# Infinite Series :-

$\sum x_n$ $\quad \sum_{i=1}^{\infty} x_n = ?$

$S_1 = x_1$

$S_2 = x_1 + x_2$

$S_n = \sum_{i=1}^{n} x_i$

$S_n \to S$

$S_n \to S$

$S_{n+1} \to S$

$\lim_n S_n - S_{n-1} = 0$

$\lim_n \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i = 0$

$\lim_n x_n = 0$

$\sum \frac{1}{n}$ $\quad \sum \frac{(-1)^n}{n}$ $\quad \sum x^n = \frac{1}{1-x}$ $\quad |x|<1$

$= \sum \frac{1}{n}$

$= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$

$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}$

$\geq 1 + \frac{1}{2}\left[1 + 1 + 1 + \cdots\right.$

$\geq \infty$

$\sum \frac{(-1)^{n+1}}{n}$

$= \underbrace{1 - \frac{1}{2}} + \underbrace{\frac{1}{3} - \frac{1}{4}} + \underbrace{\frac{1}{5} - \frac{1}{6}}$

$S_n = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots$

$= 1 - \left(\frac{1}{2 \cdot 3}\right) - \left(\frac{1}{4 \cdot 5}\right)$

$\quad\quad >0 \quad\quad >0$

$S_n \downarrow$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}} + \dots\dots + \cancel{\tfrac{1}{n}} - \frac{1}{n+1} \longrightarrow 0$$

$\Rightarrow 1$

$\sum \frac{(-1)^n}{n}$ cgt $\quad \frac{(-1)^n}{n} \to 0$

$\sum r^n \quad |r|<1$ cgt $\quad r^n \to 0 \quad 0.5^2$

$\sum \frac{1}{n^2}$ cgt $\quad \frac{1}{n^2} \to 0$

$\sum \frac{1}{n}$ dvge $\quad \frac{1}{n} \to 0$

Cauchy Criterion for convergence of series

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for any $\varepsilon>0$ $\exists\, K(\varepsilon) \in \mathbb{N}$, $\ni$ $|S_n - S_m| < \varepsilon$ $\forall\, n \geq m \geq K(\varepsilon)$

$$\Rightarrow \left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} x_i \right| < \varepsilon \quad \forall\, n \geq m \geq K(\varepsilon)$$

$$\Rightarrow \left| \sum_{i=m+1}^{n} x_i \right| < \varepsilon$$

Let xn be a sequence of non-negative real numbers then the series P \sum xn converges if and only if the sequence Sk of partial sum is bounded.

① Let $\sum_{n=1}^{\infty} x_n$ converges, $\sum_{n=1}^{\infty} x_n = s$

$\Rightarrow$ $S_n = \sum_{i=1}^{n} x_i$ = seq$^n$ of partial sums

$\Rightarrow$ $S_n$ is convergent to $s$.

$\Rightarrow$ Every convergent seq$^n$ is bounded

$\Rightarrow$ $S_n$ is bounded.

② $S_n$ is bounded $\Rightarrow$ To prove $S_n$ is cgt.

$S_{n+1} = \sum_{i=1}^{n+1} x_i = S_n + x_{n+1}$ as $x_{n+1} \geq 0$

$S_{n+1} \geq S_n$

$S_n$ is monotonically $\uparrow$ & bounded

by MCT it is cgt.

Show that $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots = \frac{1}{1-r}$ if $|r| < 1$

$S_{n+1} = \sum_{i=1}^{n+1} r^i = 1 + r + r^2 + \quad + r^{n-1} + r^n$

$S_n = \sum_{i=1}^{n} r^i = 1 + r + r^2 + \ldots + r^{n-1}$

$r S_n = r + r^2 + \ldots + r^n = S_{n+1} - 1$

$1 + r S_n = S_{n+1}$ ?

$\lim (1 + r \cdot S_n) = \lim S_{n+1}$

$1 + r \cdot S = S$ $\Rightarrow$ $S = 1/(1-r)$ ?

$p>1$

$$\sum_n \frac{1}{n^p} = 1+\underbrace{\frac{1}{2^p}+\frac{1}{3^p}}+\underbrace{\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}}+\underbrace{\frac{1}{8^p}}+\dots$$

$$\leq 1+\underbrace{\frac{1}{2^p}+\frac{1}{2^p}}+\underbrace{\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}}+\underbrace{\frac{1}{8^p}}+\dots$$

$$\leq 1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\dots$$

$$\leq 1+\frac{2}{2^p}+\frac{2^2}{2^{2p}}+\frac{2^3}{2^{3p}}+\dots$$

$$\leq 1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+\dots$$

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \leq \sum_{n=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^n$$

as $p>1$, $\frac{1}{2^{p-1}}<1 \Rightarrow \sum r^n \to \frac{1}{1-r}$ cgt.

$\Rightarrow \sum \frac{1}{n^p}$ is also cgt.

Comparison Test :- $K\in\mathbb{N} \ni \underline{0\leq x_n \leq y_n} \ \forall\, n\geq K$
$\sum y_n$ cgt. $\Rightarrow \sum x_n$ cgt

by Cauchy criterion for convergence of series
for any $\varepsilon>0$ $\exists M(\varepsilon)\in\mathbb{N}$. $|y_{m+1}+y_{m+2}+\dots+y_n|<\varepsilon$ $\forall n>m>M(\varepsilon)$

$K'(\varepsilon) = \max(M(\varepsilon), K)$

for $n, m \geq K'(\varepsilon)$

$$|y_{m+1} + \ldots + y_n| < \varepsilon$$

by $\Delta$ inequality

$$|x_{m+1} + x_{m+2} + \ldots + x_n| < |y_{m+1} + \ldots + y_n| < \varepsilon$$

by Cauchy criterion $\Sigma x$ converges.

