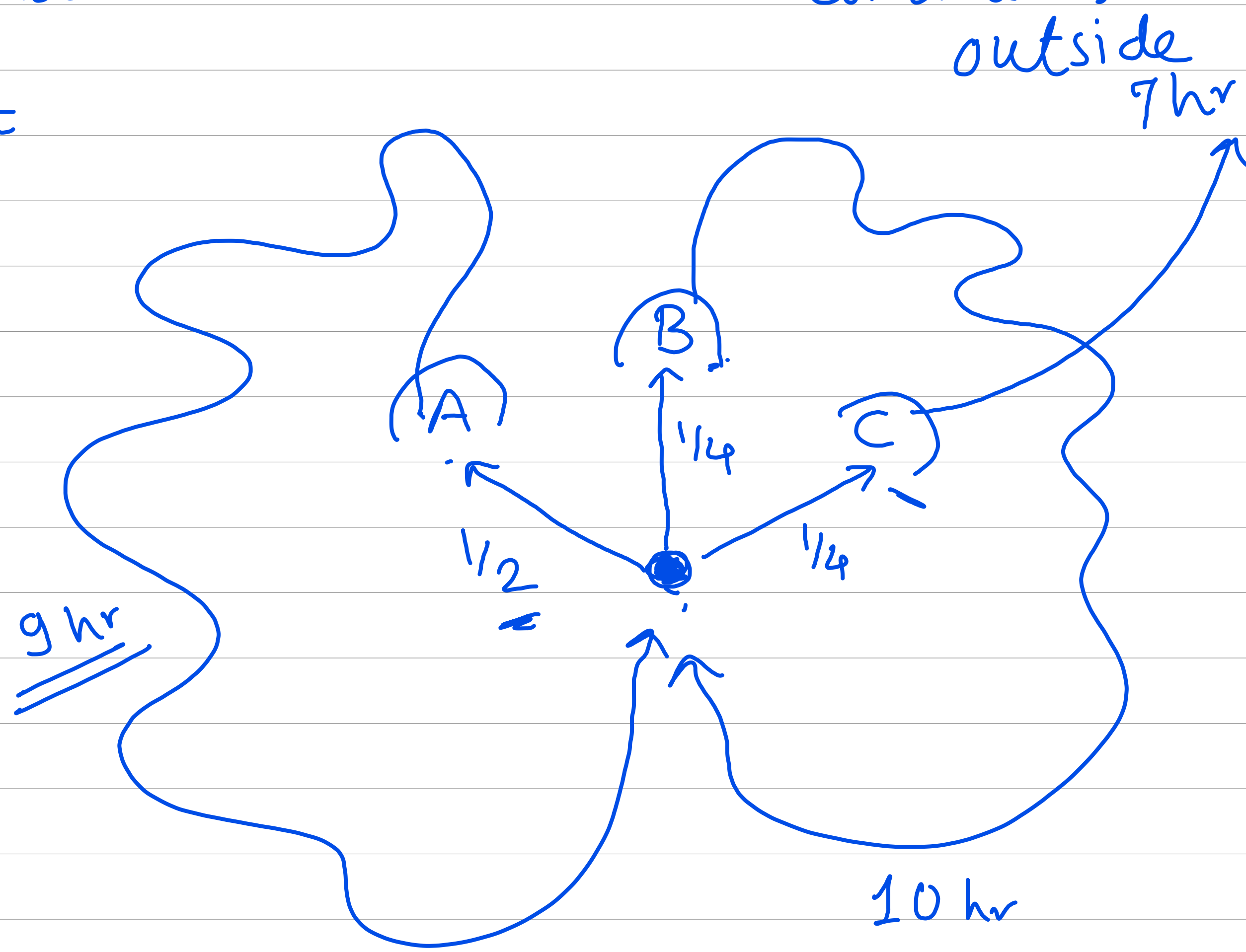
① Distributions Revision

Discrete Continuous

Rat



Rat cave

A	1/2	→	9 hr return
B	1/4	→	10 hr return
C	1/4	→	7 hr leave

X

μ

$E(X) = ?$

Assume

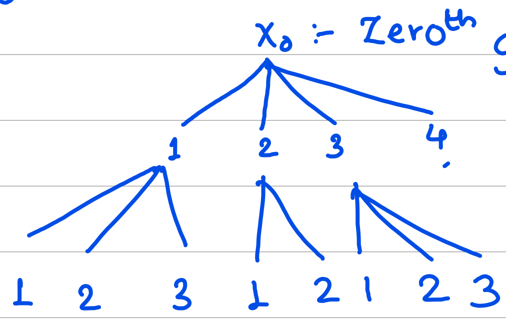
A	1/2	→	$9+\mu$
B	1/4	→	$10+\mu$
C	1/4	→	7

$$\mu = \frac{1}{2}(9+\mu) + \frac{1}{4}(10+\mu) + \mu$$

$=$

Offspring Distribution:-

X_0 :- Zeroth generation



$x_1 = 4$ 1^{st} generation size

$x_2 = 8$ 2^{nd} generation size

$x_n = 0$ $\;\; \exists\, x_m = 0 \;\; \forall\, m > n$

$x_{n+1} = 0$

Dinosaurs extints $\rightarrow x_n = 0$

$x_0 = 1$ $\quad x_1 = 3, \quad x_2 = 5$ $\qquad \lim_{n\to\infty} x_n = 0$

Sum of expectations $\qquad y = \sum_{i=1}^{n} x_i$ ($n \leftarrow$ fixed, $x_i \leftarrow$ Random) $\Rightarrow E(y) = n \cdot E(X) = \sum_{i=1}^{n} E(X_i)$

Random sum of Random number $\qquad V(y) = \sum_{i=1}^{n} V(X_i) + \sum_i \sum_{j,\, i \neq j} Cov(X_i, X_j)$

$y = \sum_{i=1}^{N} X_i$ ($N \leftarrow$ random, $X_i \leftarrow$ random)

$V(X) = E(V(X|Y)) + V(E(X|Y))$

$E(X) = E_Y(E_{X|Y}(X|Y))$

↳ compound distributions

Y ① $N \sim$ Bino $\rightarrow y$ Comp. Bino. distribution (K, p) $\quad E(Y)\; Kp \cdot E(X)$

Y ② $\sim$ Pois $\sim$ Comp. Poi. distribution $\quad E(Y)\; \lambda \cdot E(X)$

likewise $\qquad$ Comp. Geometric, Negative Binomial

$E(Y) = E\left(\sum_{i=1}^{N} X_i\right) = E_N\left(E_{Y/N}\left(\sum_{i=1}^{N} X_i / N = n\right)\right)$ $\qquad E(X_i) = \mu$

$V(X_i) = V(X)$ $\qquad E(X_i) = E(X) = \mu$

$= E_N\left(\sum_{i=1}^{N} E(X_i)\right) = E_N(n \cdot E(X)) = E(X) \cdot E(N)$ ✓

$V(Y) = V\left(\sum_{i=1}^{N} X_i\right) = E(V(X/Y)) + V(E(X/Y))$

✓ $= E\left(V\left(\sum_{i=1}^{N} X_i / N\right)\right) + V\left(E\left(\sum_{i=1}^{N} X_i / N\right)\right)$

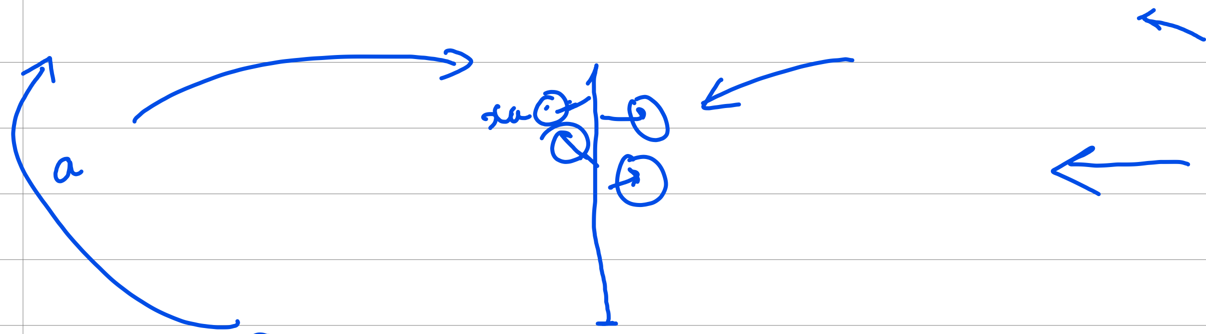
✓ $= E_N(N\, V(X)) + V_N(N\, E(X)) \Rightarrow E(N)\, V(X) + [E(X)]^2\, V(N)$

Comp. Bino. $\Rightarrow V(Y) = kp \cdot V(X) + [E(X)]^2 (kpq)$ | $\Rightarrow$ Comp. Poisson $\Rightarrow V(Y) = \lambda [V(X) + [E(X)]^2]$

$\Rightarrow \lambda \cdot E(X^2)$

$V(X) = E(X^2) - [E(X)]^2$

Matching hat problem



n

$a \to x_a \Rightarrow$ —

$b \to x_b$

$n \to$ 0, 1, 2, 2, 3, ..., n

$a_1 \quad a_2 \quad \dots \quad a_n$

$\downarrow \quad \downarrow \quad \quad \downarrow$

$1 \quad 0 \quad \quad 0$

$\Rightarrow \sum_{i=1}^{n} a_i$ matching hats

$\to n$

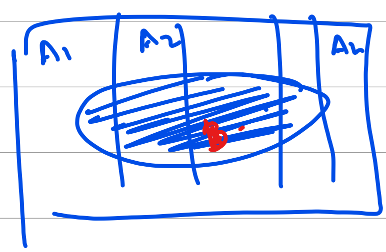
1	n	$1/n$	1
2	$n-1$	$\frac{1}{(n-1)} \times \frac{1}{n}$ +	$\frac{1}{(n-2)} (1 - 1/n)$

