MATH 108, Winter 2022
HOMEWORK 1
Due Monday, January 24

Try solve the homework on your own. If you discuss with others, please list your collaborators. You can use any theorem / lemma that was stated in class as a fact (without reproving it).

Problem 1. A magic square is an $n \times n$ array containing the integers $\{1, 2, \ldots, n^2\}$ (each exactly once) so that that the sum of each row and each column is the same. Prove that a magic square exists for every $n \neq 2, 6$. (Hint: orthogonal pairs of Latin squares exist for every $n \neq 2, 6$.)

Proof of Problem 1. Let $n \neq 2, 6$ and L, L' be an orthogonal pair of Latin squares of size $n \times n$. By definition, this means that, if we superimpose L on top of L', then the resulting square A of size $n \times n$, which will consist of n^2 ordered pairs formed from the corresponding entries of L and L', will not have any pair appearing twice. Since the number of all possible ordered pairs of numbers from 1 to n is n^2, each pair appears exactly once. Let $A_{ij} = (L_{ij}, L'_{ij})$ denote the entry of the resulting square A for row i and column j.

We will prove that, given A, we can produce a magic square M of size $n \times n$. Let $f : \mathbb{N}^2 \to \mathbb{N}$ such that $f(x, y) = (x - 1) \cdot n + y$, where $1 \leq x, y \leq n$. Consider M such that $M_{ij} = f(A_{ij})$. Function f maps each distinct pair (x, y) to a different value $1 \leq f(x, y) \leq n^2$, therefore M contains all integers from 1 to n^2.

Then, it holds that the sum of the elements of M in row i is

$$\begin{aligned}
\sum_{j=1}^{n} M_{ij} &= \sum_{j=1}^{n} \big((L_{ij} - 1) \cdot n + L'_{ij}\big) \\
&= n \cdot \left(\sum_{j=1}^{n} L_{ij} - n\right) + \sum_{j=1}^{n} L'_{ij} \\
&= n \cdot \left(\frac{n(n+1)}{2} - n\right) + \frac{n(n+1)}{2} \\
&= \frac{n^3 - n^2 + n^2 + n}{2} \\
&= \frac{n^3 + n}{2}
\end{aligned}$$

where we used the fact that the sum of the elements of a Latin square $n \times n$ in each row is $\frac{n(n+1)}{2}$.

In an analogous manner, and by using the fact that the sum of the elements of a Latin square $n \times n$ in each column is $\frac{n(n+1)}{2}$, we can prove that the elements of each column of M sum to the same value.

Therefore, since M contains all integers $\{1, \ldots, n^2\}$ and the sum of each of its rows and columns is equal, M is a $n \times n$ magic square. $\square$

Problem 2. Let $G = (V, E)$ be a graph and $\bar{G}$ its "complement": a graph on the same set of vertices V, where $E(\bar{G}) = \binom{V}{2} - E(G)$. Prove that if G is not connected, then $\bar{G}$ is connected.

Lemma 1 *If $G = (V, E)$ is not connected, then $\{u, v\} \in E(G) \implies$ there exists some $w \in V$ such that $\{u, w\}, \{v, w\} \notin E(G)$.*

Proof: Assume the opposite, therefore $\forall w \in V$, either $\{u, w\} \in E(G)$ or $\{v, w\} \in E(G)$. In that case, for $u', v' \in V$, they are either connected through an edge, or there exists a path through u (or v or both) that connects them. Thus, G is connected, contradiction. $\square$

Proof of Problem 2. Let $u, v \in V$. We will prove that there is a path in $\bar{G}$ such that u and v are connected.

- If $\{u, v\} \in E(\bar{G}) \iff \{u, v\} \notin E(G)$, they are obviously connected.
- If $\{u, v\} \notin E(\bar{G}) \iff \{u, v\} \in E(G)$, then $\exists w \in V$ such that $\{u, w\}, \{v, w\} \notin E(G)$ due to the previous lemma. Therefore, $\{u, w\}, \{v, w\} \in E(\bar{G}) \implies u, v$ are connected through w.

$\square$

Problem 3. Prove that a graph $G = (V, E)$ is a tree if and only if it has no cycles and $|V| = |E|+1$.

Proof of $\Rightarrow$. Let $G = (V, E)$ be a tree (i.e. a connected graph without any cycles). We will prove by induction that $|V| = |E| + 1$.