$\Sigma \frac{1}{n^2+n}$ $\qquad \Sigma x_n$ $\qquad \Sigma y_n = \Sigma \frac{1}{n^2}$ $\rightarrow$ cgt

$$r = \lim \frac{x_n}{y_n} = \lim \frac{1}{n^2+n} \times n^2 = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

$r \neq 0$

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$0 < x_n, \quad 0 < y_n \quad \forall n$

$$r = \lim \frac{x_n}{y_n}$$

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$ $\ni \left| \frac{x_n}{y_n} - r \right| < \varepsilon$

$$r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$$

$$y_n(r-\varepsilon) \leq x_n \leq y_n(r+\varepsilon)$$

$x_n \leq y_n$, $\quad \Sigma y_n$ cgt $\Rightarrow$ $\Sigma x_n$ cgt if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt

if $\sum x_n$ cgt $\Rightarrow$ $\sum y_n$ cgt

if $r=0$, $r-\varepsilon < \frac{x_n}{y_n} < r+\varepsilon$

$-\varepsilon < \frac{x_n}{y_n} < \varepsilon$

$-\varepsilon < 0 < \frac{x_n}{y_n} < \varepsilon$

$x_n \leq y_n \cdot \varepsilon$

$\Rightarrow$ if $\sum y_n$ cgt $\Rightarrow$ $\sum x_n$ cgt.

$\sum \frac{(-1)^n}{n}$ cgt $\qquad \sum \frac{1}{n}$ divergent

Absolute: $\sum x_n$ abs. cvgt if $\sum |x_n|$ is cvgt.

Conditional Convergence $\sum x_n$ is cgt but $\sum |x_n|$ is not cgt.

$\sum \frac{(-1)^n}{n^2}$ abs cgt $\qquad \sum \frac{1}{n^2}$ cgt

for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) \in \mathbb{N}$.

$|x_{m+1} + \ldots + x_n| \leq ||x_{m+1}| + |x_{m+2}| + \ldots + |x_n|| < \varepsilon$ $\quad \forall$ ~~$m>n>$~~ $n > m > K(\varepsilon)$

$\sum |x_n|$

Root test $\quad r = \lim |x_n|^{1/n}$ exists.

① $r<1$, $\exists$ $r < r_1 < 1$

for some $K(\varepsilon) \in \mathbb{N}$.

$$|x_n|^{1/n} \leq r_1 \quad \forall n \geq K(\varepsilon)$$

$$\Rightarrow \quad |x_n| \leq r_1^n \quad (r_1 < 1)$$

by Comparison Test

$$\sum |x_n| \leq \sum r_1^n$$

cgt. cgt.

② $r > 1$, $\exists$ $1 < r_1 < r$

$$1 \leq r_1 \leq |x_n|^{1/n}$$

$$1^n \leq r_1^n \leq |x_n|$$

Root

$$\sum 1 \leq \sum r_1^n \leq \sum |x_n|$$

$\infty$ dvgt.

Ratio $\quad r = \lim \left| \dfrac{x_{n+1}}{x_n} \right|$

$r<1$ cgt. $\quad r>1$ dvgt

$r = 1$ Test fails

① $r<1$

$$\left| \frac{x_{n+1}}{x_n} \right| < r \quad \text{for } n \geq K(\varepsilon)$$

$$|x_{n+1}| < r \cdot |x_n|$$
$$\leq r \cdot r \cdot |x_{n-1}| = r^2 |x_{n-1}|$$
$$\vdots$$
$$\leq r^n \cdot |x_1|$$

$$\sum \frac{1}{(n+1)(n+2)}$$ $x^{\cancel{1/x}}$

Ratio Test :- $r = \lim \left| \frac{x_{n+1}}{x_n} \right|$

$$= \lim \left| \frac{(n+1)\cancel{(n+2)}}{\cancel{(n+2)}(n+3)} \right|$$

$$= \lim \left| \frac{1 + 1/n}{1 + 3/n} \right|$$

$$= 1$$ Test fail

Limit Comparison Test $\sum y_n = \sum \frac{1}{n^2}$

$$r = \lim \left| \frac{x_n}{y_n} \right|$$

$$= \lim \left| \frac{n^2}{(n+1)(n+2)} \right|$$

$$= \lim \left| \frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})} \right|$$

$$= 1 \neq 0$$ $\sum |y_n|$ cgt

$\Rightarrow \sum |x_n|$ cgt.

$$\frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

$$\sum \frac{1}{n^2} \rightarrow \text{cgt}$$

cgt.

$$\sum \frac{1}{(n+1)(n+2)} \Rightarrow \quad x_n = \frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

$$S_1 = x_1 = 1 - \frac{1}{2}$$

$$S_2 = S_1 + x_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3}.$$

$$S_3 = S_2 + x_3 = 1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

$\sum x_n$ cgt if $S_n$ — cgt.

$$\lim S_n = \lim 1 - \frac{1}{n+1}$$

$S_n \to 1$ $\qquad = 1$

$\sum 2^n$ Ratio $\Rightarrow$ $\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \left|\frac{2^{n+1}}{2}\right|$

$$= 2 > 1 \quad \text{dgt.}$$

$\sum 2^r$ Root test

$$\lim |x_n|^{1/n} = \lim |2^n|^{1/n}$$

$$= 2 > 1 \quad \text{dgt.}$$

$\sum 2^{-1/n}$

Ratio $\lim_{n} \frac{2^{-1/(n+1)}}{2^{-1/n}}$

$\lim 2^{\frac{-1}{n+1} + \frac{1}{n}}$

$\lim_{n\to\infty} 2^{\frac{1}{n^2+n}} \to 1$

Test fails

$\sum 2^{-1/n}$ $\quad$ $\sum x_n$ cgt $\Rightarrow \lim x_n = 0$

$\lim x_n \neq 0 \Rightarrow \sum x_n$ not cgt.

$\lim 2^{-1/n} = 1 \quad \Rightarrow \sum 2^{-1/n}$ dvgt.

* $x_n = n/2^n$

Ratio Test $\quad r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} \right|$

$= \lim \left| \frac{n+1}{n} \right| \cdot \lim \left| \frac{2^n}{2^{n+1}} \right|$

$= \frac{1}{2} < 1$

$\Rightarrow \sum x_n$ cgt.

Root test:-

$r = \lim |x_n|^{1/n} = \lim \left| \frac{n}{2^n} \right|^{1/n} = \frac{1}{2} \underbrace{\lim n^{1/n}}_{\to 1}$

$= \frac{1}{2} < 1$

$$\sum \frac{3^n}{n!} \; ?$$

Ratio Test

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n} \right|$$

$$= \lim \frac{3}{n}$$

$$= 0 < 1 \quad \text{cgt.}$$

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$$

$$x_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$\frac{1}{\sqrt{1\cdot2}} + \frac{1}{\sqrt{2\cdot3}} + \frac{1}{\sqrt{3\cdot4}}$$

$$x_n = \frac{1}{\sqrt{n(n+1)}}$$

$$\frac{1}{n} < \frac{1}{\sqrt{n(n+1)}} \cancel{< \frac{1}{n}}$$

limit Comparison

$$\lim \left| \frac{n}{\sqrt{n(n+1)}} \right|$$

$$= \lim \left| \frac{1}{\sqrt{1(1+1/n)}} \right| = 1.$$

$\Rightarrow \sum y_n$ dvgt $\Rightarrow \sum x_n$ dvgt.

