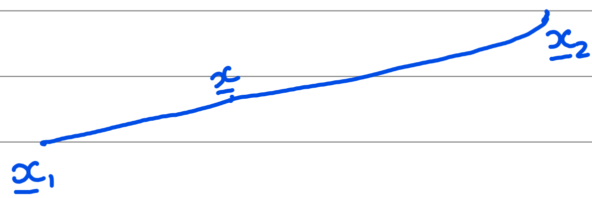
* Line segment : $\underline{x} \in \mathbb{R}^n$



scaler

$\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \qquad \alpha \in (0,1)$

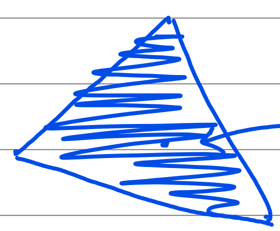
line segment joining $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$ $\{\underline{x} / \underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 ,\ \alpha \in (0,1)\}$

* Line passing $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$ $\qquad \alpha + 1 - \alpha$

line segment ← convex comb^n ✓ $\quad \underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \quad \alpha \in (0,1)$

line , linear comb^n $\quad \underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \quad \alpha \in \mathbb{R}$

vector space → $\boxed{\underline{x} = \sum \alpha_i \underline{x}_i}$ Linear Comb^n $\alpha_i \in \mathbb{R}$



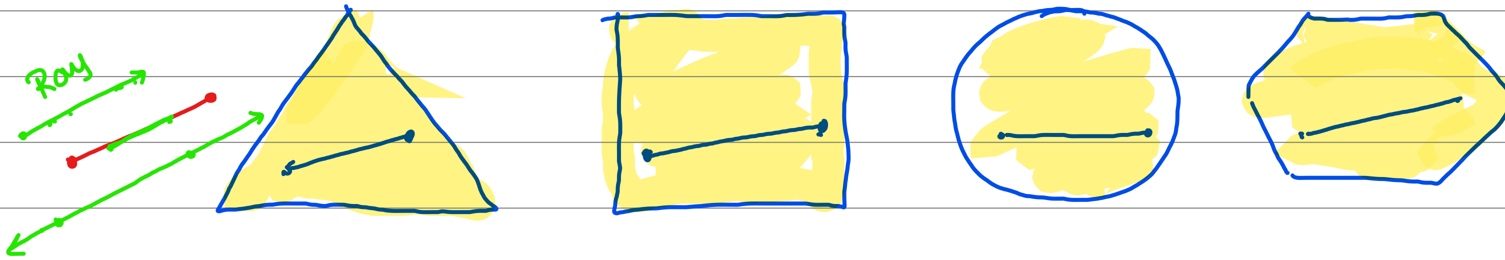
$\underline{x} = \sum \alpha_i \underline{x}_i \qquad \sum \alpha_i = 1 , \qquad \alpha_i \geq 0$

↳ Convex Combination

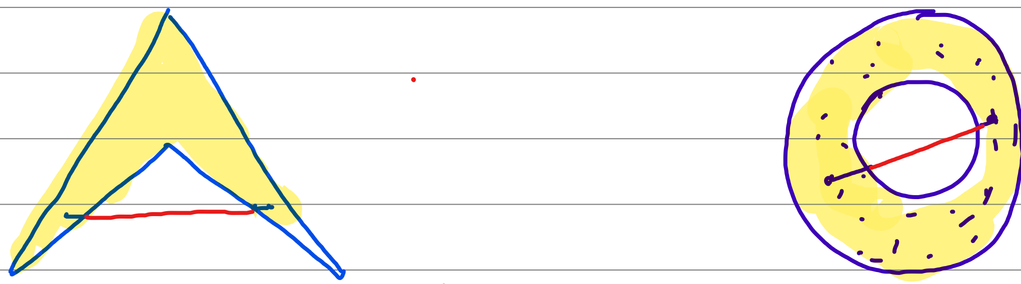
$$\left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \alpha_i x_i ,\ \alpha_i \geq 0 ,\ \sum \alpha_i = 1 \right\}$$

Convex set if $\underline{x}_1$ & $\underline{x}_2 \in A$, $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \quad \forall\, \alpha \in (0,1)$

then if $\underline{x} \in S \ \forall\, \alpha$ A is convex set



Convex sets



Non convex sets

* Ray is convex set

$$\underline{x} = \underline{x}_0 + \underline{d}\,\alpha \qquad \alpha \geq 0$$

$\underline{x}_1$ & $\underline{x}_2 \in A$

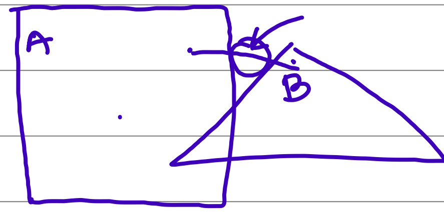
$\lambda \qquad \underline{x}_1 = \underline{x}_0 + \underline{d}\,\alpha_1$

$1-\lambda \qquad \underline{x}_2 = \underline{x}_0 + \underline{d}\,\alpha_2$

$$\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2 = (\lambda + 1 - \lambda)\underline{x}_0 + \underline{d}\,(\lambda\alpha_1 + (1-\lambda)\alpha_2)$$

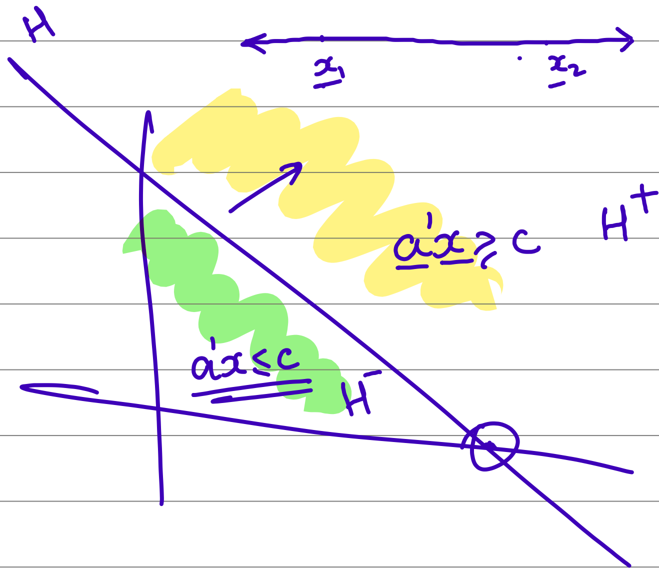
$$= x_0 + d(\qquad) \geq 0$$

$\in A$



* Union of convex sets may or may not be convex.

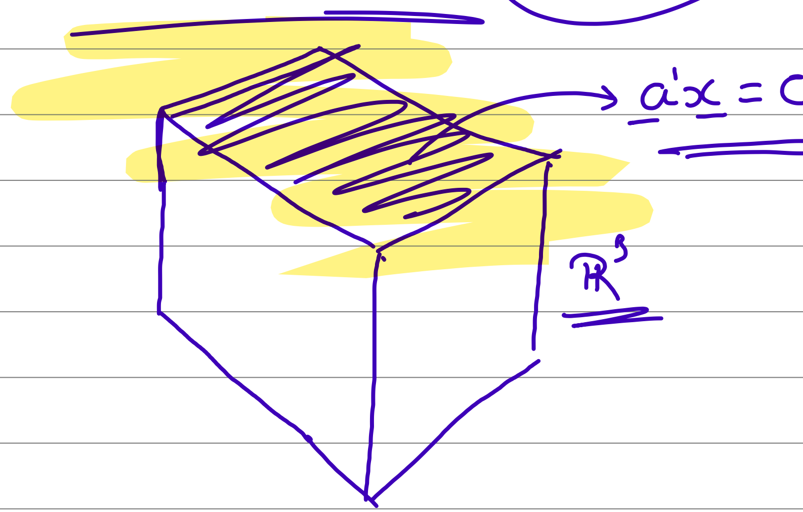
* Intersection of convex sets is also convex.



$$\sum_{i=1}^{2} \alpha_i x_i$$

$$\sum_{i=1}^{n} \alpha_i x_i = c$$

$$\underline{\alpha}'\underline{x} = c$$



Hyperplane

$H_0^+ = \{\underline{x} \,/\, a'x > c\}$ +ve open half space

$H^+ = \{\underline{x} \,/\, \underline{a}'\underline{x} \geq c\}$ ← positive closed half space

← $H = \{\underline{x} \,/\, \underline{a}'\underline{x} = c\}$ Hyperplane

$H^- = \{\underline{x} \,/\, a'x \leq c\}$ ← Negative closed half space

$H_0^- = \{\underline{x} \,/\, \underline{a}'\underline{x} < c\}$ -ve open half space

Hyperplane $H = \{\underline{x} \,/\, \underline{a}'\underline{x} = c\}$

let $\underline{x}_1, \underline{x}_2 \in H \Rightarrow \underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

$\alpha\underline{x}_1 + (1-\alpha)\underline{x}_2$

$$\left[\begin{aligned} \underline{a}'(\alpha\underline{x}_1 + (1-\alpha)\underline{x}_2) &= \alpha\,\underline{a}'\underline{x}_1 + (1-\alpha)\,\underline{a}'\underline{x}_2 \\ &= \alpha c + (1-\alpha)c \\ &= c \end{aligned}\right]$$

$\Rightarrow \alpha\underline{x}_1 + (1-\alpha)\underline{x}_2 \in H$

$\Rightarrow$ Hyperplane is convex set

H^+, H^-, H_0^+, H_0^- all are convex sets

Open ball $\underline{B = \{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| < r }\}$

To show :- B is convex set.

