PHASE-TYPE DISTRIBUTION

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by

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to the

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CERTIFICATE

This is to certify that the work contained in this report entitled **Phase type distribution** submitted by **Manoj Siyag** (**Roll No: 202123023**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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0.1.1 Introduction

In many stochastic models, accurate analytical and numerical results can only be obtained if certain random variables are assumed to have a negative exponential distribution. Thus, an analyst uses the Poisson process for analyzing queues, uses birth-death assumptions to develop models, and assigns negative exponential duration to service times. The flexibility we owe to the negative exponential distribution is due to its memorylessness property, which eliminates serious increases in dimensionality due to conditioning.

We know that exponential distributions have limitations when modeling actual durations. A large probability is assigned to shorter time intervals, and in many real situations, unimodality or multimodality is not fully represented. This was recognized by A.K. Erlang, and he introduced a probability distribution that contained his name (Erlang distribution). The Erlang distribution provides more flexibility in modeling than the exponential distribution, which only has one parameter, and it is also flexible in fitting an allocation to actual data. Now, it's common to assume Erlang or hyper exponential (finite mixture of negative exponential) distribution to model random time duration, which is more flexible than exponential.

Considering these properties, we would introduce a highly versatile class of distributions, the so-called phase-type or PH distributions. Erlang and hyper exponential distributions are very special cases of phase-type distribution.

0.2 Some preliminaries and Basic definition

0.2.1 Stochastic process

There has been the greater realization that probability (or non-deterministic) models and more realistic than deterministic models in many situations. Observations taken at different time points rather than those taken at a fixed period of time began to engage the attention of probabilistic.

Many a phenomenon occurring in physical and life science are studied now not only as a random phenomenon but also as one changing with time or space. Similar considerations are also made in other areas, such as social sciences, engineering, and management. The scope of applications of random variables, which are functions of time or space or both, has been on the increase.

Definition 1. *Let* (Ω, F, P) *be a given probability space. A collection of random variable* $(X(t), t \geq 0)$ *defined on the probability space* (Ω, F, P) *is called stochastic process.*

Alternatively, stochastic process is also defined as a function of two-argument $X(\omega, T), \omega \in \Omega, t \in T$

- *Stochastic process* $\rightarrow$ *chance process*
- *Stochastic process* $\rightarrow$ *Random process*

Note: *We can say stochastic process is a chance process or random process.*
We have a question, either we create one stochastic process or more than one stochastic process from a given probability space
The answer is, yes, you can create more than one random variable from the same probability space, which means for a different collection of a T, you can have a different stochastic process.
If I changed the σ *algebra (F), then I may land up collecting some other. stochastic process in which those real-valued function is going to be random stochastic process is going to be changed for a different collection of a* $t \in T$*. That means once you know the F, then you will have some collection of a*

state/parameter	discrete	continuous
discrete	dis. time and dis. parameter	dis. time and cont. para.
continuous	cont. time and discrete para.	cont. time and cont. parameter

Table 1: Types of stochastic process

random variable that will form a stochastic process.
If you change another F, then you may get a different stochastic process.

Parameter space:- The set T is called parameter space where t∈T may denote time, length, distance, or any other quality.

State space:- The set s is the set of all possible values of x(t) ∀ t.
One-dimensional processes can be classified, in general, into the four types of processes, as given in table (1)

0.2.2 Continuous time Markov chain

It is an stochastic process with continuous parameter and discrete state space $(X(t), t \geq 0)$

Here we have a system with a countable state space that can change its state at any point.

Definition 2. *A stochastic process* $(x(t), t \geq 0)$ *is called CTMC.If* $\forall t \geq 0, s \geq 0, i \in s, j \in$*s has markov property at each time t and it is time homogeneous or stationary transient probability.*

$$\cdot p\left\{X(t+s)=j|X(s)=i;(X(u):0\leq u\leq s)\right\}=p(X(t+s)=j|X(s)=i).$$
$$p\left\{X(t+s)=j| \quad X(s)=i\right\}=p(X(t)=j|X(0)=i)=p_{ij}(t)..$$

Provided

- $0 \leq p_{ij}(t) \leq 1 \forall i, j, t.$

- $\sum(p_{ij}(t)) = 1, \quad \forall j.$

Definition 3. ***Communicating classes:-***
A set of states $C \subseteq S$ in a CTMC is said to be a closed communicating class of the corresponding embedded DTMC.