Ratio :- $x_n = \dfrac{1}{\sqrt{n(n+1)}}$

$$\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \frac{\sqrt{n\cancel{(n+1)}}}{\sqrt{\cancel{(n+1)}(n+2)}} = \lim \sqrt{\frac{1}{1+2/n}}$$

$= 1$ Test fails.

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots$$

$x_n = \dfrac{1}{n^n}$

Root test $\lim |x_n|^{1/n} = \lim \left|\dfrac{1}{n^n}\right|^{1/n} = \lim \left|\dfrac{1}{n}\right| = 0$

$< 1$

$\sum \dfrac{n^2-1}{n^2+1}$ by necessary cond$^n$ $\lim \dfrac{n^2-1}{n^2+1} = 1 \neq 0$

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \cdots$$

$$\left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \cdots\right)$$

$\uparrow \sum (1/3^2)^n$

$$\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\dots$$

$$x_n=\frac{1}{n\cdot 2^n}$$

$$\frac{1^2\,2^2}{1!}+\frac{2^2\,3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\dots$$

$$x_n=\frac{n^2(n+1)^2}{n!}$$

$$r=\lim\left|\frac{x_{n+1}}{x_n}\right|=\lim\frac{\cancel{(n+1)^2}(n+2)^2}{(n+1)!}\;\frac{n!}{\cancel{(n+1)^2}(n^2)}$$

$$=\lim\frac{1}{(n+1)}\cdot(1+2/n)^2$$

$$=0$$

$$x_n=\frac{1\cdot 2\cdot 3\cdots n}{7\cdot 10\cdots(3n+4)}\qquad \lim\left|\frac{x_{n+1}}{x_n}\right|$$

$$\frac{1\cdot 2\cdots\; n\cdot(n+1)}{7\cdot 10\cdots(3n+4)(3(n+1)+4)}\qquad =\lim\left|\frac{n+1}{(3(n+1)+4)}\right|$$

$$=\lim\left|\frac{1}{3+4/(n+1)}\right|$$

$$=\frac{1}{3}<1$$

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{1}{n} \qquad ?$$

$$\lim\left|\frac{x_{n+1}}{x_n}\right| = \lim\left|\frac{2(n+1)-1}{2(n+1)}\left(\frac{n}{n+1}\right)\right| = 1$$

$$x_n \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{\underline{2^n}\cdot\underline{n!}}\cdot\frac{1}{n}$$

$$\lim |x_n|^{1/n} = \frac{1}{2}\left(\lim\frac{1^{1/n}\, 3^{1/n}\cdots(2n-1)^{1/n}}{(n!)^{1/n}}\cdot\frac{1}{n^{1/n}}\right)$$

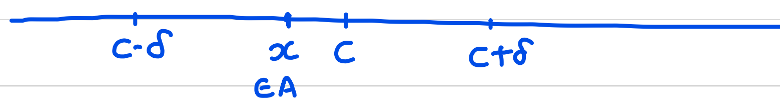
$$= \frac{1}{2} < 1$$

Functions and continuity

limit / Cluster Point :- $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ for every $\delta > 0$

$\exists\ x \in A,\ x \neq c,\ \ni\ |x - c| < \delta$

for any $\overset{\delta}{\varepsilon} > 0$, $\delta_\varepsilon(c) \cap A \neq \phi$




① $c$ cluster, $\exists\ a_n \in A$ $a_n \to c$

$\Rightarrow$ $c$ cluster pt. of $A$

for any $\frac{1}{n} > 0$ $\exists\ x_n \in A$ $\ni$ $|x_n - c| < 1/n$

$n \in \mathbb{N}$ $\Rightarrow$ $c - 1/n < x_n < c + 1/n$

$\Rightarrow$ $x_n \to c$

Suppose we have a seq$^n$ $a_n \in A$, $\ni$ $a_n \to c$

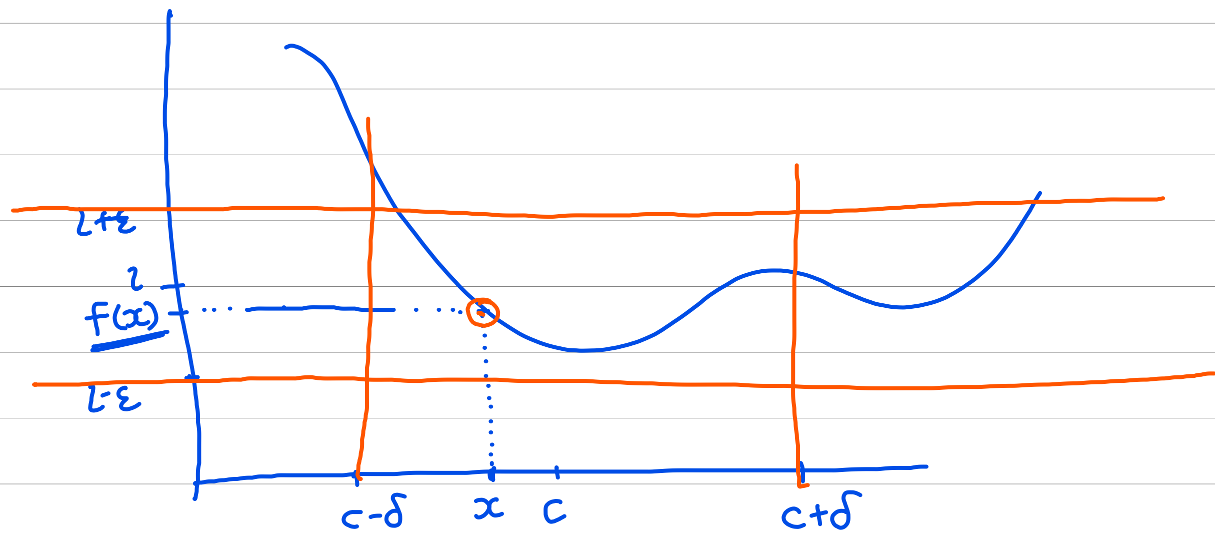
To show that :- $c$ cluster pt. of $A$

$a_n \to c$

for any $\varepsilon > 0$, $\exists\, K(\varepsilon) \in \mathbb{N}$, $\ni$ $|a_n - c| < \varepsilon$ $\forall\, n \geq K(\varepsilon)$

