9

Unit - I - Real Numbers $\mathbb{R}$

Groups Mat

$(\mathbb{R}, +, \cdot)$

* Algebraic Properties of $\mathbb{R}$:-

Add^n Mult^n

On set of $\mathbb{R}$ there are two binary operators $+$ & $\cdot$

These two operations follows few properties :-

add^n

A1 > Commutative property of addition

$$a+b = b+a \quad \forall\ a,b \in \mathbb{R}$$

A2 > Associative property of add^n

$$(a+b)+c = a+(b+c) \quad \forall\ a,b,c \in \mathbb{R}$$

A3) Existence of Zero element (Additive Identity)

$$a+0 = 0+a = a \quad \forall\ a \in \mathbb{R}$$

A4) Existence of negative element (Additive Inverse)

$$a+(-a) = (-a)+a = 0 \quad \forall\ a \in \mathbb{R}$$

$a+-a=0$

$-a+-(a)=0$

a

matric

$A \cdot B \neq B \cdot A$

M1) Commutative property of multiplication

$$a \cdot b = b \cdot a \quad \forall\ a,b \in \mathbb{R}$$

M2) Associative p of multi.

$$(a \cdot b) c = a (b \cdot c) \quad \forall\ a,b,c \in \mathbb{R}$$

M3) Existence of unit element (Multiplicative identity)

$$a \cdot 1 = 1 \cdot a = a \quad \forall\ a \in \mathbb{R}$$

Extended Real

$\infty \leftarrow \mathbb{R}$

M4) Existence of multiplicative inverse / Reciprocals

$$a \cdot \left(\frac{1}{a}\right) = \frac{1}{a} \cdot a = 1 \quad \forall\ a \in \mathbb{R} - \{0\}$$

D) Distributive property of multiplication over add^n

$$a \cdot (b+c) = a \cdot b + a \cdot c$$
$$(b+c) \cdot a = b \cdot a + c \cdot a \quad \forall\ a,b,c \in \mathbb{R}$$
$$= a \cdot b + a \cdot c$$

* Order Properties of $\mathbb{R}$

Why do we study Real Analysis?

CLT $\to \dfrac{\sum x_i - \mu}{\sigma} \to$ ____

$\bar{x} \to \mu$

seq^n of fun^c

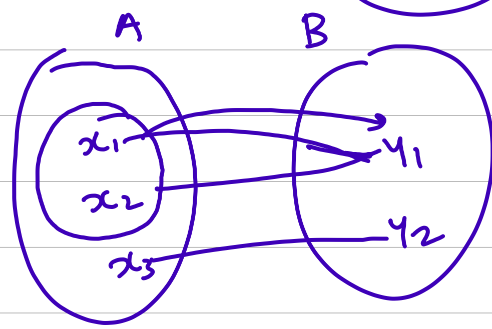
Convergence of Series of fun^{c}s

$\sum f_n(x) = \sum \dfrac{x_i}{n} \to$ ____

$\dfrac{\sum x_i}{n} \to \mu$

Set Fun^c

Fun^c



Prob.

input element $\to$ set

$\dfrac{n(A)}{n(S)}$ Classi $\in \mathbb{N}$

input Set $\to$ ____ ~~Real~~

ST-201

(A)

