Phase-Type Distribution

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Outline

Introduction

- When we study the queueing models, which are modelled by stochastic process mostly handled by Poisson process and exponential distribution.
- When we see the an practical model in exponential distribution assumptions, whether are service process or a arrival process this may be limiting.
- If we go beyond the assumption of exponential or a Poisson process, How do we handle the queue where the inter arrival time or a service time distribution are not exponential.
- We would want to introduce a highly versatile class of distributions, the so called phase-type or PH distribution. Which give more flexibility in modeling rather than Poisson or exponential although they posses exponential distribution.

Stochastic process

- The system evolves rendomly in time .
  example- Stock market index .
  - Collection of random variable. $\{X(t), t \geq 0\}$

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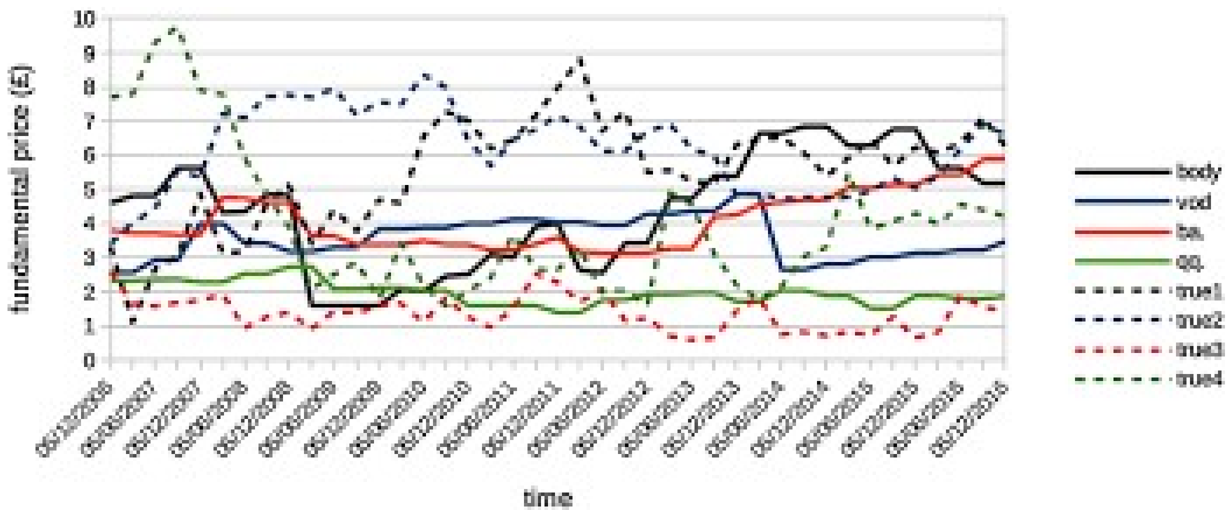

Stochastic process:Definition

Definition

Let (Ω, F, P) be a given probability space. A collection of random variable $\{X(t), t \geq 0\}$ defined on the probability space (Ω, F, P) is called stochastic process.
Alternatively, stochastic process is also defined as a function of two-argument $X(\omega, T), \omega \in \Omega, t \in T$, where ω is state space and T is parametric space.

Stochastic process:Types

- We are observe the system continuously and discretely according to time parameter and space.

state/parameter	discrete	continuous
discrete	dis. time and dis. parameter	dis. time and cont. para.
continuous	cont. time and discrete para.	cont. time and cont. parameter

Table: Types of stochastic process

Markovian propery

- prediction of process depends only on present state does not matter which path followed by process to attain present state.
  - Here contable state space that can change its state at any time .
    -
$$\cdot p\left\{X(t+s)=j|X(s)=i;(X(u):0\leq u\leq s)\right\}=p(X(t+s)=j|X(s)=i).$$

Time homogenous property

- The probability of transfer a one state to another state depends only to time taken to change states does not matter what is intial and ending time .
  - $$p\{X(t+s)=j| \quad X(s)=i\} = p(X(t)=j|X(0)=i) = p_{ij}(t)..$$