$\Rightarrow$ $|a_n - c| < \varepsilon$ $a_n \in A \cap \delta_\varepsilon(c)$

$\Rightarrow$ $c$ is also cluster pt. of $A$




Limit of fun$^c$:

$f: A \to \mathbb{R}$, $c$ cluster of $A$, $L \in \mathbb{R}$

for any $\varepsilon > 0$ $\exists\, \delta_\varepsilon > 0$ $\ni |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\ni x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(L)$

$\therefore$ $f: A \to \mathbb{R}$, $c$ cluster pt. of $A$. $\Rightarrow$ $f$ can have only one limit pt. at $c$.

Contradiction, $L$ & $L'$

for $\varepsilon > 0$ $\begin{cases} \frac{\varepsilon}{2} > 0 & \exists\, \delta'_\varepsilon > 0 \ni |x - c| < \delta'_\varepsilon \Rightarrow |f(x) - L| < \varepsilon/2 \\ \frac{\varepsilon}{2} > 0 & \exists\, \delta''_\varepsilon > 0 \ni |x - c| < \delta''_\varepsilon \Rightarrow |f(x) - L'| < \varepsilon/2 \end{cases}$

$$|L-L'| = |\underline{L-f(x)}+\underline{f(x)-L'}|$$

$$\leq |L-f(x)|+|f(x)-L'|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

$$\leq \varepsilon$$

$$L=L'$$

QED

$x_n \to c$
$f(x_n) \to L.$ } seq Criteria

def seq convergent seq$^n$ & def limit pt. of fun$^c$ at some $c$

$f: A \to \mathbb{R}$, $c$ is cluster pt. of $A$

① for any $\varepsilon>0$ $\exists\ \delta_\varepsilon>0 \ni$ if $\underline{|x-c|<\delta_\varepsilon} \Rightarrow \underline{|f(x)-L|<\varepsilon}$
$x \in A$

② $f: A \to \mathbb{R}$ $c$ $A$
$\exists\ \underline{x_n \to c} \Rightarrow \underline{f(x_n) \to L}$

$f(x)$ converges to $L$ at $c$

① $|x-c|<\delta_\varepsilon$ $|f(x)-L| > \varepsilon_0$
② $x_n \to c$ but $f(x_n) \not\to L$

does not converges $f(x_n)$ at $c$

$$\boxed{\lim_{x\to c} f(x) = L}$$

$$f(x) = \mathrm{sgn}(x) = \begin{cases} +1 & x>0 \\ -1 & x<0 \\ 0 & x=0 \end{cases}$$

$f :- \mathbb{R} \to \mathbb{R}$

$(-\delta, 0)$ $f(x) = \underline{-1}$ $(0,\delta)$ $f(x) = \underline{+1}$

$f : A \to \mathbb{R}$, $c$ cluster pt. of $A$

$f :-$ limit exists at pt. $c$. $c \in \mathbb{R}$

for any $\varepsilon>0$ $\exists\ \delta_\varepsilon>0 \ni$ if $|x-c|<\delta_\varepsilon \Rightarrow \underline{|f(x)-L|<\varepsilon}$

$$\boxed{\Rightarrow L-\varepsilon < f(x) < L+\varepsilon \quad \forall x \in V_{\delta_\varepsilon}(c)}$$

Completeness

$M = \mathrm{Sup}\{f(x), x \in V_{\delta_\varepsilon}(c)\} \Rightarrow |f(x)| \leq M$ for some $\underline{M>0}$

$f : A \to \mathbb{R}$ , $c$ cluster pt. of $A$ $\lim_{x\to c} f(x) = L$

$g : A \to \mathbb{R}$ $\lim_{x\to c} g(x) = M$

$(f+g) : A \to \mathbb{R}$ $\lim_{x\to c} (f+g)(x) = L+M$

$\lim_{x\to c} f(x) = L$ for $\varepsilon/_2 > 0$, $\exists\ \delta_\varepsilon' > 0 \ni \underline{|x-c| < \delta_\varepsilon'} \Rightarrow |f(x) - L| < \varepsilon/_2$

for $\varepsilon/_2 > 0\ \exists\ \delta_\varepsilon'' > 0 \ni \underline{|x-c| < \delta_\varepsilon''} \Rightarrow |g(x) - M| < \varepsilon/_2$

$\delta_\varepsilon = \min(\delta_\varepsilon', \delta_\varepsilon'')$ $|x-c| < \delta_\varepsilon$

$|(f+g)(x) - (L+M)|$

$= |(f(x)+g(x)) - (L+M)|$

$= |\underline{f(x) - L} + \underline{g(x) - M}| \leq |f(x) - L| + |g(x) - M|$

$\leq \varepsilon/_2 + \varepsilon/_2$

$\leq \varepsilon$

$|(f \cdot g)(x) - L \cdot M|$

$$= |f(x)\cdot g(x) - L\cdot g(x) + L\cdot g(x) - LM|$$

$$\leq |g(x)|\,|f(x)-L| + |L|\,|g(x)-M|$$

$$\leq M\cdot \frac{\varepsilon}{2M} + \cancel{|L|}\frac{\varepsilon}{2\cancel{|L|}}$$

$f : A \to \mathbb{R}$, $c$ cluster pt. of $A$

$a \leq f(x) \leq b$ , $f$ limit exists at $c$.

[for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|x-c| < \delta_\varepsilon \Rightarrow |f(x)-L| < \varepsilon$]

seqn criter

$\exists x_n \in A$ $\quad x_n \to c \Rightarrow f(x_n) \to L$

$a \leq f(x) \leq b \quad \forall\ x \in A$

$a \leq f(x_n) \leq b \quad \forall\ x_n$

by squeeze theo.

