

## Infinite Series:-

$$\underline{\sum x_n}$$

$$\sum_{i=1}^{\infty} x_n = ?$$

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

$$S_n \rightarrow S$$

$$S_{n+1} \rightarrow S$$

$$\lim_{n \rightarrow \infty} S_n - S_{n-1} = 0$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i - \sum_{i=1}^{n-1} x_i = 0$$

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$$\lim_{n \rightarrow \infty} x_n = 0$$

$$(S_n) = \sum_{i=1}^n x_i$$

$$\underline{S_n \rightarrow S}$$

$$\sum \frac{1}{n}$$

$$\sum \frac{(-1)^n}{n}$$

$$\sum x^n = \frac{1}{1-x} \quad |x| < 1$$

$$\checkmark \sum \frac{1}{n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\checkmark} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\checkmark}$$

$$\geq 1 + \frac{1}{2} [1 + 1 + 1 + \dots]$$

$$\geq \infty$$

$$\begin{aligned} & \sum \frac{(-1)^{n+1}}{n} \\ &= \underbrace{1 - \frac{1}{2}}_{\checkmark} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{\checkmark} + \underbrace{\frac{1}{5} - \frac{1}{6}}_{\checkmark} \end{aligned}$$

$$\begin{aligned} S_n &= 1 - \left( \frac{1}{2} - \frac{1}{3} \right) - \left( \frac{1}{4} - \frac{1}{5} \right) - \dots \\ &= 1 - \left( \frac{1}{2 \cdot 3} \right) - \left( \frac{1}{4 \cdot 5} \right) \end{aligned}$$

$$\underline{S_n \downarrow}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} \rightarrow 0$$

→ 1

$$\sum_n \frac{(-1)^n}{n} \quad \text{cgt} \quad \frac{(-1)^n}{n} \rightarrow 0$$

$\sum r^n$        $|r| < 1$       cgt       $r^n \rightarrow 0$        $0.5^2$

$$\sum \frac{1}{n^2} \quad \text{cgt} \quad \frac{1}{n^2} \rightarrow 0$$

$\sum \frac{1}{n}$

$\text{duge}$        $\frac{1}{n} \rightarrow 0$

Cauchy Criterion for convergence of series  
\*\*

for any  $\epsilon > 0 \exists K(\epsilon) \in \mathbb{N}, \exists |S_n - S_m| < \epsilon \Rightarrow n > m > K(\epsilon)$

$$\Rightarrow \left| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right| < \epsilon \quad \Rightarrow n > m > K(\epsilon)$$

$$\Rightarrow \left| \sum_{i=m+1}^n x_i \right| < \epsilon$$

Let  $x_n$  be a sequence of non-negative real numbers then the series P  $\sum x_n$  converges if and only if the sequence  $S_k$  of partial sum is bounded.

① Let  $\sum_{n=1}^{\infty} x_n$  converges ,  $\sum_{n=1}^{\infty} x_n = s$

$\Rightarrow S_n = \sum_{i=1}^n x_i = \text{seq of partial sums}$

$\Rightarrow S_n$  is convergent to  $s$ .

$\Rightarrow$  Every convergent seq<sup>n</sup> is bounded

$\Rightarrow S_n$  is bounded.

②  $S_n$  is bounded  $\Rightarrow$  To prove  $S_n$  is cgt.

$$S_{n+1} = \sum_{i=1}^{n+1} x_i = S_n + \underline{x_{n+1}} \quad \text{as } x_{n+1} > 0$$

$$S_{n+1} \geq S_n$$

$S_n$  is monotonically  $\uparrow$  & bounded  
by MCT it is cgt.

Show that  $\sum_{n=0}^{\infty} r^n = 1+r+r^2+\dots = \frac{1}{1-r}$  if  $|r| < 1$

$$\left| \begin{array}{l} S_{n+1} = \sum_{i=1}^{n+1} r^i = 1+r+r^2+\dots+r^{n-1}+r^n \\ S_n = \sum_{i=1}^n r^i = 1+r+r^2+\dots+r^{n-1} \\ rS_n = r+r^2+\dots+r^n = S_{n+1}-1 \end{array} \right.$$

$$1+rS_n = S_{n+1}$$

$$\lim_{n \rightarrow \infty} (1+r \cdot S_n) = \lim_{n \rightarrow \infty} S_{n+1} \quad ?$$

$$1+r \cdot \overline{S} = S \quad \Rightarrow \quad S = \frac{1}{1-r} \quad ?$$

$$\begin{aligned}
 p &> 1 \\
 \sum \frac{1}{n^p} &= 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{\leq 2^p} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}}_{\leq 4^p} + \frac{1}{8^p} + \dots \\
 &\leq 1 + \underbrace{\frac{1}{2^p} + \frac{1}{2^p}}_{\leq 2^p} + \underbrace{\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}}_{\leq 4^p} + \frac{1}{8^p} + \dots \\
 &\leq 1 + \frac{2}{2^p} + \frac{4}{2^{2p}} + \frac{8}{2^{3p}} + \dots \\
 &\leq 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \dots \\
 \sum_{n=1}^{\infty} \frac{1}{n^p} &\leq \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \\
 \text{as } p &> 1, \quad \frac{1}{2^{p-1}} < 1 \quad \Rightarrow \quad \sum r^n = \frac{1}{1-r} \quad \text{cgt.} \\
 \Rightarrow \sum \frac{1}{n^p} &\text{ is also cgt.}
 \end{aligned}$$

Comparison Test :-  $K \in \mathbb{N} \Rightarrow \frac{0 \leq x_n \leq y_n}{\sum y_n \text{ cgt.}} \Rightarrow n \geq K \Rightarrow \sum x_n \text{ cgt.}$

by Cauchy criterion for convergence of series  
for any  $\epsilon > 0 \exists M(\epsilon) \in \mathbb{N}. |y_{m+1} + y_{m+2} + \dots + y_n| < \epsilon \Rightarrow n > m > M(\epsilon)$

$$K'(\epsilon) = \max(M(\epsilon), K)$$

for  $\forall m \geq K(\epsilon)$

$$|y_{m+1} + \dots + y_n| < \epsilon$$

by  $\Delta$  inequality

$$\underline{|x_{m+1} + x_{m+2} + \dots + x_n|} < |y_{m+1} + \dots + y_n| < \epsilon$$

by Cauchy criterion  $\sum x$  converges.

$$\sum \frac{1}{n^2+n}$$

$$\sum x_n$$

$$\sum y_n = \sum \frac{1}{n}$$

$\rightarrow$  cgt

$$r = \lim \frac{x_n}{y_n} = \lim \frac{1}{n^2+n} \times n^2 = \lim_{n \rightarrow \infty} \frac{1}{1+1/n} = 1$$

$$r \neq 0$$

$$0 < x_n, 0 < y_n \rightarrow n$$

$$r = \lim \frac{x_n}{y_n}$$

for any  $\epsilon > 0 \exists K(\epsilon) \in \mathbb{N}, \exists \left| \frac{x_n}{y_n} - r \right| < \epsilon$

$$r - \epsilon < \frac{x_n}{y_n} < r + \epsilon$$

$$y_n(r - \epsilon) \leq x_n \leq y_n(r + \epsilon)$$

$$x_n \leq y_n, \sum y_n \text{ cgt} \Rightarrow \sum x_n \text{ cgt}$$

if  $\sum y_n \text{ cgt} \Rightarrow \sum x_n \text{ cgt}$

if  $\sum x_n \text{ cgt} \Rightarrow \sum y_n \text{ cgt}$

if  $r=0,$

$$r-\varepsilon < \frac{x_n}{y_n} < r+\varepsilon$$

$$-\varepsilon < \frac{x_n}{y_n} < \varepsilon$$

$$-\varepsilon < 0 < \frac{x_n}{y_n} < \varepsilon$$

$$x_n \leq y_n \cdot \varepsilon$$

$\Rightarrow$  if  $\sum y_n \text{ cgt} \Rightarrow \sum x_n \text{ cgt}.$

$$\sum \frac{(-1)^n}{n} \xrightarrow{\text{cgt}} \sum \frac{1}{n} \text{ divergent}$$

Absolute:  $\sum |x_n| \text{ abs. cvgt if } \sum |x_n| \text{ is cvgt.}$

Conditional Convergence  $\sum x_n$  is cgt but  $\sum |x_n|$  is not cgt.

$$\sum \frac{(-1)^n}{n^2} \text{ abs cgt}$$

$$\sum \frac{1}{n^2} \text{ cgt}$$

for any  $\varepsilon > 0 \exists K(\varepsilon) \in \mathbb{N}.$

$$\frac{|x_{m+1} + \dots + x_n|}{\sum |x_n|} \leq |x_{m+1}| + |x_{m+2}| + \dots + |x_n| \leq \varepsilon \quad \begin{matrix} \rightarrow n \rightarrow \\ n \rightarrow m > K(\varepsilon) \end{matrix}$$

Root test       $r = \lim |x_n|^{1/n}$  exists.

$$\textcircled{1} \quad r \leq 1, \exists r_1 \leq r < 1$$

for some  $K(\epsilon) \in \mathbb{N}$ .

