MATH 108, Winter 2022
HOMEWORK 2
Due Monday, February 7

Try solve the homework on your own. If you discuss with others, please list your collaborators. You can use any theorem / lemma that was stated in class as a fact (without reproving it).

Problem 1. Prove that every tree has at most 1 perfect matching.

Proof: Let graph $G = (V, E)$ be a tree. We will prove that G has at most one perfect matching. If G does not have a perfect matching or has a unique one, then the statement holds.

Let $M_1, M_2 \subseteq E$ be two *perfect matchings* of tree G. We will prove that $M_1 = M_2$. Let $G' = (V, E')$, where $E' = M_1 \cup M_2$. It is easy to see that $\forall v \in V$, one of the following holds

- *Case 1*: $\deg_{G'}(v) = 1$, in the case that the same edge $e = \{v, u\} \in M_1 \cap M_2$ covers vertex v in both matchings, for some $u \in V$. Furthermore, $\deg_{G'}(u) = 1$ follows.
- *Case 2*: $\deg_{G'}(v) = 2$, in the case that different edges $e_1 \neq e_2$ cover vertex v in matchings M_1 and M_2 respectively. Notice that[1] $e_1, e_2 \in M_1 \triangle M_2$, since no two edges of the same matching may cover the same vertex.

In other words, G' contains some pairs of vertices connected by a single edge, while the rest of vertices have degree 2. Suppose that $M_1 \triangle M_2 \neq \emptyset$. Then, excluding the vertices of Case 1, all of the rest of vertices have degree 2, hence they form a cycle. Since G is a tree, G' cannot have any cycles, contradiction. Therefore $M_1 \triangle M_2 = \emptyset \iff M_1 = M_2$. □

[1] $M_1 \triangle M_2 = (M_1 \cup M_2) \setminus (M_1 \cap M_2)$.

Problem 2. Prove or disprove:

- If a finite partially ordered set has a unique minimal element, then it is the minimum (smallest) element as well.

  Proof: Let $(P, \leq)$ be a finite partially ordered set and $m \in P$ its unique minimal element. We will prove that m is the minimum (smallest) element, i.e., $\forall p \in P, m \leq p$.

  Suppose that $\exists p \in P$ such that $m \leq p$ does not hold. Then, either $p < m$ or m and p are not comparable.

  (a) If $p < m$, then m is not a minimal element, contradiction.

  (b) If m and p are not comparable, then let $L_p = \{p' \in P \mid p' \leq p\}$. Obviously $m \notin L_p$, and since set P is finite, L_p is also finite.

  Claim 1 *$\exists m' \in L_p$ such that m' is minimal.*

  Proof: Assume the opposite, therefore $\forall m' \in L_p, \exists p' \in P$ such that $p' < m'$. Since $m' \leq p \implies p' \leq p \iff p' \in L_p$ follows. However, that also means that $\exists p'' \in L_p$ such that $p'' < p'$. Using the same argument $|L_p|$ times, and due to the property of transitivity, $p^* < p^*$ for some $p^* \in L_p$ follows, contradiction. □

  Therefore, there exists a minimal element $m' \neq m$, contradiction.

  Hence, $\forall p \in P, m \leq p \iff m$ is the minimum element. □

- If an infinite partially ordered set has a unique minimal element, then it is the minimum (smallest) element as well.

  Proof: We will provide an example which disproves the statement. Let $(P, \leq^*)$ be an infinite partially ordered set, where

  - $P = \{1\} \cup \{\ldots, -2, 0, +2, \ldots\}$ consist of 1 plus all even integers
  - $\leq^* = \{(x, y) \mid x, y \text{ are even} \wedge x \leq y\} \cup \{(1, 1)\}$

  The unique minimal element for this partially ordered set is element 1, since for any element $p \in P \setminus \{1\}, \exists p' \in P \setminus \{p\} : p' <^* p$. Notice however that it does *not* hold that $\forall p \in P, 1 \leq^* p$, therefore 1 is not the minimum element. □

Problem 3. An element u in a poset is called an upper bound on a set A, if $\forall a \in A; a \leq u$. An element s is called a supremum of A, if it is the smallest upper bound: it is an upper bound, and for every other upper bound s', we have $s \leq s'$.