Let $\underline{x}, \underline{y} \in B \quad \Rightarrow \quad \underline{\alpha \cdot \underline{x}} + (1-\alpha)\, \underline{y} \; \boxed{\in B}$

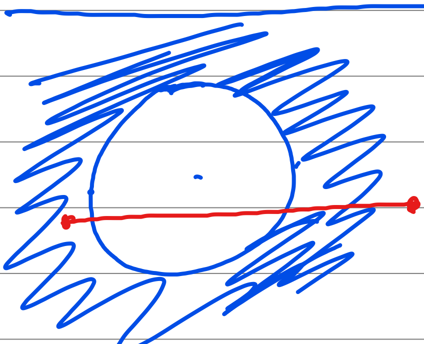
$\underline{x} \in B \Rightarrow \|\underline{x} - \underline{x}_0\| < r$

$\underline{y} \in B \Rightarrow \|\underline{y} - \underline{x}_0\| < r$

$\alpha \in (0,1)$

$$\|\alpha \underline{x} + (1-\alpha)\underline{y} - \underline{x}_0\|$$
$$= \|\alpha \underline{x} + (1-\alpha)\underline{y} - (\alpha \underline{x}_0 + (1-\alpha) \cdot x_0)\|$$
$$= \|\alpha(\underline{x} - \underline{x}_0) + (1-\alpha)(\underline{y} - x_0)\|$$
$$\leq \alpha \underbrace{\|\underline{x} - \underline{x}_0\|}_{<r} + (1-\alpha) \underbrace{\|y - x_0\|}_{<r} \qquad \alpha \in (0,1)$$
$$< \alpha r + (1-\alpha) r$$
$$< r$$

$\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| = r \}$ Convex or Not ✓



$\{ \underline{x} \mid \|\underline{x} - \underline{x}_0\| \geq r \}$ convex or not ✓

If C is convex set λC is also convex set.

$$\lambda C = \{ \underline{y} \mid \underline{y} = \lambda \underline{x}, \underline{x} \in C \}$$

to show λC as convex set

$\underline{y}_1, \underline{y}_2 \in \lambda C \Rightarrow \underline{y}_1 = \lambda \underline{x}_1, \underline{y}_2 = \lambda \underline{x}_2 \quad \underline{x}_1, \underline{x}_2 \in C$

$$\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 = \alpha \cdot \lambda \underline{x}_1 + (1-\alpha) \lambda \underline{x}_2$$

$$= \lambda (\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2)$$

$$= \lambda \underline{x} \qquad \left(\text{as } \underline{x}_1, x_2 \in C \overset{\text{as } C \text{ is convex}}{\Rightarrow} x \in C \right)$$

$$\Rightarrow \alpha \underline{y}_1 + (1-\alpha) \underline{y}_2 \in \lambda C$$

C, D are convex sets C+D is also convex

$\rightarrow \quad C+D = \{ \underline{z} \mid \underline{z} = \underline{x} + \underline{y}, \underline{x} \in C, \underline{y} \in D \}$

$\underline{z}_1, \underline{z}_2 \in C+D \Rightarrow z_1 = \underline{x}_1 + \underline{y}_1$

$z_2 = \underline{x}_2 + \underline{y}_2 \qquad \underline{x}_1, \underline{x}_2 \in C, \ \underline{y}_1, \underline{y}_2 \in D.$

$$\alpha z_1 + (1-\alpha) z_2 = \alpha \cdot (\underline{x}_1 + \underline{y}_1) + (1-\alpha)(\underline{x}_2 + \underline{y}_2)$$

$$= \underbrace{\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2}_{\in C} + \underbrace{\alpha \underline{y}_1 + (1-\alpha) \underline{y}_2}_{\in D}$$

convex sets

Intersection of any convex sets is convex

Let $\{S_i\}_{i=1}^{\infty}$ be collection of convex sets

$\cap S_i$ is convex

$\underline{x}, \underline{y} \in \cap S_i$

$\Rightarrow \underline{x}, \underline{y} \in S_i \quad \forall\, i$

$\Rightarrow \alpha \underline{x} + (1-\alpha)\, \underline{y} \in S_i \quad \forall\, i$ (S_i is convex)

$\Rightarrow \alpha \underline{x} + (1-\alpha)\, \underline{y} \in \cap S_i$

$\Rightarrow \cap S_i$ is convex.

- A set $S \in R^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S

∴ Assume that every convex combⁿ of any finite no of points of S is in S.

② → ①

$\Rightarrow$ it is also true for $n=2$

$\Rightarrow$ if $\forall\, \underline{x}_1, \underline{x}_2 \in S \Rightarrow \alpha \cdot \underline{x}_1 + (1-\alpha)\, \underline{x}_2 \in S \quad \forall\, \alpha\, (0,1)$

$\Rightarrow$ S is convex set

(II) Assume S is convex and for any finite n $\sum_{i=1}^{n} \alpha_i \underline{x}_i \in S$

$\rightarrow$ let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n \in S$ $\quad \sum_{i=1}^{n} \alpha_i = 1$

we will prove this by mathematical induction

As S is convex, $\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S$ $\quad \alpha + 1 - \alpha = 1$

$\therefore$ So the above statement is true for $n=2$

Assume it is true for $n=K$ $\Rightarrow \sum_{i=1}^{K} \alpha_i \underline{x}_i$ $= 1$

$\sum_{i=1}^{K} \alpha_i = 1$

We have to prove it for $n=K+1$

$$\sum_{i=1}^{K+1} \beta_i \underline{x}_i = \sum_{i=1}^{K} \beta_i x_i + \beta_{K+1} x_{K+1} \qquad \sum \beta_i = 1$$

imp?

$$\sum_{i=1}^{K} \beta_i = 1 - \beta_{K+1}$$

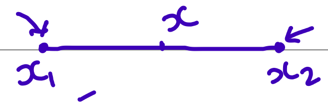
$$\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} = 1$$

$$= (1-\beta_{K+1}) \underbrace{\boxed{\sum_{i=1}^{K} \frac{\beta_i}{1-\beta_{K+1}} \cdot x_i}}_{\in S} + \beta_{K+1} x_{K+1}$$

$$= (1-\beta_{K+1})\, x^* + \beta_{K+1}\, x_{K+1}$$

$\in S$ as S is convex set

* Vertices

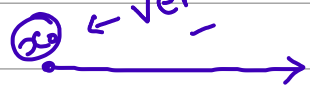


2 vertices

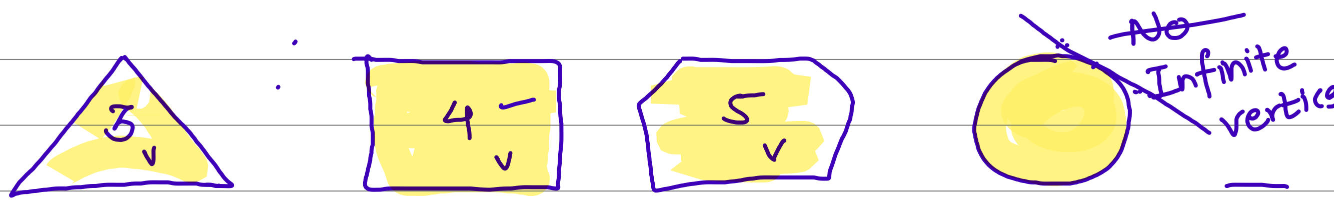
$\Rightarrow \quad x = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

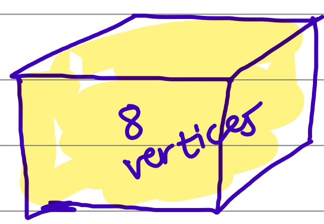
$x_2 = \alpha \underline{x}_1 + (1-\alpha)x_2 = \underline{x}_2$

$\uparrow \alpha = 0$



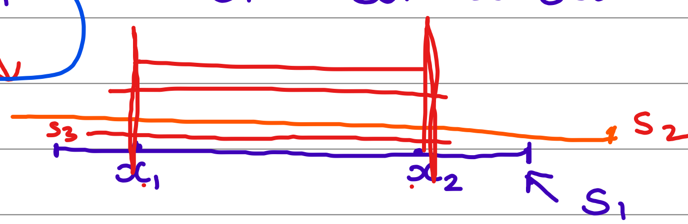
No vertex



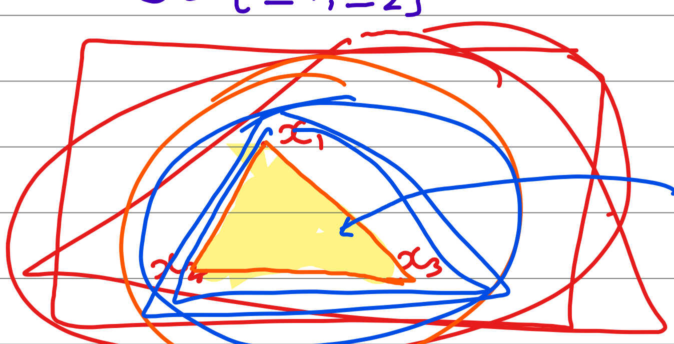


Convex Hull

$Co(S) \Rightarrow \bigcap_{i=1}^{\infty} S_i$ $\quad S_i$ is convex set containing S



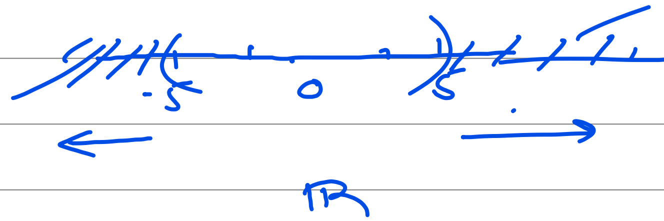
$S = \{\underline{x}_1, \underline{x}_2\}$