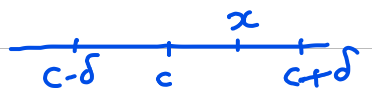
$$a \leq \lim_{x_n \to c} f(x_n) \leq b$$

$$a \leq L \leq b$$

$c$ cluster of $A$ iff $\exists\ a_n \in A$ $a_n \to c$

① $c$ cluster pt. of $A$.

for every $\delta > 0$ $\exists$ $x \in A$ $|x-c| < \delta$




$\forall\ n \in \mathbb{N}$, $\frac{1}{n} > 0$ $\exists$ $x_n \in A$ $\Rightarrow$ $|x_n - c| < 1/n$ $\ni$ $c - 1/n < x_n < c + 1/n$

$x_n \to c$

② $a_n \in A$, $a_n \to c$

for any $\varepsilon > 0$ $\exists\ K(\varepsilon) \in \mathbb{N}$, $\ni$ $|a_n - c| < \varepsilon$ $\forall\ n \geq K(\varepsilon)$

infinite $a_n \in A$

$\Rightarrow$ $c$ is cluster pt. of $A$.

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$f, g, h : A \to \mathbb{R}$, $c$ cluster pt. of $A$

$f(x) \leq g(x) \leq h(x)$ $\forall\ x \in A$ $x \neq c$

✓ $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$

To show $\lim_{x \to c} g(x) = L$

for any $\varepsilon > 0$ $\exists\ \delta'_\varepsilon > 0$ $\ni$ $|x - c| < \delta'_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$

$\Rightarrow L - \varepsilon < f(x) < L + \varepsilon$ —

-,,- $\delta''_\varepsilon > 0$ $\ni$ $|x - c| < \delta''_\varepsilon \Rightarrow |h(x) - L| < \varepsilon$

$\Rightarrow L - \varepsilon < h(x) < L + \varepsilon$ —

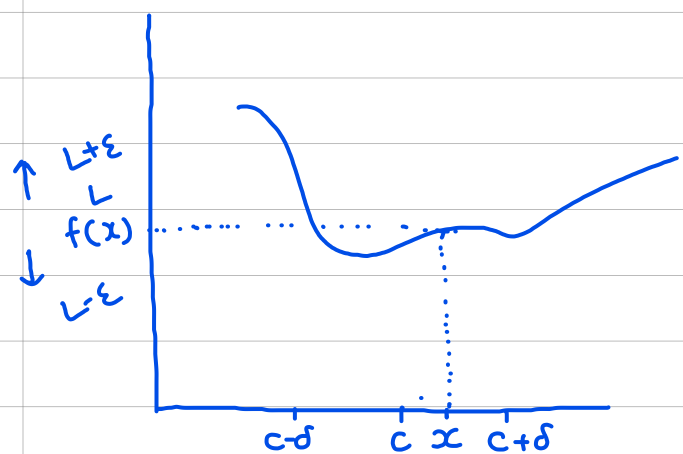
$$f(x) \le g(x) \le h(x)$$

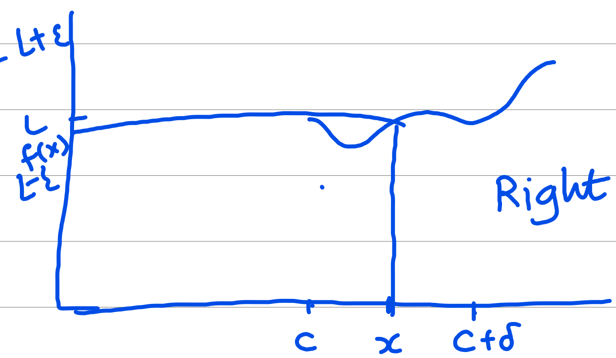
$$\Rightarrow L-\varepsilon < f(x) \le \underline{g(x)} \le h(x) < L+\varepsilon$$

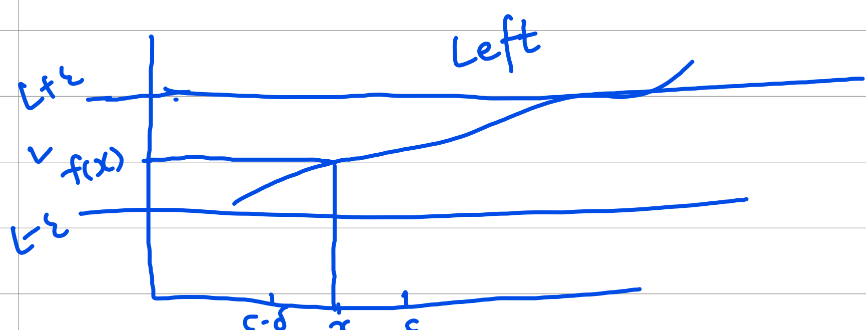
$$\Rightarrow \quad L-\varepsilon < g(x) < L+\varepsilon$$

