition probabilities

Chapman- Kolmogorov's — (CK - eqn)

$$P_{ij}^{(n+m)} = \sum_{K \in S} P_{iK}^{(n)} P_{Kj}^{(m)}$$



$\{X_n, n \geq 0\}$ is Markov chain?

$P^{(n+m)}$

$$P_{ij}^{(n+m)} = P[X_{n+m} = j / X_0 = i]$$

$$= \sum_{K \in S} P[\underbrace{X_{n+m} = j}_{\text{future}} / \underbrace{X_m = K}_{\text{present}}, \underbrace{X_0 = i}_{\text{past}}] \cdot P[X_m = K / X_0 = i] \quad K \in S$$

$$= \sum_{K \in S} P[X_{n+m} = j / X_m = K] \cdot P[X_m = K / X_0 = i]$$

stationary processes

$$= \sum_{K \in S} P[X_n = j / X_0 = K] \cdot P[X_m = K / X_0 = i]$$

$$= \sum_{K \in S} P_{Kj}^{(n)} \cdot P_{iK}^{(m)}$$

if we have used ~~(step)~~ it in Matrix form

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

if $n = m = 1$,

$$P^{(1+1)} = P^{(2)} = P^{(1)} \cdot P^{(1)}$$

$$= \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix}$$

$$= \begin{bmatrix} (1-\alpha)^2 + \alpha(1-\beta) & \alpha[(1-\alpha) + \beta] \\ (1-\alpha)(1-\beta) + \beta(1-\beta) & (1-\alpha)(1-\beta) + \beta^2 \end{bmatrix}$$

$\{X_n, n \geq 0\}$ denotes whether it rains on n^{th} day or not

$$X_n = \begin{cases} 0 \\ 1 \end{cases}$$

Monday $\Rightarrow$ 0

Monday $\rightarrow$ 1 ... S ... 5

Saturday $\rightarrow$ 1

one step transition

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

(X_n rows, X_{n+1} columns)

Five step. $P^{(5)}$

$P[X_5 = 1 / X_0 = 1]$

$$= P^{(5)} = P^{(4)} \cdot P$$

$$= P^{(3)} \cdot P^{(2)}$$

$$= P^{(3)} \cdot \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= P^{(3)} \frac{1}{6}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$$

$$= \frac{1}{6^2} P^{(3)} \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$$

$$= \frac{1}{6^2} P \cdot P^{(2)} \cdot \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$$

$$= \frac{1}{6^4} P \cdot \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix} \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$$

$7\times2\times7\times3$

49×6

$21\times(15+22)$

$=21\times(37)$

$=777$

$$= \frac{1}{6^4} P \begin{bmatrix} 778 & 518 \\ 777 & 519 \end{bmatrix}$$

1296

$=1$ ✓

Row sum 1

$$= \frac{1}{6^4} \begin{bmatrix} 778 & 518 \\ 777 & 519 \end{bmatrix} \times \frac{1}{6}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$$

$$= \frac{1}{6^5} \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 4666 & 3110 \\ 4665 & \boxed{3111} \end{bmatrix} \end{array}$$

5-step transition prob. matrix

$$P[X_5 = 1 / X_0 = 1] = \frac{3111}{6^5} = \underline{0.4}$$

MC $\{X_n, n \geq 0\}$ Markov chain with statespace $S = \{0, 1, 2\}$

One-step TPM

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1/2 & 1/3 & 1/6 \\ 1 & 0 & 1/3 & 2/3 \\ 2 & 1/2 & 0 & 1/2 \end{array}$$

initial prob. ① $P[X_0 = 0] = P[X_0 = 1] = 1/2$?

② $X_1 = ?$

$P[X_3 = 0]$, $P[X_3 = 1]$, $P[X_3 = 2]$ } $E(X_3) = ?$

$P[X_1 = 1] = ?$ $= P[X_1 = 1 / X_0 = 0] \cdot P[X_0 = 0] + P[X_1 = 1 / X_0 = 1] \cdot P[X_0 = 1] + P[X_1 = 1 / X_0 = 2] \cdot P[X_0 = 2]$

$$= \sum_k P[X_1 = 1 / X_0 = k] \cdot P[X_0 = k]$$

$1/3 \times 1/2 + 1/3 \times 1/2 + 1/2 \times 0$

$= 1/3$

$P[X_1 = ?] = \alpha \cdot P$ $\quad \alpha = [\alpha_0 \ \alpha_1 \ \alpha_2]'$

$$= [1/2 \ \ 1/2 \ \ 0] \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2}[1 \ 1 \ 0] \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

$P[X_1 = \ ] = \frac{1}{12}[3 \ \ 4 \ \ 5]$

$P(X_1 = 0) = 3/12$ $\quad P(X_1 = 1) = 4/12$ $\quad P(X_1 = 2) = 5/12$

$P[X_3 = ?] \Rightarrow [P(X_3 = 0) \ \ P(X_3 = 1) \ \ P(X_3 = 2)] = \alpha \cdot P^3$

$$= \frac{1}{2}[1 \ 1 \ 0] \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} \cdot P^2 \qquad = \frac{1}{12}[3 \ 4 \ 5] P^2$$

$1 \times 3 \quad 3 \times 3$

$$= \frac{1}{12}[3 \ 4 \ 5] \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} P$$

9, 15, 6, 8, $3 + 16 + 15$

$$= \frac{1}{72}[24 \ 14 \ 34] \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

$24 + 56$

X_3

$$= \frac{1}{432}[174 \ \ 76 \ \ 182]$$

$$E(X_3) = \frac{1}{432}\left[0\times 174 + 1\times 76 + 2\times 182\right]$$

$$= \frac{1}{432}\left[76 + 364\right] = \frac{1}{432}\left[440\right] \approx 1 \quad = 1.0185$$

Solve This

Let the TPM of two state markov chain is $P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$

Show that, by mathematical induction

$$P^{(n)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{bmatrix}$$

Put $n=1$,

$$P^{(1)} = P = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^1 & \frac{1}{2}-\frac{1}{2}(2p-1) \\ \frac{1}{2}-\frac{1}{2}(2p-1)^1 & \frac{1}{2}+\frac{1}{2}(2p-1) \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Assume that it is true for $n=k$

$$P^{(k)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{bmatrix}$$

$$P^{(k+1)} = P^{(k)} \cdot P = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$= \begin{bmatrix} p\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right) + (1-p)\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right) & (1-p)\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right) + p\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right) \\ p\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right) + (1-p)\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right) & (1-p)\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right) + p\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} \end{bmatrix}$$

$S = \{1, 2\}$ $\quad P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \\ 2 & \end{array}\!\!\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad P_{12}^{(3)} = ?$

$$P^{(3)} = P^{(2)} \cdot P = \frac{1}{3^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} P$$

$$= \frac{1}{3^2}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3^3}\begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{cc} 1 & 2 \\ \multicolumn{2}{c}{\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}} \end{array}$$

$$\left(P^{(3)}\right)_{12} = \frac{13}{27} \doteq 0.48$$

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{cc} 0 & 1 \\ \multicolumn{2}{c}{\begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}} \end{array} = \frac{1}{10}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}, \quad P_{\boxed{11}}^{(4)} = ?$$

$$P^{(2)} = \frac{1}{10^2}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix} = \frac{1}{10^2}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}$$

$$P^{(4)} = P^{(2)} \cdot P^{(2)} = \frac{1}{10^4}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}$$

$$= \frac{1}{10^4}\begin{bmatrix} 6256 & 3744 \\ 6240 & \boxed{3760} \end{bmatrix}$$

$$P_{11}^{(4)} = \frac{3760}{10^4}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{cc} 0 & 1 \\ \multicolumn{2}{c}{\begin{bmatrix} \; & \; \\ \; & \boxed{\;} \end{bmatrix}} \end{array}$$

