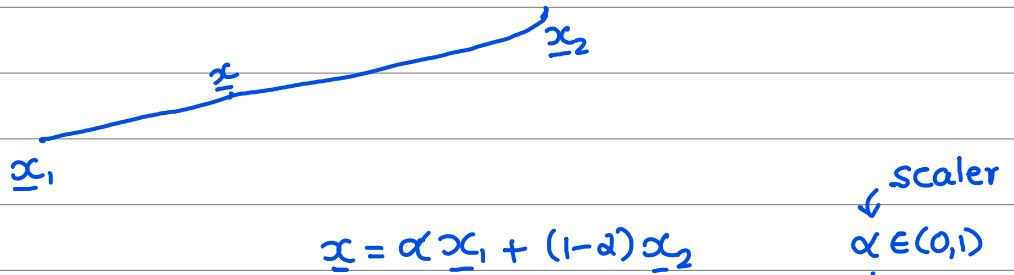


* Line segment : $\underline{x} \in \mathbb{R}^n$

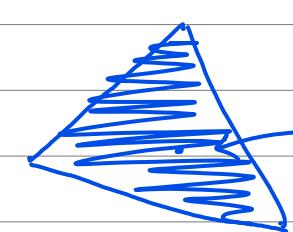


line segment $\{ \underline{x} / \underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2, \alpha \in (0,1) \}$
joining $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$

* Line passing $\underline{x}, \underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$

line segments $\underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2$ $\alpha \in (0,1)$

line 1 $\underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2$ $\alpha \in \mathbb{R}$



vector space

$$\underline{x} = \sum \alpha_i \underline{x}_i$$

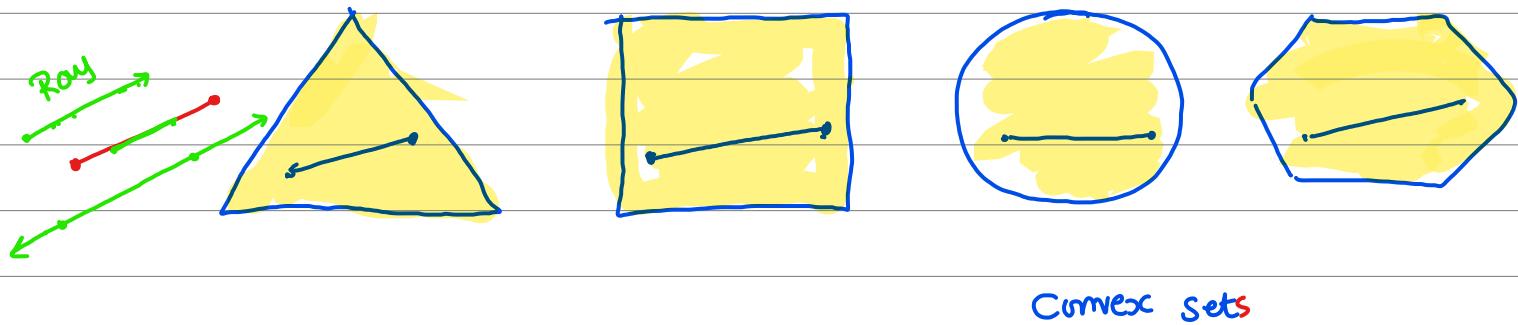
Linear Comb' $\alpha_i \in \mathbb{R}$

$$\underline{x} = \sum \alpha_i \underline{x}_i \quad \sum \alpha_i = 1, \quad \alpha_i \geq 0$$

\hookrightarrow Convex Combination

$$\{ \underline{x} / \underline{x} = \sum_{i=1}^n \alpha_i \underline{x}_i, \alpha_i \geq 0, \sum \alpha_i = 1 \}$$

Convex set if $\underline{x}_1, \underline{x}_2 \in A$, $\underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \Rightarrow \alpha \in (0,1)$
 then if $\underline{x} \in S \Rightarrow \alpha \in A$ is convex set



Non convex sets

* Ray is convex set
 $\underline{x} = \underline{x}_0 + d\alpha \quad \alpha > 0$

$$\underline{x}_1, \underline{x}_2 \in A$$

$$\lambda \quad \underline{x}_1 = \underline{x}_0 + d\alpha_1$$

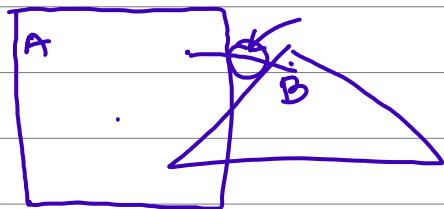
$$1-\lambda \quad \underline{x}_2 = \underline{x}_0 + d\alpha_2$$

$$\lambda \underline{x}_1 + (1-\lambda) \underline{x}_2 = (\lambda + 1 - \lambda) \underline{x}_0 + d(\lambda \alpha_1 + (1-\lambda) \alpha_2)$$

$$= \underline{x}_0 + d(\underbrace{\lambda \alpha_1 + (1-\lambda) \alpha_2}_{\geq 0}) \geq 0$$

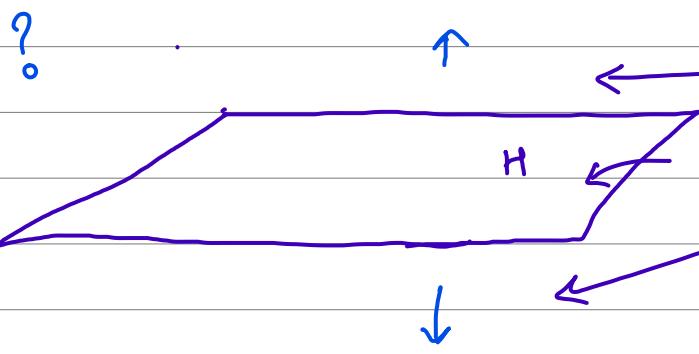
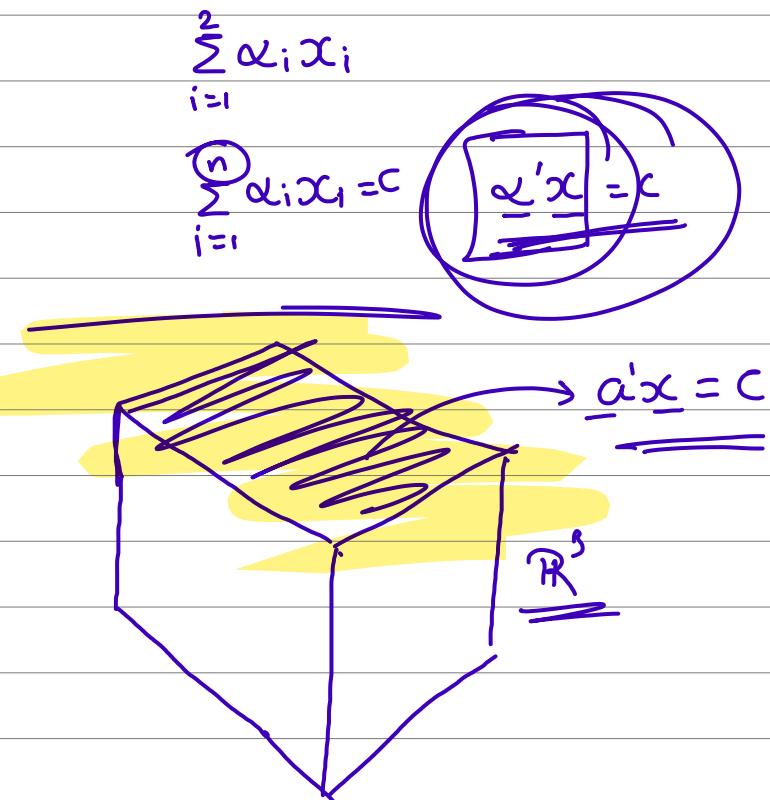
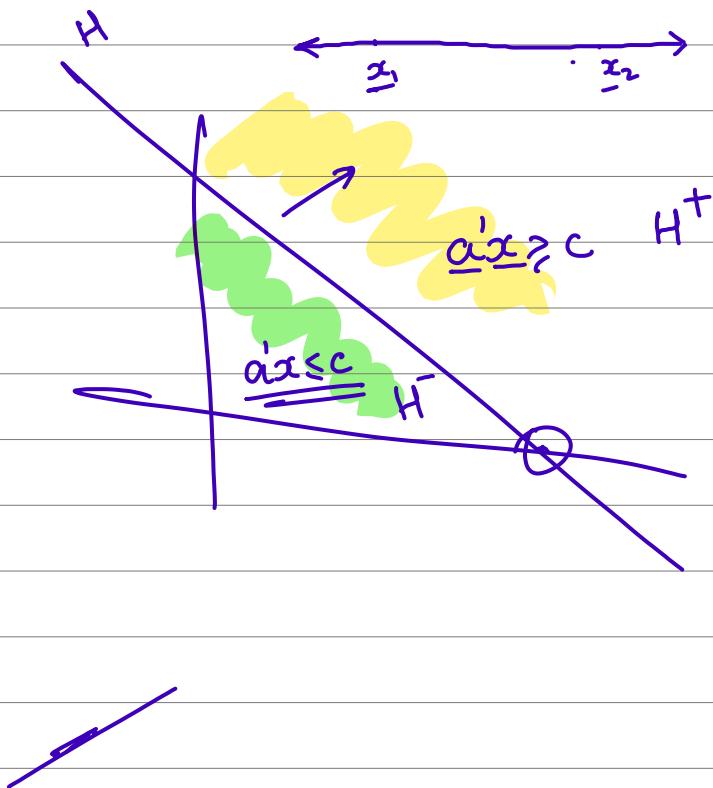


$$\in A$$



* Union of convex sets
may or may not be convex.

* Intersection of convex sets
is also convex.



