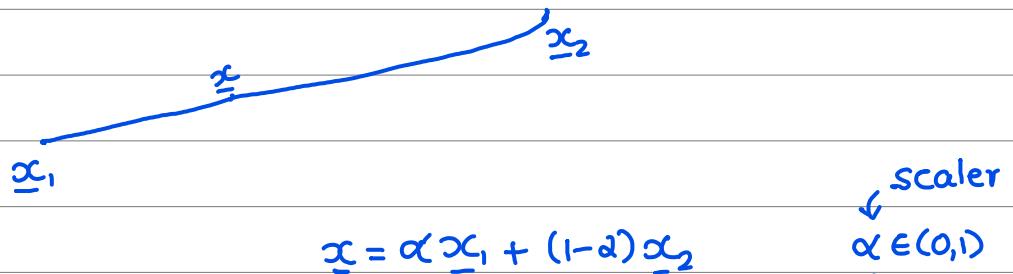


* Line segment : $\underline{x} \in \mathbb{R}^n$



line segment $\{ \underline{x} / \underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2, \alpha \in (0,1) \}$
joining $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$

* Line passing $\underline{x}, \underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$

line segments $\underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \quad \alpha \in (0,1)$

line 1 $\underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \quad \alpha \in \mathbb{R}$

vector space

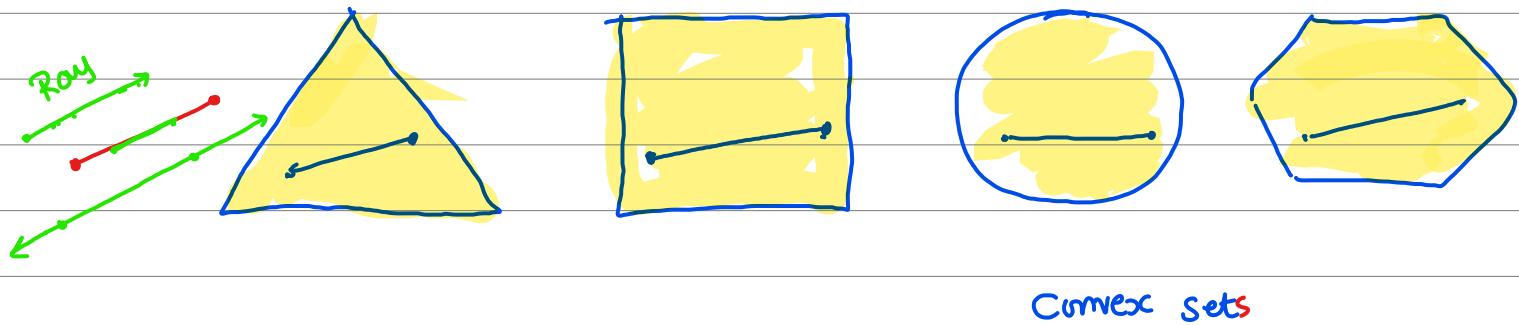
$\underline{x} = \sum \alpha_i \underline{x}_i$ Linear Comb' $\alpha_i \in \mathbb{R}$

$\underline{x} = \sum \alpha_i \underline{x}_i \quad \sum \alpha_i = 1, \alpha_i \geq 0$

↳ Convex Combination

$$\{ \underline{x} / \underline{x} = \sum_{i=1}^n \alpha_i \underline{x}_i, \alpha_i \geq 0, \sum \alpha_i = 1 \}$$

Convex set if $\underline{x}_1, \underline{x}_2 \in A$, $\underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \Rightarrow \alpha \in (0,1)$
 then if $\underline{x} \in S \Rightarrow \alpha \in A$ is convex set



Non convex sets

* Ray is convex set
 $\underline{x} = \underline{x}_0 + d\alpha \quad \alpha > 0$

$$\underline{x}_1, \underline{x}_2 \in A$$

$$\lambda \quad \underline{x}_1 = \underline{x}_0 + d\alpha_1$$

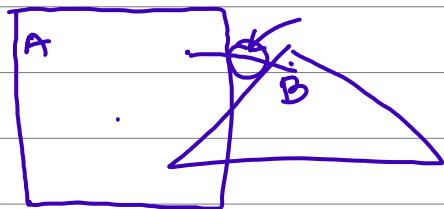
$$1-\lambda \quad \underline{x}_2 = \underline{x}_0 + d\alpha_2$$

$$\lambda \underline{x}_1 + (1-\lambda) \underline{x}_2 = (\lambda + 1 - \lambda) \underline{x}_0 + d(\lambda \alpha_1 + (1-\lambda) \alpha_2)$$

$$= \underline{x}_0 + d(\underbrace{\lambda \alpha_1 + (1-\lambda) \alpha_2}_{\geq 0}) \geq 0$$

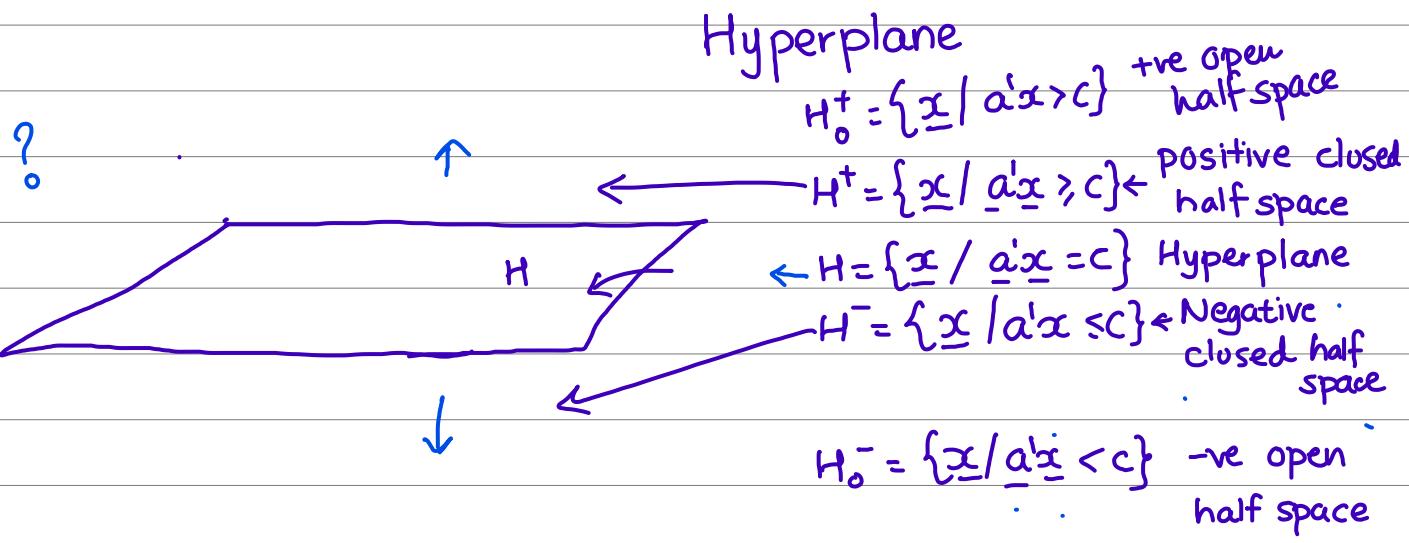
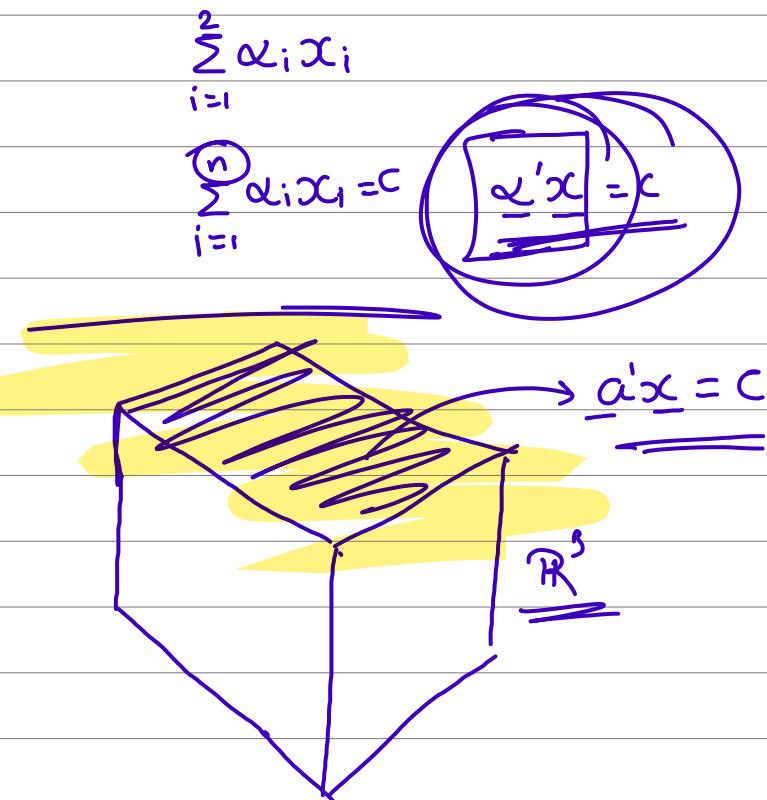
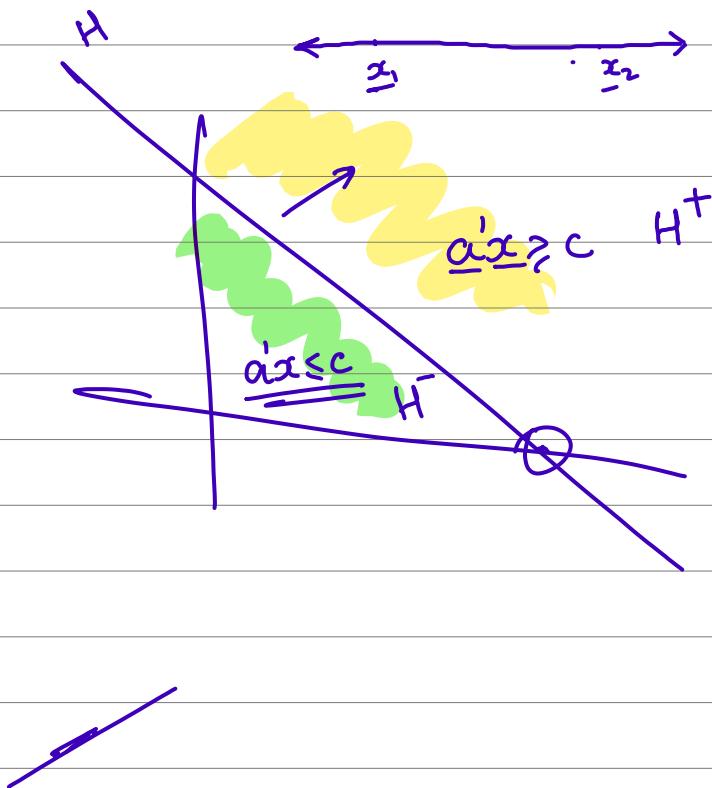