$S = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$

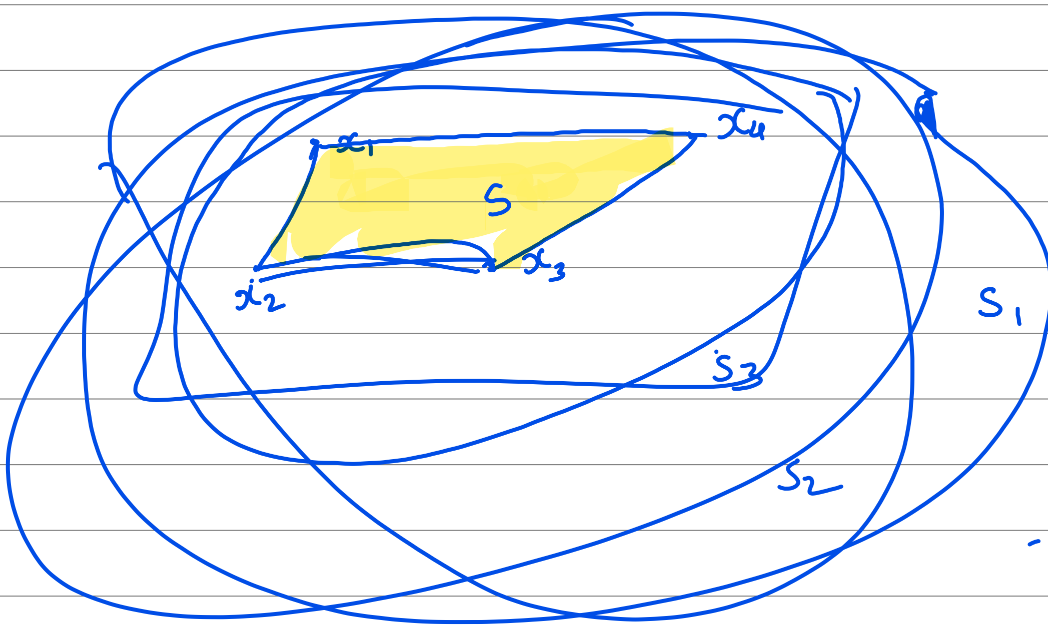
$$S = \{ \underline{x} \mid \|\underline{x}\| \geq 5 \}$$

$$Co(S) = \mathbb{R}^n$$

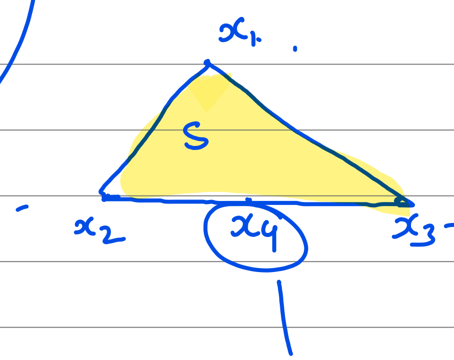


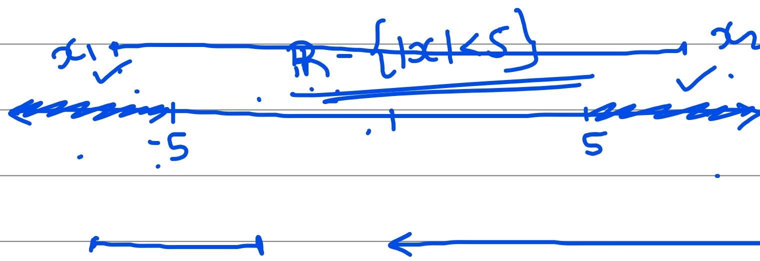
if set is convex $\underline{Co(S) = S}$

$Co(S) = \cap S_i$, S_i is convex set containing S.



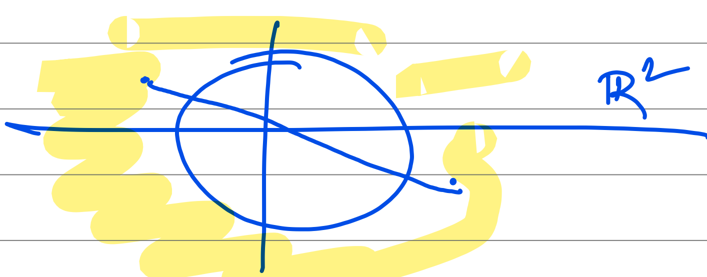
$\cap S_i = Co(S)$



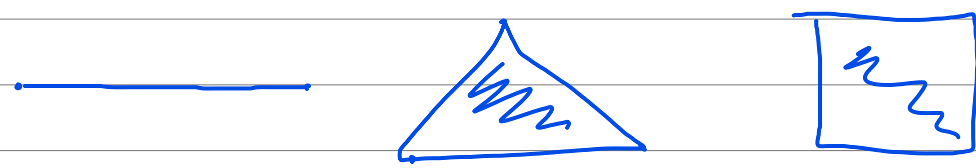


$S = \{|x| > 5\}$

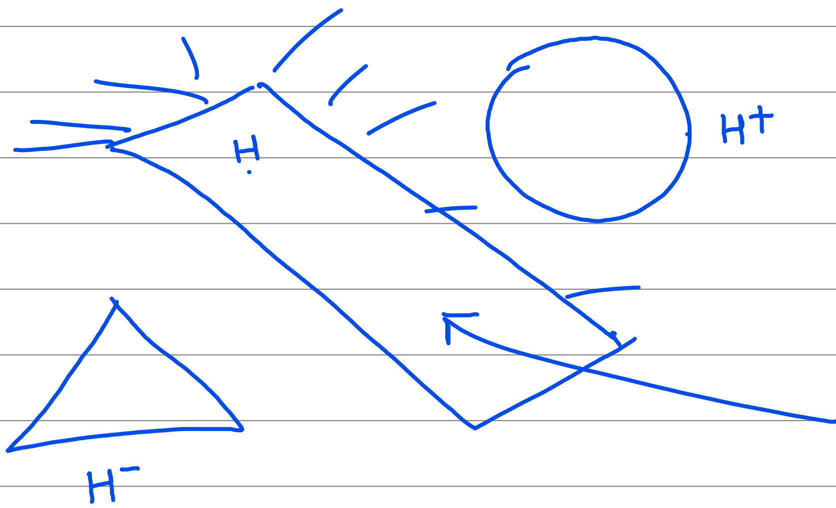
$\mathbb{R} = Co(S)$



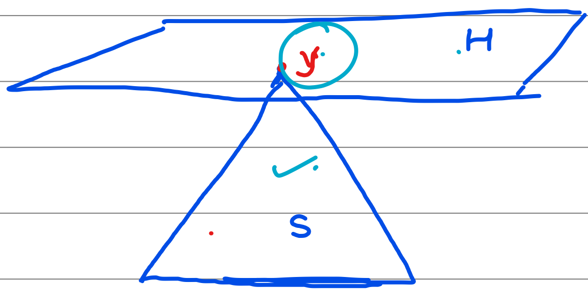
$$Co(S) = \left\{ \underline{x} \,/\, \underline{x} = \sum_{i=1}^{n} \lambda_i x_i, \quad x_i \in S, \quad \sum \lambda_i = 1, \ \lambda_i \geq 0 \right\}$$



Hyperplane:- $H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$



Separating Hyperplane



$S \in H^-$ or $S \in H^+$

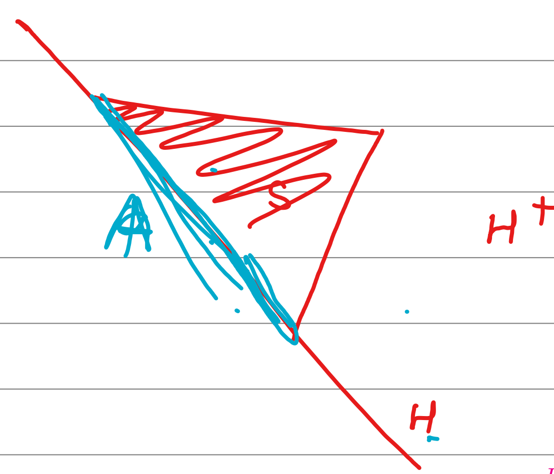
$\underline{a}'\underline{x} \leq c \ \ni x \in S$ or $\underline{a}'\underline{x} \geq c \ \ni x \in S$

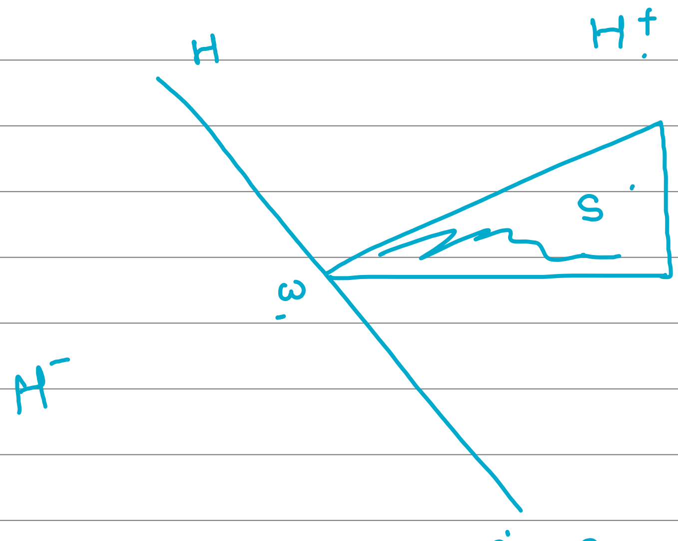
$S \in H^-$

$H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$

$\underline{a}'\underline{y} = c \Rightarrow \underline{y} \in S \ni \underline{y} \in H$

$S \in H^-$





$S \in H$

$\forall x \in S$, $a'x \geq c$

ω $a'\omega = c$

$\omega \in S \cap H$

$\min_{x \in S} a'x \Rightarrow a'\omega$

(4)