Continuous time Markov chain

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Definition

Define $S_0 = 0$ and let $\{S_n, n \geq 1\}$ denote a sequence of RVs such that $S_n > S_{n-1}$ for all $n \geq 1$ and $S_n \to \infty$ as $n \to \infty$. Further, let $\{X_n, n \geq 0\}$ be a sequence of RVs taking values in a countable state space S. A stochastic process $\{Y_t, t \geq 0\}$ with $Y_t = X_n$ for $S_n \leq t < S_{n+1}$ is said to be a pure jump process. The variable $T_n = S_{n+1} - S_n$ (resp. X_n) is called the nth holding time (resp. the nth state) of the process $\{Y_t\}$. If, further, $\{X_n, n \geq 0\}$ is a Markov chain with (stationary) transition probability matrix $P = ((p_{ij}))$. and the variables T_n are independent and distributed exponentially with parameter λ_{x_n} only depending on the state X_n, then $\{Y_i, t \geq 0\}$ is called a (time-homogeneous) continuous-time Markov chain (CTMC). The chain $\{X_n, n \geq 0\}$ is called the embedded discrete time Markov chain of the CTMC. We will always assume that $\sup\{\lambda_1, i \in S\} \leqslant \infty$.

Continuous time Markov chain:Example

Example

Two State Machine:

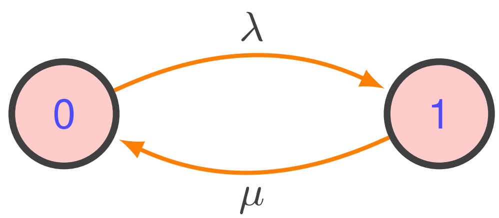
- Consider a machine that can be up or down at any time.
- If the machine is up, it fails after an $\exp(\mu)$ amount of time.
- If it is down it is repaired in an $\exp(\lambda)$ amount of time. The successive up times are iid, ans so are ythe successive down time, and the two are independent of each other.
- Define

$$X(t) = \begin{cases} 0 & \text{if the machine is down at time } t, \\ 1 & \text{if the machine is up at time } t, \end{cases}$$

- $\{X(t), t \geq 0\}$

Continuous time Markov chain:Example

- Rate transition diagram



- Infinitesimal generator matrix

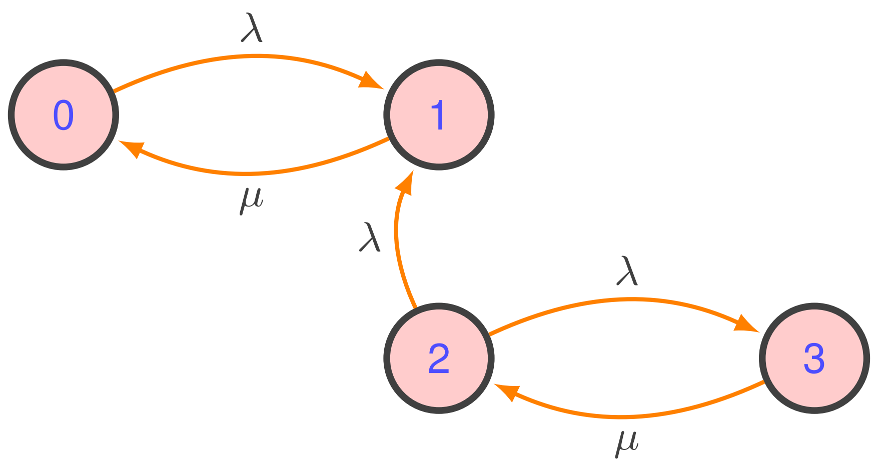
$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Commuinicating class ,Recurrent, Transient and Irreducible of markov chain states

- **communicating class** A communication class $C \in S$ is a set of states whose members communicate, i.e., $i \leftrightarrow j$ for all $i, j \in C$, and no state in C communicates with any state not in C.
- **Irreducible** A finite Markov chain (or equivalently, its transition matrix T) is irreducible, if it has a single communicating class $C = S$.
- **Recurrent state:** A recurrent state has the property that a Markov chain starting at this state returns to this state infinitely often, with probability 1.
- **Transient state:**A transient state has the property that a Markov chain starting at this state returns to this state only finitely often, with probability 1.
- A CTMC contain all of these property if their embedded DTMC posses all these properties.

