MATH 108, Winter 2022
HOMEWORK 3
Due Monday, February 21

Try solve the homework on your own. If you discuss with others, please list your collaborators. You can use any theorem / lemma that was stated in class as a fact (without reproving it).

Problem 1. Call a graph G d-orderable if there is a linear ordering of the vertices, such that every vertex has at most d neighbors preceding it in the order. Prove that every d-orderable graph can be vertex-colored with $d+1$ colors.

Proof: We will prove the statement by induction on the number of vertices of the graph.

- *Basis*: Let $G = (\{v\}, \emptyset)$ denote the graph composed of a single vertex. Obviously G is d-orderable and can be vertex colored with $d+1$ colors.
- *Inductive Hypothesis*: Suppose that the statement holds for every d-orderable graph $G = (V, E)$ with $|V| \leq n-1$, i.e. G can be vertex colored with $d+1$ colors.
- *Inductive Step*: Let $G = (V, E)$ such that $|V| = n$ and G is d-orderable, therefore, there exists a linear ordering $L = \ell_1, \ldots, \ell_{|V|}$ of G's vertices such that every vertex has at most d neighbors preceding it. Notice that $\deg_G(\ell_{|V|}) \leq d$ follows.

  Let $G' = (V', E')$ be the graph obtained by G by removing vertex $\ell_{|V|}$ and its associated edges, i.e. $V' = V \setminus \{\ell_{|V|}\}$ and $E' = \{e = \{u, v\} \in E \mid u, v \neq \ell_{|V|}\}$. Then, notice that $L' = \ell_1, \ldots, \ell_{|V|-1}$ is a linear ordering for graph G' such that every vertex has at most d neighbors preceding it, thus G' is d-orderable with $|V'| = n-1$. Due to the inductive hypothesis, graph G' can be vertex colored with $d+1$ colors, and since $\deg_G(\ell_{|V|}) \leq d$, then we can color graph G with $d+1$ colors by coloring vertex $\ell_{|V|}$ with a color not used by its at most d neighbors.

$\square$

Problem 2. Prove that every bipartite graph with degrees at most 2 is 2-list-vertex-colorable (colorable from arbitrary lists of size 2). Either prove this directly, or using a theorem from class.

Proof: Let $G = (V, E)$ be a bipartite graph, such that $\forall v \in V, \deg_G(v) \leq 2$. Moreover, let, for $v \in V, L(v)$ denote the list of possible colors for vertex v, where $|L(v)| = 2$. We will prove that then, there exists an assignment of colors which constitutes a valid coloring for graph G.

Firstly, notice that, provided we have a legal coloring for the rest of the graph, we can always legally color a vertex of degree 0 or 1, since the number of available colors exceeds the number of its neighbors.

Therefore, by repeatedly removing the vertices of degree 0 or 1, it follows that, if we can find a proper coloring for the resulting graph G', then this can be extended to the original graph G, by coloring the removed vertices in a LIFO order.

Thus, it suffices to guarantee a proper coloring for graph $G' = (V', E')$, where for all $v' \in V$, it holds that $\deg_{G'}(v') = 2$. If $V' = \emptyset$, then G' is colorable in a trivial way. Alternatively, all vertices have degree 2, and each connected component of G' is an even cycle, since G is bipartite. However, even cycles are 2-list-vertex-colorable due to the following lemma and the statement again holds. $\square$

Lemma 1 *Let graph $G = (V, E)$ be an even cycle. Then G is 2-list-vertex-colorable.*

Proof: Let $u_i, u_{i+1} \in V$ be two adjacent vertices (i.e. $e = \{u_i, u_{i+1}\} \in E$) with different color lists. If no such pair of vertices exists, all vertices have the same color list, and since G is bipartite, G is also 2-colorable.

Firstly color u_{i+1} with a color not available for u_i and proceed to color $u_{i+2}, u_{i+3}, \ldots$ greedily. Each vertex other than the last vertex (which is u_i) can be colored without problems. Moreover, u_i is guaranteed not to conflict with vertex u_{i+1}, so it also can be colored, since at most one of its options (which are both different than the color of u_{i+1}) is ruled out by the color of vertex u_{i-1}. $\square$

Problem 3. Consider the non-bipartite variant of stable matching: a complete graph K_n (assume n even), each vertex has a linear preference order on the other vertices, and a stable matching is such that there is no unstable pair (which would prefer each other to their current partners). Find an instance where no stable perfect matching exists.