Borel fun^c

$x^3 + 2x + 3 = 0$

$x = ?$

$\mathbb{R}$

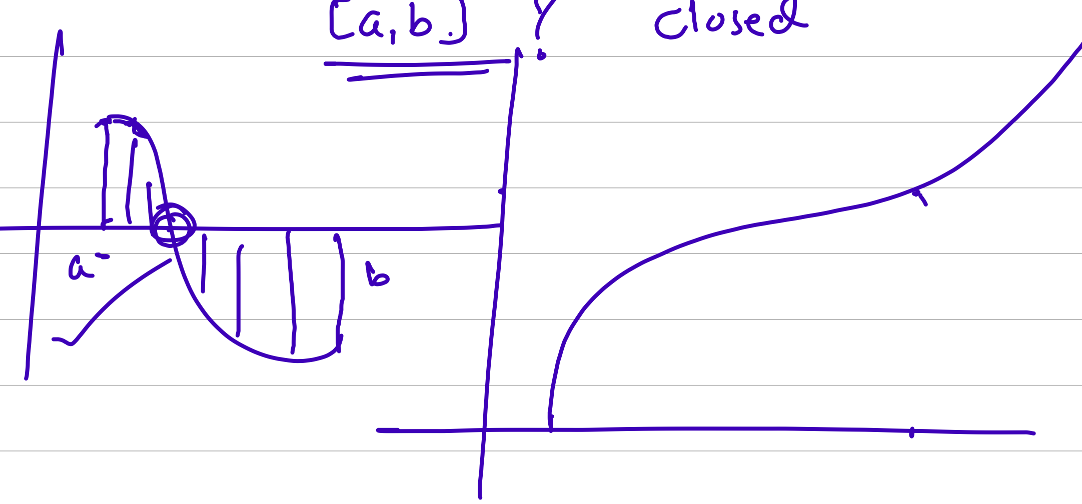
0 1

[a, b] ? Closed

ST-205

Numerical Methods

Cont?



Series $\rightarrow$ $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$$E(X) = \sum_x \frac{e^{-\lambda}\lambda^x}{x!}x$$

$$= e^{-\lambda} \sum_x \frac{\lambda^x}{x!}x$$

$$= e^{-\lambda}\lambda \sum_x \frac{\lambda^{x-1}}{(x-1)!}$$ Series

$$= \cancel{e^{-\lambda}} \lambda \cancel{e^{\lambda}}$$?

Real Analysis

Geometric $P(X=x) = pq^{x-1}$

$$E(X) = \sum x \cdot pq^{x-1}$$

$$= p \cdot \sum x \cdot q^{x-1}$$?

$$= p \frac{1}{(1-q)^2}$$

$$= \cancel{p} \frac{1}{p^2} = \frac{1}{p}$$

$$= 1/p$$

$$\sum x^n = \frac{1}{1-x} \qquad |x|<1$$

? $$\frac{d}{dx}\sum x^n = \frac{d}{dx}\frac{1}{1-x}$$

$$\cancel{d}\ \sum n x^{n-1} = \frac{1}{(1-x)^2}$$

ST-201 Probability

Inverse fun^c $\leftarrow$ X Random Variable

ST-301 Asymptotic $\rightarrow$ as $n \to \infty$

$\bar{x} \to \mu$ $\quad \frac{1}{n}\sum x_i \to \mu$

① Unbiased
② Suff
③ Eff
④ Consistent $\rightarrow$ as $n \to \infty$

$X_{(n)}$ # C.E. $\quad X \sim U(0,\theta)$

$2\bar{X}$ $\rightarrow$

* Order Properties of $\mathbb{R}$

$\mathbb{P}$ set of +ve nos $\in \mathbb{R}$, $\mathbb{P} \subseteq \mathbb{R}$, $\mathbb{R}^+$

$\mathbb{R}^+$ satisfies following properties

① if $a, b \in \mathbb{R}^+ \Rightarrow a+b \in \mathbb{R}^+$

② if $a, b \in \mathbb{R}^+ \Rightarrow a \cdot b \in \mathbb{R}^+$

③ if $a \in \mathbb{R}$ then exactly one of the following is true:-

$a \in \mathbb{R}^+$, / $a = 0$, / $-a \in \mathbb{R}^+$

× OR × OR ✓

-4.5

Law of Trichotomy

$\underline{a \in \mathbb{R}}$ >0 <0 $=0$

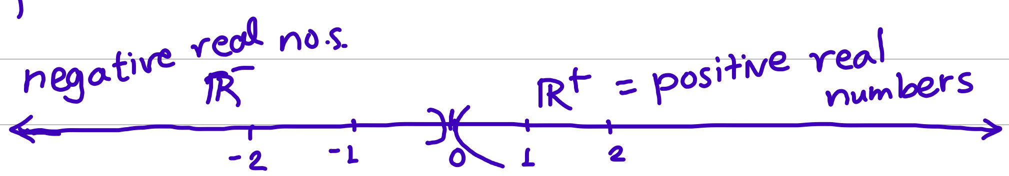
$a \in \mathbb{R}^+$ $a \in \mathbb{R}^-$

