Algebraic Reductibility Experiments of RANS inspired equations

A Preprint

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Abstract

In recent years, a new generation of wind tunnel experiments has emerged that employ an active grid to replicate turbulent airflow. We propose that this phenomenon can be explained by considering a fluctuating initial vorticity. This finding suggests an RANS-type modelling approach, in which combinations of the original velocity field and initial turbulent vorticity can be observed. Rather than presupposing the particular distribution of the fluctuation field, we can hypothesise that the wind initially behaves in a classical manner. We can then incorporate solely the initial conditions of the turbulent field to analyse the consequences of this deflection in the cross-convective terms that emerge. Specifically, rather than assuming a distribution associated with the fluctuation, algebraic simplifications can be obtained by suggesting a classical flow $\boldsymbol{v}$ and formulating a new equation for the vorticity tilde $\tilde{\omega}$ of the drift. by applying the Ronsenfeld-Groebner algorithm. This specific approach facilitates the exploration of innovative formulations of traditional fluid equations, a perspective that has largely been overlooked within the field. More elementary systems have the potential to yield fresh insights, reveal novel algebraic frameworks and significantly reduce computational costs by simplifying the problem.

Keywords Turbulence Theory, Vortex Dynamics, Gas Dynamics

Contents

1 Introduction

Olga Ladyzhenskaya, stated the: "*Problem 1. Do the Navier-Stokes equations together with the initial and boundary conditions actually give a deterministic description of the dynamics of an incompressible fluid?*"[28, p. 252]. However, there are inaccuracies in the description of flows given by ideal models and real fluids [14, 15]. In particular, there is *turbulence*. In 2021, the Intergovernmental Panel on Climate Change (IPCC) reported that "*Atmospheric models include representations of (...) turbulence, convection and gravity waves that are not fully represented by grid-scale dynamics*"[22, p. 277]. More recently, it claimed that: "*The choices and actions implemented in this decade will have impacts now and for thousands of years.*" [23]. In this context, Ladyzhenskaya's Problem 1 raises the central question guiding this study: *Could limit models derived from turbulent problems provide a better understanding of the dynamics of turbulent fluids in the atmosphere? If so, at what scale? Can general turbulent models derive valid limit formulas? Could this lead to the development of new hybrid models and mathematical analyses in the context of generating digital twins?*

The study of *turbulent flow*, is pivotal in engineering applications, specifically "*for better analytical, synthetic and reduced order models of turbulence, better model coupling methods and basic understanding of flow phenomena governing kinetic energy entrainment*" [34]. As showed in the work of Deskos, Lee, Draxl, and Sprague [16] and Jeon and Kang [24] (2024), the interactions between flow-sediment and peak wind loads [20] have become instrumental in modelling tools designed to predict and prepare for the effects of extreme events, such as the impact of hurricanes in coastal cities. In 2020, Neuhaus, Hölling, Wouter, Bos and Peinke [35] presented an experimental setup in which an active grid of 80 rotating shafts with diamond-shaped aluminium wings recreated atmospheric-like turbulence under laboratory conditions. A substantial body of research has been conducted on the validation of experiments with active-grid turbulence in different laboratories, yielding highly satisfactory results [36, 25, 26, 2].

In order to adopt a more systematic framework to describe this case, it is proposed here that the effect of the active grid in the flow be represented as an the *fluctuation's* initial vorticity to obtain a model for curl dynamics inspired by the *Reynolds-Averaged Navier-Stokes equations* (RANS) of the form:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} \; + \; (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{\omega}} + (\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{\omega}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{\omega} - \nu\Delta(\boldsymbol{\omega}+\tilde{\boldsymbol{\omega}}) = 0, \tag{1}$$

where the total momentum $\boldsymbol{v}+\tilde{\boldsymbol{v}}$ consists of two components on an open rectangular prism in Euclidean space $\mathbb{R}^3$. One, $\boldsymbol{v}$ corresponds to the classical velocity, and the other, $\tilde{\boldsymbol{v}}$, to the deflection caused by the active grid, both with null boundary conditions. In this approach, fluctuations have a clear physical interpretation. They represent deviations in flow caused by grid deflections or environmental inhomogeneities. We seek a pressureless model where the solutions allow us to understand the behavior of both curls, $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ and $\tilde{\boldsymbol{\omega}} = \nabla\times\tilde{\boldsymbol{v}}$, in terms of their initial kinetic energies and angles. In Theorem 1, we derive the expression above by applying the curl operator to the Navier–Stokes equations. We emphasise that this derivation is sufficiently detailed to yield a more general form with bounded boundary conditions, as demonstrated in Corollary **??**. This general form is not equivalent to the simpler case, and importantly provides an error estimate that is explicitly dependent on these boundary conditions.

Reynolds splits the velocity flow as the sum $\boldsymbol{v}+\tilde{\boldsymbol{v}}$ of one, $\boldsymbol{v}$, that comes directly form the forces acting on each fluid particle, and the *fluctuation* $\tilde{\boldsymbol{v}}$ that deflects it, both with zero divergence [39, p. 140]. According to Tominaga and Meneveau [42, p. 3] [34, p. 15], the RANS equations remain central in the study of turbulence. Our far-reaching objective is to establish sufficient conditions for the existence and uniqueness of periodic RANS solutions and restrictions to its *frequency spectrum* which are empirically found [1] and for which, even recently, there is a vast number of cases where, to the best of our knowledge, no mathematical proofs have been reported. For example, for a fluid flowing past a cylinder and having Reynolds numbers between 50 and 250 [41]. In general, there are very few fundamental results on local and global existence and uniqueness of solutions for models [5, p. 11] that seek to represent turbulent behaviour based on the Navier-Stokes Equations such as the LANS-α [31, 32], the Leray-α [13], and their limit cases. Given that the model in Eq. (1) above, as well as in the LANS-α and Leray-α statements, the systems are *algebraic* —involving only polynomials in variables and derivatives and no transcendental functions— this opens up a new avenue for exploring the application of methods and algorithms from *differential algebra* that have not previously been considered in this context.