Consider the following complete graph $G = (V, E)$, with vertices $V = \{a, b, c, d\}$ and edges $E = \{ab, ac, ad, bc, bd, cd\}$. There are three perfect matchings for this graph:

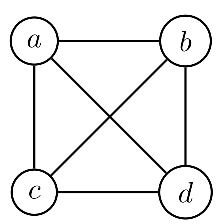
- $M_1 = \{ab, cd\}$
- $M_2 = \{ac, bd\}$
- $M_3 = \{ad, bc\}$

Denote by $L(v)$ the preference ordering over the rest of the vertices for vertex v. Then, if

- $L(a) = c, b, d$
- $L(b) = a, c, d$
- $L(c) = b, a, d$

notice that

- (a, c) is an unstable pair for matching M_1
- (b, c) is an unstable pair for matching M_2
- (a, b) is an unstable pair for matching M_3



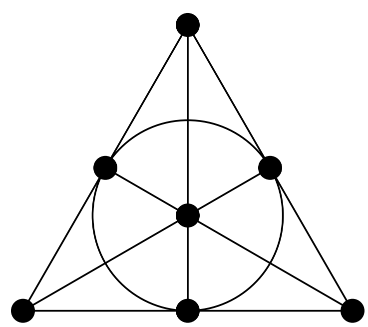
Problem 4. We say that a finite projective plane is 2-colorable, if the points can be colored with 2 colors so that each line has at least one point of each color. Prove that the Fano plane is *not* 2-colorable.

Proof: Consider the Fano plane $G = (V, E)$ as depicted in the following figure. We will prove that it is *not* 2-colorable. Suppose that it is, i.e. each vertex can be colored either *red* or *blue*, while each line has at least one vertex of each color. Let $R, B \subseteq V$ denote the set of the red and blue vertices respectively, where $R \cup B = V$ and $R \cap B = \emptyset$. W.l.o.g. assume that $|R| \geq |B|$.

- If $|R| \geq 5$, then R contains at least $\binom{5}{2} = 10$ pairs of points, each of which defines a line. However, since there are only 7 lines, it follows that there exist multiple pairs defining a single line, i.e. a line contains only same colored vertices.

- Else, $|R| = 4$. In that case, R contains $\binom{4}{2} = 6$ pairs of points, and if two different pairs correspond to the same line, then this line is monochromatic and we are done. Therefore, assume that each pair corresponds to a distinct line. Notice however that then, B contains $\binom{3}{2} = 3$ pairs of points, each defining a line. Thus, there exists a line which is defined simultaneously by a pair of red and a pair of blue points, contradiction since each line has 3 points.

Thus, we conclude that the Fano plane is not 2-colorable. $\square$



Problem 5. Prove that the Fano plane cannot be embedded in the usual Euclidean plane: there is no arrangement of 7 points and 7 lines in $\mathbb{R}^2$ such that the point-line relationships are exactly as in the Fano plane.

Proof: Suppose that there exists such an embedding, where X is the set of points and $\mathcal{L}$ the set of lines. Moreover, it holds that $|X| = |\mathcal{L}| = 7$ and each $x \in X$ belongs to exactly 3 lines $L \in \mathcal{L}$, while $|L| = 3$, for all $L \in \mathcal{L}$.

Now, call *edges* the parts of lines connecting consecutive points. Therefore, since each line has 3 points, it incurs 2 edges, for a total of 14 edges.

Each point belongs to exactly 3 lines, therefore each point has 3 adjacent edges. Let for point $x \in X, E(x)$ be the set of edges adjacent to it, where $|E(x)| = 3, \forall x \in X$. Moreover, it holds that

$$\left| \bigcup_{x \in X} E(x) \right| = \frac{3 \cdot 7}{2},$$

since each edge is adjacent to exactly 2 distinct points. However, this value is different from 14, thus contradiction. $\square$

Problem 6. Define a *liberated square* to be an $n \times n$ array of integers from $\{1, \dots, n\}$ without any additional constraints (numbers are allowed to be repeated arbitrarily). Two liberated squares A, B are orthogonal if every pair $(a, b) \in [n] \times [n]$ appears exactly once as (A_{ij}, B_{ij}) for some i, j.

Prove that t mutually orthogonal $n \times n$ Latin squares exist if and only if $t+2$ mutually orthogonal liberated $n \times n$ squares exist.

Proof of $\implies$. Let $S_1, \dots, S_t$ be t mutually orthogonal $n \times n$ Latin squares. Consider the square R such that $r_{ij} = i$, i.e. each row of the square has the same value, which is the number of the row. In an analogous manner, let C be the square where $c_{ij} = j$, i.e. each column of the square has the same value, which is the number of the column.