$$\begin{array}{cc} 0 & 1 \\ \multicolumn{2}{c}{[\,0 \quad 1\,]} \end{array} \quad P^4 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}\!\!\begin{array}{c} 0 \\ 1 \end{array}$$

$$\frac{1}{10^2}[\,0 \quad 1\,]\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix} P^2 \begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{10^2}[\,5 \quad 5\,]\, P^2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{5}{10^4}[\,1 \quad 1\,]\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{5}{10^4}\begin{bmatrix}12 & 8\end{bmatrix}\begin{bmatrix}36\\40\end{bmatrix}$$

$$= \frac{5}{10^4}\begin{bmatrix}752\end{bmatrix} = \frac{3760}{10^4}$$

$$P = \begin{array}{c|c} & \begin{array}{ccc}1 & 2 & 3\end{array} \\ \begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix}2 & 3 & 1\\1 & 2 & 3\\3 & 2 & 1\end{bmatrix}\end{array} \times \frac{1}{6}$$

$$P_{23}^{(3)}$$

$$= \begin{array}{c}\begin{array}{ccc}1 & 2 & 3\end{array}\\ \begin{bmatrix}0 & 1 & 0\end{bmatrix}\end{array} P^3 \begin{array}{cc}\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{array}{c}1\\2\\3\end{array}\end{array}$$

$$= \begin{bmatrix}3 & 2 & 2\end{bmatrix} P \begin{bmatrix}1\\3\\1\end{bmatrix}$$

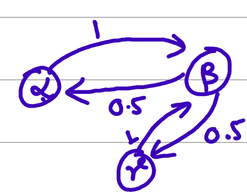
$$= \frac{1}{6^3}\begin{bmatrix}3 & 2 & 2\end{bmatrix}\begin{bmatrix}2 & 3 & 1\\1 & 2 & 3\\3 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix}14 & 17 & 11\end{bmatrix}\begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix}76\end{bmatrix} = \frac{76}{216}$$

$C(\alpha) = \{\alpha, \beta, \gamma\}$ Irreducible

$$\begin{array}{c|c} & \begin{array}{ccc}\alpha & \beta & \gamma\end{array} \\ \begin{array}{c}\alpha\\\beta\\\gamma\end{array} & \begin{bmatrix}0 & 1 & 0\\1/2 & 0 & 1/2\\0 & 1 & 0\end{bmatrix}\end{array}$$



$$\begin{array}{ccc}\alpha & \beta & \gamma\end{array}$$
$$\begin{bmatrix}0.25 & 0.25 & 0.5\end{bmatrix}$$

$X_0 \Rightarrow \; = \frac{1}{4}\begin{bmatrix}1 & 1 & 2\end{bmatrix}$

$P[X_4 = \gamma] = ? \quad \Rightarrow \frac{1}{4}\begin{bmatrix}1 & 1 & 2\end{bmatrix} P^4 \begin{bmatrix}0\\0\\1\end{bmatrix}$

PQ4

$P[X_4 = \nu / X_2 = \alpha] \Rightarrow [1\ 0\ 0]\ P^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Classifi

Periodicity is class property ie. $i \leftrightarrow j \Rightarrow d(i) = d(j)$

$\rightarrow \quad i \rightarrow j \quad P_{ij}^{n} > 0$ for some n

$j \rightarrow i \quad P_{ji}^{m} > 0$ for some m

$d(i) = \gcd\{t / P_{ii}^{t} > 0\}$ $\qquad d(j) = \gcd\{t / P_{jj}^{t} > 0\}$

$P_{ii}^{(n+m)} = \sum_{k \in S} P_{ik}^{n} P_{ki}^{m} \geq P_{ij}^{n} P_{ji}^{m} > 0$ $\qquad P_{jj}^{(n+m)} = \sum_{k \in S} P_{jk}^{m} P_{kj}^{n} \geq P_{ji}^{m} P_{ij}^{n} > 0$

$\Rightarrow$ $n+m$ is multiple of $d(i)$ $\quad \Rightarrow n+m = d(i) \times$ something

Similarly, $P_{jj}^{(n+m)} > 0 \Rightarrow n+m$ is multiple of $d(j)$

$\Rightarrow n+m = d(j) \times$ something

for some s, $P_{ii}^{(s)} > 0 \Rightarrow s = d(i) \times$ something — (1)

$P_{jj}^{(n+m+s)} \geq P_{ji}^{(m)} P_{ii}^{(s)} P_{ij}^{(n)}$, with $P_{ji}^{(m)} > 0$, $P_{ii}^{(s)} > 0$, $P_{ij}^{(n)} > 0$

> 0

$\Rightarrow n+m+s = d(j) \times$ something

$\Rightarrow s = d(j) \times$ something — (2) (as $d(j)$ divides $n+m$)

From (1) & (2)

$\Rightarrow d(i) = d(j)$

Transitivity :- $i \leftrightarrow j$, $j \leftrightarrow k$ $\Rightarrow$ $i \leftrightarrow k$

$i \leftrightarrow j$ $\quad P_{ij}^{n} > 0$, $P_{ji}^{m} > 0$ for some n, m

$j \leftrightarrow k$ $\quad P_{jk}^{s} > 0$, $P_{kj}^{t} > 0$ for some s, t

To show $i \leftrightarrow k$:→ $\quad P_{ik}^{n+s} \geq \underbrace{P_{ij}^{n}}_{>0} \underbrace{P_{jk}^{s}}_{>0}$ $\qquad P_{ki}^{t+m} \geq \underbrace{P_{kj}^{t}}_{>0} \cdot \underbrace{P_{ji}^{m}}_{>0}$

> 0 $\qquad\qquad > 0$

$i \to k$ $\qquad\qquad k \to i$

$\Rightarrow$ $i \leftrightarrow k$

Recurrent: Starting from state i, Process reenters in the same state with probability 1

① $F_i = 1$ ✓

② $\sum_n P_{ii}^{n} = \infty$

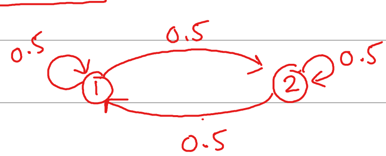
For any state i, F_i denote the probability that the process, starting from state i, will ever reaches state i.

State i is said to recurrent if $F_i = 1$, & transient if $F_i < 1$

$f_i^{(n)}$ $\Rightarrow$ Prob. that starting from state i process reaches to state i in n steps first time.

$\boxed{F_i = \sum_n f_i^{n}}$ ✓

Irreducible



$f_1^{1} = 0.5$ $\qquad$ ~~1-1-1~~

$f_1^{2} = 0.5^2$ $\qquad$ 1-2-1

$f_1^{3} = 0.5^3$ $\qquad$ 1-2-2-1

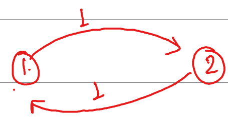
$\Rightarrow f_1^{n} = 0.5^n$

$F_1 = \sum_n f_1^{n} \Rightarrow = 0.5 + 0.5^2 + \dots$

$= 0.5 \left(1 + 0.5 + 0.5^2 + \dots\right)$

$= 0.5 \left(\frac{1}{1-0.5}\right) = \cancel{0.5} \cdot \frac{1}{\cancel{0.5}} = 1$

1 & 2 recurrent



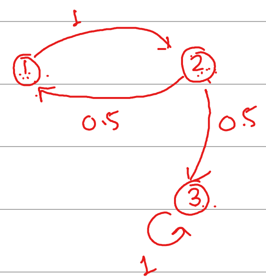
$f_1^{(1)} = 0$ $\quad F_1 = f_1^{(1)} + f_1^{(2)} + 0 + 0$

