CHAPTER IV

DIVIDE-and-CONQUER

Divide and conquer is the best known algorithm design technique. Divide and conquer algorithms work according to following general plan

1. A problem's instance is divided into several smaller problem of same size
2. The smaller instances are solved.
3. The solutions obtained for smaller instances are combined to get a solution to the original problem

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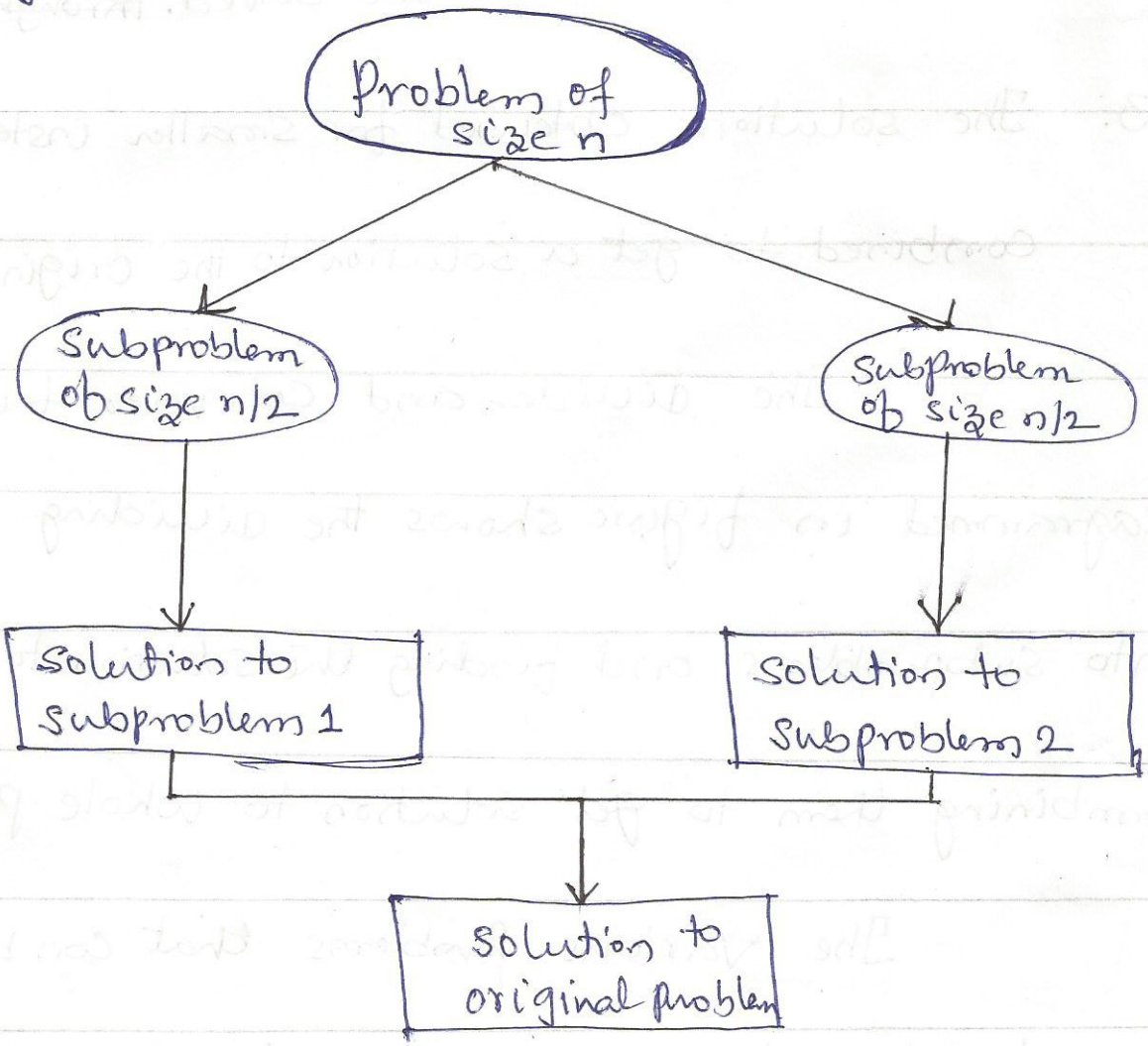
The divide and conquer technique is diagrammed in figure shows the dividing problem into subproblems and finding the solution to each and combining them to get solution to whole problem.