Hyperplane

$$H^+ = \{ \underline{x} \mid a' \underline{x} > c \} \quad \text{+ve open half space}$$

$$H^+ = \{ \underline{x} \mid a' \underline{x} \geq c \} \quad \text{positive closed half space}$$

$$H = \{ \underline{x} \mid a' \underline{x} = c \} \quad \text{Hyperplane}$$

$$H^- = \{ \underline{x} \mid a' \underline{x} \leq c \} \quad \text{negative closed half space}$$

$$H^- = \{ \underline{x} \mid a' \underline{x} < c \} \quad \text{-ve open half space}$$

Hyperplane $H = \{\underline{x} | \underline{a}'\underline{x} = c\}$

let $\underline{x}_1, \underline{x}_2 \in H \Rightarrow \underline{a}'\underline{x}_1 = c \text{ & } \underline{a}'\underline{x}_2 = c$

$$\left[\begin{aligned} \underline{a}'(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) &= \alpha \underline{a}'\underline{x}_1 + (1-\alpha) \underline{a}'\underline{x}_2 \\ &= \alpha c + (1-\alpha)c \\ &= c \end{aligned} \right]$$

$\Rightarrow \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \in H$

\Rightarrow Hyperplane is convex set

H^+, H^-, H_+^+, H_-^- all are convex sets

Open ball $\underline{B} = \{\underline{x} \mid \|\underline{x} - \underline{x}_0\| < r\}$

To show :- B is convex set.

$$\text{Let } \underline{x}, \underline{y} \in B \Rightarrow \underline{\alpha \cdot x + (1-\alpha) y} \in B$$

$$\underline{x} \in B \Rightarrow \|\underline{x} - \underline{x}_0\| < r$$

$$\underline{y} \in B \Rightarrow \|\underline{y} - \underline{x}_0\| < r$$

$$\alpha \in (0,1)$$

$$\begin{aligned} & \|\alpha \underline{x} + (1-\alpha) \underline{y} - \underline{x}_0\| \\ &= \|\alpha \underline{x} + (1-\alpha) \underline{y} - (\alpha \underline{x}_0 + (1-\alpha) \cdot \underline{x}_0)\| \end{aligned}$$

$$= \|\alpha(\underline{x} - \underline{x}_0) + (1-\alpha)(\underline{y} - \underline{x}_0)\|$$

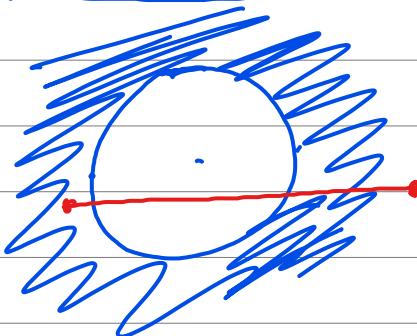
$$\leq \frac{\alpha \|\underline{x} - \underline{x}_0\|}{r} + (1-\alpha) \|\underline{y} - \underline{x}_0\| < r$$

$$< \alpha r + (1-\alpha) r$$

$$< r$$

$$\alpha \in (0,1)$$

$\cdot \left\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| = r \right\}$ Convex or Not



$\left\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| \geq r \right\}$ convex or not

If C is convex set $\underline{\lambda}C$ is also convex set.

$$\underline{\lambda}C = \{ \underline{y} \mid \underline{y} = \lambda \underline{x}, \underline{x} \in C \}$$

to show $\underline{\lambda}C$ as convex set

$$\underline{y}_1, \underline{y}_2 \in \underline{\lambda}C \Rightarrow \underline{y}_1 = \lambda \underline{x}_1, \underline{y}_2 = \lambda \underline{x}_2, \underline{x}_1, \underline{x}_2 \in C$$

$$\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 = \alpha \cdot \lambda \underline{x}_1 + (1-\alpha) \lambda \underline{x}_2$$

$$= \lambda (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$= \lambda \underline{x} \quad \begin{matrix} \text{as } C \text{ is convex} \\ (\text{as } \underline{x}_1, \underline{x}_2 \in C \Rightarrow \underline{x} \in C) \end{matrix}$$

$$\Rightarrow \alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 \in \underline{\lambda}C$$

C, D are convex sets $C+D$ is also convex

$$\rightarrow C+D = \{ \underline{z} \mid \underline{z} = \underline{x} + \underline{y}, \underline{x} \in C, \underline{y} \in D \}$$

$$\underline{z}_1, \underline{z}_2 \in C+D \Rightarrow \underline{z}_1 = \underline{x}_1 + \underline{y}_1$$

$$\underline{z}_2 = \underline{x}_2 + \underline{y}_2 \quad \underline{x}_1, \underline{x}_2 \in C, \underline{y}_1, \underline{y}_2 \in D$$

$$\alpha \underline{z}_1 + (1-\alpha) \underline{z}_2 = \alpha \cdot (\underline{x}_1 + \underline{y}_1) + (1-\alpha) (\underline{x}_2 + \underline{y}_2)$$

$$= \underbrace{\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2}_{\in C} + \underbrace{\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2}_{\in D}$$

Intersection of any convex sets is convex

Let $\{S_i\}_{i=1}^{\infty}$ be collection of convex sets

$\cap S_i$ is convex

$$\underline{x}, \underline{y} \in \cap S_i \\ \Rightarrow \underline{x}, \underline{y} \in S_i \quad \forall i$$

$$\Rightarrow \alpha \underline{x} + (1-\alpha) \underline{y} \in S_i \quad \forall i \quad (S_i \text{ is convex})$$

$$\Rightarrow \alpha \underline{x} + (1-\alpha) \underline{y} \in \cap S_i$$

$\Rightarrow \cap S_i$ is convex.

A set $S \in \mathbb{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S

∴ Assume that every convex combⁿ of (any finite no.) of points of S is in S .

⇒ it is also true for $n=2$

$$\Rightarrow \text{if } \underline{x}_1, \underline{x}_2 \in S \Rightarrow \alpha \cdot \underline{x}_1 + (1-\alpha) \underline{x}_2 \in S \Rightarrow \alpha \in (0,1)$$

⇒ S is convex set

② Assume S is convex and for any finite n

$$\sum_{i=1}^n \alpha_i \underline{x}_i \in S$$

→ let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in S$

$$\sum_{i=1}^n \alpha_i = 1$$

we will prove this by mathematical induction

As S is convex, $\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \in S$ $\underline{\alpha_i + 1 - \alpha = 1}$

∴ So the above statement is true for $n=2$

Assume it is true for $\underline{n=k} \Rightarrow \sum_{i=1}^k \alpha_i \underline{x}_i = 1$

$$\sum_{i=1}^k \alpha_i = 1$$

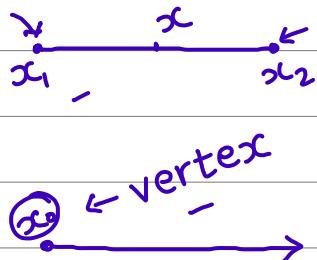
We have to prove it for $\underline{n=k+1}$

$$\begin{aligned} \sum_{i=1}^{k+1} \beta_i \underline{x}_i &= \left(\sum_{i=1}^k \beta_i \underline{x}_i \right) + \beta_{k+1} \underline{x}_{k+1} \\ &\quad \text{imp? } \left\{ \begin{array}{l} \sum_{i=1}^{k+1} \beta_i = 1 \\ \sum_{i=1}^k \beta_i = 1 - \beta_{k+1} \\ \frac{\sum_{i=1}^k \beta_i}{1 - \beta_{k+1}} = 1 \end{array} \right. \\ &= (1 - \beta_{k+1}) \left[\sum_{i=1}^k \frac{\beta_i}{1 - \beta_{k+1}} \cdot \underline{x}_i \right] + \beta_{k+1} \underline{x}_{k+1} \\ &\quad \in S. \end{aligned}$$

$$= (1 - \beta_{k+1}) \underline{x}^* + \beta_{k+1} \underline{x}_{k+1}$$

$\in S$ as S is convex set

* Vertices



2 vertices

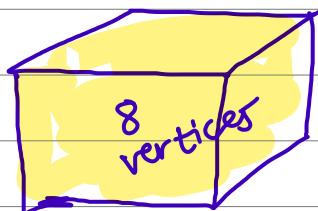
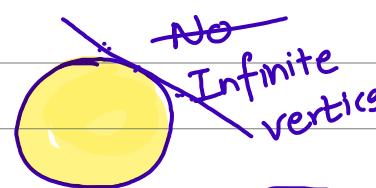
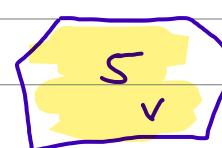
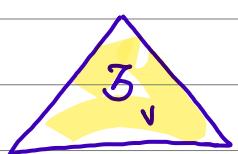
$$\Rightarrow \underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2$$

$$\underline{x}_2 = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 = \underline{x}_2$$

$\uparrow \alpha = 0$



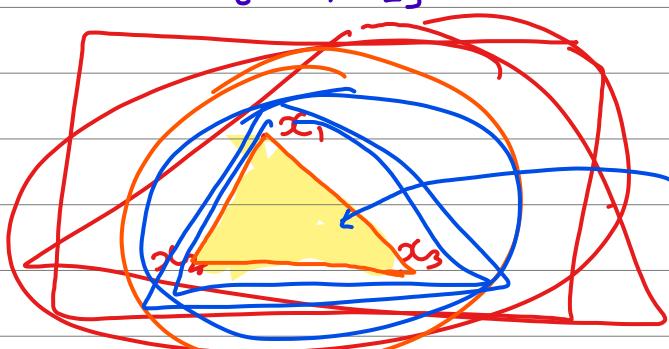
No vertex



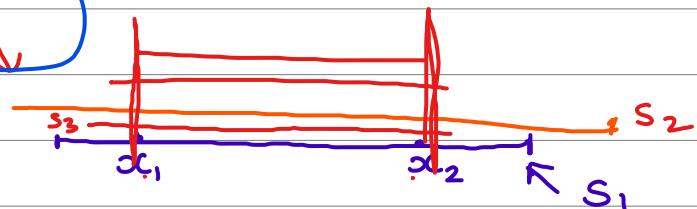
Convex Hull

$$\text{Co}(S) \Rightarrow \bigcap_{i=1}^{\infty} S_i$$

$$S = \{\underline{x}_1, \underline{x}_2\}$$



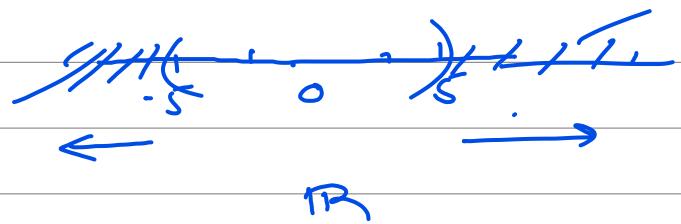
S_i is convex set containing S



$$S = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$$

$$S = \{ \underline{x} \mid \| \underline{x} \| \geq 5 \}$$

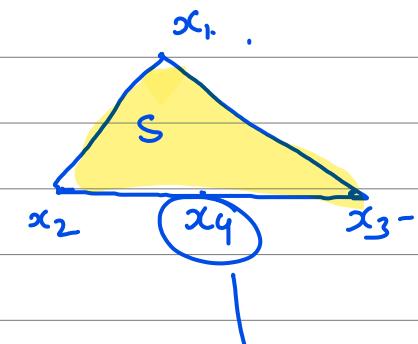
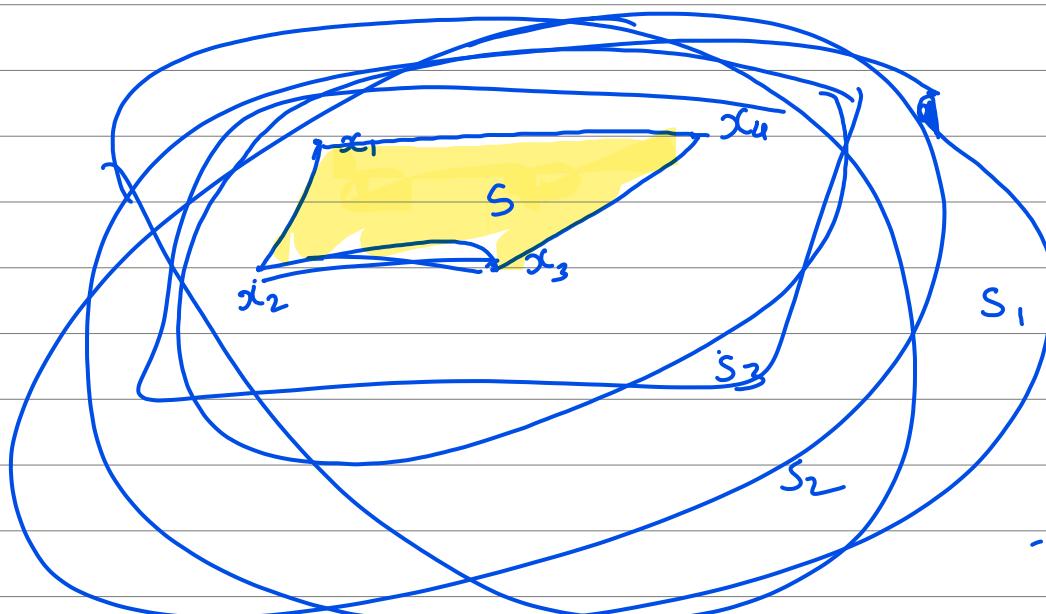
$$Co(S) = \mathbb{R}^n$$



if set is convex
 $\underline{Co(S) = S}$

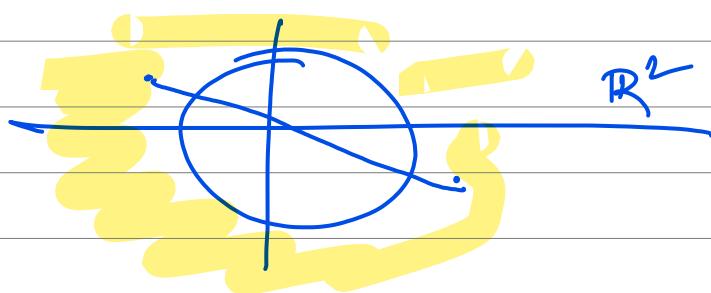
$Co(S) = \bigcap S_i$, S_i is convex set containing S .