In 1932, Joseph Fels Ritt published a book called "*Differential equations from an algebraic standpoint*", where he put forward his first ideas on the subject. He states the fundamental differential algebra theorems in 1950 [38]. Afterwards, the theory is systematized by Ellis Robert Kolchin [27]. In 1994, Francois Boulier gives alternative proofs and implements algorithms to "*make certain theorems effective*" [9, 8]. His distinguished doctoral student and colleague, François Lemaire, created the MAPLE package called *DifferentialAlgebra* [1] based on the same precepts, which makes it possible to find this system when the search time allows it. Finally, they both brought it to Python last December of 2023 [2]. It is this tool that we use to explore the algebraic restatement of different fluid models. To our knowledge, there has been no other analysis of turbulent models in fluids that includes this point of view. This is, a work that includes a step of fluid dynamics simplification from an algebraic standpoint. We experimented with the possibility of reducing the number of unknowns present in each equation by applying *Differential Algebra* algorithms. The results obtained in this research were presented, without detailed proofs, at *the Engineering Mechanics Institute Conference and Probabilistic Mechanics & Reliability Conference* (EMI/PMC 2024) held in Chicago, and at the *the* $11th$ *European Nonlinear Dynamics Conference* (ENOC) conducted in Delft.

2 Problem Statement

It is a well-known practice to use the curl operator on the Navier-Stokes equations to derive pressureless systems. However, we have not come across similar approaches for the Navier-Stokes equations that take into consideration velocities for the *two-scale decomposition of the flow*, as outlined in the RANS equations, before an average hypothesis is considered, and where the fluctuation has a clear physical meaning as the effect of a deflection caused by a a lack of homogeneity in the environment. In Theorem1, we derive the general form obtained by applying the rotational to the Navier-Stokes equations where the velocity is split as a sum of two parts, $\boldsymbol{v} + \tilde{\boldsymbol{v}}$, where $\boldsymbol{v}$, and $\tilde{\boldsymbol{v}}$ are assumed to be divergence free, smooth functions on an open rectangular prism in the Euclidean space $\mathbb{R}^3$, both with null boundary conditions. We would like to remark that the same process is sufficiently detailed to derive, in Corollary**??**, a general form with bounded boundary conditions that is not equivalent to the simpler statement.

We follow the standard procedure of applying the rotational operator to the RANS approach, before averaging over time. Next, we calculate several kinetic energy quantities based on turbulent vorticity. This way, the restriction on the time of existence and uniqueness in the 3D Navier-Stokes Equations obtained by Jean Leray [**?**] can be extended to its turbulent associated field. This extension depends on the initial kinetic energy of the turbulent vorticity., Given the classical conditions of existence and uniqueness of a classical velocity in the Navier-Stokes incompressible equations over a two dimensional domain for a finite time, the associated fluctuation properties become a consequence the original classical variables and the initial turbulent vorticity, without any other assumption over the fluctuation.

Remark 1. *The following results are stated so that they may be extended to* starshaped *real domains R (where any ray with an origin in a fixed point has a unique common point with the topological boundary of the domain) and unknown solutions in a Sobolev space $W_p^k(R)$, with $p \in \mathbb{N} - \{0\}$. This is possible thanks to the fact that the space of infinitely differentiable functions defined on the topological closure $\bar{R} = R \cup \partial R$ of starshaped domain R, denoted as $C^\infty(\bar{R})$, is dense in $W_p^k(R)$, where ∂R denotes the topological boundary of R.*

Theorem 1. *Let $R= (0, L_1) \times (0, L_2) \times (0, L_3)$, for fixed and real values denoted by $L_1, L_2, L_3 > 0$, and $p, \boldsymbol{v}, \tilde{\boldsymbol{v}} \in C^1\left([0,T), C^\infty(R;\mathbb{R}^3)\right)$, for a finite time T, $0 < T < \infty$, such that:*

(i) For each $t \in [0,\infty)$, p_t ,$\boldsymbol{v}_t$, and $\tilde{\boldsymbol{v}}_t$ have a continuous extension to the topological boundary ∂R of R.

(ii) They satisfy the following boundary conditions for $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ at ∂R:

$$\begin{cases} \boldsymbol{v}(x_1,x_2,x_3,t) = (0,0,0); \\ \tilde{\boldsymbol{v}}(x_1,x_2,x_3,t) = (0,0,0), \end{cases} \tag{2}$$

for all $(x_1, x_2, x_3) \in \partial R$ and for all $t \in [0, T)$.

Assume that $\boldsymbol{v} = (v_1, v_2, v_3)$, $\tilde{\boldsymbol{v}} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$, and p verify the following system of equations:

$$\begin{cases} \nabla \cdot \boldsymbol{v} = 0, \\ \nabla \cdot \tilde{\boldsymbol{v}} = 0, \\ \frac{D(\boldsymbol{v}+\tilde{\boldsymbol{v}})}{Dt} = \nu\Delta(\boldsymbol{v}+\tilde{\boldsymbol{v}}) - \nabla p, \end{cases} \tag{3}$$

[1] Overview of the DifferentialAlgebra Package Page Link
[2] Overview of the DifferentialAlgebra Package in Python Link

where $\nabla \cdot \boldsymbol{v} = \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}$ *denotes the divergence,* $\frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ *is the convective derivative of* $\boldsymbol{v}$ *with a convective term* $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \left(\sum_{i=1}^{3} v_i \frac{\partial v_j}{\partial x_i}\right)_{j=1}^{3}$*, and* Δ *is the Laplacian operator* $\Delta\boldsymbol{v} = \left(\sum_{m=1}^{3} \frac{\partial^2 v_j}{\partial x_j^2}\right)_{j=1}^{3}$.

Then, $\boldsymbol{v}$*,* $\tilde{\boldsymbol{v}}$*, and their corresponding curls* $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$*, and* $\tilde{\boldsymbol{\omega}} = \nabla \times \tilde{\boldsymbol{v}}$ *satisfy:*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{\omega}} + (\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{\omega}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{\omega} - \nu\Delta(\boldsymbol{\omega} + \tilde{\boldsymbol{\omega}}) = 0, \tag{4}$$

for at least one point $x_0 = (x_1^0, x_2^0, x_3^0) \in R \subset \mathbb{R}^3$.