$$\Rightarrow \quad |g(x) - L| < \varepsilon$$

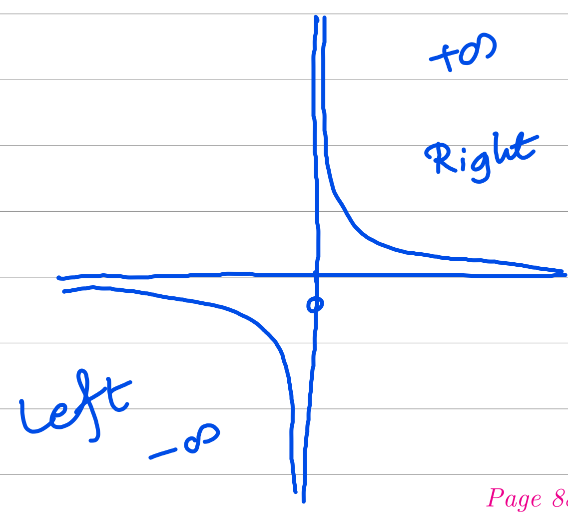
$$\Rightarrow \quad g(x) \to L$$




$f(x) = 1/x$




$(c-\delta, c)$ $\quad$ $\underline{(-\delta, 0)}$

$(c, c+\delta)$ $\quad$ $(0, \delta)$

$$\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$$

$$(c-\delta, c+\delta) \qquad (c, c+\delta) \qquad (c-\delta, c)$$

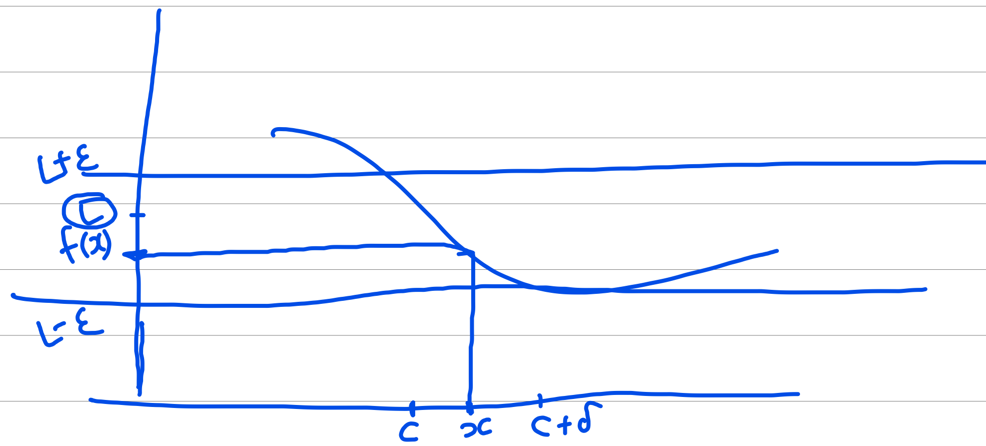
$\lim_{x\to c^+} f(x)$ for any $\varepsilon>0$ $\exists$ $\delta_\varepsilon>0$ $\ni$ $x\in(c, c+\delta)$

$\ni |f(x)-L|<\varepsilon$

$|x-c|<\delta_\varepsilon$

$c-\delta_\varepsilon < x < c+\delta_\varepsilon$

$(c-\delta_\varepsilon, c)\cup(c, c+\delta_\varepsilon)$



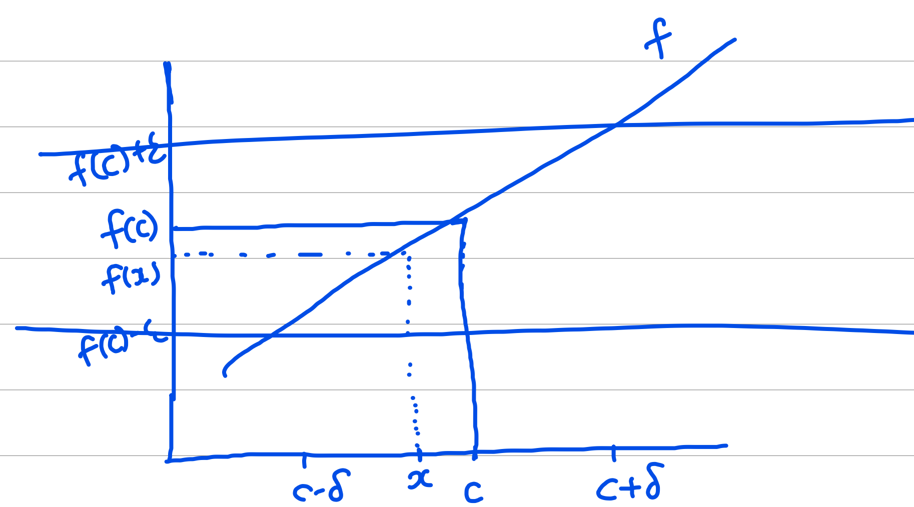

$\lim_{x\to c} f(x) = L = f(c)$

f cont. at pt. c.

Right

$\lim_{x\to c^+} f(x) = f(c)$

$\lim_{x\to c^-} f(x) = f(c)$




$\lim_{x\to c} f(x) = f(c)$

$$f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$$

(i) for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|x - c| < \delta_\varepsilon$ & $x \in A$

$x \in V_\delta(c)$ & $x \in A$ $\ni$ $x \in V_\delta(c) \cap A$

$\ni$ $|f(x) - f(c)| < \varepsilon$

$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

$f(x) \in V_\varepsilon(f(c))$

$$\ni \quad f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$$

Sequential Criteria for continuity

$x_n \to c$

$f(x_n) \to f(c)$

if $\quad x_n \to c \quad \Rightarrow f(x_n) \to f(c)$

$f, g$ cont. at $c$ $\qquad$ $f + g$ cont at $c$.

for any $\frac{\varepsilon}{2} > 0$ $\exists \delta_\varepsilon > 0$ $\ni$ $|x - c| < \delta_\varepsilon$ $\Rightarrow$ $|f(x) - f(c)| < \varepsilon/2$

$\Rightarrow$ $|g(x) - g(c)| < \varepsilon/2$

$$|f(x) + g(x) - f(c) - g(c)| = |f(x) - f(c)| + |g(x) - g(c)|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

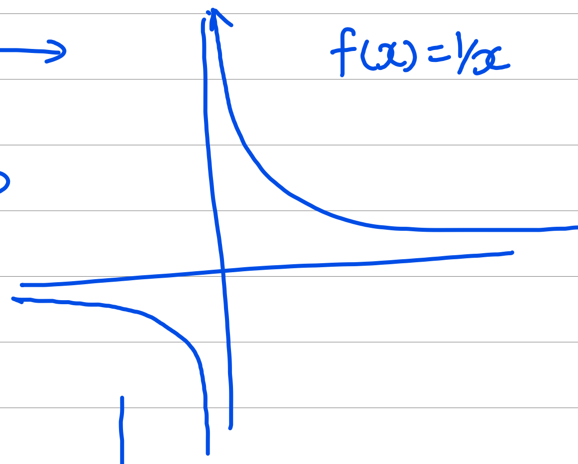
$$\leq \varepsilon$$

$\left( c \text{ cluster pt. } A \rightarrow \begin{array}{l} f: A \to \mathbb{R} \\ \lim\limits_{x \to c} f(x) = f(c) \end{array} \right)$ ✓

$\left[ \text{Conti. on set } B, \; \forall \; y \in B \quad \lim\limits_{x \to y} f(x) = f(y) \right]$

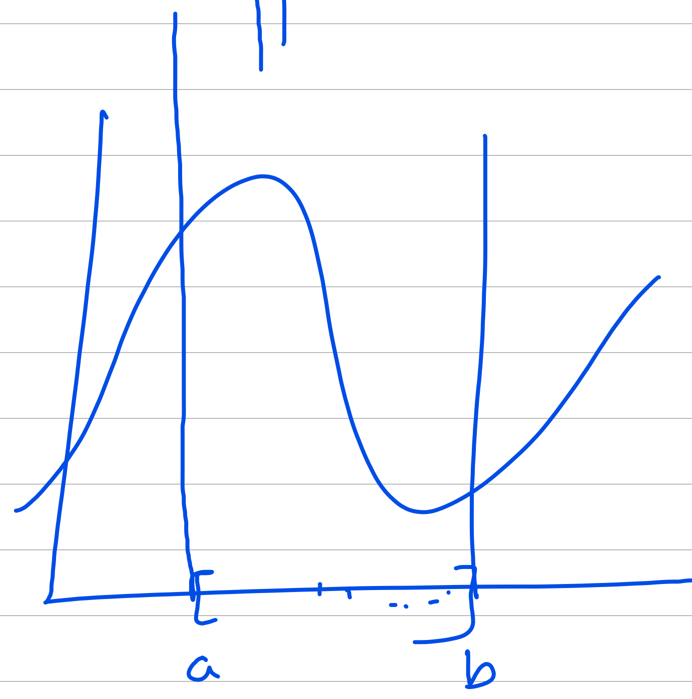
$\Rightarrow$ conti at every point of B.