Commuinicating class ,Recurrent, Transient and Irreducible of Markov chain states:Example

Example



Chapman Kolmogorov Equation: Forward and Backward

- Let p(t) be the TPM of a CTMC $\{X(t), t \geq 0\}$ then
  - (i) $p_{ij}(t) \geq 0, i, j \in s, t \geq 0$
  - (ii) $\sum_{j \in s} p_{ij}(t) = 1, i \in s, t \geq 0$.
  - (iii) $p_{ij}(s+t) = \sum_{k \in s} p_{ik}(s) p_{kj}(t); \quad i, j \in s \quad s, t, \geq 0$

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  (iii) $p_{ij}(s+t) = \sum_{k \in s} p_{ik}(s) p_{kj}(t); \quad i, j \in s \quad s, t, \geq 0$
- Let p(t) be the transition probability matrix of a CTMC with state -space $S = \{0, 1, 2, \cdots\}$ and generator matrix Q.Then p(t)is differentiable with respect t and satisfies.
- $\frac{d(p(t))}{dt} = P' = QP(t) \quad$ {Backward Equation} .
- $\frac{d(p(t))}{dt} = P' = P(t)Q \quad$ {Forward Equation} .
  with intial condition $P(0) = I$, where I is an identity matrix of appropriate size.

Phase type distribution

Definition

let $\chi = \{x_t, t \geq 0\}$ denote an homogeneous Markov process with finite state $\{1, 2, 3, \cdots, m+1\}$ and generator

$$Q = \begin{bmatrix} T & \eta \\ 0 & 0 \end{bmatrix}$$

where, T is square matrix, η is column vector, $[0] =$ row vector with m dimension. The initial distribution of χ be row vector $\widetilde{\alpha} = [\alpha, \alpha_{m+1}]$. First state $\{1, 2, 3, \cdots, m\} \rightarrow$ transient and state $m+1 \rightarrow$ absorbing state. let $Z = \inf\{t \geq 0 : x_t = m+1\}$ be the random variable of the **time until absorption** in state $m+1$. The distribution of Z is called phase type distribution with parameter (α, T).

CDF and pdf of Phase type distribution

Theorem

Let $Z \sim PH(\alpha, T)$. Then the distribution function of Z is given by

$$F(t) := \mathbb{P}(Z \leq t) = 1 - \alpha e^{T \cdot t} 1$$

where 1 is column vector for all $t \geq 0$, and the density function is

$$f(t) = \alpha e^{T \cdot t} \eta$$

*for all $t > 0$. Here, the function $e^{T \cdot t} := \exp(T, t) := \sum_{n=0}^{\infty} \frac{t^n}{n} T^n$ denotes **a matrix exponential function**.*

Erlang distribution

- **Erlang Distribution** We take here the an absorbing CTMC with 2-phase process with rate μ. We will prove it by CTMC it is an Erlang distribution of phase 2. Here State $1, 2$ is transient, and state 3 is absorbing.

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- We have rate transition diagram as

$$1 \xrightarrow{\mu} 2 \xrightarrow{\mu} 3$$

Figure: Erlang distribution

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Figure: Erlang distribution

- We have a rate transition matrix of above CTMC.

$$Q = \begin{bmatrix} -\mu & \mu & 0 \\ 0 & -\mu & \mu \\ 0 & 0 & 0 \end{bmatrix}$$

Erlang distribution

- Here, we can observe some block matrices in the rate transition matrix, given below, which can play a crucial role in phase type distribution.

$$T = \begin{bmatrix} -\mu & \mu \\ 0 & -\mu \end{bmatrix}$$

$$\eta = \begin{bmatrix} 0 \\ \mu \end{bmatrix}, 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Thus the rate transition matrix can be written in following manner-

$$Q = \begin{bmatrix} T & \eta \\ 0 & 0 \end{bmatrix}$$

$$\text{where } \eta = -T1 \tag{1}$$

Erlang distribution

- by Chapman-Kolmogorov equation, we have state transition probabilities

$$\begin{aligned} p_1'(t) &= -\mu p_1(t) \\ p_2'(t) &= \mu p_1(t) - \mu p_2(t) \\ p_3'(t) &= \mu p_2(t) \end{aligned}$$

- on solving first differential equation

$$\begin{aligned} p_1(t) &= e^{-\mu(t)}, \qquad \{\text{using intial condition } p_1(0) = 1\} \\ p_2(t) &= \mu t e^{-\mu t} \\ p_3'(t) &= \mu^2 t e^{-\mu t} \end{aligned}$$

- We observe here that $p_3'(t)$ is the PDF same as Erlang-2. So we prove here the phase-type process we consider above is the Erlang process. Hence Erlang distribution is the phase-type distribution.