$$|x_n|^{1/n} \leq r, \quad \nexists n \geq K(\epsilon)$$

$$\Rightarrow |x_n| \leq r_1^n \quad (r_1 < 1)$$

by Comparison Test

$$\sum |x_n| \leq \sum r_1^n \quad \begin{matrix} \uparrow \\ \text{cgt.} \end{matrix} \quad \begin{matrix} \nearrow \\ \text{cgt.} \end{matrix}$$

$$\textcircled{2} \quad r > 1, \exists r_1 < r$$

~~Root~~

$$\begin{aligned} & r \leq r_1 \leq |x_n|^{1/n} \\ & 1^n \leq r_1^n \leq |x_n| \\ & \sum_{\infty}^1 \leq \sum r_1^n \leq \sum |x_n| \quad \text{dvgt.} \end{aligned}$$

Ratio

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right|$$

$$\textcircled{1} \quad r < 1 \quad \text{cgt.}$$

$r > 1$       dvgt

$r = 1$       Test fails

$$\textcircled{2} \quad r \leq 1$$

$$\left| \frac{x_{n+1}}{x_n} \right| < r$$

for  $n \geq K(\epsilon)$

$$\begin{aligned}
 |x_{n+1}| &< r \cdot |x_n| \\
 &\leq r \cdot r \cdot |x_{n-1}| = r^2 |x_{n-1}| \\
 &\vdots \\
 &\leq r^n \cdot |x_1|
 \end{aligned}$$

$$\begin{aligned}
 \sum \frac{1}{(n+1)(n+2)} & \\
 \text{Ratio Test} : - \quad r &= \lim \left| \frac{x_{n+1}}{x_n} \right| \\
 &= \lim \left| \frac{(n+1)(n+2)}{(n+2)(n+3)} \right| \\
 &= \lim \left| \frac{1 + \frac{1}{n}}{1 + \frac{3}{n}} \right| \\
 &= 1 \quad \text{Test fail}
 \end{aligned}$$

$x > 1/x$

Unit Comparison Test  $\sum y_n = \sum \frac{1}{n^2}$

$$\begin{aligned}
 r &= \lim \left| \frac{x_n}{y_n} \right| \\
 &= \lim \left| \frac{n^2}{(n+1)(n+2)} \right| \\
 &= \lim \left| \frac{1}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} \right| \\
 &= 1 \neq 0 \quad \sum |y_n| \text{ cgt}
 \end{aligned}$$

$$\Rightarrow \sum |x_n| \text{ cgt.}$$

$$\sum \frac{1}{n^2} > \sum \frac{1}{(n+1)(n+2)}$$

cgt.

$$\sum \frac{1}{(n+1)(n+2)} \geq x_n = \frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

$$S_1 = x_1 = 1 - \frac{1}{2}$$

$$S_2 = S_1 + x_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$$

$$S_3 = S_2 + x_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

$\sum x_n$  cgt if  $S_n$  cgt.

$$\lim S_n = \lim 1 - \frac{1}{n+1}$$

$$S_n \rightarrow 1 = 1$$

$$\sum 2^n \quad \text{Ratio} \Rightarrow \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{2^{n+1}}{2^n} \right| = 2 > 1 \quad \text{dgt.}$$

$\sum 2^n$  Root test

$$\lim |x_n|^{1/n} = \lim [2^n]^{1/n} = 2 > 1 \quad \text{dgt.}$$

$$\sum 2^{-1/n}$$

Ratio  $\lim_{n \rightarrow \infty} \frac{2^{-1/(n+1)}}{2^{-1/n}}$

$$\lim_{n \rightarrow \infty} \frac{2^{\frac{-1}{n+1}} + \frac{1}{n}}{2^{\frac{-1}{n+1}}} \rightarrow 1$$

Test fails

$$\sum 2^{-1/n}$$

$$\sum x_n \text{ cgt} \Rightarrow \lim x_n = 0$$

$$\lim x_n \neq 0 \Rightarrow \sum x_n \text{ not cgt.}$$

$$\lim 2^{-1/n} = 1$$

$$\Rightarrow \sum 2^{-1/n} \text{ diverges.}$$

\*  $x_n = n/2^n$

$$\text{Ratio Test } r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} \right|$$

$$= \lim \left| \frac{n+1}{n} \right| \cdot \lim \left| \frac{2^n}{2^{n+1}} \right|$$

$$= \frac{1}{2} < 1$$

$$\Rightarrow \sum x_n \text{ cgt.}$$

Root test :-

$$r = \lim |x_n|^{1/n} = \lim \left| \frac{n}{2^n} \right|^{1/n}$$

$$= \frac{1}{2} \lim \frac{n^{1/n}}{2} \rightarrow 1$$

$$= \frac{1}{2} < 1$$

$$\underline{\sum \frac{3^n}{n!}} ?$$

Ratio Test

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left( \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n} \right)$$

$$= \lim \cancel{\frac{3}{n+1}} \frac{3}{n}$$

$$= 0 < 1 \text{ cgt.}$$

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \dots$$

$$x_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$\frac{1}{\sqrt{1 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 3}} + \frac{1}{\sqrt{3 \cdot 4}}$$

$$x_n = \frac{1}{\sqrt{n(n+1)}}$$

limit  
Comparison

$$\lim \left| \frac{n}{\sqrt{n(n+1)}} \right|$$

$$= \lim \left| \frac{1}{\sqrt{1 + 1/n}} \right|$$

$$= 1$$

$$\Rightarrow \sum y_n \text{ diverges} \Rightarrow \sum x_n \text{ diverges}$$

Ratio:  $x_n = \frac{1}{\sqrt{n(n+1)}}$

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \frac{\sqrt{n(n+1)}}{\sqrt{(n+1)(n+2)}} = \lim \sqrt{\frac{1}{1 + 2/n}}$$

$$= 1 \quad \text{Test fails.}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots$$

$$x_n = \frac{1}{n^n}$$

Root test  $\lim |x_n|^{1/n} = \lim \left| \frac{1}{n^n} \right|^{1/n} = \lim \left| \frac{1}{n} \right| = 0 < 1$

$$\sum \frac{n^2-1}{n^2+1} \quad \text{by necessary cond'} \quad \lim \frac{n^2-1}{n^2+1} = 1 \neq 0$$

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \dots$$

$$\underbrace{\left( \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots \right)} + \left( \frac{1}{3^2} + \frac{1}{3^4} + \dots \right)$$

$$\underbrace{\sum \left( \frac{1}{3^2} \right)^n}_{= 1}$$

$$\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \dots$$

$$x_n = \frac{1}{n \cdot 2^n}$$

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \dots$$

$$x_n = \frac{n^2(n+1)^2}{n!}$$

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \frac{(n+1)^2(n+2)^2}{(n+1)!} \cdot \frac{n!}{(n+1)^2(n^2)}$$

$$= \lim \frac{1}{(n+1)} \cdot (1 + 2/n)^2 \\ = 0$$

$$x_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{7 \cdot 10 \cdots (3n+4)}$$

$$\lim \left| \frac{x_{n+1}}{x_n} \right|$$

$$\frac{1 \cdot 2 \cdots n \cdot (n+1)}{7 \cdot 10 \cdots (3n+4)(3n+1+4)} = \lim \left| \frac{\frac{n+1}{(3(n+1)+4)}}{\frac{n+1}{(3(n+1)+4)}} \right|$$

$$\frac{1}{3+4/(n+1)} = \lim \left| \frac{1}{3+4/(n+1)} \right|$$

$$= \frac{1}{3} < 1$$

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{1}{n} ?$$

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{2(n+1)-1}{2(n+1)} \left( \frac{1}{n+1} \right) \right| = 1$$

$$x_n \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} \cdot \frac{1}{n}$$

$$\lim |x_n|^{\frac{1}{n}} = \frac{1}{2} \left( \lim \frac{1^{\frac{1}{n}} 3^{\frac{1}{n}} \cdots (2n-1)^{\frac{1}{n}}}{(n!)^{\frac{1}{n}}} \cdot \frac{1}{n^{\frac{1}{n}}} \right)$$

$$= \frac{1}{2} < 1$$









## Functions and continuity

limit / Cluster Point :-  $\underline{A \subseteq \mathbb{R}}$ ,  $c \in \mathbb{R}$  for every  $\delta > 0$   
 $\exists \underline{x \in A, x \neq c, \exists |x - c| < \delta}$

for any  $\underline{\delta > 0}$ ,  $\underline{\delta_\epsilon(c) \cap A \neq \emptyset}$