- *Basis*: If $|E| = 0$, then the graph G can have only a single vertex in order to be connected (in a trivial way), therefore, $|V| = 1 = |E| + 1$.
- *Inductive Hypothesis*: Suppose that, for every tree $G = (V, E)$, with $|E| \leq m - 1$, it holds that $|V| = |E| + 1$.
- *Inductive Step*: Let $G = (V, E)$ be a tree with $|E| = m$. G necessarily contains a leaf (vertex of degree 1). Let $u \in V$ be one such leaf, with $e \in E$ its corresponding (single) edge. Let $H = (V \setminus \{u\}, E \setminus \{e\})$ be the graph occurring after the removal of u and its corresponding edge. H cannot have any cycles, since it occurs from the removal of an edge and a vertex from G, which has no cycles. Furthermore, H is connected, since, $\forall v, w \in V \setminus \{u\}$, (v, w) are connected by a path in G. The interval vertices on the path cannot contain u, since $\deg_G(u) = 1$, therefore the path still exists in H. Consequently, H is a tree with $|E(H)| < m$, and by the induction hypothesis, $|V(H)| = |E(H)| + 1$. In that case, $|V(G)| = |V(H)| + 1 = |E(H)| + 2 = |E(G)| + 1$ and by mathematical induction the statement holds.

$\square$

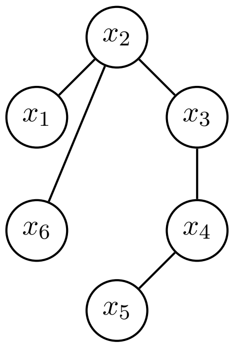
Proof of $\Leftarrow$. Let $G = (V, E)$ be a graph without any cycles and $|V| = |E| + 1$. We will prove by induction that G is a tree (i.e. a connected graph without any cycles).

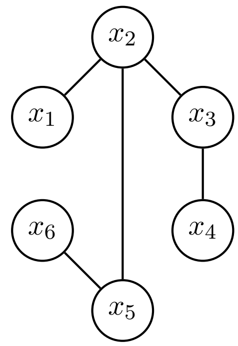
- *Basis*: If $|V| = 1$ and $|E| = 0$, then the graph G is a tree.
- *Inductive Hypothesis*: Suppose that, every graph $G = (V, E)$, with $|V| = |E| + 1 \leq n - 1$ which contains no cycles, is a tree.
- *Inductive Step*: Let $G = (V, E)$ be a graph with $|V| = |E| + 1 = n$ that contains no cycles. Let P be a path of maximum possible length in G. The endpoints v_0, v_l of P have degree 1. If not, there would be an edge $\{v_k, v_l\}$ incident to v_l. If $v_k \in V(P)$, then G contains a cycle. Alternatively, if $v_k \notin V(P)$, then P is not a path of maximum length. Let $u \in V$ be one such endpoint, with $e \in E$ its corresponding (single) edge. Let $H = (V \setminus \{u\}, E \setminus \{e\})$ be the graph occurring after the removal of u and its corresponding edge. H cannot have any cycles, since it occurs from the removal of an edge and a vertex from G, which has no cycles. Furthermore, $V(H) = V(G) - 1$ and $E(H) = E(G) - 1$ which implies that $V(H) = E(H) + 1 < n$, and by the induction hypothesis, H is a tree (i.e. connected and without any cycles). In that case, adding vertex u and edge e incurs no cycles and the graph remains connected, therefore G is also a tree and by mathematical induction the statement holds.

$\square$

Problem 4. A "score" of a graph is the sequence of degrees of its vertices, in descending order. For example, the score of a path of length 3 is $(2, 2, 1, 1)$. Find two trees with the same score which are *not* isomorphic.

Below we present two *non* isomorphic trees of the same score $(3, 2, 2, 1, 1, 1)$.





Problem 5. A k-partitite graph is such that the vertices can be partitioned into disjoint sets $V_1, V_2, \ldots, V_k$ and there is no edge with both endpoints inside the same set V_i. If the number of vertices is $|V| = k\ell$, what is the largest possible number of edges that a k-partite graph can have? Prove that your answer is correct.

In the following, assume that every possible edge is present. We will prove that, given V_l, for $l = 1, \ldots, k$, if there exist V_i, V_j, such that $|V_i| \neq |V_j|$, then the graph which occurs from balancing V_i and V_j has more edges than the previous one.