$$\bigcap S_i = Co(S)$$

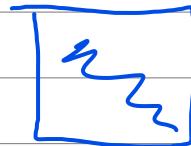
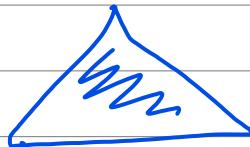


$$\underline{\mathbb{R} - \{ \underline{x} < 5 \}} \quad \underline{\mathbb{R} - \{ \underline{x} > 5 \}} \quad \underline{S = \{ \underline{x} | \underline{x} > 5 \}}$$

$\mathbb{R} = Co(S)$

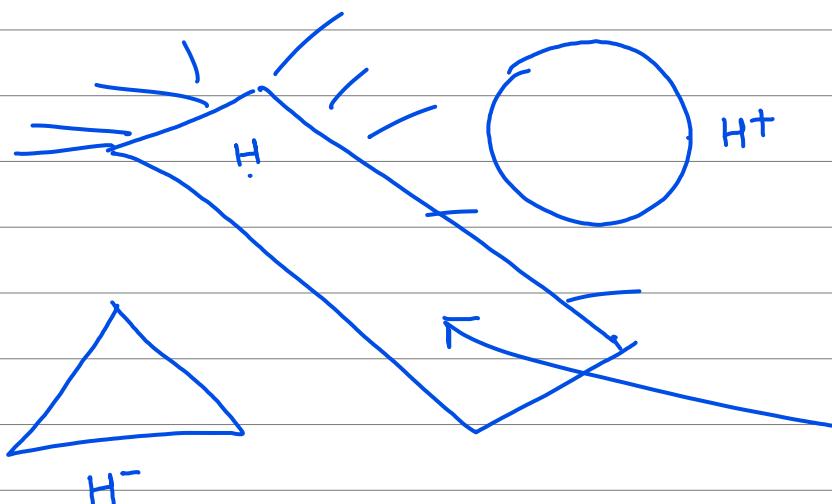


$$C_0(S) = \left\{ \underline{x} \mid \underline{x} = \sum_{i=1}^n \lambda_i x_i, \quad x_i \in S, \quad \sum \lambda_i = 1, \quad \lambda_i \geq 0 \right\}$$

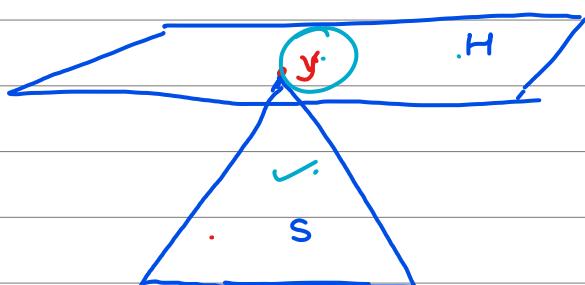


Hyperplane:-

$$H = \{ \underline{x} \mid \underline{a}' \underline{x} = c \}$$



Separating Hyperplane

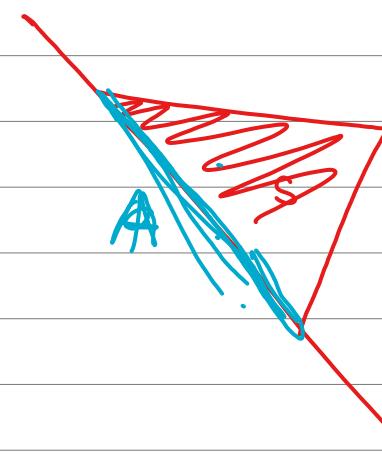


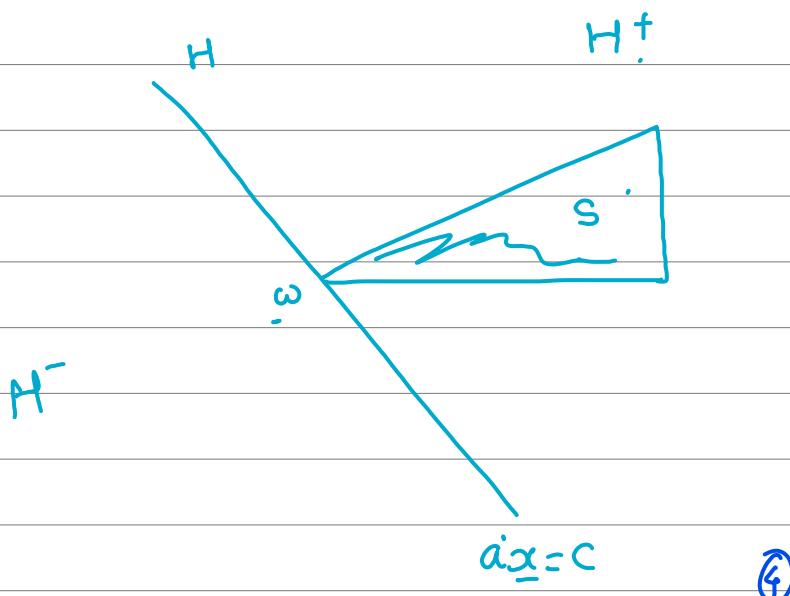
$$s \in H^-$$

$$\begin{aligned} s \in H^- \text{ or } s \in H^+ \\ \underline{a}' \underline{x} \leq c \text{ or } \underline{a}' \underline{x} \geq c \\ \Rightarrow x \in S \quad \Rightarrow x \in H \end{aligned}$$

$$H = \{ \underline{x} \mid \underline{a}' \underline{x} = c \}$$

$$\underline{a}' \underline{y} = c \Rightarrow \underline{y} \in S \Rightarrow \underline{y} \in H$$

 $s \in H^-$  H^+ H^-



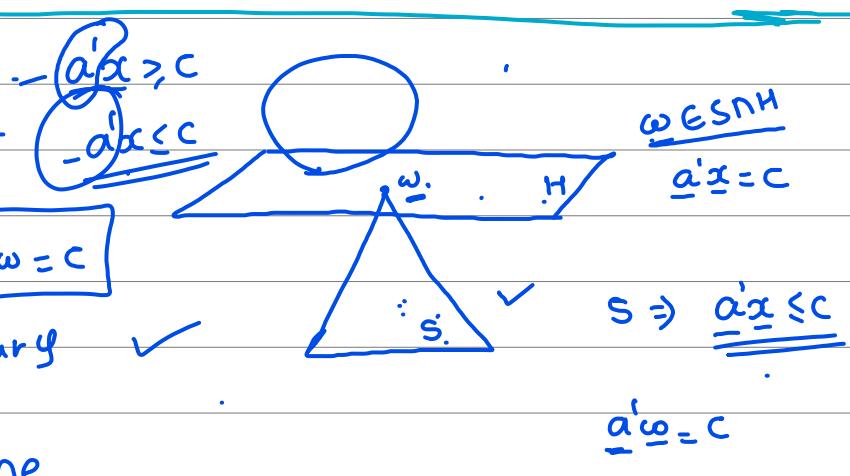
$$\begin{aligned} S &\in H \\ \forall x \in S, a'x &\geq c \\ \omega & \in S \cap H \\ a'\omega &= c \end{aligned}$$

① Supporting hyperplane :-

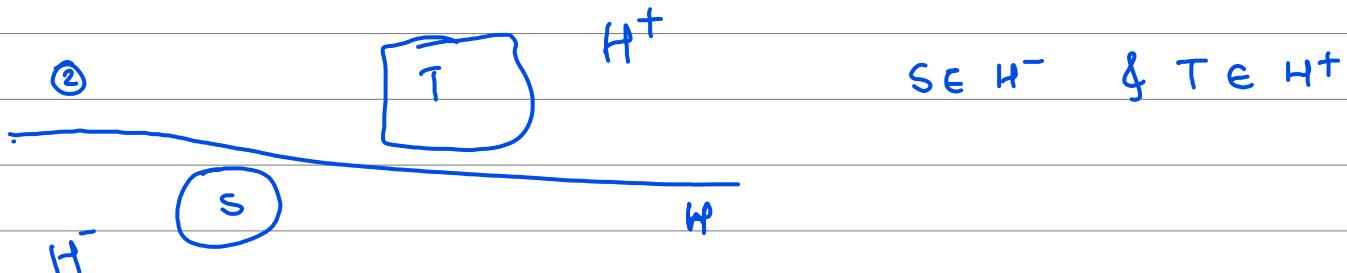
$$H: \{x \mid a'x = c\}$$

$$a'\omega = c$$

$\underline{\omega \in S}$ ← boundary S .

