9

Unit - I - Real Numbers $\mathbb{R}$

Groups Mat

$(\mathbb{R}, +, \cdot)$

* Algebraic Properties of $\mathbb{R}$:- Add^ Mult^

On set of $\mathbb{R}$ there are two binary operators $+$ & $\cdot$

These two operations follows few properties :-

add^a

A1 > Commutative property of addition

$$a+b = b+a \quad \forall\ a,b \in \mathbb{R}$$

A2 > Associative property of add^

$$(a+b)+c = a+(b+c) \quad \forall\ a,b,c \in \mathbb{R}$$

A3) Existence of Zero element (Additive Identity)

$$a+0 = 0+a = a \quad \forall\ a \in \mathbb{R}$$

A4) Existence of negative element (Additive Inverse)

$$a+(-a) = (-a)+a = 0 \quad \forall\ a \in \mathbb{R}$$

$a + -a = 0$

$-a + -(-a) = 0$

a

matric

$A \cdot B \neq B \cdot A$

M1) Commutative property of multiplication

$$a \cdot b = b \cdot a \quad \forall\ a,b \in \mathbb{R}$$

M2) Associative p of multi.

$$(a \cdot b) c = a (b \cdot c) \quad \forall\ a,b,c \in \mathbb{R}$$

M3) Existence of unit element (Multiplicative identity)

$$a \cdot 1 = 1 \cdot a = a \quad \forall\ a \in \mathbb{R}$$

Extended Real

$\infty \leftarrow \mathbb{R}$

M4) Existence of multiplicative inverse / Reciprocals

$$a \cdot \left(\frac{1}{a}\right) = \frac{1}{a} \cdot a = 1 \quad \forall\ a \in \mathbb{R} - \{0\}$$

D) Distributive property of multiplication over add^

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a \quad \forall\ a,b,c \in \mathbb{R}$$

$$= a \cdot b + a \cdot c$$

* Order Properties of $\mathbb{R}$

Why do we study Real Analysis?

CLT $\to \frac{\sum x_i - \mu}{\sigma} \to$ ___

$\bar{x} \to \mu$

seq^n of fun^c

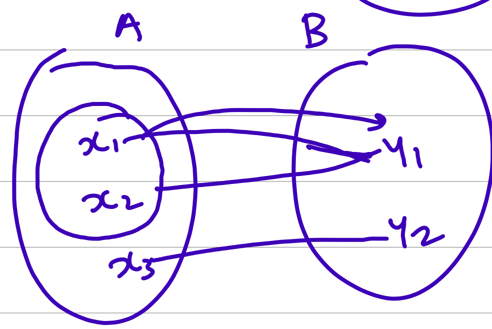
Convergence of Series of fun^{c}s

$\sum f_n(x) = \sum \frac{x_i}{n} \to$ ___

$\frac{\sum x_i}{n} \to \mu$

Set Fun^c

Fun^c



Prob.

input element $\to$ set

$\frac{n(A)}{n(S)}$ Classi $\in \mathbb{N}$

Input Set $\to$ ___ ~~Reale~~

ST-201

A

Borel fun^c

$x^3 + 2x + 3 = 0$

$x = ?$

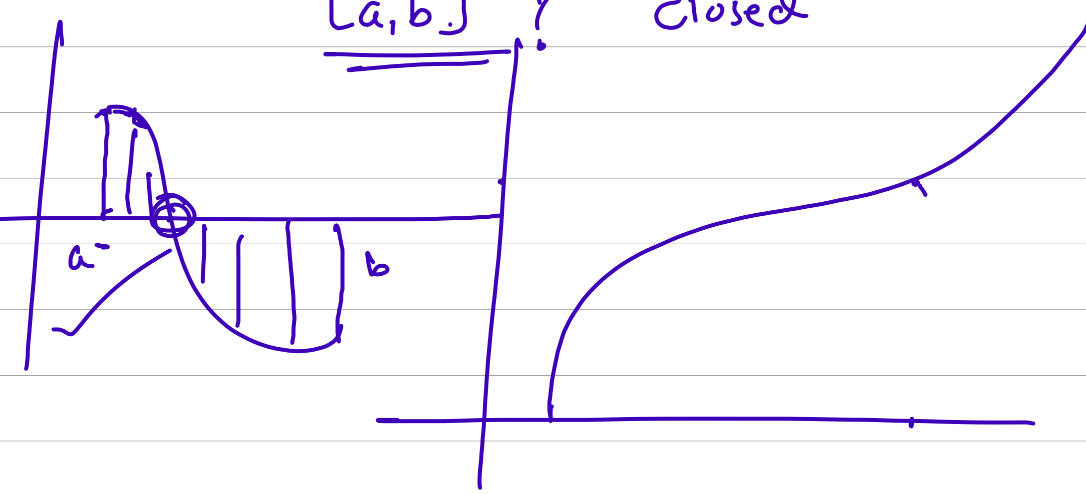
0 1 $\mathbb{R}$

[a, b] ? Closed

ST-205

Numerical Methods

Cont?



Series $\rightarrow$ $P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$E(X) = \sum_x \frac{e^{-\lambda}\lambda^x}{x!}x$$
$$= e^{-\lambda}\sum_x \frac{\lambda^x}{x!}x$$
$$= e^{-\lambda}\lambda \sum_x \frac{\lambda^{x-1}}{(x-1)!}$$
$$= \cancel{e^{-\lambda}}\,\lambda\,\cancel{e^{\lambda}}$$

? Series

Geometric $P(X=x) = pq^{x-1}$

Real Analysis

$$E(X) = \sum x \cdot pq^{x-1}$$
$$= p \cdot \sum x \cdot q^{x-1}$$

?

$$= p\frac{1}{(1-q)^2}$$
$$= \cancel{p}\frac{1}{p^2} = \frac{1}{p}$$
$$= 1/p$$

$$\sum x^n = \frac{1}{1-x} \qquad |x|<1$$

? $\dfrac{d}{dx}\sum x^n = \dfrac{d}{dx}\dfrac{1}{1-x}$

$$\sum n x^{n-1} = \frac{1}{(1-x)^2}$$

ST-201 Probability

Inverse fun^c $\leftarrow$ X Random Variable

ST-301 Asymptotic

$\rightarrow$ as $n \to \infty$ $\quad \bar{x} \to \mu \quad \dfrac{1}{n}\sum x_i \to \mu$

① Unbiased
② Suff
③ Eff
④ Consistent $\rightarrow$ as $n \to \infty$

$X_{(n)}$ # C.E. $\quad X \sim U(0,\theta)$

$2\bar{x}$ $\rightarrow$

* Order Properties of $\mathbb{R}$

$\mathbb{P}$ set of +ve nos $\in \mathbb{R}$, $\mathbb{P} \subseteq \mathbb{R}$, $\mathbb{R}^+$

$\mathbb{R}^+$ satisfies following properties

① if $a, b \in \mathbb{R}^+ \Rightarrow a + b \in \mathbb{R}^+$

② if $a, b \in \mathbb{R}^+ \Rightarrow a \cdot b \in \mathbb{R}^+$

③ if $a \in \mathbb{R}$ then exactly one of the following is true:-

$\underline{a \in \mathbb{R}^+}$, / $\underline{a = 0}$, / $-a \in \mathbb{R}^+$

× OR × OR ✓

-4.5

$\underline{\underline{a \in \mathbb{R}}}$ >0 <0 $=0$

$a \in \mathbb{R}^+$ $a \in \mathbb{R}^-$

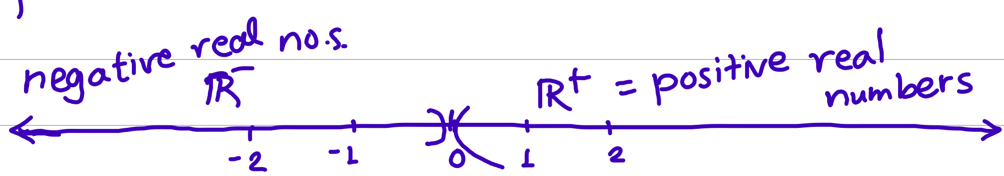
$-a \in \mathbb{R}^+$

Law of Trichotomy

Let $a, b \in \mathbb{R}$

ⓐ If $a - b \in \mathbb{R}^+ \Rightarrow a > b$ or $b < a$

ⓑ If $a - b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a \geq b$ or $b \leq a$



✓ $\mathbb{R}^+ = \{x \,/\, x > 0, x \in \mathbb{R}\}$

$\mathbb{R}^- = \{x \,/\, x < 0, x \in \mathbb{R}\}$

Proof :- ⓐ $a, b \in \mathbb{R}$, $a - b \in \mathbb{R}^+ \Rightarrow a - b > 0 \Rightarrow a > b$

ⓑ $a, b \in \mathbb{R}$ $a - b \in \mathbb{R}^+ \underline{\underline{\cup \{0\}}} \Rightarrow a - b \geq 0 \Rightarrow a \geq b$

Theo:- Let $a, b, c \in \mathbb{R}$

① If $a > b$ & $b > c \Rightarrow a > c$ (Transitivity)

② If $a > b \Rightarrow a + c > b + c$

③ If $a > b$, $c > 0 \Rightarrow a \cdot c > b \cdot c$

If $a > b$, $\underline{c < 0} \Rightarrow a \cdot c < b \cdot c$

① If $a>b$, & $b>c$

$a-b>0$ & $b-c>0$

$\Rightarrow a-b \in \mathbb{R}^+$ & $b-c \in \mathbb{R}^+$

$\Rightarrow (a-b)+(b-c) \in \mathbb{R}^+$ (by order properties)