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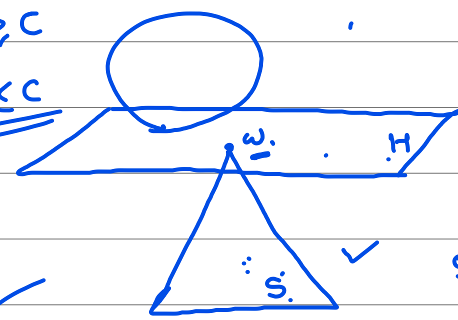
(1) Supporting hyperplane:- $a'x \geq c$, $a'x \leq c$



$\omega \in S \cap H$

$a'x = c$

$S \in H^{-}$

$x \in S$

$a'x \leq c$

$H : \{ x \mid a'x = c \}$ $\boxed{a'\omega = c}$

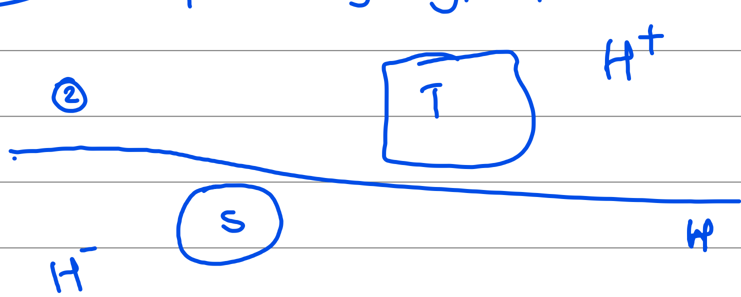
$\omega \in S$ ← boundary S.

$S \Rightarrow a'x \leq c$

$a'\omega = c$

Separating hyperplane

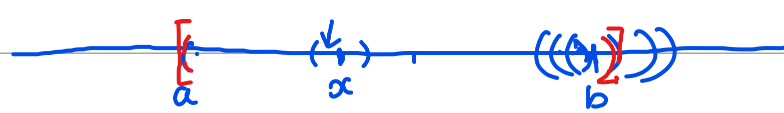
(2)

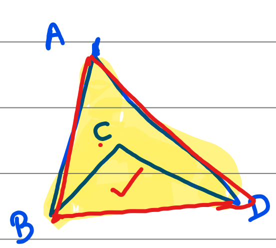


$S \in H^{-}$ & $T \in H^{+}$

$S' = (a,b) \cup \{a\} \cup \{b\} = [a,b]$

Closure. $S = (a,b)$



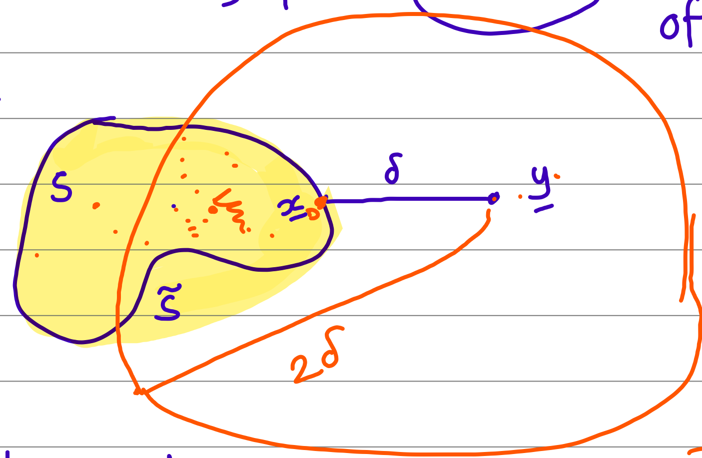


$|x - c| < \varepsilon$

$c - \varepsilon, c + \varepsilon$

$S \subseteq \mathbb{R}^n$ convex, y ~~boundary point~~ exterior to closure of S

$\exists\ \underline{a} \in \mathbb{R}^n$, $\underline{a} \neq \underline{0} \in \mathbb{R}^n$, $\underline{a}'\underline{y} < \inf\limits_{x \in S} \underline{a}'\underline{x}$



$\tilde{S}$ closure

let $\delta = \inf\limits_{x \in \tilde{S}} \|\underline{x} - \underline{y}\|$ $\delta > 0$ as y is exterior to $\tilde{S}$

$y \notin \tilde{S}$

$$B_{2\delta} = \{ \underline{x} \mid \|\underline{x} - \underline{y}\| < 2\delta \}$$

$$\delta = \inf_{x \in \tilde{S}} \|\underline{x} - \underline{y}\| = \inf_{x \in \tilde{S} \cap B_{2\delta}} \|\underline{x} - \underline{y}\|$$

$\tilde{S} \cap B_{2\delta}$ is closed & bounded.

lets define $f : \tilde{S} \cap B_{2\delta} \to \mathbb{R}$, $f(\underline{x}) = \|\underline{x} - \underline{y}\|$

f contin ,

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by max^m min^m theo., f attains its extremum in that set

$\exists$ some $\underline{x}_0 \in \tilde{S} \cap B_{2\delta} \ni$

$$\delta = \min_{\underline{x} \in \tilde{S} \cap B_{2\delta}} \|\underline{x} - \underline{y}\| = \|\underline{x}_0 - \underline{y}\|$$

$\Rightarrow \underline{x}_0$ is boundary pt. of $\tilde{S}$

To show $\underline{a} = \underline{x}_0 - \underline{y}$

$$\underline{a}'\underline{y} \le \inf_{\underline{x}} \underline{a}'\underline{x}$$

$\underline{x}, \underline{x}_0 \in \tilde{S}$, $\alpha \underline{x} + (1-\alpha)\underline{x}_0 \in \tilde{S}$

$$\|\alpha \underline{x} + (1-\alpha)\underline{x}_0 - y\| \geq \|\underline{x}_0 - \underline{y}\|$$

$$\|(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\| \geq \|\underline{x}_0 - \underline{y}\|$$

$\|\underline{x}\| = \sqrt{\underline{x}'\underline{x}}$

$$\|(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\|^2 \geq \|\underline{x}_0 - \underline{y}\|$$

$$\Rightarrow \left((\underline{x}_0 - \underline{y}) + \underline{\alpha}(\underline{x} - \underline{x}_0)\right)^T \left((x_0 - y) + \alpha(\underline{x} - \underline{x}_0)\right) \geq (\underline{x}_0 - \underline{y})^T(\underline{x}_0 - y)$$

$$\Rightarrow \left[(\underline{x}_0 - \underline{y})' + \alpha(\underline{x} - \underline{x}_0)'\right]\left[(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\right] \geq (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})$$

$$\Rightarrow (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y}) + \alpha(\underline{x}_0 - \underline{y})'(\underline{x} - x_0) + \alpha(\underline{x} - \underline{x}_0)'(\underline{x}_0 - \underline{y}) + \alpha^2(\underline{x} - \underline{x}_0)'(\underline{x} - \underline{x}_0)$$

$$\geq (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})$$

$$2\alpha(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) + \alpha^2|x - x_0|^2 \geq 0$$

$$2(\underline{x}_0 - \underline{y})'(x - x_0) + \alpha|x - x_0|^2 \geq 0$$

let $\alpha \to 0$

$$(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) \geq 0$$

$$\Rightarrow (\underline{x}_0 - \underline{y})'\underline{x} \geq (\underline{x}_0 - \underline{y})'\underline{x}_0$$

$$\Rightarrow \underline{a}'\underline{x} \geq \underline{a}'(\underline{x}_0 - \underline{y} + \underline{y})$$

let $a = \underline{x}_0 - \underline{y}$

$$\geq a'(x_0 - y) + \underline{a}'\underline{y}$$

$\|z\| = \sqrt{z'z}$

$$\underline{a}'\underline{x} \geq \underline{(x_0-y)'(x_0-y)} + a'\underline{y} \qquad \forall\, x \in S$$

$$\geq \delta^2 + \underline{a}'\underline{y} \qquad \delta > 0$$

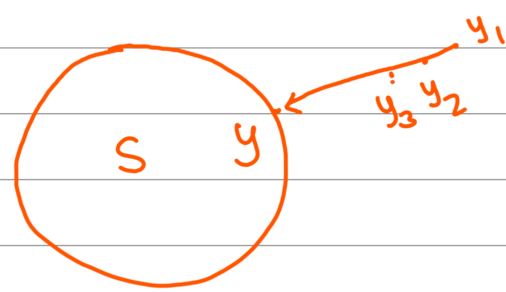
$$\inf_{\underline{x} \in S} \underline{a}'\underline{x} \geq a'y$$

$$\underline{a}'\underline{y} < \inf_{\underline{x} \in S} \underline{a}'\underline{x}$$

Theo. S convex, y boundary of S

To show $\exists\, H, \ni S \in H^+/H^-$, $y \in S \cap H$

→ [let y_n be seq^n of points exterior to closure of S
Assume $\underline{y}_n \to \underline{y}$]



$$\exists\ \underline{a}_n \in \mathbb{R}^n, \qquad a_n' y_n \leq \inf_{\underline{x} \in S} a_n' x \qquad \|a_n\| = 1$$

$$\underline{a}_n\underline{y}_n - \underline{a}_n'\underline{y} + a_n' y \leq a_n \underline{x} \qquad \forall\, \underline{x \in S}$$

for large n, $\underline{y}_n \to \underline{y} \Rightarrow a_n'\underline{y}_n \to \underline{a}'\underline{y}$

$$\underline{a}_n' \underline{y} < \underline{a}_n' \underline{x}$$

$\|a_n\| = 1$ Seqⁿ of a_n bounded.
Bolzano Weierstrauss Theo. for seqⁿs.