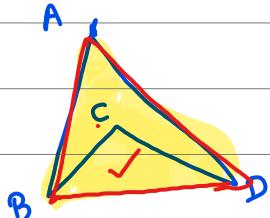


Separating hyperplane



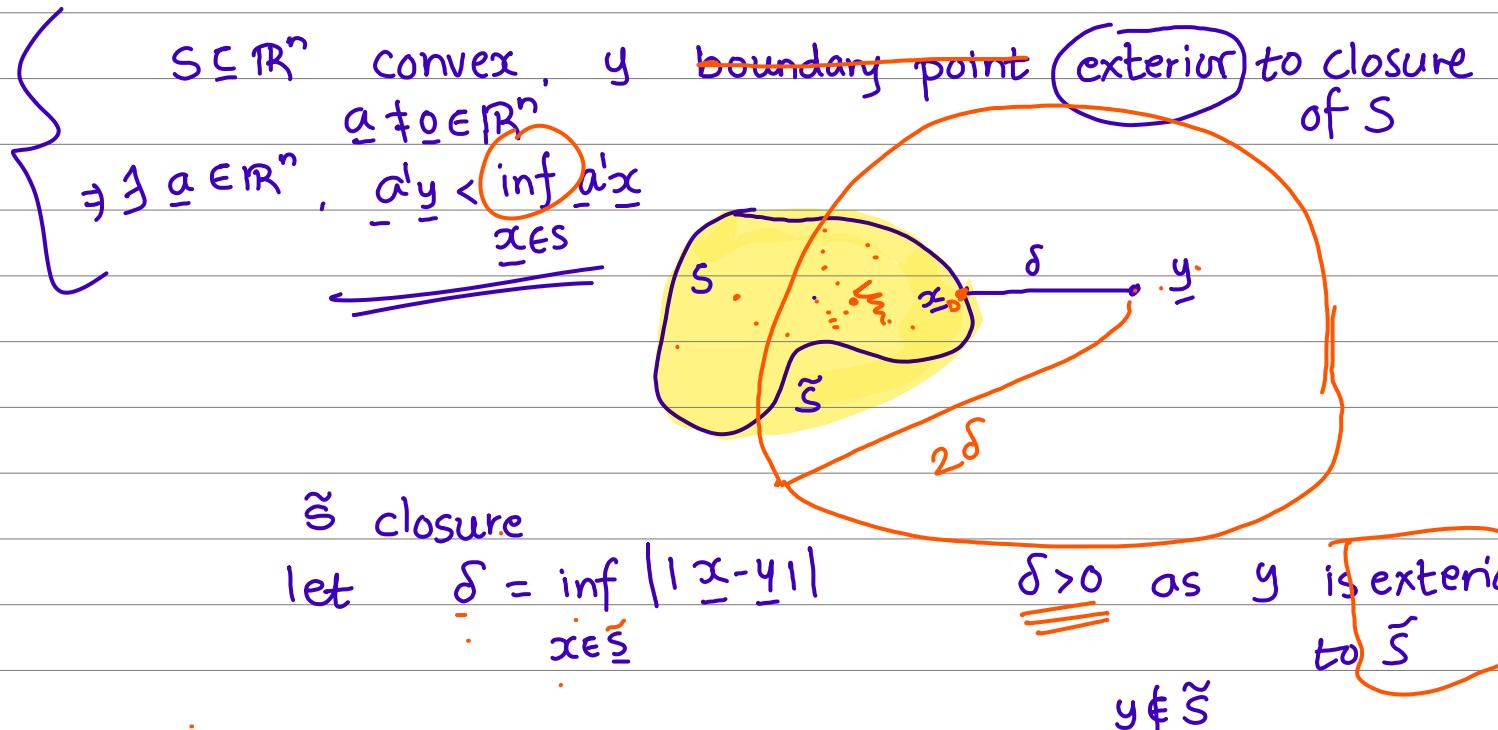
Closure. $S = (a, b) \cup \{a\} \cup \{b\} = [a, b]$

$$S' = (a, b) \cup \{a\} \cup \{b\} = [a, b]$$



$$|x - c| < \epsilon$$

$$c - \epsilon, c + \epsilon$$



$$\underline{B}_{2\underline{\delta}} = \{\underline{x} \mid \|\underline{x} - \underline{y}\| < 2\underline{\delta}\}$$

$$\underline{\delta} = \inf_{\underline{x} \in \tilde{S}} \|\underline{x} - \underline{y}\| = \inf_{\underline{x} \in \tilde{S} \cap \underline{B}_{2\underline{\delta}}} \|\underline{x} - \underline{y}\|$$

$\tilde{S} \cap \underline{B}_{2\underline{\delta}}$ is closed & bounded.
 lets define $f: \tilde{S} \cap \underline{B}_{2\underline{\delta}} \rightarrow \mathbb{R}, f(\underline{x}) = \|\underline{x} - \underline{y}\|$
 f. contin _____,
 by max^m min^m theo., f attains its extremum in that set

\exists some $\underline{x}_0 \in \tilde{S} \cap \underline{B}_{2\underline{\delta}} \Rightarrow$

$$\underline{\delta} = \min_{\underline{x} \in \tilde{S} \cap \underline{B}_{2\underline{\delta}}} \|\underline{x} - \underline{y}\| = \|\underline{x}_0 - \underline{y}\|$$

$\Rightarrow \underline{x}_0$ is boundary pt. of \tilde{S}

To show

$$\underline{a} = \underline{x}_0 - \underline{y}$$

$$\underline{a}' \leq \inf_{\underline{x}} \underline{a}' \underline{x}$$

$$\underline{x}, \underline{x}_0 \in \tilde{S}, \quad \alpha \underline{x} + (1-\alpha) \underline{x}_0 \in \tilde{S}$$

$$\left\| \underline{\alpha} \underline{x} + (1-\alpha) \underline{x}_0 - \underline{y} \right\| \geq \left\| \underline{x}_0 - \underline{y} \right\|$$

$$\left\| (\underline{x}_0 - \underline{y}) + \alpha (\underline{x} - \underline{x}_0) \right\| \geq \left\| \underline{x}_0 - \underline{y} \right\|$$

$$\left\| \underline{x}_0 - \underline{y} + \alpha (\underline{x} - \underline{x}_0) \right\|^2 \geq \left\| \underline{x}_0 - \underline{y} \right\|^2$$

$$\Rightarrow ((\underline{x}_0 - \underline{y}) + \alpha (\underline{x} - \underline{x}_0))^T ((\underline{x}_0 - \underline{y}) + \alpha (\underline{x} - \underline{x}_0)) \geq (\underline{x}_0 - \underline{y})^T (\underline{x}_0 - \underline{y})$$

$$\Rightarrow [(\underline{x}_0 - \underline{y})' + \alpha (\underline{x} - \underline{x}_0)'] [(\underline{x}_0 - \underline{y}) + \alpha (\underline{x} - \underline{x}_0)] \geq (\underline{x}_0 - \underline{y})^T (\underline{x}_0 - \underline{y})$$

$$\Rightarrow (\underline{x}_0 - \underline{y})' (\underline{x}_0 - \underline{y}) + \alpha (\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0) + \alpha (\underline{x} - \underline{x}_0)' (\underline{x}_0 - \underline{y}) + \alpha^2 (\underline{x} - \underline{x}_0)' (\underline{x} - \underline{x}_0)$$

$$\geq (\underline{x}_0 - \underline{y})' (\underline{x}_0 - \underline{y})$$

$$2\alpha (\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0) + \alpha^2 |\underline{x} - \underline{x}_0|^2 \geq 0$$

$$2(\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0) + \alpha |\underline{x} - \underline{x}_0|^2 \geq 0$$

Let $\alpha \rightarrow 0$

$$(\underline{x}_0 - \underline{y})' (\underline{x} - \underline{x}_0) \geq 0$$

$$\Rightarrow (\underline{x}_0 - \underline{y})' \underline{x} \geq (\underline{x}_0 - \underline{y})' \underline{x}_0$$

$$\Rightarrow \underline{a}' \underline{x} \geq \underline{a}' (\underline{x}_0 - \underline{y} + \underline{y})$$

let $a = \underline{x}_0 - \underline{y}$

$$\geq \underline{a}'(x_0 - y) + \underline{a}'y$$

$\|z\| = \sqrt{z'z}$

$$\begin{aligned} \underline{a}'x &\geq \frac{(x_0 - y)(x_0 - y)}{\delta^2 + \underline{a}'y} + \underline{a}'y \\ &\geq \frac{\delta^2}{\delta^2 + \underline{a}'y} \quad \delta > 0 \end{aligned} \quad \Rightarrow \underline{x} \in S$$

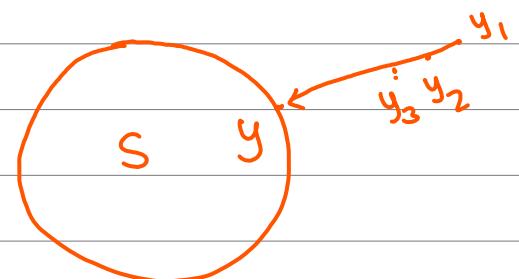
$$\inf_{\underline{x} \in S} \underline{a}'\underline{x} \geq \underline{a}'y \quad \checkmark$$

$$\underline{a}'y < \inf_{\underline{x} \in S} \underline{a}'\underline{x}$$

Theo. S convex, y boundary of S

To show $\exists H, \exists s \in H^+ / H^-$, $y \in s \cap H$

\rightarrow [let y_n be seqn of points exterior to closure of S
Assume $\underline{y}_n \rightarrow \underline{y}$]



$$\exists \underline{a}_n \in \mathbb{R}^n, \quad \underline{a}_n'y_n \leq \inf_{\underline{x} \in S} \underline{a}'\underline{x} \quad \|\underline{a}_n\|=1$$

$$\underline{a}'y_n - \underline{a}'y + \underline{a}'y \leq \underline{a}'x \quad \Rightarrow \underline{x} \in S$$

for large n , $y_n \rightarrow y \Rightarrow \underline{a}'y_n \rightarrow \underline{a}'y$

$$\underline{a_n}^T \underline{y} < \underline{a_n}^T \underline{x}$$

$\{\underline{a_m}\} \nsubseteq$ Seqⁿ of an bounded.

Bolzano Weierstrass Theorem for seqⁿ's.

\exists convergent subseqⁿ $\{\underline{a_{n_k}}\}$

Suppose it converges to $\underline{a_{n_k}} \rightarrow \underline{a}$

$$\underline{a_{n_k}}^T \underline{y} < \underline{a_{n_k}}^T \underline{x}$$

letting $k \rightarrow \infty$, $\underline{a_{n_k}} \rightarrow \underline{a}$

$$\underline{a}^T \underline{y} = \underline{a_{n_k}}^T \underline{y} < \underline{a_{n_k}}^T \underline{x} = \underline{a}^T \underline{x}$$

$$\Rightarrow \underline{a}^T \underline{y} \leq \underline{a}^T \underline{x}$$

\nexists $x \in S$

$H = \{x | \underline{a}^T \underline{x} = \underline{a}^T \underline{y}\}$ is supporting hyperplane.
at y

$$T = S \cap H = \{\omega\}$$



H Supportive Hyperplane, $T = S \cap H$
by method of contradiction

let \underline{x}_0 be extreme pt. of T but not of S .

let } some $\underline{x}_1, \underline{x}_2 \in S \Rightarrow \underline{x}_0 = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2$

$H = \{ \underline{x} \mid \underline{a}' \underline{x} = c \}$ supportive hyperplane of S

let $S \in H^+$, $\underline{a}' \underline{x} \geq c \nRightarrow \underline{x} \in S$.
 $\Rightarrow \underline{a}' \underline{x}_1 \geq c, \underline{a}' \underline{x}_2 \geq c$

$$\underline{x}_0 \in T = S \cap H \Rightarrow \underline{a}' \underline{x}_0 = c$$

$$\Rightarrow \underline{a}' (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) = c$$

$$\Rightarrow \alpha \underline{a}' \underline{x}_1 + (1-\alpha) \underline{a}' \underline{x}_2 \geq c$$

$$\alpha \in (0,1), (1-\alpha) \in (0,1)$$

$$\Rightarrow \alpha \cdot \frac{\underline{a}' \underline{x}_1}{\geq c} + (1-\alpha) \cdot \frac{\underline{a}' \underline{x}_2}{\geq c} = c$$