$-a \in \mathbb{R}^+$

Let $a, b \in \mathbb{R}$

ⓐ If $a-b \in \mathbb{R}^+ \Rightarrow a > b$ or $b < a$

ⓑ If $a-b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a \geq b$ or $b \leq a$



✓ $\mathbb{R}^+ = \{x \mid x > 0, x \in \mathbb{R}\}$

$\mathbb{R}^- = \{x \mid x < 0, x \in \mathbb{R}\}$

Proof :- ⓐ $a, b \in \mathbb{R}$, $a-b \in \mathbb{R}^+ \Rightarrow a-b > 0 \Rightarrow a > b$

ⓑ $a, b \in \mathbb{R}$ $a-b \in \mathbb{R}^+ \cup \{0\} \Rightarrow a-b \geq 0 \Rightarrow a \geq b$

Theo:- Let $a, b, c \in \mathbb{R}$

① If $a > b$ & $b > c \Rightarrow a > c$ (Transitivity)

② If $a > b \Rightarrow a+c > b+c$

③ If $a > b$, $c > 0 \Rightarrow a \cdot c > b \cdot c$

If $a > b$, $c < 0 \Rightarrow a \cdot c < b \cdot c$

(1) If $a > b$, & $b > c$

$a - b > 0$ & $b - c > 0$

$\Rightarrow a - b \in \mathbb{R}^+$ & $b - c \in \mathbb{R}^+$

$\Rightarrow (a-b) + (b-c) \in \mathbb{R}^+$ (by order properties)

$\Rightarrow (a-c) \in \mathbb{R}^+$

$\Rightarrow a - c > 0$

$\Rightarrow a > c$ QED

(2) $a - b \in \mathbb{R}^+$

$a - b + c - c \in \mathbb{R}^+$

$(a+c) - (b+c) \in \mathbb{R}^+$

$a + c > b + c$

(3) $a > b$, $c > 0$ to prove $a \cdot c > b \cdot c$

$\underline{\underline{a > b}}$, $a - b > 0$

$a - b \in \mathbb{R}^+$

$c(a-b) = c \cdot a - cb$ as $c > 0$

$ca - cb > 0$

$ca > cb$

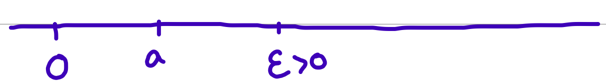
Similarly $a > b$, $c < 0 \Rightarrow ac < bc$

Theo:- If $a \in \mathbb{R}$, such that $0 \leq a < \varepsilon$ $\forall \varepsilon \in \mathbb{R}$, $\varepsilon > 0$

$\Rightarrow a = 0$

$\varepsilon = 0.005$ 08

$\varepsilon = 0.00000$



$a \to 0$

By method of contradiction.

Assume $\underline{\underline{a>0}}$ $\Rightarrow$ (as $\underline{\underline{0\leq a<\varepsilon}}$) $\Rightarrow$ a is positive

$\Rightarrow$ $\frac{a}{2}$ is positive

we can assume $\varepsilon_0 = a/2 < a$

for any $\varepsilon>0$, $0\leq a<\varepsilon$ but for $\varepsilon_0 \Rightarrow$ $0<\varepsilon_0<a$

which contradicts to our assumption. $\boxed{\varepsilon}$

Rational Nos. :- $Q = \{x \,/\, x = p/q,\ p,q \in I,\ q \neq 0\}$

* Theo. There doesnot exist any rational no. $r \in Q$ $\underline{\underline{r^2 = 2}}$

$\Rightarrow$ Assume that $\sqrt{2}$ is rational no.

$\frac{4}{6} \Rightarrow \frac{2}{3}$

$$\sqrt{2} = \frac{p}{q}$$

common divisor of p,q is 1

$(p,q) = 1$ ✓

$$\Rightarrow \quad 2 = \frac{p^2}{q^2}$$

$p=4$, $\underline{\underline{p^2=16}}$

$$\Rightarrow \quad 2q^2 = p^2 = p\cdot p$$

$\rightarrow$ $p=3$, $\underline{\underline{p^2=9}}$

$\Rightarrow$ p is divisible by 2 $\Rightarrow$ $p = 2\cdot m$

$\Rightarrow$ $p^2 = 2^2\cdot m^2 = 4m^2$

$\underline{\underline{p = \text{even}}}$

$$\Rightarrow \quad 2q^2 = \cancel{4}^{\,2}m^2$$

$$\Rightarrow \quad q^2 = 2m^2$$

$\Rightarrow$ q is divisible by 2 ——— (**)