$\exists$ convergent subseqⁿ $\{a_{n_k}\}$
Suppose it converges to $a_{n_k} \to a$

$$\underline{a}_{n_k}' \underline{y} < a_{n_k}' \underline{x}$$

letting $k \to \infty$, $\underline{a}_{n_k} \to \underline{a}$

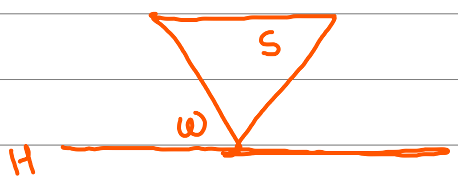
$$\underline{a}'\underline{y} = \underline{a}_{n_k}' \underline{y} < a_{n_k}' \underline{x} = \underline{a}'\underline{x}$$

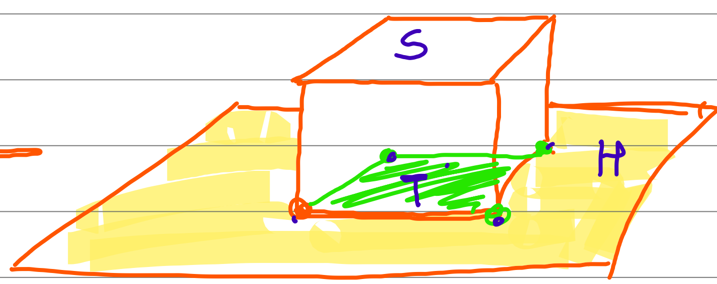
$$\Rightarrow \underline{a}'\underline{y} \leq \underline{a}'\underline{x} \qquad \forall x \in S$$

$H = \{ \underline{x} / \underline{a}'\underline{x} = \underline{a}'\underline{y} \}$ is supporting hyperplane. at y

---

$$T = S \cap H = \{\omega\}$$





H Supportive Hyperplane, $T = S \cap H$
by method of contradiction

let $\underline{x}_0$ be extreme pt. of T but not of S.

let $\exists$ some $x_1, x_2 \in S \ni \underline{x}_0 = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$H = \{ \underline{x} \mid \underline{a}'\underline{x} = c \}$ supportive hyperplane of S

let $S \in H^+$, $\underline{a}'\underline{x} \geq c \ \forall\ \underline{x} \in S$.

$\Rightarrow \underline{a}'\underline{x}_1 \geq c, \ \underline{a}'\underline{x}_2 \geq c$

$\underline{x}_0 \in T = S \cap H \qquad \Rightarrow \underline{a}'\underline{x}_0 = c$

$\Rightarrow \underline{a}'(\alpha x_1 + (1-\alpha)\underline{x}_2) = c$

$\Rightarrow \alpha \underline{a}'\underline{x}_1 + (1-\alpha)\underline{a}'\underline{x}_2 = c$

$\alpha \in (0,1), (1-\alpha) \in (0,1)$

$\Rightarrow \alpha \cdot \underbrace{\qquad}_{\geq c} + (1-\alpha)\underbrace{(\qquad)}_{\geq c} = c$

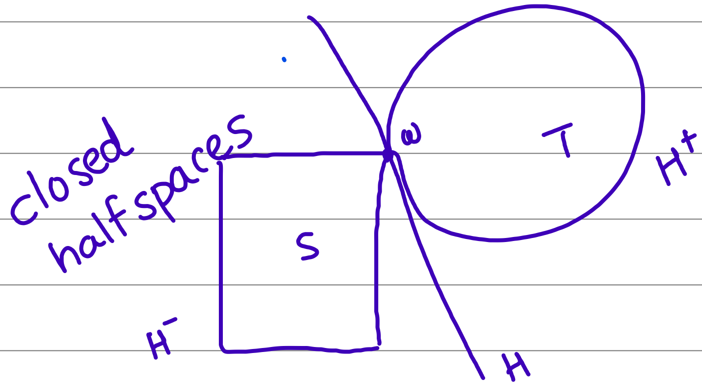
it is only possible if $\underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

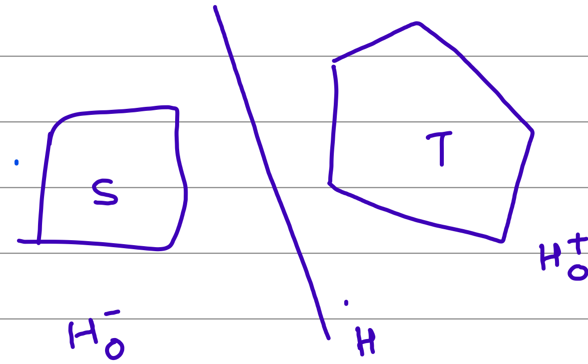
$\Rightarrow \underline{x}_1, \underline{x}_2 \in H$ & $\Rightarrow x_1, x_2 \in S \cap H$

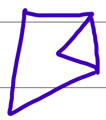
$\Rightarrow x_1, x_2 \in T$

$\therefore$ Which contradicts to our assumption that x_0 is extreme pt. of T.

Separating hyperplane

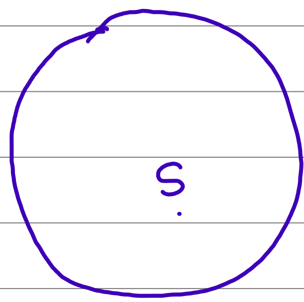






S convex

$\underline{0} \notin S$



$\cdot\ \underline{0}$

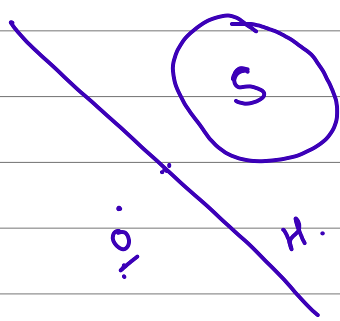
$\tilde{S} = S$

$\rightarrow \underline{0} \notin \tilde{S}$

convex set

exterior pt.

$$\left[\underline{0} = \underline{a}'\underline{0} < \inf_{\underline{x} \in S} \underline{a}'\underline{x} \right] \Rightarrow \underline{a}'\underline{x} > 0 \quad \forall\ \underline{x} \in S \qquad \text{— (1)}$$

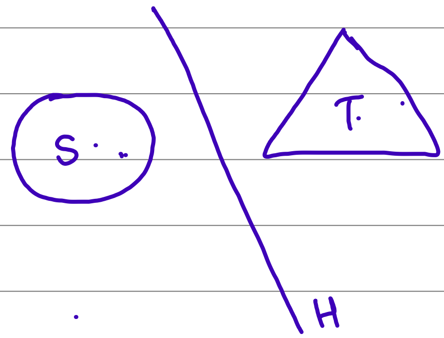


if $H = \{ \underline{x} \mid \underline{a}'\underline{x} = c \}$ — (2)

$0 < c < \inf_{x \in S} \underline{a}'\underline{x}$

$S \in H_0^+$

$\underline{0} \in H$

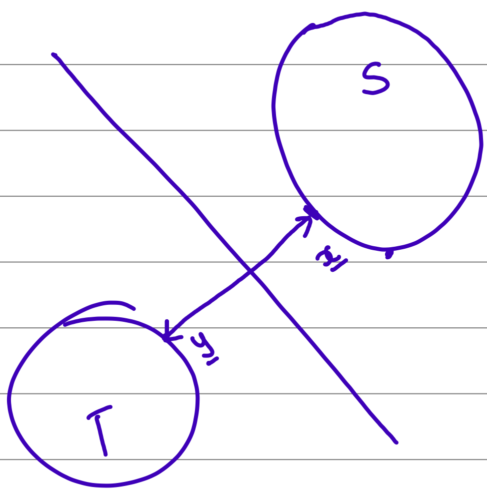


$\underline{0} \notin S-T$

$S-T = \{ \underline{z} \,/\, \underline{z} = \underline{x} - \underline{y},\ \underline{x} \in S,\ \underline{y} \in T \}$

$\underline{a}'\underline{z} > 0 \quad \forall\ \underline{z} \in S-T$

$\underline{a}'\underline{x} - a'\underline{y} > 0$



$\inf_{z \in S-T} a'\underline{z} > 0$

$\inf_{\substack{x \in S \\ y \in T}} \underline{a}'\underline{x} - \underline{a}'\underline{y} > 0$

$\inf_{\underline{x} \in S} \underline{a}'\underline{x} > \sup_{y \in T} \underline{a}'\underline{y}.$