$f_1^{(2)} = 1$ $\quad = 0 + 1 + 0$

$f_1^{(3)} = 0$ $\quad = 1$

$\Rightarrow$ 1 & 2 recurrent

Reducible



$c(1) = \{1,2\} = c(2)$ $\qquad c(3) = \{3\}$

$d(1) = \gcd\{2,4,6,\dots\}$ $\qquad d(3) = 1$

$= 2$

$= d(2)$

(Periodicity is class property)

$f_2^1 = 0, \quad f_2^2 = 0.5, \quad f_2^3 = 0, \quad f_2^4 = 0 \quad \dots\dots$

$f_1^1 = 0 \quad f_1^2 = 0.5 \quad f_1^3 = 0 \quad f_1^4 = 0 \quad \dots\dots$

$F_{2} = 0.5 < 1 \quad \Rightarrow$ 1&2 ~~is~~ are transient states

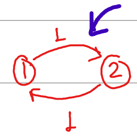
$F_3 = f_3^{(1)} + f_3^{(2)} + f_3^{(3)} + \dots = 1 + 0 + 0 + \dots = 1 \Rightarrow$ 3 is ~~recurrent~~ persistent

$P_{ii}^{(n)} \rightarrow$ Starting from i, process reaches to state i, in n steps

To prove

i recurrent iff $\sum_n P_{ii}^n = \infty$ $\quad$ | Transient iff $\sum P_{ii}^n < \infty$

$c(1) = \{1,2\} = c(2)$
Irreducible MC



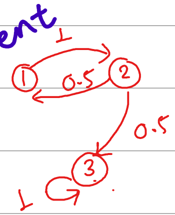
Recurrent ~~Transient~~

$\sum_n P_{11}^{(n)} = P_{11}^1 + P_{11}^2 + P_{11}^3 + \dots \qquad = \sum_n P_{11}^{2n}$

$= 0 + 1 + 0 + 1 + \qquad$ infinite - Recurrent

$= \infty$

$c(1) = \{1,2\} \rightarrow$ ~~recurrent~~ transient
$c(3) = \{3\}$
Reducible MC



$\sum_n P_{11}^{(n)} = P_{11}^1 + P_{11}^2 + P_{11}^3 + P_{11}^4 + \dots \qquad = \sum_n P_{11}^{2n}$

$= 0 + 0.5 + 0 + 0.5^2 + \dots \qquad = \sum_n 0.5^n$

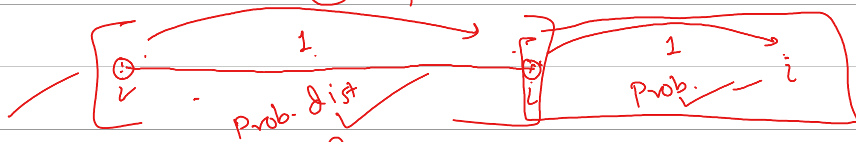
$= \frac{1}{1-0.5} = \frac{1}{0.5} = 2 < \infty$

Finite $\Rightarrow$ Transient

i Recurrent iff $\sum P_{ii}^{(n)} = \infty$

n steps

Markov property



infinite → enters

infinite Process probabilistic replica — $F_{ii} = 1$, 0

i state transient, starting from i ~~stat~~ process reaches to state i w.p. < 1

$F_{ii} < 1$

$1 - F_{ii} > 0$ it will never return to state i

No. visits Finite no.

$\sim$ Geo

$E(\text{No of visits}) \Rightarrow \dfrac{1}{1 - F_{ii}}$

- State i is recurrent if, with probability 1, a process starting from state i, will eventually return.
- However, by Markovian Property, the process probabilistically restarts itself upon returning to state i. Hence with prob. 1 it will return to i.
- Repeating this argument, with probability 1, the no. of visits to state i will be infinite & will thus have infinite expectation.

On the other hand, if state i is transient, there is positive prob., $1 - F_{ii} > 0$, that it will never return again. Hence the no. of visits is Geometric with $E(n) = \dfrac{1}{1 - F_{ii}}$

$\therefore$ State i is recurrent iff $E(\text{no. of visits to } i / X_0 = i) = \infty$

Let $I_n = \begin{cases} 1 & \text{if } x_n = i \\ 0 & \text{if } x_n \neq i \end{cases}$

$E(I_n) = P(I_n = 1 / X_0 = i)$
$= P(X_n = i / X_0 = i)$
$= P_{ii}^n$

$\therefore \sum_n I_n$ denotes no. of visits to i. ✓

$E\left(\sum_n I_n / X_0 = i\right) = \sum_n E(I_n / X_0 = i)$

$= \sum_n P_{ii}^n$

$\Rightarrow$ i is recurrent iff $\sum_n P_{ii}^n = \infty$ ✓

Theo: $\boxed{\text{Recurrence is class property so does transience}}$

Recurrence	Transience
$\hookrightarrow$ $F_{ii} = 1$ or/and $\sum P_{ii}^n = \infty$	$F_{ii} < 1$ $\quad \sum P_{ii}^n < \infty$

$i \leftrightarrow j$. $\Rightarrow$ i is recurrent $\Rightarrow$ j is also recurrent

$\rightarrow$ $i \leftrightarrow j$, $\odot$ $i \rightarrow j$, for some s, $P_{ij}^s > 0$
$j \rightarrow i$ for some t, $P_{ji}^t > 0$

i is recurrent $\Rightarrow \sum_n P_{ii}^n = \infty$

$P_{jj}^{s+t+n} = \sum_{k \in S} P_{jk}^t P_{kk}^n P_{kj}^s \qquad < \sum_{n=0}^{\infty} P_{jj}^n = \infty$

$$\geq P_{ji}^{t} P_{ii}^{n} P_{ij}^{s}$$

$$\sum_n P_{jj}^{s+t+n} \geq \sum_n P_{ji}^{t} P_{ii}^{n} P_{ij}^{s}$$

$$\geq \underset{>0}{P_{ji}^{t}} \underset{\infty}{\left(\sum_n P_{ii}^{n}\right)} \underset{>0}{P_{ij}^{s}}$$

$$\Rightarrow \sum_n P_{jj}^{n} \geq \sum_n P_{jj}^{s+t+n} \geq \infty$$

$< \infty$

$$\Rightarrow \sum_n P_{jj}^{n} = \infty$$

Transience is class property.

$i \leftrightarrow j$, if i is transient $\Rightarrow$ j is also transient.

$i \leftrightarrow j$ $\quad$ $\cdot\, i \to j$, $\quad 1 \geq P_{ij}^{s} > 0$

$\quad j \to i$, $\quad 1 \geq P_{ji}^{t} > 0$

i is transient $\Rightarrow$ $\sum_n P_{ii}^{n} < \infty$

$$P_{jj}^{s+t+n} \geq P_{ji}^{t} P_{ii}^{n} P_{ij}^{s}$$

$$\sum P_{jj}^{s+t+n} \geq \boxed{\underset{\leq 1}{P_{ji}^{t}} \underset{<\infty}{\left(\sum P_{ii}^{n}\right)} \underset{\leq 1}{P_{ij}^{s}}} \leq \infty$$

$$\sum P_{jj}^{s+t+n} \leq \infty$$

Recurrence

$i \leftrightarrow j$ $\quad$ i transient $\Rightarrow$ j transient

By method of contradiction

i transient but j is recurrent

j recurrent $\Rightarrow$ i recurrent

which is contradiction

Ergodic

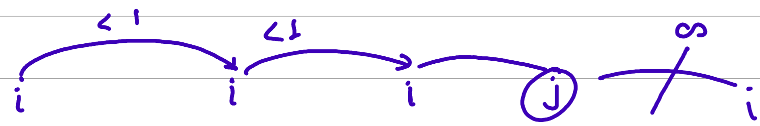
Recurrent

$\circ\ F_i = 1$

$\circ\ \sum P_{ii} = \infty$

Irreducible MC :- Only one class, All states are communicating with each other.