$(p,q)=1$ but here $(p,q)=2$

which contradicts to our assumption.

even^2 odd^2

$(2n)^2$ $\qquad$ $(2n+1)^2$

$=2(2n^2)$ $\qquad$ $=4n^2+2n+1$

even $\qquad$ $=2(2n^2+n)+1$

$=$ odd

Theo:- If $ab>0$ then either ① $a>0, b>0$
② $a<0, b<0$

(*) ① If $a,b \in \mathbb{R}$ show that $a^2+b^2=0 \Leftrightarrow a=0, b=0$ ✓

② If $0<c<1$ show that $0<c^2<c<1$ ✓

③ If $x,y \in Q$, $x+y \in Q$, $x\cdot y \in Q$ ✓

If $x \in Q$, $y \in Q^c$, $x+y \in Q^c$

① $a^2+b^2=0 \Rightarrow (a+b)^2 = a^2+b^2+2ab = 0$

$2ab=0$

$\Rightarrow a\cdot b=0$

$\Rightarrow a=0$ or $b=0$

(if $a=0$ & $b\neq 0 \Rightarrow a^2+b^2>0$

$(a^2+b^2=0 \Rightarrow b=0$

$a,b \in \mathbb{R}^+, \ c>0, \qquad a>b \Rightarrow c\cdot a > c\cdot b$

$0<c<1, \ \text{as } c>0$

$c<1$

$c\cdot c < 1\cdot c$

$0<c^2<c<1$

$x,y \in Q \Rightarrow x+y \in Q$

$x=\frac{p_1}{q_1}, \ y=\frac{p_2}{q_2} \Rightarrow x+y = \frac{p_1}{q_1}+\frac{p_2}{q_2}$

$= \frac{p_1q_2+p_2q_1}{q_1\cdot q_2}$

as $p_1,q_1,p_2,q_2 \in I$, $\quad p_1q_2 \in I, \ p_2\cdot q_2 \in I, \ q_1\cdot q_2 \in I$

$p_1q_1+p_2q_2 \in I$

$= \frac{p^*}{q^*} \in Q$

Absolute values, $\forall a \in \mathbb{R}$ $\quad |a| = \begin{cases} +a & \text{if } a>0 \\ -a & \text{if } a<0 \\ 0 & \text{if } a=0 \end{cases}$

$\boxed{|a| = \max(a,-a)}$

Theo :- (a) $|ab| = |a|\cdot|b|$ $\quad \forall a,b \in \mathbb{R}$ ✓

(b) $|a|^2 = a^2$ $\quad \forall a \in \mathbb{R}$ $\qquad -c \le a+b \le c \Rightarrow |a+b| \le c$

(c) If $c \geq 0$ then $|a| \le c$ iff $-c \le a \le c$

(d) $-|a| \le a \le |a|$

Proof :- (a) If $a,b>0 \Rightarrow ab>0 \Rightarrow |a|=a, |b|=b,$ $|a\cdot b| = ab = |a|\cdot|b|$

$a>0, b<0 \Rightarrow ab<0 \Rightarrow |a|=a, |b|=-b,$ $|ab| = -ab = a\cdot(-b) = |a||b|$

$a<0, b>0 \Rightarrow ab<0 \Rightarrow$ Simillarly

$a,b<0 \Rightarrow ab>0 \Rightarrow |a|=-a, |b|=-b,$ $|ab| = ab = (-a)\cdot(-b)$

$= |a|\cdot|b|$ ✓

(b) $|a|^2 = a^2$

If $a>0$, $|a| = a \Rightarrow |a|^2 = a^2$ ✓

$a<0$ $\quad |a| = -a \Rightarrow |a| = (-a)^2 = a^2$ ✓

$a=0$ $\quad |a| = a = 0 \Rightarrow |a|^2 = a^2 = 0$ ✓

(c) If $c \geq 0$ then

$|a| \le c$ iff $-c \le a \le c$.

i) $|a| \le c \Leftrightarrow |a| = \max(a,-a) \le c$

$\Leftrightarrow a \le c$ & $-a \le c$

$\Leftrightarrow a \le c$ & $a \geq -c$.