Definition 4. ***Irreducibility:-***
A CTMC is called irreducible if the corresponding embedded DTMC is irreducible. Otherwise, it is called reducible.

Definition 5. ***Recurrent(transient):-***
A state i is recurrent in CTMC if and only if it is recurrent in the corresponding embedded DTMC.

Instantaneous transition rates

- For any pair of states i,j.
  Let $q_{ij} = \lambda_i p_{ij}$ Rate ,when in state i, at which process makes a transition into state j.
  $\lambda_i \rightarrow$ Rate at which the process makes a transition when in state i.
  $p_{ij} \rightarrow$ probability that this transition is into state j.
  $\sum_j p_{ij} = \lambda_i p_{ij} = \lambda_i \sum_j p_{ij} = \lambda_i$
  $p_{ij} = \frac{q_{ij}}{\lambda_i} = \frac{q_{ij}}{\sum_j p_{ij}}$

- so specifying the instantaneous transition rates determines the parameter of the CTMC.

- For finite state space $s = \{0, 1, 2, 3, \cdots, m\}$

$$
Q_{m\times m} = \begin{bmatrix} -q_0 & q_{01} & \cdots & q_{0m} \\ q_{10} & -q_1 & \cdots & q_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m0} & q_{m1} & \cdots & -q_{mm} \end{bmatrix}
$$
$$
= \begin{bmatrix} -\lambda_0 & \lambda_0 p_{01} & \cdots & \lambda_0 p_{0m} \\ \lambda_1 p_{10} & -\lambda_1 & \cdots & \lambda_1 p_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_m p_{m0} & \lambda_m p_{m1} & \cdots & -\lambda_m \end{bmatrix}
$$

0.2.3 Chapman kolmogorov equation:-

Let p(t) be the TPM of a CTMC$\{X(t), t \geq 0\}$ then

(i) $p_{ij}(t) \geq 0, i, j \in s, t \geq 0$

(ii) $\sum_{j\in s} p_{ij}(t) = 1, i \in s, t \geq 0$.

(iii) $p_{ij}(s+t) = \sum_{k\in s} p_{ik}(s)p_{kj}(t); \quad i, j \in s \quad s, t, \geq 0$

0.2.4 Forward and backward Equation:-

- Let p(t) be the transition probability matrix of a CTMC with state -space $S = \{0, 1, 2, \cdots\}$ and generator matrix Q.Then p(t)is differentiable with respect t and satisfies.

- $\frac{d(p(t))}{dt} = P' = QP(t) \quad \{backwardEquation\}$.

- $\frac{d(p(t))}{dt} = P' = P(t)Q \quad \{ForwardEquation\}$. with intial condition P(0)=1 Where I is an identity matrix of appropriate size.

0.3 Phase type

A phase-type distribution is a probability distribution constructed by a convolution or mixture of exponential distributions. It results from a system of one or more inter-related Poisson processes occurring in sequence or phases. The sequence in which each of the phases occurs may be a stochastic process. The distribution can be represented by a random variable describing the time until absorption of a Markov process with one absorbing state. Each of the states of the Markov process represents one of the phases.

0.3.1 Initial Motivation

- **Erlang Distribution** We take here the Erlang distribution, and we prove that it is a K-phase phase-type distribution. We take here 2-phase process with rate μ. We will prove it by CTMC it is an Erlang distribution of phase 2. Here State $1, 2$ is transient, and state 3 is absorbing.

  We have a rate transition matrix of above CTMC.

  $$Q = \begin{bmatrix} -\mu & \mu & 0 \\ 0 & -\mu & \mu \\ 0 & 0 & 0 \end{bmatrix}$$

  Here, we can observe some block matrices in the rate transition matrix, given below, which can play a crucial role in phase type distribution.

  $$T = \begin{bmatrix} -\mu & \mu \\ 0 & -\mu \end{bmatrix}$$

  $$\eta = \begin{bmatrix} 0 \\ \mu \end{bmatrix}, 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

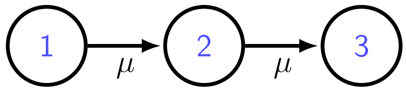


Figure 1: Erlang distribution

Thus the rate transition matrix can be written in following manner-

$$Q = \begin{bmatrix} T & \eta \\ 0 & 0 \end{bmatrix}$$
$$\text{where} \quad \eta = -T1 \tag{1}$$