Baye's Rule

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A_i/B) = \frac{P(B/A_i)}{\sum_{i=1} P(B/A_i)}$$

ΣA_i



$\cup A_i = \Omega$

mutually exclusive & exhaustive

e.g. Family 2 children

$X_i = 0$, if i is female $1/2$

$= 1$ if i is male $1/2$

$$P\left[\sum_{i=1}^{2} X_i = 2 / \sum_{i=1}^{2} X_i \geq 1\right] = ?$$

E $\Omega = \{BB, BG, GB, GG\}$

B $A = \{BB, BG, GB\}$ ✓

$B = \{BB\}$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

e.g. H/T (coin)

Maths → A → 1/2

Stats → A → 1/3

$P[\text{Stats}/A] = ?$

$P[A/\text{stats}] = 1/3$ $\quad$ $P[A/\text{Maths}] = 1/2$

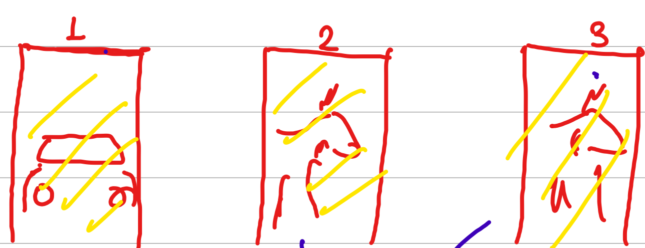
$P[\text{Stats}] = 1/2$ $\quad$ $P[\text{Maths}] = 1/2$

$$P[\text{Stats}/A] = \frac{P[\text{Stats}, A]}{P[\text{Stats}, A] + P[\text{Maths}, A]}$$

$$P[\text{Stats}/A] = \frac{1/3}{1/3 + 1/2} = \frac{2}{2+3} = \frac{2}{5}$$

Monty Hall Problem / Paradox ? Show 80

Monty Hall →



R → 2 ✓

M → 3 open

1/2 1/2 ✓ ×

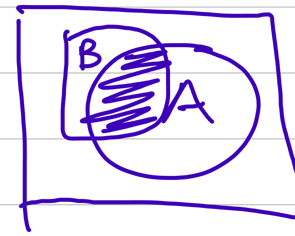
Paradox

1000 Papers	Letters	Original	Rashmi → 2	M (open)		
W	1	1	2	3	2	N
N	3	2	2	1, 3	2	W
W	3	3	2	1	2	N
2/3						1/3

$E(X/Y)$

$$= V(X/Y)\cdot E(X) + E(X/Y)\cdot V(X)$$

$$P(B/A) = \frac{P(B\cap A)}{P(A)}$$



X, Y ind

x/y	1	2	3
Prob	1/4	1/2	1/4

$S = x + y \Rightarrow$

y \ x	1	2	3
1	2	$\boxed{3}$	4
2	$\boxed{3}$	4	5
3	4	5	6

$E(X/S)$ $\quad \underline{S=3} \rightarrow (x=1)\ 1/4$

$(x=2)\ 1/2$

$(x=3)\ 1/4$

	X	1/4	1/2	1/4
	Y	1	2	3
1/4	1	2	3	4
1/2	2	3	4	5
1/4	3	4	5	6

S	2	3	4
P(S)	1/16	$2\times\frac{1}{2}\times\frac{1}{4}$ = 1/4	$\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{4}$

$$\frac{1}{16} + \frac{1}{4} + \frac{1}{16}$$

$$\frac{2}{16} + \frac{2}{4} = \frac{3}{8}$$

5	6
1/8	1/16

	1/16	1/4	3/8	1/8	1/16
S =	2	3	4	5	6
x =	1	1, 2	1, 2, 3.	2, 3	3
x = 1	1	1/2	1/6		
x = 2		1/2	2/3		
			1/6		

$$\frac{P(X=1, S=3)}{P(S=3)} = \frac{P(X=1, Y=2)}{1/4} = P(X=1)\,(P(Y=2))$$

$$= \frac{1/4 \times 1/2}{1/4} = 1/2$$

$$\frac{P(x=2, S=4)}{P(S=4)} = \frac{1/2 \times 1/2}{3/8} = 2/3$$

$$\frac{P(X=1, S=4)}{P(S=4)} = \frac{1/4 \quad 1/4}{3/8} = \frac{3}{8} \qquad \frac{1/16}{3/8}$$

$$\frac{P(X=1, S=4)}{P(S=4)} = \frac{P(X=1) \cdot P(Y=3)}{3/8} \quad \frac{1/4 \times 1/4}{3/8} = 2 \quad \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$\frac{P(X=2, S=4)}{P(S=4)} = \frac{P(X=2) \cdot P(Y=2)}{P(S=4)} = \frac{\frac{1}{2} \times \frac{1}{2}}{3/8} = \frac{1/4}{3/8_2} = \frac{2}{3}$$

$$\frac{1}{6} + \frac{2}{3} = \frac{1+4}{6} = \frac{5}{6} \to P(X=3/S=4)\ 1/6$$

$$E(X/S=2) = 1$$

$$E(X/S=3) = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5$$

$$E(\quad /S=4) = \_\_\_\_\_\_$$

$$E(X) = \cancel{E_X[E(X/Y)]} = E_Y[E_{X/Y}(X/Y)]$$

MGF of compound distributions $\quad Y = \sum_{i=1}^{N} X_i$

$E(X_i) = E(X)$

$$M_Y(t) = E(e^{tY}) = E_N\left(E_{Y/N}(e^{tY}/N=n)\right)$$

$$= E_N\left(E\left(e^{t\sum_{i=1}^{N} X_i} / N=n\right)\right)$$

$X_i =$ i.i.d.

$$= E_N\left(\prod_{i=1}^{N} E(e^{tX_i})\right)$$

$E(e^{tX}) = t$

$E(t^{X})$

$e^{\log t}$

$$= E_N\left([E(e^{tX})]^N\right)$$

$$M_Y(t) = E_N\left([M_X(t)]^N\right) = P_N(M_X(t))$$

$$= E_N\left(e^{N \cdot \log M_X(t)}\right) = M_N(\log M_X(t))$$

Ballot Problem

n, m

$\underline{\underline{n > m}}$

A		B
1	>	0
2	>	0,1
⋮	>	
	>	
$\underline{n}$	>	$\underline{m}$

$$P_{n,m} = \frac{n}{n+m} \cdot P_{n-1,m} + \frac{m}{n+m} P_{n,m-1} \qquad \boxed{\frac{n-m}{n+m}}$$

Ballot Problem

$$= \frac{n}{n+m} \cdot \frac{(n-1)-m}{(n-1)+m} + \frac{m}{n+m} \frac{(n)-(m-1)}{n+m-1}$$

$$= \frac{n^2 - n - \cancel{mn} + \cancel{mn} - m^2 + m}{(n+m)(m+n-1)} = \frac{n^2 - n - m^2 + m}{(n+m)(m+n-1)}$$

$$= \frac{(n^2 - m^2) - (n-m)}{(n+m)(n+m-1)} = \frac{(n-m)(n+m) - (n-m)}{(n+m)(n+m-1)}$$

$$= \frac{(n-m)}{(n+m)} \frac{\cancel{(n+m-1)}}{\cancel{(n+m-1)}}$$

Excercise 1.8) $X_1 \sim Poi(\lambda_1) \quad X_2 \sim Poi(\lambda_2) \quad X_1 + X_2 \sim ?$

$Z = X_1 + X_2$

$$M_Z(t) = M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^t - 1)}$$

$$= e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

$Z \sim Poi(\lambda_1 + \lambda_2)$

Conditional distribution of $X_1 / X_1 + X_2$

$$P(X_1 = x / X_1 + X_2) = \frac{P(X_1 = x,\ X_1 + X_2 = Z)}{P(X_1 + X_2 = Z)} = \frac{P(X_1 = x, X_2 = Z - x)}{P(Z)}$$

$$= \frac{P(X_1 = x) \cdot P(X_2 = Z - x)}{P(Z)}$$

$$= \frac{\cancel{e^{-\lambda_1}} \lambda_1^x}{x!} \cdot \frac{\cancel{e^{-\lambda_2}} \lambda_2^{Z-x}}{(Z-x)!} \Big/ \frac{\cancel{e^{-(\lambda_1+\lambda_2)}} (\lambda_1 + \lambda_2)^Z}{Z!}$$

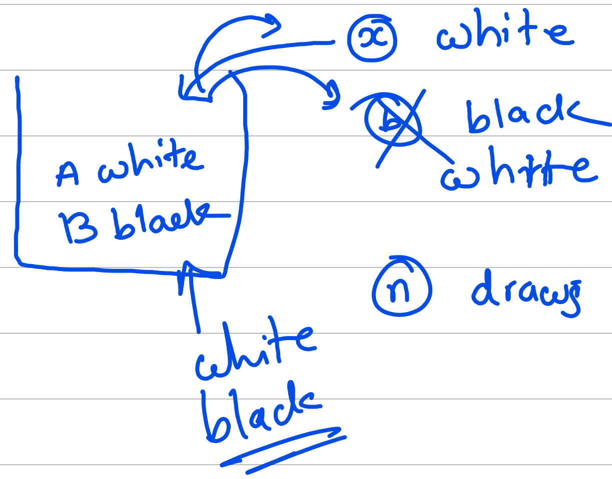
$$= \frac{z!}{x!(z-x)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-x}$$

$$= {}^zC_x$$

$$X_1 / X_1+X_2 \sim \text{Bino}\left(z, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

1.19 excercise

Urn



$a \quad b \qquad M_0 = a$

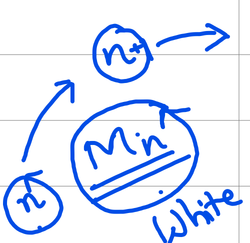
1 W|B W B W $\quad M_1 = a \times \frac{a}{a+b} + (a+1)\frac{b}{a+b} = \frac{a(a+b)}{(a+b)} + \frac{b}{(a+b)}$