$\Leftrightarrow -c \le a \le c$ ✓

(d) If $c \geq 0$, then $|a| \le c$ iff $-c \le a \le c$ ✓

now assume $c = |a| \geq 0$, $\quad |a| \le |a|$

$\Rightarrow -|a| \le a \le |a|$

* Triangular inequality :- If $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$

proof :- If $a, b \in \mathbb{R}$, then

$$\begin{aligned} -|a| &\leq a \leq |a| \\ + \quad -|b| &\leq b \leq |b| \\ \hline -|a|-|b| &\leq a+b \leq |a|+|b| \end{aligned}$$

put $c = |a| + |b|$

$\Rightarrow -(|a|+|b|) \leq a+b \leq |a|+|b|$

$\Rightarrow -c \leq a+b \leq c$

$\Rightarrow |a+b| \leq c$

$\Rightarrow |a+b| \leq |a| + |b|$

$|a| = |-a|$

* Corollary : If $a, b \in \mathbb{R}$, then (a) $||a| - |b|| \leq |a-b|$
(b) $|a-b| \leq |a| + |b|$

(a) If $a, b \in \mathbb{R}$

$a = a - b + b$

$|a| = |a-b+b| \leq |a-b| + |b| \Rightarrow$ — (*) (by triangular inequality)

$b = b - a + a$

$|b| = |b-a+a| \leq |b-a| + |a| = |a-b| + |a|$ (by tri. inequality) — (*)

From (*) $|a| - |b| \leq |a-b|$.

(**) $|b| - |a| \leq |a-b| \Rightarrow |a| - |b| \geq -|a-b|$

(*)

$\boxed{-c \leq a \leq c \Rightarrow |a| \leq c}$

$\Rightarrow -|a-b| \leq |a| - |b| \leq |a-b|$

put $c = |a-b| \Rightarrow ||a|-|b|| \leq c \Rightarrow ||a|-|b|| \leq |a-b|$

$$\underbrace{-|a-b|}_{c} \le |a|-|b| \le |a-b| \quad \Rightarrow \big||a|-|b|\big| \le |a-b|$$

$$\boxed{-c \le a \le c \quad \Rightarrow \quad |a| \le c}$$

(b) $|a-b| \le |a|+|b|$

We have triangular inequality $|a+b| \le |a|+|b|$

replace b by $-b$

$|a+(-b)| \le |a|+|(-b)|$

$\Rightarrow \quad |a-b| \le |a|+|-b| \qquad (|b|=|-b|)$

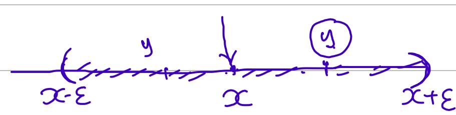
$\Rightarrow \quad |a-b| \le |a|+|b|$

$a-b \le a+b$? $\qquad a,b \in \mathbb{R}$

Triangular inequality $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$

* Real line $\qquad$ Extended Real Numbers

ε-nbhd. $\quad V_\varepsilon(x) = \{y \,/\, y \in (x-\varepsilon, x+\varepsilon)\} \subseteq \mathbb{R}$



$(x-\varepsilon, x+\varepsilon) = V_\varepsilon(x)$

interval length $\Rightarrow 2\varepsilon$

$|x-y| \le \varepsilon \quad \not\Rightarrow y \in V_\varepsilon(x)$

$$\left.\begin{array}{l} V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \\ y \in V_\varepsilon(x) \Rightarrow |x-y| \le \varepsilon \quad \not\Leftarrow \\ \Rightarrow x-\varepsilon \le y \le x+\varepsilon \end{array}\right\}$$

