

Unit - I - Real Numbers  $\mathbb{R}$ 

~~Groups Mat~~  
~~( $\mathbb{R}, +, \cdot$ )~~

\* Algebraic Properties of  $\mathbb{R}$  :-Add<sup>~</sup> Mult<sup>~</sup>On set of  $\mathbb{R}$  there are two binary operators  $\underline{+}$  &  $\underline{\cdot}$ 

These two operations follows few properties :-

## A1) Commutative property of addition

$$a+b = b+a \quad \forall a, b \in \mathbb{R}$$

A2) Associative property of add<sup>~</sup>

$$(a+b)+c = a+(b+c) \quad \forall a, b, c \in \mathbb{R}$$

## A3) Existence of zero element ( Additive Identity ) ✓

$$a+0 = 0+a = a \quad \forall a \in \mathbb{R}$$

## A4) Existence of negative element ( Additive Inverse )

$$a+(-a) = (-a)+a = 0 \quad \forall a \in \mathbb{R}$$

## ✓ M1) Commutative property of multiplication

$$a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$$

## M2) Associative p of multi.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in \mathbb{R}$$

## M3) Existence of unit element ( Multiplicative identity )

$$a \cdot 1 = 1 \cdot a = a \quad \forall a \in \mathbb{R}$$

## M4) Existence of multiplicative inverse / Reciprocals ✓

$$a \cdot \left(\frac{1}{a}\right) = \frac{1}{a} \cdot a = 1 \quad \forall a \in \mathbb{R} - \{0\}$$

D) Distributive property of multiplication over add<sup>~</sup>

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a$$

$$= a \cdot b + a \cdot c$$

$$\forall a, b, c \in \mathbb{R}$$

## \* Order Properties of IR

**Why do we study Real Analysis?**

seq of func  
 $\sum f_n(x) = \sum \frac{x_i}{n}$

Convergence of Series of func's

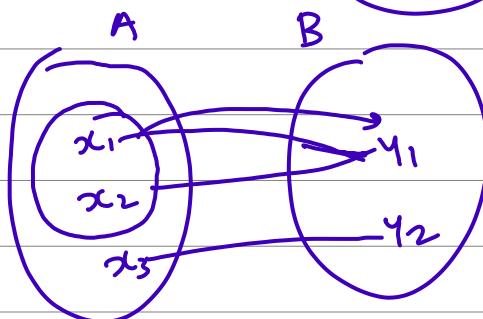
$$\text{CLT} \rightarrow \frac{\sum x_i - \mu}{\sigma} \rightarrow \text{N}(0, 1)$$



$$\left( \frac{\sum x_i}{n} \right) \rightarrow \bar{x}$$

Set Func

func



Prob.

input element  $\rightarrow$  Set  $n(A) \in \mathbb{N}$   
 classi  $n(S)$

input Set  $\rightarrow$  Real

ST-201

(A)

Borel func

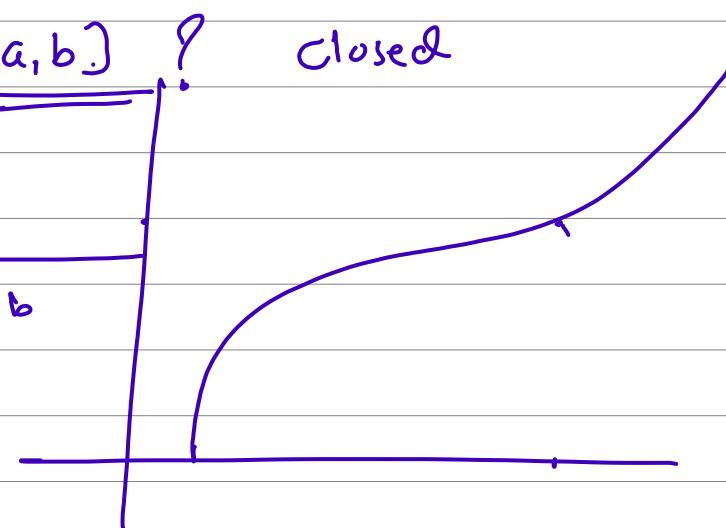
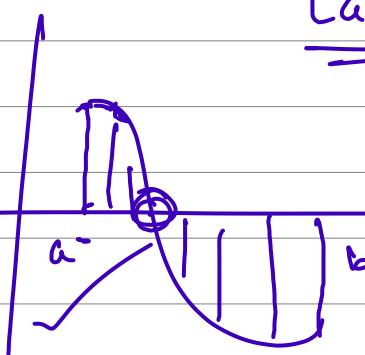
$$x^3 + 2x + 3 = 0 \quad x=? \quad \mathbb{R}$$