Remark 2. *We may assume that the domain is a rectangular prism* R *and* $\boldsymbol{v}$ *is an element of the space* $C^\infty(\bar{R})$ *that satisfies both the Leibinz rule for the product derivative, the Gauss-Green formula [**?**, p. 699], and the generalisation of Schwarz's Theorem [**?**, p. 280] to exchange the order of the generalised second partial derivatives with respect to the spatial and temporal independent variables [33, **?**], and perform an integration by parts process when needed.*

Proof. In order to obtain Eq. (4), we will integrate each term on both sides of the Conservation of Momentum Law of Eq. (3) over the entire domain R:

$$\iiint_R \left(\frac{D(\boldsymbol{v}+\tilde{\boldsymbol{v}})}{Dt}\right) d\boldsymbol{x} = \iiint_R \left(\nu\Delta(\boldsymbol{v}+\tilde{\boldsymbol{v}}) - \nabla p\right) d\boldsymbol{x},$$

where $d\boldsymbol{x} = dx_1 dx_2 dx_3$. By the monotone convergence theorem [**?**], if:

$$\nabla \times \left(\iiint_R \left(\frac{D(\boldsymbol{v}+\tilde{\boldsymbol{v}})}{Dt}\right) d\boldsymbol{x}\right) = \nabla \times \left(\iiint_R \left(\nu\Delta(\boldsymbol{v}+\tilde{\boldsymbol{v}}) - \nabla p\right) d\boldsymbol{x}\right),$$

then,

$$\iiint_R \left(\nabla \times \frac{D(\boldsymbol{v}+\tilde{\boldsymbol{v}})}{Dt}\right) d\boldsymbol{x} = \iiint_R \left(\nabla \times \left(\nu\Delta(\boldsymbol{v}+\tilde{\boldsymbol{v}}) - \nabla p\right)\right) d\boldsymbol{x}. \tag{5}$$

The left term is developed as follows:

$$\nabla \times \frac{D(\boldsymbol{v}+\tilde{\boldsymbol{v}})}{Dt} = \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\frac{\partial(v_3+\tilde{v}_3)}{\partial t} + \sum_{m=1}^{3}(v_m+\tilde{v}_m)\frac{\partial(v_3+\tilde{v}_3)}{\partial x_m}\right) \\ \frac{\partial}{\partial x_3}\left(\frac{\partial(v_1+\tilde{v}_1)}{\partial t} + \sum_{m=1}^{3}(v_m+\tilde{v}_m)\frac{\partial(v_1+\tilde{v}_1)}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial(v_2+\tilde{v}_2)}{\partial t} + \sum_{m=1}^{3}(v_m+\tilde{v}_m)\frac{\partial(v_2+\tilde{v}_2)}{\partial x_m}\right) \end{pmatrix}^t - $$
$$- \begin{pmatrix} \frac{\partial}{\partial x_3}\left(\frac{\partial(v_2+\tilde{v}_2)}{\partial t} + \sum_{m=1}^{3}(v_m+\tilde{v}_m)\frac{\partial(v_2+\tilde{v}_2)}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial(v_3+\tilde{v}_3)}{\partial t} + \sum_{m=1}^{3}(v_m+\tilde{v}_m)\frac{(v_3+\tilde{v}_3)}{\partial x_m}\right) \\ \frac{\partial}{\partial x_2}\left(\frac{\partial(v_1+\tilde{v}_1)}{\partial t} + \sum_{m=1}^{3}(v_m+\tilde{v}_m)\frac{\partial(v_1+\tilde{v}_1)}{\partial x_m}\right) \end{pmatrix}^t,$$

where $(a_{ij})^t = (a_{ji})$ denotes the matrix transpose of $(a_{ij}) \in M_{m\times n}(\mathbb{R})$, $m, n \in \mathbb{N}$. Then, a vector that contains a time derivative can be extracted:

$$A = \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\frac{\partial(v_3+\tilde{v}_3)}{\partial t}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial(v_2+\tilde{v}_2)}{\partial t}\right) \\ \frac{\partial}{\partial x_3}\left(\frac{\partial(v_1+\tilde{v}_1)}{\partial t}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial(v_3+\tilde{v}_3)}{\partial t}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial(v_2+\tilde{v}_2)}{\partial t}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial(v_1+\tilde{v}_1)}{\partial t}\right) \end{pmatrix}^t.$$

With the help the Schwarz's Theorem, we are able to change the order of the derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_j}$, for $j = 1, 2, 3$, so that the last expression A is equal to the sum of:

$$\frac{\partial}{\partial t}\begin{pmatrix} \frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3} \\ -\frac{\partial v_3}{\partial x_1}+\frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2} \end{pmatrix}^t = \frac{\partial \boldsymbol{\omega}}{\partial t}; \quad \text{and} \quad \frac{\partial}{\partial t}\begin{pmatrix} \frac{\partial \tilde{v}_3}{\partial x_2}-\frac{\partial \tilde{v}_2}{\partial x_3} \\ -\frac{\partial \tilde{v}_3}{\partial x_1}+\frac{\partial \tilde{v}_1}{\partial x_3} \\ \frac{\partial \tilde{v}_2}{\partial x_1}-\frac{\partial \tilde{v}_1}{\partial x_2} \end{pmatrix}^t = \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t}.$$

This is done analogously to the classic Navier-Stokes equations for the curl. In this case, we obtain: $A = \frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t}$, as expected. We are left with a sum over m given by the curl of the convective term $(\boldsymbol{v} + \tilde{\boldsymbol{v}} \cdot \nabla)\, \boldsymbol{v} + \tilde{\boldsymbol{v}}$ of $\boldsymbol{v} + \tilde{\boldsymbol{v}}$:

$$\begin{aligned} M &= \nabla \times \left[(\boldsymbol{v} + \tilde{\boldsymbol{v}} \cdot \nabla)\, \boldsymbol{v} + \tilde{\boldsymbol{v}}\right]^t \\ &= \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\sum_{m=1}^3 (v_m + \tilde{v}_m)\frac{\partial (v_3+\tilde{v}_3)}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(\sum_{m=1}^3 (v_m + \tilde{v}_m)\frac{\partial (v_2+\tilde{v}_2)}{\partial x_m}\right) \\ \frac{\partial}{\partial x_3}\left(\sum_{m=1}^3 (v_m + \tilde{v}_m)\frac{\partial (v_1+\tilde{v}_1)}{\partial x_m}\right) - \frac{\partial}{\partial x_1}\left(\sum_{m=1}^3 (v_m + \tilde{v}_m)\frac{(v_3+\tilde{v}_3)}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(\sum_{m=1}^3 (v_m + \tilde{v}_m)\frac{\partial (v_2+\tilde{v}_2)}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(\sum_{m=1}^3 (v_m + \tilde{v}_m)\frac{\partial (v_1+\tilde{v}_1)}{\partial x_m}\right) \end{pmatrix}; \\ &= \sum_{m=1}^3 \begin{pmatrix} \frac{\partial}{\partial x_2}\left(v_m\frac{\partial v_3}{\partial x_m} + v_m\frac{\partial \tilde{v}_3}{\partial x_m} + \tilde{v}_m\frac{\partial v_3}{\partial x_m} + \tilde{v}_m\frac{\partial \tilde{v}_3}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(v_m\frac{\partial v_2}{\partial x_m} + v_m\frac{\partial \tilde{v}_2}{\partial x_m} + \tilde{v}_m\frac{\partial v_2}{\partial x_m} + \tilde{v}_m\frac{\partial \tilde{v}_2}{\partial x_m}\right) \\ -\frac{\partial}{\partial x_1}\left(v_m\frac{\partial v_3}{\partial x_m} + v_m\frac{\partial \tilde{v}_3}{\partial x_m} + \tilde{v}_m\frac{\partial v_3}{\partial x_m} + \tilde{v}_m\frac{\partial \tilde{v}_3}{\partial x_m}\right) + \frac{\partial}{\partial x_3}\left(v_m\frac{\partial v_1}{\partial x_m} + v_m\frac{\partial \tilde{v}_1}{\partial x_m} + \tilde{v}_m\frac{\partial v_1}{\partial x_m} + \tilde{v}_m\frac{\partial \tilde{v}_1}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(v_m\frac{\partial v_2}{\partial x_m} + v_m\frac{\partial \tilde{v}_2}{\partial x_m} + \tilde{v}_m\frac{\partial v_2}{\partial x_m} + \tilde{v}_m\frac{\partial \tilde{v}_2}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(v_m\frac{\partial v_1}{\partial x_m} + v_m\frac{\partial \tilde{v}_1}{\partial x_m} + \tilde{v}_m\frac{\partial v_1}{\partial x_m} + \tilde{v}_m\frac{\partial \tilde{v}_1}{\partial x_m}\right) \end{pmatrix}. \end{aligned}$$

We will assign a name for each term and manipulate them separately; then we'll rejoin them. From the vector M and the linearity of the differential operators, we get four terms, M_1, M_2, M_3, and M_4:

$$M_1 = \sum_{m=1}^3 \begin{pmatrix} \frac{\partial}{\partial x_2}\left(v_m\frac{\partial v_3}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(v_m\frac{\partial v_2}{\partial x_m}\right) \\ -\frac{\partial}{\partial x_1}\left(v_m\frac{\partial v_3}{\partial x_m}\right) + \frac{\partial}{\partial x_3}\left(v_m\frac{\partial v_1}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(v_m\frac{\partial v_2}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(v_m\frac{\partial v_1}{\partial x_m}\right) \end{pmatrix}; \quad M_2 = \sum_{m=1}^3 \begin{pmatrix} \frac{\partial}{\partial x_2}\left(v_m\frac{\partial \tilde{v}_3}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(v_m\frac{\partial \tilde{v}_2}{\partial x_m}\right) \\ -\frac{\partial}{\partial x_1}\left(v_m\frac{\partial \tilde{v}_3}{\partial x_m}\right) + \frac{\partial}{\partial x_3}\left(v_m\frac{\partial \tilde{v}_1}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(v_m\frac{\partial \tilde{v}_2}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(v_m\frac{\partial \tilde{v}_1}{\partial x_m}\right) \end{pmatrix};$$

$$M_3 = \sum_{m=1}^3 \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\tilde{v}_m\frac{\partial v_3}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(\tilde{v}_m\frac{\partial v_2}{\partial x_m}\right) \\ -\frac{\partial}{\partial x_1}\left(\tilde{v}_m\frac{\partial v_3}{\partial x_m}\right) + \frac{\partial}{\partial x_3}\left(\tilde{v}_m\frac{\partial v_1}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(\tilde{v}_m\frac{\partial v_2}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(\tilde{v}_m\frac{\partial v_1}{\partial x_m}\right) \end{pmatrix}; \quad M_4 = \sum_{m=1}^3 \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\tilde{v}_m\frac{\partial \tilde{v}_3}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(\tilde{v}_m\frac{\partial \tilde{v}_2}{\partial x_m}\right) \\ -\frac{\partial}{\partial x_1}\left(\tilde{v}_m\frac{\partial \tilde{v}_3}{\partial x_m}\right) + \frac{\partial}{\partial x_3}\left(\tilde{v}_m\frac{\partial \tilde{v}_1}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(\tilde{v}_m\frac{\partial \tilde{v}_2}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(\tilde{v}_m\frac{\partial \tilde{v}_1}{\partial x_m}\right) \end{pmatrix},$$

such that:

$$\iiint_R \left(\nabla \times \frac{D(\boldsymbol{v} + \tilde{\boldsymbol{v}})}{Dt}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + M_1 + M_2 + M_3 + M_4\right) d\boldsymbol{x}.$$

Development of M_1: In order to split M_1 into two terms, we use the Leibniz product rule and the Schwarz theorem:

$$M_1 = \sum_{m=1}^3 \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} + v_m\frac{\partial^2 v_3}{\partial x_m \partial x_2} - \frac{\partial v_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} - v_m\frac{\partial^2 v_2}{\partial x_m \partial x_3} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} - v_m\frac{\partial^2 v_3}{\partial x_m \partial x_1} + \frac{\partial v_m}{\partial x_3}\frac{\partial v_1}{\partial x_m} + v_m\frac{\partial^2 v_1}{\partial x_m \partial x_3} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial v_2}{\partial x_m} + v_m\frac{\partial^2 v_2}{\partial x_m \partial x_1} - \frac{\partial v_m}{\partial x_2}\frac{\partial v_1}{\partial x_m} - v_m\frac{\partial^2 v_1}{\partial x_m \partial x_2} \end{pmatrix}.$$

