Q.1 a) Compute the likelyhood Function (LF)

Ans: Let $x_1, x_2 \ldots x_n$ be a sample of n iid random variable following poisson distribution.

Thus: PMF

$$P(X_i = k) = (\lambda^k * e^{(-\lambda)}) / k!$$

where $k = 0, 1, 2 \ldots$ so on

$$= P(X_1 = x_1) * P(X_2 = x_2) * \ldots * P(X_n = x_n)$$

$$= (\lambda^{x_1} * e^{(-\lambda)}) / x_1! * (\lambda^{x_2} * e^{(-\lambda)}) / x_2! * \ldots * (\lambda^{x_n} * e^{(-\lambda)}) / x_n!$$

$$\boxed{= \lambda^{(x_1 + x_2 + \ldots + x_n)} * e^{(-n * \lambda)} / (x_1! * x_2! \ldots * x_n!)}$$

OR

$$f(x_i|\lambda) = \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \Rightarrow f((x_1, x_2, x_3, \ldots x_9, x_{10}) / \lambda)$$

$$= \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \times \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \times \ldots \times \frac{e^{-\lambda} \lambda^{x_{10}}}{x_{10}!}$$

$$\boxed{= e^{-10\lambda} \times \lambda^{(x_1 + x_2 + x_3 + \ldots + x_{10})}}$$

Q2b) Adopt the appropriate conjugate prior to Parameter

Ans: Appropriate conjugate prior to poisson dist is gamma distribution.

PDF:

$$F(x \mid A, \beta) = (\beta^{A} * x^{(A-1)} * e^{(-\beta * x)}) / \Gamma(A)$$

x = Random variable

A, β = parameters

Γ = Gamma function.

Assuming I choose a gamma prior with hyperparameter $a0$ & $\beta 0$

Thus PDF:

$$F(\lambda \mid a0 \beta 0) = (\beta 0^{a0} * \lambda^{(a0-1)}) * e^{(-\beta 0 * \lambda)} / \Gamma(a0)$$

Note: λ (lambda) is rate parameter of Poisson Distribution.

$$\boxed{F(\lambda \mid x_1, x_2 \ldots x_n), a0, \beta 0) = (\beta^{a} * \lambda^{(a-1)}) * e^{(-\beta * \lambda)}) / \Gamma(a')}$$

Q2 c) Using (a) & (b), Find the posterior distribution of

Ans: Combining Likelihood Function with Gamma Prior

$$LF(\lambda|x_1, x_2 \ldots x_n) = e^{(-n*\lambda)} * \lambda^{\Sigma x_i} / P(x_i!)$$

PDF for Gamma prior Distribution

$$F(\lambda | a_0 \beta_0) = (\beta_0^{a_0} * \lambda^{(a_0-1)}) * e^{(-\beta_0 * \lambda)} / \Gamma(a_0)$$

$$\boxed{F(\lambda|x_1, x_2, \ldots, x_n, a_0, \beta_0) = (\beta^{a} * \lambda^{(a-1)} * e^{(-\beta*\lambda)}) / \Gamma(a)}$$

Q2 d) compute the minimum Bayesian risk estimator of

Ans: Minimize loss function (Squared Error Loss).

$$L(\lambda, d) = (\lambda - d)^2 \rightarrow d \text{ is true value of } \lambda$$

Expected loss:

$$E[L(\lambda, d) | x_1, x_2 \ldots x_n, a_0, \beta_0] =$$

$$\int L(\lambda, d) * F(\lambda|x_1, x_2, x_n, a_0, \beta_0)\, d\lambda$$

$$= F(\lambda - a)^2 * (\beta^a * \lambda^{(a-1)} * e^{(-\beta * \lambda)}) / \Gamma(a)\, d\lambda$$

∴ Taking derivative & setting it to zero to minimize loss

$$d/d\lambda\, [(\lambda - a)^2 * (\beta^a * \lambda^{(a-1)} * e^{(-\beta * \lambda)}) / \Gamma(a)$$

$$= 2a(\beta^a * \lambda^{(a-1)} * e^{(-\beta * a)}) / \Gamma(a)$$