[a, b]

closed

ST-705  
Numerical Methods

Cont?



$$\underline{\text{Series}} \rightarrow P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$E(X) = \sum_x \frac{e^{-\lambda} \lambda^x}{x!} x$$

$$= e^{-\lambda} \sum_x \frac{\lambda^x}{x!} x$$

$$= e^{-\lambda} \lambda \sum_x \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \lambda e^{\lambda} \quad ?$$

Geometric

$$P(X=x) = pq^{x-1}$$

$$E(X) = \sum x \cdot pq^{x-1}$$

$$= 1/p$$

$$= p \cdot \sum x \cdot q^{x-1} \quad ?$$

$$= p \cdot \frac{1}{(1-q)^2}$$

$$= p \cdot \frac{1}{p^2} = \frac{1}{p}$$

$$\sum x^n = \frac{1}{1-x} \quad |x| < 1$$

$$\text{?} \quad \frac{d}{dx} \sum x^n = \frac{d}{dx} \frac{1}{1-x}$$

$$\text{?} \quad \frac{d}{dx} n x^{n-1} = \frac{1}{(1-x)^2}$$

Real  
AnalysisInverse fun<sup>c</sup>

X

Random Variable

ST-201  
Probability  
ST-301  
Asymptotic

- ① Unbiased
- ② Suff
- ③ Eff
- ④ Consistent  $\rightarrow$  as  $n \rightarrow \infty$

as  $n \rightarrow \infty$  $\bar{x} \rightarrow \mu$ 

$$\frac{1}{n} \sum x_i \rightarrow \mu$$

$$x \sim U(0, \infty)$$

$$2\bar{x}$$

## \* Order Properties of R

$P$  set of tve nos  $\in \mathbb{R}$ ,  $P \subseteq \mathbb{R}$ ,  $\mathbb{R}^+$

$\mathbb{R}^+$  satisfies following properties

- ① if  $a, b \in \mathbb{R}^+$   $\Rightarrow a+b \in \mathbb{R}^+$   
② if  $a, b \in \mathbb{R}^+$   $\Rightarrow a \cdot b \in \mathbb{R}^+$

③ if  $a \in \mathbb{R}$  then exactly one of the following is true:-

$$\frac{a \in \mathbb{R}^+}{X}, \text{ OR } \frac{a=0}{X}, \text{ OR } \frac{-a \in \mathbb{R}^+}{\checkmark}$$

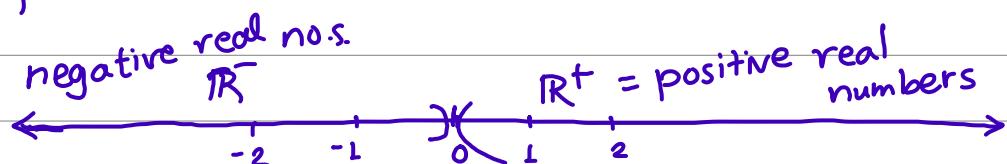
$$\begin{array}{c} \underline{a \in \mathbb{R}} \\ \begin{matrix} > 0 & < 0 & = 0 \\ a \in \mathbb{R}^+ & a \in \mathbb{R}^- \\ -a \in \mathbb{R}^+ \end{matrix} \end{array}$$

## Law of Trichotomy

Let  $a, b \in \mathbb{R}$

- a) If  $\underline{a} - b \in \mathbb{R}^+$   $\Rightarrow a > b$  or  $b < a$

b) If  $\underline{a} - b \in \mathbb{R}^+ \cup \{0\}$   $\Rightarrow a \geq b$  or  $b \leq a$



$$\checkmark \quad \mathbb{R}^+ = \{x / x > 0, x \in \mathbb{R}\}$$

$$R^- = \{x / x < 0, x \in R\}$$

**Proof :-** @  $a, b \in \mathbb{R}$ ,  $a - b \in \mathbb{R}^+$   $\Rightarrow a - b > 0 \Rightarrow a > b$

$$\textcircled{5} \quad a, b \in \mathbb{R} \quad a - b \in \mathbb{R}^+ \cup \underline{\underline{\{0\}}} \Rightarrow a - b \geq 0 \Rightarrow a \geq b$$

Theo:- Let  $a, b, c \in \mathbb{R}$

- ① If  $a > b$  &  $b > c \Rightarrow a > c$  (Transitivity)
  - ② If  $a > b \Rightarrow a+c > b+c$
  - ③ If  $a > b, c > 0 \Rightarrow a \cdot c > b \cdot c$   
If  $a > b, c < 0 \Rightarrow a \cdot c < b \cdot c$