The various problems that can be solved using divide and conquer approach are

1. Quick sort
2. Merge sort
3. Binary search
4. Binary tree traversals.
5. Multiplication of numbers
6. Matrix Multiplication using Strassen's algorithm

The divide and conquer technique is shown pictorially as shown below



QUICK SORT

Quicksort is the divide and conquer method used to arrange numbers in Ascending order.

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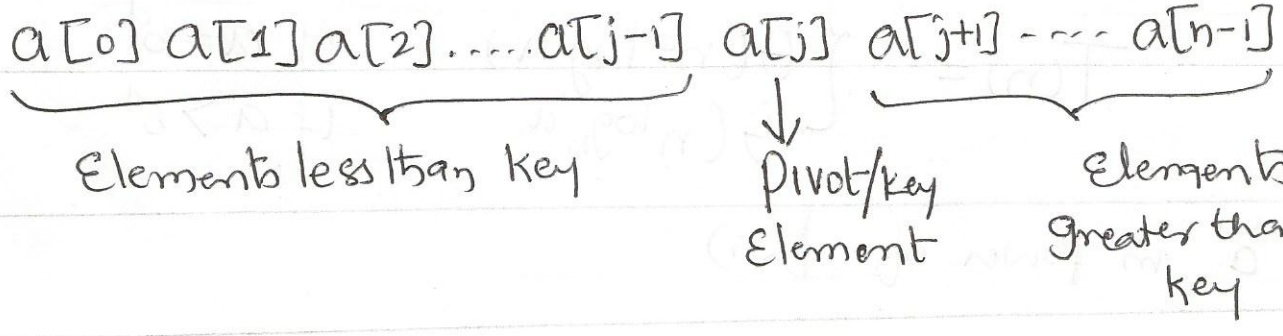
Method:

The Quicksort method assumes the first element of the list of 'n' numbers as key or Pivot element. Then the key element is compared with the elements one by one from second element until the key is greater. Then the key element is compared from n^{th} element one by one until the key element is less. Then compare the position of two pointer i and j, if i<j swap i^{th} and j^{th} elements else swap Pivot element and j^{th} element. After swaping key element is at j^{th} position, to left are less and to right are greater than key elements.



Now sort left part of the array a[0] ---- a[j-1] recursively using the same method and Sort right part of the array

The time efficiency of divide and conquer algorithms can be computed using

1. Master theorem
2. Back substitution method.

Master Theorem

An instance of size n can be divided into several instances say a of size n/b then general divide and conquer recurrance relation is

$$T(n) = aT(n/b) + f(n)$$

* where a and b are positive constants
* f(n) is time required to divide and combine subproblems

Time Efficiency can be calculated using following relation

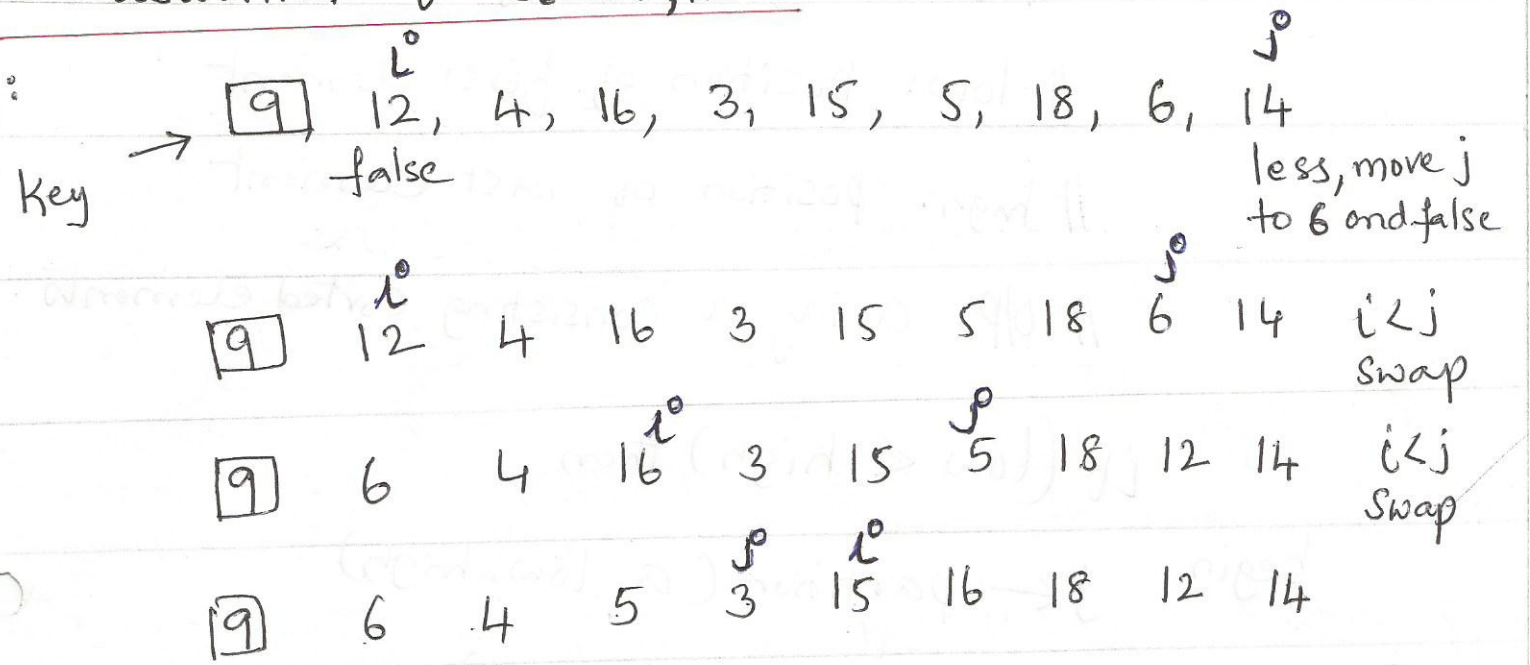
$$T(n) = \begin{cases} \theta(n^d) & \text{if } a < b^d \\ \theta(n^d \log n) & \text{if } a = b^d \\ \theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