M

$M_0 = a$ white ball

$$M_1 = a\left(\frac{a}{a+b}\right) + (a+1)\left(\frac{b}{a+b}\right)$$

$3+7$



$$M_{n+1} = M_n\left(\frac{?}{a+b}\right) + (M_n+1)\left(\frac{(a+b)-?}{a+b}\right)$$

$$= M_n \cdot \left(\frac{M_n}{a+b}\right) + (M_n+1)\left(\frac{(a+b)-M_n}{a+b}\right) \qquad 1-\frac{M_n}{a+b}$$

$$= \cancel{\frac{M_n^2}{(a+b)}} + (M_n+1) - \cancel{\frac{M_n^2}{(a+b)}} - \frac{M_n}{(a+b)}$$

① $M_{n+1} = 1 + M_n\left(1 - \frac{1}{a+b}\right)$ ② $M_0 = a$

$$M_n = a+b - b\left(1 - \frac{1}{a+b}\right)^n$$

$$M_1 = 1 + a\left(1 - \frac{1}{a+b}\right) = 1 + ax$$

$$M_2 = 1 + (1+ax)\cdot x = 1 + x + ax^2$$

$$M_3 = 1 + (1 + x + ax^2)x = 1 + x + x^2 + ax^3$$

$$= 1 + x + x^2 + x^3 + (a-1)x^3$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad = \left(\frac{1-x^3}{1-x}\right) + (a-1)x^3$$

$$\sum_{n=0}^{j} x^n = \frac{1-x^{j+1}}{1-x} \qquad M_n = \left(\frac{1-x^{n+1}}{1-x}\right) + (a-1)x^n$$

$$x = 1 - \frac{1}{a+b} \Rightarrow \frac{1}{1-x} = (a+b)$$

$$M_n = (a+b) - \frac{x^{n+1}}{1-x} + (a-1)x^n$$

$$= (a+b) - (a+b)x^{n+1} + (a-1)x^n$$

$$= (a+b) - x^n\left[(a+b)\cdot x - a + 1\right]$$

$$= (a+b) - \left(1 - \frac{1}{a+b}\right)^n\left[(a+b)\left(1 - \frac{1}{a+b}\right) - a + 1\right]$$

$$\cancel{a} + b - \cancel{1} - \cancel{a} + \cancel{1}$$

Sheldon Ross

$$= (a+b) - b\left(1 - \frac{1}{a+b}\right)^n$$

J. Medhi ↓

Stochastic Process

Seq^n of Random Variable

✓ X_n = fortune of gambler ~~a~~ after n^{th} game

$X_n = X_{n-1} + Z_n$ $\qquad$ $Z_n = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$

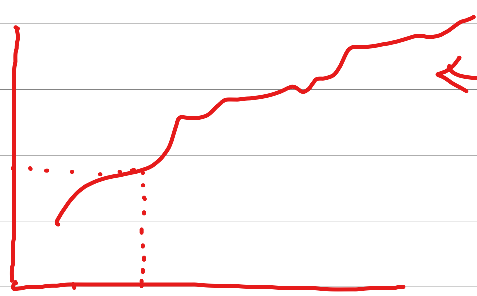
100 → 99, 101

Discrete Seq^n of Random

$X_0 \quad X_1 \quad X_2 \; \ldots \quad X_n \ldots$ $\qquad$ ✓ $\{X_i\}_{i=1}^{\infty}$

9.00 → 9.12 $\quad$ 9.12 — 9.15 $\quad$ 9-30 —— 3:30

SENSEX

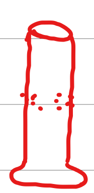


Time

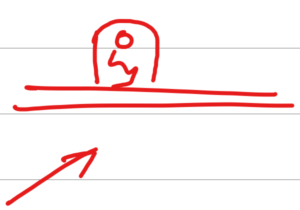
continous $\cancel{X_n}$ $\qquad$ $\{X_t\}_{t\in I}$

I = Time index set

Rain



Cash counter



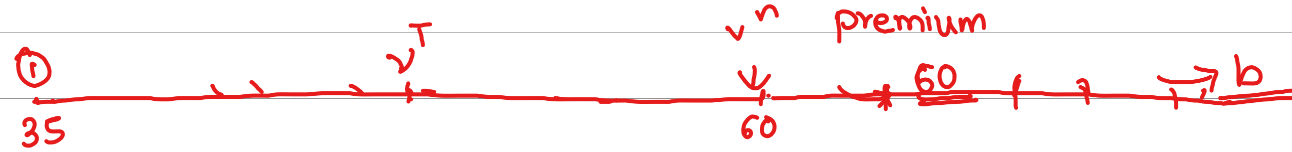
7th 8th 9th

20 10 0 Sm

t

	Statespace	Time domain	Example
1	Discrete	Discrete	* No. of covid patients on n^{th} day * No. of customers on a shop · " -
2	Discrete	Conti	* Pop^n size → conti time discrete pop^n * No. of accidents up to time t
3	Continuous	Discrete	* Milk by cow on n^{th} day * Min^m/Max^m temp on n^{th} day
4	Continuous	Contin.	* Fever - body temp on t time * Blood conc up to time t
5	Mixture	Discrete Cont	* Speed of internet * Rainfall up to time t

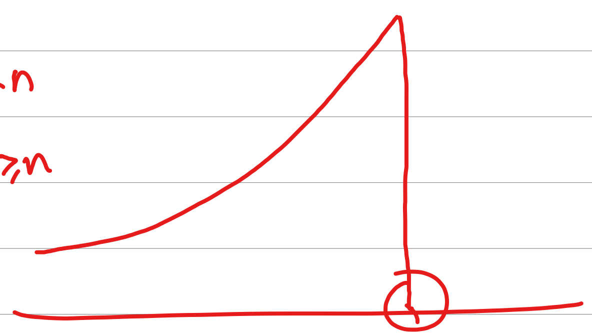
Endowment → $x \to n$ 35 → 60 → b



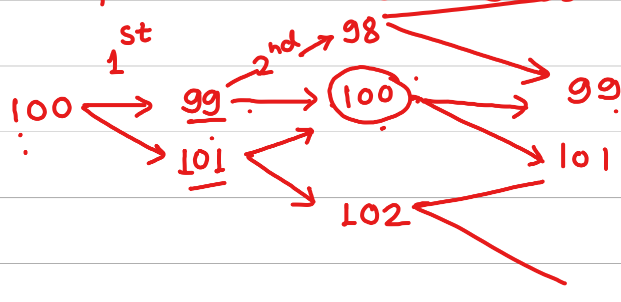
mixture

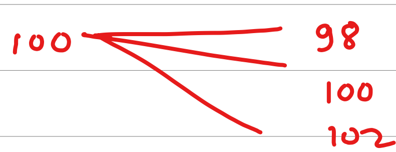
Present = v^T $T < n$

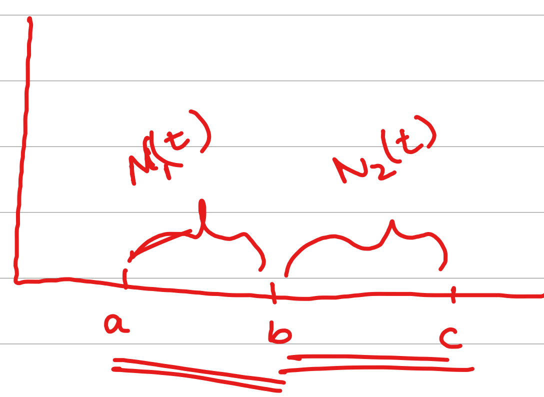
v^n $T \geq n$



Independent Increments







disjoint time interval

$N_1(t)$ & $N_2(t)$

dist^n indep

independent increments

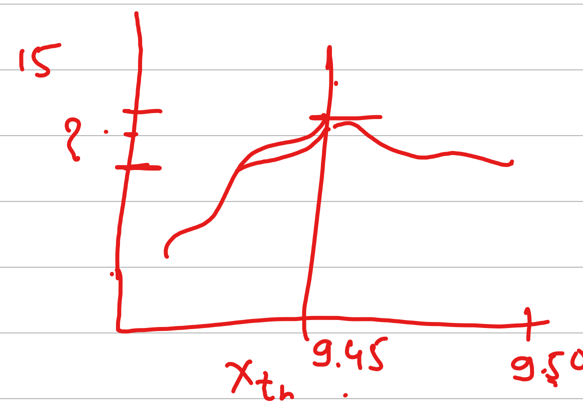
$t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n$

$\rightarrow X_{t_0}, X_{t_1}, X_{t_2}, \dots\dots < X_{t_{n-1}} < X_{t_n}$

$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1} \dots\dots, X_{t_n} - X_{t_{n-1}}$ indep. inde-

12 - 2

12 - 1 $\rightarrow$ 12.30 - 1.30



Joint same $\quad X_{t_0+u} \quad X_{t_1+u} \quad X_{t_2+u} \qquad X_{t_n}$

Joint $\quad t_0 \quad t_1 \quad t_2 \qquad t_{n-1} \quad t_n$

$(X_{t_0}, X_{t_1}, X_{t_2}, \dots X_{t_{n-1}}, X_{t_n})$

$X_{t_1} - X_{t_0} \quad X_{t_2} - X_{t_1} \qquad X_{t_n} - X_{t_{n-1}}$

independent $\qquad$ Stationarity $\quad$ strictly

$s < t \qquad u > 0$

increment $(s, t] \qquad X(t) - X(s) \qquad$ 12-2 . No. of students

✓ interval length $(t-s)$

$(s+u, t+u] \qquad X(t+u) - X(s+u) \qquad$ 7-9.