①  $c$  cluster,  $\underline{\exists a_n \in A} \quad a_n \rightarrow c$

$\Rightarrow c$  cluster pt. of A

$$\left\{ \begin{array}{l} \text{for any } \frac{1}{n} > 0 \quad \exists x_n \in A \quad \Rightarrow \quad |x_n - c| < \frac{1}{n} \\ \quad \uparrow n \in \mathbb{N} \end{array} \right. \Rightarrow c - \frac{1}{n} < x_n < c + \frac{1}{n}$$

$$\Rightarrow x_n \rightarrow c$$

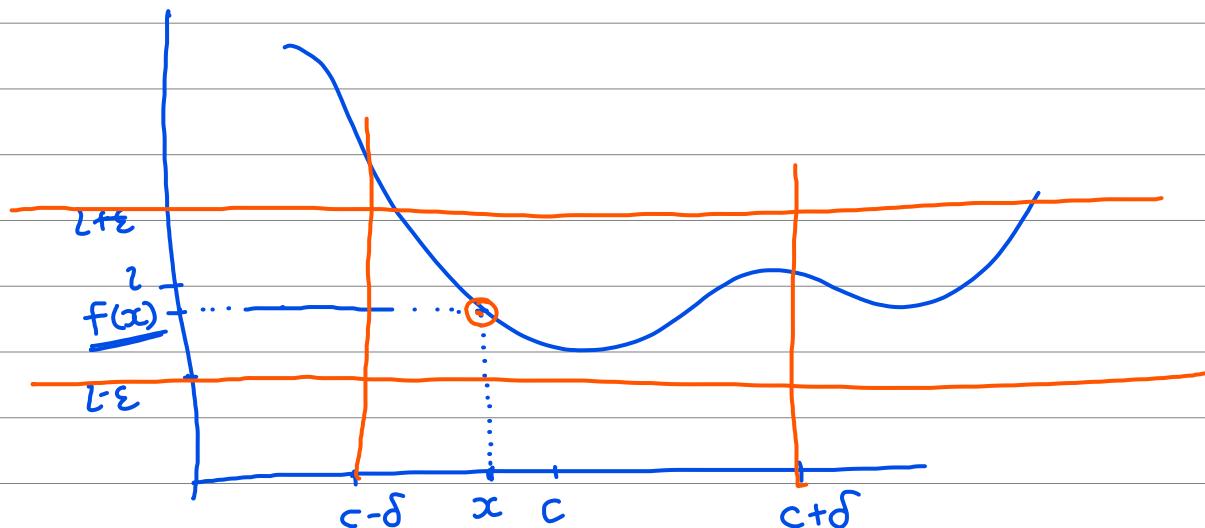
Suppose we have a seq<sup>n</sup>  $a_n \in A, \exists a_n \rightarrow c$ .  
To show that :-  $c$  cluster pt. of A

✓  $a_n \rightarrow c$

for any  $\epsilon > 0$ ,  $\exists K(\epsilon) \in \mathbb{N}$ ,  $\exists |a_n - c| < \epsilon$   $\Rightarrow n > K(\epsilon)$

$\Rightarrow |a_n - c| < \epsilon$   $a_n \in A \cap S_\epsilon(c)$

$\Rightarrow c$  is also cluster pt. of A



Limit of func :

$f: A \rightarrow \mathbb{R}$ ,  $c$  cluster of  $A$ ,  $L \in \mathbb{R}$

for any  $\epsilon > 0$   $\exists \delta_\epsilon > 0$   $\exists |x - c| < \delta_\epsilon \Rightarrow |f(x) - L| < \epsilon$   
 $\Rightarrow x \in V_\delta(c) \Rightarrow f(x) \in V_\epsilon(L)$

$\therefore f: A \rightarrow \mathbb{R}$ ,  $c$  cluster pt. of  $A$ .  $\Rightarrow f$  can have only one limit pt. at  $C$ .

Contradiction .  $L$  &  $L'$

for  $\epsilon > 0$   $\left\{ \begin{array}{l} \frac{\epsilon}{2} > 0 \quad \exists \delta'_\epsilon > 0 \quad \exists |x - c| < \delta'_\epsilon \Rightarrow |f(x) - L| < \epsilon/2 \\ \frac{\epsilon}{2} > 0 \quad \exists \delta''_\epsilon > 0 \quad \exists |x - c| < \delta''_\epsilon \Rightarrow |f(x) - L'| < \epsilon/2 \end{array} \right.$

$$\begin{aligned}
 |L - L'| &= \underline{|L - f(x)|} + \underline{|f(x) - L'|} \\
 &\leq |L - f(x)| + |f(x) - L'| \\
 &\leq \varepsilon_1 + \varepsilon_2 \\
 &\leq \varepsilon
 \end{aligned}$$

$L = L'$

QED

def seq convergent seqn    & def limit pt. of func  
at some  $c$

$f: A \rightarrow \mathbb{R}$ ,  $c$  is cluster pt. of  $A$

① for any  $\varepsilon > 0 \exists \delta_\varepsilon > 0 \ni \text{if } \underline{|x - c| < \delta_\varepsilon} \Rightarrow \underline{|f(x) - L| < \varepsilon} \quad x \in A$

②  $f: A \rightarrow \mathbb{R} \quad c \quad A \quad f(x) \text{ converges to } L \text{ at } c$

$\ni \underline{x_n \rightarrow c} \Rightarrow \underline{f(x_n) \rightarrow L}$

①  $|x - c| < \delta_\varepsilon \quad |f(x) - L| > \varepsilon_0$

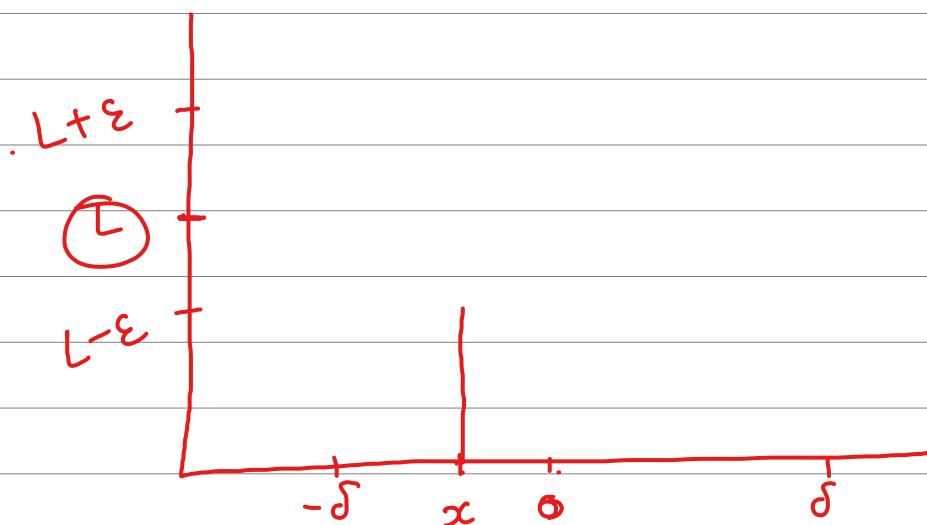
②  $x_n \rightarrow c \quad \text{but} \quad f(x_n) \not\rightarrow L$

does not converges  
 $f(x_n) \rightarrow \text{out } c$

$$\boxed{\lim_{x \rightarrow c} f(x) = L}$$

$$f(x) = \operatorname{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$$

$f: \mathbb{R} \rightarrow \mathbb{R}$



$$(-\delta, 0) \quad f(x) = -1 \quad (0, \delta) \quad f(x) = +1$$

$f: A \rightarrow \mathbb{R}$ ,  $c$  cluster pt. of  $A$

✓  $f$ : limit exists at pt.  $c$ ,  $c \in \mathbb{R}$   
 for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni$  if  $|x - c| < \delta_\epsilon \Rightarrow |f(x) - L| < \epsilon$

$$\Rightarrow (L - \epsilon < f(x) < L + \epsilon) \Rightarrow x \in V_{\delta_\epsilon}(c)$$

Completeness

$$M = \sup \{ f(x), x \in V_{\delta_\epsilon}(c) \}$$

$$\Rightarrow |f(x)| \leq M$$

for some  $M > 0$

$$\begin{array}{l}
 f: A \rightarrow \mathbb{R}, c \text{ cluster pt. of } A \quad \left. \begin{array}{l} \lim_{x \rightarrow c} f(x) = L \\ \lim_{x \rightarrow c} g(x) = M \end{array} \right\} \\
 g: A \rightarrow \mathbb{R} \\
 \downarrow \\
 (f+g): A \rightarrow \mathbb{R} \\
 \left. \begin{array}{l} \lim_{x \rightarrow c} (f+g)(x) = L+M \end{array} \right\}
 \end{array}$$

$$\lim_{x \rightarrow c} f(x) = L \quad \text{for } \varepsilon_1 > 0, \exists \delta'_\varepsilon > 0 \ni |x - c| < \delta'_\varepsilon \Rightarrow |f(x) - L| < \varepsilon_1$$

$$\text{for } \varepsilon_2 > 0, \exists \delta''_\varepsilon > 0 \ni |x - c| < \delta''_\varepsilon \Rightarrow |g(x) - M| < \varepsilon_2$$

$$\delta_\varepsilon = \min(\delta'_\varepsilon, \delta''_\varepsilon) \quad |x - c| < \delta_\varepsilon$$

$$|(f+g)(x) - (L+M)|$$

$$= |(f(x) + g(x)) - (L+M)|$$

$$= |\underline{f(x)-L} + \underline{g(x)-M}| \leq |f(x)-L| + |g(x)-M|$$

$$\leq \varepsilon_1 + \varepsilon_2$$

$$\leq \varepsilon$$

$$|(f+g)(x) - (L+M)|$$

$$= |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - LM|$$

$$\leq |g(x)| |f(x)-L| + |L| |g(x)-M|$$

$$\leq M \cdot \frac{\epsilon}{2M} + L \cdot \frac{\epsilon}{2L}$$

$f: A \rightarrow \mathbb{R}$ ,  $c$  cluster pt. of  $A$   
 $a \leq f(x) \leq b$ ,  $f$  limit exists at  $c$ .

for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |x - c| < \delta_\epsilon \Rightarrow |f(x) - L| < \epsilon$

seq Criter  $\exists x_n \in A$   $x_n \rightarrow c \Rightarrow f(x_n) \rightarrow L$

$a \leq f(x) \leq b \Rightarrow x \in A$   
 $a \leq f(x_n) \leq b \Rightarrow x_n$

by squeeze theo.