$$\in A$$



* Union of convex sets
may or may not be convex.

* Intersection of convex sets
is also convex.



Hyperplane $H = \{\underline{x} | \underline{a}'\underline{x} = c\}$

let $\underline{x}_1, \underline{x}_2 \in H \Rightarrow \underline{a}'\underline{x}_1 = c \text{ & } \underline{a}'\underline{x}_2 = c$

$$\left[\begin{aligned} \underline{a}'(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) &= \alpha \underline{a}'\underline{x}_1 + (1-\alpha) \underline{a}'\underline{x}_2 \\ &= \alpha c + (1-\alpha)c \\ &= c \end{aligned} \right]$$

$\Rightarrow \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \in H$

\Rightarrow Hyperplane is convex set

H^+, H^-, H_+^+, H_-^- all are convex sets

Open ball $\underline{B} = \{\underline{x} \mid \|\underline{x} - \underline{x}_0\| < r\}$

To show :- B is convex set.

$$\text{Let } \underline{x}, \underline{y} \in B \Rightarrow \underline{\alpha \cdot x + (1-\alpha) y} \in B$$

$$\underline{x} \in B \Rightarrow \|\underline{x} - \underline{x}_0\| < r$$

$$\underline{y} \in B \Rightarrow \|\underline{y} - \underline{x}_0\| < r$$

$$\alpha \in (0,1)$$

$$\begin{aligned} & \|\alpha \underline{x} + (1-\alpha) \underline{y} - \underline{x}_0\| \\ &= \|\alpha \underline{x} + (1-\alpha) \underline{y} - (\alpha \underline{x}_0 + (1-\alpha) \cdot \underline{x}_0)\| \end{aligned}$$

$$= \|\alpha(\underline{x} - \underline{x}_0) + (1-\alpha)(\underline{y} - \underline{x}_0)\|$$

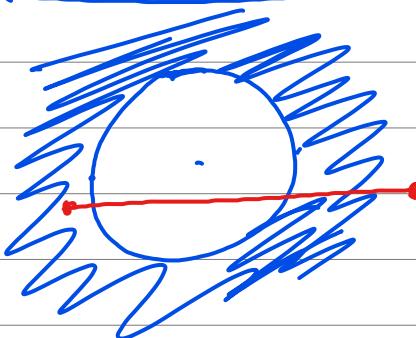
$$\leq \frac{\alpha \|\underline{x} - \underline{x}_0\|}{r} + (1-\alpha) \|\underline{y} - \underline{x}_0\| < r$$

$$< \alpha r + (1-\alpha) r$$

$$< r$$

$$\alpha \in (0,1)$$

$\cdot \left\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| = r \right\}$ Convex or Not



$\left\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| \geq r \right\}$ convex or not

If C is convex set $\underline{\lambda}C$ is also convex set.

$$\underline{\lambda}C = \{ \underline{y} \mid \underline{y} = \lambda \underline{x}, \underline{x} \in C \}$$

to show $\underline{\lambda}C$ as convex set

$$\underline{y}_1, \underline{y}_2 \in \underline{\lambda}C \Rightarrow \underline{y}_1 = \lambda \underline{x}_1, \underline{y}_2 = \lambda \underline{x}_2, \underline{x}_1, \underline{x}_2 \in C$$

$$\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 = \alpha \cdot \lambda \underline{x}_1 + (1-\alpha) \lambda \underline{x}_2$$

$$= \lambda (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$= \lambda \underline{x} \quad \begin{matrix} \text{as } C \text{ is convex} \\ (\text{as } \underline{x}_1, \underline{x}_2 \in C \Rightarrow \underline{x} \in C) \end{matrix}$$

$$\Rightarrow \alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 \in \underline{\lambda}C$$

C, D are convex sets $C+D$ is also convex

$$\rightarrow C+D = \{ \underline{z} \mid \underline{z} = \underline{x} + \underline{y}, \underline{x} \in C, \underline{y} \in D \}$$

$$\underline{z}_1, \underline{z}_2 \in C+D \Rightarrow \underline{z}_1 = \underline{x}_1 + \underline{y}_1$$

$$\underline{z}_2 = \underline{x}_2 + \underline{y}_2 \quad \underline{x}_1, \underline{x}_2 \in C, \underline{y}_1, \underline{y}_2 \in D$$

$$\alpha \underline{z}_1 + (1-\alpha) \underline{z}_2 = \alpha \cdot (\underline{x}_1 + \underline{y}_1) + (1-\alpha) (\underline{x}_2 + \underline{y}_2)$$

$$= \underbrace{\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2}_{\in C} + \underbrace{\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2}_{\in D}$$

Intersection of any convex sets is convex

Let $\{S_i\}_{i=1}^{\infty}$ be collection of convex sets

$\cap S_i$ is convex

$$\underline{x}, \underline{y} \in \cap S_i \\ \Rightarrow \underline{x}, \underline{y} \in S_i \quad \forall i$$

$$\Rightarrow \alpha \underline{x} + (1-\alpha) \underline{y} \in S_i \quad \forall i \quad (S_i \text{ is convex})$$

$$\Rightarrow \alpha \underline{x} + (1-\alpha) \underline{y} \in \cap S_i$$

$\Rightarrow \cap S_i$ is convex.

A set $S \in \mathbb{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S

∴ Assume that every convex combⁿ of (any finite no.) of points of S is in S .

⇒ it is also true for $n=2$

$$\Rightarrow \text{if } \underline{x}_1, \underline{x}_2 \in S \Rightarrow \alpha \cdot \underline{x}_1 + (1-\alpha) \underline{x}_2 \in S \Rightarrow \alpha \in (0,1)$$

⇒ S is convex set

II Assume S is convex and for any finite n

$$\sum_{i=1}^n \alpha_i \underline{x}_i \in S$$

\rightarrow let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in S$

$$\sum_{i=1}^n \alpha_i = 1$$

we will prove this by mathematical induction

As S is convex, $\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \in S$ $\underline{\alpha_i + 1 - \alpha = 1}$

\therefore So the above statement is true for $n=2$

Assume it is true for $\underline{n=k} \Rightarrow \sum_{i=1}^k \alpha_i \underline{x}_i = 1$

$$\sum_{i=1}^k \alpha_i = 1$$

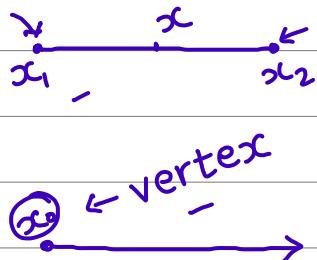
We have to prove it for $\underline{n=k+1}$

$$\begin{aligned} \sum_{i=1}^{k+1} \beta_i \underline{x}_i &= \left(\sum_{i=1}^k \beta_i \underline{x}_i \right) + \beta_{k+1} \underline{x}_{k+1} \\ &\quad \text{imp? } \left\{ \begin{array}{l} \sum_{i=1}^k \beta_i = 1 - \beta_{k+1} \\ \frac{\sum_{i=1}^k \beta_i}{1 - \beta_{k+1}} = 1 \end{array} \right. \\ &= (1 - \beta_{k+1}) \left[\sum_{i=1}^k \frac{\beta_i}{1 - \beta_{k+1}} \cdot \underline{x}_i \right] + \beta_{k+1} \underline{x}_{k+1} \end{aligned}$$

$$= (1 - \beta_{k+1}) \underline{x}^* + \beta_{k+1} \underline{x}_{k+1}$$

$\in S$ as S is convex set

* Vertices



2 vertices

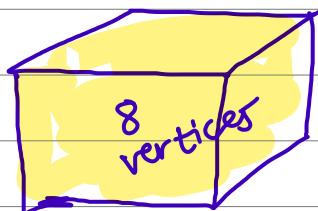
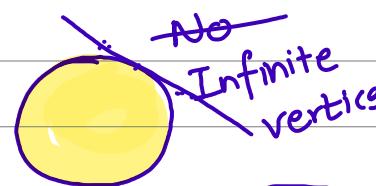
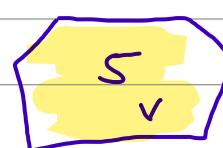
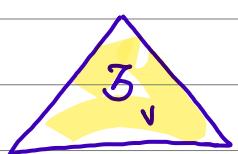
$$\Rightarrow \underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2$$

$$\underline{x}_2 = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 = \underline{x}_2$$