$B = [-1, 1] \longrightarrow$

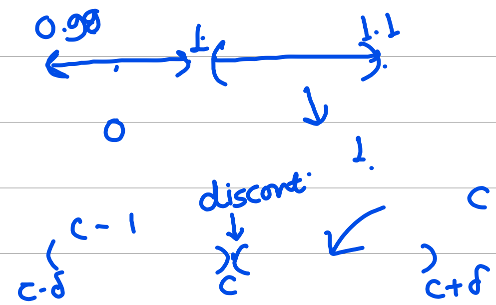



$$\begin{cases} \lim\limits_{x \to 0^+} f(x) = +\infty \\ \lim\limits_{x \to 0^-} f(x) = -\infty \end{cases}$$

$f(x) = 2x^2 + 3$ ✓




$f(x) = [x]$



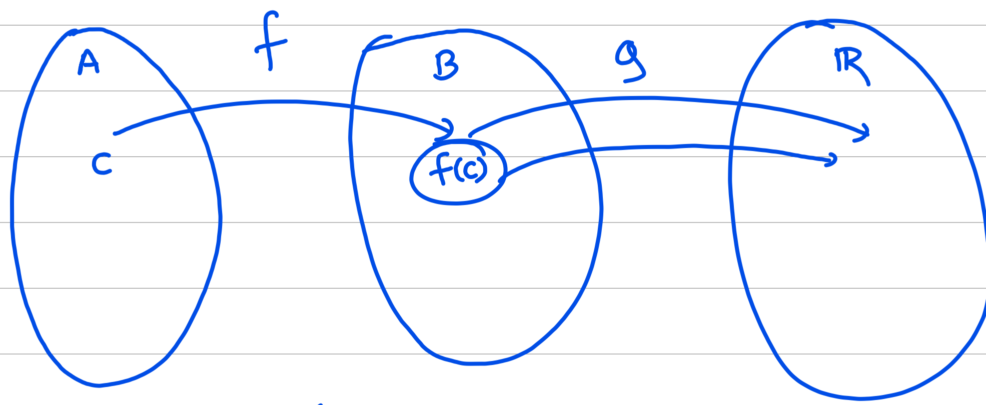

$[x]$

$f$ cont. at pt. $c$.

for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

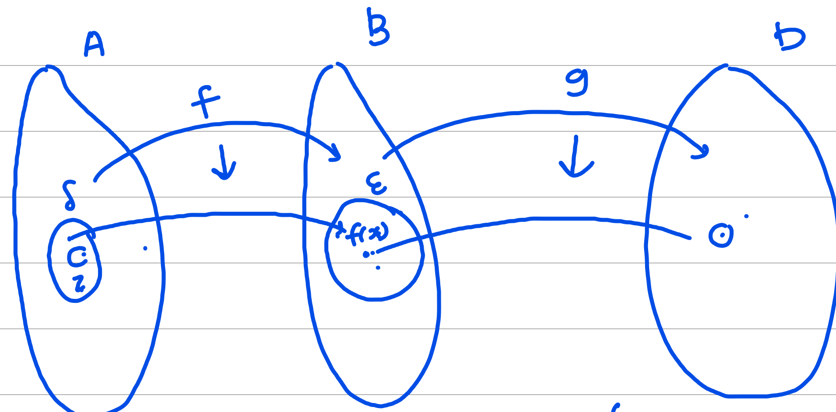
$$\big|\, |f(x)| - |f(c)| \,\big| \leq |f(x) - f(c)| < \varepsilon$$

$\Rightarrow$ $|f|$ is cont. at $c$.




$g \circ f$ $\quad\quad$ $g(f(x))$.

Let A,B  R & let f : A  R & g : B  R be function such that f (A)  B if f is continuous at point c  A and g is continuous at b = f (c)  B then the composition g  f : A  R is continuous at c.




$f : A \to B$ cont $\qquad g : B \to D$ cont $\qquad g \circ f : A \to D$ cont

for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

$x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$

for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|y - c'| < \delta \Rightarrow |g(y) - g(c')| < \varepsilon'$

$y \in V_\delta(c') \Rightarrow g(y) \in V_\varepsilon(g(c'))$

As we have assumed that $f(A) \subseteq B$

$\exists$ some $\underline{\underline{c' = f(c)}}$

cont def cont

[ for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon > 0$ $\ni$ $|z - c| < \delta \Rightarrow$ ]

$|(g \circ f)(z) - (g \circ f)(c)| = |g(f(z)) - g(f(c))|$

$(\ |z - c| < \delta \Rightarrow |f(z) - f(c)| < \varepsilon\ ,\ \varepsilon > 0)$

$|y - c'| < \delta' \cancel{\Rightarrow |g(y) - g(c')| < \varepsilon}$

$\boxed{\Rightarrow |g(f(z)) - g(f(c))| < \varepsilon}$

Boundedness Theo.

$I = [a,b]$ closed bounded interval -

$f : I \to \mathbb{R}$ cont. ✓

Toshow:- $f$ bounded. $|f(x)| \le M$

Assume $f(x)$ is not ~~bounded on~~ $I$

$\exists$ some $n \in \mathbb{N}$ $|f(x_n)| > M$ ✓

- I closed bounded interval..

Let $\exists$ some seq$^n$ $x_n \in I$,

$I$ bounded $\Rightarrow$ $x_n$ is bounded.

[bounded seq$^n$

cgt subseq$^n$

By Bolzano Weierstrass theo. $\exists$ subseq$^n$ $x_{n_k} \in I$ which is convergent.