Proof by Chapman–Kolmogorov equation, we have state transition probabilities

$$\begin{aligned} p_1'(t) &= -\mu p_1(t) \\ p_2'(t) &= \mu p_1(t) - \mu p_2(t) \\ p_3'(t) &= \mu p_2(t) \end{aligned}$$

on solving first differential equation

$$\begin{aligned} p_1(t) &= e^{-\mu(t)}, \qquad \{\text{using intial condition} \;\; p_1(0) = 1\} \\ p_2(t) &= \mu t e^{-\mu t} \\ p_3'(t) &= \mu^2 t e^{-\mu t} \end{aligned}$$

We observe here that $p_3'(t)$ is the PDF same as Erlang-2. So we prove here the phase-type process we consider above is the Erlang process. Hence Erlang distribution is the phase-type distribution.

- **Note** We get different phase-type distributions by choosing different Markov chains with different initial conditions. These distributions are not (in general) exponential, but they can be analyzed using the theory of continuous-time Markov chain, taking advantage of the underlying

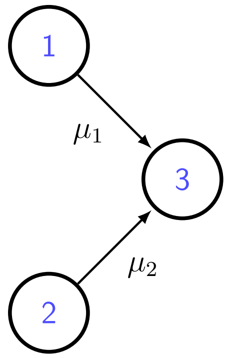


Figure 2: Hyperexponential distribution

distribution. Some more examples are given below, which we can prove are not, in general, exponential.

- **Hyperexponential Distribution** We take here the Hyperexponetial distribution, and we prove that it is a K-phase phase-type distribution. We take her 2-phase process with rate μ_1 and μ_2. We will prove it by CTMC it is an Erlang distribution of phase 2. Here States $1, 2$ are transient, state 3 is absorbing.

$$Q = \begin{bmatrix} -\mu_1 & 0 & \mu_1 \\ 0 & -\mu_2 & \mu_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix}$$

$$\eta = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} T & \eta \\ 0 & 0 \end{bmatrix} \qquad \eta = -T1$$

From Chapman-Kolmogorov we have

$$p'(t) = p(t)Q$$
$$[p_1'(t) : \ p_1'(t) : \ p_1'(t)] = [-\mu_1 p_1(t) : \ -\mu_2 p_2(t) : \ \mu_1 p_1(t) + \mu_2 p_2(t)]$$

On solving for $p_1'(t), p_2'(t), p_3'(t)$

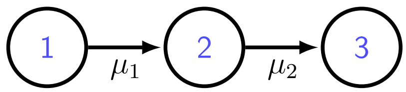
$$p_1(t) = qe^{(-\mu_1 t)}, \quad \text{using initial condition}(p_1(0) = q, p_2(0) = (1-q))$$
$$p_2(t) = (1-q)\exp(-\mu t)$$
$$p_3'(t) = q\mu_1 \exp(-\mu t) + (1-q)\mu_2 \exp(-\mu_2 t)$$

We observe here that $p_3'(t)$ is the PDF same as Hyperexponential-2. So we prove here the phase-type process we consider above is the Hyperexponential process.

- **Hypo-exponential distribution** $T \rightarrow$ time of absorption is convolution of 2 non identical exponential variable consider the following continuous time markov chain where μ_1 not equal to μ_2 and the system starts in state 1.

The associated Q matrix and initial conditions are

$p(0) = (1, 0, 0)$



$$Q = \begin{bmatrix} -\mu_1 & \mu_1 & 0 \\ 0 & -\mu_2 & \mu_2 \\ 0 & 0 & 0 \end{bmatrix}$$

The time to absorbation is the convolution of two nonidentical exponetial random variable .(This is not Erlang , since Erlang reuire IID exponetials) .

$$\begin{aligned} p_1(t) &= e^{(-\mu_1 t)}, \\ p_2(t) &= \frac{\mu_1}{\mu_1 + \mu_2} \exp(-\mu t) - \frac{\mu_1}{\mu_2 - \mu_1} \exp(-\mu_2 t) \\ p_3'(t) &= \mu_2 p_2(t) = \frac{\mu_1 \mu_2}{\mu_2 - \mu_1} \exp(-\mu_1 t) + \frac{\mu_1 \mu_2}{\mu_2 - \mu_1} \exp(-\mu_2 t) \end{aligned}$$

Where $p_3'(t)$ is the density funtion of the time to absorbation. which is a two-term hypoexponetial distribution.