✓ interval length $(t-s)$

X_{t_0}

weak stationarity μ

$$\begin{cases} E(X(t)) = \mu & \text{constant } \forall t \in I \\ \operatorname{cov}(X(t), X(s)) = \operatorname{fun}(t-s) \end{cases}$$

Gambler → Random walk

Renewal

Stationarity

Probabilistic Replica

Find the p.g.f. of the sum Sn = X1 + + Xn of n independent and identical zero-truncated Poisson variates. Find E (Sn) and Pr {Sn = m}, m = n, n + 1, n + 2..

$X_i \sim$ zero-truncated poisson

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!} \Big/ (1-e^{-\lambda}) \rightarrow E(X)$$

$$S_n = \sum_{i=1}^{n} X_i \Rightarrow E(S_n) = n \cdot E(X) = \frac{n\cdot\lambda}{1-e^{-\lambda}}$$

$$P_S(t) = E(t^S) = E\left(t^{\sum_{i=1}^{n} x_i}\right) = [P_X(t)]^n$$

$$P_X(t) = \frac{1}{1-e^{-\lambda}} e^{-\lambda} \sum_{x=1}^{\infty} \frac{(\lambda t)^x}{x!}$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left[e^{\lambda t} \underbrace{\sum_{x=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^x}{x!}}_{\rightarrow 1} - e^{-\lambda t} \right]$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left[e^{\lambda t} - e^{-\lambda t} \right]$$

Markov Chains:-

discrete statespace & discrete time domain stochastic process following Markov property.

Markove Property:- Future depends on present not on past

Notation $P_{ij}^{(n)} = P[X_n = j / X_0 = i]$

Starting from state i process reaches to state j in n-steps

$\underset{0}{i}$ — 1 — 2 — 3 — ... — n-1 — $\underset{n}{j}$

Stationary → Time-homonegenous process

& Non-stationary

Non-stationary

$P_{ij}^{(m,n)}$ = Starting from i process reaches to state j from m to n steps (No. of steps - n-m)

$P_{ij}^{(m,n+m)} = P[X_{n+m} = j / X_m = i]$

statespace = $S = \{1, 0\}$ ⟹ 1 if it rains today 0 o.w.

0	0		1	1	1		0
1	2	3	8	9	15		30

$P_{10}^{(15)}$ P_{ij} $P_{10}^{(15,30)}$

~~Pf~~ ~~P_{ij}~~ $P[X_{n+1} = j / X_n = i] = P_{ij}^{1} = P_{ij}$

$X_n, n \geq 0$

$P[X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0]$

$P[\underbrace{X_{30} = 1}_{\text{future}} / \underbrace{X_{29} = 1}_{\text{present}}, \underbrace{X_{28} = 1, \ldots\ldots X_0 = 0}_{\text{past}}] = P[X_{30} = 1 / X_{29} = 1]$

$P_{ij}^{(1)} = P[X_{n+1}=j / X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] = P[X_{n+1}=j / X_n = i]$

$\{X_n, n \geq 0\}$ as Markov chain.

one-step transition prob. matrix

$$\begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \leftarrow \text{future state} \\ \begin{array}{c} \text{Current state} \end{array} \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \end{array}$$

$P_{00} \Rightarrow P[X_{n+1}=0 / X_n = 0]$

$P_{10} \Rightarrow P[X_{n+1}=0 / X_n = 1]$

starting ending

$P^{(2)}$ = Two step transition prob. matrix

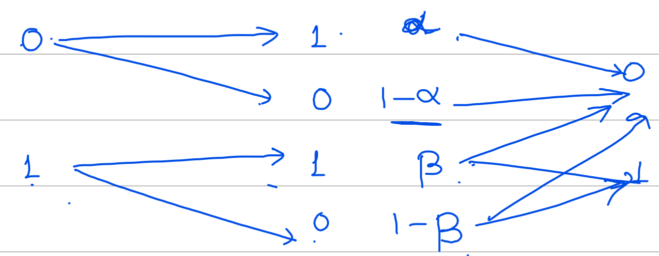
$$P^{(2)} = \begin{array}{c c} & \text{Future} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ \text{current state} \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} P_{00}^{(2)} & P_{01}^{(2)} \\ P_{10}^{(2)} & P_{11}^{(2)} \end{bmatrix} \end{array}$$

n-step

$S = \{0,1\}$ $P^{(n)} = \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} \\ P_{10}^{(n)} & P_{11}^{(n)} \end{bmatrix}$

$S = \{1, 2, \ldots, K\}$

Yesterday Today Tommarow

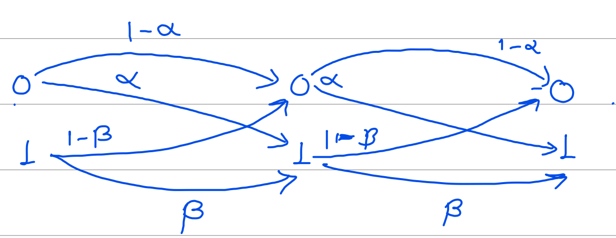


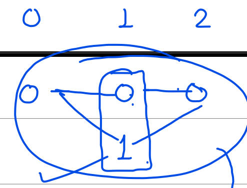
$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \end{array}$$

one-step

Yest Toda Tomm

Two-step





$= P_{00} \cdot P_{00} + P_{01} P_{10}$

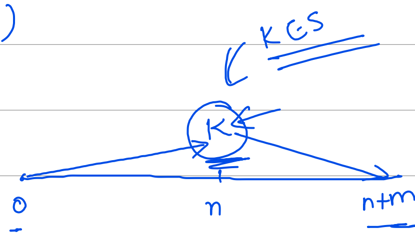
$= (1-\alpha)^2 + \alpha(1-\beta)$

$$\begin{array}{c|cc} & \text{Tommor} & \\ \text{Yest} & 0 & 1 \\ \hline 0 & & \\ 1 & & \end{array} \quad \begin{bmatrix} P_{00}^{(2)} & P_{01}^{(2)} \\ P_{10}^{(2)} & P_{11}^{(2)} \end{bmatrix}$$

n-step transition probabilities

Chapman- Kolmogorov's — (CK - eqn)

$$P_{ij}^{(n+m)} = \sum_{K \in S} P_{ik}^{(n)} P_{kj}^{(m)}$$



$\{X_n, n \geq 0\}$ is Markov chain?

$p^{(n+m)}$

$$P_{ij}^{(n+m)} = P[X_{n+m}=j / X_0 = i]$$

$$= \sum_{K \in S} P[\underbrace{X_{n+m}=j}_{\text{future}} / \underbrace{X_m = k}_{\text{present}}, \underbrace{X_0 = i}_{\text{past}}] \cdot P[X_m = k / X_0 = i]$$

$K \in S$

$$= \sum_{K \in S} P[X_{n+m}=j / X_m = k] \cdot P[X_m = k / X_0 = i]$$

stationary processes

$$= \sum_{k \in S} P[X_n = j / X_0 = k] \cdot P[X_m = k / X_0 = i]$$

$$= \sum_{k \in S} P_{kj}^{(n)} \cdot P_{ik}^{(m)}$$

if we have used ~~(~~ ~~it~~ in Matrix form

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

if $n = m = 1$,

$$P^{(1+1)} = P^{(2)} = P^{(1)} \cdot P^{(1)}$$

$$= \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix}$$

$$= \begin{bmatrix} (1-\alpha)^2 + \alpha(1-\beta) & \alpha[(1-\alpha)+\beta] \\ (1-\alpha)(1-\beta) + \beta(1-\beta) & (1-\alpha)(1-\beta) + \beta^2 \end{bmatrix}$$

$\{X_n, n \geq 0\}$ denotes whether it rains on n^{th} day or not

$$X_n = \begin{cases} 0 \\ 1 \end{cases}$$

Monday $\Rightarrow$ 0 ... 5

Monday $\to$ 1

Saturday $\to$ 1

one step transition

$$\begin{array}{c|cc} & \multicolumn{2}{c}{X_{n+1}} \\ X_n & 0 & 1 \\ \hline 0 & & \\ 1 & & \end{array} \quad \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}$$

Five step. $P^{(5)}$

$P[X_5 = 1 / X_0 = 1]$

$= P^{(5)} = P^{(4)} \cdot P$

$= P^{(3)} \cdot P^{(2)}$

$= P^{(3)} \cdot \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}$

$= P^{(3)} \frac{1}{6} \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

$= \frac{1}{6^2} P^{(3)} \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$

$= \frac{1}{6^2} P \cdot P^{(2)} \cdot \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$

$7 \times 2 \times 7 \times 3$

49×6

$= \frac{1}{6^4} P \cdot \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix} \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix}$

$21 \times (15 + 22)$

$21 \times (37)$

$= \frac{1}{6^4} P \begin{bmatrix} 778 & 518 \\ 777 & 519 \end{bmatrix}$ 1296 $= 1$ Row sum 1

217

$= \frac{1}{6^4} \begin{bmatrix} 778 & 518 \\ 777 & 519 \end{bmatrix} \times \frac{1}{6} \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

5-step transition prob. matrix

$$= \frac{1}{6^5} \begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} 4666 & 3110 \\ 4665 & \boxed{3111} \end{bmatrix}$$

(columns labelled 0, 1)

$P[X_5 = 1 / X_0 = 1] = \frac{3111}{6^5} = 0.4$

MC $\{X_n, n \geq 0\}$ Markov chain with statespace $S = \{0, 1, 2\}$

One-step TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} \end{array}$$

initial prob. ① $P[X_0 = 0] = P[X_0 = 1] = 1/2$?