S.T

$0 \notin S-T$

$0 = a'\underline{0} < \inf_{z \in S-T} a'z$

$\inf_{z \in S-T} a'z > 0$

$\inf_{x \in S} a'\underline{x} - \sup_{y \in T} a'y > 0$

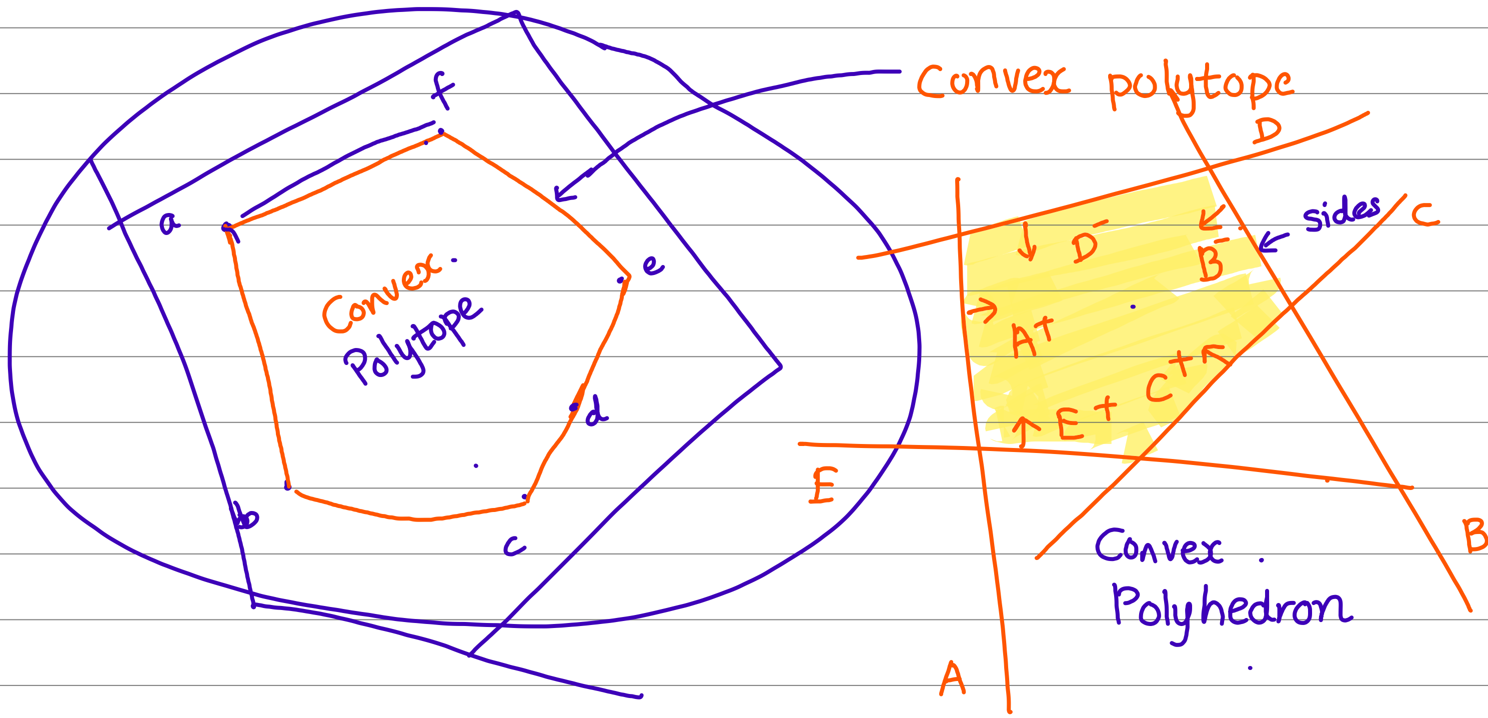
$\inf a'x > c > \sup a'y.$

$\inf_{\underline{x} \in S} \underline{a}'\underline{x} > c > \sup_{\underline{y} \in T} \underline{a}'\underline{y}$

$H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$ ✓

$S \in H_0^+ \qquad T \in H_0^-$

strictly separating.



Convex Polyhedron

$S = \{a, b, c, d, e, f\}$ $Co(S)$ = Convex hull / Polytope

$Co(S) = \{\underline{z} \mid z = \sum \alpha_i x_i,\ x_i \in S\}$ Vertices $(Co(S)) = \{a, b, c, e, f\} = V$

$\underline{x}$

$u \in V \Rightarrow u \notin S$

$S = \{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m\}$ $Co(S) = \{\underline{x} \mid \underline{x} = \sum_{i=1}^{m} \alpha_i x_i\}$

V = vertices $(Co(S))$

$\underline{u} \in V$ but $u \notin S$

$\underline{u} \in Co(S)$

$\Rightarrow \underline{u} = \sum_{i=1}^{m} \alpha_i x_i$ $\alpha_i \in (0,1)$ $\sum \alpha_i = 1$ $\alpha_i < 1$

$\underline{u} = \alpha_1 x_1 + \sum_{i=2}^{m} \alpha_i x_i = \alpha_1 x_1 + (1-\alpha_1) \underbrace{\sum_{i=2}^{m} \frac{\alpha_i x_i}{(1-\alpha_1)}}_{x^*} = \alpha_1 x_1 + (1-\alpha_1) x^*$

$A\underline{x} \leq \underline{b}$ $\quad a_1'\underline{x} = a_{11}x_1 + a_{12}x_2 \leq b_1$ $\quad H \quad H^+ \quad H^-$

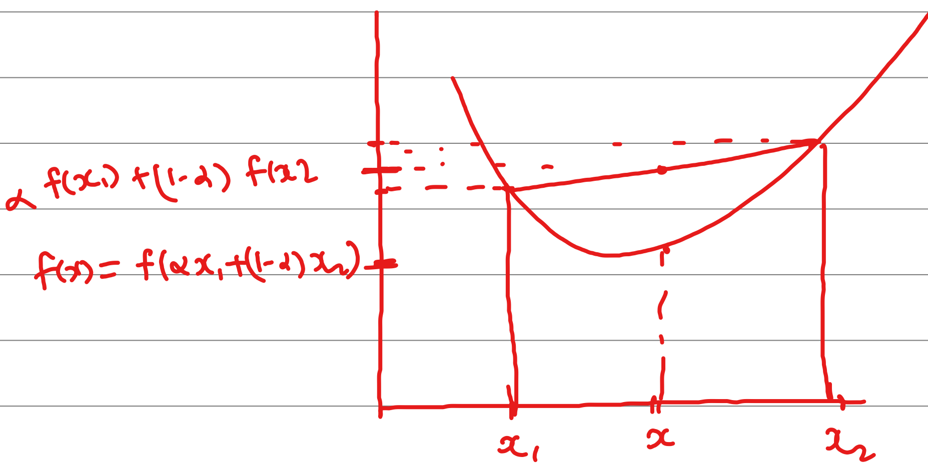
$a_2'\underline{x} = a_{21}x_1 + a_{22}x_2 \leq b_2$ $\quad H \quad H^+ \quad H^-$

$\underline{x} \geq \underline{0}$ $\quad H^+$

$H = \{ \underline{x} \mid \underline{a}_1'\underline{x} = b_1 \}$

$\{ x : / A x \geq b \}$

Polyhedron / Polyhedral



convex

e.g x^2

concave

eg $-x^2$

$f(\underline{x}) = \underline{c}'\underline{x} + \underline{d}$

$\underline{x}_1, \underline{x}_2 \in S$

$f(\alpha\underline{x}_1 + (1-\alpha)\underline{x}_2) = \underline{c}'(\alpha\underline{x}_1 + (1-\alpha)\cdot\underline{x}_2) + \underline{d}$

$= \alpha\underline{c}'\underline{x}_1 + (1-\alpha)\underline{c}'\underline{x}_2 + \alpha\underline{d} + (1-\alpha)\underline{d}$

$= \alpha(\underline{c}'\underline{x}_1 + \underline{d}) + (1-\alpha)(\underline{c}'\underline{x}_2 + \underline{d})$

$= \alpha \cdot f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$

$\leq$ $\Rightarrow$ convex

$\geq$ $\Rightarrow$ concave

f convex fun^c on T, $\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in T$.

$$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

$$T_K = \{ \underline{x} / \underline{x} \in T,\ f(\underline{x}) \leq K \}$$

let $\underline{x}_1, \underline{x}_2 \in T_K$

$\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in T$ and $f(\underline{x}_1) \leq K$, $f(x_2) \leq K$

$\Rightarrow$ as T is convex, & $\underline{x}_1, x_2 \in T$ $\Rightarrow$ $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in T$

$$f(x) = f(\alpha x_1 + (1-\alpha) x_2)$$

$$\leq \alpha \underbrace{f(x_1)}_{\leq K} + (1-\alpha) \underbrace{f(x_2)}_{K} \qquad (\text{as } f \text{ is convex}) \quad \alpha \in (0,1)$$

$$\leq K$$

$\underline{x} \in T$ & $f(\underline{x}) \leq K$ $\Rightarrow$ $\underline{x} \in T_K$

$\Rightarrow$ T_K is convex set.

---

f, g are convex functions

f is convex fun^c $\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in S$
$$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$$

$\mu \geq 0$

$(\mu \cdot f)(\underline{x}) = \mu \cdot f(\underline{x})$

$(\mu f)(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \mu \cdot f(\alpha x_1 + (1-\alpha)\underline{x}_2)$

$\leq \mu \cdot [\alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)]$

$\leq \alpha \cdot \mu \cdot f(x_1) + (1-\alpha)\mu \cdot f(x_2)$

$\leq \alpha \cdot (\mu f)(x_1) + (1-\alpha)(\mu f)(x_2)$

μf is also convex.

$(f+g)(\underline{x}) = f(\underline{x}) + g(\underline{x})$

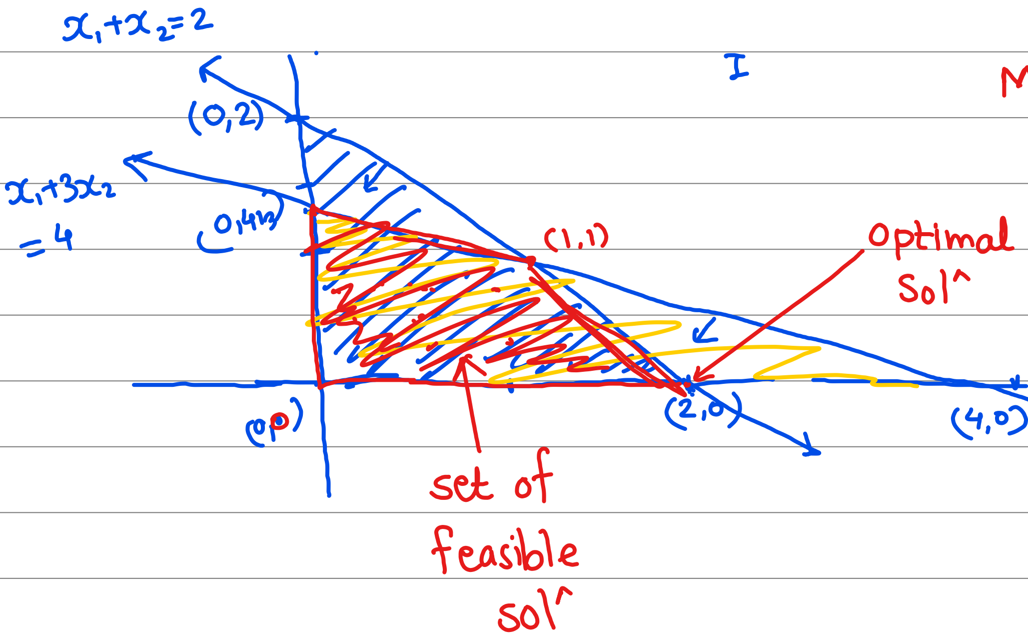
$(f+g)(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) + g(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2)$