① If  $a > b$ , &  $b > c$

$$a - b > 0 \text{ & } b - c > 0$$

$$\Rightarrow a - b \in \mathbb{R}^+ \text{ & } b - c \in \mathbb{R}^+$$

$$\Rightarrow (a - b) + (b - c) \in \mathbb{R}^+$$

$$\Rightarrow (a - c) \in \mathbb{R}^+$$

$$\Rightarrow a - c > 0$$

$$\Rightarrow a > c$$

(by order properties)

QED

②  $a - b \in \mathbb{R}^+$

$$a - b + c - c \in \mathbb{R}^+$$

$$(a + c) - (b + c) \in \mathbb{R}^+$$

$$a + c > b + c$$

③  $a > b$ ,  $c > 0$

to prove  $a \cdot c > b \cdot c$

$$\begin{array}{c} a > b, \\ \underline{a - b \in \mathbb{R}^+} \end{array}$$

$$c(a - b) = c \cdot a - c \cdot b \Leftarrow \text{as } c > 0$$

$$\cancel{c a - c b > 0} \quad \Leftarrow$$

$$ca > cb$$

similarly  $a > b$ ,  $c < 0 \Rightarrow ac < bc$

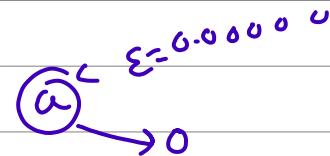
Theo:-

If  $a \in \mathbb{R}$ , such that  $0 \leq a < \varepsilon \Rightarrow \varepsilon \in \mathbb{R}$ ,  $\varepsilon > 0$

$$\Rightarrow a = 0$$

$$\varepsilon = 0.005$$

08



By method of contradiction.

Assume  $a > 0 \Rightarrow (as 0 \leq a < \varepsilon) \Rightarrow a$  is positive  
 $\Rightarrow \frac{a}{2}$  is positive

we can assume  $\varepsilon = a/2 < a$

for any  $\varepsilon > 0$ ,  $0 \leq a < \varepsilon$  but for  $\varepsilon_0 \Rightarrow 0 < \varepsilon_0 < a$

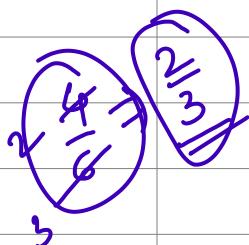
which contradicts to our assumption.  $\square$

Rational Nos. :-  $Q = \{x / x = p/q, p, q \in \mathbb{Z}, q \neq 0\}$

\* Theo. There doesnot exist any rational no.  $\sqrt{2} \in Q$

$$\underline{\underline{r^2 = 2}}$$

$\Rightarrow$  Assume that  $\sqrt{2}$  is rational no.



$$\sqrt{2} = \frac{p}{q}$$

common divisor of  $p, q$  is 1  
 $\hookrightarrow (p, q) = 1$ .

$$\Rightarrow 2 = \frac{p^2}{q^2}$$

$$\Rightarrow 2q^2 = p^2 = p \cdot p$$

$$\rightarrow p = 3 \quad p^2 = 9 \quad p = 4 \quad p^2 = 16$$

$\Rightarrow p$  is divisible by 2

$$\Rightarrow p = 2 \cdot m$$

$$\Rightarrow p^2 = 2^2 \cdot m^2 = 4m^2$$

~~$p = \text{even}$~~

$$\Rightarrow 2q^2 = 4m^2$$

$$\Rightarrow q^2 = 2m^2$$

$\Rightarrow q$  is divisible by 2 (\*\*\*\*)

$(p,q) = 1$  but here  $(p,q) = 2$

which contradicts to our assumption -

$$\begin{aligned}
 & \text{Q} \quad \frac{\text{even}^2}{(2 \cdot n)^2} \quad \frac{\text{odd}^2}{(2n+1)^2} \\
 &= 2^{\text{even}} \cdot (2^n)^2 \quad = 2^{\text{odd}} \cdot \underline{(2n+1)^2} \\
 & \qquad \qquad \qquad = 4n^2 + 2n + 1 \\
 & \qquad \qquad \qquad = 2 \cdot \underline{(2n^2+n)} + 1 \\
 & \qquad \qquad \qquad = \text{odd}
 \end{aligned}$$

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Theo:- If  $ab > 0$  then either ①  $a > 0, b > 0$   
 ②  $a < 0, b < 0$

① If  $a, b \in \mathbb{R}$  show that  $a^2 + b^2 = 0 \Leftrightarrow a = 0, b = 0$