$\Rightarrow (a-c) \in \mathbb{R}^+$

$\Rightarrow a-c>0$

$\Rightarrow a>c$ QED

② $a-b \in \mathbb{R}^+$

$a-b+c-c \in \mathbb{R}^+$

$(a+c)-(b+c) \in \mathbb{R}^+$

$a+c > b+c$

③ $a>b$, $c>0$ to prove $a \cdot c > b \cdot c$

$a>b$, $a-b>0$

$a-b \in \mathbb{R}^+$

$c(a-b) = c \cdot a - cb$ as $c>0$

$ca-cb>0$

$ca>cb$

Similarly $a>b$, $c<0 \Rightarrow ac<bc$

Theo:- If $a \in \mathbb{R}$, such that $0 \leq a < \varepsilon$ $\forall$ $\varepsilon \in \mathbb{R}$, $\varepsilon>0$

$\Rightarrow a=0$

$\varepsilon=0.005$ 08

$\varepsilon=0.00000$

$a \to 0$

By method of contradiction,

Assume $\underline{\underline{a>0}}$ $\Rightarrow$ (as $\underline{\underline{0 \le a < \varepsilon}}$) $\Rightarrow$ a is positive

$\Rightarrow$ $\frac{a}{2}$ is positive

we can assume $\varepsilon_0 = a/2 < a$

for any $\varepsilon > 0$, $0 \le a < \varepsilon$ but for $\varepsilon_0 \Rightarrow$ $0 < \varepsilon_0 < a$

which contradicts to our assumption. $\boxed{\varepsilon}$

Rational Nos. :- $Q = \{x \,/\, x = p/q,\ p, q \in I,\ q \neq 0\}$

* Theo. There doesnot exist any rational no. $r \in Q$ $\underline{\underline{r^2 = 2}}$

$\Rightarrow$ Assume that $\sqrt{2}$ is rational no.

$$\sqrt{2} = \frac{p}{q}$$

common divisor of p, q is 1

$(p, q) = 1$

$\Rightarrow$ $2 = \frac{p^2}{q^2}$

$p = 4$, $\underline{\underline{p^2 = 16}}$

$\Rightarrow$ $2q^2 = p^2 = p \cdot p$ $\rightarrow$ $p = 3$, $\underline{\underline{p^2 = 9}}$

$\frac{4}{6} \Rightarrow \underline{\underline{\frac{2}{3}}}$

$\Rightarrow$ p is divisible by 2 $\Rightarrow$ $p = 2 \cdot m$

$\Rightarrow$ $p^2 = 2^2 \cdot m^2 = 4m^2$

$\underline{\underline{p = \text{even}}}$

$\Rightarrow$ $2q^2 = 4m^2$

$\Rightarrow$ $q^2 = 2m^2$

$\Rightarrow$ q is divisible by 2 ———— (**)

$(p,q) = 1$ but here $(p,q) = 2$

which contradicts to our assumption.

$\underline{\underline{\text{even}^2}}$ $\underline{\underline{\text{odd}^2}}$ $\rightarrow$ $(\underline{\underline{2n+1}})^2$

$(2\cdot n)^2$ $= 4n^2 + 2n + 1$

$= 2(2n^2)$ $= 2\underline{\underline{(2n^2+n)}} + 1$.

even $=$ odd

Theo:- If $ab > 0$ then either ① $a > 0, b > 0$
② $a < 0, b < 0$

(*) ① If $a, b \in \mathbb{R}$ show that $a^2+b^2 = 0 \Leftrightarrow a = 0, b = 0$ ✓

② If $0 < c < 1$ show that $0 < c^2 < c < 1$ ✓

③ If $x, y \in Q$, $x+y \in Q$, $x \cdot y \in Q$ ✓

If $x \in Q$, $y \in Q^c$, $x+y \in Q^c$

① $(a+b)^2 = a^2 + b^2 + 2ab = 0$

$a^2+b^2 = 0 \Rightarrow$ $2ab = 0$

$\Rightarrow a \cdot b = 0$

$\Rightarrow a = 0$ or $b = 0$

(if $a = 0$ & $b \neq 0 \Rightarrow a^2+b^2 > 0$

$\lfloor a^2+b^2=0 \Rightarrow b=0$

$a,b \in \mathbb{R}^+,\ c>0,\qquad a>b \Rightarrow c\cdot a > c\cdot b$

$0<c<1$, as $c>0$

$c<1$

$c\cdot c < 1\cdot c$

$0<c^2<c<1$

$x,y\in Q \Rightarrow x+y \in Q$

$x=\frac{p_1}{q_1},\ y=\frac{p_2}{q_2} \Rightarrow x+y=\frac{p_1}{q_1}+\frac{p_2}{q_2}$

$=\frac{p_1q_2+p_2q_1}{q_1\cdot q_2}$

as $p_1,q_1,p_2,q_2\in I$, $p_1q_2\in I,\ p_2\cdot q_2 \in I,\ q_1\cdot q_2 \in I$

$p_1q_1+p_2q_2\in I$

$=\frac{p^*}{q^*} \in Q$

Absolute values, $\forall a \in \mathbb{R}$ $|a| = \begin{cases} +a & \text{if } a>0 \\ -a & \text{if } a<0 \\ 0 & \text{if } a=0 \end{cases}$

$\boxed{|a| = \max(a,-a)}$

Theo:- (a) $|ab| = |a|\cdot|b|$ $\forall a,b \in \mathbb{R}$ ✓

(b) $|a|^2 = a^2$ $\forall a \in \mathbb{R}$ $-c \le a+b \le c \Rightarrow |a+b| \le c$

(c) If $c \geq 0$ then $|a| \le c$ iff $-c \le a \le c$

(d) $-|a| \le a \le |a|$

Proof:- (a) If $a,b>0 \Rightarrow ab>0 \Rightarrow |a|=a, |b|=b, |a\cdot b| = ab = |a|\cdot|b|$

$a>0, b<0 \Rightarrow ab<0 \Rightarrow |a|=a, |b|=-b, |ab| = -ab = a\cdot(-b) = |a||b|$

$a<0, b>0 \Rightarrow ab<0 \Rightarrow$ Similarly

$a,b<0 \Rightarrow ab>0 \Rightarrow |a|=-a, |b|=-b, |ab| = ab = (-a)\cdot(-b)$

$= |a|\cdot|b|$ ✓

(b) $|a|^2 = a^2$

If $a>0$, $|a|=a \Rightarrow |a|^2 = a^2$ ✓

$a<0$ $|a|=-a \Rightarrow |a| = (-a)^2 = a^2$ ✓

$a=0$ $|a|=a=0 \Rightarrow |a|^2 = a^2 = 0$ ✓

(c) If $c \geq 0$ then

$|a| \le c$ iff $-c \le a \le c$.

i) $|a| \le c \Leftrightarrow |a| = \max(a,-a) \le c$

$\Leftrightarrow a \le c$ & $-a \le c$

$\Leftrightarrow a \le c$ & $a \geq -c$.

$\Leftrightarrow -c \le a \le c$ ✓

(d) If $c \geq 0$, then $|a| \le c$ iff $-c \le a \le c$ ✓

now assume $c = |a| \geq 0$, $|a| \le |a|$

$\Rightarrow -|a| \le a \le |a|$

* Triangular inequality :- If $a, b \in \mathbb{R}$, $|a+b| \le |a| + |b|$

proof :- If $a, b \in \mathbb{R}$, then

$$\begin{aligned} -|a| &\le a \le |a| \\ + \quad -|b| &\le b \le |b| \\ \hline -|a| - |b| &\le a+b \le |a| + |b| \end{aligned}$$

put $c = |a| + |b|$

$$\Rightarrow \quad -(|a|+|b|) \le a+b \le |a|+|b|$$

$$\Rightarrow \quad -c \le a+b \le c$$

$$\Rightarrow \quad |a+b| \le c$$

$$\Rightarrow \quad |a+b| \le |a| + |b|$$

$|a| = |-a|$

* Corollary : If $a, b \in \mathbb{R}$, then ⓐ $||a| - |b|| \le |a-b|$
ⓑ $|a-b| \le |a| + |b|$

ⓐ If $a, b \in \mathbb{R}$

$a = a - b + b$

$|a| = |a-b+b| \le |a-b| + |b| \quad — (*)$ (by triangular inequality)

$b = b - a + a$

$|b| = |b-a+a| \le |b-a| + |a| = |a-b| + |a| \quad — (*)$ (by tri. inequality)

From $(*)$ $|a| - |b| \le |a-b|$.

$(**)$ $|b| - |a| \le |a-b| \Rightarrow |a| - |b| \ge -|a-b|$

$(*)$

$$\boxed{-c \le a \le c \quad \Rightarrow \quad |a| \le c}$$

$$\Rightarrow \quad -|a-b| \le |a| - |b| \le |a-b|$$

put $c = |a-b| \Rightarrow ||a| - |b|| \le c \Rightarrow ||a| - |b|| \le |a-b|$

$$\underset{c}{-|a-b|} \leq |a|-|b| \leq \underset{c}{|a-b|} \qquad \Rightarrow \big||a|-|b|\big| \leq |a-b|$$

$$\boxed{-c \leq a \leq c \quad \Rightarrow \quad |a| \leq c}$$

(b) $|a-b| \leq |a|+|b|$

We have triangular inequality $|a+b| \leq |a|+|b|$

replace b by $-b$

$$|a+(-b)| \leq |a|+|(-b)|$$

$$\Rightarrow \quad |a-b| \leq |a|+|-b| \qquad (|b| = |-b|)$$

$$\Rightarrow \quad |a-b| \leq |a|+|b|$$

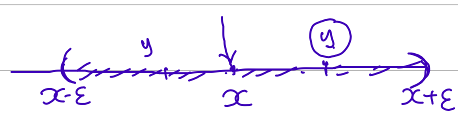
$$a-b \leq a+b \quad ? \qquad a,b \in \mathbb{R}$$

Triangular inequality $\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$

* Real line Extended Real Numbers

ε-nbhd. $V_\varepsilon(x) = \{y \,/\, y \in (x-\varepsilon, x+\varepsilon)\} \subseteq \mathbb{R}$