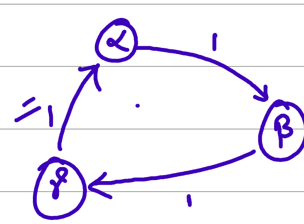
Let μ_{jj} denotes the expected no. of transitions need to return to state j



$$\mu_{jj} = \begin{cases} \infty, & \text{if } j \text{ is transient.} \\ \sum n f_i^n \checkmark & \text{if } j \text{ is recurrent.} \end{cases}$$

$\mu_{jj} = \infty$ null recurrent $\checkmark$ $\qquad \mu_{jj} < \infty$ +ve recurrent

$$\begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline \alpha & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \\ \gamma & 1 & 0 & 0 \end{array}$$



First time

$\mu_{\alpha\alpha} = \sum_n n \cdot f_\alpha^n$

$= 1\cdot 0 + 2\cdot 0 + 3\cdot 1 + 4\cdot 0 + 0 \ldots$

$= 3$

$\underline{\mu_{\alpha\alpha} = 3 < \infty}$

positive recurrent

~~aperiodic~~

$\Rightarrow$ ~~ergodic~~

$C(\alpha) = \{\alpha, \beta, \gamma\} \quad \Rightarrow$ Irreducible

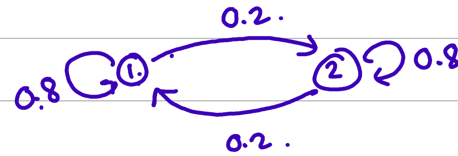
$F_\alpha = f_\alpha^1 + f_\alpha^2 + f_\alpha^3 + f_\alpha^4 \ldots$

$= 0 + 0 + 1 + 0 \ldots$

$= 1 \qquad \Rightarrow \quad \alpha, \beta, \gamma$ all are recurrent

$\mu_{\alpha\alpha} = 3 < \infty \quad \Rightarrow \quad \alpha$ is +ve recurrent

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.8 & 0.2 \\ 2 & 0.2 & 0.8 \end{array}$$



$C(1) = \{1,2\} \Rightarrow$ Irreducible $\qquad 1 \to 2 \to 2 \to 1$ $\quad$ 4

First time 1 $\to$ 2 $\qquad$ 2

$F_1 = f_1^1 + f_1^2 + f_1^3 + f_1^4 + \dots$

$= \underline{0.8} + 0.2^2 + 0.2^2 \cdot 0.8^1 + 0.2^2 \cdot 0.8^2 + \dots + \underline{0.2^2 (0.8)^{n-2}} + \dots$

$= \underline{0.8} + 0.2^2 [1 + 0.8 + 0.8^2 + \dots] \quad < \infty \qquad 0.8 < 1$

$= 0.8 + 0.2^2 \left[\frac{1}{1-0.8}\right]$

$= 0.8 + 0.2$

$= \underline{1} \qquad \Rightarrow$ 1, 2 are recurrent states

$\mu_{11} = \sum_n n \cdot f_1^n = 1 \cdot 0.8 + 2 \cdot 0.2^2 + 3 \cdot 0.2^2 \cdot 0.8 + \dots + n \cdot 0.2^2 \cdot 0.8^{n-2} + \dots$

$= 0.8 + 0.2^2 [2 + 3 \cdot 0.8 + \dots + n \cdot 0.8^{n-2} + \dots]$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \underline{\frac{d}{dx} \sum_{n=0}^{\infty} x^n} = \frac{d}{dx} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2}$$

$\therefore = 0.8 + 0.2^2 \left[\sum_n (n+1) x^{n-1}\right] = 0.8 + 0.2^2 \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1} + 0.2^2 \sum_{n=1}^{\infty} x^{n-1}$

$= 0.8 + \cancel{(0.2)^2} \cdot \frac{1}{\cancel{(1-0.8)^2}} + \cancel{0.2^2 \sum_{n=0}^{\infty} x^n} = 0.8 + 1 + 0.2 \quad \underline{< \infty}$

1, 2 +ve recurrent.

* Ergodic state :- Aperiodic, positive recurrent

  i ergodic state $\Rightarrow$ $d(i) = 1$, $F_i = 1$ or $\sum P_{ii}^{n} = \infty$

  $\mu_{ii} < \infty$

* Ergodic MC $\Rightarrow$ All states are ergodic.

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$C(1) = \{1,2\}$ 1,2, positive recurrent

$d(1) = \gcd(1,2,\ldots) = 1 = d(2)$

1,2 positive recurrent, aperiodic $\Rightarrow$ ergodic

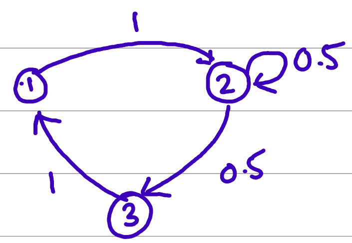
$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

states:- Aperiodic Recurrent Transient

+ve null

Ergodic

$\Rightarrow$ MC Ergodic ?

$\Rightarrow$ 

$C(1) = \{1,2,3\}$ $\Rightarrow$ Irreducible

$d(2) = \gcd\{1,2,\ldots\} = 1$ $\Rightarrow$ Aperiodic

Periodicity is class property $\Rightarrow$ All states are aperiodic.

$F_2 = f_2^1 + f_2^2 + \ldots = 0.5 + 0 + 0.5 \cdot 1 \cdot 1 + 0 + \ldots$

$= 0.5 + 0.5$

$= 1$ $\Rightarrow$ Recurrent

$\because$ Recurrence is class property. $\Rightarrow$ All states are recurrent.

$\mu_2 = 1 \cdot f_2^1 + 2 \cdot f_2^2 + 3 \cdot f^3 + 4 \cdot f_2^4 + \ldots$

$= 1 \cdot 0.5 + 0 + 3 \cdot 0.5 + 0 + \ldots$

$= 0.5 + 1.5$

$= 2 < \infty$ $\Rightarrow$ Positive Recurrent

All states are positive recurrent & aperiodic, i.e. all states are ergodic

$\therefore$ MC is ergodic MC.

Limit Theo. :-

If j is transient, then $\sum_n P_{jj}^n < \infty$

$P_{jj}^n \to 0$ as $n \to \infty$

We have only finite no. of transitions returning to j.

$\pi_j = \lim_{n\to\infty} P_{jj}^n$ $\pi_{ij} = \lim_{n\to\infty} P_{ij}^n$

+ve recurrent state $\pi_j > 0$

null recurrent $\pi_j = 0$?

Stationary

CK eq^n

$$P^{n+1} = P^n \cdot P$$

$$\lim_{n\to\infty} P_{ij}^{(n+1)} = \lim_{n\to\infty} \sum_{k\in S} P_{ik}^{(n)} \cdot P_{kj}^{(1)}$$

$$\pi_{ij} = \sum_{k \in S} \pi_{ik} P_{kj}$$

In Matrix form

$$P^{n+1} = P^n \cdot P$$

$$\left(\lim_{n\to\infty} P^{n+1}\right) = \left(\lim_{n\to\infty} P^n\right) \cdot P$$

$$\underline{\pi} = \underline{\pi} \cdot P$$

Recurrent limiting $\rightarrow$ π = Stationary dist^n

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad \underline{\underline{\pi}} = [\pi_1, \pi_2, \pi_3]'$$

$$\underline{\pi}' = \underline{\pi}' P$$

$$[\pi_1 \ \pi_2 \ \pi_3] = [\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\pi_1 = \pi_3$$

$$\pi_2 = \pi_1 + \pi_2/2 \Rightarrow \pi_1 = \pi_2/2$$

$$\pi_3 = \pi_2/2$$

$$\underline{\pi} = \underline{\underline{\left[\frac{\pi_2}{2}, \pi_2, \frac{\pi_2}{2}\right]}} \Rightarrow \sum_{i=1}^{3} \pi_i = 1 \Rightarrow$$