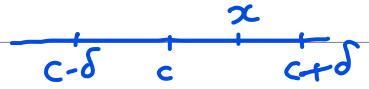
$$a \leq \lim_{x_n \rightarrow c} f(x_n) \leq b$$

$$a \leq L \leq b$$

$c$  cluster of  $A$  iff  $\exists a_n \in A \quad a_n \rightarrow c$

i)  $c$  cluster pt. of  $A$ .

for every  $\delta > 0 \quad \exists x \in A \quad |x - c| < \delta$



$\nexists n \in \mathbb{N}, \frac{1}{n} > 0 \quad \exists x_n \in A \rightarrow |x_n - c| < \frac{1}{n} \Rightarrow c - \frac{1}{n} < x_n < c + \frac{1}{n}$

$x_n \rightarrow c$

ii)  $a_n \in A, a_n \rightarrow c$

for any  $\varepsilon > 0 \quad \exists k(\varepsilon) \in \mathbb{N}, \quad \exists \frac{|a_n - c| < \varepsilon}{\text{infinite}} \quad \nexists n, k(\varepsilon)$

$\Rightarrow c$  is cluster pt. of  $A$ .

$f, g, h : A \rightarrow \mathbb{R}, \quad c$  cluster pt. of  $A$

$f(x) \leq g(x) \leq h(x) \quad \Rightarrow x \in A \quad x \neq c$

$$\checkmark \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$$

To show  $\lim_{x \rightarrow c} g(x) = L$

for any  $\varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \exists |x - c| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$

$$\begin{aligned} \dots \quad \delta_\varepsilon > 0 \quad \exists |x - c| < \delta_\varepsilon \Rightarrow & |f(x) - L| < \varepsilon \\ & \Rightarrow L - \varepsilon < f(x) < L + \varepsilon \quad \dots \\ & \Rightarrow |h(x) - L| < \varepsilon \\ & \Rightarrow L - \varepsilon < h(x) < L + \varepsilon \quad \dots \end{aligned}$$

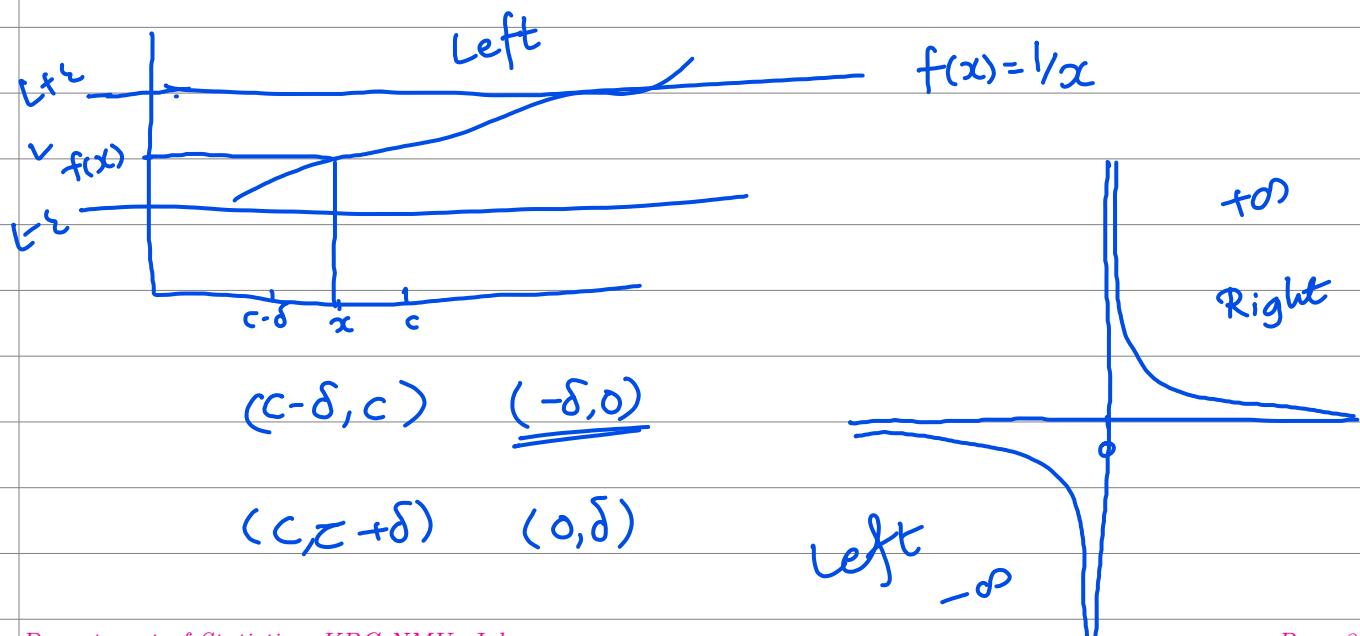
$$f(x) \leq g(x) \leq h(x)$$

$$\Rightarrow L - \varepsilon < f(x) \leq \underline{g(x)} \leq h(x) < L + \varepsilon$$

$$\Rightarrow L - \varepsilon < g(x) < L + \varepsilon$$

$$\Rightarrow |g(x) - L| < \varepsilon$$

$$\Rightarrow g(x) \rightarrow L$$



$$\checkmark \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x)$$

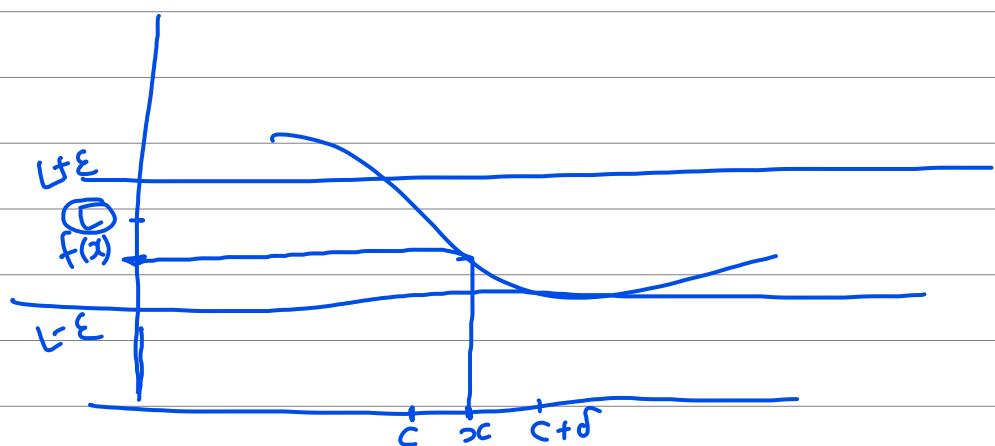
$(c-\delta, c+\delta)$                              $(c, c+\delta)$                              $(c-\delta, c)$

$\lim_{x \rightarrow c^+} f(x)$  for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni x \in (c, c+\delta) \Rightarrow |f(x) - L| < \epsilon$

$$(x-c) < \delta_\epsilon$$

$$x < c + \delta_\epsilon$$

$$(c - \delta_\epsilon, c) \cup (c, c + \delta_\epsilon)$$



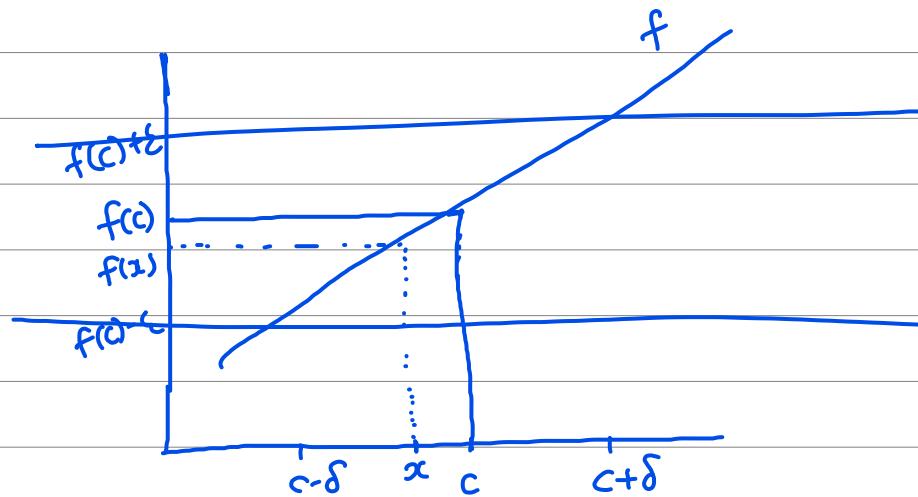
$$\lim_{x \rightarrow c} f(x) = L = \underline{f(c)}$$

f cont. at pt. c.