$\leq \alpha f(x_1) + (1-\alpha) f(x_2) + \alpha g(x_1) + (1-\alpha) g(x_2)$ $\quad$ f & g conv

$\leq \alpha(f(x_1) + g(x_1)) + (1-\alpha)(f(x_2) + g(x_2))$

$\leq \alpha \cdot (f+g)(\underline{x}_1) + (1-\alpha)(f+g)(\underline{x}_2)$

I



Max $2x_1+x_2$

$x_1+x_2 \le 2$

$x_1+3x_2 \le 4$

$x_1, x_2 \geq 0$

$x_1+x_2=2$

(0,2)   (2,0)

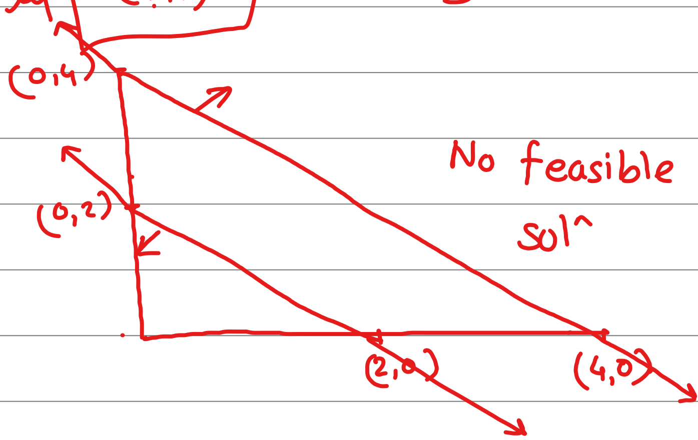
$x_1+3x_2=4$

(0,4/3)   (4,0)

$Z = 2x_1+x_2$

x_1	(0,0)	0
x_2	(0,4/3)	4/3
x_3	(2,0)	4
x_4	(1,1)	3

$Z_0 = \min\{ \underline{c}'\underline{x}_i, \ i=1:k \}$



No feasible sol^

$x_1+x_2 \le 2$

$x_1+x_2 \geq 4$

Let   Max $\underline{Z} = \underline{c}'\underline{x}$

$A\underline{x} \le b$

$T = \{ \underline{x} \mid A\underline{x} \le b \}$ constraint set.

let $\underline{x}^1$ & $\underline{x}^2$ are to optimum sol^n to LP

$$\text{Max } Z = \underset{\underline{x} \in T}{\text{Max}} \; \underline{c}'\underline{x} = \underline{c}'\underline{x}^1 = \underline{c}'\underline{x}^2 = Z_0 \text{ (say)}$$

optimum value

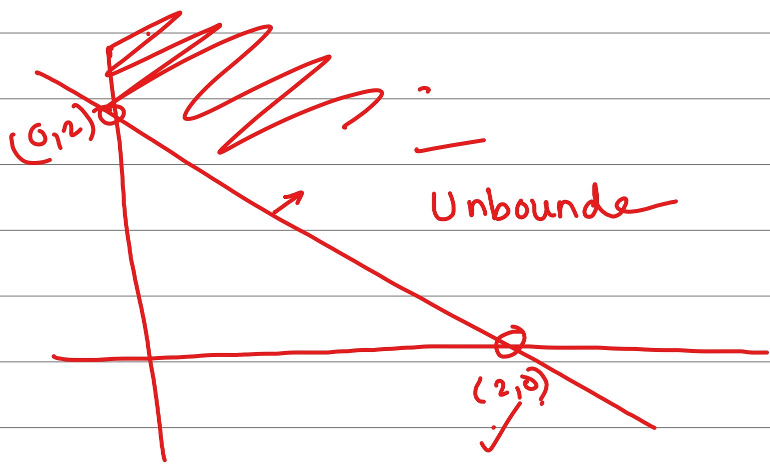
$\underline{x}^0 = \alpha \underline{x}^1 + (1-\alpha)\, \underline{x}^2$

To show $\underline{x}^0$ is also optimum sol^n

$$\underline{c}'\underline{x}^0 = c'(\alpha \underline{x}^1 + (1-\alpha)\, \underline{x}^2)$$
$$= \alpha\, \underline{c}'\underline{x}^1 + (1-\alpha)\; \underline{c}'\underline{x}^2$$
$$= \alpha Z_0 + (1-\alpha) Z_0$$
$$= Z_0$$

$\Rightarrow$ $\underline{x}^0$ is also optimum sol^n

$\Rightarrow$ Set of optimum sol^n is also convex set.



Min $Z = 2x_1 + 3x_2$
$x_1 + x_2 \geq 2$

$Z_0 = \min\{\underline{c}'\underline{x}_i \;, i = 1:K\}$

$\underline{x} \in T \Rightarrow \quad \underline{x} = \sum_{i=1}^{K} \alpha_i x_i \quad . \quad \alpha_i \geq 0 \quad \sum \alpha_i = 1$

$$\underline{c}'\underline{x} = \sum \alpha_i \underline{c}'\underline{x}_i$$
$$\geq \sum \alpha_i Z_0 = Z_0 \quad \Rightarrow$$

$\underline{c}'\underline{x} \geq z_0 \qquad \forall\ \underline{x} \in T$

$\min_{\underline{x} \in T} \underline{c}'\underline{x} \geq z_0 = \min \{ \underline{c}_i'\underline{x}_i, \quad i = 1:k \}$

max $5x + 8y$

subject to

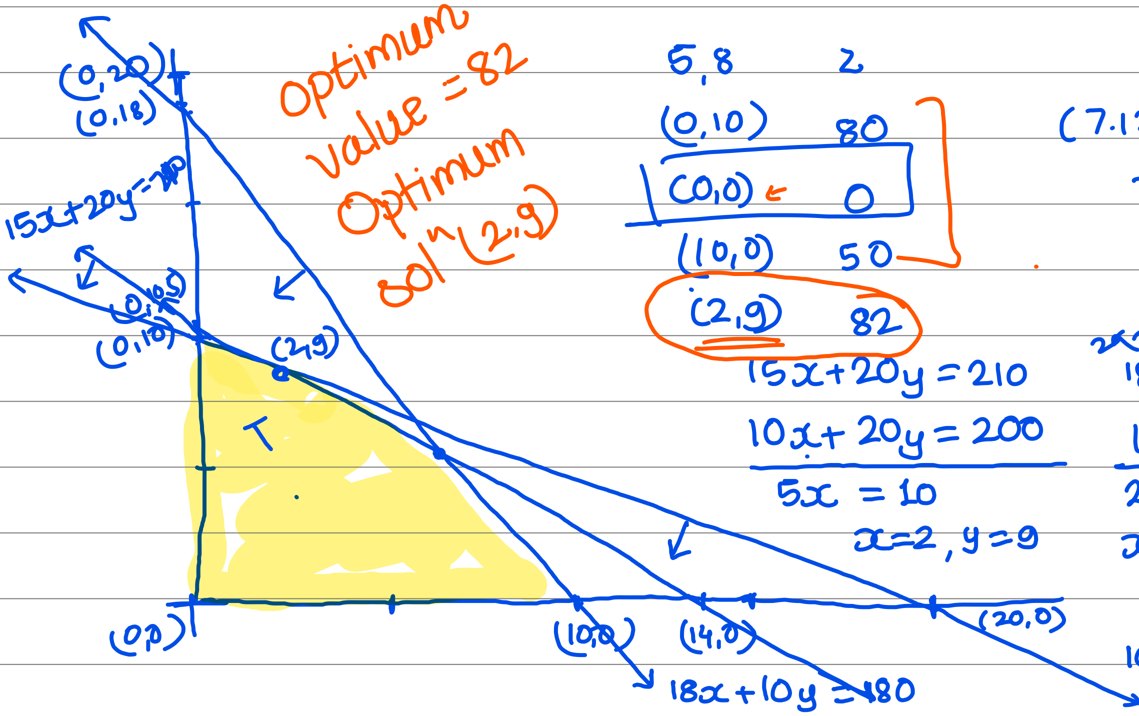
$18x + 10y \leq 180$

$10x + 20y \leq 200$

$15x + 20y \leq 210$

$\underline{x, y \geq 0}$

$18x + 10y = 180$	(0,18)	(10,0)
$10x + 20y = 200$	(0,10)	(20,0)
$15x + 20y = 210$	(0,10.5)	(14,0)
$x, y \geq 0$		



Optimum value = 82

Optimum sol^n (2,9)

5, 8	z
(0,10)	80
$\boxed{(0,0)}$	0
(10,0)	50
(2,9)	82

$15x + 20y = 210$

$\underline{10x + 20y = 200}$

$5x = 10$

$x = 2,\ y = 9$

(7.12, 5.14)