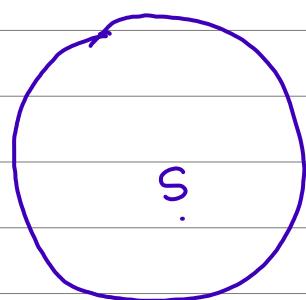
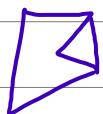
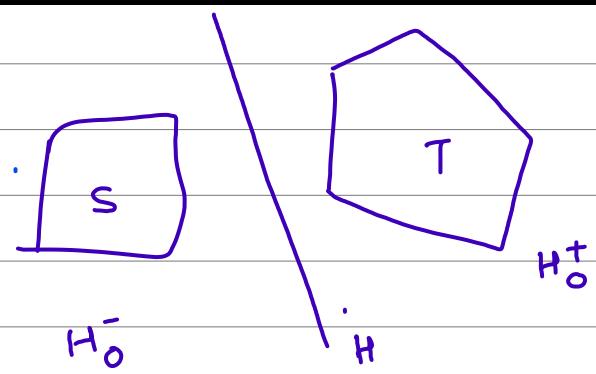
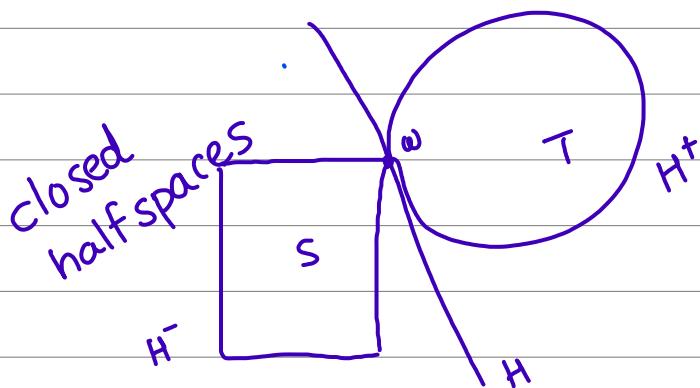
it is only possible if $\underline{a}' \underline{x}_1 = c$ & $\underline{a}' \underline{x}_2 = c$

$$\Rightarrow \underline{x}_1, \underline{x}_2 \in H \quad \& \Rightarrow \underline{x}_1, \underline{x}_2 \in S \cap H$$

$$\Rightarrow \underline{x}_1, \underline{x}_2 \in T$$

\therefore which contradicts to our assumption that
 \underline{x}_0 is extreme pt. of T .

Separating hyperplane



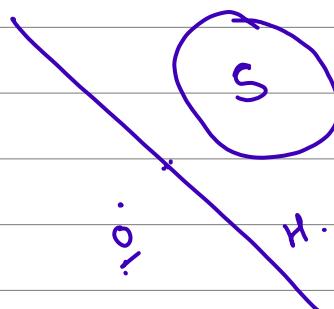
convex set

S convex
 $\underline{o} \notin S$

$$\underline{\underline{S}} = S$$

$\underline{o} \notin \underline{\underline{S}}$
exterior pt.

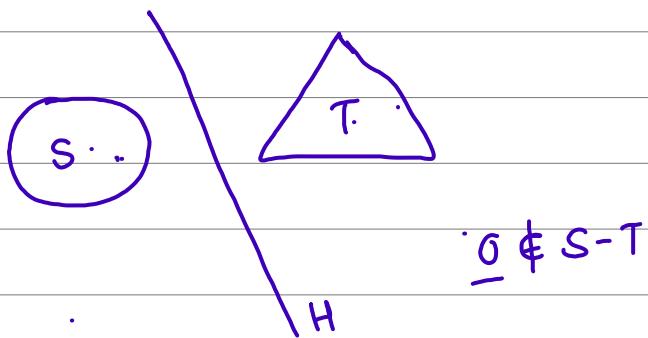
$$\left[\underline{o} = \underline{\underline{o}} < \inf_{\underline{x} \in S} \underline{a}' \underline{x} \right] \Rightarrow \underline{a}' \underline{x} > 0. \quad \forall \underline{x} \in S. \quad \text{---(1)}$$



$$\text{if } H = \{ \underline{x} \mid \underline{a}' \underline{x} = c \} \quad \text{---(2)}$$

$0 < c < \inf_{\underline{x} \in S} \underline{a}' \underline{x}$

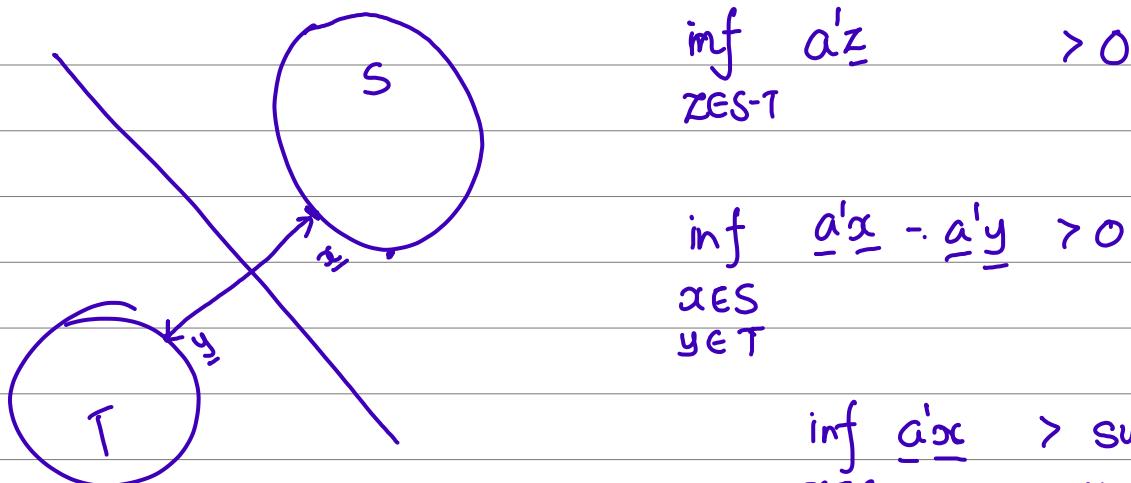
$$\begin{aligned} S &\in H^+_0 \\ \underline{o} &\in H \end{aligned}$$



$$S-T = \{ \underline{z} \mid \underline{z} = \underline{x} - \underline{y}, \underline{x} \in S, \underline{y} \in T \}$$

$$\underline{a}' \underline{z} \geq 0 \quad \forall \underline{z} \in S-T$$

$$\underline{a}' \underline{x} - \underline{a}' \underline{y} \geq 0$$



$0 \notin S-T$
 $\inf_{\underline{x} \in S} \underline{a}' \underline{z} > 0$
 $\inf_{\underline{y} \in T} \underline{a}' \underline{z} < 0$

$$\inf_{\underline{x} \in S} \underline{a}' \underline{z} > 0$$

$$\inf_{\underline{x} \in S} \underline{a}' \underline{x} - \sup_{\underline{y} \in T} \underline{a}' \underline{y} > 0$$

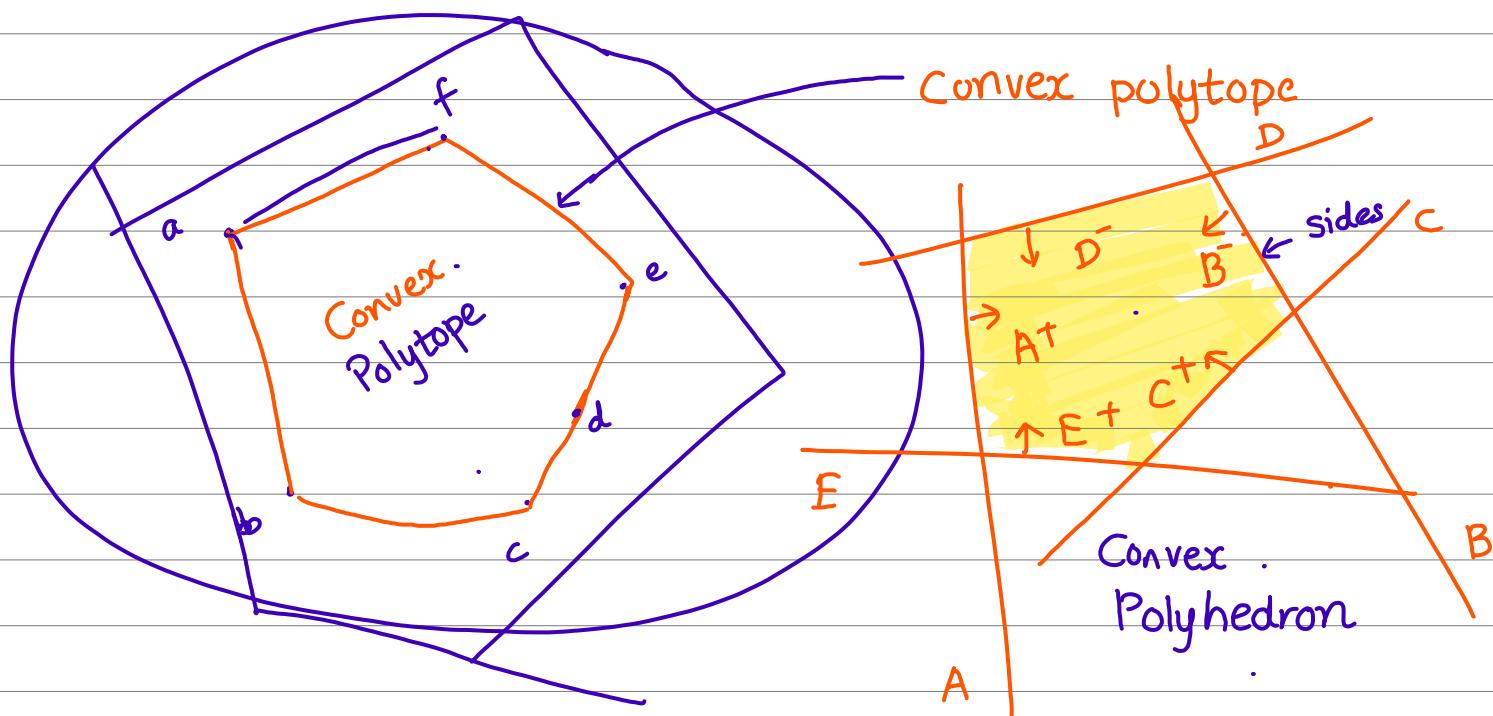
$$\inf_{\underline{x} \in S} \underline{a}' \underline{x} > \sup_{\underline{y} \in T} \underline{a}' \underline{y}$$

$$\inf_{\underline{x} \in S} \underline{a}' \underline{x} > c > \sup_{\underline{y} \in T} \underline{a}' \underline{y}$$

$H = \{ \underline{z} \mid \underline{a}' \underline{z} = c \}$

$S \in H_0^+$ $T \in H_0^-$

strictly separating.



Convex Polyhedron

$$S = \{a, b, c, d, e, f\}$$

Co(S) = Convex hull / Polytope

$$Co(S) = \{z \mid z = \sum \alpha_i x_i, x_i \in S\}$$

Vertices (Co(S)) = {a, b, c, e, f}.