$(x-\varepsilon, x+\varepsilon) = V_\varepsilon(x)$



interval length $\Rightarrow 2\varepsilon$

$|x-y| \leq \varepsilon \quad \not\Rightarrow y \in V_\varepsilon(x)$

$$\left.\begin{aligned} V_\varepsilon(x) &= (x-\varepsilon, x+\varepsilon) \\ y \in V_\varepsilon(x) &\Rightarrow |x-y| \leq \varepsilon \qquad \not\Longleftarrow \\ &\Rightarrow x-\varepsilon \leq y \leq x+\varepsilon \end{aligned}\right\}$$

$\delta_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) - \{x\}$ deleted nbhd of x

e.g. we know $|a+b| \leq |a|+|b|$ but if $|a+b| = |a|+|b|$ iff $ab \geq 0$

If ~~a~~ $ab \geq 0$ to prove $|a+b| = |a|+|b|$

i) If $a>0, b>0, a+b>0 \Rightarrow |a+b| = a+b = |a|+|b|$

$a<0, b<0, a+b<0 \Rightarrow |a+b| = -(a+b) = (-a)+(-b) = |a|+|b|$

~~$a>0, b<0$~~

If $|a+b| = |a|+|b|$ to prove $ab \geq 0$

$$|a+b|^2 = (|a|+|b|)^2$$

$$(a+b)^2 = |a|^2+|b|^2+2|a|\cdot|b|$$

$$\Rightarrow \cancel{a^2}+\cancel{b^2}+\cancel{2}ab = \cancel{a^2}+\cancel{b^2}+\cancel{2}|a||b|$$

$$\Rightarrow ab = |a|\cdot|b| = |a\cdot b|$$

$$\Rightarrow ab \geq 0$$

e.g. If $x,y,z \in \mathbb{R}$ & $x \leq z$ show that $x \leq y \leq z$ iff $|x-y|+|y-z| = |x-z|$

i) Let $x \leq y \leq z$.

$x-y$ & $y-z$ negative.

LHS $= |x-y|+|y-z|$

$= (y-x)+(z-y)$

$= z-x$

$= |x-z|$ ✓

$=$ R.H.S.

$x \leq y \Rightarrow x-y \leq 0$, $y \leq z \Rightarrow y-z \leq 0$

$|x-y| = -(x-y) = y-x$

$|y-z| = -(y-z) = z-y$

$x \leq z \Rightarrow x-z \leq 0$

$|x-z| = z-x$

$|x-y| \neq (x-y)$

ii) Let $|x-y|+|y-z| = |x-z|$, $x \leq z$

To prove $x \leq y \leq z$

$x \leq z \Rightarrow x-z \leq 0$, $z-x \geq 0$

$x-z = x-y+y+z$

$x-z = \underline{x-y}+\underline{y-z}$

$|x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z|$

but $|x-z| = |x-y|+|y-z|$ So. we can use

$[\ |a+b| = |a|+|b| \quad \text{iff} \quad \underline{\underline{ab \geq 0}}\ ]$

$x-y \geq 0$

$x \geq y$

$(x-y)\ (y-z) \geq 0$

Only two possibilities

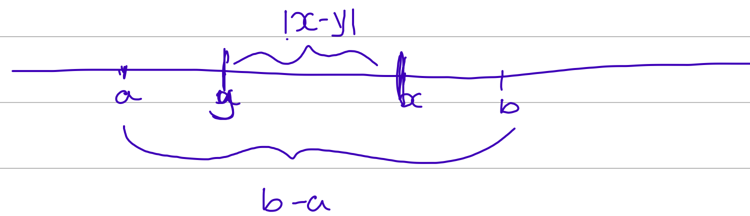
$(x-y),(y-z) \geq 0 \Rightarrow x \geq y,\ y \geq z \Rightarrow x \geq y \geq z$, which is not possible

$(x-y)\ (y-z) \leq 0 \Rightarrow x \leq y,\ y \leq z \Rightarrow \underline{x \leq y \leq z}$

if $a<x<b$

$a<y<b$

$|x-y| \leq b-a$



?

$-a > -x > -b \Rightarrow -b < -x < -a$

$a < y < b \qquad + \quad a < y < b$

$-(b-a) < y-x < b-a$

$\Rightarrow \quad |x-y| \leq b-a$

e.g. $|x-1| > |x+1|$ $\qquad x \in ?$ $\qquad \underline{\underline{x<0}}$

$\underline{\underline{-3}} \qquad \underline{\underline{\leq 0}} \qquad -0.01$

$|x+1|+|x-2| = 7$ $\qquad x=-3 \qquad x=4$

e.g. $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$ $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$

→ ① Let $\underline{a<b}$, $\min(a,b) = a$ $\max(a,b) = b$

$|a-b| = b-a$

$RHS = \frac{1}{2}(a+b+|a-b|)$

$= \frac{1}{2}(\cancel{a}+b+b-\cancel{a})$

$= \frac{2b}{2} = \max(a,b)$

$RHS = \frac{1}{2}(a+b-|a-b|)$

$= \frac{1}{2}(a+b-(b-a))$

$= \frac{2a}{2} = \min(a,b)$

github.com/manojcpatil/Lecture-scribbles

{ Absent Aditya, Dipali, Anushree, Aakash, Lalit, Vishal, Bhavesh, Nikita, Sopan Nikhil Pranit Bhanuja

* Set Operations, $A,B \subseteq \mathbb{R}$

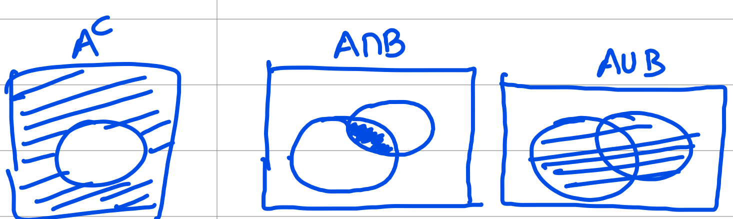
Union : $A\cup B = \{x / x\in A \text{ or } x \in B\}$

Intersection : $A\cap B = \{x / x\in A \ \& \ x\in B\}$

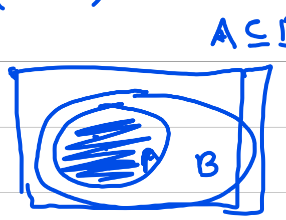
Complement $A^c = \{x / x \notin A, x\in \mathbb{R}\}$

Substract $A-B = A\cap B^c = A\setminus B$

$= A-(A\cap B)$



① Subset $A\subseteq B$ :: $x\in A \Rightarrow x\in B$ $\forall x\in A$



Theo:- ① $A\setminus(B\cup C) = (A\setminus B)\cap(A\setminus C)$

② $A\setminus(B\cap C) = (A\setminus B)\cup(A\setminus C)$

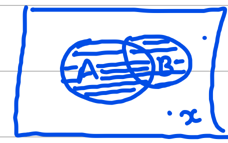
$A = B \Rightarrow A \subseteq B$ & $B \subseteq A$? Can we say that?

$(A \cup B)^c = A^c \cap B^c$ Can you prove this?

To prove ① $(A \cup B)^c \subseteq A^c \cap B^c$

② $A^c \cap B^c \subseteq (A \cup B)^c$

→ Let $x \in (A \cup B)^c$



$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A$ and $x \notin B$

$\Leftrightarrow x \in A^c$ and $x \in B^c$

$\Leftrightarrow x \in A^c \cap B^c$.

$\Rightarrow (A \cup B)^c \subseteq A^c \cap B^c$

$\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c$

$\Rightarrow \quad =$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law)

→ To prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ✓

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Let $x \in A \cap (B \cup C)$ $\boxed{\Rightarrow x \in (A \cap B) \cup (A \cap C)}$

$\Leftrightarrow x \in A$ and $x \in B \cup C$

$\Leftrightarrow x \in A$ and $(x \in B$ or $C)$

$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ → equal $=$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

e.g. If A and B are two sets such that
$A \subseteq B$ iff $A \cap B = A$

i) $A \subseteq B \Rightarrow x \in A \Rightarrow x \in B \quad \forall x \in A$
$\Rightarrow$ each and every element of A is is B.
$\Rightarrow A \cap B = A \subseteq B$

ii) $A \cap B = A \Rightarrow$ To prove $A \subseteq B$
$A \cap B = \{x / x \in A \text{ and } x \in B\} \subseteq B$

$A \cap B \subseteq B$
$\Rightarrow A \subseteq B$

$A \setminus (B \cup C) = A \cap (B \cup C)^c$. $\qquad A \setminus B \cap A \setminus C$
$= (A \cap B^c) \cap (A \cap C^c)$

$A \cap (B \cup C)^c \subseteq (A \cap B^c) \cap (A \cap C^c)$
$\supseteq$

Let $x \in A \cap (B \cup C)^c$ $\qquad \cup$ - or/and
$\Leftrightarrow x \in A$ and $x \in (B \cup C)^c$
$\Leftrightarrow x \in A$ and $x \notin B \cup C$
$\Leftrightarrow x \in A$ and $[x \notin B$ and $x \notin C]$

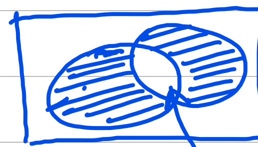
$\Leftrightarrow x \in A$ & $x \notin B$ and $x \in A$ & $x \notin C$
$\Leftrightarrow x \in A \setminus B$ and $x \in A \setminus C$

$\Leftrightarrow \quad x \in A \setminus B \;\cap\; A \setminus C$

Similarly we can prove ② part too

* Symmetric Difference :- $\qquad x \in (A \cup B) \cap (A \cap B)^c$

$$\left\{\begin{aligned} A \triangle B &= \{x \,/\, \underline{x \in A \text{ or } x \in B} \text{ but } \underline{x \notin A \cap B}\} = (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) - (A \cap B) \\ &= (A \cup B) \setminus (A \cap B) \end{aligned}\right.$$



$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

$$= (A \cap B^c) \cup (B \cap A^c)$$

$$= \underline{(A \cap B^c) \cup B} \;\cap\; \underline{(A \cap B^c) \cup A^c}$$

$$= (A \cup B) \cap \underbrace{(B^c \cup B)}_{\mathbb{R}} \;\cap\; \underbrace{(A \cup A^c)}_{\mathbb{R}} \cap (B^c \cup A^c)$$

B^c

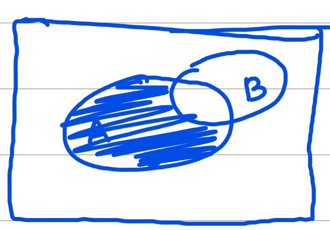
$$= (A \cup B) \cap (B^c \cup A^c)$$

$$= (A \cup B) \cap (A \cap B)^c$$

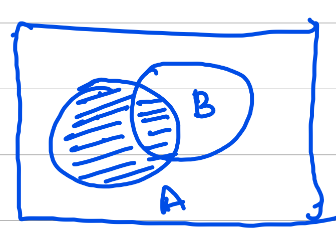
$$= (A \cup B) \setminus (A \cap B)$$

$A \setminus B$

$= A \cap B^c$



$A \setminus (B \setminus A)$



$$A \setminus (B \setminus A) = B \cap A^c$$

$$\to A \cap (B \setminus A)^c$$

$$= A \cap (B \cap A^c)^c$$

$$= A \cap (A \cup B^c)$$

$$= \underline{\underline{A}}$$

Lower bound :- Upper bound

$S = \{\frac{1}{n}, n \in \mathbb{N}\} \ni$ low $S = 0 \notin S$ Upper bound $(S) = 1 \in S$

Bounded below. Bounded Above.