Then:

$$M_1 = \sum_{m=1}^3 \begin{pmatrix} v_m\frac{\partial^2 v_3}{\partial x_m \partial x_2} - v_m\frac{\partial^2 v_2}{\partial x_m \partial x_3} \\ -v_m\frac{\partial^2 v_3}{\partial x_m \partial x_1} + v_m\frac{\partial^2 v_1}{\partial x_m \partial x_3} \\ v_m\frac{\partial^2 v_2}{\partial x_m \partial x_1} - v_m\frac{\partial^2 v_1}{\partial x_m \partial x_2} \end{pmatrix} + \sum_{m=1}^3 \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} + \frac{\partial v_m}{\partial x_3}\frac{\partial v_1}{\partial x_m} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial v_2}{\partial x_m} - \frac{\partial v_m}{\partial x_2}\frac{\partial v_1}{\partial x_m} \end{pmatrix}.$$

We can factor out the term v_m in the first vector of the right side of this last expression. By definition of the curl $\nabla \times \boldsymbol{v} = \boldsymbol{\omega}$ and the convective term $(\boldsymbol{v} \cdot \nabla)\,\boldsymbol{\omega}$, we obtain:

$$M_1 = \sum_{m=1}^{3} v_m \begin{pmatrix} \frac{\partial}{\partial x_m}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) \\ \frac{\partial}{\partial x_m}\left(-\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) \\ \frac{\partial}{\partial x_m}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) \end{pmatrix} + \sum_{m=1}^{3} \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} + \frac{\partial v_m}{\partial x_3}\frac{\partial v_1}{\partial x_m} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial v_2}{\partial x_m} - \frac{\partial v_m}{\partial x_2}\frac{\partial v_1}{\partial x_m} \end{pmatrix}, \tag{7a}$$

$$= \left[(\boldsymbol{v} \cdot \nabla)\,\boldsymbol{\omega}\right]^t + \sum_{m=1}^{3} \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} + \frac{\partial v_m}{\partial x_3}\frac{\partial v_1}{\partial x_m} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial v_2}{\partial x_m} - \frac{\partial v_m}{\partial x_2}\frac{\partial v_1}{\partial x_m} \end{pmatrix}. \tag{7b}$$

We will look at the first coordinate of the second vector on the right side of Eq. (7a), and then follow the same process with its other two components:

$$\sum_{m=1}^{3} \iiint_R \left(\frac{\partial v_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial v_2}{\partial x_m}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_1}{\partial x_2}\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\frac{\partial v_2}{\partial x_1}\right) d\boldsymbol{x} + \tag{7c}$$

$$+ \iiint_R \left(\frac{\partial v_2}{\partial x_2}\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_2}\frac{\partial v_3}{\partial x_3} - \frac{\partial v_3}{\partial x_3}\frac{\partial v_2}{\partial x_3}\right) d\boldsymbol{x}. \tag{7d}$$

The second integral (7d) at the right side of the last equation is:

$$\iiint_R \left(\frac{\partial v_2}{\partial x_2}\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_2}\frac{\partial v_3}{\partial x_3} - \frac{\partial v_3}{\partial x_3}\frac{\partial v_2}{\partial x_3}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) d\boldsymbol{x}. \tag{7e}$$

We notice that we have almost completed the divergence $\nabla \cdot \boldsymbol{v}$ in the right side of Eq. (7e). To obtain the remaining term, $\boxed{\frac{\partial v_1}{\partial x_1}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)}$, it is necessary to integrate by parts the integral on the right side of line (7c). We recall the boundary conditions of Eq. (2), then:

$$\iint_{\partial R} \left(v_1 \frac{\partial v_3}{\partial x_2}, 0, 0\right) \cdot n \; ds = 0 = \iint_{\partial R} \left(0, v_1 \frac{\partial v_3}{\partial x_1}, 0\right) \cdot n \; ds. \tag{7f}$$

This way, by the Ostrogradskii-Gauss theorem, we obtain:

$$\iiint_R \left(\frac{\partial v_1}{\partial x_2}\frac{\partial v_3}{\partial x_1}\right) d\boldsymbol{x} + \iiint_R \left(v_1 \frac{\partial^2 v_3}{\partial x_1 \partial x_2}\right) d\boldsymbol{x} = 0 = \iiint_R \left(\frac{\partial v_1}{\partial x_1}\frac{\partial v_3}{\partial x_2}\right) d\boldsymbol{x} + \iiint_R \left(v_1 \frac{\partial^2 v_3}{\partial x_2 \partial x_1}\right) d\boldsymbol{x}.$$

Finally, by the Swcharz theorem, the second integral on each side of the last equation is equal to the other, and:

$$\boxed{\iiint_R \left(\frac{\partial v_1}{\partial x_2}\frac{\partial v_3}{\partial x_1}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_1}{\partial x_1}\frac{\partial v_3}{\partial x_2}\right) d\boldsymbol{x}.} \tag{7g}$$

Analogously, suppose that:

$$\iint_{\partial R} \left(v_1 \frac{\partial v_2}{\partial x_3}, 0, 0\right) \cdot n \; ds = 0 = \iint_{\partial R} \left(0, 0, v_1 \frac{\partial v_2}{\partial x_1}\right) \cdot n \; ds. \tag{7h}$$

Thus:

$$\iiint_R \left(\frac{\partial v_1}{\partial x_3}\frac{\partial v_2}{\partial x_1}\right) d\boldsymbol{x} + \iiint_R \left(v_1 \frac{\partial^2 v_2}{\partial x_3 \partial x_1}\right) d\boldsymbol{x} = 0 = \iiint_R \left(\frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_3}\right) d\boldsymbol{x} + \iiint_R \left(v_1 \frac{\partial^2 v_2}{\partial x_1 \partial x_3}\right) d\boldsymbol{x}, \tag{7i}$$

and:

$$\boxed{\iiint_R \left(\frac{\partial v_1}{\partial x_3}\frac{\partial v_2}{\partial x_1}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_3}\right) d\boldsymbol{x}.} \tag{7j}$$

Together, Eq. (7g) and (7j), allow us to rewrite the integral at the right side of line (7c):

$$\iiint_R \left(\frac{\partial v_1}{\partial x_2}\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\frac{\partial v_2}{\partial x_1}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_1}{\partial x_1}\right)\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) d\boldsymbol{x}. \tag{7k}$$

Because $\nabla \cdot \boldsymbol{v} = 0$, we obtain that:

$$\sum_{m=1}^{3} \iiint_R \left(\frac{\partial v_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial v_2}{\partial x_m}\right) d\boldsymbol{x} = \iiint_R \left(\sum_{m=1}^{3}\frac{\partial v_m}{\partial x_m}\right)\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) d\boldsymbol{x} = 0. \tag{7l}$$

This is, the first coordinate of the second vector of Eq. (7a) is equal to zero.