$\uparrow \alpha = 0$



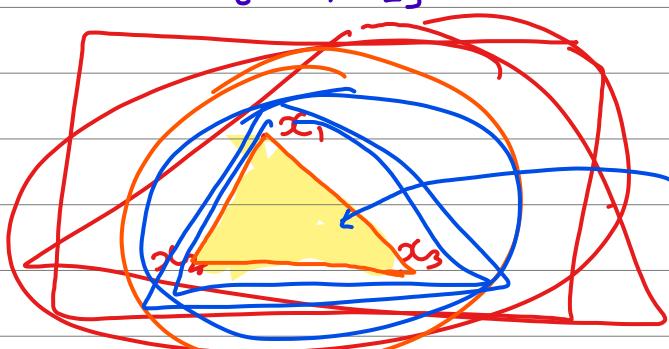
No vertex



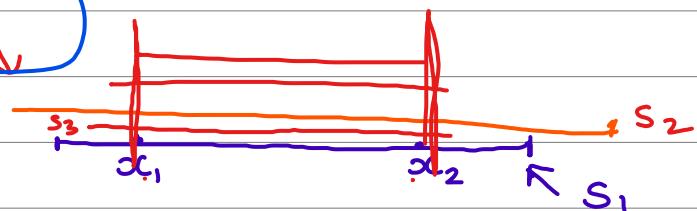
Convex Hull

$$\text{Co}(S) \Rightarrow \bigcap_{i=1}^{\infty} S_i$$

$$S = \{\underline{x}_1, \underline{x}_2\}$$



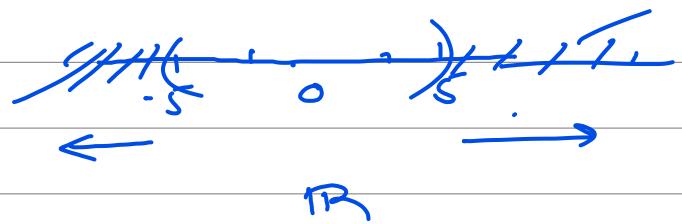
S_i is convex set containing S



$$S = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$$

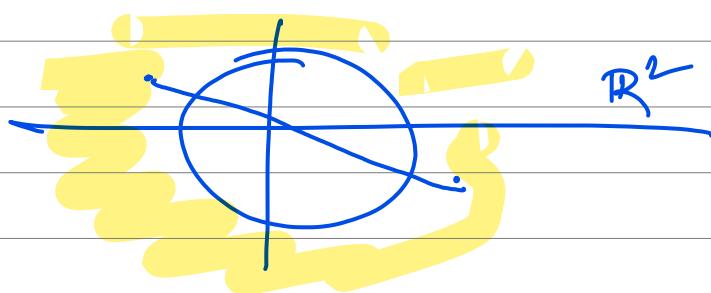
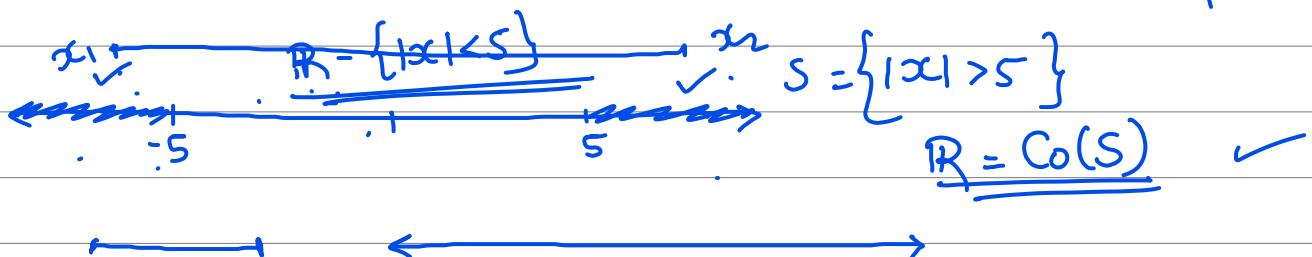
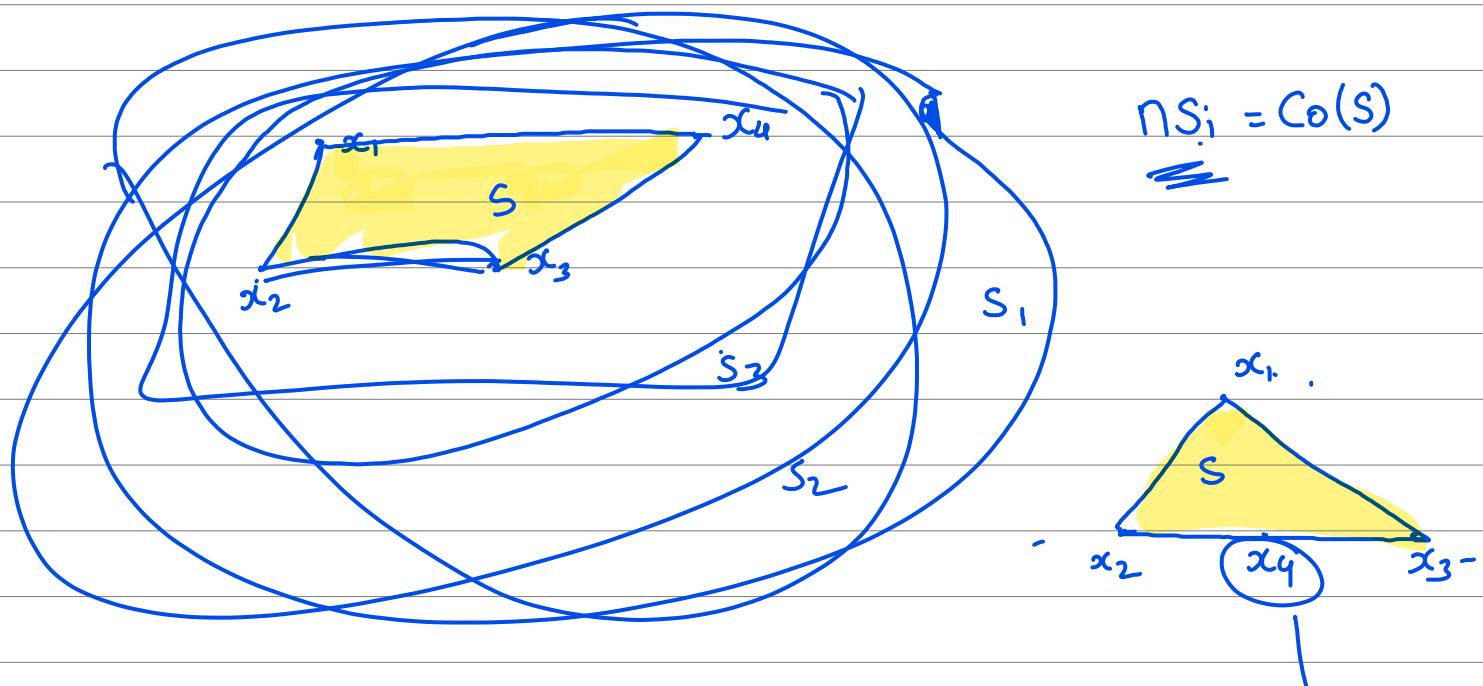
$$S = \{ \underline{x} \mid \| \underline{x} \| \geq 5 \}$$

$$C_0(S) = \mathbb{R}^n$$

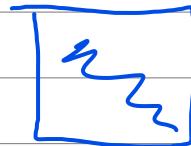
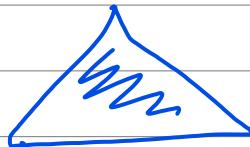


if set is convex
 $\underline{C_0(S) = S}$

$C_0(S) = \bigcap S_i$, S_i is convex set containing S .

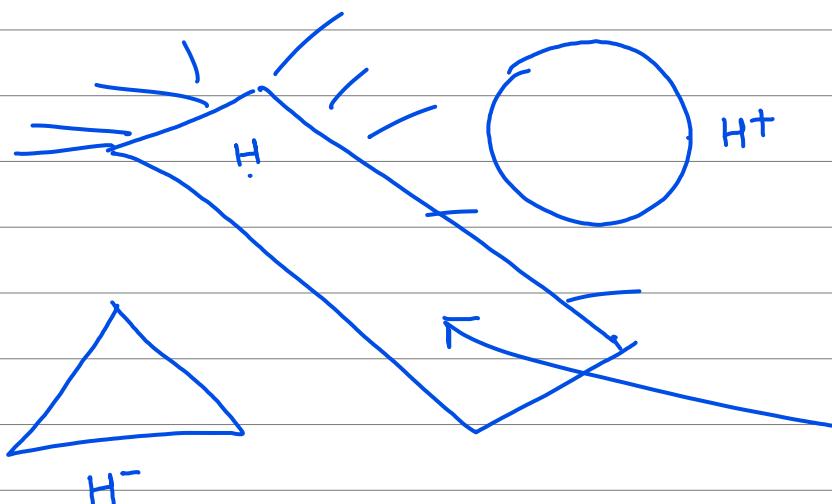


$$C_0(S) = \left\{ \underline{x} \mid \underline{x} = \sum_{i=1}^n \lambda_i x_i, \quad \lambda_i \in S \quad \sum \lambda_i = 1, \quad \lambda_i \geq 0 \right\}$$