Consider the natural numbers ordered by divisibility. Does every non-empty set have a supremum? Does every non-empty finite set have a supremum? Prove your answers.

Proof: Let $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n > 0\}$ and $(\mathbb{Z}_+, |)$ be a partially ordered set. Additionally, let $A \subseteq \mathbb{Z}_+$ and for an element $n \in \mathbb{Z}_+$, denote $A|n \iff \forall a \in A, a|n$. Furthermore, let $U = \{n \in \mathbb{Z}_+ \mid A|n\}$ denote the set comprised of all the upper bounds of subset A.

Let $\ell = \text{lcm}\{a \in A\}$ denote the least common multiple of all the elements of A (which may not necessarily exist). Thus, $\forall n \in \mathbb{Z}_+, A|n \implies \ell \leq n$. Obviously $\ell \in U$, hence, if it exists, it is an upper bound. It remains to prove that it is a supremum.

Claim 2 *$\forall u \in U, \ell|u$.*

Proof of Claim. Let $u \in U$ such that $\ell|u$ does not hold. Since $u > \ell$, it follows that ℓ and u are not comparable. In that case, $\exists q, r : u = q \cdot \ell + r$, with $q \geq 1$ and $1 \leq r < \ell$. Then, $u \in U \iff A|u \iff A|(q \cdot \ell + r) \overset{A|l}{\Longrightarrow} A|r$, contradiction, since $r < \ell$. □

Therefore, we seek to determine whether every subset A has a least common multiple $\ell \in \mathbb{Z}_+$. When set A is finite, we can indeed compute ℓ. However, that is not possible for every infinite set A. Consider for instance the set $A = \{p \in \mathbb{Z}_+ \mid p \text{ is prime}\} \subseteq \mathbb{Z}_+$ comprised of all the prime numbers. In that case, ℓ is the product of all the prime numbers p_i, i.e.

$$\ell = \prod_{i=1}^{\infty} p_i = +\infty$$

□

Problem 4. Let B_n denote the poset of $2^{[n]}$ ordered by inclusion. We already know that every finite poset $(X, \leq)$ can be embedded in B_n for $n = |X|$. Find a finite poset $(X, \leq)$ which can be embedded in B_n for some $n < |X|$.

Let $(X, \leq)$ be a finite partially ordered set, where X contains all the $0-1$ bit strings of size $\log n$, therefore $X = \{0,1\}^{\log n} \implies |X| = 2^{\log n} = n$. For $x \in X$, denote its i-th bit by x_i, where $i \in [\log n] = \{1, \dots, \log n\}$, and let $\leq = \{(x,y) \mid x, y \in X \wedge \forall i \in [\log n], x_i \leq y_i\}$. We will prove that $(X, \leq)$ can be embedded in $B_{\log n}$, where $\log n < n = |X|$.

For each element $x \in X$, define $L_x = \{i \in [\log n] \mid x_i = 1\}$. Therefore, the corresponding subset in $B_{\log n}$ of a given string is comprised of the elements denoting the position of its 1's. Also, notice that each $x \in X$ uniquely defines a corresponding subset L_x and vice versa.

Claim 3 $L_x \subseteq L_y \iff x \leq y$

Proof of $\Rightarrow$. Let $x, y \in X$, such that $L_x \subseteq L_y$. Then, for every $i \in [\log n]$, it holds that

$$x_i = 1 \iff i \in L_x \implies i \in L_y \iff y_i = 1,$$

therefore if a bit of element x is 1, the corresponding bit of element y is also 1, hence $x \leq y$. $\square$

Proof of $\Leftarrow$. Let $x, y \in X$, such that $x \leq y$. Then, for every $i \in [\log n]$, it holds that

$$i \in L_x \iff x_i = 1 \implies y_i = 1 \iff i \in L_y,$$

therefore $L_x \subseteq L_y$. $\square$

Problem 5. An *edge cover* of a graph $G = (V, E)$ is a set of edges $F \subseteq E$ such that every vertex in V is incident to some edge in F. Prove that if G has no isolated vertices (every degree is at least 1), then a minimum-cardinality edge cover has $|V| - |M|$ edges, where M is a maximum-cardinality matching.