② If  $0 < c < 1$  show that  $0 < c^2 < c < 1$  ✓

③ If  $x, y \in \mathbb{Q}$ ,  $x+y \in \mathbb{Q}$ ,  $\underline{x \cdot y \in \mathbb{Q}}$  ✓

If  $x \in \mathbb{Q}, y \in \mathbb{Q}^c$ ,  $x+y \in \mathbb{Q}^c$

$$\begin{aligned}
 & (a+b)^2 = a^2 + b^2 + 2ab = 0 \\
 & a^2 + b^2 = 0 \Rightarrow 2ab = 0 \\
 & \Rightarrow a \cdot b = 0
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{?}{=} a = 0 \quad \text{or} \quad b = 0 \\
 & \text{if } a = 0 \quad \& \quad b \neq 0 \Rightarrow a^2 + b^2 > 0
 \end{aligned}$$

$$a^2 + b^2 = 0 \Rightarrow b = 0$$

$$a, b \in \mathbb{R}^+, c > 0,$$

$$a > b \Rightarrow c \cdot a > c \cdot b$$

$$0 < c < 1$$

, as  $c > 0$

$$c < 1$$

$$c \cdot c < 1 \cdot c$$

$$\underline{0 < c^2 < c < 1}$$



$$x, y \in \mathbb{Q} \Rightarrow x+y \in \mathbb{Q}$$

$$x = \frac{p_1}{q_1}, y = \frac{p_2}{q_2} \Rightarrow x+y = \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$$

$$\text{as } p_1 q_1, p_2 q_2 \in \mathbb{Z},$$

$$\underline{p_1 q_2 \in \mathbb{Z}, p_2 q_1 \in \mathbb{Z}}, q_1 q_2 \in \mathbb{Z}$$

$$p_1 q_2 + p_2 q_1 \in \mathbb{Z}$$

$$= \frac{p^*}{q^*} \in \mathbb{Q}$$

Absolute values,  $\forall a \in \mathbb{R}$

$$|a| = \begin{cases} +a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

$|a| = \max(a, -a)$

Theo:- ①  $|ab| = |a||b| \Rightarrow a, b \in \mathbb{R}$  ✓

②  $|a|^2 = a^2 \Rightarrow a \in \mathbb{R}$   $-c \leq a+b \leq c \Rightarrow |a+b| \leq c$

③ If  $c \geq 0$  then  $|a| \leq c$  iff  $-c \leq a \leq c$

④  $-|a| \leq a \leq |a|$

Proof:- ① If  $a, b > 0 \Rightarrow ab > 0 \Rightarrow |a|=a, |b|=b, |a \cdot b|=ab=|a||b|$

$a > 0, b < 0 \Rightarrow ab < 0 \Rightarrow |a|=a, |b|=-b, |ab|=-ab=a \cdot (-b)=|a||b|$

$a < 0, b > 0 \Rightarrow ab < 0 \Rightarrow$  Simillarly

$a \cdot b < 0 \Rightarrow ab > 0 \Rightarrow |a|=-a, |b|=-b, |ab|=ab=(-a) \cdot (-b)$

$= |a||b|$  ✓

②  $|a|^2 = a^2$

If  $a > 0, |a|=a \Rightarrow |a|^2 = a^2$  ✓

$a < 0 \quad |a| = -a \Rightarrow |a| = (-a)^2 = a^2$  ✓

$a = 0 \quad |a| = a = 0 \Rightarrow |a|^2 = a^2 = 0$  ✓

③ If  $c \geq 0$  then

$|a| \leq c$  iff  $-c \leq a \leq c$

i)  $|a| \leq c \Leftrightarrow |a| = \max(a, -a) \leq c$

$\Leftrightarrow a \leq c \text{ & } -a \leq c$

$\Leftrightarrow a \leq c \text{ & } a \geq -c$

$\Leftrightarrow -c \leq a \leq c$  ✓

ii)  $|a| \leq c$  iff  $-c \leq a \leq c$  ✓

now assume  $c = |a| \geq 0, |a| \leq |a|$

$\Rightarrow -|a| \leq a \leq |a|$

\* Triangular inequality :- If  $a, b \in \mathbb{R}$ ,  $|a+b| \leq |a| + |b|$

proof :- If  $a, b \in \mathbb{R}$ , then

$$\begin{array}{r} -|a| \leq a \leq |a| \\ + \quad -|b| \leq b \leq |b| \\ \hline -|a|-|b| \leq a+b \leq |a|+|b| \end{array}$$