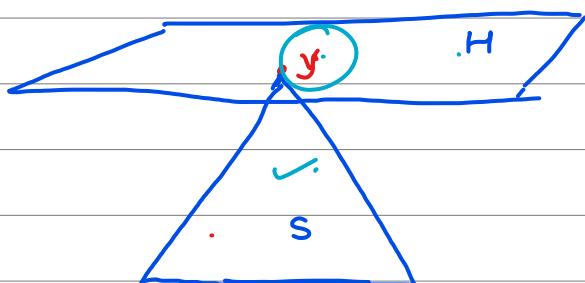


Hyperplane:-

$$H = \{ \underline{x} \mid \underline{a}' \underline{x} = c \}$$



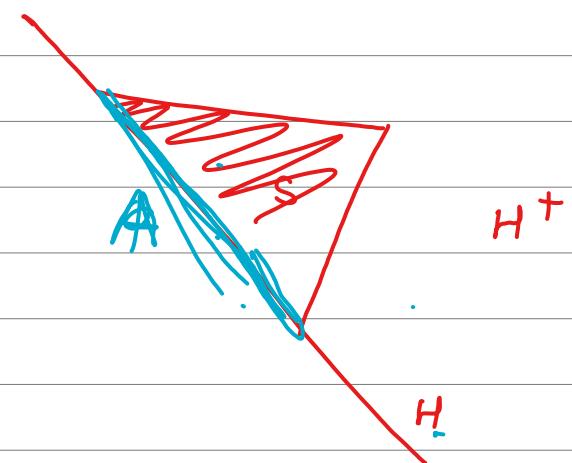
Separating Hyperplane

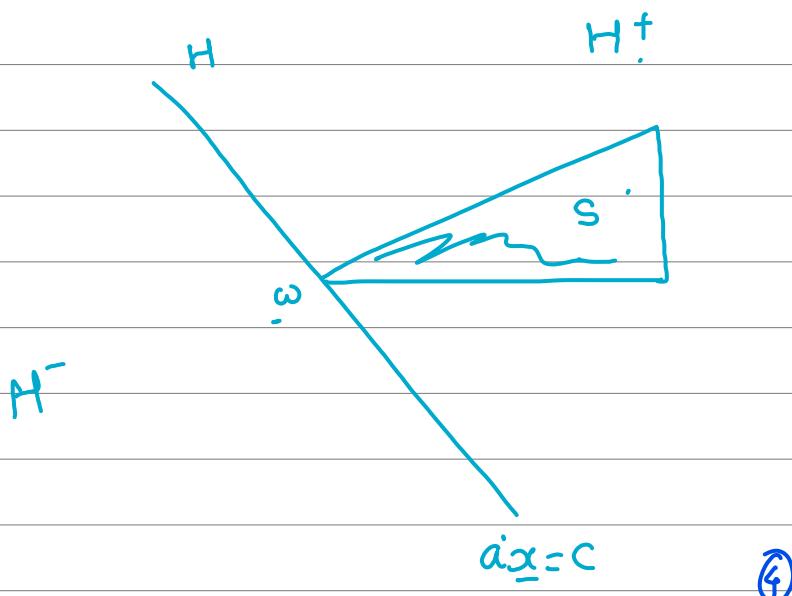


$$\begin{aligned} & s \in H^- \text{ or } s \in H^+ \\ & \underline{a}' \underline{x} \leq c \text{ or } \underline{a}' \underline{x} \geq c \\ & \Rightarrow x \in s \quad \Rightarrow x \in H \end{aligned}$$

$$H = \{ \underline{x} \mid \underline{a}' \underline{x} = c \}$$

$$\underline{a}' \underline{y} = c \Rightarrow \underline{y} \in s \Rightarrow \underline{y} \in H$$

 $s \in H^-$  H^-



$$\begin{aligned} S &\subset H \\ \forall x \in S, a^T x &\geq c \\ \omega & \in S \cap H \\ a^T \omega &= c \end{aligned}$$

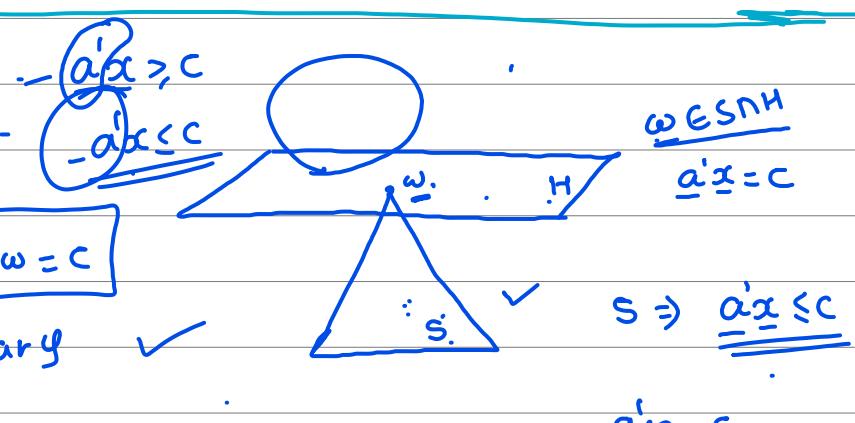
① Supporting hyperplane :-



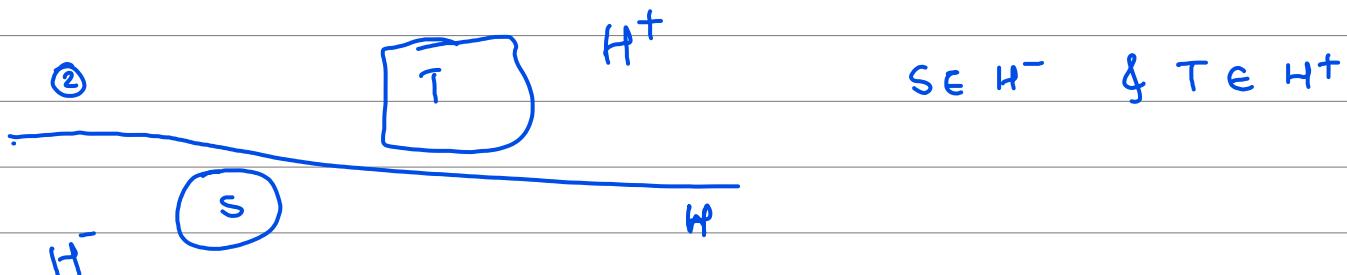
$$H: \{x \mid a^T x = c\}$$

$$a^T \omega = c$$

$\underline{w \in S} \leftarrow \text{boundary } S$

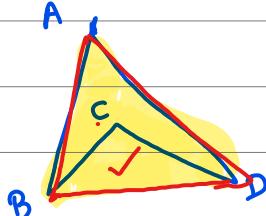


Separating hyperplane



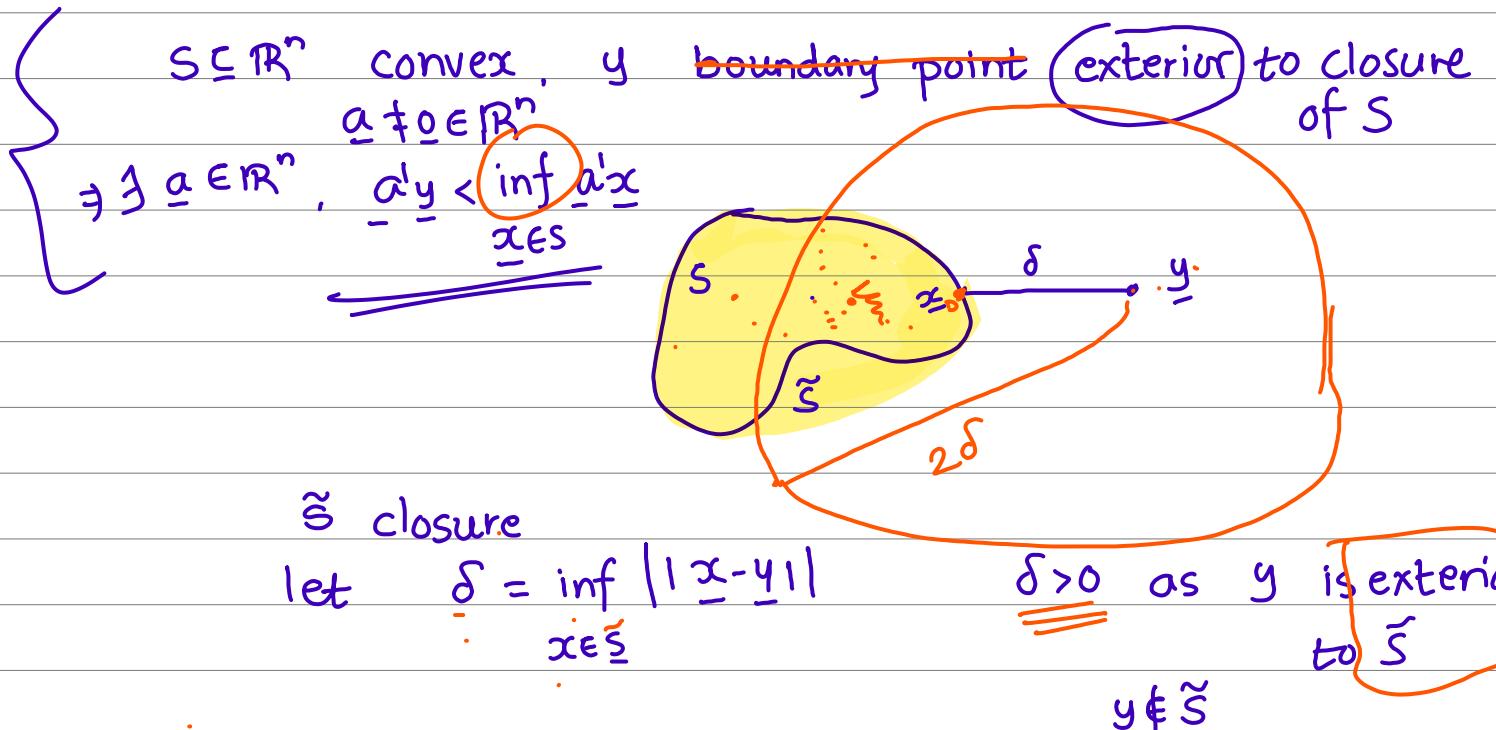
Closure. $S = (a, b) \cup \{a\} \cup \{b\} = [a, b]$