Right

$$\checkmark \lim_{x \rightarrow c^+} f(x) = f(c)$$

$$\checkmark \lim_{x \rightarrow c^-} f(x) = f(c)$$



$$\lim_{x \rightarrow c} f(x) = \underline{f(c)}$$

$$\checkmark f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c)) \quad \checkmark$$

i) for any  $\varepsilon > 0 \exists \delta_\varepsilon > 0 \ni |x - c| < \delta_\varepsilon \& x \in A$   
 $x \in V_\delta(c) \& x \in A \Rightarrow x \in V_\delta(c) \cap A$

$\Rightarrow |f(x) - f(c)| < \varepsilon$   
 $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$   
 $f(x) \in V_\varepsilon(f(c))$

$$\checkmark \Rightarrow f(\underline{A \cap V_\delta(c)}) \subseteq V_\varepsilon(f(c))$$

Sequential Criteria  
for continuity

$$\begin{cases} x_n \rightarrow c \\ f(x_n) \rightarrow f(c) \end{cases}$$

$$\text{if } x_n \rightarrow c \Rightarrow f(x_n) \rightarrow f(c)$$

f, g cont. at c      f+g cont at c.

for any  $\varepsilon_2 > 0 \exists \delta_\varepsilon > 0 \ni |x - c| < \delta_\varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon_1$   
 $\Rightarrow |g(x) - g(c)| < \varepsilon_2$

$$\begin{aligned} |f(x) + g(x) - f(c) - g(c)| &= |f(x) - f(c)| + |g(x) - g(c)| \\ &\leq \varepsilon_1 + \varepsilon_2 \\ &\leq \varepsilon \end{aligned}$$

( $c$  cluster pt. A)  $\rightarrow f: A \rightarrow \mathbb{R}$   
 $\lim_{x \rightarrow c} f(x) = f(c)$ ) ✓

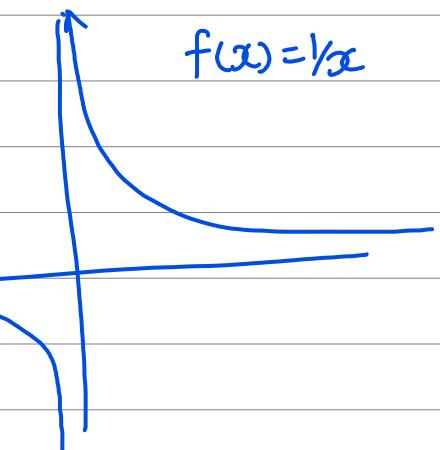
Conti. on set  $B$ ,  $\Rightarrow \forall y \in B$   $\lim_{x \rightarrow y} f(x) = f(y)$  ]

$\Rightarrow$  conti at every point of  $B$ .

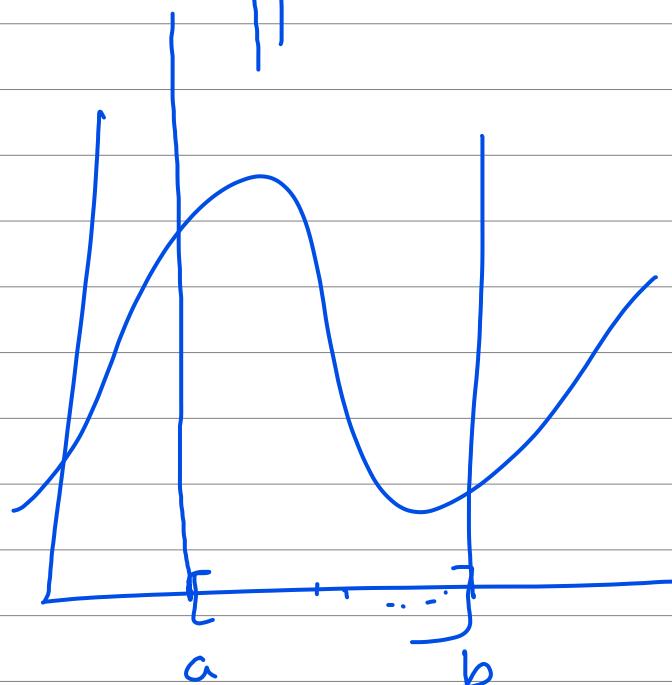
$$\underline{B = [-1, 1]}$$

$$f(x) = 1/x$$

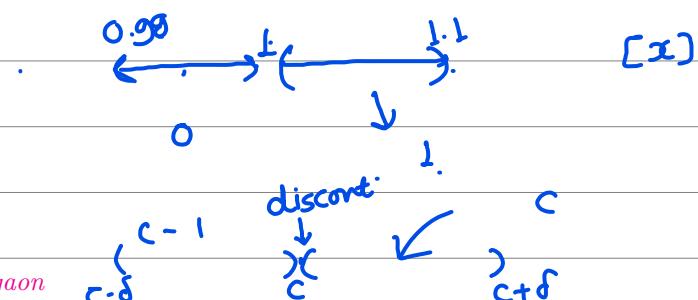
$$\begin{cases} \lim_{x \rightarrow 0^+} f(x) = +\infty \\ \lim_{x \rightarrow 0^-} f(x) = -\infty \end{cases}$$



$$f(x) = 2x^2 + 3 \checkmark$$



$$f(x) = \underline{\underline{\lfloor x \rfloor}}$$

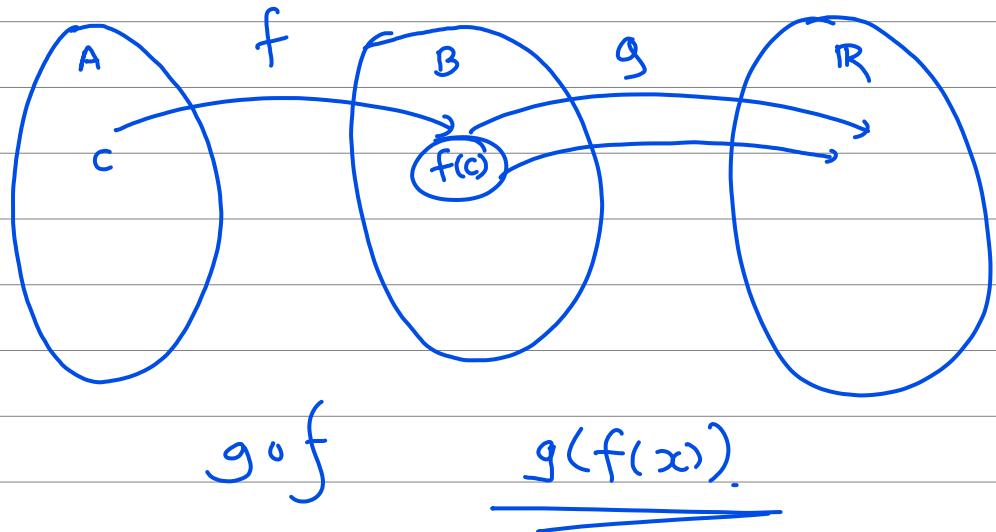


$f$  cont. at pt.  $c$ .

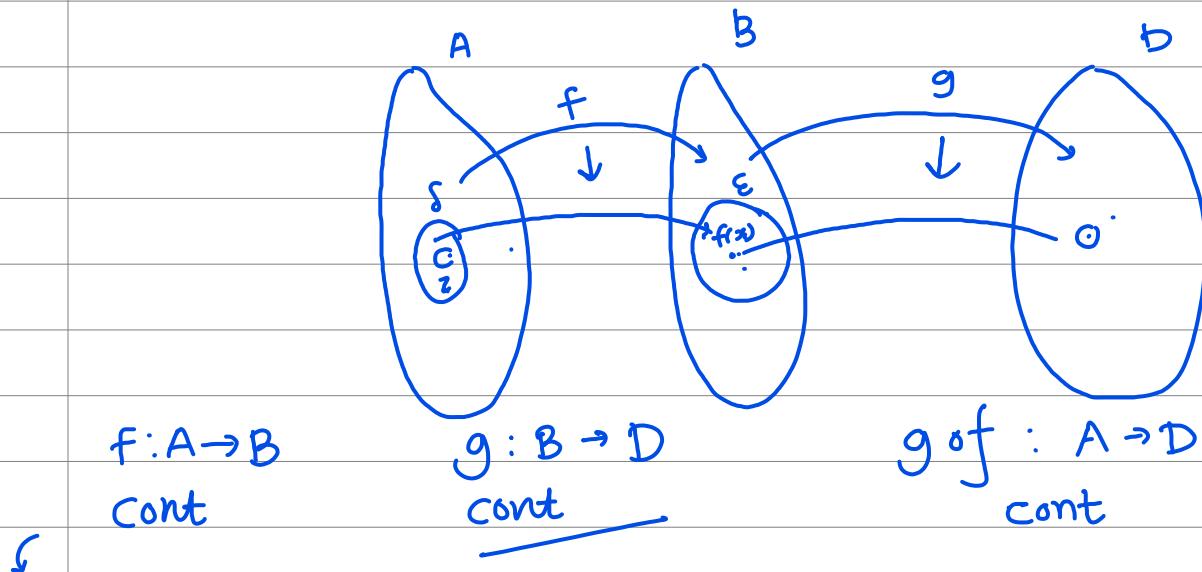
for any  $\varepsilon > 0$   $\exists \delta_\varepsilon > 0 \Rightarrow |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

$$|f(x) - f(c)| \leq |f(x) - f(c)| < \varepsilon$$

$\Rightarrow |f|$  is cont. at  $c$ .