'd' is power of f(n)

a[j+1] to a[n-1] recursively.

Illustration of Quicksort

Eg:



Now i is more than j, so swap pivot and j item

3, 6, 4, 5, [9] 15, 16, 18, 12, 14

j

After swapping key is exactly in middle, to left of key are less and right are greater than key.

Now sort Low to j-1 recursively and j+1 to high elements recursively.

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Since Quicksort method works on partitioning the set of elements, it is also called as PARTITION EXCHANGE Sort. The partitioning of should be done until each subpartition can have maximum two elements.

i.e. low < high

Algorithm: Quicksort (a, low, high)

// a: array containing unsorted elements

// low: position of first element

// high: position of last element

// o/p: array a consisting sorted elements.

```
if (low < high) then
begin   j ← partition (a, low, high)
        Quicksort (a, low, j-1)
        Quicksort (a, j+1, high)
endif
```

Algorithm: partition (a, low, high)

```
begin
        key ← a[low]
        i ← low+1
        j ← high
for
        while (key > a[i]) do
        i ← i+1
        while (key < a[j]) do
        j = j-1
        if (i<j) then swap (a[i], a[j])
Else    swap (a[low], a[j]), return j
Endwhile
```

TIME EFFICIENCY: The Time complexity of Quick Sort can be computed using

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① Master theorem

② Substitution method.

The recurrance relation for this algorithm can be written as

$$T(n) = T(n/2) + T(n/2) + n$$

n ← Time required to partition n elements

$T(n/2)$ ↑ Time required to Sort left part of the array

$T(n/2)$ ↖ Time required to Sort right part of the array

So

$$T(n) = \begin{cases} 0 & \text{if } n=1 \\ 2T(n/2)+n & \text{otherwise} \end{cases}$$

1. By master theorem: The general relation is

$T(n) = aT(n/b) + f(n)$, the recurrance relation of Quick Sort is

$$T(n) = 2T(n/2) + n$$

By comparing, we have $a=2$, $b=2$, $f(n)=n^1=n^d$, $d=1$

$$T(n) = \begin{cases} \theta(n^d) & \text{if } (a < b^d) \\ \theta(n^d \log_b n) & \text{if } a = b^d \\ \theta(n^{\log_b a} & \text{if } a > b^d \end{cases}$$