$x_{n_k} \to x^*$ (say)

if $x_{n_k}$ is subseq$^n$ in $I$ & $I$ is closed

$\Rightarrow x^* \in I$

$\therefore$ $f$ is cont.

$x_{n_k} \to x^* \Rightarrow f(x_{n_k}) \to f(x^*)$

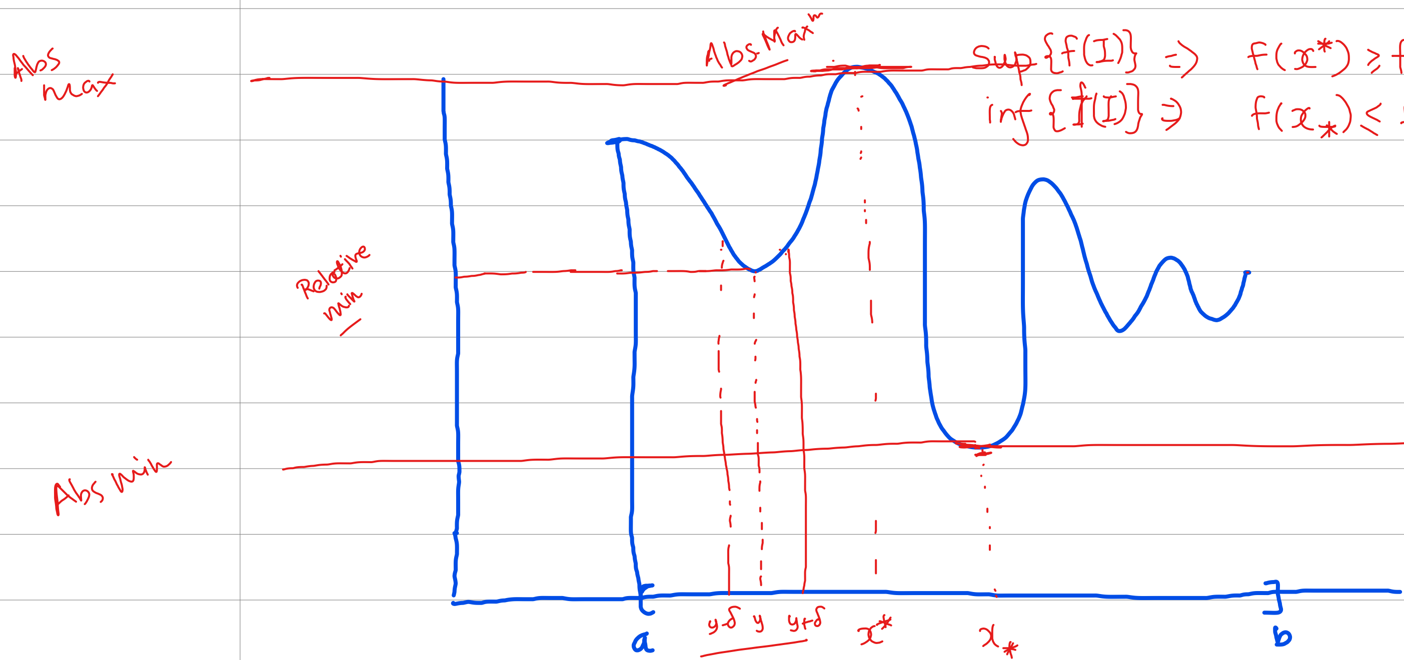
$\Rightarrow$ $f(x_{n_k})$ is convergent subseq$^n$

$\Rightarrow$ bounded seq$^n$

$\Rightarrow$ Our assumption is wrong.

Absolute Extremum — $\max^m$ min

Relative extremum — $\max^m$ min




$\operatorname{Sup}\{f(I)\} \Rightarrow f(x^*) \geq f(x) \ \forall\, x \in I$

$\inf\{f(I)\} \Rightarrow f(x_*) \leq f(x) \ \forall\, x \in I$

Abs. min $\quad f(x_*) \leq f(x) \quad \forall\, x \in I$ $\qquad$ Abs. max$^m$ $\quad f(x^*) \geq f(x) \quad \forall\, x \in I$

Rel. min $\quad f(y) \leq f(x) \quad \forall\, x \in (y-\delta, y+\delta)$ $\qquad$ Rel. min $\quad f(y) \geq f(x) \quad \forall\, x \in V_\delta(y)$

Max$^m$ & Min$^m$ the

$f: I \to \mathbb{R}$, $\quad$ I closed bounded, f cont.

To show :- $\exists\, x^*, x_*$ as abs. max$^m$ & abs. min resp.

$\because$ I closed bounded, f cont. $f: I \to \mathbb{R}$

by Boundedness Theo. $\quad |f(x)| \leq M \quad \forall\, x \in I$

$f(I) = \{f(x), x \in I\}$ bounded.

by Completeness property.

$\exists\ s^* = \operatorname{Sup}\{f(I)\}$ and $s_* = \operatorname{Inf}\{f(I)\}$

If $s^*$ is sup. then for $n \in \mathbb{N}$. $s^* - \frac{1}{n}$ can't be sup.

$$\checkmark \left[\ s^* - \frac{1}{n} < f(x_n) < s^* \qquad \forall\, n \in \mathbb{N} \right]$$

So we got seq$^n$ $x_n \in I$, $I$ closed bounded
by Bolzano weierstrass $\exists$ $x_{n_k} \to x^*$
and as $x_{n_k} \in I$, $\because$ $I$ closed $\Rightarrow$ $x^* \in I$

Now $x_{n_k} \to x^*$
but $f$ is cont on $I$, by seq$^n$ criteria

$$f(x_{n_k}) \to f(x^*)$$

$$s^* - \frac{1}{n_k} < f(x_{n_k}) < s^*$$

by squeez theo.

$$\lim s^* - \frac{1}{n_k} < \lim f(x_{n_k}) \leq \lim s^*$$

$$s^* \leq f(x^*) \leq s^*$$

$$\Rightarrow \quad s^* = f(x^*)$$

$\Rightarrow$ $s^* = \sup\{f(I)\}$ $\Rightarrow$ $f(x^*) \geq f(x)$ $\forall x \in I$

Simillarly we can obtain $x_*$ $\ni$ $f(x_*) \leq f(x)$ $\forall$ $x \in I$.
$x^*, x_* \in I$