② $X_1 = ?$

$P[X_3 = 0]$, $P[\quad = 1]$, $P[\quad 2]$ $\Big\}$ $E(X_3) = ?$

$P[X_1 = 1] = ?$ $= P[X_1 = 1 / X_0 = 0] \cdot P[X_0 = 0] + P[X_1 = 1 / X_0 = 1] \cdot P[X_0 = 1] + P[X_1 = 1 / X_0 = 2] \cdot P[X_0 = 2]$ $\Big\}$ $1/3 \times 1/2 + 1/3 \times 1/2 + 1/2 \times 0$

$= \sum_K P[X_1 = 1 / X_0 = K] \cdot P[X_0 = K]$

$=$

$= 1/3$

$P[X_1 = ?] = \alpha \cdot P$ $\qquad \alpha = [\alpha_0 \ \alpha_1 \ \alpha_2]'$

$$= [1/2 \ 1/2 \ 0] \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2}[1 \ 1 \ 0] \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

$P[X_1 = \quad] = \frac{1}{12}[3 \ 4 \ 5]$

$P[X_1 = 0] = 3/12$ $\qquad P(X_1 = 1)$ $4/12$ $\qquad P(X_1 = 2)$ $5/12$

$P[X_3 = ?] \Rightarrow [P(X_3) = 0 \quad P(X_3) = 1 \quad P(X_3) = 2] = \alpha \cdot P^3$

$$= \frac{1}{2}[1 \ 1 \ 0] \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} \cdot P^2 \qquad = \frac{1}{12}[3 \ 4 \ 5] P^2$$

$1 \times 3 \quad 3 \times 3$

$$= \frac{1}{12}[3 \ 4 \ 5] \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} P$$

9, 15, 6, 8, ~~3+16+15~~

$$= \frac{1}{72}[24 \ 14 \ 34] \frac{1}{6} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}$$

24 + 56

X_3

$$= \frac{1}{432}[174 \ 76 \ 182]$$

$$E(X_3) = \frac{1}{432}\left[0 \times 174 + 1 \times 76 + 2 \times 182\right]$$

$$= \frac{1}{432}\left[76 + 364\right] = \frac{1}{432}\left[440\right] > 1. = 1.0185$$

Solve This :)

Let the TPM of two state markov chain is $P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$

Show that, by mathematical induction

$$P^{(n)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{bmatrix}$$

Put $n=1$,

$$P^{(1)} = P = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^1 & \frac{1}{2}-\frac{1}{2}(2p-1) \\ \frac{1}{2}-\frac{1}{2}(2p-1)^1 & \frac{1}{2}+\frac{1}{2}(2p-1) \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Assume that it is true for $n=k$

$$P^{(k)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{bmatrix}$$

$$P^{(k+1)} = P^{(k)} \cdot P = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$= \begin{bmatrix} p\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right)+(1-p)\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right) & (1-p)\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right)+p\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right) \\ p\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right)+(1-p)\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right) & (1-p)\left(\frac{1}{2}-\frac{1}{2}(2p-1)^k\right)+p\left(\frac{1}{2}+\frac{1}{2}(2p-1)^k\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} \end{bmatrix}$$

$S = \{1, 2\}$

$$P = \begin{array}{c|cc} & 1 & 2 \\ 1 & 2/3 & 1/3 \\ 2 & 1/3 & 2/3 \end{array} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad P_{12}^{(3)} = ?$$

$$P^{(3)} = P^{(2)} \cdot P = \frac{1}{3^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad P$$

$$= \frac{1}{3^2}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3^3}\begin{array}{c|cc} & 1 & 2 \\ 1 & 14 & 13 \\ 2 & 13 & 14 \end{array}$$

$$\left(P^{(3)}\right)_{12} = \frac{13}{27} = 0.48$$

$$P = \begin{array}{c|cc} & 0 & 1 \\ 0 & 0.7 & 0.3 \\ 1 & 0.5 & 0.5 \end{array} = \frac{1}{10}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}, \quad P_{11}^{(4)} = ?$$

$$P^{(2)} = \frac{1}{10^2}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix} = \frac{1}{10^2}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}$$

$$P^{(4)} = P^{(2)} \cdot P^{(2)} = \frac{1}{10^4}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}\begin{bmatrix} 64 & 36 \\ 60 & 40 \end{bmatrix}$$

$$= \frac{1}{10^4}\begin{bmatrix} 6256 & 3744 \\ 6240 & \boxed{3760} \end{bmatrix}$$

$$P_{11}^{(4)} = \frac{3760}{10^4}$$

$$P = \begin{array}{c|cc} & 0 & 1 \\ 0 & & \\ 1 & & \boxed{} \end{array}$$

$$\begin{array}{cc} 0 & 1 \end{array}$$
$$\begin{bmatrix} 0 & 1 \end{bmatrix} \quad P^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{array}{c} 0 \\ 1 \end{array}$$

$$\frac{1}{10^2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix} P^2 \begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{10^2}\begin{bmatrix} 5 & 5 \end{bmatrix} P^2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{5}{10^4}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{5}{10^4}\begin{bmatrix}12 & 8\end{bmatrix}\begin{bmatrix}36\\40\end{bmatrix}$$

$$= \frac{5}{10^4}\begin{bmatrix}752\end{bmatrix} = \frac{3760}{10^4}$$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \end{array} \times \frac{1}{6}$$

$$P_{23}^{(3)}$$

$$= \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array} P^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$= \begin{bmatrix} 3 & 2 & 2 \end{bmatrix} P \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix} 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix} 14 & 17 & 11 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix} 76 \end{bmatrix} = \frac{76}{216}$$

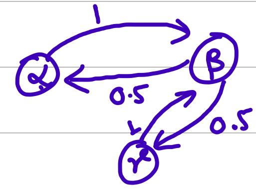
$C(\alpha) = \{\alpha, \beta, \gamma\}$ Irreducible

$$\begin{array}{ccc} \alpha & \beta & \gamma \end{array}$$

$$\begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$X_0 \Rightarrow \; = \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{c} \\ \alpha \\ \beta \\ \gamma \end{array}\begin{array}{c} \begin{array}{ccc} \alpha & \beta & \gamma \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$



$$P[X_4 = \gamma] = ? \Rightarrow \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} P^4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

P.Q.4.

$P[X_4 = \gamma / X_2 = \alpha] \Rightarrow [1\ 0\ 0]\ P^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Classifi

Periodicity is class property ie. $i \leftrightarrow j \Rightarrow d(i) = d(j)$

$\rightarrow \quad i \rightarrow j \quad P_{ij}^{n} > 0$ .. for some n

$j \rightarrow i \quad P_{ji}^{m} > 0$ for some m

$d(i) = \gcd\{t / P_{ii}^{t} > 0\}$ $\quad d(j) = \gcd\{t / P_{jj}^{t} > 0\}$

$P_{ii}^{(n+m)} = \sum_{k \in S} P_{ik}^{n} P_{ki}^{m} \geq P_{ij}^{n} P_{ji}^{m} > 0$

$P_{jj}^{(n+m)} = \sum_{k \in S} P_{jk}^{m} P_{kj}^{n} \geq P_{ji}^{m} P_{ij}^{n} > 0$

$\Rightarrow$ $n+m$ is multiple of $d(i)$ $\Rightarrow n+m = d(i) \times$ something

Similarly, $P_{jj}^{(n+m)} > 0 \Rightarrow n+m$ is multiple of $d(j)$

$\Rightarrow n+m = d(j) \times$ something

for some s, $P_{ii}^{(s)} > 0 \Rightarrow s = d(i) \times$ something —— (1)

$P_{jj}^{(n+m+s)} \geq P_{ji}^{(m)} P_{ii}^{(s)} P_{ij}^{(n)}$

$>0 \quad >0 \quad >0$

> 0

$\Rightarrow n+m+s = d(j) \times$ something

$\Rightarrow s = d(j) \times$ something —— (2) (as $d(j)$ divides $n+m$)

From (1) & (2)

$\Rightarrow d(i) = d(j)$

Transitivity :- $i \leftrightarrow j,\ j \leftrightarrow k \Rightarrow i \leftrightarrow k$

$i \leftrightarrow j$ $\quad P_{ij}^{n} > 0,\ P_{ji}^{m} > 0$ for some n, m

$j \leftrightarrow k$ $\quad P_{jk}^{s} > 0,\ P_{kj}^{t} > 0$ for some s, t

To show $i \leftrightarrow k$:-

$P_{ik}^{n+s} \geq \underbrace{P_{ij}^{n}}_{>0}\ \underbrace{P_{jk}^{s}}_{>0}$ $\qquad P_{ki}^{t+m} \geq \underbrace{P_{kj}^{t}}_{>0} \cdot \underbrace{P_{ji}^{m}}_{>0}$

> 0 $\qquad\qquad > 0$

$i \to k$ $\qquad\qquad k \to i$

$\Rightarrow i \leftrightarrow k$

Recurrent

① $F_i = 1$

② $\sum_n P_{ii}^{n} = \infty$

Starting from state i, Process reenters in the same state with probability 1

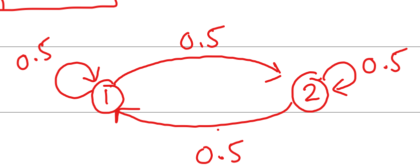
For any state i, F_i denote the probability that the process, starting from state i, will ever reaches state i.