Let  $A, B \subseteq \mathbb{R}$  & let  $f : A \rightarrow B$  &  $g : B \rightarrow D$  be functions such that  $f(A) \subseteq B$  if  $f$  is continuous at point  $c \in A$  and  $g$  is continuous at  $b = f(c) \in B$  then the composition  $g \circ f : A \rightarrow D$  is continuous at  $c$ .



for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$   
 $x \in V_\delta(c) \Rightarrow f(x) \in V_\epsilon(f(c))$

for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |y - c'| < \delta \Rightarrow |g(y) - g(c')| < \epsilon$   
 $y \in V_\delta(c') \Rightarrow g(y) \in V_\epsilon(g(c'))$

As we have assumed that  $f(A) \subseteq B$

$\exists$  some  $c' = f(c)$

[ for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |z - c| < \delta \Rightarrow$  ]

cont  
def  
cont

$$|(g \circ f)(z) - (g \circ f)(c)| = |g(f(z)) - g(f(c))|$$

$$( |z - c| < \delta \Rightarrow |f(z) - f(c)| < \epsilon, \epsilon > 0 )$$

$$|y - c'| < \delta \Rightarrow |g(y) - g(c')| < \epsilon$$

$$\Rightarrow |g(f(z)) - g(f(c))| < \epsilon$$

## Boundedness Theo.

$I = [a, b]$  closed bounded interval -

$f: I \rightarrow \mathbb{R}$  cont. ✓

To show:  $f$  bounded.

$$\underline{|f(x)| \leq M}$$

{ Assume  $f(x)$  is not bounded on  $I$

$\exists$  some  $n \in \mathbb{N}$   $|f(x_n)| > M$  ✓

-  $I$  closed bounded interval..

Let  $\exists$  some seq  $\underline{x_n} \in I$ ,  
 $I$  bounded  $\Rightarrow \underline{x_n}$  is bounded.

By Bolzano Weierstrass Theo.  $\exists$  subseq  $\underline{x_{n_k}} \in I$   
 which is convergent.

$$x_{n_k} \downarrow \rightarrow x^* \quad (\text{say})$$

if  $x_{n_k}$  is subseq in  $I$  &  $I$  is closed  
 $\Rightarrow x^* \in I$

$\therefore f$  is cont.

$$x_{n_k} \rightarrow x^* \Rightarrow \underline{\underline{f(x_{n_k}) \rightarrow f(x^*)}}$$

$\Rightarrow f(x_{n_k})$  is convergent subseq

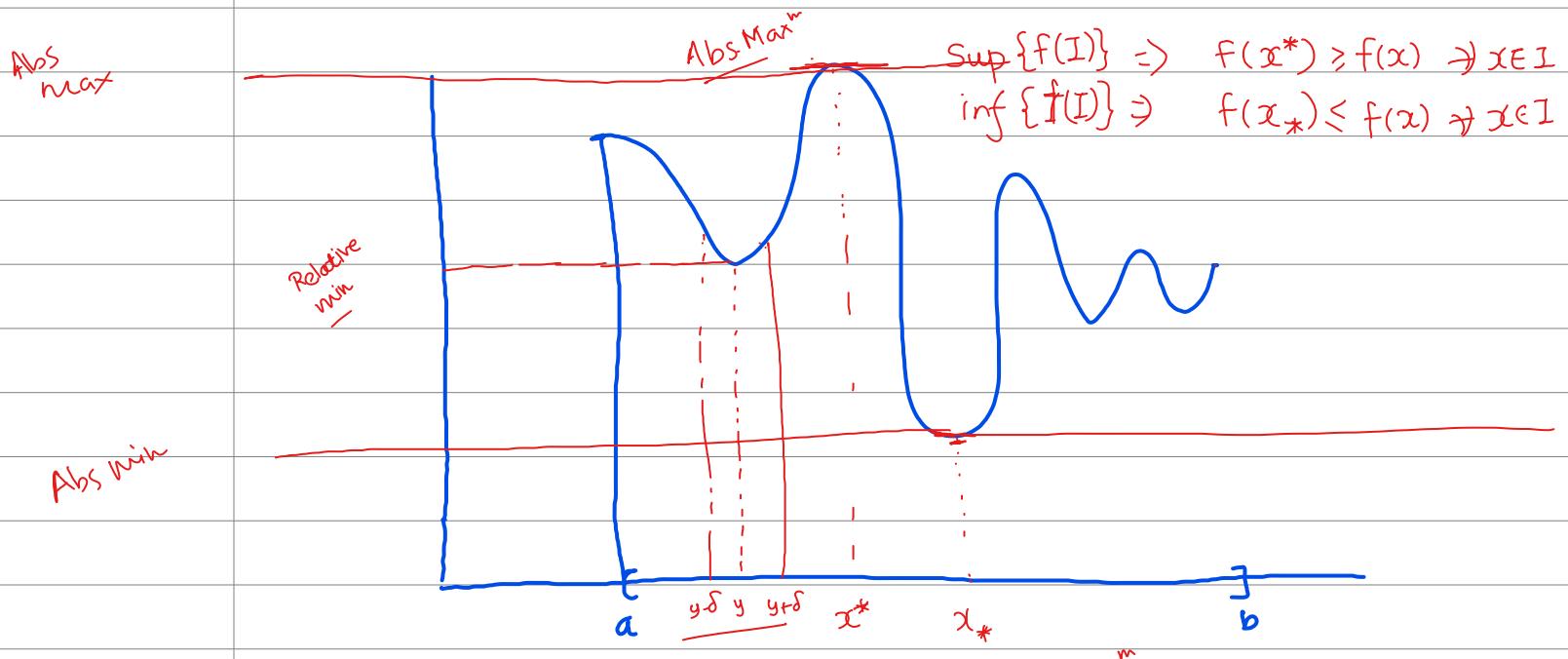
$\Rightarrow$  bounded seq

$\Rightarrow$  Our assumption is wrong.

## Absolute Extremum

max<sup>m</sup> min

## Relative extremum

max<sup>m</sup> min

$\text{Max}^m \& \text{Min}^m$  the  
 $f: I \rightarrow \mathbb{R}$ ,  $I$  closed bounded,  $f$  cont.

To show:  $\exists x^*, x_*$  as abs. max<sup>m</sup> & abs. min resp.

$\therefore I$  closed bounded,  $f$  cont.  $f: I \rightarrow \mathbb{R}$

by Boundedness Theo.  $|f(x)| \leq M \Rightarrow x \in I$

$f(I) = \{f(x), x \in I\}$  bounded.

by Completeness property.

$\exists S^* = \sup \{f(I)\}$  and  $S_* = \inf \{f(I)\}$

If  $S^*$  is sup. then for  $n \in \mathbb{N}$ .  $S^* - \frac{1}{n}$  can't be sup.

$$\checkmark \left[ S^* - \frac{1}{n} < f(x_n) < S^* \quad \forall n \in \mathbb{N} \right]$$

So we got seq<sup>n</sup>  $x_n \in I$ ,  $I$  closed bounded by Bolzano weierstrass  $\Rightarrow x_{n_k} \rightarrow x^*$   
 and as  $x_{n_k} \in I$ ,  $I$  closed  $\Rightarrow x^* \in I$

Now  $x_{n_k} \rightarrow x^*$

but  $f$  is cont on  $I$ , by seq<sup>n</sup> criteria

$$f(x_{n_k}) \rightarrow f(x^*)$$

$$s^* - \frac{1}{n_k} < f(x_{n_k}) < s^*$$

by squeez theo.