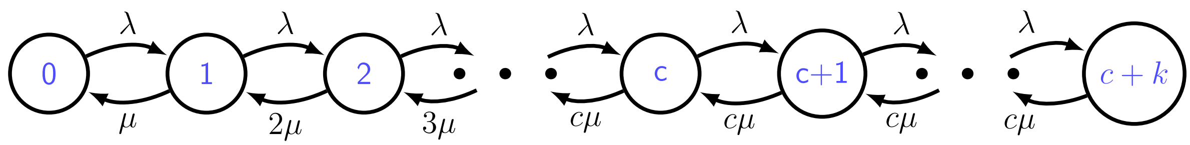


Figure 3: $M/M/c/c+k$ queue

- Now we begin with **practical example**. Consider the $M/M/C/C+K$ queue capacity of waiting room k total customer in system c+k
  If c+k users are already in the system, then new arrival (c+k+1) nth arrival is not admitted into the system. arrival user lost, this queueing system call loss system.

  The generator matrix for the above queue is

$$Q_{(c+k)\times(c+k)} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 \\ \mu & -(\mu+\lambda) & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & c\mu & -(\lambda+\mu) & \lambda \\ 0 & \cdots & \cdots & c\mu & -c\mu \end{bmatrix}$$

  From a system administrator's point of view. The loss of a user is regarded as a bad event .

  question arises:- how the distribute the time up to the first loss might be expressed(he wants to know probability when system absorption)

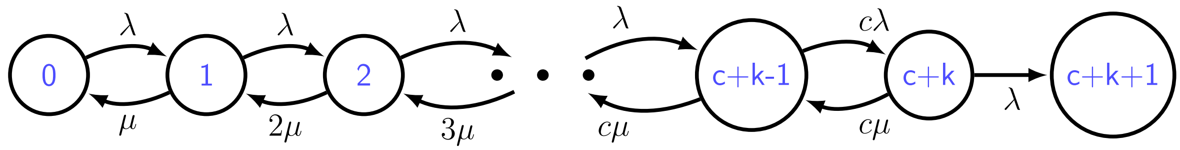


Figure 4: $M/M/c/c+k$ queue with one lost customer

Now, we have to include the loss event into our model of the queue.

$$Q_{(c+k+1)\times(c+k+1)} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & c\mu & -(\lambda+c\mu) & \lambda & 0 \\ 0 & 0 & \cdots & c\mu & -(\lambda+c\mu) & \lambda \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Now, the system administration queue can be formulated mathematically as the distribution of the time until the Markov process with generator Q' enters the absorbing state c+k+1.

Exactly, this problem is addressed (in a general form) by the concept of phase-type distribution.

Definition 6. *let* $\chi = \{x_t, t \geq 0\}$ *denote an homogeneous Markov process with finite state* $\{1, 2, 3, \cdots, m+1\}$ *and generator*

$$Q = \begin{bmatrix} T & \eta \\ 0 & 0 \end{bmatrix}$$

where, T is square matrix, η is column vector, $[0]$ = row vector with m dimension. The initial distribution of χ be row vector $\widetilde{\alpha} = [\alpha, \alpha_{m+1}]$. First state $\{1, 2, 3, \cdots, m\} \rightarrow$ transient and state $m+1 \rightarrow$ absorbing state. let $Z = \inf\{t \geq 0 : x_t = m+1\}$ be the random variable of the ***time until absorption*** *in state $m+1$. The distribution of Z is called phase type distribution with parameter (α, T). We write $Z \sim PH(\alpha, T)$. The dimension m of T is called the* ***order*** *of the distribution $PH(\alpha, T)$. The states $\{1, \ldots, m\}$ are also called phases, which gives rise to the name phase-type distribution.*

Let 1 denote the column vector m with all entries equal to one. The observation from the above definition are $\eta \rightarrow$ exit vector ; $\eta = -T1$, and $\alpha_{m+1} = 1 - \alpha 1 = 1 - \sum_{i=1}^{m} \alpha_i \left\{\sum_{i=1}^{m+1} \alpha_i = 1\right\}$. The vector η is called the ***exit vector*** *of the PH distribution.*

Theorem 1. *Let $Z \sim PH(\alpha, T)$. Then the distribution function of Z is given by*

$$F(t) := \mathbb{P}(Z \leq t) = 1 - \alpha e^{T \cdot t} 1$$

for all $t \geq 0$, and the density function is

$$f(t) = \alpha e^{T \cdot t} \eta$$

for all $t > 0$. Here, the function $e^{T \cdot t} := \exp(T, t) := \sum_{n=0}^{\infty} \frac{t^n}{n} T^n$ denotes ***a matrix exponential function****.*