State i is said to recurrent if $F_i = 1$, & transient if $F_i < 1$

$f_i^{(n)} \Rightarrow$ Prob. that starting from state i process reaches to state i in n steps first time

$$\boxed{F_i = \sum_n f_i^{n}}$$

Irreducible



$f_1^{1} = 0.5$ $\qquad$ ~~1-1-1~~

$f_1^{2} = 0.5^2$ $\qquad$ 1-2-1

$f_1^{3} = 0.5^3$ $\qquad$ 1-2-2-1

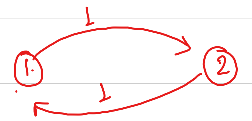
$\Rightarrow f_1^{n} = 0.5^n$

$$F_i = \sum_n f_i^{n} \Rightarrow = 0.5 + 0.5^2 + \ldots$$

$$= 0.5\left(1 + 0.5 + 0.5^2 + \ldots\right)$$

$$= 0.5\left(\frac{1}{1-0.5}\right) = \cancel{0.5} \cdot \frac{1}{\cancel{0.5}} = 1$$

1 & 2 recurrent



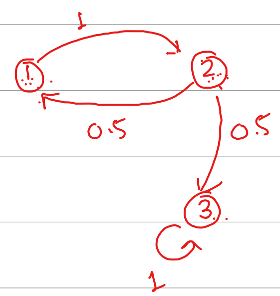
$f_1^{(1)} = 0$ $\qquad F_1 = f_1^{(1)} + f_1^{(2)} + 0 + 0$

$f_1^{(2)} = 1$ $\qquad = 0 + 1 + 0$

$f_1^{(3)} = 0$ $\qquad = 1$

$\Rightarrow$ 1 & 2 recurrent

Reducible



$c(1) = \{1,2\} = c(2)$ $\qquad c(3) = \{3\}$

$d(1) = \gcd\{2,4,6,\dots\}$ $\qquad d(3) = 1$

$= 2$

$= d(2)$

(Periodicity is class property)

$f_2^{1} = 0, \quad f_2^{2} = 0.5, \quad f_2^{3} = 0, \quad f_2^{4} = 0 \quad \dots$

$f_1^{1} = 0 \quad f_1^{2} = 0.5 \quad f_1^{3} = 0 \quad f_1^{4} = 0 \quad \dots$

$F_{2i} = 0.5 < 1 \quad \Rightarrow$ 1&2 ~~is~~ are transient states

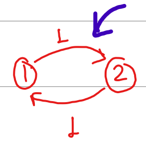
$F_3 = f_3^{(1)} + f_3^{(2)} + f_3^{(3)} + \dots = 1 + 0 + 0 + \dots = 1 \Rightarrow$ 3 is ~~recurrent~~ persistent

$P_{ii}^{(n)} \rightarrow$ Starting from i, process reaches to state i, in n steps

To prove

i recurrent iff $\sum_n P_{ii}^{n} = \infty$ | Transient iff $\sum P_{ii}^{n} < \infty$ ✓

$c(1) = \{1,2\} = c(2)$ Irreducible MC



← Recurrent ~~Transient~~

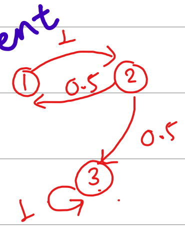
$\sum_n P_{11}^{(n)} = P_{11}^{1} + P_{11}^{2} + P_{11}^{3} + \dots = \sum_n P_{11}^{2n}$

$= 0 + 1 + 0 + 1 + \dots$ $\qquad$ infinite - Recurrent

$= \infty$

$c(1) = \{1,2\}$ ~~recurrent~~ transient

$c(3) = \{3\}$ Reducible MC



$\sum_n P_{11}^{(n)} = P_{11}^{1} + P_{11}^{2} + P_{11}^{3} + P_{11}^{4} + \dots = \sum_n P_{11}^{2n}$

$= 0 + 0.5 + 0 + 0.5^2 + \dots = \sum_n 0.5^n$

$= \dfrac{1}{1-0.5} = \dfrac{1}{0.5} = 2 < \infty$ $\Rightarrow$ Transient

Finite

(i) Recurrent iff $\sum P_{ii}^{(n)} = \infty$

Markov property

infinite → enters

infinite Process probabilistic replica — $F_{ii} = 1$, 0

i state transient, { starting from i ~~stat~~ process reaches to state i w.p. (<1)

$F_{ii} < 1$

$1 - F_{ii} > 0$ it will never return to state i

No. visits Finite no.

~ Geo

E(No of visits) ⇒ $\frac{1}{1-F_{ii}}$

- State i is recurrent if, with probability 1, a process starting from state i, will eventually return.
- However, by Markovian Property, the process probabilistically restarts itself upon returning to state i. Hence with prob. 1 it will return to i.
- Repeating this argument, with probability 1, the no. of visits to state i will be infinite & will thus have infinite expectation.

On the other hand, if state i is transient, there is positive prob., $1-F_{ii} > 0$, that it will never return again. Hence the no. of visits is Geometric with $E(n) = \frac{1}{1-F_{ii}}$

$\therefore$ State i is recurrent iff $E(\text{no. of visits to } i / X_0 = i) = \infty$

Let $I_n = \begin{cases} 1 & \text{if } x_n = i \\ 0 & \text{if } x_n \neq i \end{cases}$

$E(I_n) = P(I_n = 1 / X_0 = i)$
$= P(X_n = i / X_0 = i)$
$= P_{ii}^n$

$\therefore \sum_n I_n$ denotes no. of visits to i. ✓

$$E\left(\sum_n I_n / X_0 = i\right) = \sum_n E(I_n / X_0 = i)$$

$$= \sum_n P_{ii}^n$$

$\Rightarrow$ i is recurrent iff $\sum_n P_{ii}^n = \infty$ ✓

Theo $\boxed{\text{Recurrence is class property so does transience}}$

Recurrence
$\hookrightarrow$ $F_{ii} = 1$ or/and $\sum P_{ii}^n = \infty$

Transience
$F_{ii} < 1$ $\sum P_{ii}^n < \infty$

$i \leftrightarrow j \Rightarrow i$ is recurrent $\Rightarrow j$ is also recurrent

→ $i \leftrightarrow j$, $\odot$ $i \to j$ for some s, $P_{ij}^s > 0$
$j \to i$ for some t, $P_{ji}^t > 0$

i is recurrent $\Rightarrow \sum_n P_{ii}^n = \infty$

$$P_{jj}^{s+t+n} = \sum_{k \in S} P_{jk}^t P_{kk}^n P_{kj}^s \qquad < \sum_{n=0}^{\infty} P_{jj}^n = \infty$$

$$\geq \quad P_{ji}^{t} \; P_{ii}^{n} \; P_{ij}^{s}$$

$$\sum_n P_{jj}^{s+t+n} \geq \sum_n P_{ji}^{t} \; P_{ii}^{n} \; P_{ij}^{s}$$

$$\geq \underset{>0}{P_{ji}^{t}} \underset{\infty}{\left(\sum_n P_{ii}^{n}\right)} \underset{>0}{P_{ij}^{s}}$$

$$\Rightarrow \sum_n P_{jj}^{n} \geq \sum_n P_{jj}^{s+t+n} \geq \infty \qquad \cancel{<\infty}$$

$$\Rightarrow \sum_n P_{jj}^{n} = \infty$$

Transience is class property.

$i \leftrightarrow j$, if i is transient $\Rightarrow$ j is also transient.

$i \leftrightarrow j$ $\quad \cdot i \rightarrow j, \quad 1 \geq P_{ij}^{s} > 0$

$\quad j \rightarrow i, \quad 1 \geq P_{ji}^{t} > 0$

i is transient $\Rightarrow$ $\sum_n P_{ii}^{n} < \infty$

$$P_{jj}^{s+t+n} \geq P_{ji}^{t} \; P_{ii}^{n} \; P_{ij}^{s}$$

$$\sum P_{jj}^{s+t+n} \geq \boxed{\underset{\leq 1}{P_{ji}^{t}} \underset{<\infty}{\left(\sum P_{ii}^{n}\right)} \underset{\leq 1}{P_{ij}^{s}}} < \infty$$

$$\sum P_{jj}^{s+t+n} \leq \infty$$

Recurrence

$i \leftrightarrow j$ $\quad i$ transient $\Rightarrow j$ transient

By method of contradiction

i transient but j is recurrent

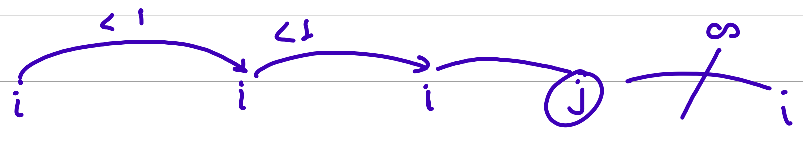
j recurrent $\Rightarrow$ i recurrent

which is contradiction

Irreducible MC :- Only one class, All states are communicating with each other.

Ergodic

Let μ_{jj} denotes the expected no. of transitions need to return to state j



Recurrent

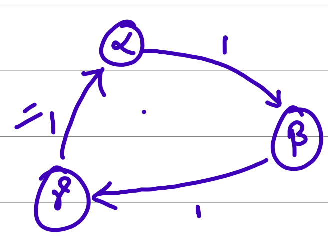
∘ $F_i = 1$

∘ $\sum P_{ii} = \infty$

$$\mu_{jj} = \begin{cases} \infty & \text{if } j \text{ is transient.} \\ \sum n f_i^n & \text{if } j \text{ is recurrent.} \end{cases}$$

$\mu_{jj} = \infty$ null recurrent $\qquad \mu_{jj} < \infty$ +ve recurrent

$$\begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline \alpha & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \\ \gamma & 1 & 0 & 0 \end{array}$$



First time

$\mu_{\alpha\alpha} = \sum_n n \cdot f_\alpha^n$

$= 1\cdot0 + 2\cdot0 + 3\cdot1 + 4\cdot0 + 0 \ldots$

$= 3$

$\mu_{\alpha\alpha} = 3 < \infty$

$\leftrightarrow$

positive recurrent

~~aperiodic~~

⇒ ~~ergodic~~

$C(\alpha) = \{\alpha, \beta, \gamma\} \Rightarrow$ Irreducible

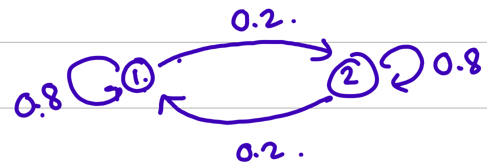
$F_\alpha = f_\alpha^1 + f_\alpha^2 + f_\alpha^3 + f_\alpha^4$

$= 0 + 0 + 1 + 0 \ldots$

$= 1 \quad \Rightarrow \quad \alpha, \beta, \gamma$ all are recurrent

$\mu_{\alpha\alpha} = 3 < \infty \quad \Rightarrow \quad \alpha$ is +ve recurrent

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.8 & 0.2 \\ 2 & 0.2 & 0.8 \end{array}$$