Claim 2 *Let $V_1, \ldots, V_k$ be the disjoint sets of a given partition. Let V_i, V_j be two of those disjoint sets, with $|V_i| \neq |V_j|$ and $|V_i| + |V_j| = n$. Additionally, let $C = |V| - n$. Then, if we balance V_i and V_j, in the sense of $|V_i'| = |V_j'|$, then the occurring graph has an increased number of edges.*

Proof:

$$|E| = |V_i| \cdot C + |V_j| \cdot C + |V_i| \cdot |V_j| = n \cdot C + |V_i| \cdot |V_j| \leq n \cdot C + (n/2)^2$$

Where we used the fact that

$$\left(\frac{n}{2} + \varepsilon\right) \cdot \left(\frac{n}{2} - \varepsilon\right) = \frac{n^2}{4} - \varepsilon^2 \leq \frac{n^2}{4} = \left(\frac{n}{2}\right)^2$$

□

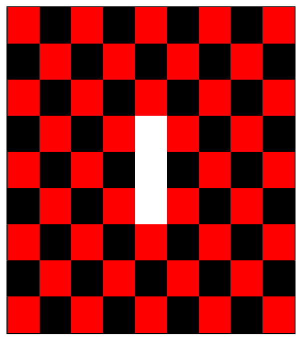
Then, it follows that as long as there exist two unbalanced sets V_i and V_j, we can increase the number of edges by balancing them. Therefore, we conclude that the maximum number of edges is attained when $|V_i| = l$, for all i, and that is

$$l \cdot (k-1)l + l \cdot (k-2)l + \cdots + l^2 = l^2 \cdot \sum_{i=1}^{k-1} i = l^2 \cdot \frac{(k-1)k}{2}$$

Problem 6. Consider an area obtained by taking a 9×9 square and cutting out a 3×1 rectangular area, located symmetrically in the middle of the square. Is it possible to cover this area perfectly by tiles of size 2×1? Prove that your answer is correct.

Proof of Problem 6.

Color the square as depicted in the following figure. In that case, the covering of our square by tiles of size 2×1 reduces to finding whether, by using tiles covering two adjacent differently colored tiles, we can cover the whole square. Consider an edge connecting each possible pair of differently colored adjacent tiles, while the tiles themselves are vertices of a bipartite graph. Then, the initial problem is equivalent to searching for a perfect matching on the graph described.



In particular, consider the graph $G = (V, E)$ corresponding to our square. Let $V = R \cup B$, where R depicts the set of *red* colored vertices, while B the set of *black* colored vertices. Each vertex corresponds to a tile of the square, while the set of edges E is composed of pairs of vertices $\{r, b\}$, where $r \in R, b \in B$ and r and b correspond to tiles adjacent to each other. Therefore, G is a bipartite graph.

Notice that $|R| = 40$ and $|B| = 38$. In that case, it is obvious that there can be no perfect matching, since $|R| \neq |B|$, therefore we *cannot* cover the initial square using tiles of size 2×1. $\square$

Problem 7. Consider the following graph orientation problem: we would like to orient the edges of a graph G in such a way that each vertex has at most k incoming edges. Prove that this is possible if and only if $|E[W]| \leq k|W|$ for each subset of vertices W. ($E[W]$ denotes the edges with both endpoints in W.) *Hint:* Formulate the problem as a bipartite matching problem.

Proof of $\Rightarrow$. Let $G = (V, E)$ be a graph for which we can orient its edges in such a way such that $\forall v \in V$, v has at most k incoming edges. We will prove that $|E[W]| \leq k|W|, \forall W \subseteq V$, where $E[W]$ denotes the edges with both endpoints in W.

Let $G' = (V, E')$ be the directed graph for which the aforementioned property holds and $W \subseteq V$ a subset of the vertices of the graph. Let, for $u \in V$, $E'_{in}(u) = \{e' \in E' \mid e' \text{ is incoming to } u\}$ denote the set of edges that are incoming to u. Let $E'_{in}(W) = \bigcup_{w \in W} E'_{in}(w)$.

Now, the aforementioned property translates to $|E'_{in}(w)| \leq k \implies |E'_{in}(W)| \leq k \cdot |W|$. Additionally, let $E'[W]$ contain the *directed* edges belonging in $E[W]$, with $|E[W]| = |E'[W]|$. Then, $E'[W] \subseteq E'_{in}(W) \implies |E'[W]| \leq |E'_{in}(W)|$, since if an edge $\in E$ has both its endpoints in W, then in G' it is necessarily incoming to one of those endpoints, both of which belong to W. Hence, $|E[W]| \leq k|W|$. $\square$

Proof of $\Leftarrow$. Let $G = (V, E)$ be a graph for which $|E[W]| \leq k|W|, \forall W \subseteq V$, where $E[W]$ denotes the edges with both endpoints in W. We will prove that we can orient its edges in such a way that $\forall v \in V$, v has at most k incoming edges.

We will formulate the problem as a bipartite matching problem. In particular, we will construct a bipartite graph $G' = (V', E')$, with $V' = A' \cup B'$, as follows:

- A' contains a vertex a_e for each edge $e = \{v, w\} \in E$ of graph G, therefore $|A'| = |E|$.
- B' contains vertices $v_1, \ldots, v_k$ for each vertex $v \in V$ of graph G, therefore $|B'| = k \cdot |V|$.
- If $e = \{u, v\} \in E$, then $\{a_e, u_i\}, \{a_e, v_i\} \in E'$, for all $i = \{1, \ldots, k\}$, therefore each vertex of A' has $2k$ edges connecting it to B'.