$\underline{\underline{\mathbb{N}}}, \mathbb{Z}, \underline{\underline{\{\frac{1}{n}, n \in \mathbb{N}\}}} \longrightarrow \mathbb{I}^-, \mathbb{Z}^-, \mathbb{R}^-$

Bounded set bounded below & Above

$\{(-1)^{2n}, n \in \mathbb{N}\} = \{1\}$ singleton set

$\underline{\text{Unique?}}$

$\{(-1)^n, n \in \mathbb{N}\} = \{-1, 1\}$ lower bound $= -1 \in S$

Upper bound $= +1 \in S$

$S = \underline{\underline{(0, 1]}}$ $\{-10000, -10, -0.5\}$ Lower bound $= 0 \notin S$

$x \geq 0 \ \forall x \in S$ Upper bound $= 1 \in S$

$x \leq 1 \ \forall x \in S$

$\underline{\underline{u}}$ l.b. of $S \Rightarrow \forall x \in S, \underline{x \geq u}$

Sup:- u is sup of S if ① u is upper bound
② v is any other upper bound of S then $v \geq u$.

u must be least upper bound.

* Inf :- if v is inf of S if ① v is lower bound
② u is any other lower bound of S then $v \geq u$

v must be greatest lower bound.

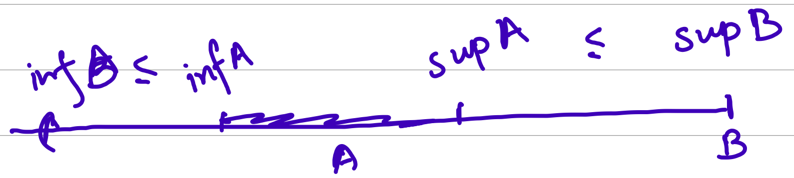
No upper bound / Supremum :- $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{I}, \mathbb{Z} \ldots$

No lower bound / Inf :- $-\mathbb{N}, \mathbb{Q}^-, \mathbb{R}^-, \mathbb{I}^-, \mathbb{Z}^-$

$S = \{1, 2, 3, 4\}$. $\sup(S) = 4$ set of Upper bounds $= [4, \infty)$
$\inf(S) = 1$ set of lower bounds $= (-\infty, 1]$

* Completeness property :- Lower bound exists $\Rightarrow$ Inf exists
Upper Sup

$A \subseteq B \Rightarrow \inf A \geq \inf B$



Infr Suppose

Absent :- Adity, Dipali, Anushree, Pradip, Aakash, Lalit, vishal, Komal, Samiya, Govinda, Atul, Chetan, Yogita. ___

Completeness Property :-

Lower bound → inf exists $\mathbb{N} \Rightarrow$ 0 lower - 1 inf

Upper → sup exists $\mathbb{R}^- \Rightarrow$ 1 upper, 0 sup

Supremum — least upper bound

Infimum — Greatest Lower bound

LUB :- If u is ~~up~~ least upper bound of set S then

① u is upper bound ✓

② if v is any other upper bound of S the $v \geq u$.

GLB :-

Theo :- If sup exists, then it is unique.

proof :- We will prove this by method of contradiction.
Assume for set S $\exists$ u_1 & u_2 be two sup.
by def^n

u_1 is sup
① u_1 is upper bound of S
② If $\exists$ u_2 as any other u.b. $\exists$
then $u_2 \geq u_1$ ✓

u_2 is sup.
① u_2 is upper bound of S
② If $\exists$ u_1 as any other
then $u_1 \geq u_2$ ✓

$\Rightarrow$ $u_1 = u_2$

$\Rightarrow$ Sup is unique

Theo :- If $A, B \subseteq \mathbb{R}$ and $A \subseteq B$ then if inf & Sup of B exists then

① $\inf(A) \geq \inf(B)$

② $\sup(A) \leq \sup(B)$

Proof :- ① Assume $\inf(A) = u_A$ $\inf(B) = u_B$

$A \subseteq B$ to prove $\inf(A) \geq \inf(B)$

$u_A \geq u_B$.

We will prove this by method of contradiction

$\boxed{u_A \leq u_B}$

by def of infimum

① u_B is lower bound of B.

② u_x any lower bound of B $\Rightarrow$ $u_x \leq u_B$ (Greatest Lower bound)

u_B Lower bound of $B \Rightarrow \forall x \in B,\ u_B \le x$

$A \subseteq B \Rightarrow \forall x \in A \Rightarrow x \in B$

u_B is also lower bound of A as $\forall\, \underline{x \in A} \subseteq B,\ u_B \le x$

$\left\{ \begin{array}{l} u_A \text{ is inf of } A \Rightarrow u_A \text{ is also lower bound of } A \\ \quad\& \text{ if } \exists \text{ some other lower bound of } A \text{ then } u_A \text{ is greater than that lower bound} \end{array} \right.$

$\Rightarrow u_B \le u_A$

which contradicts to our assumption.

Sup?

$A \subseteq B \quad \inf A \geq \inf B$

Assume $u_A \le u_B$.

(1) u_B inf $B \Rightarrow \left\{ \begin{array}{l} u_B \text{ lower bound} \rightarrow x \geq u_B \ \forall x \in B \\ u_x \text{ lower bound of } B \Rightarrow u_x \le u_B \end{array} \right.$ GLB

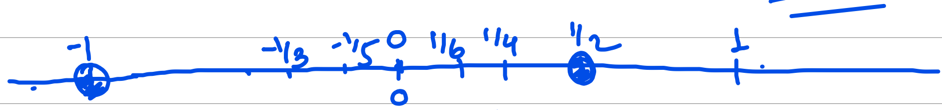
$x \geq u_B \quad \forall x \in B$

$\Rightarrow x \geq u_B. \quad \forall x \in A \subseteq B$

$\Rightarrow \underline{u_B}$ is lower bound of A.

$\left\{ \begin{array}{l} u_A \text{ is } \inf(A) \\ \text{i) } u_A \text{ l.b.} \\ \text{2) } u_y \text{ l.b. of } A,\ u_y \le u_A \end{array} \right.$

$\Rightarrow \underline{u_B \le u_A}$



$S = \left\{ \dfrac{(-1)^n}{n},\ n \in \mathbb{N} \right\}$

$S = \left\{ -1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \dots \right\}$

Set of Lower bounds $= (-\infty, -1]$

Upper bounds $= [1/2, \infty)$

$\inf = -1 \in S$

$\sup = 1/2 \in S$

$S = \left\{ 1 - \frac{(-1)^n}{n},\ n \in \mathbb{N} \right\}$ inf $S = 1/2$

set of LBs = sup $S = 2$

UBs =

$S = \left\{ 1 - \frac{(-1)^1}{1},\ 1 - \frac{(-1)^2}{2}, \dots \right\} = \left\{ 2, \frac{1}{2}, \frac{4}{3}, \dots \right\}$

$S = \left\{ \frac{1}{m} + \frac{1}{n},\ m, n \in \mathbb{N} \right\}$

2 sup inf 0

? $\left\{ \frac{(-1)^m}{m} + \frac{(-1)^n}{n},\ m, n \in \mathbb{N} \right\}$? -2 1

* Cartesian Product

$A \times B = \{ \langle x, y \rangle \,/\, x \in A,\ y \in B \}$

$A = \{2, 3, 4\}$ $B = \{1, 5\}$.