$$\text{put } c = |a| + |b|$$

$$\Rightarrow -(|a| + |b|) \leq a+b \leq |a| + |b|$$

$$\Rightarrow -c \leq a+b \leq c$$

$$\Rightarrow |a+b| \leq c$$

$$\Rightarrow |a+b| \leq |a| + |b|$$

$$|a| = |-a|$$

\* Corollary : If  $a, b \in \mathbb{R}$ , then (a)  $||a|-|b|| \leq |a-b|$

$$(b) |a-b| \leq |a| + |b|$$

(a) If  $a, b \in \mathbb{R}$

$$\checkmark a = \underline{a-b+b}$$

$$|a| = |\underline{a-b+b}| \leq |a-b| + |b| \quad \text{--- (by triangular inequality)}$$

$$\checkmark b = b-a+a$$

$$|b| = |b-a+a| \leq |\underline{b-a}| + |a| = |\underline{a-b}| + |a| \quad (\text{by tri. inequality})$$

$$\text{From (a)} \quad |a| - |b| \leq |a-b|.$$

$$(**) \quad |b| - |a| \leq |a-b| \Rightarrow |a| - |b| \geq -|a-b|$$

$$\Rightarrow \boxed{\begin{array}{l} -c \leq a \leq c \\ \Rightarrow |a| \leq c \end{array}} \quad \boxed{-|a-b| \leq |a| - |b| \leq |a-b|}$$

$$\text{put } c = |a-b| \Rightarrow ||a|-|b|| \leq c \Rightarrow ||a|-|b|| \leq |a-b|$$

$$\frac{-|a-b|}{c} \leq \frac{|a|-|b|}{c} \leq \frac{|a-b|}{c}$$

$\Rightarrow |a|-|b| \leq c$

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$$\textcircled{b} \quad |a-b| \leq |a| + |b|$$

We have triangular inequality  $|a+b| \leq |a| + |b|$

replace  $b$  by  $-b$

$$|a+(-b)| \leq |a| + |-b|$$

$$\Rightarrow |a-b| \leq |a| + |-b| \quad (|-b| = |b|)$$

$$\Rightarrow |a-b| \leq |a| + |b|$$

$$a-b \leq a+b \quad ?$$

$$a, b \in \mathbb{R}$$

Triangular inequality

$$|\sum_{i=1}^n a_i| \leq \sum_{i=1}^n |a_i|$$

\* Real line

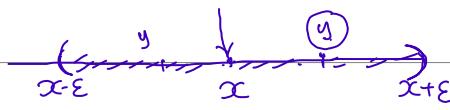
Extended Real Numbers



$\varepsilon$ -nbhd.

$$V_\varepsilon(x) = \{y / y \in (x-\varepsilon, x+\varepsilon)\} \subseteq \mathbb{R}$$

$$(x-\varepsilon, x+\varepsilon) = V_\varepsilon(x)$$



interval length  $\Rightarrow 2\varepsilon$

$$|x-y| \leq \varepsilon \Rightarrow y \in V_\varepsilon(x)$$

$$V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$$

$$y \in V_\varepsilon(x) \Rightarrow |x-y| \leq \varepsilon$$

$$\Rightarrow x-\varepsilon \leq y \leq x+\varepsilon$$

$$\delta_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) - \{x\}$$

deleted nbhd of  $x$

e.g. we know  $|a+b| \leq |a| + |b|$  but if  $|a+b| = |a| + |b|$  iff  $ab > 0$

i) If  $ab > 0$  to prove  $|a+b| = |a| + |b|$   
 If  $a > 0, b > 0, a+b > 0 \Rightarrow |a+b| = a+b = |a| + |b|$   
 $a < 0, b < 0, a+b < 0 \Rightarrow |a+b| = -(a+b) = (-a) + (-b) = |a| + |b|$

$a > 0, b < 0$

If  $|a+b| = |a| + |b|$  to prove  $ab > 0$

$$\begin{aligned} |a+b|^2 &= (|a| + |b|)^2 \\ (a+b)^2 &= |a|^2 + |b|^2 + 2|a||b| \\ \Rightarrow a^2 + b^2 + 2ab &= a^2 + b^2 + 2|a||b| \\ \Rightarrow ab &= |a||b| = |a \cdot b| \quad \text{on} \\ \Rightarrow ab &> 0 \end{aligned}$$

e.g. If  $x, y, z \in \mathbb{R}$  &  $\underline{x \leq z}$  show that  
 $x \leq y \leq z$  iff  $|x-y| + |y-z| = |x-z|$

i) Let  $\underline{x \leq y \leq z}$ .  $x-y$  &  $y-z$  negative.