Obviously R and C are mutually orthogonal, since on position (i, j), the pair produced is (i, j). Moreover, since $S_1, \dots, S_t$ are Latin squares, there is no column or row containing the same number more than once, i.e., $s_{ij} = s_{i'j'} \implies i \neq i'$ and $j \neq j'$. Therefore, we can conclude that R and C are mutually orthogonal with all S_i, for $i = 1, \dots, t$. □

Proof of $\impliedby$. Let $L_1, \dots, L_{t+2}$ be $t+2$ mutually orthogonal $n \times n$ liberated squares. Suppose that, for some L_i, there exists $k \in \{1, \dots, n\}$ appearing at least $n+1$ times. In that case, since $\exists L_j$ mutually orthogonal with L_i, on the corresponding $n+1$ different positions, square L_j contains $n+1$ different numbers, contradiction since we have at most n different values to choose from. Thus, each value appears at most n times, therefore we conclude that each value appears *exactly* n times.

It remains to make some necessary permutations in order to produce *Latin* squares from these liberated squares. We firstly note that by simultaneously permuting all squares in the same way, mutual orthogonality remains.

Now, make the necessary permutations in order to have the first square become square R, which has $r_{ij} = i$. Due to orthogonality, all the rest squares $L'_2, \dots, L'_{t+2}$ have on each row a single appearance of each $k \in \{1, \dots, n\}$. Then, simultaneously permute the entries inside each row for all squares, so that the second square becomes square C, where $c_{ij} = j$. Square R remains as is, and squares $L''_3, \dots, L''_{t+2}$ remain mutually orthogonal, while also being orthogonal to both R and C. In that case however, for some $S \in \{L''_3, \dots, L''_{t+2}\}$, it holds that

- $s_{ij} = s_{i'j'} \implies i \neq i'$, due to orthogonality with R
- $s_{ij} = s_{i'j'} \implies j \neq j'$, due to orthogonality with C

In other words, each $k \in \{1, \dots, n\}$ appears exactly once in each row and column, i.e. S is a Latin square. □

Problem 7. Let $\mathcal{F} \subset \binom{X}{3}$ be a set of triples on a ground set X, where $|X| = 9$, such that for every $S, T \in \mathcal{F}$, $S \neq T$, we have $|S \cap T| \leq 1$. Prove that the maximum number of such triples is 12, and find an example of such a system.

Proof: We will firstly prove that any given point $x \in X$ is contained to at most 4 sets. Suppose that there exist $S_i \in \mathcal{F}$ such that $x \in S_i$, for $i = 1, \ldots, 5$. Then, it holds that $|S_i \cap S_j| \leq 1$, for $i \neq j$, therefore the rest of the elements of X appearing in S_i's (apart from x) appear at most once. Thus,

$$\left| \bigcup_{i=1}^{5} S_i \right| = 1 + 2 \cdot 5 = 11 > |X|,$$

contradiction.

This means that each given element appears at most 4 times in $\mathcal{F}$. Therefore, since there are 9 distinct elements, an upper bound on the total number of appearances (i.e. $\sum_{S \in \mathcal{F}} |S|$) is $4 \cdot 9 = 36$. Since $\forall S \in \mathcal{F}, |S| = 3$, it follows that $|\mathcal{F}| \leq 12$. □

Below we present an example where $X = \{a_{i,j} \mid 1 \leq i, j \leq 3\}$ and $|\mathcal{F}| = 12$. In this case, the triples are

- For $1 \leq i \leq 3, R_i = \{a_{i1}, a_{i2}, a_{i3}\}$ and $C_i = \{a_{1i}, a_{2i}, a_{3i}\}$.
- $D_1 = \{a_{11}, a_{22}, a_{33}\}$ and $D_2 = \{a_{13}, a_{22}, a_{31}\}$.
- $F_1 = \{a_{21}, a_{32}, a_{13}\}$ and $F_2 = \{a_{31}, a_{12}, a_{23}\}$.
- $F_3 = \{a_{11}, a_{32}, a_{23}\}$ and $F_4 = \{a_{21}, a_{12}, a_{33}\}$.

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}$$

Bonus problem. Let A be a subset of points of a finite projective plane of order n such that no 3 points in A lie on the same line. Prove that $|A| \leq n+2$.

Proof: Let $a \in A$ be a point belonging to $A \subseteq X$, where X denotes the set of points. Since the finite projective plane is of order n, there are exactly $n+1$ lines $L_1, \ldots, L_{n+1}$ passing through point a. Moreover, there is no point $x' \in X$ such that it does not belong to one of these lines, since then there would be $n+2$ lines passing through a. Therefore, the lines L_i cover the set X, i.e.

$$\bigcup_{i=1}^{n+1} L_i = X.$$

Thus, apart from point a, set A may contain at most *one* element from each line L_i, hence it holds that $|A| \leq 1 + (n+1) = n+2$. $\square$