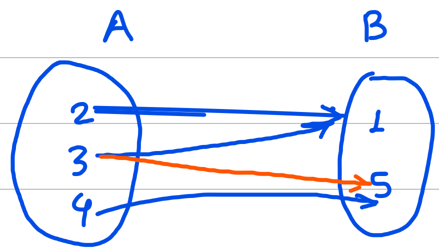
$A \times B = \{ \langle 2,1 \rangle, \langle 2,5 \rangle, \langle 3,1 \rangle, \langle 3,5 \rangle, \langle 4,1 \rangle, \langle 4,5 \rangle \}$

$f = \{ \langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 4, 5 \rangle \}$

$f(2) = 1$

$f(3) = 1$

$f(4) = 5$

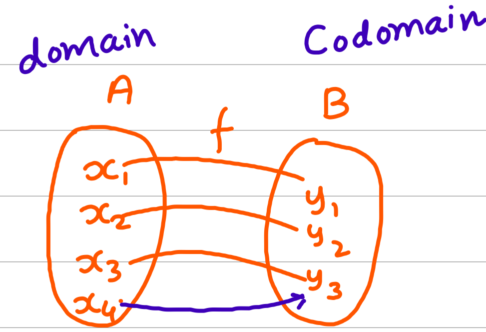


One-one fun^c (Injective)

$x_1, x_2 \in A$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

not one-one $\quad x_3 \neq x_4 \Rightarrow f(x_3) = f(x_4) = y_3$

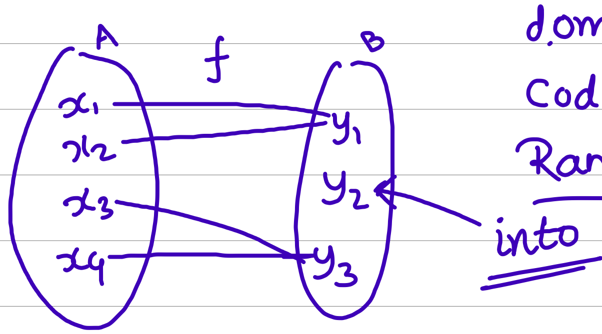


one-one

$f(x) = x^3 \quad \forall x \in \mathbb{I}$ $\qquad f(x) = x^2 \quad \forall x \in \mathbb{N}$ ✓

*

Into



domain $= \{x_1, x_2, x_3, x_4\}$

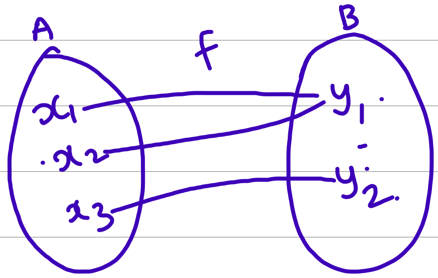
Codomain $= \{y_1, y_2, y_3\}$

Range$(f) = \{y_1, y_3\}$

Range $\subset$ Codomain

Onto

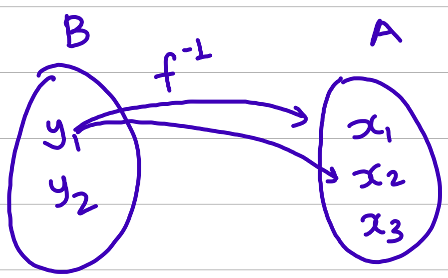
Surjective



Codomain = Range

onto

set fun^c f^{-1}



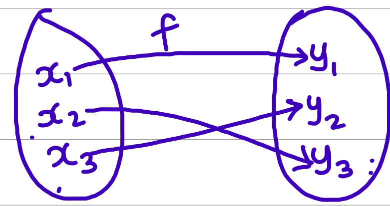
$f^{-1}(y_1) = \{x_1, x_2\}$

$f^{-1}(y_2) = \{x_3\}$.

set fun^c

$$f^{-1}(y) = \{x \mid x \in A,\ f(x) = y \in B\}$$ ✓

One-One & onto $\Rightarrow$ (Injective + Surjective) $\Rightarrow$ Bijective fun^c

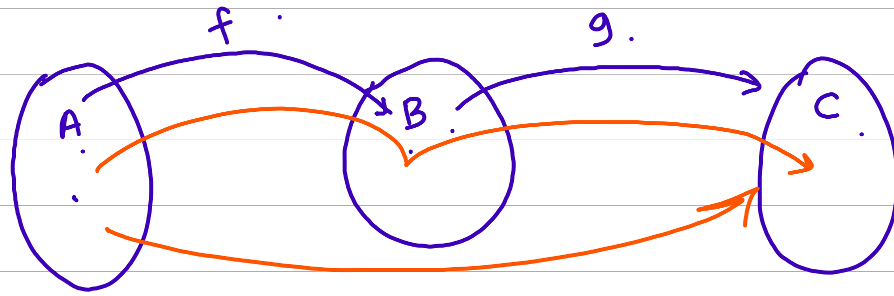


one-one & onto
In Sur
& Bijective

$f(x) = 2x \quad \ni \; x \in \mathbb{I}$

$f(x) = x^2 \quad \ni \; x \in \mathbb{N}$

Composite fun^c

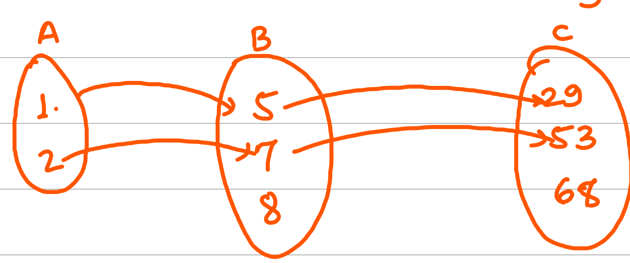


$D(f) = A \qquad R(f) \subseteq B \qquad D(g) = B$

$R(g) \subseteq C$

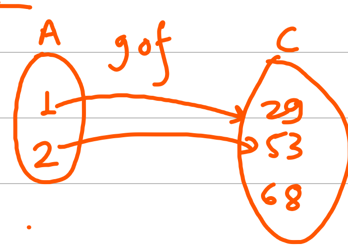
$(2x+3)^2 + 4 \qquad g \circ f$

$f(x) = 2x+3 \qquad g(y) = y^2 + 4$



$D(f) = A, \quad R(f) \subseteq D(g)$

$g \circ f$



$$g\circ f \underset{\text{may or may not}}{\neq} f\circ g$$

$f : \mathbb{R} \to \mathbb{R}$ & $g : \mathbb{R} \to \mathbb{R}$

$f(x) = 2x+3$ $\qquad$ $g(y) = y^2+4$

$$g\circ f(x) = g(f(x)) = g(2x+3) = (2x+3)^2+4$$
$$= 4x^2+12x+13$$

$$f\circ g(x) = f(g(x)) = f(x^2+4) = 2(x^2+4)+3$$
$$= 2x^2+11$$

$$g\circ f \neq f\circ g$$

$f : A \to B$, $E, F \subseteq A$

$$\underset{\text{set}}{f(E\cup F)} = \underset{\text{set}}{f(E)\cup f(F)}$$

$f(E) = \{y / y = f(x),\ x \in E \subseteq A\}$ ✓

$f(F) = \{y / y = f(x),\ x \in F \subseteq A\}$

$\underline{f(E\cup F)} = \{y / \underline{y = f(x),\ x \in E\cup F} \subseteq A\}$ ✓

To prove : $\quad$ - $f(E)\cup f(F) = f(E\cup F)$

$\subseteq$

$\supseteq$

Let $y \in f(E) \cup f(F)$

$\Leftrightarrow$ $\underline{y \in f(E)}$ or $y \in f(F)$

$\Leftrightarrow$ $y = f(x)$ and

$x \in E$ or $x \in F$

$\Leftrightarrow$ $y = f(x)$ and $x \in E$ or F

$\Leftrightarrow$ $y = f(x)$ and $x \in E \cup F$

$\Leftrightarrow$ $y \in f(E \cup F)$

As $y \in f(E) \cup f(F) \Rightarrow y \in f(E \cup F)$

So $f(E) \cup f(F) \subseteq f(E \cup F)$

Simillarly we can obtain,

$f(E \cup F) \subseteq f(E) \cup f(F)$

$\Rightarrow$ $f(E \cup F) = f(E) \cup f(F)$

$f(E \cap F) \subseteq f(E) \cap f(F)$

let $y \in f(E \cap F)$

$\Rightarrow$ $y = f(x)$ and $x \in E \cap F$

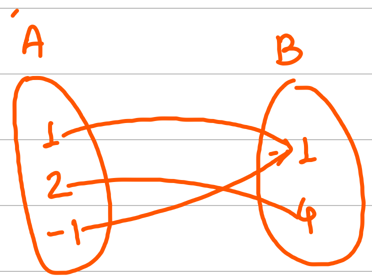
$\Rightarrow$ $y = f(x)$ & $x \in E$ and $y = f(x)$ & $x \in F$

$\Rightarrow$ $y \in f(E)$ and $y \in f(F)$

$\Rightarrow$ $y \in f(E) \cap f(F)$

$\Rightarrow$ $f(E \cap F) \subseteq f(E) \cap f(F)$

✓ $f(E \cap F) \not\supseteq f(E) \cap f(F)$



$E = \{1, 2\}$ $f(E) = \{1, 4\}$

$F = \{-1, 2\}$ $f(F) = \{1, 4\}$

$E \cap F = \{2\}$ $f(E \cap F) = \{4\}$

$f(E) \cap f(F) = \{1, 4\}$

$\Rightarrow$ $f(E \cap F) \not\supseteq f(E) \cap f(F)$

$f : A \to B$ $G, H \subseteq B$

i) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

$f^{-1}(G) = \{x \,/\, f(x) = y \in G, x \in A\}$

$f^{-1}(H) = \{x \,/\, f(x) = y \in H, x \in A\}$

$f^{-1}(G \cup H) = \{x \,/\, f(x) \in G \cup H, x \in A\}$

To prove $f^{-1}(G \cup H) \underset{\supseteq}{\subseteq} f^{-1}(G) \cup f^{-1}(H)$

Let $x \in f^{-1}(G \cup H)$

$\Rightarrow f(x) \in G \cup H$ $\quad x \in A$

$\Rightarrow f(x) \in G$ or $f(x) \in H$ $\quad x \in A$

$\Rightarrow x \in A$ & $f(x) \in G$ or $x \in A$ & $f(x) \in H$

$\Rightarrow x \in f^{-1}(G)$ or $x \in f^{-1}(H)$

$\Rightarrow x \in f^{-1}(G) \cup f^{-1}(H)$

$\Rightarrow f^{-1}(G \cup H) \subseteq f^{-1}(G) \cup f^{-1}(H)$

Simillarly we can obtain $f^{-1}(G \cup H) \supseteq f^{-1}(G) \cup f^{-1}(H)$

Thm. $f: A \to B$ is injective $E \subseteq A$

$f^{-1}(f(E)) = E$

$\longrightarrow$

$\therefore$ $f: A \to B$ as injective fun^c

for $x_1, x_2 \in A$, if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f^{-1}(D) = \{x \,/\, f(x) \in D, x \in A\}$ ✓

$f(E) = \{y \,/\, y = f(x), x \in E\}$

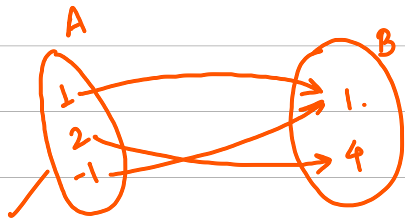
Let $x \in f^{-1}(f(E))$

$\Rightarrow y = f(x) \in f(E)$ & $x \in A$, $y \in B$

$\exists$ some $x \in E$? ($x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$) one-one

$\Rightarrow f^{-1}(f(E)) \subseteq E$

Not one-one



$E = \{1,2\}$ ✓

$f(E) = \{1,4\}$

$f^{-1}(f(E)) = f^{-1}(\{1,4\})$

$= \{1,-1,2\}$

$E \neq f^{-1}(f(E))$

Finite Set :- $A = \{x_1, x_2, x_3, \dots, x_n\}$

$B = \{1, 2, 3, \dots, n\} \subseteq \mathbb{N}$, n is some fixed $\in \mathbb{N}$

Infinite ✓ $\mathbb{Q}$, $\mathbb{N}$. $\{1,2,3,\dots, \_\_\_\_\}$

even odd square ____

complex Real

$S, T \subseteq \mathbb{R}$, $T \subseteq S$

To prove If S is finite then T is also finite.