$$S' = (a, b) \cup \{a\} \cup \{b\} = [a, b]$$



$$|x - c| < \epsilon$$

$$c - \epsilon, c + \epsilon$$



$$\underline{B}_{2\underline{\delta}} = \{\underline{x} \mid \|\underline{x} - \underline{y}\| < 2\underline{\delta}\}$$

$$\underline{\delta} = \inf_{\underline{x} \in \tilde{S}} \|\underline{x} - \underline{y}\| = \inf_{\underline{x} \in \tilde{S} \cap \underline{B}_{2\underline{\delta}}} \|\underline{x} - \underline{y}\|$$

$\tilde{S} \cap \underline{B}_{2\underline{\delta}}$ is closed & bounded.
lets define $f: \tilde{S} \cap \underline{B}_{2\underline{\delta}} \rightarrow \mathbb{R}, f(\underline{x}) = \|\underline{x} - \underline{y}\|$
f. contin _____,
by max^m min^m theo., f attains its extremum in that set

\exists some $\underline{x}_0 \in \tilde{S} \cap \underline{B}_{2\underline{\delta}} \Rightarrow$

$$\underline{\delta} = \min_{\underline{x} \in \tilde{S} \cap \underline{B}_{2\underline{\delta}}} \|\underline{x} - \underline{y}\| = \|\underline{x}_0 - \underline{y}\|$$

$\Rightarrow \underline{x}_0$ is boundary pt. of \tilde{S}

To show

$$\underline{a} = \underline{x}_0 - \underline{y}$$

$$\underline{a}' \leq \inf_{\underline{x}} \underline{a}' \underline{x}$$

$$\underline{x}, \underline{x}_0 \in \tilde{S}, \quad \alpha \underline{x} + (1-\alpha) \underline{x}_0 \in \tilde{S}$$

$$\left\| \underline{\alpha} \underline{x} + (1-\alpha) \underline{x}_0 - \underline{y} \right\| \geq \left\| \underline{x}_0 - \underline{y} \right\|$$

$$\left\| (\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0) \right\| \geq \left\| \underline{x}_0 - \underline{y} \right\|$$

$$\left\| \underline{x}_0 - \underline{y} + \alpha(\underline{x} - \underline{x}_0) \right\|^2 \geq \left\| \underline{x}_0 - \underline{y} \right\|^2$$

$$\Rightarrow ((\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0))^T ((\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)) \geq (\underline{x}_0 - \underline{y})^T (\underline{x}_0 - \underline{y})$$

$$\Rightarrow [(\underline{x}_0 - \underline{y})' + \alpha(\underline{x} - \underline{x}_0)'] [(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)] \geq (\underline{x}_0 - \underline{y})^T (\underline{x}_0 - \underline{y})$$

$$\Rightarrow (\underline{x}_0 - \underline{y})' (\underline{x}_0 - \underline{y}) + \alpha(\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0) + \alpha(\underline{x} - \underline{x}_0)' (\underline{x}_0 - \underline{y}) + \alpha^2 (\underline{x} - \underline{x}_0)' (\underline{x} - \underline{x}_0)$$

$$\geq (\underline{x}_0 - \underline{y})' (\underline{x}_0 - \underline{y})$$

$$2\alpha(\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0) + \alpha^2 |\underline{x} - \underline{x}_0|^2 \geq 0$$

$$2\underline{(\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0)} + \underline{\alpha |\underline{x} - \underline{x}_0|^2} \geq 0$$

Let $\alpha \rightarrow 0$

$$(\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0) \geq 0$$

$$\Rightarrow (\underline{x}_0 - \underline{y})' \underline{x} \geq (\underline{x}_0 - \underline{y})' \underline{x}_0$$

$$\Rightarrow \underline{a}' \underline{x} \geq \underline{a}' (\underline{x}_0 - \underline{y} + \underline{y})$$

let $a = \underline{x}_0 - \underline{y}$

$$\geq \underline{a}'(x_0 - y) + \underline{a}'y$$

$\|z\| = \sqrt{z'z}$

$$\begin{aligned} \underline{a}'x &\geq \frac{(x_0 - y)(x_0 - y)}{\delta^2 + \underline{a}'y} + \underline{a}'y \\ &\geq \frac{\delta^2}{\delta^2 + \underline{a}'y} \quad \delta > 0 \end{aligned} \quad \Rightarrow \underline{x} \in S$$

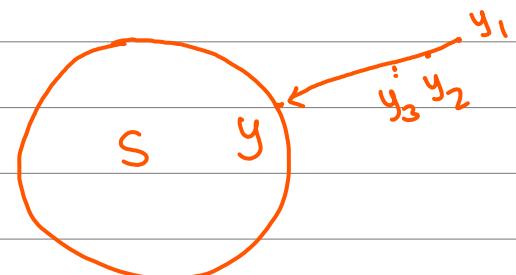
$$\inf_{\underline{x} \in S} \underline{a}'\underline{x} \geq \underline{a}'y \quad \checkmark$$

$$\underline{a}'y < \inf_{\underline{x} \in S} \underline{a}'\underline{x}$$

Theo. S convex, y boundary of S

To show $\exists H, \exists s \in H^+ / H^-$, $y \in s \cap H$

\rightarrow [let y_n be seqn of points exterior to closure of S
Assume $\underline{y}_n \rightarrow \underline{y}$]



$$\exists \underline{a}_n \in \mathbb{R}^n, \quad \underline{a}_n'y_n \leq \inf_{\underline{x} \in S} \underline{a}'\underline{x} \quad \|\underline{a}_n\|=1$$

$$\underline{a}'y_n - \underline{a}'y + \underline{a}'y \leq \underline{a}'x \quad \Rightarrow \underline{x} \in S$$

for large n , $y_n \rightarrow y \Rightarrow \underline{a}'y_n \rightarrow \underline{a}'y$

$$\underline{a_n}^T \underline{y} < \underline{a_n}^T \underline{x}$$

$\{\underline{a_m}\} \nsubseteq$ Seqⁿ of an bounded.

Bolzano Weierstrass Theorem for seqⁿ's.

\exists convergent subseqⁿ $\{\underline{a_{n_k}}\}$

Suppose it converges to $\underline{a_{n_k}} \rightarrow \underline{a}$

$$\underline{a_{n_k}}^T \underline{y} < \underline{a_{n_k}}^T \underline{x}$$

letting $k \rightarrow \infty$, $\underline{a_{n_k}} \rightarrow \underline{a}$

$$\underline{a}^T \underline{y} = \underline{a_{n_k}}^T \underline{y} < \underline{a_{n_k}}^T \underline{x} = \underline{a}^T \underline{x}$$

$$\Rightarrow \underline{a}^T \underline{y} \leq \underline{a}^T \underline{x}$$

\nexists $x \in S$

$H = \{x | \underline{a}^T \underline{x} = \underline{a}^T \underline{y}\}$ is supporting hyperplane.
at y

$$T = S \cap H = \{\omega\}$$



H Supportive Hyperplane , $T = S \cap H$
by method of contradiction

let \underline{x}_0 be extreme pt. of T but not of S .

let } some $\underline{x}_1, \underline{x}_2 \in S \Rightarrow \underline{x}_0 = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2$

$H = \{ \underline{x} \mid \underline{a}' \underline{x} = c \}$ supportive hyperplane of S

let $S \in H^+$, $\underline{a}' \underline{x} \geq c \nRightarrow \underline{x} \in S$.
 $\Rightarrow \underline{a}' \underline{x}_1 \geq c, \underline{a}' \underline{x}_2 \geq c$

$$\underline{x}_0 \in T = S \cap H \Rightarrow \underline{a}' \underline{x}_0 = c$$

$$\Rightarrow \underline{a}' (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) = c$$

$$\Rightarrow \alpha \underline{a}' \underline{x}_1 + (1-\alpha) \underline{a}' \underline{x}_2 \geq c$$

$$\alpha \in (0,1), (1-\alpha) \in (0,1)$$

$$\Rightarrow \alpha \cdot \frac{\underline{a}' \underline{x}_1}{\geq c} + (1-\alpha) \cdot \frac{\underline{a}' \underline{x}_2}{\geq c} = c$$