Hyper-Exponential Distribution

- Suppose system starts is state 1 with probability p and state 2 with probability $1-p$.
- Initial probability vector $p(0) = (p, 1-p)$.
- The rate transition diagram

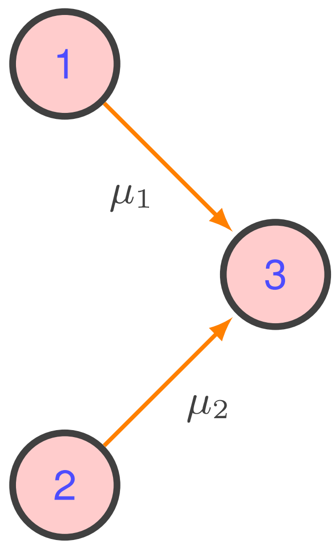

Figure: Hyperexponential distribution

Hyper-Exponential Distribution

- The generator matrix is

$$Q = \begin{bmatrix} -\mu_1 & 0 & \mu_1 \\ 0 & -\mu_2 & \mu_2 \\ 0 & 0 & 0 \end{bmatrix}$$

- We can observe that

$$T = \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix}$$

$$\eta = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} T & \eta \\ 0 & 0 \end{bmatrix} \quad \eta = -T1$$

- Using concept of CTMC we can prove that this phase type distribution is Hyper-exponential distribution.

Practical Example

- Now we begin with **practical example**. Consider the $M/M/C/C+K$ queue capacity of waiting room k total customer in system c+k
  If c+k users are already in the system, then new arrival (c+k+1) nth arrival is not admitted into the system. arrival user lost, this queueing system call loss system.

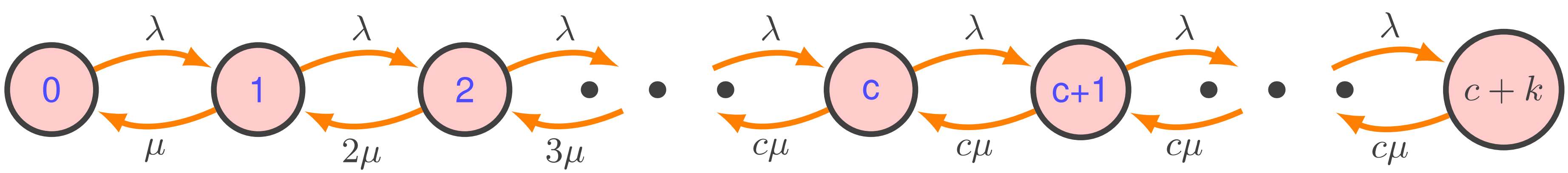

Figure: $M/M/c/c+k$ queue

Practical Example

- Rate transition matrix $Q_{(c+k)\times(c+k)} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 \\ \mu & -(\mu+\lambda) & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & c\mu & -(\lambda+\mu) & \lambda \\ 0 & \cdots & \cdots & c\mu & -c\mu \end{bmatrix}$
- From a system administrator's point of view. The loss of a user is regarded as a bad event .