$$\Rightarrow \frac{\pi_2}{2} + \pi_2 + \frac{\pi_2}{2} = 2\pi_2 = 1$$

$$\pi_2 = 1/2$$

✓ $\underline{\pi} = [1/4, 1/2, 1/4]$

$$P = \frac{1}{10}\begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix} \qquad \pi = ?$$

$$\underline{\pi} = \underline{\pi} P$$

$$[\pi_1 \ \pi_2 \ \pi_3] = [\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix} \frac{1}{10}$$

$$10\pi_1 = 3\pi_1 + 4\pi_2 + 3\pi_3 \Rightarrow 7\pi_1 = 4\pi_2 + 3\pi_3 \Rightarrow 7\pi_1 = 4\pi_2 + \frac{3}{2}(\pi_1 + \pi_2)$$

$$10\pi_2 = 4\pi_1 + 3\pi_2 + 3\pi_3 \Rightarrow 7\pi_2 = 4\pi_1 + 3\pi_3$$

$$10\pi_3 = 3\pi_1 + 3\pi_2 + 4\pi_3 \Rightarrow 6\pi_3 = 3\pi_1 + 3\pi_2 \Rightarrow \underline{2\pi_3} = \pi_1 + \pi_2$$

$$11\pi_1 \ \cancel{14\pi_1} = 8\pi_2 + \cancel{3\pi_1} + 3\pi_2 \Rightarrow \pi_1 = \pi_2$$

$$\underline{\Pi} = [\Pi_1 \ \Pi_2 \ \Pi_3] = [\Pi_1 \ \Pi_1 \ \Pi_1]$$

$$\sum \Pi_i = 1, \quad \Pi_1 + \Pi_1 + \Pi_1 = 1 \Rightarrow \Pi_1 = 1/3$$

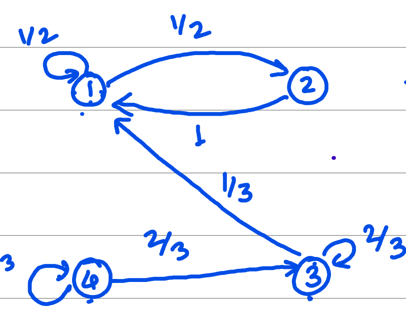
$$\Pi = [1/3 \ 1/3 \ 1/3]$$

①
$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix} \Rightarrow ? \quad \underline{[x, x, y, y]} \quad \underline{\underline{x, y \in (0,1)}}$$

$$\underline{\underline{x + y = 1/2}}$$

Stationary distribution may or may not be unique.

②
$$\frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$



Transition Graph

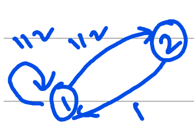
$C(1) = \{1,2\}, \quad C(3) = \{3\}, \quad C(4) = \{4\} \Rightarrow$ Reducible MC

$d(1) = 1 \quad d(3) = 1 \quad d(4) = 1 \Rightarrow$ aperiodic

$$F_1 = f_1^1 + f_1^2 + f_1^n + \dots$$
$$= \frac{1}{2} + \frac{1}{2}\cdot 1 + 0 + \dots$$
$$= 1 \qquad \Rightarrow F_1 = 1 \Rightarrow 1,2 \text{ are recurrent}$$

$$F_3 = f_3^1 + f_3^2 + f_3^3 + \dots$$
$$= \frac{2}{3} + 0 + 0 + \dots$$
$$= \frac{2}{3} < 1 \Rightarrow \text{state } 3 \text{ is transient}$$

$$F_4 = f_4^1 + f_4^2 + f_4^3 + \dots$$
$$= \frac{1}{3} + 0 + \dots$$
$$= \frac{1}{3} < 1 \Rightarrow 4 \text{ is transient}$$



$$\mu_{11} = \sum_n n f_1^n = 1\cdot f_1^1 + 2\cdot f_1^2 + 0 = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5 < \infty$$

Recurrence is class property

$$F_2 = f_2^1 + f_2^2 + f_2^3 + f_2^4 + \dots + f_2^n + \dots$$
$$= 0 + \frac{1}{2} + \frac{1}{2}\cdot 1\cdot\frac{1}{2} \qquad + \frac{1}{2}\cdot 1\cdot\left(\frac{1}{2}\right)^{n-2} = \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\dots\right)$$
$$= \frac{1}{2}\cdot\left(\frac{1}{1-1/2}\right) = 1 \quad \Rightarrow \quad 2 \text{ is recurrent.}$$

$$\mu_2 = \sum n f_2^n \Rightarrow 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2^2} + \dots + n\cdot\frac{1}{2^{n-1}} + (n+1)\frac{1}{2^n}$$
$$\Rightarrow \sum (n+1)\frac{1}{2^n} = \sum n\cdot\frac{1}{2^n} + \sum\frac{1}{2^n} < \infty$$

$\Rightarrow$ 2 is positive recurrent

1,2, aperiodic, positive recurrent $\Rightarrow$ Ergodic States

Lets find out limiting distribution / Stationary

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4] = [\pi_1\ \pi_2\ \pi_3\ \pi_4]\ \frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$6\pi_1 = 3\pi_1 + 6\pi_2 + 2\pi_3 \qquad \cancel{6\pi_1 = 3\pi_1 + 3\pi_1}$$
$$6\pi_2 = 3\pi_1 \qquad \Rightarrow \pi_1 = 2\pi_2$$
$$6\pi_3 = 4\pi_3 + 4\pi_4 \Rightarrow \pi_3 = 0$$
$$6\pi_4 = 2\pi_4 \qquad \Rightarrow \pi_4 = 0 \Rightarrow \qquad 4\pi_4 = 0 \Rightarrow \pi_4 = 0$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \Rightarrow 2\pi_2 + \pi_2 + 0 + 0 = 1 \Rightarrow \pi_2 = 1/3$$
$$\Rightarrow \pi_1 = 2/3$$

Stationary Distribution $[1/3 \quad 2/3 \quad 0 \quad 0]$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 1 & 1 \\ & & 1 & 1 \end{bmatrix}$$

$\underline{\underline{\pi}} = [\pi_1\ \pi_1\ \ \pi_3\ \pi_3]$

$\pi_1 + \pi_1 + \pi_3 + \pi_3 = 1$

$\Rightarrow \underline{\underline{\pi_1 + \pi_3 = 1}}$

$\underline{\underline{\pi_1 = x}}$

unique

$\underline{[x\ \ x\ \ 1-x\ \ 1-x]}$

$\lim_{n\to\infty} P_{ij}^n$ $i \to j$ in nsteps

$\to \underline{\underline{\pi_j}}$ ✓

$i \to$ doesn't matter

$\pi' = \pi' P$ eq^n $\to$ ✓

$\pi' = \pi' P$

$A\underline{x} = \underline{b}$,

$I\underline{\pi} = P'\underline{\pi}$

$\Rightarrow (P' - I)\underline{\pi} = \underline{0}$

$\pi_1 + \pi_2 + \pi_3 = 1$

$$[1\ 1\ 1]\begin{bmatrix}\pi_1\\ \pi_3\\ \pi_3\end{bmatrix} = 1$$

$$A = \begin{bmatrix} P' - I \\ \text{ones} \end{bmatrix} \qquad b = \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$$

No limiting distribution exists

$$\begin{array}{c c} & \\ 1 & \\ 2 & \end{array}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$$

$P^n \Rightarrow$ $P^{2n} = I$, $P^{2n+1} = P$

$\pi' = \pi' P$

initial $= \pi$ π

~~$P_2 = \pi P P^2$~~

$P(X_0 = x) = \alpha$ $\lim_{n\to\infty} P(X_n = x) \to \pi$

x_1 αP x_2 $\alpha . P^2$

$\alpha = \pi$ x $\pi P = P$ $\pi . P^2$

$\pi . P . P^2$

$P P^2$

Theo.:- An irreducible aperiodic MC belongs to one of the following classes :-

j transient $\forall i \in S$ $P_{ij}^n \to 0$

① Either the states are all transient or all null recurrent in this case, $P_{ij}^n \to 0 \ \forall i, j$ and $\nexists$ no stationary distⁿ.