$C(1) = \{1,2\}$ $\Rightarrow$ Irreducible $\quad 1 \to 2 \to 2 \to 1$

First time ④ · ②

$$F_1 = f_1^1 + f_1^2 + f_1^3 + f_1^4 + \dots$$

④ ②

$$= 0.8 + 0.2^2 + 0.2^2 \cdot 0.8^1 + 0.2^2 \cdot 0.8^2 + \dots + 0.2^2 (0.8)^{n-2} + \dots$$

$$= 0.8 + 0.2^2 [1 + 0.8 + 0.8^2 + \dots] < \infty \qquad 0.8 < 1$$

$$= 0.8 + 0.2^2 \left[\frac{1}{1-0.8}\right]$$

$$= 0.8 + 0.2$$

$$= 1 \qquad \Rightarrow \quad 1, 2 \text{ are recurrent states}$$

$$\mu_{11} = \sum_n n \cdot f_1^n = 1 \cdot 0.8 + 2 \cdot 0.2^2 + 3 \cdot 0.2^2 \cdot 0.8 + \dots + n \cdot 0.2^2 \cdot 0.8^{n-2} + \dots$$

$$= 0.8 + 0.2^2 [2 + 3 \cdot 0.8 + \dots + n \cdot 0.8^{n-2} + \dots]$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{d}{dx} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2}$$

$$= 0.8 + 0.2^2 \left[\sum_n (n+1) x^{n-1}\right] = 0.8 + 0.2^2 \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1} + 0.2^2 \sum_{n=1}^{\infty} x^{n-1}$$

$$= 0.8 + \cancel{(0.2)^2} \cdot \frac{1}{\cancel{(1-0.8)^2}} + 0.2^2 \cdot \sum_{n=0}^{\infty} x^n = 0.8 + 1 + 0.2$$

$$< \infty$$

1, 2 +ve recurrent.

* Ergodic state :- Aperiodic, positive recurrent

i ergodic state $\Rightarrow$ $d(i) = 1$, $\bar{F}_i = 1$ or $\sum P_{ii}^n = \infty$

$\mu_{ii} < \infty$

* Ergodic MC $\Rightarrow$ All states are ergodic.

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$C(1) = \{1,2\}$ 1,2, positive recurrent

$d(1) = \gcd(1,2,\ldots) = 1 = d(2)$

1,2 positive recurrent, aperiodic $\Rightarrow$ ergodic

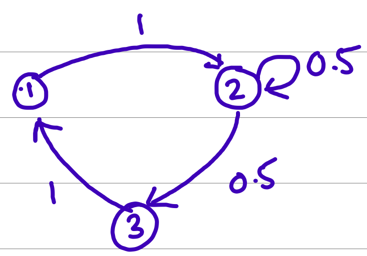
$$\begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

states:- Aperiodic   Recurrent   Transient

+ve   null

Ergodic

$\Rightarrow$ MC Ergodic ?

$\Rightarrow$ 

$C(1) = \{1,2,3\}$ $\Rightarrow$ Irreducible

$d(2) = \gcd\{1,2,\ldots\} = 1$ $\Rightarrow$ Aperiodic

Periodicity is class property $\Rightarrow$ All states are aperiodic.

$$F_2 = f_2^1 + f_2^2 + \ldots = 0.5 + 0 + 0.5\cdot 1\cdot 1 + 0 + \ldots$$
$$= 0.5 + 0.5$$
$$= 1 \quad \Rightarrow \text{Recurrent}$$

$\because$ Recurrence is class property. $\Rightarrow$ All states are recurrent.

$$\mu_2 = 1\cdot f_2^1 + 2\cdot f_2^2 + 3\cdot f^3 + 4\cdot f_2^4 + \ldots$$
$$= 1\cdot 0.5 + 0 + 3\cdot 0.5 + 0 + \ldots$$
$$= 0.5 + 1.5$$
$$= 2 < \infty \quad \Rightarrow \text{Positive Recurrent}$$

All states are positive recurrent & aperiodic, i.e. all states are ergodic
$\therefore$ MC is ergodic MC.

Limit Theo. :-

If j is transient, then $\sum_n P_{jj}^n < \infty$

$P_{jj}^n \to 0$ as $n \to \infty$

We have only finite no. of transitions returning to j.

$\pi_j = \lim_{n\to\infty} P_{jj}^n$ $\qquad$ $\pi_{ij} = \lim_{n\to\infty} P_{ij}^n$

✓ +ve recurrent state $\pi_j > 0$

null recurrent $\pi_j = 0$?

Stationary

CK eq^n

$$P^{n+1} = P^n \cdot P$$

$$\lim_{n\to\infty} P_{ij}^{(n+1)} = \lim_{n\to\infty} \sum_{k\in S} P_{ik}^{(n)} \cdot P_{kj}^{(1)}$$

$$\pi_{ij} = \sum_{k\in S} \pi_{ik} P_{kj}$$

In Matrix form

$$P^{n+1} = P^n \cdot P$$

$$\left(\lim_{n\to\infty} P^{n+1}\right) = \left(\lim_{n\to\infty} P^n\right) \cdot P$$

$$\pi = \pi \cdot P$$

Recurrent limiting $\longrightarrow$ π = Stationary dist^n

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad \underline{\underline{\pi}} = [\pi_1, \pi_2, \pi_3]'$$

$$\underline{\pi}' = \underline{\pi}' P$$

$$[\pi_1 \ \pi_2 \ \pi_3] = [\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\pi_1 = \pi_3$$

$$\pi_2 = \pi_1 + \pi_2/2 \Rightarrow \pi_1 = \pi_2/2$$

$$\pi_3 = \pi_2/2$$

$$\underline{\pi} = \underline{\underline{\left[\frac{\pi_2}{2}, \pi_2, \frac{\pi_2}{2}\right]}} \Rightarrow \sum_{i=1}^{3} \pi_i = 1 \Rightarrow$$

$$\Rightarrow \frac{\pi_2}{2} + \pi_2 + \frac{\pi_2}{2} = 2\pi_2 = 1$$

$$\pi_2 = 1/2$$

✓ $$\underline{\pi} = [1/4, 1/2, 1/4]$$

$$P = \frac{1}{10}\begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix} \qquad \pi = ?$$

$$\underline{\pi} = \underline{\pi} P$$

$$[\pi_1 \ \pi_2 \ \pi_3] = [\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix} \frac{1}{10}$$

$$10\pi_1 = 3\pi_1 + 4\pi_2 + 3\pi_3 \Rightarrow 7\pi_1 = 4\pi_2 + 3\pi_3 \Rightarrow 7\pi_1 = 4\pi_2 + \frac{3}{2}(\pi_1 + \pi_2)$$

$$10\pi_2 = 4\pi_1 + 3\pi_2 + 3\pi_3 \Rightarrow 7\pi_2 = 4\pi_1 + 3\pi_3$$

$$10\pi_3 = 3\pi_1 + 3\pi_2 + 4\pi_3 \Rightarrow 6\pi_3 = 3\pi_1 + 3\pi_2 \Rightarrow \underline{2\pi_3} = \pi_1 + \pi_2$$

$$11\pi_1 \ \cancel{14\pi_1} = 8\pi_2 + \cancel{3\pi_1} + 3\pi_2 \Rightarrow \pi_1 = \pi_2$$

$$\underline{\pi} = [\pi_1 \;\; \pi_2 \;\; \pi_3] = [\pi_1 \;\; \pi_1 \;\; \pi_1]$$

$$\sum \pi_i = 1 \;, \quad \pi_1 + \pi_1 + \pi_1 = 1 \;\Rightarrow\; \pi_1 = 1/3$$

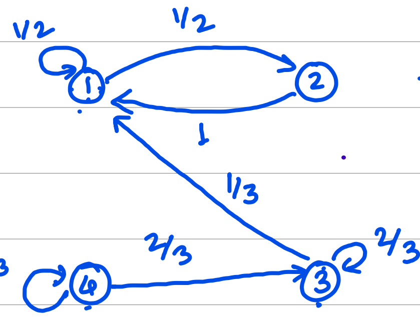
$$\pi = [1/3 \;\; 1/3 \;\; 1/3]$$

① $$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 6 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix} \Rightarrow ? \quad \underline{[x, x, y, y]} \quad \underline{\underline{x, y}} \in \underline{\underline{(0,1)}}$$

$$\underline{x + y = 1/2}$$

Stationary distribution may or may not be unique.

② $$\frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$



Transition Graph

$c(1) = \{1, 2\}$, $c(3) = \{3\}$, $c(4) = \{4\}$ $\Rightarrow$ Reducible MC

$d(1) = 1$ $\quad d(3) = 1$ $\quad d(4) = 1$ $\Rightarrow$ aperiodic

$$F_1 = f_1^1 + f_1^2 + f_1^n + \dots$$
$$= \frac{1}{2} + \frac{1}{2}\cdot 1 + 0 + \dots$$
$$= 1$$

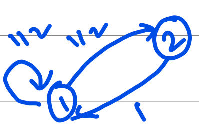
$\Rightarrow F_1 = 1 \Rightarrow$ 1,2 are recurrent

$$F_3 = f_3^1 + f_3^2 + f_3^3 + \dots$$
$$= \frac{2}{3} + 0 + 0 + \dots$$
$$= \frac{2}{3} < 1$$

$\Rightarrow$ State 3 is transient

$$F_4 = f_4^1 + f_4^2 + f_4^3 + \dots$$
$$= \frac{1}{3} + 0 + \dots$$
$$= \frac{1}{3} < 1$$

$\Rightarrow$ 4 is transient



$$\mu_{11} = \sum_n n f_1^n = 1\cdot f_1^1 + 2\cdot f_1^2 + 0 = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5 < \infty$$