→ S is finite if it is either empty or it has n element

① $S = \phi$ & $T \subseteq S$ ⇒ $T = \phi$ ⇒ T is finite.

② We will prove this by mathematical induction

✓ S is finite ✓

$\#(S) = 1$ & $T \subseteq S$ (2.7) $\#(T) = 0$ or $\#(T) = 1$

$T = \phi$ or $T = S$

T is finite

✓ $\#(S) = K$ & $T \subseteq S$, ⇒ T is finite.

$\#(S) = K+1$ ⇒ $S = \{x_1, x_2, \ldots, x_{K+1}\}$

$\{1, 2, \ldots, K+1\}$.

$S_1 = S - \{\overset{x_{K+1}}{f(K+1)}\}$ & $T \subseteq S$

$f(K+1) \in T$ or

$T \nsubseteq S_1$

$T_1 = T - \{f(K+1)\} \subseteq S_1$

$f(K+1) \notin T$

$T \subseteq S_1 \subseteq S$

⇒ T is finite

$T_1 \subseteq S_1$ & $\#(S_1) = K$

= T_1 is finite set

$\#(T_1) + 1 = \#(T)$

⇒ T is also finite

* Countable Set?

even $\{2n, n \in \mathbb{N}\} = \{2,4,6,8,10,\dots\}$] Bijection with $\mathbb{N}$
odd $\{1,2,3,4,5,\dots\}$ of subset of $\mathbb{N}$

countable set :- finite or bijection with $\mathbb{N}^m$ $m \in \mathbb{N}$.

$\mathbb{N}^2$
$(1,1)\ (1,2)\ (1,3)\ \dots$
$(2,1)\ (2,2)\ (2,3)\ \dots$
$\vdots$

$1,2,3,4,\dots$

countable finite / bijection $\mathbb{N}^m$

$\mathbb{N}^3$ $\langle 1,2,2\rangle$

$S, T \subseteq \mathbb{R}$, $T \subseteq S$

① S is countable $\Rightarrow$ T is also countable

· S is countable $\Rightarrow$ S is either finite or bijection with $\mathbb{N}$ (denumerable)

① if S is finite $\Rightarrow$ T is finite $\Rightarrow$ T is countable
(by previous theo.)

② if S has bijection with $\mathbb{N}$ (denumerable)
$\mathbb{N} = \{1,2,3,\dots\}$

We can write $S = \{f(1), f(2), \dots\}$

$T = \phi$, $T =$ finite, if T is infinite

$T \subseteq S \Rightarrow T = \{f(n_1), f(n_2), f(n_3) \dots\}$

$B = \{n_1, n_2, n_3, \dots\}$
$\mathbb{N} = \{1, 2, 3, \dots\}$

T denumerable set

countable

$\mathbb{N}^2$ 1, 2, 3, 4, 5, ... countably infinite / denumerable

$\mathbb{N}^2$

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \dots \\ (2,1) & (2,2) & (2,3) & (2,4) & \dots \\ (3,1) & (3,2) & (3,3) & (3,4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

countable

Set of Rational Numbers is denumerable.

$$Q = \left\{ \frac{p}{q},\ p,q \in \mathbb{Z},\ q \neq 0 \right\}$$

Lets discuss Q^+

2.5 $\frac{5}{2}$. $\frac{10}{4}$

$\exists$ $\mathbb{N}^2$

$$\begin{matrix} \cancel{1/1} & 1/2 & 1/3 & 1/4 & \dots \\ 2/1 & \cancel{2/2} & 2/3 & 2/4 & \dots \\ 3/1 & 3/2 & \cancel{3/3} & 3/4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

countable

< 5000

Archemedian Property

$$x \in \mathbb{R} \quad \exists\ n_x \in \mathbb{N}, \quad x < n_x$$

$$23.56 \in \mathbb{R} \quad \exists\ 24, 25 \in \mathbb{N}, \quad 23.56 < 24$$

Set of $\mathbb{N}$ is bounded below $\Rightarrow$ It has no upper bound.
$\Rightarrow$ It doesnt have supremum.

We will prove this by method of contradiction

$x \in \mathbb{R}$ $\nexists$ any $n_x \in \mathbb{N}$, $\ni$ $x < n_x$

$\Rightarrow$ $n \leq x$ $\forall$ $n \in \mathbb{N}$

$\Rightarrow$ x is upper bound for $\mathbb{N}$

Completeness $\curvearrowright$ it has some supr let u.

$\Rightarrow$ $n \leq u$ $\forall$ $n \in \mathbb{N}$.

$\Rightarrow$ $n+1 \leq u$ $\forall$ $n \in \mathbb{N}$

$\Rightarrow$ $n \leq u-1$ $\forall$ $n \in \mathbb{N}$ $\Rightarrow$ $u-1$ upper bound of $\mathbb{N}$

by def^n of supremum if u is sup $\Rightarrow$ ① u upper bound
② There exist no upper bound less than u

but $u-1$ is upper bound
u -"- but u is sup.

$\therefore$ This contradicts to our assumption that u is sup.

$\therefore$ $\exists$ $n_x \in \mathbb{N}$, $\ni$ $x < n_x$ (QED)

* If $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ then $\inf S = 0$

let $\varepsilon > 0$, $1/\varepsilon \in \mathbb{R}$ $0 < 1/\varepsilon$

by Archemedian property

$\varepsilon > 0$, $\quad 1/\varepsilon \in \mathbb{R}$, $\quad \exists\, n_\varepsilon \in \mathbb{N} \ni 0 < 1/\varepsilon < n_\varepsilon$

Any $\varepsilon > 0$. $\quad \Rightarrow 0 < 1/n_\varepsilon < \varepsilon$

$\Rightarrow$ for set S, 0 is lower bound

$\inf S \geq 0$

Assume $\inf S > 0$, $\quad \varepsilon = \inf S > 0$

$\Rightarrow \exists\, n_\varepsilon, \ni \quad 0 < 1/n_\varepsilon < \varepsilon$

$\boxed{0 < 1/n_\varepsilon < \inf S}$

$\inf S$ is infimum of S $\ni$ S is Lower bound

$\ni \quad \inf S \leq 1/n \quad n \in \mathbb{N}$

$\therefore$ Our assumption is wrong $\inf S > 0$

$\Rightarrow \quad \inf S = 0$

If $t > 0$, $\exists\, n_t \in \mathbb{N}, \ni \quad 0 < \dfrac{1}{n_t} < t$

* Let $y \in \mathbb{R}$ $\quad \exists\, n_y \in \mathbb{N}$, $\quad \boxed{n_y - 1 \leq y \leq n_y}$

$2.34 \in \mathbb{R}$ $\quad \exists\, 3 \in \mathbb{N}$ $\quad 3 - 1 \leq y \leq 3 \quad \Rightarrow 2 \leq y \leq 3$

Let $y \in \mathbb{R}$, $E_y = \{n \mid n > y, n \in \mathbb{N}\}$

if $y = 2.34$ $E_y = \{3, 4, 5, 6, \ldots\}$

$y \leq n \quad \forall\, n \in E_y$

$\Rightarrow$ y is Lower bound of E_y

by completeness property, $\exists$ inf of $E_y = u$

$\therefore \quad u \leq n \quad \forall\, n \in \mathbb{N}$

if u is inf , $u+1$ cannot be lower bound of S

$\exists$ some $n \in \mathbb{N}$, $\quad n < u+1$

$\Rightarrow \quad n-1 < u$.

Density Theo:- $x, y \in \mathbb{R}$, $x < y$, $\exists\, r \in \mathbb{Q}$, $x < r < y$

$x < y$, $y - x > 0$ $\exists\, n \in \mathbb{N}$, $\quad 0 < \frac{1}{n} < y - x$

$\Rightarrow \quad 1 < ny - nx$

$\Rightarrow \quad nx + 1 \leq ny$.