② Or else, all states are positive recurrent, that is ($F_j = 1$, $\mu_{jj} < \infty$)

$$\pi_j = \lim_{n \to \infty} P_{ij}^n > 0$$

unique stationary distⁿ limiting distⁿ

In this case, $\{\pi_j, j = 0, 1, 2 \ldots\}$ is stationary distⁿ & $\nexists$ no other stationary distribution.

Proof: We will prove (ii) $S = \{0, 1, 2, \ldots\} = \{0, 1, 2, \ldots, M, \ldots\}$ for some $M \in S$

$\sum_{j=0}^{\infty} P_{ij}^n = 1$ $j \in S$

$$\sum_{j=0}^{M} P_{ij}^n \le \sum_{j=0}^{\infty} P_{ij}^n = 1 \quad \forall \text{ all } M \in S$$

$\sum_{j=0}^{M} P_{ij}^n \le 1$

letting $n \to \infty$ yields $\sum_{j=0}^{M} P_{ij}^n \le 1$

$$\sum_{j=0}^{\infty} \lim_{n\to\infty} P_{ij}^{n} \le 1$$

π_j

$$\sum_{j=0}^{M} \pi_j \le 1 \qquad \forall\ M \in S$$

$M \to \infty$ $\quad S = \{0, 1, \ldots\}$

implying that

$$\sum_{j=0}^{\infty} \pi_j \le 1$$

$\geq$ $\quad S = \{0, 1, \ldots, M, \ldots\}.$

Now

$$P_{ij}^{n+1} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{1} \geq \sum_{k=0}^{M} P_{ik}^{n} P_{kj} \qquad \forall\ M \in S \qquad \text{C-K eq}^n$$

CK $P_{ij}^{m+n} = \sum_{k \in S} P_{ik}^{m} P_{kj}$

letting $n \to \infty$ yields

$$\lim_{n\to\infty} P_{ij}^{n+1} \geq \lim_{n\to\infty} \sum_{k=0}^{M} P_{ik}^{n} P_{kj} \qquad \forall\ M \in S$$

π_j $\quad \lim_{n\to\infty} P_{ik}^{n} \Rightarrow \pi_k$

$\lim_{n\to\infty} P_{ij}^{n} = \lim_{n\to\infty} P_{ij}^{n+1} = \pi_j$

implying that

$$\pi_j \geq \sum_{k=0}^{M} \left(\lim_{n\to\infty} P_{ik}^{n}\right) P_{kj} \qquad \forall\ M \in S$$

$$\pi_j \geq \sum_{k=0}^{M} \pi_k \cdot P_{kj} \qquad \forall\ M \in S$$

$$\Rightarrow \quad \pi_j \geq \sum_{k=0}^{\infty} \pi_k P_{kj} \qquad \forall\ j \geq 0$$

$>$: $\otimes$

Q.

To prove equality, assume strict inequality for some j

$$\Rightarrow \quad \pi_j > \sum_{k=0}^{\infty} \pi_k P_{kj} \qquad k \to j \qquad j = 0, 1, \ldots \infty$$

summation

$$\Rightarrow \quad \sum_{j=0}^{\infty} \pi_j > \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \pi_k P_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \pi_k \quad 1$$

$$\Rightarrow \quad \sum_{j=0}^{\infty} \pi_j > \sum_{k=0}^{\infty} \pi_k \qquad \text{[which is contradiction]}$$

$P_j = P(X_n = j)$

$p^{(2)} = P.P$

$p^{(1)} = \alpha.P$

$\pi' = \pi P$

$$\Rightarrow \quad \boxed{\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}} \qquad \forall\, j \in S = \{0,1,2,\ldots\}$$

Putting $P_j = \dfrac{\pi_j}{\sum_k \pi_k}$ $\Rightarrow$ P_j is stationary dist^n

$X_1 \rightarrow \alpha.P$

$X_1 \quad X_0$

$\alpha = \pi$

$\therefore$ At least one stationary dist^n exists.

Now let $\{P_j, j=0,1,2\ldots\}$ be any stationary distribution.

Then if $\{P_j, j=0,1,2,\ldots\}$ is the prob. dist^n of X_0 (Initial Prob.dist)

$X_1 \sim \alpha P$

$X_1 \sim \pi.P.$

$\pi.P = \pi'$ $\quad X_1$

$$P_j = P\{X_n = j\}$$

$$= \sum_{i=0}^{\infty} P\{X_n = j / X_0 = i\} \; P\{X_0 = i\}$$

$$P_j = \sum_{i=0}^{\infty} P_{ij}^{n} P_i \qquad \text{——— (1)}$$

$$\Rightarrow \quad P_j \geq \sum_{i=0}^{M} P_{ij}^{n} P_i \qquad \forall\, M \in S$$

Let $n \to \infty$ and $M \to \infty$ $\quad P_j \geq \sum_{i=0}^{M} \left(\lim_{n\to\infty} P_{ij}^{n}\right) . P_i$ $\quad \pi_j$

stationary

$$P_j \geq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j \underbrace{\left(\sum_{i=0}^{\infty} P_i\right)}_{1} = \pi_j \qquad \text{——— (2)}$$

limiting

$$\Rightarrow \quad P_j \geq \pi_j$$

Lets prove other side.

CK

$$P_j = \sum_{i=0}^{\infty} P_{ij}^{n} P_i = \sum_{i=0}^{M} P_{ij}^{n} P_i + \sum_{i=M+1}^{\infty} P_{ij}^{n} . P_i$$

$$\leq \sum_{i=0}^{M} P_{ij}^{n} P_i + \sum_{i=M+1}^{\infty} P_i \qquad (\text{as } P_{ij}^{n} \leq 1)$$

$$\lim_{n\to\infty} P_j \leq \sum_{i=0}^{M} \left(\lim_{n\to\infty} P_{ij}^{n}\right) . P_i + \sum_{i=M+1}^{\infty} P_i$$

$$P_j \leq \sum_{i=0}^{M} \pi_j \cdot P_i + \sum_{i=M+1}^{\infty} P_i$$

$$\leq \pi_j \left(\sum_{i=0}^{M} P_i\right) + \sum_{i=M+1}^{\infty} P_i$$

letting $M \to \infty$

$$P_j \leq \pi_j \underbrace{\left(\sum_{i=0}^{\infty} P_i\right)}_{1} + 0$$

$$P_j \leq \pi_j \qquad \text{—③}$$

from ② & ③ $\Rightarrow$ $P_j = \pi_j \quad j = 0, 1, 2, \ldots,$ is only stationary distⁿ.

If all states are transient or null recurrent and $\{P_j, j = 0, 1, 2, \ldots\}$ is stationary distⁿ, then ① eqⁿ holds i.e.

$$P_j = \sum_{i=0}^{\infty} P_{ij}^{n} P_i$$

$\searrow 0 \quad \nearrow 1 \quad \to 0$

and $P_{ij}^{n} \to 0$, which is impossible.
Thus for case ①, no stationary distⁿ exists.