Recurrence is class property

$$F_2 = f_2^1 + f_2^2 + f_2^3 + f_2^4 + \dots + f_2^n + \dots$$
$$= 0 + \frac{1}{2} + \frac{1}{2}\cdot 1\cdot \frac{1}{2} \qquad + \frac{1}{2}\cdot 1\cdot \left(\frac{1}{2}\right)^{n-2} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right)$$
$$= \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) = 1 \quad \Rightarrow \quad 2 \text{ is recurrent.}$$

$$\mu_2 = \sum n f_2^n \Rightarrow \quad 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2^2} + \dots + n\cdot\frac{1}{2^{n-1}} + (n+1)\frac{1}{2^n}$$
$$\Rightarrow \sum (n+1)\frac{1}{2^n} = \sum n\cdot\frac{1}{2^n} + \sum\frac{1}{2^n} < \infty$$

$\Rightarrow$ 2 is positive recurrent

1,2, aperiodic, positive recurrent $\Rightarrow$ Ergodic States

Lets find out limiting distribution / Stationary

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4] = [\pi_1\ \pi_2\ \pi_3\ \pi_4]\ \frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$6\pi_1 = 3\pi_1 + 6\pi_2 + 2\pi_3 \quad \cancel{6\pi_1 = 3\pi_1 + 3\pi_1}$$
$$6\pi_2 = 3\pi_1 \qquad \Rightarrow \pi_1 = 2\pi_2$$
$$6\pi_3 = 4\pi_3 + 4\pi_4 \Rightarrow \pi_3 = 0$$
$$6\pi_4 = 2\pi_4 \qquad \Rightarrow \pi_4 = 0 \quad \Rightarrow \quad 4\pi_4 = 0 \Rightarrow \pi_4 = 0$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \Rightarrow \quad 2\pi_2 + \pi_2 + 0 + 0 = 1 \Rightarrow \pi_2 = 1/3$$
$$\Rightarrow \pi_1 = 2/3$$

Stationary Distribution $[1/3 \quad 2/3 \quad 0 \quad 0]$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 1 & 1 \\ & & 1 & 1 \end{bmatrix}$$

$\underline{\underline{\pi}} = [\pi_1 \ \pi_1 \ \pi_3 \ \pi_3]$

$\pi_1 + \pi_1 + \pi_3 + \pi_3 = 1$

$\Rightarrow \underline{\underline{\pi_1 + \pi_3 = 1}}$

$\underline{\underline{\pi_1 = x}}$

unique

$[x \ x \ 1-x \ 1-x]$

$\lim_{n\to\infty} P_{ij}^{n}$ $\quad$ $i \to j$ in nsteps

$\to \underline{\underline{\pi_j}}$ ✓ $\quad$ $i \to$ doesn't matter

$\pi' = \pi' P$ eq^n $\to$ ✓

$\pi' = \pi' P$ $\quad$ $\underset{\cdot}{A}\underset{\cdot}{\underline{x}} = \underline{b}$,

$I\underline{\pi} = P'\underline{\pi}$

$\Rightarrow (P' - I)\underline{\pi} = \underline{0}$ $\quad$ $\pi_1 + \pi_2 + \pi_3 = 1$

$$[1 \ 1 \ 1]\begin{matrix}\pi_1 \\ \pi_3 \\ \pi_3\end{matrix} = 1$$

$$A = \begin{bmatrix} P' - I \\ \text{ones} \end{bmatrix} \qquad b = \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$$

No limiting distribution exists

$$\begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$$

$P^n \Rightarrow$ $P^{2n} = I$, $P^{2n+1} = P$

$\pi' = \pi' P$

initial $= \pi$ π

$\cancel{P_2 = \pi P P^2}$

$P(X_0 = x) = \alpha$ $\lim_{n\to\infty} P(X_n = x) \to \pi$

x_1 αP x_2 $\alpha . P^2$

$\alpha = \pi$ x $\pi P = P$ $\pi . P^2$

$\pi . P . P^2$

$P P^2$

1 0 0 1

Theo.:- An irreducible aperiodic MC belongs to one of the following classes :-

① Either the states are all transient or all null recurrent in this case, $P_{ij}^n \to 0$ $\forall$ i, j and $\exists$ no stationary distⁿ

② Or else, all states are positive recurrent, that is ($F_j = 1$, $\mu_{jj} < \infty$)

$$\pi_j = \lim_{n\to\infty} P_{ij}^n > 0$$

In this case, $\{\pi_j, j = 0, 1, 2 \ldots\}$ is stationary distⁿ & $\exists$ no other stationary distribution.

j $n\to\infty$ an $P_{ij}^n \to 0$ MC

Transient $S = \{1, 2, 3\}$ ∞ time state

Proof :- We will prove (ii)

$$\sum_{j=0}^{M} P_{ij}^n \le \sum_{j=0}^{\infty} P_{ij} = 1 \qquad \forall \text{ all } M \in S$$

letting $n\to\infty$ yields

$$\sum_{j=0}^{M} \pi_j \leq 1 \qquad \forall\, M$$

implying that

$$\sum_{j=0}^{\infty} \pi_j \leq 1$$

Now

$$P_{ij}^{n+1} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj} \geq \sum_{k=0}^{M} P_{ik}^{n} P_{kj} \qquad \forall\, M \in S$$

letting $n \to \infty$ yields

$$\lim_{n\to\infty} P_{ij}^{n+1} \geq \lim_{n\to\infty} \sum_{k=0}^{M} P_{ik}^{n} P_{kj} \qquad \forall\, M \in S$$

implying that

$$\pi_j \geq \sum_{k=0}^{M} \left(\lim_{n\to\infty} P_{ik}^{n}\right) P_{kj} \qquad \forall\, M \in S$$

$$\pi_j \geq \sum_{k=0}^{M} \pi_k \cdot P_{kj} \qquad \forall\, M \in S$$

$$\Rightarrow \quad \pi_j \geq \sum_{k=0}^{\infty} \pi_k P_{kj} \qquad \forall\, j \geq 0$$

To prove equality, assume strict inequality for some j

$$\Rightarrow \quad \pi_j > \sum_{k=0}^{\infty} \pi_k P_{kj}$$

$$\Rightarrow \quad \sum_{j=0}^{\infty} \pi_j > \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \pi_k$$

$$\Rightarrow \quad \sum_{j=0}^{\infty} \pi_j > \sum_{k=0}^{\infty} \pi_k \qquad \text{[Which is contradiction]}$$

$$\Rightarrow \boxed{\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}} \quad \forall\, j \in S = \{0,1,2,\ldots\}$$

Putting $P_j = \dfrac{\pi_j}{\sum_k \pi_k}$ $\Rightarrow$ P_j is stationary dist^n

$\therefore$ At least one stationary dist^n exists.

Now let $\{P_j, j=0,1,2,\ldots\}$ be any stationary distribution.

Then if $\{P_j, j=0,1,2,\ldots\}$ is the prob. dist^n of X_0 (Initial Prob. dist^n)

$$P_j = P\{X_n = j\}$$

$$= \sum_{i=0}^{\infty} P\{X_n = j / X_0 = i\} \; P\{X_0 = i\}$$

$$P_j = \sum_{i=0}^{\infty} P_{ij}^{n} P_i \qquad \text{——— (1)}$$

$$\Rightarrow \quad P_j \geq \sum_{i=0}^{M} P_{ij}^{n} P_i \qquad \forall\, M \in S$$

Let $n \to \infty$ and $M \to \infty$

$$P_j \geq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j \left(\sum_{i=0}^{\infty} P_i\right) = \pi_j \qquad \text{——— (2)}$$

$$\Rightarrow \quad P_j \geq \pi_j$$

Lets prove other side.

$$P_j = \sum_{i=0}^{\infty} P_{ij}^{n} P_i = \sum_{i=0}^{M} P_{ij}^{n} P_i + \sum_{i=M+1}^{\infty} P_{ij}^{n} \cdot P_i$$

$$\leq \sum_{i=0}^{M} P_{ij}^{n} P_i + \sum_{i=M+1}^{\infty} P_i \qquad (\text{as } P_{ij}^{n} \leq 1)$$

$$\lim_{n\to\infty} P_j \leq \sum_{i=0}^{M} \left(\lim_{n\to\infty} P_{ij}^{n}\right) \cdot P_i + \sum_{i=M+1}^{\infty} P_i$$

$$P_j \leq \sum_{i=0}^{M} \Pi_j \cdot P_i + \sum_{i=M+1}^{\infty} P_i$$

$$\leq \Pi_j \left(\sum_{i=0}^{M} P_i\right) + \sum_{i=M+1}^{\infty} P_i$$

letting $M \to \infty$

$$P_j \leq \Pi_j \left(\sum_{i=0}^{\infty} P_i\right) + 0$$

$$P_j \leq \Pi_j \quad \text{--- (3)}$$

from (2) & (3) $\Rightarrow$ $P_j = \Pi_j \quad j = 0, 1, 2, \ldots,$ is only stationary distⁿ.

If all states are transient or null recurrent and $\{P_j, j = 0, 1, 2, \ldots\}$ is stationary distⁿ, then (1) eqⁿ holds i.e.

$$P_j = \sum_{i=0}^{\infty} P_{ij}^{n} P_i$$

and $P_{ij}^{n} \to 0$, which is impossible.

Thus for case (1), no stationary distⁿ exists.

Note:-

* In the irreducible, positive recurrent, we have that $\Pi_j, j \geq 0$ are unique non-negative solution of

$$\Pi_j = \sum_i \Pi_i P_{ij} \quad \& \quad \sum_j \Pi_j = 1$$

i.e. In long run, Π_j is the proportion of time that the MC is in state j.

$$\Rightarrow \quad \pi_j = \frac{1}{\mu_{jj}}$$

$$\Rightarrow \quad \lim_{n\to\infty} P_{jj}^{n \cdot d(j)} = \frac{d}{\mu_{jj}} = d \cdot \pi_j \qquad (d = d(j) \ \forall j)$$

Prob. Distⁿ $\{P_j, j \geq 0\}$ is said to be stationary for MC if

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij} \qquad j \geq 0$$