Proof

$$P_T = (P(t), P_{m+1}(t))$$

by C-K,

$$P_T'(t) = P_{(t)} Q$$

$$p' = P_{(t)} Q$$

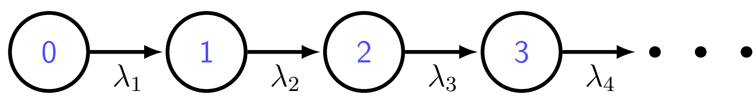
solution to

$$P' = p(t)T \quad is \quad p(t) = \alpha e^{tT}$$

If X is a time of absorbing, thus the probability that CTMC is not yet absorbed at time t

$$
\begin{aligned}
p(t)1 &= \alpha e^{tT} = p\{t < x\} \\
\text{so } p(X \geq x) &= 1 - p(X \leq x) \\
&= 1 - \alpha e^{Tx} = F_x(x) \\
F(x) &= F'(x) = -\alpha e^{Tx} T1 \quad \{T1 + \eta = 0\} \\
&= \alpha e^{Tx}(-T1) \\
&= \alpha e^{Tx}\eta = f(x) \\
e^T t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} T^n = I + Tt + \frac{(Tt)^2}{2!} + \cdots
\end{aligned}
$$

Example Generalized Erlang distribution



A slight generalization of the Erlang distribution is obtained if one admits the exponential stages to have different parameters. Then we talk about a generalized Erlang (or a hypo-exponential) distribution. The representation as a PH distribution results in the figure and leads to a PH representation

$$
\alpha = (1, 0, \ldots, 0), \quad T = \begin{pmatrix} -\lambda_1 & \lambda_1 & & \\ & \ddots & \ddots & \\ & & -\lambda_{n-1} & \lambda_{n-1} \\ & & & -\lambda_n \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_n \end{pmatrix}
$$

with all non-specified entries in T being zero. For this family of distributions, a closed formula for the density function is already rather complex.

Theorem 2. *Let $Z \sim PH(\pi, T)$. If $\eta = -Te = \lambda e$ for some constant $\lambda > 0$, Then $Z \sim \exp(\lambda)$, i.e, an exponential distribution with rate λ.*

The density f of t is given by

$$\begin{aligned} f(x) &= \pi e^{Tx_t} \\ &= \lambda \pi e^{Tx_e} \\ &= \lambda S(x) \end{aligned}$$

where $S(x) = 1 - F(x) = \mathbb{P}(Z > x)$ is the survival function of Z. But $f(x) = -S'(x)$ and $S(0) = 1$, so we have that

$$S'(x) = -\lambda S(x), \quad S(0) = 1$$

the solution of which is $S(x) = \exp(-\lambda x)$. Hence $Z \sim \exp(\lambda)$. Hence the dimension of the representation p does not necessarily reflect the "true" order of a phase-type distribution. To this end, we make the following definition.

Definition 7. *For a phase-type representation $PH(\alpha, T)$, m is referred to as it's dimension. The order of a phase-type distribution is the smallest dimension among all its representations.*

Thus the order of the exponential distribution seen as phase-type distribution is one, while the dimension of a representation for the exponential distribution can be arbitrarily large.
Example.
Let $Z \sim PH(\alpha, T), M = 3$, with $\alpha = [\alpha_1, \alpha_2, \alpha_3]$

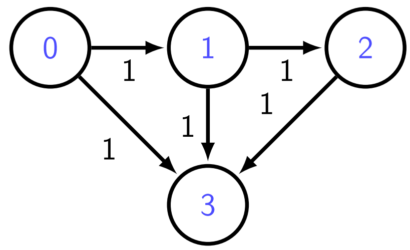


Figure 5: Order of PH-type

$$Q = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where we can observe

$$T = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\eta = -T1 = (1, 1, 1)'$ and $Z \sim \exp(1)$

So here order of the phase type distribution is 1 and dimension is 3.

Definition 8. *A phase-type representation PH(α, T) is called irreducible if there is a positive probability of any of the transient states starting from the initial distribution α,i.e., if*
$(\alpha P(t))_i \geq 0 \implies (\alpha e^{(Tt)})_i \geq 0$ *for i=1,2,3,... ,p and* $\forall t \geq 0$

Theorem 3. *A phase-type distribution with an irreducible representation has strictly positive density f(x)* $\forall x \geq 0$.