We may repeat the process on the second and third components of the second vector at the right hand side of Eq. (7a). In the second coordinate, the boundary conditions from Eq. (2) imply that:

$$\iint_{\partial R} \left(v_1\frac{\partial v_3}{\partial x_3}, 0, 0\right) \cdot n\, ds = 0 = \iint_{\partial R} \left(0, 0, v_1\frac{\partial v_3}{\partial x_1}\right) \cdot n\, ds, \tag{7m}$$

$$\iint_{\partial R} \left(v_2\frac{\partial v_3}{\partial x_2}, 0, 0\right) \cdot n\, ds = 0 = \iint_{\partial R} \left(0, v_2\frac{\partial v_3}{\partial x_1}, 0\right) \cdot n\, ds, \tag{7n}$$

$$\iint_{\partial R} \left(0, v_1\frac{\partial v_2}{\partial x_3}, 0\right) \cdot n\, ds = 0 = \iint_{\partial R} \left(0, 0, v_1\frac{\partial v_2}{\partial x_2}\right) \cdot n\, ds. \tag{7o}$$

From Eq. (7m), we have that:

$$\boxed{\iiint_R \left(\frac{\partial v_1}{\partial x_3}\frac{\partial v_3}{\partial x_1}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_1}{\partial x_1}\frac{\partial v_3}{\partial x_3}\right) d\boldsymbol{x};} \tag{7p}$$

from Eq. (7n):

$$\boxed{\iiint_R \left(\frac{\partial v_2}{\partial x_3}\frac{\partial v_3}{\partial x_2}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_2}{\partial x_2}\frac{\partial v_3}{\partial x_3}\right) d\boldsymbol{x};} \tag{7q}$$

and from Eq. (7o)

$$\boxed{\iiint_R \left(\frac{\partial v_2}{\partial x_1}\frac{\partial v_3}{\partial x_2}\right) d\boldsymbol{x} = \iiint_R \left(\frac{\partial v_2}{\partial x_2}\frac{\partial v_3}{\partial x_1}\right) d\boldsymbol{x}.} \tag{7r}$$

So that, the second coordinate of the second vector in M_1 can be rewritten as:

$$\sum_{m=1}^{3} \iiint_R \left(-\frac{\partial v_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} + \frac{\partial v_m}{\partial x_3}\frac{\partial v_1}{\partial x_m}\right) d\boldsymbol{x} = \iiint_R \left(\sum_{m=1}^{3}\frac{\partial v_m}{\partial x_m}\right)\left(\frac{\partial v_3}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) d\boldsymbol{x} = 0. \tag{7s}$$

Similarly, the restrictions over the boundary conditions Eq. (2) lead to the elimination of the third coordinate: If

$$\iint_{\partial R} \left(0, v_1\frac{\partial v_2}{\partial x_1}, 0\right) \cdot n\, ds = 0 = \iint_{\partial R} \left(v_1\frac{\partial v_2}{\partial x_2}, 0, 0\right) \cdot n\, ds,$$

$$\iint_{\partial R} \left(v_3\frac{\partial v_2}{\partial x_3}, 0, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, 0, v_3\frac{\partial v_2}{\partial x_1}\right) \cdot n\, ds = 0,$$

and,

$$\iint_{\partial R} \left(0, v_3\frac{\partial v_2}{\partial x_3}, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, 0, v_3\frac{\partial v_2}{\partial x_2}\right) \cdot n\, ds = 0.$$

then:

$$\sum_{m=1}^{3} \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} + \frac{\partial v_m}{\partial x_3}\frac{\partial v_1}{\partial x_m} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial v_2}{\partial x_m} - \frac{\partial v_m}{\partial x_2}\frac{\partial v_1}{\partial x_m} \end{pmatrix} = \begin{pmatrix} \left(\sum_{m=1}^{3}\frac{\partial v_m}{\partial x_m}\right)\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) \\ \left(\sum_{m=1}^{3}\frac{\partial v_m}{\partial x_m}\right)\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) \\ \left(\sum_{m=1}^{3}\frac{\partial v_m}{\partial x_m}\right)\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7t}$$

Hence,

$$M_1 = \left[(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{\omega}\right]^t. \tag{7u}$$

Development of M_4. Analogously to M_1,

$$M_4 = \sum_{m=1}^{3} \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\tilde{v}_m \frac{\partial \tilde{v}_3}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(\tilde{v}_m \frac{\partial \tilde{v}_2}{\partial x_m}\right) \\ -\frac{\partial}{\partial x_1}\left(\tilde{v}_m \frac{\partial \tilde{v}_3}{\partial x_m}\right) + \frac{\partial}{\partial x_3}\left(\tilde{v}_m \frac{\partial \tilde{v}_1}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(\tilde{v}_m \frac{\partial \tilde{v}_2}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(\tilde{v}_m \frac{\partial \tilde{v}_1}{\partial x_m}\right) \end{pmatrix}, \tag{8a}$$

and M_4 is the same vector as M_1 but with $\tilde{\boldsymbol{v}}$ as its argument, instead of $\boldsymbol{v}$. Thus, we have:

$$M_4 = \left[(\tilde{\boldsymbol{v}} \cdot \nabla)\, \tilde{\boldsymbol{\omega}}\right]^t, \tag{8b}$$

given the boundary conditions in Eq. (2) such that:

$$\iint_{\partial R} \left(\tilde{v}_1 \frac{\partial \tilde{v}_3}{\partial x_2}, 0, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, \tilde{v}_1 \frac{\partial \tilde{v}_3}{\partial x_1}, 0\right) \cdot n\, ds = 0. \tag{8c}$$

$$\iint_{\partial R} \left(\tilde{v}_1 \frac{\partial \tilde{v}_2}{\partial x_3}, 0, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, 0, \tilde{v}_1 \frac{\partial \tilde{v}_2}{\partial x_1}\right) \cdot n\, ds = 0; \tag{8d}$$

$$\iint_{\partial R} \left(0, \tilde{v}_1 \frac{\partial \tilde{v}_2}{\partial x_3}, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, 0, \tilde{v}_1 \frac{\partial \tilde{v}_2}{\partial x_2}\right) \cdot n\, ds = 0; \tag{8e}$$