Proof: Let $G = (V, E)$ such that $\forall v \in V, \deg_G(v) \geq 1$. Moreover, let $F \subseteq E$ be a minimum-cardinality edge cover of G and $G' = (V, F)$ the corresponding subgraph. Notice that G' cannot contain any cycles, since if that is the case, we can remove an edge from the cycle and still cover all the vertices. Therefore, all the connected components of G' are *trees*. Furthermore, each connected component contains at least 2 vertices, since if otherwise, the single vertex of the component will not be covered by any edge. Remember that a tree with n vertices has $n - 1$ edges, and notice that if G' has k connected components, with $n_1, \ldots, n_k$ vertices each (where $n_i \geq 2$), it follows that $|F| = n_1 - 1 + \ldots + n_k - 1 = |V| - k$. Thus, it suffices to prove that graph G can be partitioned to at most $|M|$ connected components, each of which contains at least 2 vertices, where $M \subseteq E$ is a maximum cardinality matching of G.

Suppose that it is possible to partition graph G to connected components $C_1, \ldots, C_k \subseteq V$ with $k > |M|$, such that

- for $i \neq j, C_i \cap C_j = \emptyset$
- $\forall i, |C_i| \geq 2$
- $\bigcup C_i = V$

In that case, if we keep an edge from each C_i (which always exists since $|C_i| \geq 2$), we get a matching of cardinality $k > |M|$, contradiction. Hence, $|F| \geq |V| - |M|$.

Finally, it remains to prove that there exists an F for which equality holds. Let $V(M)$ be the set of vertices covered by matching M, where $|V(M)| = 2|M|$. Initially, include the edges of M in our edge cover, thereby covering all but $|V \setminus V(M)| = |V| - 2|M|$ vertices. For each of the remaining vertices, just add one of its edges into edge cover F. Notice that each of those edges covers just this single uncovered vertex, since if otherwise, M would not be a maximum matching. Therefore, $|F| = |M| + (|V| - 2|M|) = |V| - |M|$. $\square$

Problem 6. Prove that if $A \subseteq \{1, 2, \ldots, 2n\}$, $|A| = n + 1$, then there are two numbers $a, b \in A$ such that $a|b$.

Proof: Our proof will closely follow the proof of Dilworth's Theorem presented in class. Let $\Omega = \{1, \ldots, 2n\}$ and define the poset $(\Omega, |)$. We will prove that the size of the maximum antichain is at most n, by proving that Ω can be decomposed into n chains. Therefore, if $A \subseteq \Omega$, with $|A| = n + 1$, then A is not an antichain, thus $\exists a, b \in A : a|b$, with $a \neq b$.

Define the bipartite graph $G = (V, E)$ such that $V = \Omega \cup \Omega'$ where each side contains the elements of Ω and $(u, v') \in E \iff u \neq v$ and $u|v$. Let M be a maximum matching on the aforementioned graph. Then, $|M| = n$, since each $i \in \Omega$ with $1 \leq i \leq n$, can be matched with $(2i)' \in \Omega'$ and for all $v \in \Omega$ with $v > n$, it holds that $\deg_G(v) = 0$.

In order to cover set Ω, start with $2n$ chains $C_i = \{i\}, i \in \Omega$, where C_i denotes the chain covering element i. Start from $i = 1$. If $e = (i, j') \in M$, then merge the two corresponding chains C_i and C_j. The result remains a chain, since $\forall i' \in C_i, i'|i$ and since $i|j$, $i'|j$ follows. Each edge in our matching decreases the number of chains required by 1. Since $|M| = n$, it follows that $2n - n = n$ chains cover Ω. $\square$

Problem 7. Prove that for any $k, \ell \geq 1$, every sequence of $k\ell+1$ distinct real numbers has either an increasing subsequence of length $k+1$, or a decreasing subsequence of length $\ell+1$. Prove also that this is optimal: there is a sequence of $k\ell$ distinct real numbers, which has neither an increasing subsequence of length $k+1$, nor a decreasing subsequence of length $\ell+1$.