Here $a=2$, $b^d = b^1 = 2^1 = 2$. So second relation $a = b^d$ holds good and time efficiency is given by

$$T(n) = \theta(n^d \log_b n)$$

$$= \theta(n \log_2 n)$$

2. By Substitution method

$$T(n) = T(n/2) + T(n/2) + n$$

$$= 2T(n/2) + n$$

Let, number of elements 'n' $= 2^k$

So $T(2^k) = 2T\left(\frac{2^k}{2}\right) + 2^k$

$T(2^k) = 2T(2^{k-1}) + 2^k$, Substitute for $T(2^{k-1})$

$$= 2\left[2T(2^{k-2}) + 2^{k-1}\right] + 2^k$$

$$= 2^2 T(2^{k-2}) + 2 \cdot 2^{k-1} + 2^k$$

$$= 2^2 T(2^{k-2}) + 2^k + 2^k$$

$$= 2^2 T(2^{k-2}) + 2 \cdot 2^k$$

Again, $T(2^{k-2})$ can be substituted

$$T(2^k) = 2^3 T(2^{k-3}) + 3 \cdot 2^k$$

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When k^{th} value is substituted

$$T(2^k) = 2^k T(2^{k-k}) + k \cdot 2^k$$

$$= 2^k T(2^0) + k \cdot 2^k$$

$$= 2^k T(01) + k \cdot 2^k$$

Time required to sort 1 element is zero, so

$$T(2^k) = k \cdot 2^k, \quad \text{but} \quad n = 2^k$$

$$= n \cdot k, \quad k = \log_2 n$$

$$T(n) = \theta(n \log_2 n)$$

So, time complexity of Quick Sort is

$$T(n) = \theta(n \log_2 n)$$

Merge Sort

Merge Sort is divide and conquer technique used to arrange numbers in ascending order. The different steps of merge sort are

1. Divide array of n elements to two subarrays of n/2 each; left part and right part
2. Sort left array and right array recursively
3. Merge sorted left part and sorted right part to get single sorted array.

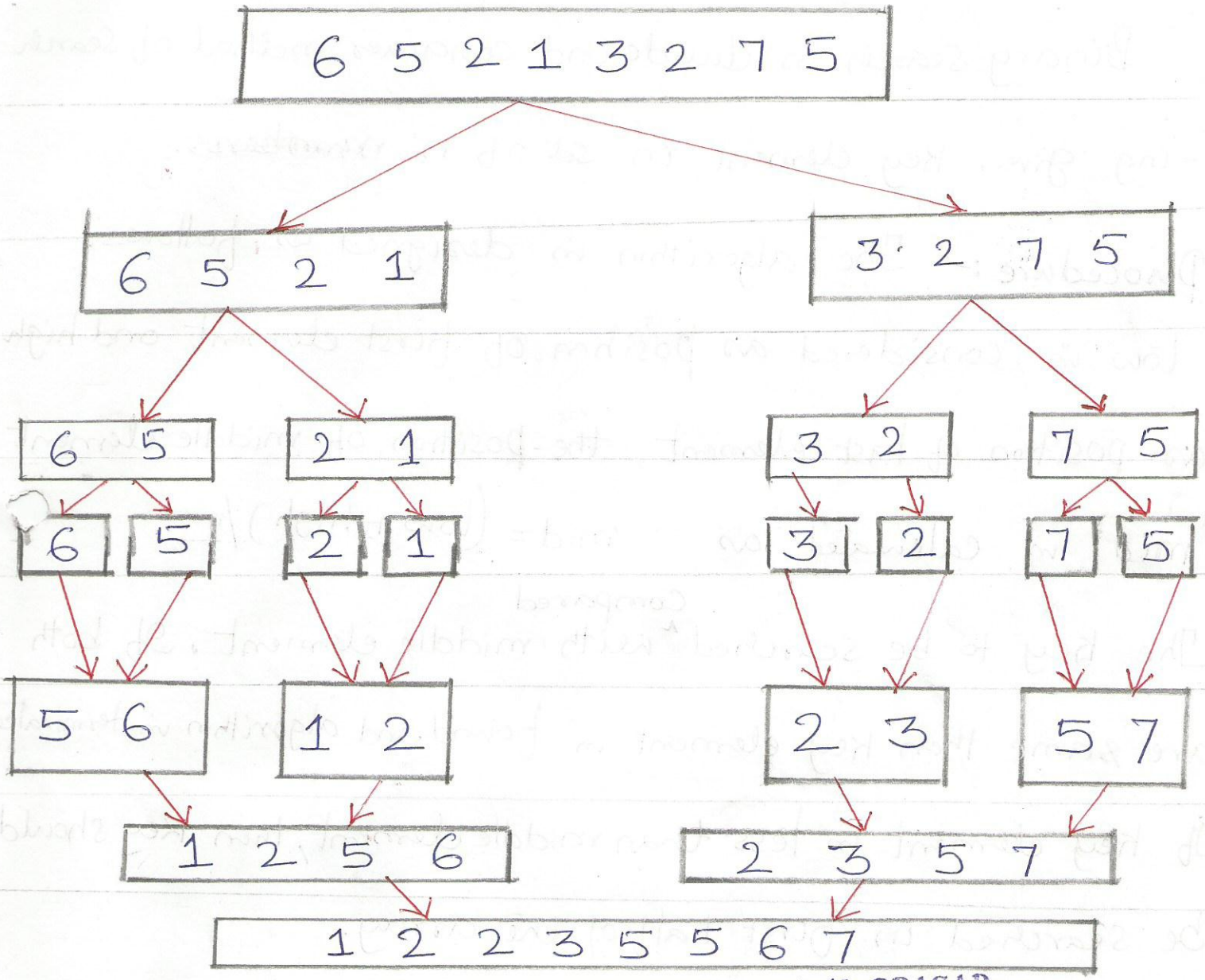
The array is divided into subarrays until only one element is in subarray. ie low <= high