\approx

$u \in V$ $\Rightarrow u \notin S$

$$S = \{x_1, x_2, \dots, x_m\}$$

$$Co(S) = \{x \mid x = \sum_{i=1}^m \alpha_i x_i\}$$

$u \in V$ but

$u \notin S$

$u \in Co(S)$

$$\therefore u = \sum_{i=1}^m \alpha_i x_i$$

$$\alpha_i \in (0, 1)$$

$$\sum \alpha_i = 1$$

$$\alpha_i < 1$$

$$u = \alpha_1 x_1 + \sum_{i=2}^m \alpha_i x_i = \alpha_1 x_1 + (1 - \alpha_1)$$

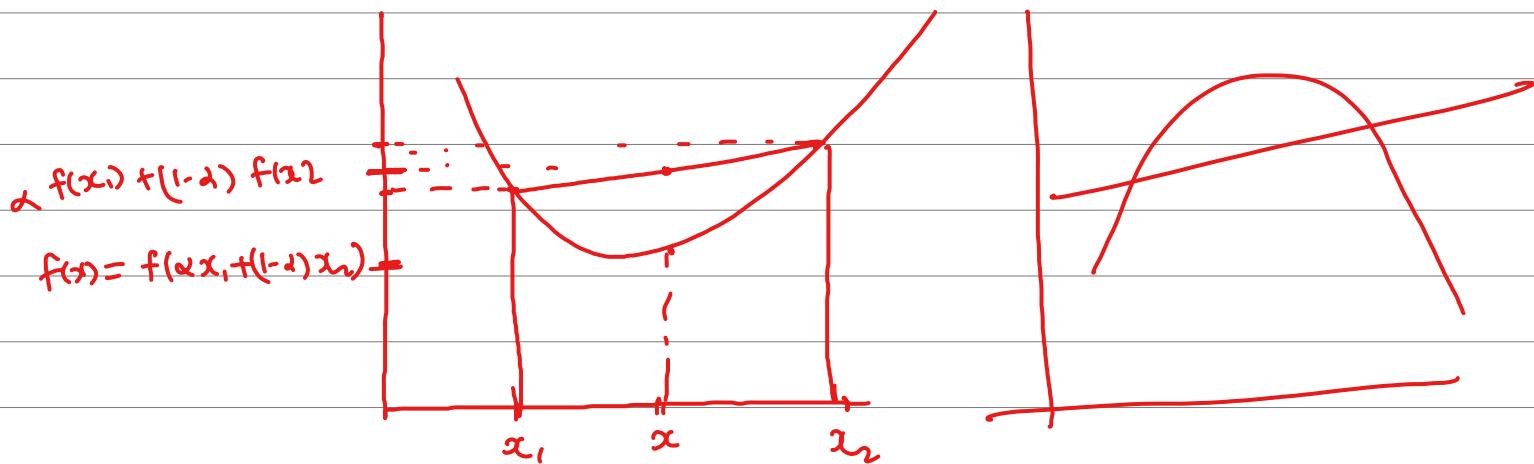
$$\boxed{\sum_{i=2}^m \alpha_i x_i} \quad x^*$$

$$= \alpha_1 x_1 + (1 - \alpha_1) x^*$$

$$\begin{aligned}
 & \text{Ax} \leq b \\
 & a_1^T x = a_{11}x_1 + a_{12}x_2 \leq b_1 \\
 & a_2^T x = a_{21}x_1 + a_{22}x_2 \leq b_2 \\
 & x \geq 0 \\
 H = \{ x \mid a_i^T x = b_i \} & \quad \underline{\underline{\quad}}
 \end{aligned}$$

$$\{ x : Ax \leq b \}$$

Polyhedron / Polyhedral



Convex
e.g. x^2

Concave
e.g. $-x^2$

$$f(x) = c^T x + d$$

$$x_1, x_2 \in S$$

$$f(\alpha x_1 + (1-\alpha)x_2) = c^T (\alpha x_1 + (1-\alpha)x_2) + d$$

$$= \alpha c^T x_1 + (1-\alpha) c^T x_2 + \alpha d + (1-\alpha)d$$

$$= \alpha (c^T x_1 + d) + (1-\alpha) (c^T x_2 + d)$$

$$= \alpha \cdot f(x_1) + (1-\alpha) f(x_2)$$

$$\leq$$

⇒ Convex

⇒ Concave

f convex fun^c on T, $\Rightarrow \underline{x}_1, \underline{x}_2 \in T$.

$$f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$$

$$T_K = \{ \underline{x} \mid \underline{x} \in T, f(\underline{x}) \leq K \}$$

let $\underline{x}_1, \underline{x}_2 \in T_K$

$\Rightarrow \underline{x}_1, \underline{x}_2 \in T$ and $f(\underline{x}_1) \leq K, f(\underline{x}_2) \leq K$

\Rightarrow as T is convex, & $\underline{x}_1, \underline{x}_2 \in T \Rightarrow \underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 \in T$

$$f(\underline{x}) = f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$\leq \underline{\underline{\alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)}} \quad (\text{as } f \text{ is convex})$$

$$\leq \underline{\underline{K}} \quad \underline{\underline{K}} \quad \alpha \in (0,1)$$

$$\leq K$$

$$\underline{\underline{\underline{x}}} \in T \& f(\underline{\underline{\underline{x}}}) \leq K \Rightarrow \underline{\underline{\underline{x}}} \in T_K$$

$\Rightarrow T_K$ is convex set.

f, g are convex functions

$$\left[\begin{array}{l} f \text{ is convex fun}^c \Rightarrow \underline{x}_1, \underline{x}_2 \in S \\ f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2) \end{array} \right]$$

$$(\alpha f)(\underline{x}) = \alpha \cdot f(\underline{x})$$

$$(\alpha f)(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) = \alpha \cdot f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

\equiv

$$\leq \alpha [\alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)]$$

$$\leq \alpha \cdot \alpha \cdot f(\underline{x}_1) + (1-\alpha) \alpha \cdot f(\underline{x}_2)$$

$$\leq \underbrace{\alpha \cdot (\alpha f)(\underline{x}_1)}_{<} + (1-\alpha) \alpha \cdot f(\underline{x}_2)$$

αf is also convex.

$$(f+g)(\underline{x}) = f(\underline{x}) + g(\underline{x})$$

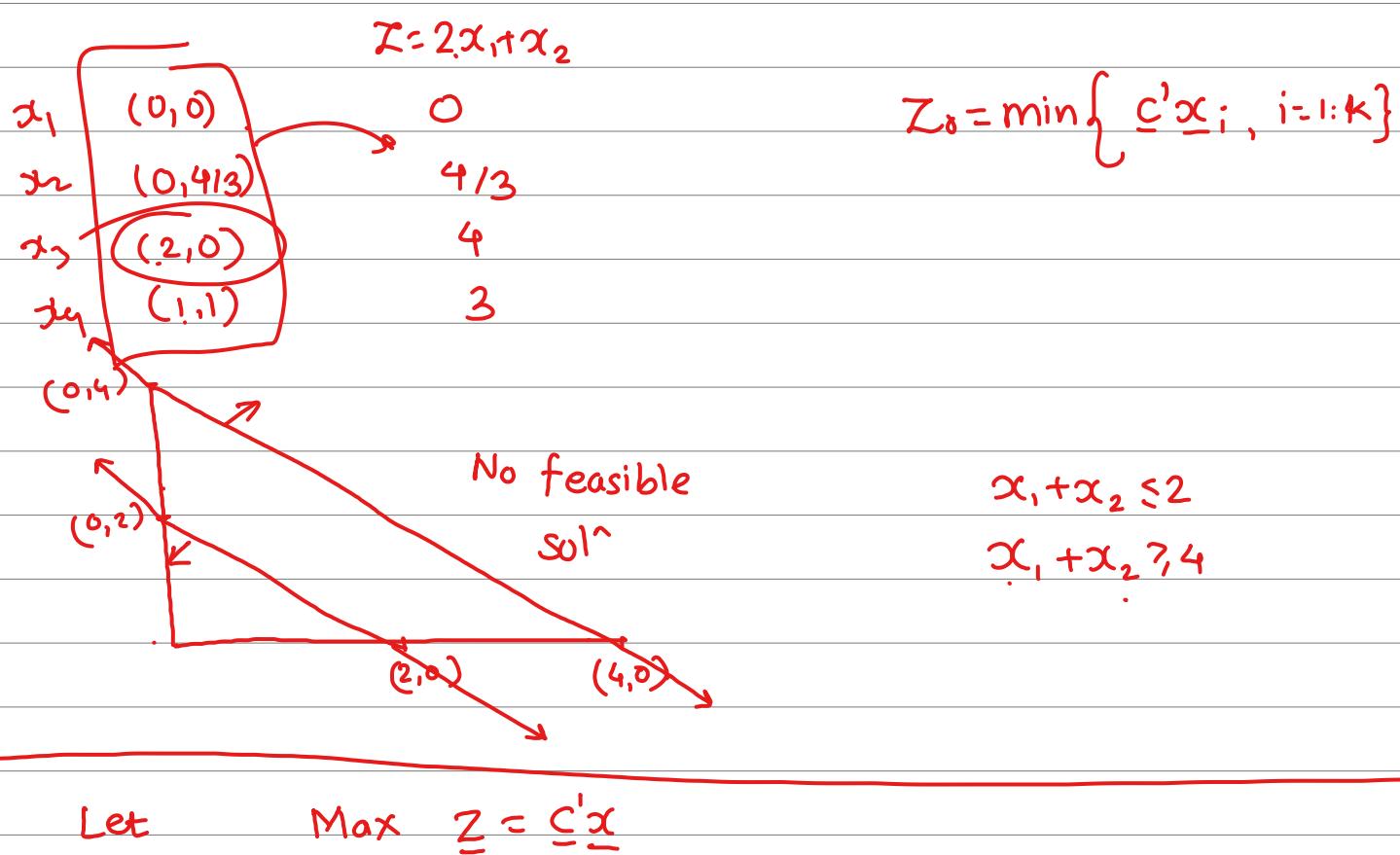
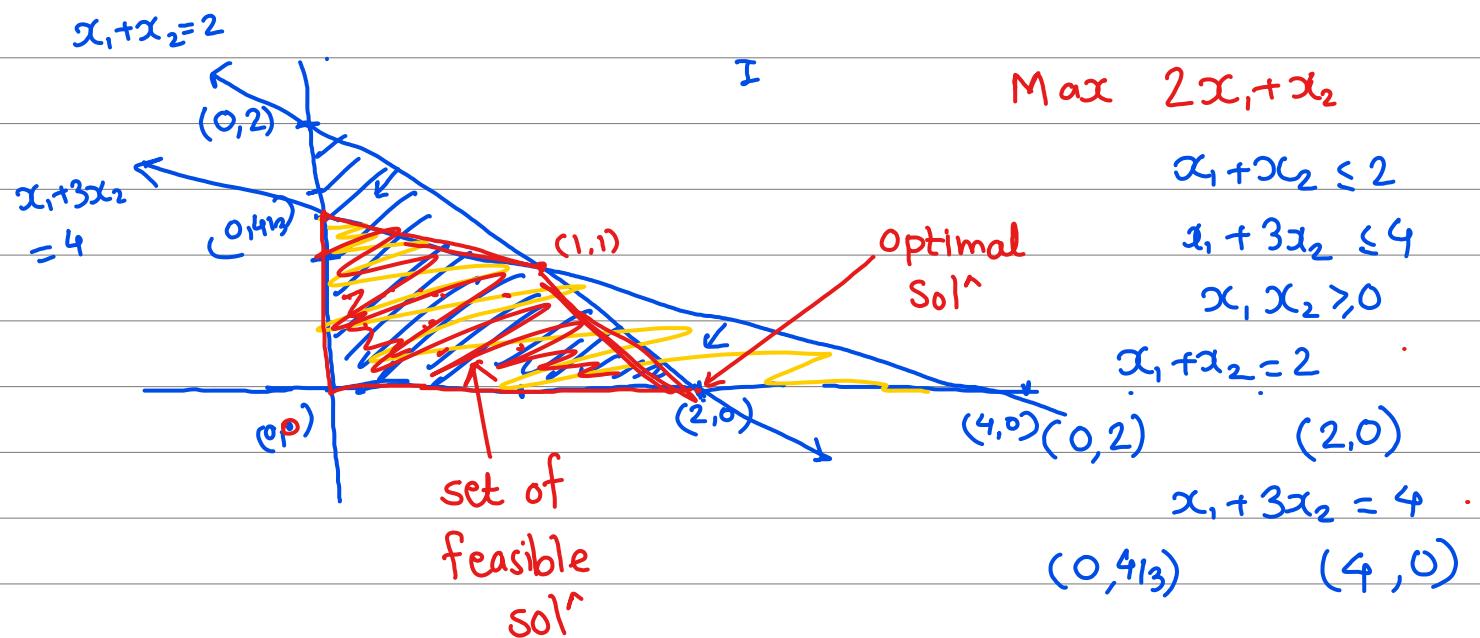
$$(f+g)(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) = f(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2) + g(\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$\leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2) + \alpha g(\underline{x}_1) + (1-\alpha) g(\underline{x}_2)$$

f & g conv

$$\leq \alpha (f(\underline{x}_1) + g(\underline{x}_1)) + (1-\alpha) (f(\underline{x}_2) + g(\underline{x}_2))$$

$$\leq \alpha \cdot (f+g)(\underline{x}_1) + (1-\alpha) (f+g)(\underline{x}_2)$$



$$A\underline{x} \leq b$$

$T = \{ \underline{x} \mid \underline{A}\underline{x} \leq b \}$ constraint set.