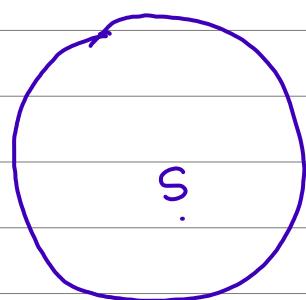
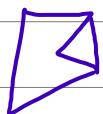
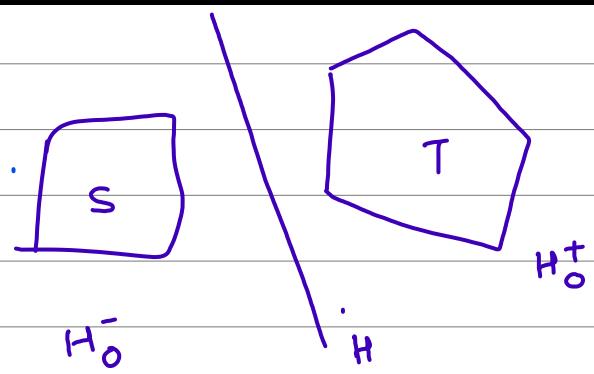
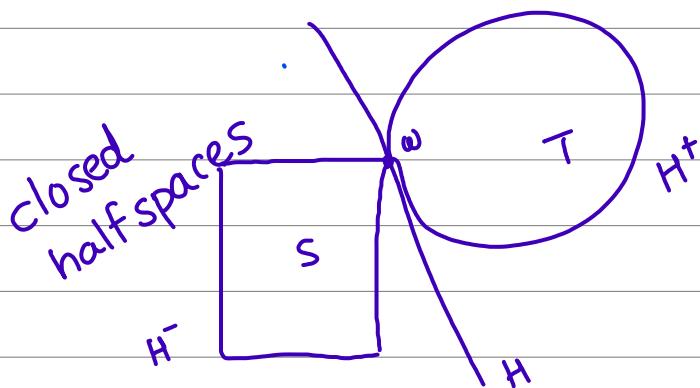
it is only possible if $\underline{a}' \underline{x}_1 = c$ & $\underline{a}' \underline{x}_2 = c$

$$\Rightarrow \underline{x}_1, \underline{x}_2 \in H \quad \& \Rightarrow \underline{x}_1, \underline{x}_2 \in S \cap H$$

$$\Rightarrow \underline{x}_1, \underline{x}_2 \in T$$

\therefore which contradicts to our assumption that
 \underline{x}_0 is extreme pt. of T .

Separating hyperplane



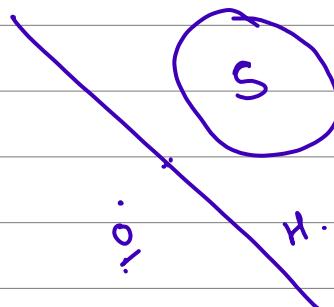
convex set

S convex
 $\underline{o} \notin S$

$$\underline{\underline{S}} = S$$

$\underline{o} \notin \underline{\underline{S}}$
exterior pt.

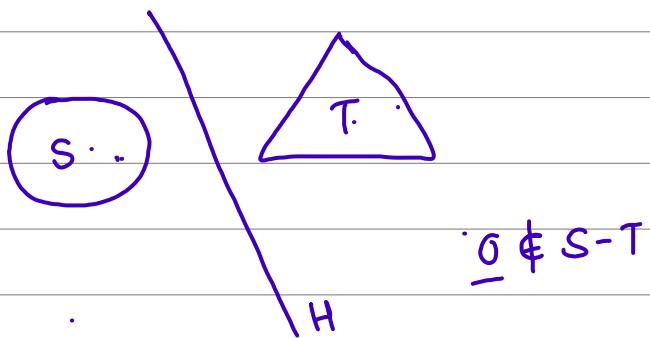
$$\left[\underline{o} = \underline{\underline{a'}}\underline{o} < \inf_{x \in S} \underline{a'}x \right] \Rightarrow \underline{a'}x > 0. \quad \forall x \in S. \quad \text{---(1)}$$



$$\text{if } H = \{x \mid \underline{a'}x = c\} \quad \text{---(2)}$$

$0 < c < \inf_{x \in S} \underline{a'}x$

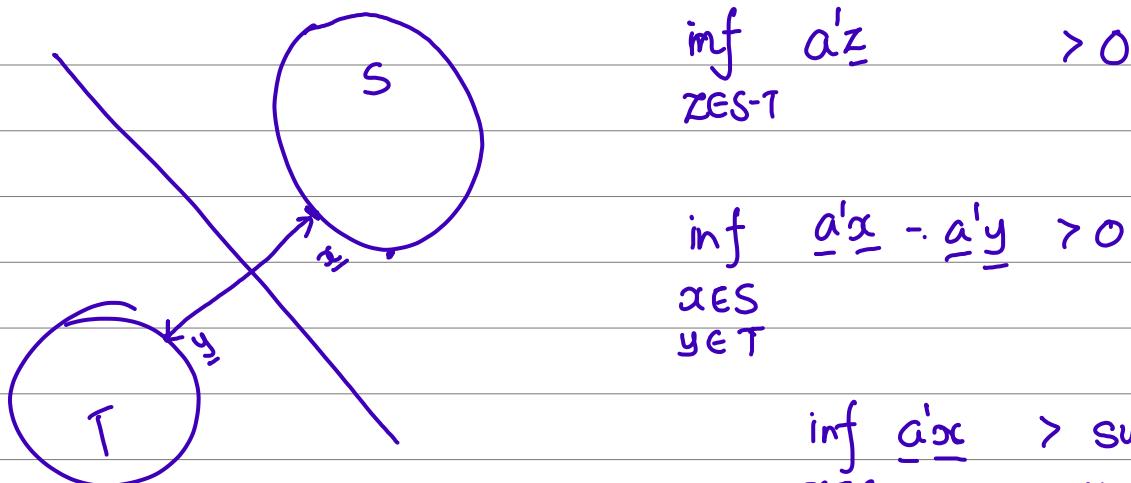
$$\begin{aligned} S &\in H^+_0 \\ \underline{o} &\in H \end{aligned}$$



$$S-T = \{ \underline{z} \mid \underline{z} = \underline{x} - \underline{y}, \underline{x} \in S, \underline{y} \in T \}$$

$$\underline{a}' \underline{z} \geq 0 \quad \forall \underline{z} \in S-T$$

$$\underline{a}' \underline{x} - \underline{a}' \underline{y} \geq 0$$



$0 \notin S-T$
 $\inf_{\underline{x} \in S} \underline{a}' \underline{z} > 0$
 $\inf_{\underline{y} \in T} \underline{a}' \underline{z} < 0$

$$\inf_{\underline{x} \in S} \underline{a}' \underline{z} > 0$$

$$\inf_{\underline{x} \in S} \underline{a}' \underline{x} - \sup_{\underline{y} \in T} \underline{a}' \underline{y} > 0$$

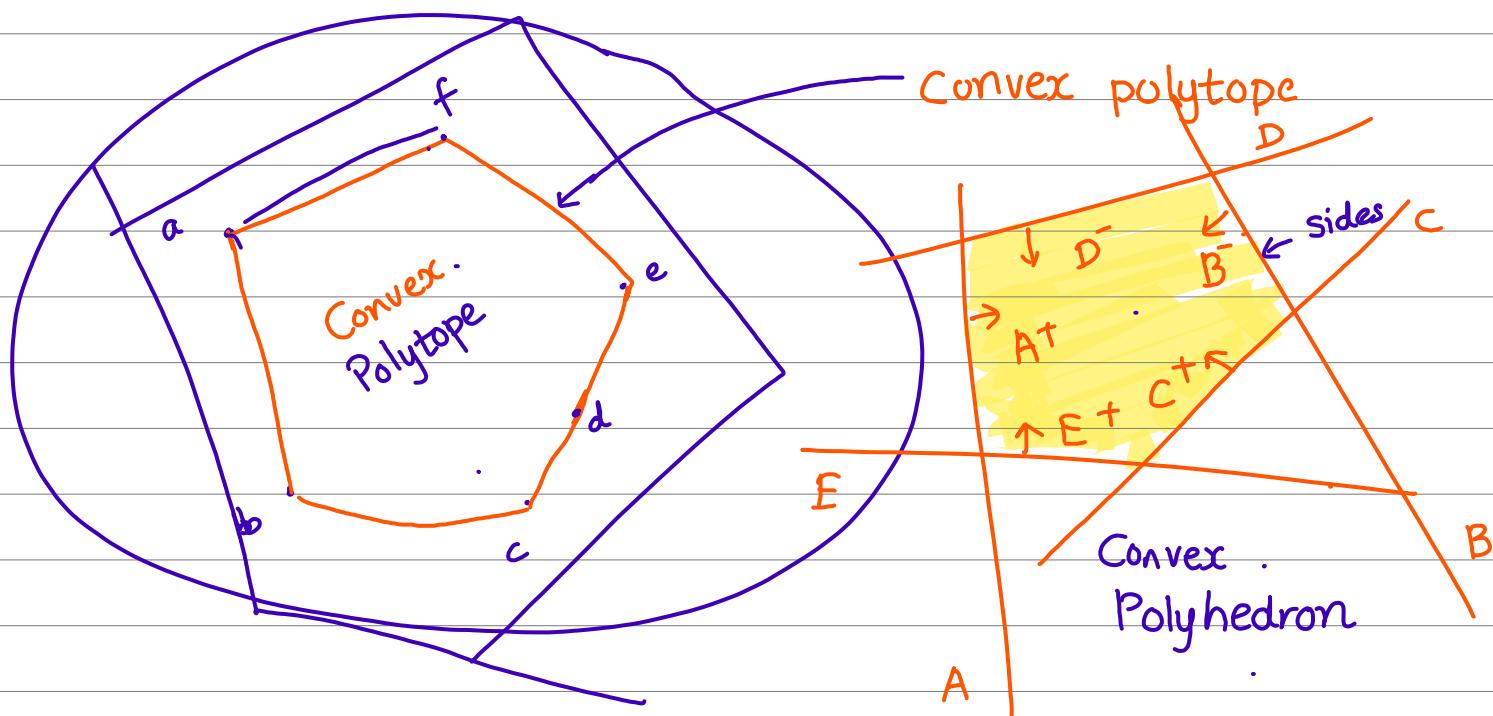
$$\inf_{\underline{x} \in S} \underline{a}' \underline{x} > \sup_{\underline{y} \in T} \underline{a}' \underline{y}$$

$$\inf_{\underline{x} \in S} \underline{a}' \underline{x} > c > \sup_{\underline{y} \in T} \underline{a}' \underline{y}$$

$H = \{ \underline{z} \mid \underline{a}' \underline{z} = c \}$

$S \in H_0^+$ $T \in H_0^-$

strictly separating.