Assume $x > 0$, $nx > 0$ $\quad m \in \mathbb{N}$ $\quad m - 1 \leq nx \leq m$

$nx \leq m \leq nx + 1 \leq ny$ $\quad$? $\mathbb{Q}$ $\quad r = \frac{m}{n}$

$\Rightarrow \quad nx \leq m \leq ny \quad \Rightarrow \quad x \leq \frac{m}{n} \leq y$

by density theo.

if $x, y \in \mathbb{R}$, $x < y$ then $\exists\ r \in \mathbb{Q}$, $x < r < y$

$\sqrt{2}$ is irrational no. so does $x\sqrt{2}$ & $y\sqrt{2}$

$\therefore$ if $x < y$

$$\sqrt{2}x < \sqrt{2}y$$

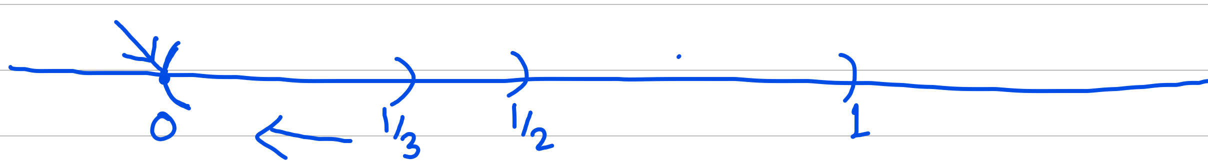
$\because$ by density theo.

$$\sqrt{2}x < r < \sqrt{2}y$$

$$x < \frac{r}{\sqrt{2}} < y$$

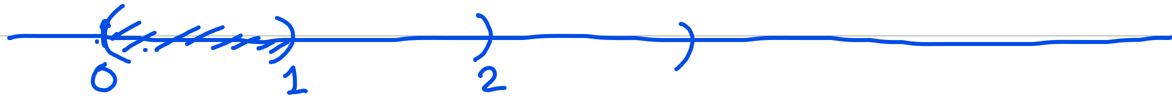
Arbitrary intersection

$\bigcap_{n=1}^{k} \left(0, \frac{1}{n}\right)$ finite intersection $\Rightarrow$ $\left(0, \frac{1}{k}\right)$



$$\bigcap_{n=1}^{\infty} (0, 1/n) = \phi$$

$$\bigcap_{n=1}^{\infty} (0,n) = (0,1) \qquad \bigcup_{n=1}^{\infty} (0,n) = (0,\infty)$$

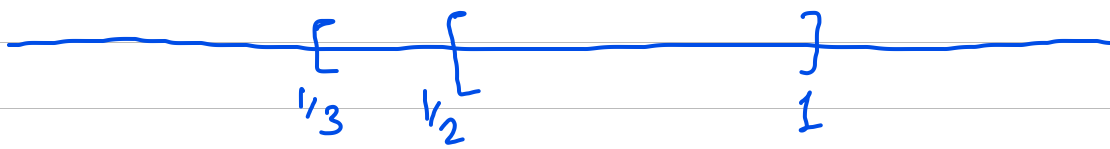


$$(0,1) \cup (0,2) = (0,2)$$

$$(0,1) \cup (0,2) \cup (0,\dots) \cup (0,n) = (0,n)$$

$$\left[\frac{1}{k},1\right] \Leftarrow \bigcup_{n=1}^{k} \left[\frac{1}{n},1\right] \qquad n=1 \Rightarrow \{1\} \qquad n=2 \Rightarrow \left[\frac{1}{2},1\right]$$

$$n=3 \Rightarrow \left[\frac{1}{3},1\right]$$

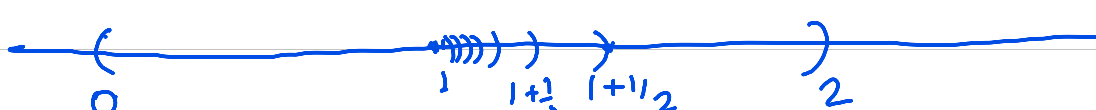


$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n},1\right] = (0,1]$$

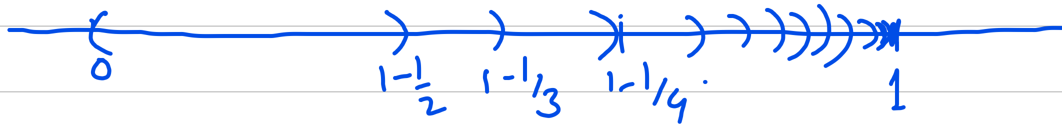
closed

Arbitrary union of closed intervals ~~is~~ may or maynot again closed. interval

$$(0,1] = \bigcap_{n=1}^{\infty} (0, 1+1/n) \qquad (0,2) \cap (0,1+\tfrac{1}{2}) = (0, 1+\tfrac{1}{2})$$



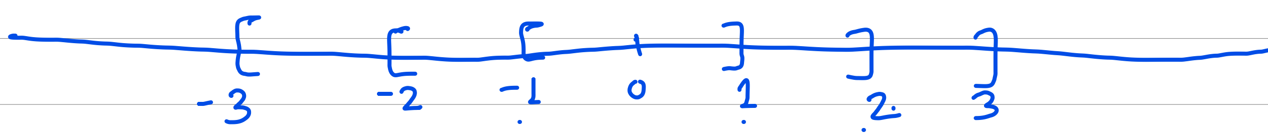
$$\bigcup_{n=1}^{\infty}\left(0,1-1/n\right) = (0,1-1) = \phi \quad (0,1-1/2)$$



$$= (0,1)$$

$$\bigcup_{n=1}^{\infty}[0,1-1/n] = [0,1)$$ half closed half open interval

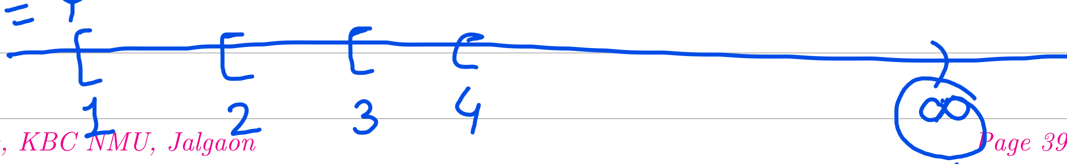
$$\bigcup_{n=1}^{\infty}[-n,n] \checkmark = (-\infty,\infty)$$



$$\bigcup_{n=1}^{K}[-n,n] = [-K,K]$$

$$\bigcap_{n=1}^{\infty}[-n,n] = [-1,1] \checkmark$$

$$\bigcap_{n=1}^{\infty}[n,\infty) = \phi$$



$\mathbb{R}$ is uncountable

$\Rightarrow$ $(0,1)$ is uncountable

Assume $(0,1)$ is countable.

$\Rightarrow$ $(0,1)$ is denumerable

$\Rightarrow$ $(0,1)$ has one-one & onto ~~corr~~ relation with $\mathbb{N}$

$\Rightarrow$ $S=(0,1)=\{b_1,b_2,b_3,\dots\}$

$b_i \in (0,1)$ it can be written in the form of $0,1,2,\dots 9.$

$b_1 = 0.\underline{a_{11}}\,a_{12}a_{13}\dots$ $\quad c_1 \neq a_{11}$ $\quad 0.121234$

$b_2 = 0.a_{21}\,\underline{a_{22}}\,a_{23}\dots$ $\quad c_2 \neq a_{22}$

$b_3 = 0.a_{31}\,a_{32}\,a_{33}\dots$ $\quad c_3 \neq a_{33}$

$\vdots$

$c = 0.c_1c_2c_3\dots$

$c \in (0,1)$ and as $c_i \neq a_{ii} \ \forall i$

$\Rightarrow$ $c \neq b_i \ \forall i$

but as $c \in (0,1)$ & $c \neq b_i \ \forall i$

$\therefore$ So our assumption that $(0,1)$ is countable is wrong

$\Rightarrow$ $(0,1) \subseteq \mathbb{R}$, $(0,1)$ is uncountable $\Rightarrow$ $\mathbb{R}$ is uncountable.

* Cauch Schwartzs inequality

If $a_i, b_i \in \mathbb{R}$ $\quad i=1:n$

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

$a_i, b_i \in \mathbb{R}$, some $x \in \mathbb{R}$

$$a_i x + b_i \in \mathbb{R} \qquad \forall i$$

$$\boxed{(a_i x + b_i)^2 \geq 0} \qquad \forall i$$

$$a_i^2 x^2 + 2 a_i b_i x + b_i^2 \geq 0 \qquad \forall i$$

$$x^2 \sum_{i=1}^{n} a_i^2 + 2x \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2 \geq 0$$

This statement is true for any $x \in \mathbb{R}$

$A = \sum a_i^2 \qquad B = \sum a_i b_i \qquad C = \sum b_i^2$

$$A x^2 + 2Bx + C \geq 0$$

Assume $\boxed{x = \frac{-B}{A}}$ ✓

$$A \cdot \frac{B^2}{A^2} + \frac{2B(-B)}{A} + C \geq 0$$

$$\frac{B^2}{A} - \frac{2B^2}{A} + C \geq 0$$

$$\frac{-B^2}{A} + C \geq 0$$

$$C \geq \frac{B^2}{A}$$

$$AC \geq B^2$$

$$B^2 \leq AC$$

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right) \qquad \text{(QED)}$$

$$(\sum a_i b_i)^2 = (\sum a_i^2)(\sum b_i^2)$$

?

a_i, b_i

$\boxed{a_i = b_i}$

$$(\sum a_i^2)^2 = (\sum a_i^2)(\sum a_i^2) \checkmark$$

* Set Topology

Open Set

(a,b)

$[a,b]$

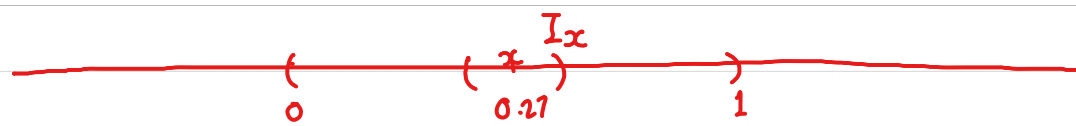
$\boxed{[a,\infty) \quad \text{Open | Closed Set}}$ It is actually closed set

$\{1,2,3,4,5\}$

$\mathbb{R}$ $\quad (-\infty,\infty)$ open / closed both (?)