Note:-

* In the irreducible, positive recurrent, we have that $\pi_j, j \geq 0$ are unique non-negative solution of

$$\pi_j = \sum_i \pi_i P_{ij} \quad \& \quad \sum_j \pi_j = 1$$

i.e. In long run, π_j is the proportion of time that the MC is in state j.

$$\Rightarrow \quad \pi_j = \frac{1}{\mu_{jj}}$$

$$\Rightarrow \quad \lim_{n\to\infty} P_{jj}^{n \cdot d(j)} = \frac{d}{\mu_{jj}} = d \cdot \pi_j \qquad (d = d(j) \ \forall j)$$

Prob. Dist^n $\{P_j, j \geq 0\}$ is said to be stationary for MC if

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij} \qquad j \geq 0$$

* Random Walk

X_0 = initial position $\qquad x_0 = 100$

$$Z_i = \begin{cases} +1 & \text{with } p \\ -1 & \text{with } 1-p \end{cases}$$

$x_1 = x_0 + z_1$ ← outcome of 1^{st} trial/game

$x_2 = x_1 + z_2$

$x_n = x_{n-1} + z_n$ ← outcom of n^{th} game/trial

$x_1 = x_0 + z_1$

$x_2 = x_0 + \sum_{i=1}^{2} z_i$

$x_n = x_0 + \sum_{i=1}^{n} z_i$

$\{x_i, i \in I\} \Rightarrow \underset{\text{Statespace}}{S} = \{-\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \infty\}$

↑ Random Walk

R.W. $X_n = X_0 + \sum_{i=1}^{n} Z_i$

if Z_i takes ± 1 values

$$Z_i = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$$

↳ Simple ~~Rand~~ RW

If

$$Z_i = \begin{cases} +\omega.. & \text{w.p. } 1/2.. \\ -\omega.. & 1/2.. \end{cases}$$

↳ symmetric r.w.

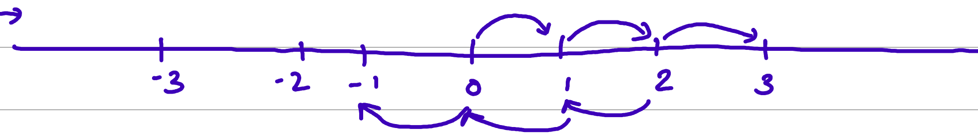
$$Z_i = \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & 1/2 \end{cases}$$

Simple Symmetric RW.

Ireducible MC

$C(0) = \{0, 1, -1, 2, -2, \dots\}$ Single communicating class

S.R.W →



Does random walk follows markov property?

Random Walk $X_n = X_0 + \sum_{i=1}^{n} Z_i$ $Z_i \sim$ i.i.d.

$$P[X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \dots X_0 = i_0]$$
$$= P[X_{n+1} = j / X_n = i]$$

$$P[X_0 + \sum_{i=1}^{n} Z_i = j / X_0 + \sum_{i=1}^{n-1} Z_i = i, \dots]$$

$$= P[\underline{X_0 + \sum_{i=1}^{n-1} Z_i} + Z_i = j / \underline{X_0 + \sum_{i=1}^{n-1} Z_i = i}, \dots]$$

$$= P[Z_i = j - i / X_0 + \sum Z_i = i] \_\ . \ ]$$

as Z_i are i.i.d

$\underline{X_n \text{ is MC}}$

$$X_n = X_0 + \sum_{i=1}^{n} Z_i \quad \text{or } X_n = X_{n-1} + Z_i$$

* TPM for random walk

$$\begin{array}{c|ccccccc}
 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
-3 & 0 & p & & & & & \\
-2 & q & 0 & p & & & & \\
-1 & & q & 0 & p & & & \\
0 & & & q & 0 & p & & \\
1 & & & & q & 0 & p & \\
2 & & & & & q & 0 & p \\
3 & & & & & & &
\end{array}$$

* Gambler's Ruin Problem

X_0 = initial amount $\qquad X_n$ = Gambler's Fortune at n^{th} game

$$Z_i = \begin{cases} +1 & p \\ -1 & q \end{cases} \quad \text{i.i.d games}$$

Interest $\underline{X_n \to 0}$ Gambler $\underline{\text{Ruined}}$

A B

x_0 $N-X_0$

Markov chain $S = \{0, 1, 2, \dots, N\}$

0

N

1 +1

-1

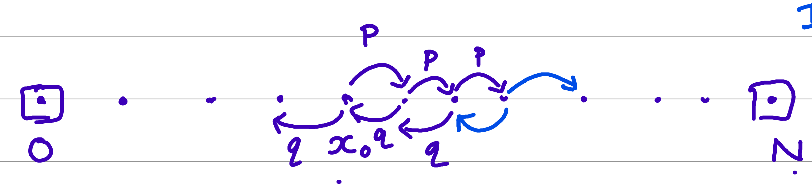
Absorbing State

$$\begin{array}{c|cccccc}
 & 0 & 1 & 2 & \cdots & N-1 & N \\
\hline
\to 0 & 1 & 0 & & & & \\
1 & q & 0 & p & & & \\
2 & & q & 0 & p & & \\
N-1 & & & & q & 0 & p \\
N & & & & & & 1
\end{array}$$

Absorbing State

$d(i) = \gcd\{n / P_{ii}^{n} > 0\}$

Transition graph



Irreducible

period i

$d_i = 1$? ✗

$= 2$? ✓

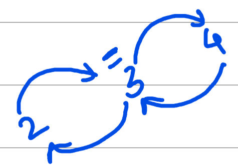
✓

	A	B
	x_0	$N - x_0$
+1	$x_0 + 1$	$N - x_0 - 1$
−1	x_0	$N - x_0$

let $N = 4$

$2n$

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & q & 0 & p & & \\ 2 & & q & 0 & p & \\ 3 & & & q & 0 & p \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$



Classes $\{0\}, \{N\}, \underbrace{\{1, 2, \ldots, N-1\}}$

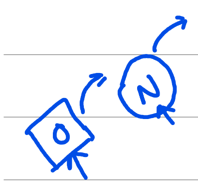
$\{0\}, \{N\}$ ↓ Recurrent

$\{1, 2, \ldots, N-1\}$ ↓ Transient

Long run probabilities

P_{ij}^{1} $\quad i \to j$ $\qquad j = i-1, i+1$

$\qquad\qquad\qquad\qquad\qquad\quad\;\; q \quad\; p$



✓ $P_{00} = 1 = P_{NN}$

✓ $P_{i,i+1} = p = 1 - P_{i,i-1} \qquad i = 1, 2, \ldots, N-1$

Let f_i denote the prob. that starting from i gambler's fortune reaches to state N before reaching 0.

f_i
$i \to N$

$i \to N$ $\quad$ $i+1 \to N$ $\quad$ $i-1 \to N$

$i < \begin{matrix} i+1 \cdot p \\ i-1 \cdot q \end{matrix}$

$$f_i = p \cdot f_{i+1} + q \cdot f_{i-1} \qquad i = 1, 2, \ldots, N-1$$

$p+q=1$ $\qquad$ $p+q+r=1$ $\quad$ r $\quad$ tie $\quad$ $r>0$

$$(p+q) \cdot f_i = p \cdot f_{i+1} + q f_{i-1}$$

$$q(f_i - f_{i-1}) = p(f_{i+1} - f_i)$$

$$\left[(f_{i+1} - f_i) = \frac{q}{p}(f_i - f_{i-1}) \right] \qquad i = 1, 2, \ldots, N-1$$

$f_0 = ?$
$f_0 = 0$

$i=1, \quad f_2 - f_1 = \frac{q}{p}(f_1 - f_0) = \frac{q}{p}(f_1)$

$i=2, \quad f_3 - f_2 = \frac{q}{p}(f_2 - f_1) = \frac{q}{p}\left(\frac{q}{p} f_1\right) = \left(\frac{q}{p}\right)^2 f_1$

