MATH 108, Winter 2022
HOMEWORK 4
Due Friday, March 11

Try solve the homework on your own. If you discuss with others, please list your collaborators. You can use any theorem / lemma that was stated in class as a fact (without reproving it).

Problem 1. What is the number of subsets of $[n]$ containing no two consecutive integers?

Equivalently, we search for the number of $0-1$ strings of size n, such that no two subsequent positions are 1. Suppose that $T(n)$ denotes the number of strings of size n that satisfy the aforementioned constraint. Then, it holds that

$$T(n) = T(n-1) + T(n-2),$$

since if the first bit is 0, then there is no additional restriction regarding the rest of the string, while if the first bit is 1, then necessarily the second bit is 0. Moreover, $T(1) = 2$ and $T(2) = 3$ and we do not count any string multiple times, since they differ on at least one bit. Notice however that, if F_i denotes the i-th Fibonacci number, then $T(n) = F_{n+2}$ follows.

Problem 2. Prove that the number of subsets of $[n]$ of cardinality divisible by 4 is

$$2^{n-2} + \frac{1}{2} Re\left((1+i)^n\right)$$

(here, i is the imaginary unit and Re denotes the real part).

Proof: Let Σ denote the number of subsets of $[n]$ of cardinality divisible by 4. Notice that

$$\Sigma = \sum_{\substack{0 \le k \le n \\ k \text{ is divisible by } 4}} \binom{n}{k}$$

From the binomial theorem, we have that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Therefore, it holds that

$$(1+1)^n + (1-1)^n + (1+i)^n + (1-i)^n = \sum_{k=0}^{n} \binom{n}{k} \left(1^k + (-1)^k + i^k + (-i)^k\right).$$

Let k' be a non negative integer and remember that $i^2 = -1$ and $i^4 = 1$. Then, one of the following holds:

1. If $k = 4k' + 0$, then $1^k + (-1)^k + i^k + (-i)^k = 1 + 1 + 1 + 1 = 4$.
2. If $k = 4k' + 1$, then $1^k + (-1)^k + i^k + (-i)^k = 1 - 1 + i - i = 0$.
3. If $k = 4k' + 2$, then $1^k + (-1)^k + i^k + (-i)^k = 1 + 1 - 1 - 1 = 0$.
4. If $k = 4k' + 3$, then $1^k + (-1)^k + i^k + (-i)^k = 1 - 1 - i + i = 0$.

In other words, it holds that

$$\begin{aligned} 4\Sigma &= (1+1)^n + (1-1)^n + (1+i)^n + (1-i)^n \\ &= 2^n + (1+i)^n + (1-i)^n \\ &= 2^n + (1+i)^n + \left(\overline{1+i}\right)^n \\ &= 2^n + (1+i)^n + \overline{(1+i)^n} \\ &= 2^n + 2Re\left((1+i)^n\right), \end{aligned}$$

thus

$$\Sigma = 2^{n-2} + \frac{1}{2} Re\left((1+i)^n\right)$$

follows. $\square$

Problem 3. For $m \leq n$, find a simple formula (not involving summation) for the expression $\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}$.

For any fixed k, where $m \leq k \leq n$, it holds that $\binom{n}{k}\binom{k}{m} = \binom{n}{n-k}\binom{k}{m}$. Essentially, this value is the number of ways we can color n balls, such that $(n-k)$ balls are of the first color (say red), m balls are of the second color (say green) and the rest are of the third color (say blue). Notice that it doesn't really matter the order we pick to color the balls, thus we could firstly choose the m balls we would like to color green and from the rest the $(n-k)$ balls we would like to color red. Therefore, it holds that

$$\begin{aligned}
\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} &= \sum_{k=m}^{n} \binom{n}{m}\binom{n-m}{n-k} \\
&= \binom{n}{m} \sum_{k=m}^{n} \binom{n-m}{n-k} \\
&= \binom{n}{m} \sum_{k=0}^{n-m} \binom{n-m}{k} \\
&= \binom{n}{m} 2^{n-m}
\end{aligned}$$

Problem 4. In the problem with n gentlemen and n hats (under a uniformly random assignment of hats to gentlemen), what is the probability that exactly one gentleman gets his own hat?

We will compute $D(n)$, which is the number of assignments of n hats to n gentlemen, such that *no* gentleman gets his own hat. In that case, the number of assignments such that *exactly* one gentleman gets his own hat (for n gentlemen) is $T(n) = n \cdot D(n-1)$, since there are n possible pairs of (hat, gentleman) that may match, and for the rest of the $n-1$ gentlemen, no one should take his own hat.

Therefore, if $A_i = \{\text{assignments such that Sir } i \text{ gets his own hat}\}$, $A_1 \cup \ldots \cup A_n =$ the set of assignments such that there exists at least 1 sir who gets his hat, and

$$\begin{aligned}
D(n) &= n! - \left|\bigcup_{i=1}^{n} A_i\right| \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} \\
&= n! \left(1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!}\right) \\
&= n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\end{aligned}$$

where the second equality is due to the inclusion-exclusion formula.

Finally, we have that the probability of exactly 1 gentleman among n to get his hat is

$$\begin{aligned}
\frac{T(n)}{n!} &= \frac{1}{n!} \cdot n \cdot (n-1)! \cdot \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \\
&= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}
\end{aligned}$$

Problem 5. How many ways are there to form n dancing pairs among n married couples, such that no married couple dances together?

We will compute $D(n)$, which is the number of assignments of n men to n women, such that *no* married couple dances together. If $A_i = \{\text{assignments such that the } i\text{-th couple dances together}\}$, $A_1 \cup \ldots \cup A_n =$ the set of assignments such that there exists at least 1 couple which dances together, and

$$\begin{aligned}
D(n) &= n! - \left|\bigcup_{i=1}^{n} A_i\right| \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} \\
&= n! \left(1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!}\right) \\
&= n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\end{aligned}$$

where the second equality is due to the inclusion-exclusion formula.