Proof: Let $(a_1, \ldots, a_{k\ell+1})$ be a given sequence of distinct real numbers. We will prove that it either contains an increasing subsequence of length $k+1$, or a decreasing one of length $\ell+1$. Assume the opposite, i.e. that the given sequence of length $k\ell+1$ does not have neither an increasing subsequence of length $k+1$ nor a decreasing one of length $\ell+1$. Define a poset on $P = (1, \ldots, k\ell+1)$ such that $i \leq_P j \iff i \leq j$ and $a_i \leq a_j$. A chain on this poset is essentially an increasing subsequence. Moreover, for $i < j$, elements a_i and a_j comprise an antichain iff $a_i > a_j$, i.e., they form a (strictly) decreasing subsequence. Then, our assumption translates to

$$a(P) = \max|\text{antichain in } P| \leq \ell \quad \text{and} \quad \omega(P) = \max|\text{chain in } P| \leq k$$

However, it must hold that

$$k\ell + 1 = |P| \leq a(P) \cdot \omega(P) \leq k\ell,$$

contradiction, hence the statement holds.

Additionally, we will prove that there is a sequence of $k\ell$ distinct real numbers, which has neither an increasing subsequence of length $k+1$, nor a decreasing subsequence of length $\ell+1$.

Given k and ℓ, we will construct a sequence consisting of elements $1, \ldots, k\ell$ that satisfies the aforementioned statement. Consider the subsequences $A_i = (\ell \cdot i, \ell \cdot i - 1, \ldots, \ell \cdot (i-1) + 1)$, for $i = 1, \ldots, k$.

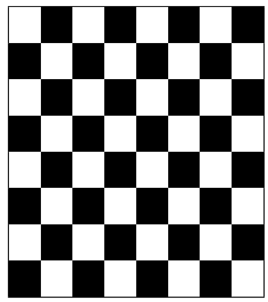
For the concatenation of A_i's, namely the sequence $A_1 A_2 \ldots A_k$, it holds that

- If $x \in A_i$ and $y \in A_j$ with $i < j$, then $x < y$, therefore the maximum length of a decreasing subsequence is $|A_i| = \ell$.
- There is no increasing subsequence of size greater than 1 in any of the A_i's, therefore the maximum increasing subsequence contains a single element from each A_i, hence has size k.

$\square$

Bonus problem. How many knights can you place on a chessboard without attacking one another?

Let $\mathbb{N}_0 = \{0, 1, \ldots\}$ be the set of natural numbers, including 0. A knight attacks another knight if they are both placed in the same diagonal, i.e., if the first one is placed on (i, j) and the second on (i', j'), then $\exists k \in \mathbb{N}_0$ such that $|i' - i| = |j' - j| = k$. Let the 8×8 chessboard as depicted in the following figure. Denote the tile placed on (i, j) as B_{ij} or W_{ij}, depending on its color, and let the upper left square of the following figure be W_{11}.



We firstly notice that no knight placed on a white square can attack a knight placed on a black square, and vice versa. Therefore, it suffices to find bounds for the white and the black squares separately.

Define a poset on $B = \{B_{ij}\}$ such that $B_{ij} \leq_B B_{i'j'} \iff \exists k \in \mathbb{N}_0 : i - i' = j' - j = k$. Essentially, we consider that a knight attacks another knight only if the second is placed on the same diagonal, but only to the upper right. In that case, we can cover B by taking the union of 7 chains, since each ($\nearrow$) diagonal is a chain. Thus, the maximum number of elements in an antichain is upper bounded by 7.

Notice that by restricting the possible "attacks" of a knight only to ($\nearrow$) movements, while completely ignoring ($\searrow$) movements, we can express this relationship as a partial order. Moreover, since we restricted the movements of the knight, the number of "complete" knights that can be placed on the chessboard is upper bounded by the number of "restricted" knights that can be placed, hence is upper bounded by 7.

In an analogous manner, we define a poset on $W = \{W_{ij}\}$ such that $W_{ij} \leq_W W_{i'j'} \iff \exists k \in \mathbb{N}_0 : i' - i = j' - j = k$. As in the previous case, by taking the union of the 7 upper left to lower right ($\searrow$) chains, we can cover the whole set W, thus the maximum number of elements in an antichain is upper bounded by 7.

Therefore, we have established an upper bound of $7 + 7 = 14$ on the number of knights that can be placed on the chessboard without any knight attacking another. It remains to prove that this is actually tight. Indeed, if we place our knights on the whole first row and on every tile of the last row but the tiles B_{81} and W_{88}, then no knight among the $8 + 6 = 14$ we placed attacks another.