$\delta_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) - \{x\}$ $\quad$ deleted nbhd of x

e.g. we know $|a+b| \leq |a|+|b|$ but if $|a+b| = |a|+|b|$ iff $ab \geq 0$

If $ab \geq 0$ to prove $|a+b| = |a|+|b|$

i) If $a>0, b>0, a+b>0 \Rightarrow |a+b| = a+b = |a|+|b|$

$a<0, b<0, a+b<0 \Rightarrow |a+b| = -(a+b) = (-a)+(-b) = |a|+|b|$

~~$a>0, b<0$~~

If $|a+b| = |a|+|b|$ to prove $ab \geq 0$

$$|a+b|^2 = (|a|+|b|)^2$$

$$(a+b)^2 = |a|^2 + |b|^2 + 2|a|\cdot|b|$$

$$\Rightarrow \cancel{a^2} + \cancel{b^2} + \cancel{2}ab = \cancel{a^2} + \cancel{b^2} + \cancel{2}|a||b|$$

$$\Rightarrow ab = |a|\cdot|b| = |a\cdot b|$$

$$\Rightarrow ab \geq 0$$

e.g. If $x, y, z \in \mathbb{R}$ & $x \leq z$ show that

$x \leq y \leq z$ iff $|x-y| + |y-z| = |x-z|$

i) Let $x \leq y \leq z$. $x-y$ & $y-z$ negative.

LHS $= |x-y| + |y-z|$

$= (y-x) + (z-y)$

$= z-x$

$= |x-z|$

$=$ RHS.

$x \leq y \Rightarrow x-y \leq 0$, $y \leq z \Rightarrow y-z \leq 0$

$|x-y| = -(x-y) = y-x$ $|y-z| = -(y-z) = z-y$.

$x \leq z \Rightarrow x-z \leq 0$

$|x-z| = z-x$

$|x-y| \neq (x-y)$

ii) Let $|x-y| + |y-z| = |x-z|$, $x \leq z$

To prove $x \leq y \leq z$

$x \leq z \Rightarrow x-z \leq 0$, $z-x \geq 0$

$x-z = x-y+y+z$

$x-z = \underline{x-y}+\underline{y-z}$

$|x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z|$ ✓

but $|x-z| = |x-y|+|y-z|$ so. we can use

$\left[\ |a+b| = |a|+|b| \quad \text{iff} \quad \underline{\underline{ab \geq 0}}\ \right]$

$(x-y)\,(y-z) \geq 0$

$x-y \geq 0$

$x \geq y$

Only two possibilities

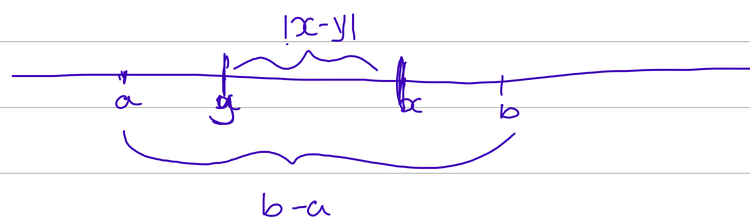
$(x-y),(y-z) \geq 0 \Rightarrow x \geq y,\ y \geq z \Rightarrow x \geq y \geq z$, which is not possible

$(x-y)\,(y-z) \leq 0 \Rightarrow x \leq y,\ y \leq z \Rightarrow \underline{x \leq y \leq z}$

if $a < x < b$

$a < y < b$

$|x-y| \leq b-a$



?

$-a > -x > -b \Rightarrow -b < -x < -a$

$a < y < b \qquad + \quad \underline{a < y < b}$

$-(b-a) < y-x < b-a$

$\Rightarrow |x-y| \leq b-a$

e.g. $|x-1| > |x+1|$ $\qquad x \in ?$ $\qquad \underline{\underline{x<0}}$

$\underline{\underline{-3}}$ $\qquad \underline{\underline{\leq 0}}$ $\qquad -0.01$

$|x+1|+|x-2| = 7$ $\qquad x=-3 \quad x=4$

e.g. $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$ $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$

→ (1) Let $\underline{\underline{a<b}}$, $\min(a,b) = a$ $\max(a,b) = b$

$|a-b| = b-a$

RHS $= \frac{1}{2}(a+b+|a-b|)$

$= \frac{1}{2}(\cancel{a}+b+b-\cancel{a})$

$= \frac{2b}{2} = \max(a,b)$

RHS $= \frac{1}{2}(a+b-|a-b|)$

$= \frac{1}{2}(a+b-(b-a))$

$= \frac{\cancel{2}a}{\cancel{2}} = \min(a,b)$