$i=j \quad f_{j+1} - f_j = \left(\frac{q}{p}\right)^j f_1$

$i=N-1 \quad f_N - f_{N-1} = \left(\frac{q}{p}\right)^{N-1} f_1$

sum of first 2 terms $\quad f_3 - \cancel{f_2} + \cancel{f_2} - f_1 = \frac{q}{p} f_1 + \left(\frac{q}{p}\right)^2 f_1$

j terms $\quad f_{j+1} - f_1 = f_1 \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^j \right]$

$$f_{j+1} = f_1 \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \ldots + \left(\frac{q}{p}\right)^j \right]$$

if $q/p = 1$
$\Rightarrow q = p = 1/2$ $\quad f_{j+1} = f_1 [1 + 1 + 1 + \ldots + 1] = (j+1) f_1 \Rightarrow f_j = j \cdot f_1$

geometric series

$$\frac{1}{1-x} = 1+x+x^2+\dots+x^n+x^{n+1}+\dots+\dots \qquad x<1$$

$$\frac{1}{1-x} = \text{sum} + x^{n+1}[1+x+\dots]$$

$$\frac{1}{1-x} = \text{sum?} + \frac{x^{n+1}}{1-x}$$

if $\frac{q}{p} \neq 1 < 1$

$$\text{sum?} = \frac{1-x^{n+1}}{1-x} \quad \checkmark$$

$f_0 = 0$ ~~$\to N$~~

$f_i \quad i \to N$

$N \to N$

$f_N = 1$

$$f_j = f_1\left[1+\frac{q}{p}+\dots+\left(\frac{q}{p}\right)^{j-1}\right] = \begin{cases} f_1 \dfrac{1-(q/p)^j}{1-q/p} & q/p \neq 1 \\ j \cdot f_1 & q/p = 1 \end{cases}$$

$$f_N = 1, \quad j=N \Rightarrow \quad f_N = \begin{cases} f_1 \dfrac{1-(q/p)^N}{1-q/p} & q/p<1 \\ N \cdot f_1 & q/p=1 \end{cases}$$

But we are not going to discuss ✓

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & & & \\ 1 & q & r & p & \\ 2 & & q & r & p \\ 3 & & & & 1 \end{array}$$

$r>0 \qquad p+q+r=1$

$$1 = \begin{cases} f_1 \dfrac{1-(q/p)^N}{1-q/p} \\ f_1 \, N \end{cases}$$

$$\Rightarrow \quad f_1 = \begin{cases} \dfrac{1-q/p}{1-(q/p)^N} & q/p<1 \\ 1/N & q/p=1 \end{cases}$$

$$f_j = \begin{cases} \dfrac{\cancel{1-q/p}}{1-(q/p)^N} \cdot \dfrac{1-(q/p)^j}{\cancel{1-q/p}} = \dfrac{1-(q/p)^j}{1-(q/p)^N} & q/p<1 \\ = j \cdot \dfrac{1}{N} = \dfrac{j}{N} & \dfrac{q}{p}=1 \end{cases}$$

Gambler's Ruin prob.

$$1-f_j = \begin{cases} 1 - \dfrac{1-(q/p)^j}{1-(q/p)^N} & q/p<1 \\ 1 - \dfrac{j}{N} & q/p=1 \end{cases}$$

limitting $N\to\infty$

$$\lim_{N\to\infty} f_j = \begin{cases} \lim_{N\to\infty} \dfrac{1-(q/p)^j}{1-(q/p)^N} = 1-(q/p)^j & q/p<1 \quad (q/p)^N<1 \to 0 \\ \lim_{N\to\infty} \dfrac{j}{N} = 0 \end{cases}$$

Thaughtful.

Determine the expected number of steps/games that gambler makes, starting from state i, before reaching 0 or N.

$$(i) \longrightarrow (0) \longrightarrow -i$$
$$(i) \longrightarrow (N) \longrightarrow N-i$$

$$B = \min\left\{ m \;\middle|\; \sum_{i=1}^{m} X_i = -i \text{ or } N-i \right\}$$

$$X_i = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } q \end{cases}$$

$$E(X_i) = +1(p) + (-1)(1-p)$$
$$= p - 1 + p$$
$$= 2p - 1$$

by Wald's eq^{ns}

$$E\left(\sum_{i=1}^{B} X_i\right) = E(B)\cdot E(X_i) = (2p-1)\cdot E(B)$$

started from i, to reach N, $f_i = \dfrac{1-(q/p)^i}{1-(q/p)^N}$ ✓

i, to reach 0, $1-f_i$.

$$\sum_{j=1}^{B} x_j = \begin{cases} -i & 1-f_i \\ n-i & f_i \end{cases}$$

$$E\left(\sum_{j=1}^{B} x_j\right) = -i\cdot(1-f_i) + (N-i)\cdot f_i = -i + \cancel{if_i} + N f_i - \cancel{if_i}$$

$$(2p-1)\cdot E(B) = -i + N\left(\frac{1-(q/p)^i}{1-(q/p)^N}\right)$$

$$E(B) = \frac{1}{(2p-1)}\left[-i + N\left(\frac{1-(q/p)^i}{1-(q/p)^N}\right)\right]$$

* $N=4$, $i=2$ $p=0.6$ $q=0.4$ $\dfrac{q}{p} = \dfrac{0.4}{0.6} = \dfrac{2}{3}$

$$f_2 = \frac{1-(q/p)^i}{1-(q/p)^N} = \frac{1-(2/3)^2}{1-(2/3)^4} = \frac{3^2-2^2}{3^4-2^4} \times \frac{\cancel{3^4}\ 3^2}{\cancel{3^2}}$$

winning $$= \frac{9-4}{81-16} \times 9 = \frac{45}{65} = \frac{9}{13}$$

Ruin $1-f_2 \Rightarrow 4/13$

~~Irreducible~~ MC

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.4 & 0.1 \\ 2 & 0.5 & 0.5 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

Reduces

3 - absorbing state

stationary / limiting distⁿ $[0 \; 0 \; 1]$

$C(1) = \{1,2\} = C(2)$, $C(3) = \{3\}$

absorbing state

After sufficiently large no. of steps process $\xrightarrow{\text{is in}}$ 3

$$P = \begin{array}{c|cc} & S-A & A \\ \hline S-A & N & B \\ A & O & I \end{array}$$

$N = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$ $B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$

$\underline{0} = [0 \; 0]$

$A = \{3\}$

$$\boxed{j \in A \quad P[X_n = j, X_{n-1} \notin A, X_{n-2} \notin A, \ldots x_0] = \underline{\alpha}' N^{n-1} B_j}$$

j^{th} column of B

$x_0 = 1$

$$\alpha = \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ [1 \; 0 \; 0] \end{array}$$

$$P[X_n = 3, X_{n-1} \neq 3, X_{n-2} \neq 3, \ldots, X_0 = 1] = [1 \; 0 \; 0] \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}^{n-1} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

* Simulation of RW

$$Z_i = \begin{cases} +1 & p \\ -1 & q \end{cases}$$

simulate?

$$X_n = X_0 + \sum_{i=1}^{n} Z_i = X_{n-1} + Z_n$$

$X_0 = 100$

$n = 7$ simulate?

$X_n = X_{n-1} + \underline{Z_n}$ $\quad X_n = X_0 + \Sigma Z_i$ $\quad Z_i = \begin{cases} +1 & p = \\ -1 & q \end{cases}$

$\underline{n=7}$

$Z_1 = \text{rbinom}(\overset{1,1,p.}{\underline{1,p}})$ $\quad X_0 = 100$ $\quad$ 7 R.S.

$Z_2 =$ $\quad X_1 = X_0 + Z_1$

$=$ $\quad X_2 = X_1 + \underline{Z_2}$ $\quad$ sample(

?

$=$

$=$

$=$

General R.W

$$Z_i = \begin{cases} +2 & 0.25 \\ -1 & 0.35 \\ 0 & 0.2 \\ 1 & 0.2 \end{cases}$$

Absent: 2001, 3, 6, 10, 16, 17, 22, 23, 33, 35, 38, 39, 43, 44, 45, 47, 50, 51, 55