The existence of a matching M of size $|A'|$ for graph G' implies that we can orient the edges of graph G in such a way that $\forall v \in V$, v has at most k incoming edges. Indeed, for each edge $\{u, v\} \in E$, the corresponding edge $\in M$ which covers $a_e \in A'$ denotes whether e is incoming to u or v for our orientation. Moreover, since in graph G' there are at most k copies of each vertex $v \in V$, each such vertex has at most k incoming edges in our orientation.

It remains to prove that there exists a matching of size $|A'|$ for graph G', or equivalently (due to Hall's Theorem), that $\forall A^* \subseteq A', |A^*| \leq |N_{G'}(A^*)|$.

Let $E^* \subseteq E$ be the set of edges of G corresponding to some set of vertices $A^* \subseteq A'$ of G', where obviously $|A^*| = |E^*|$. Additionally, let $W \subseteq V$ be the set of vertices covered by E^* (i.e. each vertex $w \in W$ is an endpoint for some edge $e^* \in E^*$). Then, $E^* \subseteq E[W] \implies |E^*| \leq |E[W]| \leq k|W|$. Furthermore, notice that $w \in W \iff \exists v \in V : e^* = \{v, w\} \in E^* \iff \{w_1, \ldots, w_k\} \in N_{G'}(a_{e^*})$. Therefore, $|N_{G'}(A^*)| = k \cdot |W|$, and $|A^*| \leq |N_{G'}(A^*)|$ follows. $\square$

Bonus problem. Let G be a bipartite graph with parts A, B. Let $S \subseteq A$ be such that there is a matching covering S, and $T \subseteq B$ such that there is a matching covering T. Prove that there is a matching covering both S and T.

Proof of Bonus Problem. Let $M_S \subseteq E$ be the set of edges (colored *black*) contained in the matching covering $S \subseteq A$ and $M_T \subseteq E$ the set of edges (colored *white*) contained in the matching covering $T \subseteq B$. If an edge belongs to both M_S and M_T, let its color be *gray.* We will search for a matching $M \subseteq M_S \cup M_T$, such that M covers $S \cup T$. Obviously, if $M_S = M_T$, then $M = M_S = M_T$ is a solution.

Assume that $M_S \neq M_T$ (or equivalently, not all edges are gray). Let $G' = (V, E')$, where $E' = M_S \cup M_T$. It is easy to see that $\forall v \in V, \deg_{G'}(v) \leq 2$. That is due to the fact that there is at most one edge covering v in M_S and M_T respectively. Consider the *connected components* of G', which have the aforementioned property, and each of which is connected. Our target is to cover the vertices of $S \cup T$ belonging to those connected components. For each connected component, one of the the following cases holds:

1. The component includes just a single vertex v. In this case, $v \notin S \cup T$ and we do not need to cover this vertex.

2. The component includes 2 vertices and a single edge e connecting them. In this case, e is gray and belongs to $M_S \cap M_T$. By including e into matching M, we successfully cover both vertices.

3. The component is a *path* where there exist two vertices of degree 1 and the rest are of degree 2.

4. The component is a *cycle* where each vertex has degree 2.

For the last two cases, it is easy to see that the edges of the corresponding connected component are alternatively colored black and white (since no two edges belonging to the same matching cover the same vertex). Therefore, each cycle has even length, thus, by including all its black (or white) edges, we successfully cover it in its entirety. Similarly, if a path has odd length (an even number of vertices), we can also cover it by taking the edges of the same color as those adjacent to the vertices of degree 1.

It remains to consider the case where the connected component is a path of even length. Let P be one such path, containing vertices $v_0, \ldots, v_k$, where k is even. In that case, it holds that $v_0, v_2, \ldots, v_k \in X$ and $v_1, \ldots, v_{k-1} \in V \setminus X$, where either $X = A$ or $X = B$. If $X = A \iff v_0, v_k \in A$, then add to our matching M the black edges of this component (which comprise a subset of M_S). Alternatively, add the white ones (which comprise a subset of M_T respectively). In both cases, only a single vertex of P remains uncovered, i.e. either v_0 or v_k.

Assume w.l.o.g. that $X = A$ (the proof is analogous for $X = B$). If the first edge of P is black, then the last (which is the single edge of v_k) is white, therefore $\nexists e \in M_S$ which covers $v_k \iff v_k \notin S$. Consequently, we successfully cover all edges of $S \cup T$ belonging to P. Similarly, if the first edge is white, it follows that the single uncovered vertex v_0 does not belong to S. $\square$