$$\lim s^* - \frac{1}{n_k} < \lim f(x_{n_k}) \leq \lim s^*$$

$$s^* \leq f(x^*) \leq s^*$$

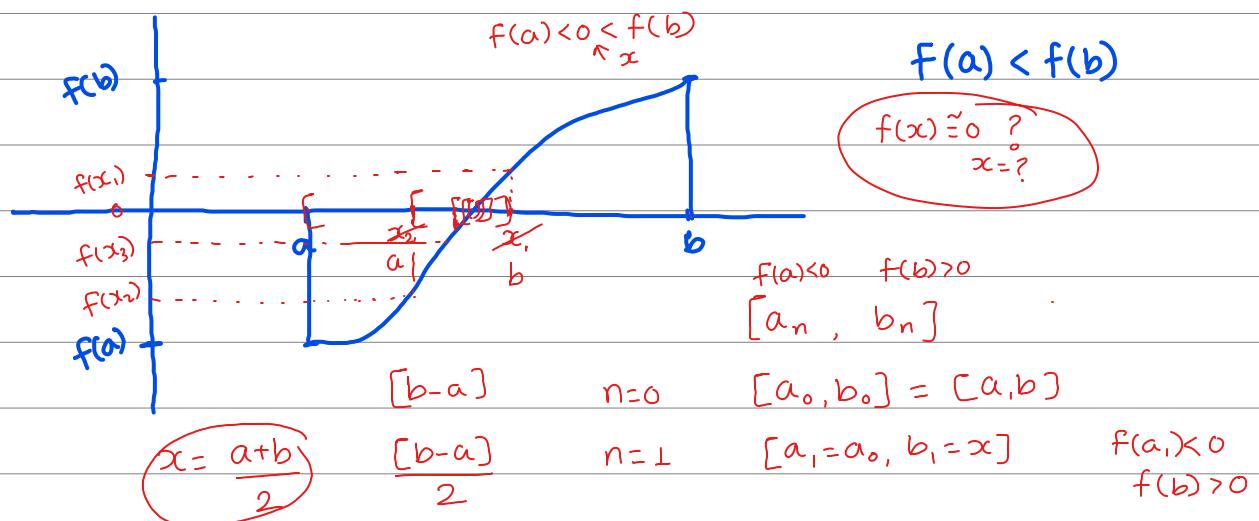
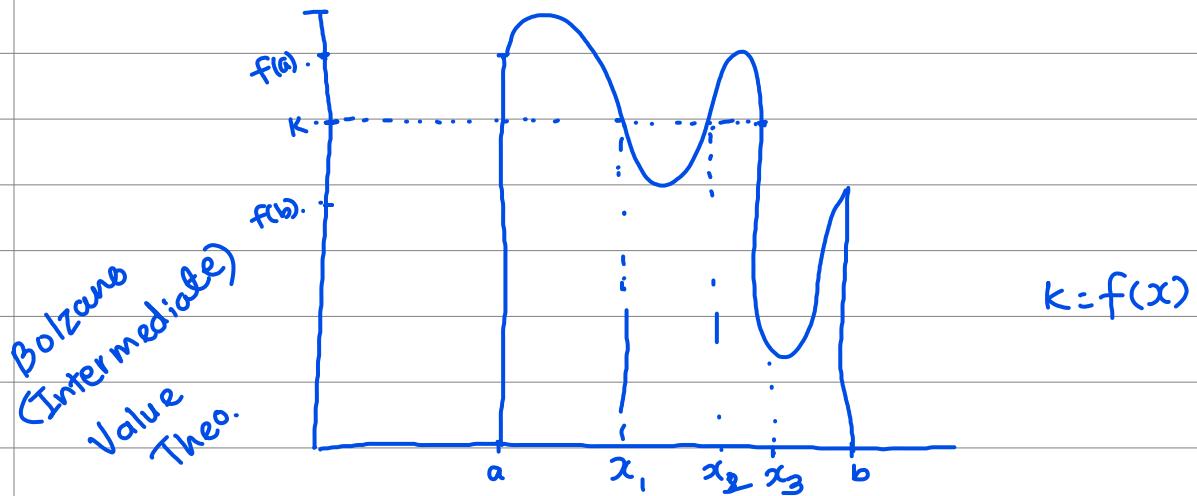
$$\Rightarrow s^* = f(x^*)$$

$$\Rightarrow \cancel{s^*} = \sup\{f(I)\} \Rightarrow f(x^*) \geq f(x) \Rightarrow x \in I$$

Simillarly we can obtain  $x_* \ni f(x_*) \leq f(x) \Rightarrow x \in I$ .

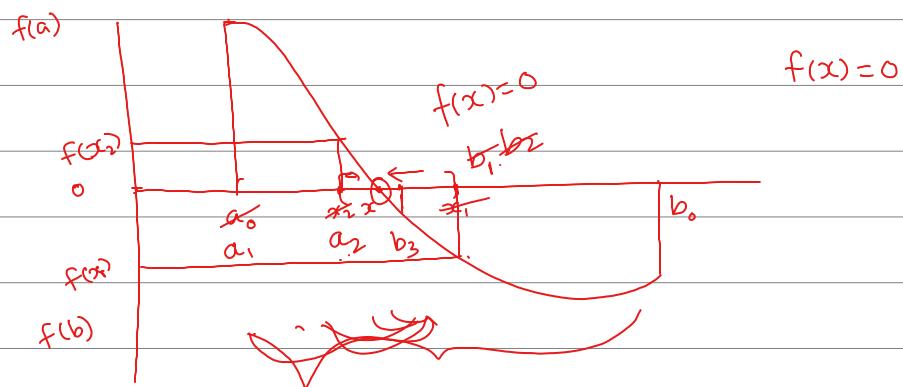
$$x^*, x_* \in I$$

## \* Location of Root



$$n=2 \quad x = \frac{a+b}{2} \quad \frac{[b-a]}{2^2}$$

$$n=3 \quad x = \frac{a+b}{2} \quad \frac{[b-a]}{2^n}$$



~~Bolzano Intermediate Value Theo.~~  
~~Existence of Root~~  $f: [a,b] \rightarrow \mathbb{R}$  cont.  
assume  $f(a) < k < f(b)$  or  $f(b) < k < f(a)$   
 $\exists c \in [a,b] \ni f(c) = k$

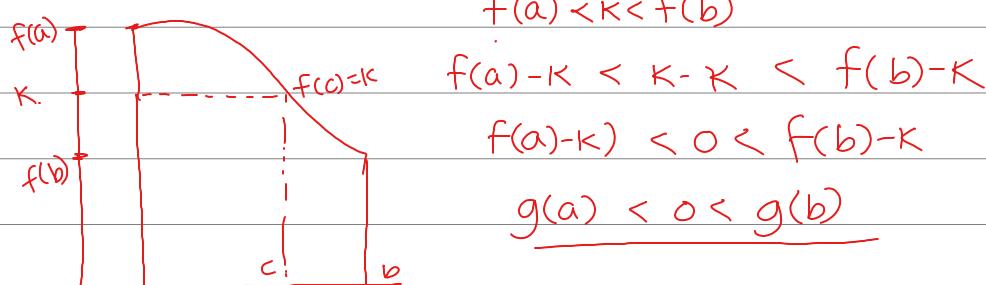
$f: I \rightarrow \mathbb{R}$  cont.

$a < b, [a,b] \subseteq I$

$$\boxed{\begin{array}{l} f(a) < 0 < f(b) \\ \exists c \in [a,b] \ni f(c) = 0 \end{array}}$$

$f: [a,b] \rightarrow \mathbb{R}$  cont. Assume  $f(a) < k < f(b)$

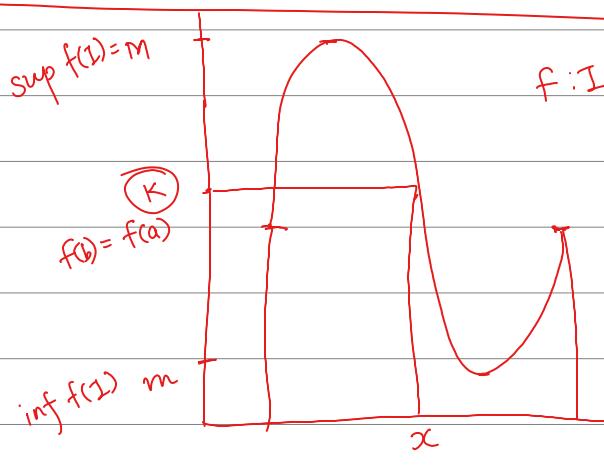
$$g(x) = f(x) - k \Rightarrow g: [a,b] \rightarrow \mathbb{R} \text{ cont.}$$



$\therefore$  By existence of Root Theo.  $\exists c \in [a,b] \ni g(c) = 0$

$$\begin{aligned} g(c) &= 0 \\ \Rightarrow f(c) - k &= 0 \\ \Rightarrow f(c) &= k \end{aligned}$$

(QED)



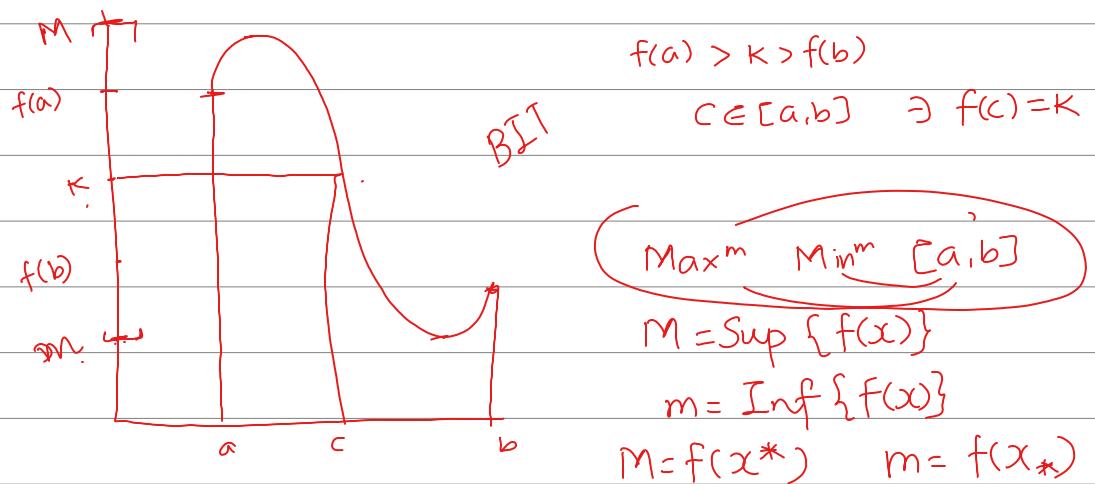
$f: I \rightarrow \mathbb{R}$   $f: [a,b] \rightarrow \mathbb{R}$  cont.