Problem 6. Find the generating function for the recurrence $a_0 = 1, a_1 = 1, a_{n+2} = a_{n+1} + 2a_n$, and using it find an explicit formula for a_n.

Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$\begin{aligned}
A(x) = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} 2a_{n-2} x^n \\
&= a_0 + a_1 x + \sum_{n=1}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+2} \\
&= a_0 + a_1 x + xA(x) - a_0 x + 2 \cdot x^2 A(x),
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
A(x)(2x^2 + x - 1) &= -1 \iff \\
A(x) \cdot 2(x - \frac{1}{2})(x+1) &= -1 \iff \\
A(x) &= -\frac{1}{2} \cdot \frac{1}{x - 1/2} \cdot \frac{1}{x+1} \\
&= -\frac{1}{2} \cdot \frac{2}{3} \left(\frac{1}{x - 1/2} - \frac{1}{x+1} \right) \\
&= \frac{1}{3} \left(\frac{2}{1 - 2x} + \frac{1}{1 - (-1)x} \right)
\end{aligned}$$

Now remember that

$$\frac{1}{1 - ax} = \sum_{k=0}^{\infty} a^k x^k,$$

and from this we can infer that

$$\begin{aligned}
A(x) = \sum_{k=0}^{\infty} \left(\frac{2^{k+1} + (-1)^k}{3} \right) x^k &\implies \\
a_n &= \frac{2^{n+1} + (-1)^n}{3}
\end{aligned}$$

Problem 7. $2n$ people want to buy one ticket each. One ticket costs \$5; n people have a \$5 bill and n people have a \$10 bill. Initially, the cashier has no money. In how many ways can the people line up so that at every moment, the next person either has a \$5 bill, or the cashier has change for a \$10 bill?

Equivalently, if we denote the n people having 5\$ as '(', and those having 10\$ as ')', we want to count the number a_n of possible permutations of n '(' and n ')', such that every substring $s_1, \ldots, s_k$ of size $1 \leq k \leq 2n$ of a permutation has at least as many '(' as ')'. Let A_n be the set of those strings. In that case, $a_1 = 1$, since '()' is the only string $\in A_1$, while $a_2 = 2$, since the only strings in A_2 are '(())' and '()()'.

Notice that any legal string $s \in A_{n+1}$ can be *uniquely* expressed as the following concatenation

$$s = (c_1)c_2,$$

where c_1, c_2 are strings in A_k and A_{n-k} respectively, for some k in $\{0, 1, \ldots, n\}$, where $c \in A_0 \iff c$ is the empty string, where $|A_0| = a_0 = 1$. Thus, it holds that

$$a_{n+1} = \sum_{k=0}^{n} a_k \cdot a_{n-k}.$$

Now, consider

$$\begin{aligned}
A(x) = \sum_{n=0}^{\infty} a_n x^n &= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= a_0 + x \cdot \sum_{n=1}^{\infty} a_n x^{n-1} = a_0 + x \cdot \sum_{n=0}^{\infty} a_{n+1} x^n \\
&= a_0 + x \cdot \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \cdot a_{n-k} \right) x^n \\
&= a_0 + x \cdot (A(x))^2
\end{aligned}$$

Therefore it holds that $x \cdot A^2(x) - A(x) + 1 = 0$. Interpreting this as a quadratic equation of $A(x)$, we get two possible solutions

$$A(x) = \frac{1 \pm \sqrt{1-4x}}{2x},$$

where only the one with $-$ has

$$a_0 = \lim_{x \to 0} A(x) = 1.$$

Also, due to the generalized binomial theorem, it holds that

$$\begin{aligned}
\sqrt{1-4x} &= \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} (-1)^n 4^n x^n \\
&= \sum_{n=0}^{\infty} \frac{-1}{2n-1} \binom{2n}{n} x^n = 1 + \sum_{n=1}^{\infty} \frac{-1}{2n-1} \binom{2n}{n} x^n
\end{aligned}$$

Thus,

$$
\begin{aligned}
A(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \sum_{n=1}^{\infty} \frac{1}{2(2n-1)} \binom{2n}{n} x^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{1}{2(2n+2-1)} \binom{2n+2}{n+1} x^{n} \\
&= \sum_{n=0}^{\infty} \frac{1}{2(2n+1)} \frac{(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!} x^{n} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)(2n)!}{(n+1)!(n+1)!} x^{n} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!n!} \frac{1}{n+1} x^{n} \\
&= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}
\end{aligned}
$$

and

$$
a_n = \binom{2n}{n} \frac{1}{n+1}
$$

follows.

Bonus problem. Prove that

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{m} = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} 2^k.$$

Proof: Let sets M, N, where $|M| = m, |N| = n$. Moreover, let $X \subseteq M$ and $Y \subseteq N \cup X$, where $|Y| = m$. Now notice the following:

- The left side counts the number of ordered pairs (X, Y). We firstly choose X and then Y, for any possible cardinality of X which is indicated by the value of k.
- As for the right side, the counter k is equal to $|Y \cap N|$, i.e. the number of elements of set N which have been chosen for Y. After choosing k elements from N, it remains to choose $m-k$ elements from M, with $\binom{m}{m-k} = \binom{m}{k}$ ways. Lastly, given that we have already fixed $m-k$ elements to belong to X (which have been chosen for set Y), it remains to count the number of sets $X \subseteq M$ which have those chosen elements. Their number is simply the number of possible subsets of the rest of the elements, i.e. $2^{m-(m-k)} = 2^k$.

$\square$