Practical Example :Loss Study

- **question arises:- how the distribute the time up to the first loss might be expressed(he wants to know probability when system absorption)**

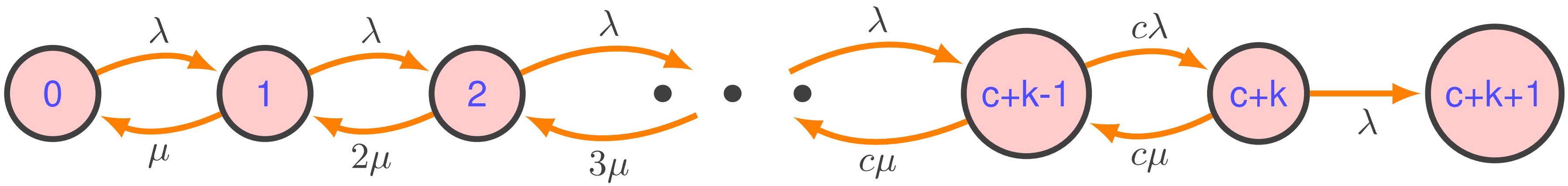

Figure: $M/M/c/c+k$ queue with one lost customer

Practical Example:Loss Study

- Rate transition matrix, when we add loss event

$$Q'_{(c+k+1)\times(c+k+1)} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & c\mu & -(\lambda+c\mu) & \lambda & 0 \\ 0 & 0 & \cdots & c\mu & -(\lambda+c\mu) & \lambda \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

- Now, the system administration queue can be formulated mathematically as the distribution of the time until the Markov process with generator Q' enters the absorbing state c+k+1.
- Exactly, this problem is addressed (in a general form) by the concept of phase-type distribution.

Dimension and order of phase type distribution

For a phase-type representation $PH(\alpha, T)$, m is referred to as it's dimension. The order of a phase-type distribution is the smallest dimension among all its representations.
Thus the order of the exponential distribution seen as phase-type distribution is one, while the dimension of a representation for the exponential distribution can be arbitrarily large.

Example

Let $Z \sim PH(\alpha, T), M = 3$, with $\alpha = [\alpha_1, \alpha_2, \alpha_3]$
Thus the order of the exponential distribution seen as phase-type distribution is one, while the dimension of a representation for the exponential distribution can be arbitrarily large.

Dimension and order of phase type distribution:Figure

- The generator transition diagram is

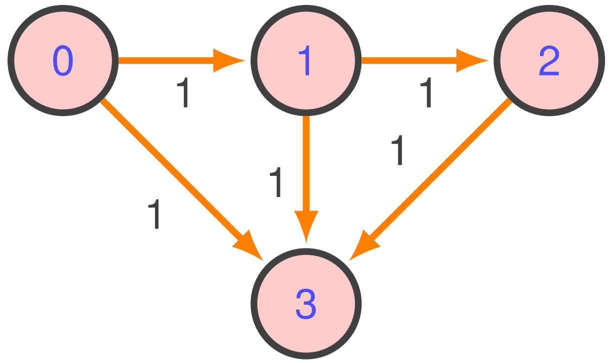

Figure: Order of PH-type

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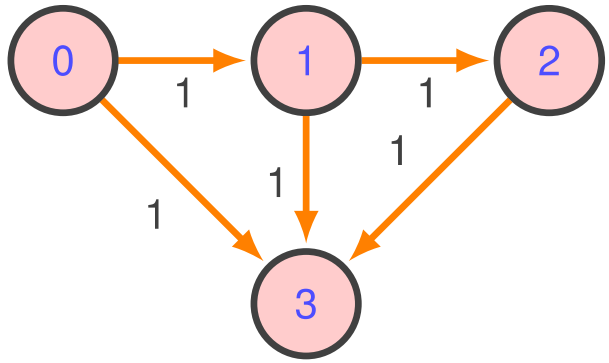


Figure: Order of PH-type

- The generator matrix

$$Q = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Dimension and order of phase type distribution

- where we can observe

$$T = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\eta = -T1 = (1, 1, 1)'$ and $Z \sim \exp(1)$

- So order of the phase type distribution is 1 and dimension is 3.