$$\begin{aligned} \text{LHS} &= |x-y| + |y-z| \\ &= (\cancel{y-x}) + (\cancel{z-y}) \\ &= \cancel{z-x} \\ &= |x-z| \checkmark \\ &= \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} x \leq y &\Rightarrow x-y \leq 0, y \leq z \Rightarrow y-z \leq 0 \\ |x-y| &= -(x-y) \\ &= y-x \\ &= z-y. \end{aligned}$$

$$\cancel{x \leq z} \Rightarrow x-z \leq 0$$

$$|x-z| = z-x$$

$$|x-y| \neq (x-y)$$

ii) Let  $|x-y| + |y-z| = |x-z|$ ,

$$x \leq z \Rightarrow x-z \leq 0, z-x \geq 0$$

$$x-z = x-y + y-z$$

$$\text{Top} \rightarrow x \leq y \leq z$$

$$x-z = \underline{x-y+y-z}$$

$$|x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z|$$

but  $|x-z| = |x-y| + |y-z|$  so. we can use

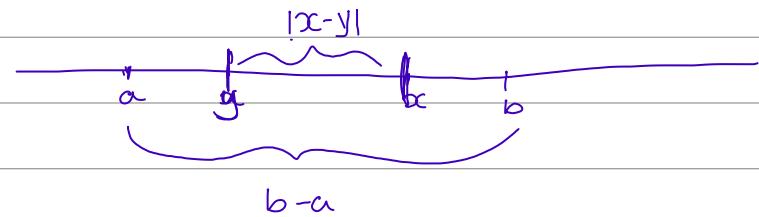
$$\boxed{|a+b| = |a| + |b| \text{ iff } \underline{ab \geq 0}}$$

$x-y \geq 0$   
 $x \geq y$

Only two possibilities

$(x-y)(y-z) > 0$	$\Rightarrow x > y, y > z \Rightarrow x > y \geq z$ , which is not possible
$(x-y)(y-z) \leq 0$	$\Rightarrow x \leq y, y \leq z \Rightarrow x \leq y \leq z$

if  $a < x < b$   
 $a < y < b$   
 $|x-y| \leq b-a$



?

$$\begin{aligned} -a > -x > -b &\Rightarrow -b < -x < -a \\ a < y < b &\quad \underline{\quad a < y < b} \\ -(b-a) < y-x < b-a & \\ \Rightarrow |x-y| &\leq b-a \end{aligned}$$

e.g.  $|x-1| > |x+1|$   $x \in ?$   $\underline{x < 0}$

$\underline{-3} \leq x < 0$   $\leq 0$   $-0.01$

$$|x+1| + |x-2| = 7 \quad x = -3 \quad x = 4$$

$$\text{e.g. } \min\{a,b\} = \frac{1}{2}(a+b - |a-b|) \quad \max\{a,b\} = \frac{1}{2}(a+b + |a-b|)$$

$$\rightarrow \text{① Let } \underline{a < b}, \quad \min(a,b) = a \quad \max(a,b) = b$$

$$|a-b| = b-a$$

$$\text{RHS} = \frac{1}{2}(a+b + |a-b|)$$

$$= \frac{1}{2}(a+b + b-a)$$

$$= \frac{2b}{2} = \max(a,b)$$

$$\text{RHS} = \frac{1}{2}(a+b - |a-b|)$$

$$= \frac{1}{2}(a+b - (b-a))$$

$$= \frac{2a}{2} = \min(a,b)$$

[github.com/manojcpatil/Lecture-scribbles](https://github.com/manojcpatil/Lecture-scribbles)

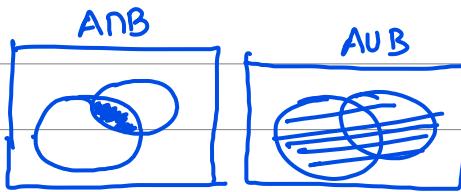
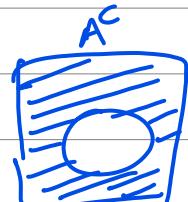
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Nikhil, Pranit, Bhanuja

\* Set Operations,  $A, B \subseteq \mathbb{R}$ . Union :  $A \cup B = \{x / x \in A \text{ or } x \in B\}$

Intersection :  $A \cap B = \{x / x \in A \text{ & } x \in B\}$

Complement  $A^c = \{x / x \notin A, x \in \mathbb{R}\}$

Subtract  $A - B = \underline{A \cap B^c} = A \setminus B$   
 $= A - (A \cap B)$



① subset  $A \subseteq B$  :  $\underline{x \in A} \Rightarrow x \in B \Rightarrow x \in A$

Theo:- ①  $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$   
 ②  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$A \subseteq B$



$A=B$   $\Rightarrow A \subseteq B \& B \subseteq A$  ? Can we say that?  
 $(A \cup B)^c = A^c \cap B^c$  Can you prove this?