76.85

$2 \times 36 \quad 20 \quad 360$

$18x + 10y = 180$

$\underline{15x + 20y = 210}$

$21x = 150$

$x = 7.12$

$y = 5.14$

$$\boxed{\begin{aligned} 2x_1+3x_2 &\le 10\\ 3x_1+2x_2 &\ge 5\\ 3x_1+5x_2 &\le 15 \end{aligned}}$$
$$x_1, x_2 \ge 0$$

$$\begin{aligned} 2x_1+3x_2+x_3 &= 10\\ 3x_1+2x_2-x_4 &= 5\\ 3x_1+5x_2+x_5 &= 15 \end{aligned}$$

$$\begin{aligned} 2x_1+3x_2 &\le 10\\ x_1+5x_2 &\le 15\\ x_1\ge 0, &\ \boxed{x_2 \le 0} \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x_1-3x_2' &\le 10\\ x_1-5x_2' &\le 15\\ x_1\ge 0 &\quad x_2' \ge 0 \end{aligned}$$

$$x_2' = -x_2$$

$$x_B = [x_1\ x_2\ x_3]$$
Basic

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 & 0 & 0\\ 3 & 2 & 0 & -1 & 0\\ 3 & 5 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} 10\\ 5\\ 15 \end{bmatrix}$$

B (Basis), R

$$A\underline{x} = \underline{b}$$
$m\times n$
3×5

$$m=3, \quad n=5$$

$$\rho(A) \le \min(m,n)$$

$$A = [B \quad R] \qquad \rho(A) = \rho(B)$$

$$A\underline{x} = \underline{b}$$

$$\underset{m\times m}{[B}\ \ \underset{m\times(n-m)}{R]} \begin{bmatrix} \underline{x}_B\\ \underline{x}_{NB} \end{bmatrix} = \underline{b}$$

$$B\underline{x}_B + R\underline{x}_{NB} = \underline{b}$$
$$B^{-1}B\underline{x}_B + B^{-1}R\underline{x}_{NB} = B^{-1}\underline{b}$$

$$\underline{x}_B = B^{-1}\underline{b} - B^{-1}R\underline{x}_{NB}$$

Basic Solⁿ $\rightarrow \underline{x}_{NB} = \underline{0}, \quad \underline{x}_B = B^{-1}\underline{b}$

Max $3x_1+5x_2$

$x_1+2x_2 \le 6$ $\quad$ $x_1+2x_2+x_3 = 6$

$3x_1+x_2 \le 5$ $\quad$ $3x_1+x_2+x_4=5$

$x_1,x_2 \ge 0$ $\quad$ $x_1,x_2,x_3,x_4 \ge 0$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$m=2, \quad n=4$

$\rho(A) = \underline{m} = 2$

$x_1+x_2 \le 2$

$\rightarrow \boxed{\cancel{2x_1+2x_2 \le 4}}$

(1) $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad R = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $\quad \underline{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$ $\quad x_{NB} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$B^{-1}\underline{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$x_B = [6,5]$

$x_{NB} = [0,0]$

$$\underline{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 0, & 0, & 6, & 5 \end{pmatrix}$$ Basic Sol^n

(2) $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $\quad \underline{x}_B = [x_1, x_2]$ $\quad \underline{x}_{NB} = [x_3\ x_4]$

$$B^{-1} = -\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \quad B^{-1}\underline{b} = -\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -4 \\ -13 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 13/5 \end{bmatrix}$$

Basic Sol^n $\quad \underline{x} = [4/5, 13/5, 0, 0]$

$a_1 \; a_2 \ldots a_p$ $\quad \sum_{i=1}^{p} \alpha_i a_i = 0 \Rightarrow \;\not\Rightarrow \alpha_i = 0$ l.i.

for some i $\alpha_i \neq 0$

↳ linearly dependent

$$\underline{x} = (\underbrace{x_1 x_2 \ldots x_p}_{>0}, \underbrace{x_{p+1} \cdots x_n}_{=\underline{0}})$$

$$\underset{m\times n}{A}\underline{x} = \underline{b}_{m\times 1}$$

$$x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n = \underline{b}_{m\times 1}$$

$$\Rightarrow \quad x_1\underline{a}_1 + \underline{x}_2\underline{a}_2 + \cdots + x_p\underline{a}_p = \underline{b}_{m\times 1}$$

① If p vectors are linearly independent

$p \le m$ ✓

↳ i) $p = m$, $\underline{x}$ is f.s. is also b.f.s.

ii) $p < m$

we can add $m-p$ linearly independent columns

$$x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_p\underline{a}_p + 0\,\underline{a}_{p+1} + 0\cdot\underline{a}_{p+2} + \cdots + 0\cdot\underline{a}_m = \underline{b}$$

$\Rightarrow$ we get degenerate b.f.s.

② If p. ~~vectors~~ columns corresponding to feasible sol^n are linearly dependent.

$$\sum_{i=1}^{p} \alpha_i a_i = 0 \Rightarrow \text{for some } \alpha_j \neq 0$$

Assume $\alpha_j \neq 0$

$$\frac{x_j}{\alpha_j} \sum_{i=1}^{p} \alpha_i a_i = 0 .$$

$$\sum_{i=1}^{p} x_i \underline{a}_i = \underline{b}. \qquad (A\underline{x} = \underline{b}$$

$$\sum_{i=1}^{p} x_i a_i - \frac{x_j}{\alpha_j} \sum_{i=1}^{p} \alpha_i a_i = \underline{b} - \underline{0}$$

$$\underbrace{\sum_{\substack{i=1 \\ i \neq j}}^{p} x_i a_i}\ + \underline{x_j a_j - x_j a_j} = \underline{b}$$

$\underline{x} = [1, 1, 1]$

$\sum \underline{\alpha}_i \underline{a}_i = 0 \Rightarrow \cancel{\alpha_1}\begin{bmatrix}1\\2\end{bmatrix} + \cancel{\alpha_2}\begin{bmatrix}1\\-1\end{bmatrix} + \cancel{\alpha_3}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$

choosing $\alpha_1 = 1, \ \alpha_2 = 1, \ \alpha_3 = -1$

$\cancel{x_1}\begin{bmatrix}1\\2\end{bmatrix} + \cancel{x_2}\begin{bmatrix}1\\-1\end{bmatrix} + \cancel{x_3}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}.$

$A\underline{x} = b$

$\underline{x} = [1, 1, 1]$

$[2, 2, 0]$

$\underline{\underline{x}}$ extreme

~~depend~~

$2\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix} + 0\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$

$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_K \underline{a}_K = \underline{b}.$ $\qquad 0 < z < \min \frac{x_i}{|\alpha_i|}$

$\pm z \left[\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \dots + \alpha_K \underline{a}_K = \underline{0}\right]$ $\qquad$ for some $\alpha_j \neq 0$

$z > 0$

$x_i - z\alpha_i \geq 0$

$\frac{1}{2}\left(\underline{x} + z\underline{\alpha}\right) + \frac{1}{2}\left(\underline{x} - z\underline{\alpha}\right) = \underline{x}$ $\qquad \min\left(\frac{x_i}{\alpha_i}\right) \geq z \quad \forall i$

$(x_1 + z\alpha_1)\underline{a}_1 + \dots + (x_K + z\alpha_K)\underline{a}_K = \underline{b}$

Max $z = 6x_1 + 5x_2$

s.t. $x_1 + x_2 \le 5$

$3x_1 + 2x_2 \le 12$

$x_1, x_2 \ge 0$

$x_1 + x_2 + x_3 = 5$

$3x_1 + 2x_2 + x_4 = 12$

$\Rightarrow x_i \ge 0$

$x_3 = 5 - x_1 - x_2 \quad — ①$

$x_4 = 12 - 3x_1 - 2x_2 \quad — ②$

$z = 0$

min $(5, 4)$

0, 0 , 6, 5

$(5, 12, 0, 0)$

(x_3, x_4, x_1, x_2)

Basic Nonbasic

Basic Nonbasic

(x_1, x_3, x_2, x_4)

6 0 5 0

$(4, 1, 0, 0)$

$z = 24$

$3x_1 = 12 - 2x_2 - x_4$

$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 \quad — ③$

$x_3 = 5 - 4 + \frac{2}{3}x_2 + \frac{1}{3}x_4 - x_2$

$\rightarrow x_3 = 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 \quad — ④$

$z = 6x_1 + 5x_2$

$= 6\cdot\left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) + 5x_2$

$z = 24 + x_2 - 2x_4$

min$(6, 3)$

$\frac{1}{3}x_2 = 1 - x_3 + \frac{1}{3}x_4 \Rightarrow x_2 = 3 - 3x_3 + x_4$

$x_1 = 4 - \frac{2}{3}(3 - 3x_3 + x_4) - \frac{1}{3}x_4$

Basic NonB

$x_1\ x_2 \quad x_3\ x_4$

2 3 0 0

6 5 0 0

$z = 27$

$x_1 = 2 + 2x_3 - x_4$

$z = 6x$

$$z = 24 + x_2 - 2x_4$$
$$= 24 + 3 - 3x_3 + x_4 - 2x_4$$
$$z = 27 - 3x_3 - x_4$$

$(x_1\ x_2\ x_3\ x_4) = (2\ \ 3\ \ 0\ \ 0)$ $\quad z = 27$

Simplex Method.

C_j		6	5	0	0		
C_j	Basic	x_1	x_2	x_3	x_4	b	θ
0	x_3	1	1	1	0	5	5/1 min
0	x_4	$\boxed{3}$	2	0	1	12	12/3 = 4 → leaving
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	6 ↑ most +ve entering	5	0	0		
0	x_3	0	$\boxed{1/3}$	1	-1/3	1	3 min $R_1 - R_2'$
6	x_1	1	2/3	0	1/3	4	6 $\frac{1}{3}R_2$
	Z_j	6	4	0	2	24	
	$C_j - Z_j$	0	1 ↑ most +ve	0	-2		
$3R_1$ 5	x_2	0	1	3	-1	3	3
$R_2 - \frac{2}{3}R_1'$ 6	x_1	1	0	-2	1	2	
	Z_j	6	5	3	1	27	
	$C_j - Z_j$	0	0	-3	-1		