Convex Polyhedron

$$S = \{a, b, c, d, e, f\}$$

Co(S) = Convex hull / Polytope

$$Co(S) = \{z \mid z = \sum \alpha_i x_i, x_i \in S\}$$

Vertices (Co(S)) = {a, b, c, e, f}.

\approx

$u \in V$ $\Rightarrow u \notin S$

$$S = \{x_1, x_2, \dots, x_m\}$$

$$Co(S) = \{x \mid x = \sum_{i=1}^m \alpha_i x_i\}$$

$u \in V$ but

$u \notin S$

$u \in Co(S)$

$$\therefore u = \sum_{i=1}^m \alpha_i x_i$$

$$\alpha_i \in (0, 1)$$

$$\sum \alpha_i = 1$$

$$\alpha_i < 1$$

$$u = \alpha_1 x_1 + \sum_{i=2}^m \alpha_i x_i = \alpha_1 x_1 + (1 - \alpha_1) \sum_{i=2}^m \alpha_i x_i$$

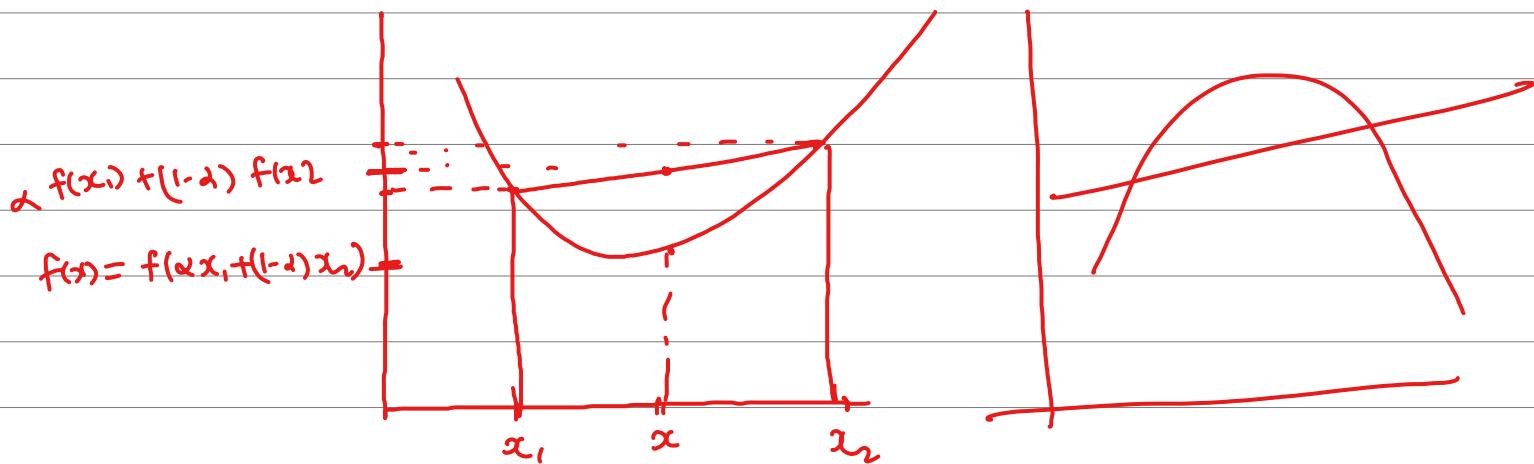
$$\begin{aligned} & \sum_{i=2}^m \alpha_i x_i \\ & \frac{\sum_{i=2}^m \alpha_i x_i}{(1 - \alpha_1)} \end{aligned}$$

$$= \alpha_1 x_1 + (1 - \alpha_1) x^*$$

$$\begin{aligned}
 & \text{Ax} \leq b \\
 & a_1^T x = a_{11}x_1 + a_{12}x_2 \leq b_1 \\
 & a_2^T x = a_{21}x_1 + a_{22}x_2 \leq b_2 \\
 & x \geq 0 \\
 H = \{ x \mid a_i^T x = b_i \} & \quad \underline{\underline{\quad}}
 \end{aligned}
 \quad \left. \begin{array}{l} H^- \\ H^+ \end{array} \right] \quad \left. \begin{array}{l} H^- \\ H^+ \end{array} \right]$$

$$\{ x : Ax \leq b \}$$

Polyhedron / Polyhedral



Convex
e.g. x^2

Concave
e.g. $-x^2$

$$f(x) = c^T x + d$$

$$x_1, x_2 \in S$$

$$f(\alpha x_1 + (1-\alpha)x_2) = c^T (\alpha x_1 + (1-\alpha)x_2) + d$$

$$= \alpha c^T x_1 + (1-\alpha) c^T x_2 + \alpha d + (1-\alpha)d$$

$$= \alpha (c^T x_1 + d) + (1-\alpha) (c^T x_2 + d)$$

$$= \alpha \cdot f(x_1) + (1-\alpha) f(x_2)$$

$$\leq$$

⇒ Convex

⇒ Concave

f convex fun^c on T, $\Rightarrow \underline{x}_1, \underline{x}_2 \in T$.

$$f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$$

$$T_K = \{ \underline{x} \mid \underline{x} \in T, f(\underline{x}) \leq K \}$$

let $\underline{x}_1, \underline{x}_2 \in T_K$

$$\Rightarrow \underline{x}_1, \underline{x}_2 \in T \text{ and } f(\underline{x}_1) \leq K, f(\underline{x}_2) \leq K$$

\Rightarrow as T is convex, & $\underline{x}_1, \underline{x}_2 \in T \Rightarrow \underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \in T$

$$f(\underline{x}) = f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$\leq \alpha \underline{\underline{f(x_1)}} + (1-\alpha) \underline{\underline{f(x_2)}} \quad (\text{as } f \text{ is convex})$$

$$\leq K \quad K \quad \alpha \in (0,1)$$

$$\leq K$$

$$\underline{x} \in T \text{ & } f(\underline{x}) \leq K \Rightarrow \underline{\underline{x}} \in T_K$$

$\Rightarrow T_K$ is convex set.

f, g are convex functions

$$\left[\begin{array}{l} f \text{ is convex fun}^c \Rightarrow \underline{x}_1, \underline{x}_2 \in S \\ f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2) \end{array} \right]$$

$$(\alpha f)(\underline{x}) = \alpha \cdot f(\underline{x})$$

$$(\alpha f)(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) = \alpha \cdot f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

\equiv

$$\leq \alpha [\alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)]$$

$$\leq \alpha \cdot \alpha \cdot f(\underline{x}_1) + (1-\alpha) \alpha \cdot f(\underline{x}_2)$$

$$\leq \underbrace{\alpha \cdot (\alpha f)(\underline{x}_1)}_{<} + (1-\alpha) \alpha \cdot f(\underline{x}_2)$$

αf is also convex.