Proof:- Consider $X \sim PH(\alpha, S)$ where (α, S) is an representation .We have $\alpha e^{Sx} \geq 0$and $s = Se \geq 0$, with strict equality at least one position .So far $x > 0$We have $\alpha e^{Sx} > 0, \forall x > 0$. Let $z \sim ph(\alpha, T)$ where (α, T) is an irreducible representation d.

$$f(t) = \alpha e^{Tt}\eta$$

$f(t) = \alpha e^{Tt}\eta$

Corollary 1. *Let $PH(\alpha, T)$ be a phase type representation.then T is invertible.*
definition Let $ph(\alpha, T)$ be a phase type representation, then $U = (-T)^{-1}$is called the it's ***Green matrix****.*

Theorem 4. *A phase-type representation .Then T is invertible*

Laplace-Stieltjes Transform

We know LST

$$\phi(t) = \int_0^\infty \exp(-st)dF(t), \quad s \in \text{add complex symbol} \quad withRe(s) \geq 0$$

$$\int_0^\infty \exp(-st)dt = \frac{-1}{s}\left[\exp(Tt)\exp() - st) - \int_0^\infty \exp(-st)dt\right]$$

T $\rightarrow$ assumed to be Invertible and we know $\int_0^\infty e^{Tt}dt \geq 0$ (Entrywise) and hence $\lim_{t\to\infty} \exp Tt = 0$
value $e_{ij}^{Tt}, p(t) = e^{Qt} \rightarrow$ probability that the process χ (phase type) is in state j at time t given that it started in state i so at t $\rightarrow \infty$ process moves to absorbing state then there probability should be zero.

$$
\begin{aligned}
\int_0^\infty e^{-st}e^{Tt}dt &= \frac{I}{s} + \frac{1}{s}\int_0^\infty e^{-st}e^{Tt}dt.T \\
&= \int_0^\infty e^{-st}e^{\iota t}dt - \frac{1}{\iota}\int_0^\infty e^{-st}e^{\iota t}dt = \frac{I}{s} \\
&= \int_0^\infty e^{-st}e^{\iota t}dt(s - I - T) = I \\
&= \int_0^\infty e^{-st}e^{\iota t}dt = (s - I - T)^{-1} \quad \forall s \neq 0 \quad withRe(s) \geq 0
\end{aligned}
$$

so,

$$
\begin{aligned}
\phi(s) = \int_0^\infty e^{-st}f(t)dt &= \alpha_{m+1} + \int_0^\infty e^{-st}\alpha e^{Tt}\eta dt \\
&= \alpha_{m+1} + \alpha(s.I - T)^{-1}\eta
\end{aligned}
$$

<u>Moments of phase type(Z)</u>:-

$$
\begin{aligned}
Z &\sim ph(\alpha, T) \\
E(Z^n) &= (-1)^n \left[\frac{d^n}{ds^n}\phi(s)\right]|_{s=0} \\
&= (-1)^n \left[\alpha\frac{d^n}{ds^n}(sI - T)^{-}1\eta\right]| \\
&= (-1)^n \left[\alpha(-1 \times -2 \times -3 \cdots \times -n)\eta(sI - T)^{-(n+1)}\right] \\
&= \alpha(n!)(-T)^{-n}(-T)^{-1}1 \quad \eta = -T1 \\
&= n!\alpha(-T)^{-n}(-T)^{-1}(-T)1 \\
E(z^n) &= n!\alpha(-T)^{-n}1
\end{aligned}
$$

This is the nth order moment for PH-type distribution.

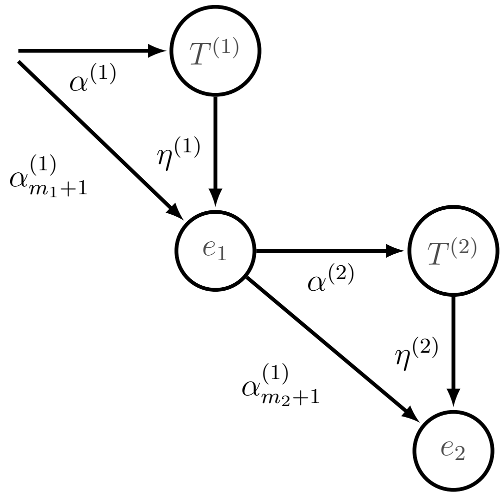
A valuable advantage of phase-type distribution is that certain compositions of PH distributions result in PH distribution again. This means that the class of PH distribution is closed under these compositions. For PH distributed random variables.