To prove ①  $(A \cup B)^c \subseteq A^c \cap B^c$   
 ②  $A^c \cap B^c \subseteq (A \cup B)^c$

→ Let  $x \in (A \cup B)^c$

$$\Leftrightarrow x \notin A \cup B$$

$$\Leftrightarrow x \notin A \text{ and } x \notin B$$

$$\Leftrightarrow x \in A^c \text{ and } x \in B^c$$

$$\Leftrightarrow x \in A^c \cap B^c.$$

$$\Rightarrow (A \cup B)^c \subseteq A^c \cap B^c$$

$$\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c$$

$$\Rightarrow =$$



$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (\text{Distributive law})$$

$$\rightarrow \text{To prove } A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \checkmark$$

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Let  $x \in \underline{A \cap (B \cup C)}$

$\Rightarrow x \in \underline{(A \cap B) \cup (A \cap C)}$

$$\Leftrightarrow x \in A \text{ and } x \in B \cup C$$

$$\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } C)$$

$$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$$

$$\Leftrightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$$

$$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$$

$$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad ] \rightarrow$$

$$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

equal

e.g. If A and B are two sets such that

$$A \subseteq B \text{ iff } A \cap B = A$$

$$\text{i) } A \subseteq B \Rightarrow x \in A \Rightarrow x \in B \Rightarrow x \in A$$

$\Rightarrow$  each and every element of A is in B.

$$\Rightarrow A \cap B = A \subseteq B$$

$$\text{ii) } A \cap B = A \Rightarrow \text{To prove } A \subseteq B$$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} \subseteq B$$

$$A \cap B \subseteq B$$

$$\Rightarrow A \subseteq B$$

$$A \setminus (B \cup C) = A \cap (B \cup C)^c.$$

$$\begin{aligned} & A \setminus B \cap A \setminus C \\ &= (A \cap B^c) \cap (A \cap C^c) \end{aligned}$$

$$A \cap (B \cup C)^c \subseteq (A \cap B^c) \cap (A \cap C^c)$$

$\subseteq$

$$\text{Let } x \in A \cap (B \cup C)^c$$

$$\Leftrightarrow x \in A \text{ and } x \notin (B \cup C)^c$$

$\cup$ - or/and

$$\Leftrightarrow x \in A \text{ and } x \notin B \cup C$$

$$\Leftrightarrow x \in A \text{ and } [x \notin B \text{ and } x \notin C]$$

$$\Leftrightarrow x \in A \text{ and } x \notin B \quad \text{and} \quad x \in A \text{ and } x \notin C$$

$$\Leftrightarrow x \in A \setminus B \quad \text{and} \quad x \in A \setminus C$$

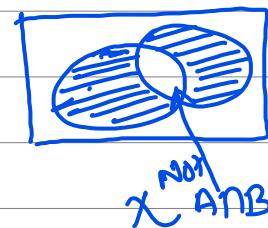
$$\Leftrightarrow x \in A \setminus B \cap A^c$$

Similarly we can prove ② part too

\* Symmetric Difference :-

$$\begin{cases} A \Delta B = \{x / x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\} = (A \cup B) \cap (A \cap B)^c \\ = (A \cup B) - (A \cap B) \\ = (A \cup B) \setminus (A \cap B) \end{cases}$$

$$x \in (A \cup B) \cap (A \cap B)^c$$



$$\begin{aligned} - (A \cup B) \setminus (A \cap B) &= (A \setminus B) \cup (B \setminus A) \\ &= (A \cap B^c) \cup (B \cap A^c) \end{aligned}$$

$$= \underline{(A \cap B^c)} \cup \underline{B} \cap \underline{(B \cap A^c)} \cup \underline{A^c}$$

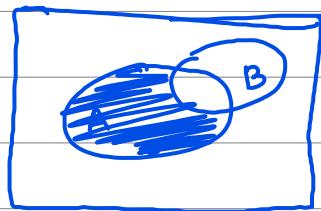
$$= (A \cup B) \cap \underbrace{B^c \cup B}_{B} \cap \underbrace{A \cup A^c}_{B} \cap \underbrace{B^c \cup A^c}_{B}$$

$$= (A \cup B) \cap (B^c \cup A^c)$$

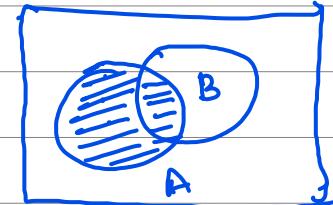
$$= (A \cup B) \cap (A \cap B)^c$$

$$= (A \cup B) \setminus (A \cap B)$$

$$\begin{aligned} A \setminus B \\ = A \cap B^c \end{aligned}$$



$$A \setminus (B \setminus A)$$



$$A \setminus (B \setminus A) = B \cap A^c$$

$$\downarrow \\ A \cap (B \setminus A)^c$$

$$= A \cap (B \cap A^c)^c$$

$$= A \cap (\underline{A \cup B^c})^c$$

$$= \underline{\underline{A}}$$

Lower bound :-

Upper bound

$$S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \Rightarrow \text{low } S = 0 \notin S$$

$$\text{Upper bound (S)} = 1 \in S$$

Bounded below.