$$(f+g)(\underline{x}) = f(\underline{x}) + g(\underline{x})$$

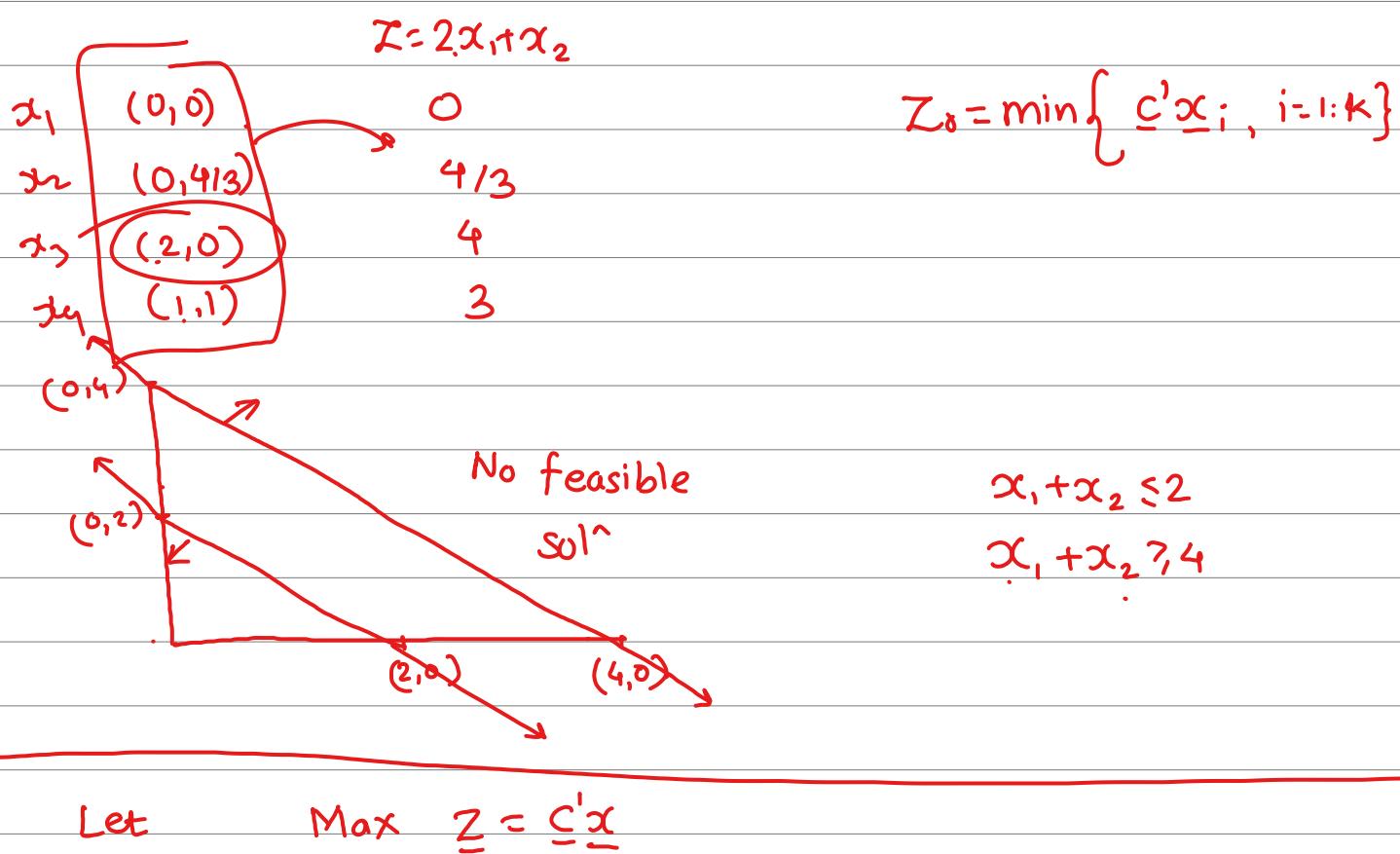
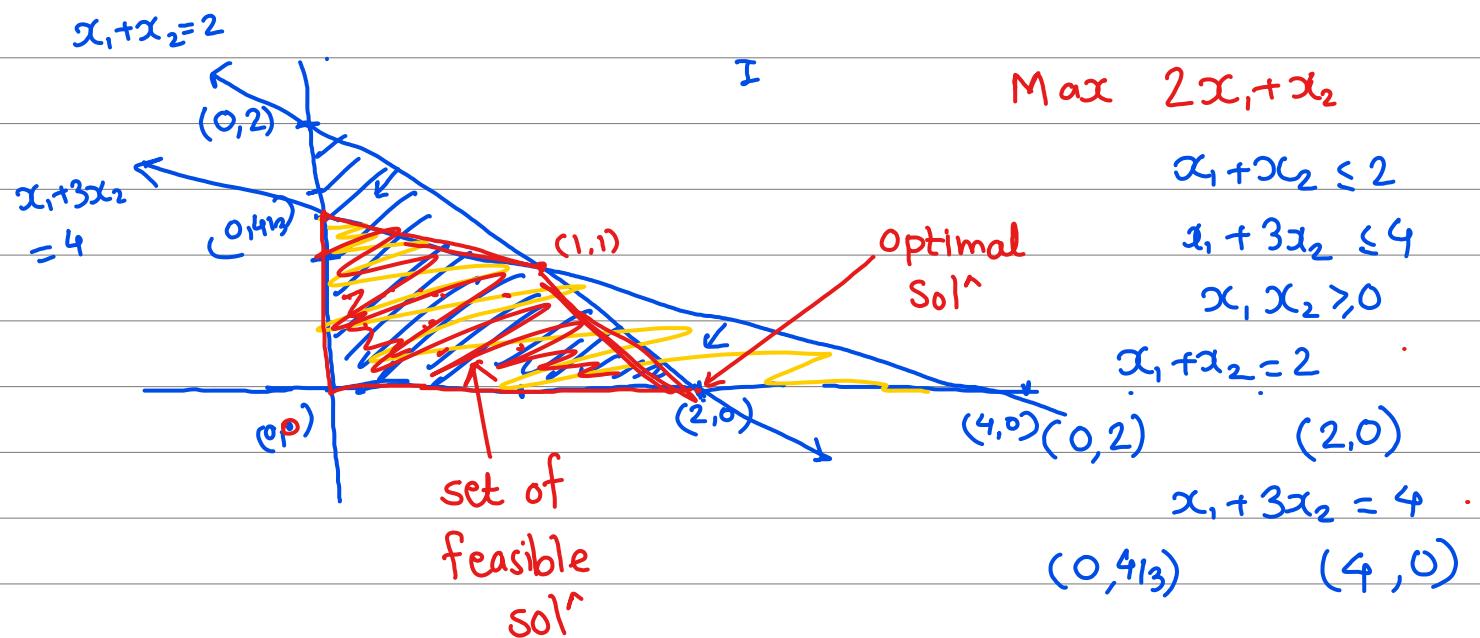
$$(f+g)(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) = f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) + g(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$\leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2) + \alpha g(\underline{x}_1) + (1-\alpha) g(\underline{x}_2)$$

f & g conv

$$\leq \alpha (f(\underline{x}_1) + g(\underline{x}_1)) + (1-\alpha) (f(\underline{x}_2) + g(\underline{x}_2))$$

$$\leq \alpha \cdot (f+g)(\underline{x}_1) + (1-\alpha) (f+g)(\underline{x}_2)$$



$$A\underline{x} \leq b$$

$T = \{ \underline{x} \mid A\underline{x} \leq b \}$ constraint set.

Let \underline{x}^1 & \underline{x}^2 are to optimum sol^r to LP.

$$\text{Max } Z = \underset{\underline{x} \in T}{\text{Max}} \underline{c}' \underline{x} = \underline{c}' \underline{x}^1 = \underline{c}' \underline{x}^2 = Z_0 \text{ (say)}^{\text{optimum value}}$$

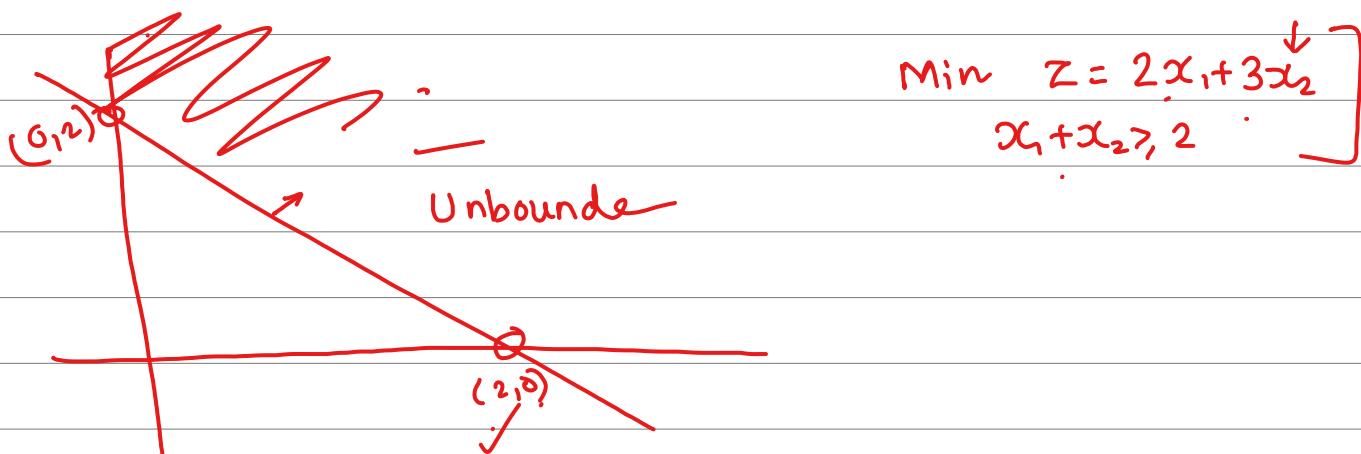
$$\underline{x}^0 = \alpha \underline{x}^1 + (1-\alpha) \underline{x}^2$$

To show \underline{x}^0 is also optimum sol^r

$$\begin{aligned}\underline{c}' \underline{x}^0 &= \underline{c}' (\alpha \underline{x}^1 + (1-\alpha) \underline{x}^2) \\ &= \alpha \underline{c}' \underline{x}^1 + (1-\alpha) \underline{c}' \underline{x}^2 \\ &= \alpha Z_0 + (1-\alpha) Z_0 \\ &= Z_0\end{aligned}$$

$\Rightarrow \underline{x}^0$ is also optimum sol^r

\Rightarrow Set of optimum sol^r is also convex set.



$$Z_0 = \min \left\{ \underline{c}' \underline{x}_i ; i=1:K \right\} \dots$$

$$\underline{x} \in T \Rightarrow \underline{c}' \underline{x} = \sum_{i=1}^K \alpha_i x_i \quad . \quad \alpha_i \geq 0 \quad \sum \alpha_i = 1$$

$$\begin{aligned}\underline{c}' \underline{x} &= \sum \alpha_i \underline{c}' \underline{x}_i \\ &\geq \sum \alpha_i Z_0 = Z_0 \Rightarrow\end{aligned}$$

$$\underline{c}' \underline{x} \geq z_0 \quad \nexists \underline{x} \in T$$

$$\min_{\underline{x} \in T} \underline{c}' \underline{x} \geq z_0 = \min \{ \underline{c}' \underline{x}_i, i=1:k \}$$