Laplace-Stieltjes Transform

- We know LST

$$\phi(t) = \int_0^\infty \exp(-st)dF(t), \quad s \in \mathbb{C} \quad \text{with } Re(s) \geq 0$$

$$\int_0^\infty e^{-st}e^{Tt}dt = \frac{-1}{s}\left[e^{Tt}e^{-st} - \int_0^\infty e^{-st}e^{Tt}T dt\right]$$

- On solving above expressions we get the following results

$$\begin{aligned}\phi(s) = \int_0^\infty e^{-st}f(t)dt &= \alpha_{m+1} + \int_0^\infty e^{-st}\alpha e^{Tt}\eta dt \\ &= \alpha_{m+1} + \alpha(s.I - T)^{-1}\eta\end{aligned}$$

Moments of Phase type distribution

- We can find nth order moment for PH-type distribution using Laplace transform.

$$
\begin{aligned}
Z &\sim ph(\alpha, T) \\
E(Z^n) &= (-1)^n \left[\frac{d^n}{ds^n}\phi(s)\right] |_{s=0} \\
&= (-1)^n \left[\alpha \frac{d^n}{ds^n}(sI - T)^{-}1\eta\right] | \\
&= (-1)^n \left[\alpha(-1 \times -2 \times -3 \cdots \times -n)\eta(sI - T)^{-(n+1)}\right] \\
&= \alpha(n!)(-T)^{-n}(-T)^{-1}1 \times \eta : \quad \eta = -T1 \\
&= n!\alpha(-T)^{-n}(-T)^{-1}(-T)1 \\
E(z^n) &= n!\alpha(-T)^{-n}1
\end{aligned}
$$

This is the nth order moment for PH-type distribution.

Closure properties of Phase-type distribution

A valuable advantage of phase-type distribution is that certain compositions of PH distributions result in PH distribution again. This means that the class of PH distribution is closed under these compositions. For PH distributed random variables. Z_1 and Z_2(convolution)$PZ_1 + (1-P)Z_2$ with $P \in [0,1]$(mixture) and $Z = Z_1 + Z_2$

Sum of two Phase-type random variables

Theorem

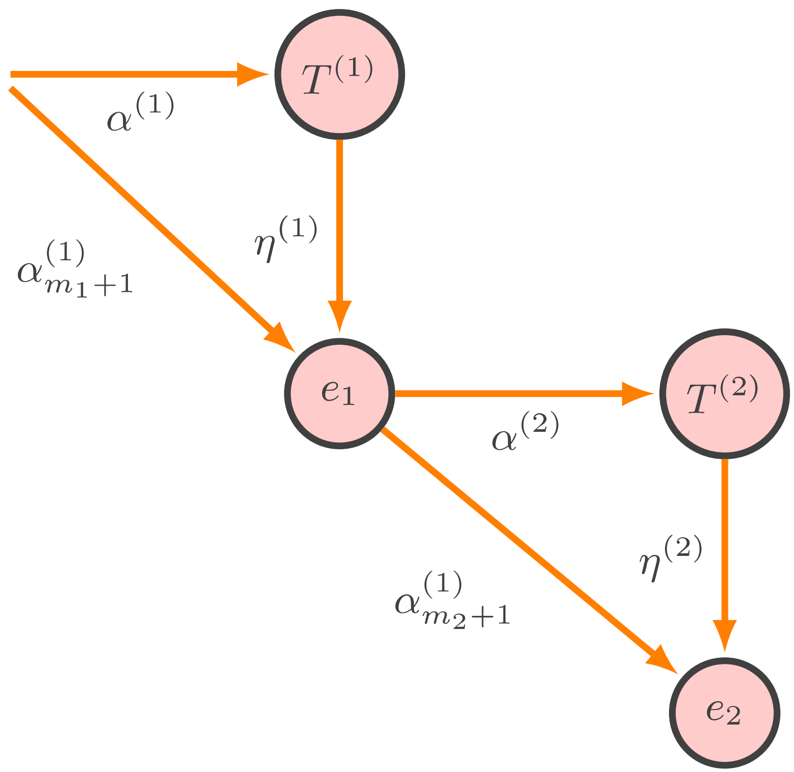
Let $Z_i \sim$ PH($\alpha^{(i)}, T^{(i)}$) of orderm_i for $i = 1, 2$. Then $Z = (Z_1 + Z_2) \sim PH(\alpha, T)$ of order $m = m_1 + m_2$ with representation..

$$\alpha_k = \begin{cases} \alpha_k^{(1)}, & 1 \leq k \leq m_1 \\ \alpha_{(m_1+1)}^{(1)} \alpha_{k-m_1}^{(2)}, & m_1 + 1 \leq k \leq m \end{cases}$$

$$Q = \begin{bmatrix} T^{(1)} & \eta^{(1)} \alpha^{(2)} \\ 0 & T^{(2)} \end{bmatrix}$$

where $\eta^{(1)} = -T^{(1)} 1_{m_1}$ and 0 denotes a zero matrix of appropriate dimension.