Theorem 5. *Z_1 and Z_2(convolution)$PZ_1+(1-P)Z_2$ with $P \to [0,1]$(mixture) and $min(Z_1, Z_2)$.*

Let $Z_i \sim \mathrm{PH}(\alpha^{(i)}, T^{(i)})$ of orderm_i for $i = 1, 2$. Then $Z = (Z_1, , Z_2) \sim PH(\alpha, T)$ of order $m = m_1 + m_2$ with representation .

$$Q = \begin{bmatrix} T^{(1)} & \eta^{(1)}\alpha^{(2)} \\ 0 & T^{(2)} \end{bmatrix}$$

proof:
By definition Z_i is the random variable of the time until absorbing in a Markov process χ with transitient states $(1, \cdots, m_i)$ and an absorbing staes which shall be denoted by $\exp_i$ in the proof .The transition rates of χ within the set of transient states are given by the matrix $T^{(i)}$and the absorption rates fromm the transient states to the absorbing state are given by the vector $\eta^{(i)}$. Then the random variable $Z = Z_1 + Z_2$ is the total time duration of first entering $\exp_1$ and then $\exp_2$in the Markov process which is structured as follows.



- **Point of joining** Here the point $e_1$is not a state of the Markov process described above but only an auxiliary construction aid for a better illustration. In particular, there is no holding time in e_2.

- **Analysis for first phase type** With probabilities $\alpha^{(1)}_{m_1+1}$ we enter the first absorbing states e_1 immediately ,while the vector $\alpha^{(1)}$ contains the probabilities that we first enter the set of transient states of X_1.In the latter case,the matrix $T^{(1)}$ and then the vecor $\eta^{(1)}$ determine the time until the first absorbing in e_1.

- **Analysis for Second phase type** After having reaches e_1 ,the chain immediately (i.e. with no holding time in e_1) proceeds to the second stage,which is completely analogous to the first with probabilities $\alpha^{(2)}_{m_2+1}$we enter the second absorbing state $e_2$immediately ,while the vector $\alpha^{(2)}$ contain the probabilities that we first enter the set of transient states of X_2 .In the latter case .the matrix $T^{(2)}$ and the the vector $\eta^{(2)}$ Determine the time until the first absorption in e_2.

- **Structure of States and initial distribution** Thus we get second absorbing states e_2 immediately with probabilities $\alpha^{(1)}_{m_1+1}.\alpha^{(2)}_{m_2+1}$. There are transient states $\{1, \cdots, m_1, m_1+1, \cdots, m_1+m_2\}$ .The first m_1 of these are reached with probabilities $\alpha_1, \cdots$ while the last m_2 of these states can only be reached via an immediate first absorption in e_1 and thus with probabilities $\alpha^{(1)}_{m_1+1}.\alpha^{(2)}_i$ for $i = 1, 2, \cdots, m_2$. This explains the expression for α.

- **Structure for matrix T and exist vector** In order to explain the structure of T ,we observe first that there is no path from the second set of transient states to the first whence the lower left entry of T is zero .The diagonal entries of T describe the transition rates within the two sets of transient states.respectively ,and thus are given in $T^{(1)}$ and $T^{(2)}$.The only way to get from the first to the second set of transient states is the path via $\exp_1$ for which we first need the rates given in $\eta^{(1)}$ and then the probabilities contained in $\alpha^{(2)}$.Hence the upper right entry of T.

Theorem 6. *Let $Z_i \sim PH(\alpha^{(i)}, T^{(i)})$ of orderm$_i$ for $i = 1, 2$. as well as $P \to [0, 1]$. ThenZ $= PZ_1 + (1-P)Z_2 \sim PH(\alpha, T)$ of order $m = m_1 + m_2$ with representation.*