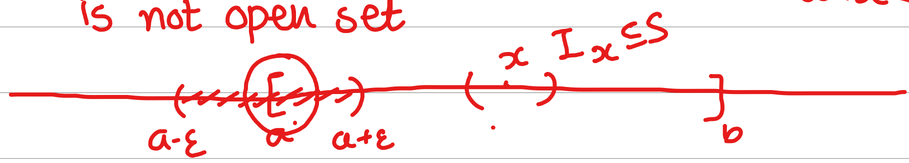
① Open S, $\forall x \in S \; \exists I_x \subseteq S \ni x \in I_x \subseteq S$

if $x = 0.27$, $I_x = (0.25, 0.30)$ $\quad x \in I_x \subseteq S$



ε-nbhd of x, $I_x = (x-\varepsilon, x+\varepsilon)$ $\quad \varepsilon > 0$

$S = [a,b]$ is not open set $\quad a < x < b$



$(a-\varepsilon, a+\varepsilon) \subseteq [a,b]$ $\qquad (a-\varepsilon, a)$

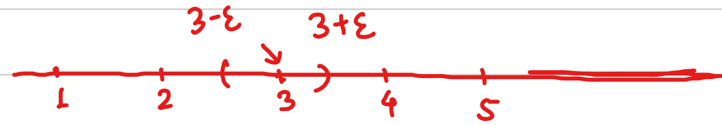
open set

[$\mathbb{R}$ ✓ $\quad \exists\ x \in \mathbb{R},\ I_x \subseteq \mathbb{R} \ni x \in I_x \subseteq \mathbb{R}$

ϕ

~~Closed set~~

Set of ① $\mathbb{N}$



$\varepsilon > 0 \qquad (3-\varepsilon, 3+\varepsilon) \nsubseteq \mathbb{N}$

$\mathbb{N}$ is not open, $\mathbb{I}$,

② $\mathbb{Q}$ is ~~not~~ open

let $x = 0.25$. $\varepsilon = 0.1 > 0$, $I_x = (x-\varepsilon, x+\varepsilon)$
$= (0.15, 0.35)$

$\overset{\text{IN}}{(0.15, 0.35)} \nsubseteq \mathbb{Q}$ ← Set of Rational Nos.

[Density Theo. :- ? any $x, y \in \mathbb{R}$, $x < y$, $\exists\ r \in \mathbb{Q} \ni x < r < y$
similarly $\exists\ r_1 \in \mathbb{Q}^c \ni x < r_1 < y$]

$x = 0.15$, $y = 0.25 \in \mathbb{R}\ \exists\ r_1 \in \mathbb{Q}^c \Rightarrow 0.15 < r_1 < 0.25$

$(0.15, 0.35) \nsubseteq \mathbb{Q}^c$

$(0.15, 0.35) = \{x \mid 0.15 < x < 0.35\} \overset{\subseteq \mathbb{R}}{\nsubseteq \mathbb{Q}}$

$\exists$ irrational

$[a, \infty)$, $\nexists$ any nbhd of $a \in S$

Interior point $\qquad (a,b)$ for every pt. in (a,b)

$\exists\ x \in (a,b)$, $\exists\ x \in I_x \subseteq (a,b)$

every pt. of (a,b) is interior pt.

$S = [a,b)$, $\therefore$ for $a \in S$, any $\varepsilon > 0$, $(a-\varepsilon, a+\varepsilon) \nsubseteq S$.
$\therefore$ a is not interior pt. of S.

$S_1 = S - \{a\} = (a,b)$

S- ← every pt. of S_1 is — interior pt.

Interior Set : Collection of Interior points of sets S.

$S \Rightarrow S^i$

$[a,b) \Rightarrow (a,b)$ ← set of interior pts.

$\mathbb{R} \Rightarrow \mathbb{R}$

$\phi \Rightarrow \phi$

$[a,b] \Rightarrow (a,b)$

$[a,b) \Rightarrow (a,b)$

$[a,b] \Rightarrow (a,b)$

$[a,\infty) \Rightarrow (a,\infty)$

$(-\infty,b] \Rightarrow (-\infty,b)$

Finite union of open sets is again open set.

Let A, B as open sets

① If A is open set, then $\forall x \in A \ \exists I_x \subseteq A \ \ni . x \in I_x \subseteq A$

if $x \in A \cup B$ and $x \in I_x \subseteq A$

$A \subseteq A \cup B$

$\Rightarrow x \in I_x \subseteq A \subseteq A \cup B$

$\Rightarrow A \cup B$ is again open set

$S = [a,b]$ S^i = interior set of $S = (a,b)$

* Finite intersection of open sets is again open

Let A & B are two open sets.

then $\forall x \in A$ $\exists$ some $I_x \subseteq A$ $\ni$ $x \in I_x \subseteq A$

$\forall x \in B$ $\exists$ some $I_x \subseteq B$ $\ni$ $x \in I_x \subseteq B$

if $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

as A, B are open set $\Rightarrow x \in I_x \subseteq A$ and $x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A$ & B

$\Rightarrow x \in I_x \subseteq A \cap B$

✓ $\Rightarrow A \cap B$ is also open

(1.3) * Arbitrary Union of open sets is open

prove: $\{A_i\}_i$ is collection of open sets, $\bigcup_{i=1}^{\infty} A_i$ is open

Let $x \in \bigcup A_i$

$\Rightarrow x \in A_i$ for some i $(0,1)$ $(0.5, 2)$

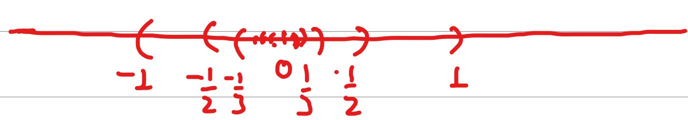
$\Rightarrow x \in I_x \subseteq A_i$ as A_i are all open sets $(0.5, 1)$ open

$\Rightarrow x \in I_x \subseteq A_i \subseteq \bigcup_{i=1}^{\infty} A_i$ $\Rightarrow$ $\bigcup A_i$ is also open set.

* Arbitrary intersection of open sets may or may not be open

counter example:

$A_i = \left(\frac{-1}{n}, \frac{1}{n}\right)$



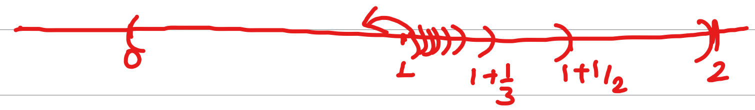
$A_1 = (-1, 1)$

$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\bigcap_{i=1}^{\infty} A_i = \{0\}$ $0 \in I_x \nsubseteq \{0\} \Rightarrow$ ~~not~~ $\bigcap A_i$ is not open here.

$A_n = (0, 1+1/n)$

$A_1 = (0, 2)$



$A_2 = (0, 1+\frac{1}{2})$

$A_3 = (0, 1+\frac{1}{3})$

$\cap A_i = (0, 1]$ is not open set

Closed Set :- S is closed if S^c is open.

$S = [a, \infty) \Rightarrow S^c = (-\infty, a)$ open $\Rightarrow S$ is closed

$S = (-\infty, b] \Rightarrow S^c = (b, \infty)$ open $\Rightarrow S$ is closed

$S = \{1,2,3\} \Rightarrow S^c = (-\infty, 1) \cup (1,2) \cup (2,3) \cup (3, \infty)$:- finite union of open set is again open

$\mathbb{N} = \{1,2,3,\ldots\} \Rightarrow S^c = (-\infty, 1) \cup \left[\bigcup_{n=1}^{\infty} (n, n+1)\right]$ $\therefore$ Arbitrary union of open is open

open.

$\mathbb{N}$ is closed set $\Rightarrow \mathbb{I}, \mathbb{Z}, \ldots$ closed sets.

$\mathbb{Q}$ $\Rightarrow \mathbb{Q}^c =$ irrational nos. $\Rightarrow \mathbb{Q}^c$ is not open

$\mathbb{Q}$ is not closed

$\mathbb{Q}, \mathbb{Q}^c$ both are not closed sets.

$\mathbb{R}$ open closed $\mathbb{R}^c = \phi \Rightarrow$ open set
$\Rightarrow \mathbb{R}$ is closed.

$\therefore \phi, \mathbb{R}$ ~~is~~ are open and closed

ϕ $\phi^c = \mathbb{R}$ is open $\Rightarrow \phi$ is closed.

* Finite union of closed set is closed
intersection

$A, B \subseteq \mathbb{R}$ closed, A^c & B^c opensets   limit pt. of set $\{0\} \cup \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ not open ~~$I_x \subseteq S$~~   not closed

To prove $A \cup B$ is closed
$\to (A\cup B)^c$ is open

$(A \cup B)^c = A^c \cap B^c$
A^c, B^c are open sets & Finite intersectⁿ of opensets is open
$(A\cup B)^c$ is open
$\Rightarrow A \cup B$ is closed

$(A \cap B)^c = A^c \cup B^c$
A^c, B^c open & Finite union of open sets is open
$\Rightarrow (A \cap B)^c$ is open.

$\frac{1}{n} \to 0$

Theo:- Arbitrary intersection of closed sets is closed.
(We have discussed,
$\to$ Arbitrary union of open sets is open. ✓

$\{A_i\}$ arbitrary collection of closed sets.

$\Rightarrow A_i^c$ is open $\forall i$

$\bigcup_{i=1}^{\infty} A_i^c$ is open set.

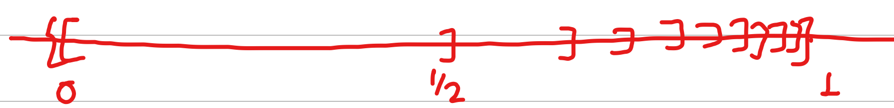
$\bigcup_i A_i^c = \left(\bigcap_{i=1}^{\infty} A_i\right)^c$ open

$\therefore \bigcap_{i=1}^{\infty} A_i$ is closed set

Arbitrary union of closed sets may or may not be closed.

counter examples

$[0, 1-1/n]$ is collection of closed sets



$A_1 = \cancel{[0,0}\ \{0\}$

$A_2 = [0, 1-1/2]$

$A_3 = [0, 1-1/3]$

$S = \bigcup [0, 1-1/n] = [0, 1)$

$S^c = \underbrace{(-\infty, 0)}_{\text{open}} \cup \underbrace{[1, \infty)}_{\text{not open}}$

$\Rightarrow S^c$ is not open $\Rightarrow$ S is not closed.