$\Rightarrow$  Bounded  $\Rightarrow$  Max<sup>m</sup> Min<sup>M</sup>

$$\underline{f(I)} = \{f(x), x \in I\}$$

$$= [m, M] \checkmark$$

$x^* \quad x^*$



C f

$$\lim_{x \rightarrow c} f(x) = L$$

Cont. at pt. c

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Cont. on set

$$\nexists c \in B$$

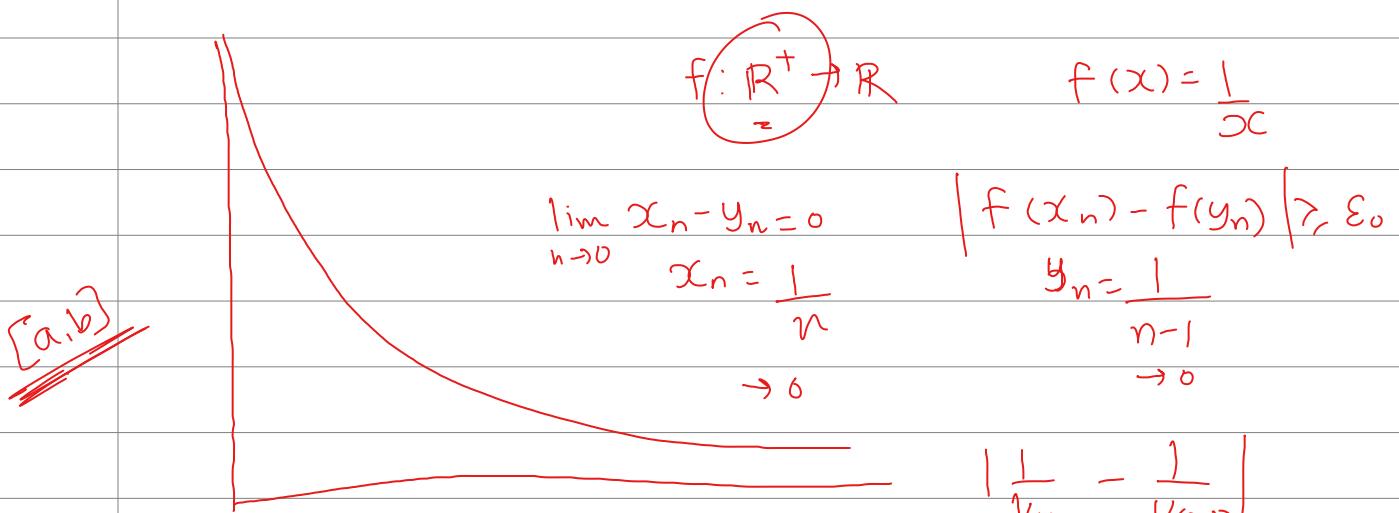
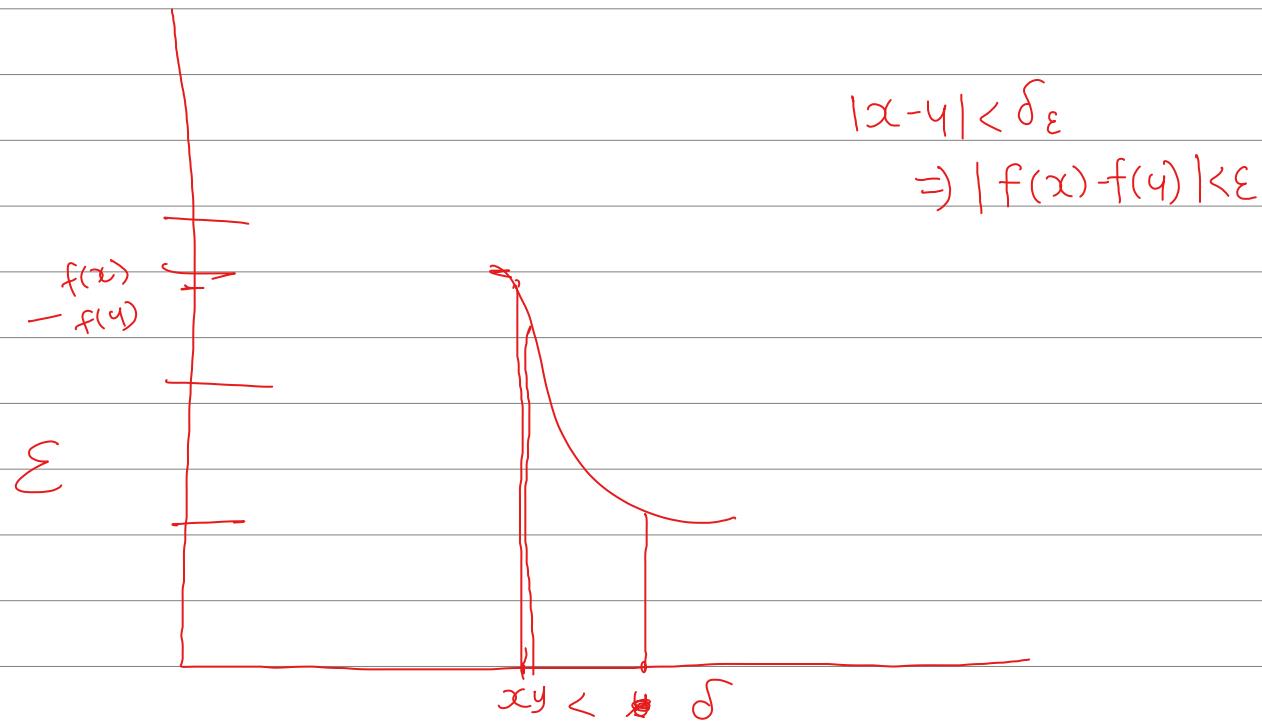
$$\lim_{x \rightarrow c} f(x) = f(c)$$

Uniform Conti

for any  $\varepsilon > 0 \exists \delta_\varepsilon > 0 \exists |x - c| < \delta_\varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon$

$$\exists |x - y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$$





$\frac{1}{x}$  is cont. but not uniformly cont on  $\mathbb{R}^+ \setminus (0, 0)$

$= |n - (n-1)|$

$= 1 > \epsilon_0$

$$\checkmark \quad |f(x) - f(y)| \leq K|x-y| \quad x, y \in I$$

/ Lipschitz fun<sup>c</sup>

cont.? Uniformly cont.?



$$f: I \rightarrow \mathbb{R}, \text{ for any } \varepsilon > 0 \exists \delta_\varepsilon > 0 \ni |x-y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$$

Now for Lipschitz fun<sup>c</sup>

$$\text{Assume}^{\leftarrow ?^0} \text{ for any } \varepsilon > 0 \exists \delta_\varepsilon = \varepsilon/K > 0$$

$$|x-y| < \delta_\varepsilon = \varepsilon/K$$

$$\text{but } |f(x) - f(y)| < K \underline{|x-y|}$$

$$\leq K \cdot \delta_\varepsilon$$

$$\leq K \cdot \varepsilon/K$$

$$\leq \varepsilon \quad \checkmark$$

Lipschitz fun is uniformly cont.

$f: A \rightarrow \mathbb{R}, A \subseteq \mathbb{R}$ , f uniformly cont.

$x_n$  Cauchy seq<sup>n</sup>

To prove,  $f(x_n)$  Cauchy.

o  $f: A \rightarrow \mathbb{R}$  uniformly cont.

~~for any  $\epsilon > 0 \exists \delta_\epsilon > 0 \ni |x-y| < \delta_\epsilon \Rightarrow |f(x)-f(y)| < \epsilon$~~

②  $x_n \in A$ ,  $x_n$  is cauchy seq<sup>n</sup>

for any  $\delta_\epsilon > 0, \exists K(\epsilon) \in \mathbb{N} \ni |x_n - x_m| < \delta_\epsilon \Rightarrow n, m > K(\epsilon)$

$|x_n - x_m| < \delta_\epsilon \Rightarrow |f(x_n) - f(x_m)| < \epsilon$

$\Rightarrow f(x_n)$  is cauchy seq<sup>n</sup>



































































































































































