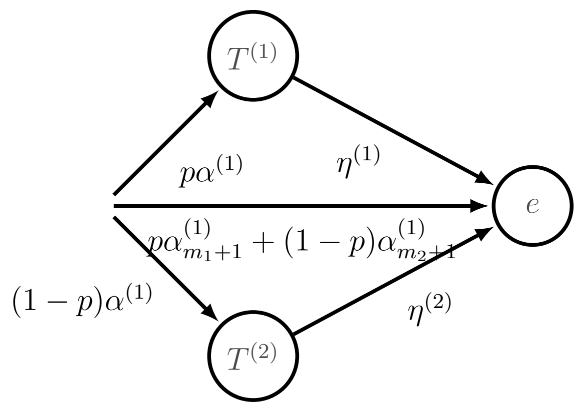
$\alpha = (p.\alpha^{(1)}, (1-p).\alpha^{(2)})$

$$T = \begin{bmatrix} T^{(1)} & 0 \\ 0 & T^{(2)} \end{bmatrix}$$

Where 0 denote zero matrices of appropriate dimensions.

proof:

We were going along the line of reasoning of the last proof. We observe that Z is equal to$Z_1$with probability p and equal toZ_2 with probability 1-p. Hence we obtain the following construction of a Markov process.



Here,we enter the first set of transient states with probabilities $p.\alpha_i^{(1)}$for $i = 1, 2, \cdots, m_1$ and the second set with probabilities $(1-p).\alpha_i^{(2)}$ for phases $i = m_1 + 1, \cdots, m_2$,This explains the expression for α .
From either of these sets, we proceed with transition matrices $T^{(i)}$and exit vector $\eta_i, i = 1, 2,$ in order to reach the absorbing state e.there is no path from one set of transition states to the other, which explains the structure of T. The absorbing state exp can be reached immediately (Without entering any transient state) with probability $p\alpha^{(1)}_{m_1+1} + (1-p)\alpha^{(2)}_{(m_2+1)}$.

0.4 Use in Queueing theory

Earlier, we give service by an exponential distribution which has less flexibility. Now we can provide a service phase-type, which is more flexible than exponential, and the model becomes more precise.
An illustrative $M/\mathrm{PH}_2/1$ generator matrix Q with mean service rates μ_1 and μ_2 is as follows, analogous to that provided for the $M/E_2/1$
Generator matrix has more stages and looks like given below-

$$\begin{pmatrix} 0 & 0 & 1,2 & 1,1 & 2,2 & 2,1 & 3,2 & 3,1 & \dots \\ 1,2 & -\Sigma b_0 & \lambda & 0 & 0 & \dots & & & \\ 0 & -\Sigma b_1 & \mu_2 & \lambda & 0 & 0 & \dots & & \\ 1,1 & \mu_1 & 0 & -\Sigma b_2 & 0 & \lambda & 0 & 0 & \cdots \\ 2,2 & 0 & 0 & 0 & \Sigma b_3 & \mu_2 & \lambda & 0 & \cdots \\ 2,1 & 0 & \mu_1 & 0 & 0 & -\Sigma b_4 & 0 & \lambda & \cdots \\ 3,2 & 0 & 0 & 0 & 0 & 0 & -\Sigma b_5 & \mu_2 & \cdots \\ 3,1 & 0 & 0 & 0 & \mu_1 & 0 & 0 & -\Sigma b_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where $\sum b_i$ denotes the sum of the quantities in row i. When this matrix is rewritten in block form, we find that

$$\boldsymbol{B}00 = [-\sum b0] = \begin{bmatrix} -\lambda \end{bmatrix}, \quad \boldsymbol{B}01 = \begin{bmatrix} \lambda & 0 \end{bmatrix}, \quad \boldsymbol{B}10 = \begin{bmatrix} 0 & \mu_1 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{B}1 = \begin{bmatrix} -\sum 0b_i & \mu_2 \\ 0 & -\sum b_i \end{bmatrix}, \quad \boldsymbol{B}2 = \begin{bmatrix} 0 & 0 \\ \mu 1 & 0 \end{bmatrix}$$

$$\begin{pmatrix} \boldsymbol{B}00 & \boldsymbol{B}01 & \mathbf{0} & \dots & & & \\ \boldsymbol{B}10 & \boldsymbol{B}1 & \boldsymbol{B}0 & \mathbf{0} & \dots & & \\ \mathbf{0} & \boldsymbol{B}2 & \boldsymbol{B}1 & \boldsymbol{B}0 & \mathbf{0} & \dots & \\ \mathbf{0} & \mathbf{0} & \boldsymbol{B}2 & \boldsymbol{B}1 & \boldsymbol{B}_0 & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Note that this block matrix representation of the Q matrix has a tridiagonal structure reminiscent of the usual birth -death generator seen back name quasi -birth-death (QBD) process.

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