Let \underline{x}^1 & \underline{x}^2 are to optimum sol^r to LP.

$$\text{Max } Z = \underset{\underline{x} \in T}{\text{Max}} \underline{c}' \underline{x} = \underline{c}' \underline{x}^1 = \underline{c}' \underline{x}^2 = Z_0 \text{ (say)}^{\text{optimum value}}$$

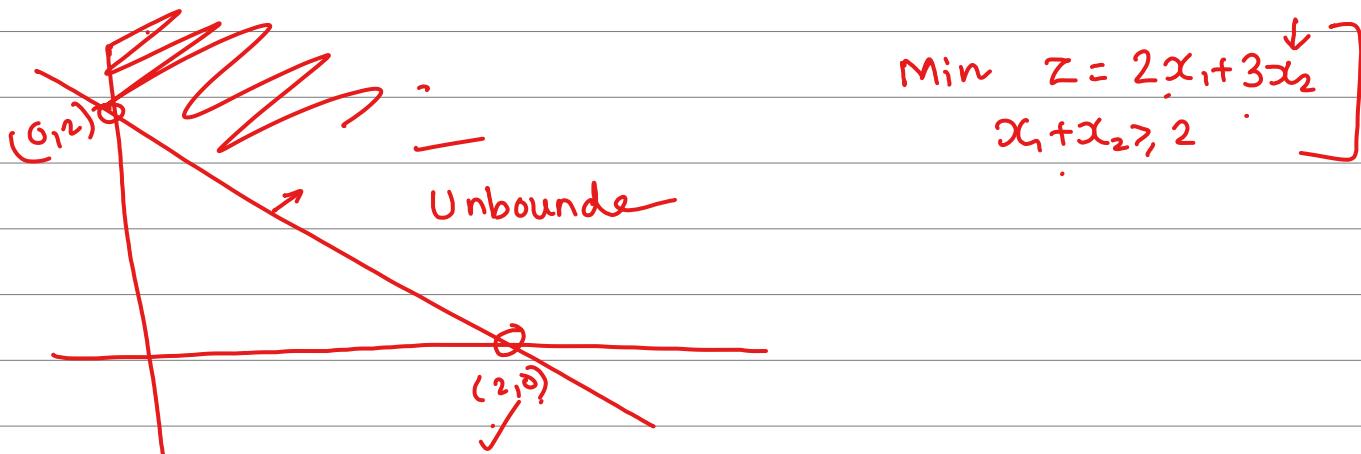
$$\underline{x}^0 = \alpha \underline{x}^1 + (1-\alpha) \underline{x}^2$$

To show \underline{x}^0 is also optimum sol^r

$$\begin{aligned}\underline{c}' \underline{x}^0 &= \underline{c}' (\alpha \underline{x}^1 + (1-\alpha) \underline{x}^2) \\ &= \alpha \underline{c}' \underline{x}^1 + (1-\alpha) \underline{c}' \underline{x}^2 \\ &= \alpha Z_0 + (1-\alpha) Z_0 \\ &= Z_0\end{aligned}$$

$\Rightarrow \underline{x}^0$ is also optimum sol^r

\Rightarrow Set of optimum sol^r is also convex set.



$$Z_0 = \min \left\{ \underline{c}' \underline{x}_i ; i=1:K \right\} \dots$$

$$\underline{x} \in T \Rightarrow \underline{x} = \sum_{i=1}^K \alpha_i \underline{x}_i \quad . \quad \alpha_i \geq 0 \quad \sum \alpha_i = 1$$

$$\begin{aligned}\underline{c}' \underline{x} &= \sum \alpha_i \underline{c}' \underline{x}_i \\ \therefore \sum \alpha_i Z_0 &= Z_0 \quad \Rightarrow\end{aligned}$$

$$\underline{c}' \underline{x} \geq z_0$$

$\nexists \underline{x} \in T$

$$\min_{\underline{x} \in T} \underline{c}' \underline{x} \geq z_0 = \min \{ \underline{c}' \underline{x}_i, i=1:k \}$$

$$\begin{aligned} & \max 5x + 8y \\ & \text{Subject to} \end{aligned}$$

$$18x + 10y \leq 180$$

$$10x + 20y \leq 200$$

$$15x + 20y \leq 210$$

$$\underline{x}, \underline{y} \geq 0$$

$$18x + 10y = 180$$

$$10x + 20y = 200$$

$$15x + 20y = 210$$

$$\underline{x}, \underline{y} \geq 0$$

$$(0, 18)$$

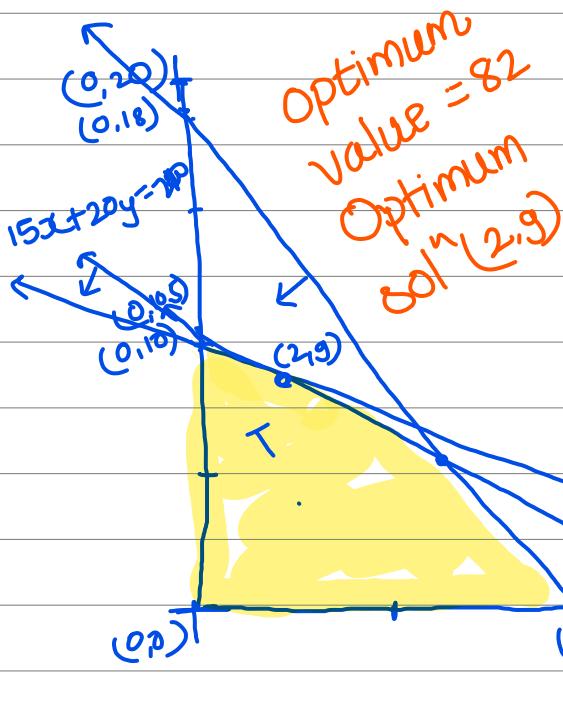
$$(10, 0)$$

$$(0, 10)$$

$$(20, 0)$$

$$(0, 10.5)$$

$$(14, 0)$$



5, 8	z
(0, 10)	80
(0, 0)	0
(10, 0)	50
(2, 9)	82

$$(7.12, 5.14)$$

$$76.85$$

$$\begin{aligned} 2x + 3y &= 20 & 36 \\ 18x + 10y &= 180 & \end{aligned}$$

$$\begin{aligned} 15x + 20y &= 210 \\ 21x &= 150 \end{aligned}$$

$$x = 7.12$$

$$y = 5.14$$

$$10x + 20y = 200$$

$$5x = 10$$

$$10x + 20y = 200$$

$$5x = 10$$

$$x = 2, y = 9$$

$$18x + 10y = 180$$

$$18x + 10y = 180$$

$$\begin{array}{l}
 \boxed{2x_1 + 3x_2 \leq 10} \\
 \boxed{3x_1 + 2x_2 \geq 5} \\
 \boxed{3x_1 + 5x_2 \leq 15} \\
 \boxed{x_1, x_2 \geq 0}
 \end{array}
 \quad
 \begin{array}{l}
 \boxed{2x_1 + 3x_2 + x_3 = 10} \\
 \boxed{3x_1 + 2x_2 - x_4 = 5} \\
 \boxed{3x_1 + 5x_2 + x_5 = 15}
 \end{array}$$

$$\begin{array}{l}
 2x_1 + 3x_2 \leq 0 \Rightarrow 2x_1 - 3x_2' \leq 10 \\
 x_1 + 5x_2 \leq 15 \quad x_1 - 5x_2' \leq 15 \\
 x_1 > 0, \quad x_2 \leq 0 \quad x_1 > 0, \quad x_2' > 0
 \end{array}$$

$$\begin{aligned}
 x_2' &= -x_2 \\
 x_B &= [x_1, x_2, x_3] \\
 \text{Basic} &\downarrow
 \end{aligned}$$

$$\left[\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 0 & -1 & 0 \\ 3 & 5 & 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right] = \left[\begin{array}{c} 10 \\ 5 \\ 15 \end{array} \right]$$

$$\begin{array}{l}
 A \underline{x} = b \\
 m \times n \\
 3 \times 5
 \end{array}$$

$$m=3, n=5$$

$$S(A) \leq \min(m, n)$$

$$A = [B \ R]$$

$$S(A) = S(B)$$

$$A \underline{x} = \underline{b}$$

$$\begin{bmatrix} B & R \end{bmatrix} \begin{bmatrix} \underline{x}_B \\ \underline{x}_{NB} \end{bmatrix} = \underline{b}$$

$$\begin{aligned}
 B \underline{x}_B + R \underline{x}_{NB} &= \underline{b} \\
 B^{-1} B \underline{x}_B + B^{-1} R \underline{x}_{NB} &= B^{-1} \underline{b}
 \end{aligned}$$

$$\begin{array}{l}
 \underline{x}_B = B^{-1} \underline{b} - B^{-1} R \underline{x}_{NB}^L \\
 \uparrow
 \end{array}$$

$$\text{Basic Soln} \rightarrow \underline{x}_{NB} = \underline{0}, \underline{x}_B = B^{-1} \underline{b}$$

$$\text{Max } 3x_1 + 5x_2$$

$$x_1 + 2x_2 \leq 6 \quad x_1 + 2x_2 + x_3 = 6$$

$$3x_1 + x_2 \leq 5 \quad 3x_1 + x_2 + x_4 = 5$$

$$x_1, x_2 \geq 0 \quad x_1, x_2, x_3, x_4 \geq 0$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$m=2, \quad n=4$$

$$S(A) = \underline{\underline{m}} = 2$$

$$\textcircled{1} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\underline{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad \underline{x}_{NB} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

~~$x_1 + x_2 \leq 2$~~
 ~~$2x_1 + 2x_2 \leq 4$~~

$$B^{-1} \underline{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \quad x_B = \{6, 5\} \\ x_{NB} = \{0, 0\}$$

$$\underline{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 6 & 5 \end{pmatrix} \quad \text{Basic Soln}$$

$$\textcircled{2} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad \underline{x}_B = [x_1, x_2] \quad \underline{x}_{NB} = [x_3, x_4]$$

$$B^{-1} = -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \quad B^{-1} \underline{b} = -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -4 \\ -13 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 13/5 \end{bmatrix}$$