$$\underline{\underline{\mathbb{N}}}, \underline{\underline{\mathbb{Z}}}, \underline{\underline{\left\{ \frac{1}{n}, n \in \mathbb{N} \right\}}}$$

Bounded Above.

$$\overline{\mathbb{N}}, \overline{\mathbb{Z}}, \overline{\mathbb{R}}$$

Bounded set bounded below &amp; Above

$$\left\{ (-1)^{2n}, n \in \mathbb{N} \right\} = \{1\} \text{ singleton set}$$

Unique?

$$\left\{ (-1)^n, n \in \mathbb{N} \right\} = \{-1, 1\}$$

$$\text{lower bound} = -1 \in S$$

$$\text{Upper bound} = +1 \in S$$

$$\underline{\underline{S = (0, 1]}}$$

$$x > 0 \Rightarrow x \in S$$

$$x \leq 1 \Rightarrow x \in S$$

$$\left\{ \begin{array}{l} 10^{10000} \\ -10 \\ -0.5 \end{array} \right\}$$

$$\text{Lower bound} = 0 \notin S$$

$$\text{Upper bound} = 1 \in S$$

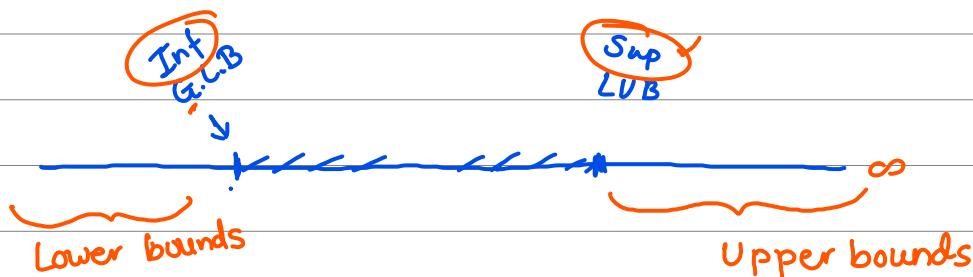
$$\underline{\underline{u}} \text{ l.b. of } S \Rightarrow \forall x \in S, \underline{x} \geq \underline{u}$$

Sup:-  $u$  is  $\sup$  of  $S$  if ①  $u$  is upper bound  
 ②  $v$  is any other upper bound of  $S$  then  $v \geq u$ .

$u$  must be least upper bound.

Inf:- if  $v$  is inf of  $S$  if ①  $v$  is lower bound  
 ②  $u$  is any other lower bound of  $S$  then  $v \leq u$

$v$  must be greatest lower bound.



No upper bound / Supremum :-  $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{I}, \mathbb{Z} \dots$

No lower bound / Inf :-  $-\mathbb{N}, \bar{\mathbb{Q}}, \bar{\mathbb{R}}, \bar{\mathbb{I}}, \mathbb{Z}^-$

$$S = \{1, 2, 3, 4\}$$

$$\sup(S) = 4$$

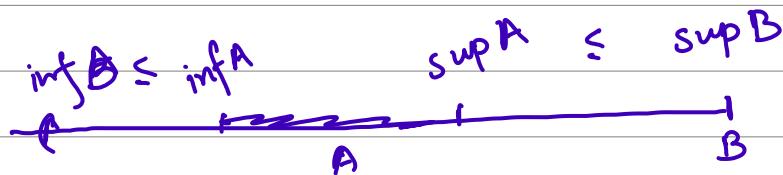
$$\inf(S) = 1$$

set of Upper bounds =  $[4, \infty)$

set of lower bounds =  $(-\infty, 1]$

\* Completeness property:- Lower bound exists  $\Rightarrow$  Inf exists  
 Upper  $\Rightarrow$  Sup

$$A \subseteq B \Rightarrow \inf A \geq \inf B$$



Infr Suppose

Absent:- Adity, Dipali, Anushree, Pradip, Akash Lalit  
vishal, komal, Samiya, Groninda, Atul, Chetan,  
Yogita .







































































































































































































































































































































