Sum of two Phase-type random variables:Figure



Sum of two Phase-type random variables:Proof

- **Point of joining**
- **Analysis for first phase type**
- **Analysis for Second phase type**
- **Structure of States and initial distribution**
- **Structure for matrix T and exist vector**

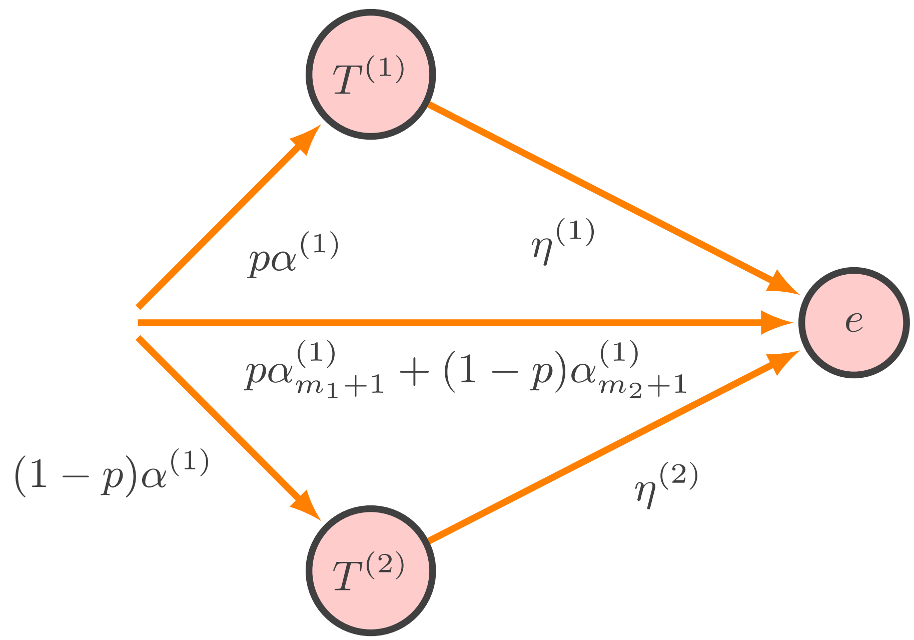
Convolution of two phase type distribution

Let $Z_i \sim$ PH($\alpha^{(i)}, T^{(i)}$) of orderm_i for $i = 1, 2.$ as well as $P \rightarrow [0, 1]$.Then$Z = pZ_1 + (1 - p)Z_2 \sim PH(\alpha, T)$ of order $m = m_1 + m_2$ with representation.
$\alpha = (p.\alpha^{(1)}, (1 - p).\alpha^{(2)})$

$$T = \begin{bmatrix} T^{(1)} & 0 \\ 0 & T^{(2)} \end{bmatrix}$$

Where 0 denote zero matrices of appropriate dimensions.

Convolution of two phase type distribution:Figure



Conclusion

- Phase-type distributions constitute a very versatile class of distributions. These are defined as the distributions of absorption times of certain continuous-time Markov chains. Indeed, any positive distribution may be approximated arbitrarily closely by phase-type distributions, whereas exact solutions to many complex problems in stochastic modelling can be obtained either explicitly or numerically.
- They have been used in a wide range of stochastic modelling applications in areas as diverse as telecommunications, finance, biostatistics, queueing theory, drug kinetics, and survival analysis.

Conclusion

- Since it contains exponential distribution, we can deduct markovian property from it. Due to the containment of Markovian property, we can make several generalisations of it and use it as an arrival process or service distribution.
- Nowadays, many Queueing theory models use phase-type distribution instead of exponential distribution because it gives more freedom for using more dimensions. For example, if we take arrival process, Erlang or service process Erlang distribution in queue models, that is, if we take models such as $M/E_r/1$ or $E_r/M/1$ these models give better results than $M/M/1$.

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