Basic Soln

$$\underline{x} = \begin{bmatrix} 4/5 & 13/5 & 0 & 0 \end{bmatrix}$$

$\alpha_1, \alpha_2, \dots, \alpha_p$ $\sum_{i=1}^p \alpha_i a_i = 0 \Rightarrow \alpha_i = 0 \text{ l.i.}$
 for some $i \quad \alpha_i \neq 0$
 \hookrightarrow linearly dependent

$$\underline{x} = (\underbrace{x_1, x_2, \dots, x_p}_{\geq 0}, \underbrace{x_{p+1}, \dots, x_n}_{=0})$$

$$A\underline{x} = b \cdot \\ \underbrace{x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n}_{m \times n} = \underline{b}_{m \times 1}$$

$$\Rightarrow x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_p \underline{a}_p = \underline{b}_{m \times 1}$$

① If p vectors are linearly independent

$$p \leq m \checkmark$$

\hookrightarrow i) $p=m$, \underline{x} is f.s. is also b.f.s.

ii) $p < m$

we can add $m-p$ linearly independent columns

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_p \underline{a}_p + 0 \cdot \underline{a}_{p+1} + 0 \cdot \underline{a}_{p+2} + \dots + 0 \cdot \underline{a}_m = \underline{b}$$

\Rightarrow we get degenerate b.f.s.

columns

② If p vectors corresponding to feasible solⁿ are linearly dependent.

$$\sum_{i=1}^p \alpha_i a_i = 0 \Rightarrow \text{for some } \alpha_j \neq 0$$

Assume $\alpha_j \neq 0$

$$\frac{x_j}{\alpha_j} \sum_{i=1}^p \alpha_i a_i = 0 \cdot$$

$$\sum_{i=1}^p \underline{x}_i \underline{a}_i = \underline{b} . \quad (A\underline{x} = \underline{b})$$

$$\sum_{i=1}^p \underline{x}_i \underline{a}_i - \frac{\underline{x}_j \cdot \sum_{i \neq j} \underline{a}_i}{\underline{x}_j \cdot \underline{a}_j} \underline{x}_j \underline{a}_j = \underline{b} - \underline{0}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^p \underline{x}_i \underline{a}_i + \underline{x}_j \underline{a}_j - \underline{x}_j \underline{a}_j = \underline{b}$$

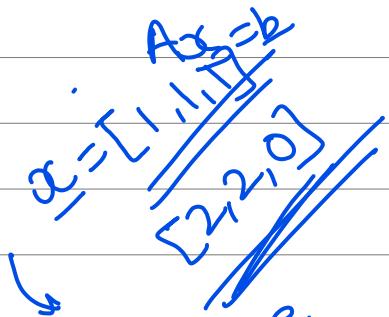
$$\underline{x} = [1, 1, 1]$$

$$\sum \underline{x}_i \underline{a}_i = 0 \Rightarrow \underline{x}_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underline{x}_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underline{x}_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

choosing $\underline{x}_1 = 1, \underline{x}_2 = 1, \underline{x}_3 = -1$

$$\underline{x}_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underline{x}_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underline{x}_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



\underline{x} extreme
points

$$\underline{x}_1 \underline{a}_1 + \underline{x}_2 \underline{a}_2 + \dots + \underline{x}_K \underline{a}_K = \underline{b}.$$

$$0 < z < \min \frac{\underline{x}_i}{(\underline{a}_i)}.$$

~~$$\pm \underline{x} (\underline{a}_1 \underline{a}_1 + \underline{a}_2 \underline{a}_2 + \dots + \underline{a}_K \underline{a}_K = 0)$$~~

for some $\underline{a}_j \neq 0$

$$\frac{1}{2} (\underline{x} + z \underline{a}) + \frac{1}{2} (\underline{x} - z \underline{a}) = \underline{x}$$

$$\min \left(\frac{\underline{x}_i}{\underline{a}_i}, z \right) > 0 \quad \forall i$$

$$(\underline{x}_1 + z \underline{a}_1) \underline{a}_1 + \dots + (\underline{x}_K + z \underline{a}_K) \underline{a}_K = \underline{b}$$

$$\begin{array}{ll}
 \text{Max} & Z = 6x_1 + 5x_2 \\
 \text{s.t.} & x_1 + x_2 \leq 5 \\
 & 3x_1 + 2x_2 \leq 12 \\
 & x_1, x_2 \geq 0
 \end{array}
 \quad \left| \begin{array}{l}
 x_1 + x_2 + x_3 = 5 \\
 3x_1 + 2x_2 + x_4 = 12 \\
 \Rightarrow x_i \geq 0
 \end{array} \right.$$

$$\begin{array}{ll}
 (0, 0, 6, 5) \\
 (5, 12, 0, 0) \\
 (x_3, x_4, x_1, x_2) \\
 \text{Basic} \quad \text{Nonbasic}
 \end{array}$$

$$\begin{array}{l}
 x_3 = 5 - x_1 - x_2 \quad \dots \text{---(1)} \\
 x_4 = 12 - 3x_1 - 2x_2 \quad \dots \text{---(2)} \\
 z = 0 \\
 \min(5, 4)
 \end{array}$$

$$\begin{array}{ll}
 \text{Basic} & \text{Nonbasic} \\
 (x_1, x_3, x_2, x_4) \\
 (6, 0, 5, 0) \\
 (4, 1, 0, 0)
 \end{array}$$

$$\begin{array}{l}
 3x_1 = 12 - 2x_2 - x_4 \\
 x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 \quad \dots \text{---(3)} \\
 x_3 = 5 - 4 + \frac{2}{3}x_2 + \frac{1}{3}x_4 - x_2 \\
 \rightarrow x_3 = 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 \quad \dots \text{---(4)}
 \end{array}$$

$$\begin{aligned}
 z &= 6x_1 + 5x_2 \\
 &= 6 \cdot \left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) + 5x_2
 \end{aligned}$$

$$\begin{aligned}
 z &= 24 + x_2 - 2x_4 \\
 &\min(6, 3)
 \end{aligned}$$

$$\frac{1}{3}x_2 = 1 - x_3 + \frac{1}{3}x_4 \Rightarrow x_2 = 3 - 3x_3 + x_4$$

$$x_1 = 4 - \frac{2}{3}(3 - 3x_3 + x_4) - \frac{1}{3}x_4$$

$$\begin{array}{ll}
 \text{Basic} & \text{NonB} \\
 x_1, x_2 & x_3, x_4 \\
 2 \ 3 & 0 \ 0 \\
 6 \ 5 & 0 \ 0
 \end{array}$$

$$z = 6x$$

$$\boxed{z = 27}$$

$$x_1 = 2 + 2x_3 - x_4$$

$$Z = 24 + x_2 - 2x_4$$

$$= 24 + 3 - 3x_3 + x_4 - 2x_4$$

$$Z = 27 - 3x_3 - x_4$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ \left(\begin{matrix} 2 & 3 & 0 & 0 \end{matrix} \right) \end{matrix}$$

$$Z = 27$$

Simplex Method.

C_j		6	5	0	0		
C_j	Basic	x_1	x_2	x_3	x_4	b	0
0.	x_3	1.	1.	1.	0.	5.	$5/1$
0.	x_4	3.	2.	0.	1.	12	$12/3 = 4 \rightarrow$ leaving
	Z_j	0.	0	0	0	0	
	$C_j - Z_j$	6	5	0	0		
		↑ most +ve entering					
0.	x_2	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	-1	$3 \xrightarrow{R_1 - R'_2}$
6.	x_1	1	$\frac{2}{3}$	0.	$\frac{1}{3}$	4	$\frac{1}{3} R_2$
	Z_j	6	4	0	2	24	
	$C_j - Z_j$	0	1	most +ve	0	-2	
		↑					
$3R_1$	5.	x_2	0	1	3	-1	3
$R_2 - \frac{2}{3}R_1$	6.	x_1	1	0	-2	1	2
	Z_j	6	5	3	1	27	
	$C_j - Z_j$	0	0	-3	-1		

$$\text{Min } Z = -x_1 - x_2$$

$$\text{s.t. } x_1 + x_2 \leq 2$$

$$x_1 - x_2 \leq 1$$

$$x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

$$\text{Max } Z = x_1 + x_2$$

$$\text{s.t. } x_1 + x_2 + x_3 = 2.$$

$$x_1 - x_2 + x_4 = 1$$

$$x_2 + x_5 = 1$$

$$x_i \geq 0 \quad \forall i=1:5$$

		Cost	1	1	0	0	0	.	
Cost	Basic		x_1	x_2	x_3	x_4	x_5	b	0
0	x_3		1.	1	1	0	0	2	2
0	x_4			-1	0	1	0	1	
0	x_5		0	1	0	0	1	1	
	Z_j		↑ 0	0	0	0	0	0	
	$C_j - Z_j$		↑ 1 max we	1	0.	0.	0.	0.	
			entering variable,		0	0	0	b	0
$R_1 - R_2$	0	x_3	0	2	1	-1	0	1	$\frac{1}{2}$ leaving
1	x_1	1.	-1.	0	1	0		1	-
0	x_5	0.	1	0.	0	1		1	
	Z_j		1.	-1	0	1	0		1
	$C_j - Z_j$		0	2 ↑	0	-1	0		
			entering		0	0	0		0
$\frac{1}{2}R_1$	1	x_2	0.	1.	$\frac{1}{2}$	$-\frac{1}{2}$	0.	$\frac{1}{2}$	-
$R_2 + R_1$	1	x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	3
$R_3 - R_1$	0	x_5	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1 →
	Z_j		1	1	1	0	0		2
	$C_j - Z_j$		0	0	-1	0 ↑	0		

Optimal Solⁿ $(\frac{3}{2}, \frac{1}{2}, 0, 0, \frac{1}{2})$ ✓

Optimal Value $Z=2$

$$\begin{array}{l}
 R_1 + \frac{1}{2}R_3 \\
 R_2 - \frac{1}{2}R_3 \\
 2R_3
 \end{array}
 \left| \begin{array}{c|ccccc|c}
 & x_2 & & & & & \\
 & 0 & 1 & 0 & 0 & 0 & 1 \\
 & 1 & 0 & 1 & 0 & -1 & 1 \\
 0 & x_4 & 0 & 0 & -1 & 1 & 2 \\
 z_j & 1 & 1 & 1 & 0 & 0 & 1 \\
 c_j - z_j & 0 & 0 & -1 & 0 & 0 & 2
 \end{array} \right| \quad \text{Q}$$

Alternate \rightarrow Optimal Solⁿ: (1,1,0,1,0) ✓

Optimal Value: 2

$$\text{Min } Z = -x_1 - x_2$$

$$x_1 + x_2 \leq 2$$

$$x_1 + x_2 = 2$$

$$(0,2), (2,0)$$

$$x_1 - x_2 \leq 1$$

$$x_1 - x_2 = 1$$

$$(0, -1), (1, 0)$$

$$x_2 \leq 1$$

$$x_2 = 1$$

$$x_1, x_2 \geq 0$$

Set of optimal solutions

Extreme Z value.
pt.s.

$$\min Z = x_1 - x_2$$

$$(0,0)$$

$$0$$

$$x_2=1$$

$$(0,1)$$

$$-1$$

$$(1,1)$$

$$(-2) \text{ min}$$

$$(\frac{3}{2}, \frac{1}{2})$$

$$(-2)$$