Illustration: Consider the following elements

6, 5, 2, 1, 3, 2, 7, 5

The trace of merge sort can be shown in tree.

S: Analysis is same as QuickSort




```
Algorithm Mergesort(a, low, high)
Begin if(low<=high)
      mid=low+high/2
      Mergesort(a,low,mid)
      Mergesort(a,mid+1,high)
      Merge(a,low,mid,high)
Endif
```

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```
merge(low, mid, high)
begin i=low, j=mid+1, k= low
   while(i<=mid and j<=high)
   if (a[i]<a[j])
      b[k]=a[i], i<-i+1, k<-k+1
   else
      b[k]=a[j], j<-j+1, k<-k+1
   endwhile
      while(i<=mid) b[k]=a[i]
                    i<-i+1, k<-k+1
      while(j<=high) b[k]=a[j]
                     j<-j+1, k<-k+1
```

BINARY SEARCH

Binary search is divide and conquer method of search-ing given key element in set of 'n' numbers.

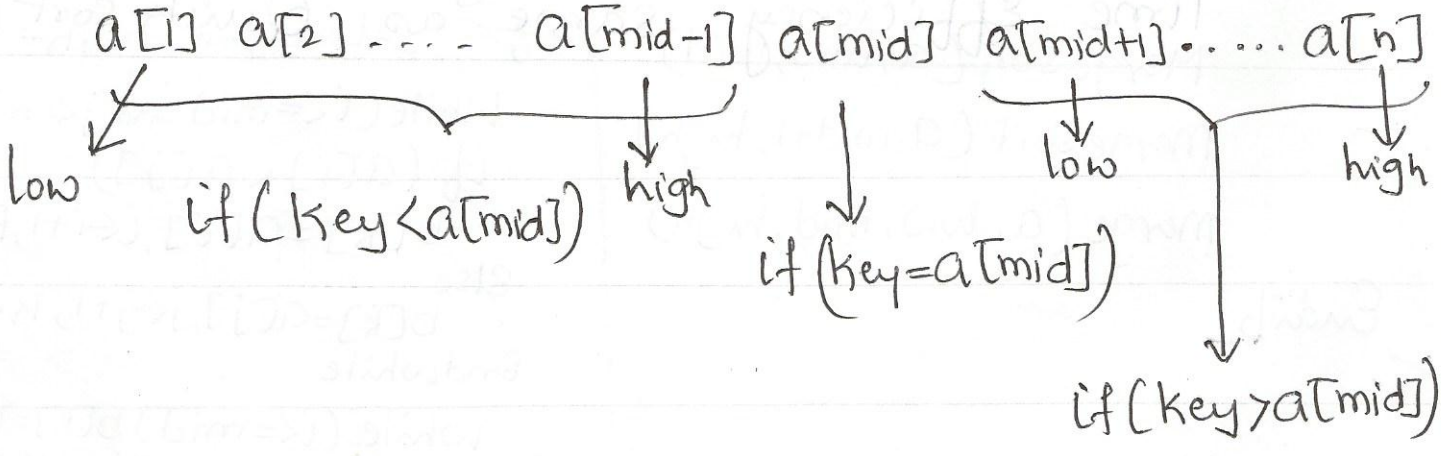
Procedure:- The algorithm is designed as follows,

low is considered as position of first element and high as position of last element, the position of middle element 'mid' is calculated as $mid = (low + high)/2$

The key to be searched (compared) with middle element. If both are same then key element is found and algorithm is terminated.

If key element is less than middle element, then key should be searched in first half of the array.

If key element is greater than middle element, then key should be searched in second half of the array.



Algorithm:

```
Algorithm binary Search (a, n, key)
// key is the element to be Searched
// n is the number of elements
// a is the array which contains elements
begin
    low ← 1
    high ← n
    while ( low <= High) do
        mid ← (low + high)/2
        if ( key = a[mid] return mid
        if ( key < a[mid]) then
                high = mid - 1
        else
                low = mid + 1
        Endif
    Endwhile
    return -1
End
```

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TIME EFFICIENCY:

The Best case occurs if the item is found in the first comparision. So the time efficiency is

$T(n) = \Omega(1)$ ie if item is in middle

The worst case occurs if the item is present in first or last position. The recurrence relation is

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(n/2)+1 & \text{otherwise} \end{cases}$$

$T(n/2)$ → Time required to search either upper or lower part

$+1$ → Time required to compare middle element

$$T(n) = T(n/2) + 1$$

Let $n = 2^k$

$$T(2^k) = T\left(\frac{2^k}{2}\right) + 1$$

$$= T(2^{k-1}) + 1$$

Now substitute for $T(2^{k-1})$

$$T(2^k) = [T(2^{k-2}) + 1] + 1$$

$$= T(2^{k-2}) + 2$$

Now we can substitute for $T(2^{k-2})$

$$= T(2^{k-3}) + 3$$

by substituting again for function $T(2^{k-i})$ till i reaches k, when $i=k$.

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$$T(n) = T(2^{k-k}) + k$$

$$= T(2^0) + k$$

$$= T(1) + k$$

$T(1)$ is time taken to search one element ie 1

$$= 1 + k$$

But $n = 2^k$ so $k = \log_2 n$

$$= 1 + \log_2 n, \quad \text{constant cannot be considered}$$

$$T(n) = \log_2 n$$

So

$$\boxed{T(n) = O(\log_2 n)}$$

The average case is when $n = 1, 2, 4, 8, 16$ etc.

When $n = 1, 2^0$ no of comparision is 1

$n = 2, 2^1$ —"— —"— —"— 2

$n = 4, 2^2$ —"— —"— —"— 3

$n = 8, 2^3$ —"— —"— —"— 4

$n = 16, 2^4$ —"— —"— —"— 5

$n = 2^{k-1}$ —"— —"— —"— k

In general $T(n) = \frac{1}{n} \sum_{i=1}^{k} i \cdot 2^{i-1}$

$$2^{i-1} = \frac{2^i}{2} = 2^i - \frac{2^i}{2} = 2^i - 2^{i-1}$$

$$= \frac{1}{n} \sum_{i=1}^{k} i (2^i - 2^{i-1})$$

$$= \frac{1}{n} \left\{ \sum_{i=1}^{k} i 2^i - \sum_{i=1}^{k} i \, 2^{i-1} \right.$$

$$= \frac{1}{n} \left\{ \sum_{i=1}^{k} i \cdot 2^i - \sum_{i=1}^{k} (i-1) 2^i \right.$$

$$= \frac{1}{n} \left\{ \sum_{i=1}^{k} i \cdot 2^i - \sum_{i=1}^{k} i \cdot 2^i - \sum_{i=1}^{k} 2^i \right.$$

$$= \frac{1}{n} k \cdot 2^k - \sum_{i=1}^{k} 2^i$$

$$= \frac{1}{n} k 2^k \left[1 + 2^1 + 2^2 + \cdots + 2^{k-1}\right]$$

$$= \frac{1}{n} \left\{ k \cdot 2^k - 1 \left[\frac{2^k - 1}{2 - 1}\right] \right. \quad \text{using } S = a\left[\frac{r^n - 1}{r - 1}\right]$$

where $a = 1, r = 2, n = k$

$$= \frac{1}{n} \{ kn - 1 (n - 1) \}$$

$$= \frac{1}{n} \{ kn - n + 1 \}$$

$$= k - 1 + \frac{1}{n}$$

it is approximated to $k - 1$

$T(n) = k - 1$ But $n = 2^{k-1}$

Substituting $k - 1$ $(k - 1) = \log_2 n$

$$T(n) = \theta(\log_2 n)$$

So average case is

$$T(n) = \theta(\log_2 n)$$

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