$$\iint_{\partial R} \left(\tilde{v}_2 \frac{\partial \tilde{v}_3}{\partial x_2}, 0, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, \tilde{v}_2 \frac{\partial \tilde{v}_3}{\partial x_1}, 0\right) \cdot n\, ds = 0, \tag{8f}$$

$$\iint_{\partial R} \left(0, \tilde{v}_1 \frac{\partial \tilde{v}_2}{\partial x_1}, 0\right) \cdot n\, ds = \iint_{\partial R} \left(\tilde{v}_1 \frac{\partial \tilde{v}_2}{\partial x_2}, 0, 0\right) \cdot n\, ds = 0, \tag{8g}$$

$$\iint_{\partial R} \left(\tilde{v}_3 \frac{\partial \tilde{v}_2}{\partial x_3}, 0, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, 0, \tilde{v}_3 \frac{\partial \tilde{v}_2}{\partial x_1}\right) \cdot n\, ds = 0, \tag{8h}$$

and,

$$\iint_{\partial R} \left(0, \tilde{v}_3 \frac{\partial \tilde{v}_2}{\partial x_3}, 0\right) \cdot n\, ds = \iint_{\partial R} \left(0, 0, \tilde{v}_3 \frac{\partial \tilde{v}_2}{\partial x_2}\right) \cdot n\, ds = 0, \tag{8i}$$

Development of M_2 and M_3. At this step, we will apply the Lebiniz product rule, and an exchange in the order of the second partial derivatives:

$$M_2 = \sum_{m=1}^{3} \begin{pmatrix} \frac{\partial}{\partial x_2}\left(v_m \frac{\partial \tilde{v}_3}{\partial x_m}\right) - \frac{\partial}{\partial x_3}\left(v_m \frac{\partial \tilde{v}_2}{\partial x_m}\right) \\ -\frac{\partial}{\partial x_1}\left(v_m \frac{\partial \tilde{v}_3}{\partial x_m}\right) + \frac{\partial}{\partial x_3}\left(v_m \frac{\partial \tilde{v}_1}{\partial x_m}\right) \\ \frac{\partial}{\partial x_1}\left(v_m \frac{\partial \tilde{v}_2}{\partial x_m}\right) - \frac{\partial}{\partial x_2}\left(v_m \frac{\partial \tilde{v}_1}{\partial x_m}\right) \end{pmatrix}; \tag{9a}$$

$$= \sum_{m=1}^{3} \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial \tilde{v}_3}{\partial x_m} + v_m \frac{\partial^2 \tilde{v}_3}{\partial x_m \partial x_2} - \frac{\partial v_m}{\partial x_3}\frac{\partial \tilde{v}_2}{\partial x_m} - v_m \frac{\partial^2 \tilde{v}_2}{\partial x_m \partial x_3} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial \tilde{v}_3}{\partial x_m} - v_m \frac{\partial^2 \tilde{v}_3}{\partial x_m \partial x_1} + \frac{\partial v_m}{\partial x_3}\frac{\partial \tilde{v}_1}{\partial x_m} + v_m \frac{\partial^2 \tilde{v}_1}{\partial x_m \partial x_3} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial \tilde{v}_2}{\partial x_m} + v_m \frac{\partial^2 \tilde{v}_2}{\partial x_m \partial x_1} - \frac{\partial v_m}{\partial x_2}\frac{\partial \tilde{v}_1}{\partial x_m} - v_m \frac{\partial^2 \tilde{v}_1}{\partial x_m \partial x_2} \end{pmatrix}. \tag{9b}$$

Now, M_2 will be split into the sum of two vectors, $M_2 = K_2 + N_2$, where K_2 will have a common factor v_m:

$$K_2 = \sum_{m=1}^{3} v_m \frac{\partial}{\partial x_m} \begin{pmatrix} \frac{\partial \tilde{v}_3}{\partial x_2} - \frac{\partial \tilde{v}_2}{\partial x_3} \\ -\frac{\partial \tilde{v}_3}{\partial x_1} + \frac{\partial \tilde{v}_1}{\partial x_3} \\ \frac{\partial \tilde{v}_2}{\partial x_1} - \frac{\partial \tilde{v}_1}{\partial x_2} \end{pmatrix} = \left[(\boldsymbol{v} \cdot \nabla)\, \tilde{\boldsymbol{w}}\right]^t. \tag{9c}$$

As for N_2, we will hold it while we get its partner from M_3:

$$N_2 = \sum_{m=1}^{3} \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial \tilde{v}_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial \tilde{v}_2}{\partial x_m} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial \tilde{v}_3}{\partial x_m} + \frac{\partial v_m}{\partial x_3}\frac{\partial \tilde{v}_1}{\partial x_m} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial \tilde{v}_2}{\partial x_m} - \frac{\partial v_m}{\partial x_2}\frac{\partial \tilde{v}_1}{\partial x_m} \end{pmatrix}. \tag{9d}$$

It is easy to see how M_3 is practically identical to M_2, with the only difference of $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ having switched places. This is, $M_3 = K_3 + N_3$, where the vectors $K_3 = [(\tilde{\boldsymbol{v}} \cdot \nabla)\, \boldsymbol{\omega}]^t$ and N_3 will hold the same structure but with those two variables exchanged:

$$M_3 = [(\tilde{\boldsymbol{v}} \cdot \nabla)\, \boldsymbol{\omega}]^t + \sum_{m=1}^{3} \begin{pmatrix} \frac{\partial \tilde{v}_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial \tilde{v}_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} \\ -\frac{\partial \tilde{v}_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} + \frac{\partial \tilde{v}_m}{\partial x_3}\frac{\partial v_1}{\partial x_m} \\ \frac{\partial \tilde{v}_m}{\partial x_1}\frac{\partial v_2}{\partial x_m} - \frac{\partial \tilde{v}_m}{\partial x_2}\frac{\partial v_1}{\partial x_m} \end{pmatrix}; \tag{9e}$$

$$= K_3 + N_3. \tag{9f}$$

Adding N_2 from M_2, Eq. (9d), and N_3 from M_3, Eq. (9e), we obtain N:

$$N = N_2 + N_3 = \sum_{m=1}^{3} \left\{ \begin{pmatrix} \frac{\partial v_m}{\partial x_2}\frac{\partial \tilde{v}_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial \tilde{v}_2}{\partial x_m} \\ -\frac{\partial v_m}{\partial x_1}\frac{\partial \tilde{v}_3}{\partial x_m} + \frac{\partial v_m}{\partial x_3}\frac{\partial \tilde{v}_1}{\partial x_m} \\ \frac{\partial v_m}{\partial x_1}\frac{\partial \tilde{v}_2}{\partial x_m} - \frac{\partial v_m}{\partial x_2}\frac{\partial \tilde{v}_1}{\partial x_m} \end{pmatrix} + \begin{pmatrix} \frac{\partial \tilde{v}_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial \tilde{v}_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} \\ -\frac{\partial \tilde{v}_m}{\partial x_1}\frac{\partial v_3}{\partial x_m} + \frac{\partial \tilde{v}_m}{\partial x_3}\frac{\partial v_1}{\partial x_m} \\ \frac{\partial \tilde{v}_m}{\partial x_1}\frac{\partial v_2}{\partial x_m} - \frac{\partial \tilde{v}_m}{\partial x_2}\frac{\partial v_1}{\partial x_m} \end{pmatrix} \right\}. \tag{9g}$$

This way, the first coordinate of the vector N is:

$$\begin{aligned} N_{11} &= \sum_{m=1}^{3} \left\{ \frac{\partial v_m}{\partial x_2}\frac{\partial \tilde{v}_3}{\partial x_m} - \frac{\partial v_m}{\partial x_3}\frac{\partial \tilde{v}_2}{\partial x_m} + \frac{\partial \tilde{v}_m}{\partial x_2}\frac{\partial v_3}{\partial x_m} - \frac{\partial \tilde{v}_m}{\partial x_3}\frac{\partial v_2}{\partial x_m} \right\} \\ &= \left(\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)\left(\frac{\partial \tilde{v}_3}{\partial x_2} - \frac{\partial \tilde{v}_2}{\partial x_3}\right) + \frac{\partial v_1}{\partial x_2}\frac{\partial \tilde{v}_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\frac{\partial \tilde{v}_2}{\partial x_3} + \\ &\quad + \left(\frac{\partial \tilde{v}_2}{\partial x_2} + \frac{\partial \tilde{v}_3}{\partial x_3}\right)\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) + \frac{\partial \tilde{v}_1}{\partial x_2}\frac{\partial v_3}{\partial x_1} - \frac{\partial \tilde{v}_1}{\partial x_3}\frac{\partial v_2}{\partial x_1}. \end{aligned}$$

It can be seen that the integration by parts may be used to complete each divergence $\nabla \cdot \boldsymbol{v}$ and $\nabla \cdot \tilde{\boldsymbol{v}}$ on the last terms of both lines on the right side of the last expression, so that:

$$N_{11} = \left(\sum_{m=1}^{3} \frac{\partial v_m}{\partial x_m}\right)\left(\frac{\partial \tilde{v}_3}{\partial x_2} - \frac{\partial \tilde{v}_2}{\partial x_3}\right) + \left(\sum_{m=1}^{3} \frac{\partial \tilde{v}_m}{\partial x_m}\right)\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) = 0. \tag{9h}$$

When we perform the same process for the other two coordinates, N_{22} and N_{33}, we obtain:

$$N = \left(\sum_{m=1}^{3} \frac{\partial v_m}{\partial x_m}\right) \begin{pmatrix} \frac{\partial \tilde{v}_3}{\partial x_2} - \frac{\partial \tilde{v}_2}{\partial x_3} \\ -\frac{\partial \tilde{v}_3}{\partial x_1} + \frac{\partial \tilde{v}_1}{\partial x_3} \\ \frac{\partial \tilde{v}_2}{\partial x_1} - \frac{\partial \tilde{v}_1}{\partial x_2} \end{pmatrix} + \left(\sum_{m=1}^{3} \frac{\partial \tilde{v}_m}{\partial x_m}\right) \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ -\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix} = 0, \tag{9i}$$

because both $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are incompressible. This is, $\nabla \cdot \boldsymbol{v} = \left(\sum_{m=1}^{3} \frac{\partial v_m}{\partial x_m}\right) = 0$ and $\nabla \cdot \tilde{\boldsymbol{v}} = \left(\sum_{m=1}^{3} \frac{\partial \tilde{v}_m}{\partial x_m}\right) = 0$.

Having defined all the vectors that compose M_2 and M_3, the sum:

$$\begin{aligned} M_2 + M_3 &= K_2 + K_3 + N_2 + N_3; \\ &= K_2 + K_3; \\ &= [(\boldsymbol{v} \cdot \nabla)\, \tilde{\boldsymbol{\omega}}]^t + [(\tilde{\boldsymbol{v}} \cdot \nabla)\, \boldsymbol{\omega}]^t . \end{aligned}$$

Hence, the left side of Eq. (5) is:

$$\iiint_R \left(\frac{D(\boldsymbol{v}+\tilde{\boldsymbol{v}})}{Dt}\right) d\boldsymbol{x} = \iiint_R (A+M)\, d\boldsymbol{x}$$
$$= \iiint_R \left(\frac{\partial\boldsymbol{\omega}}{\partial t} + \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial t} + M\right) d\boldsymbol{x}$$
$$= \iiint_R \left(\frac{\partial\boldsymbol{\omega}}{\partial t} + \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial t} + M_1 + M_2 + M_3 + M_4\right) d\boldsymbol{x}$$
$$= \iiint_R \left(\frac{\partial\boldsymbol{\omega}}{\partial t} + \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial t} + [(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\omega}]^t + [(\tilde{\boldsymbol{v}}\cdot\nabla)\,\tilde{\boldsymbol{\omega}}]^t + [(\boldsymbol{v}\cdot\nabla)\,\tilde{\boldsymbol{\omega}}]^t + [(\tilde{\boldsymbol{v}}\cdot\nabla)\,\boldsymbol{\omega}]^t\right) d\boldsymbol{x}.$$

Right side expansion. The curl of a gradient is equal to 0, so that:

$$\nabla \times \{\nu\Delta\,(\boldsymbol{v}+\tilde{\boldsymbol{v}}) - \nabla p\} = \nu\Delta\,(\boldsymbol{v}+\tilde{\boldsymbol{v}}) = \nu\Delta\boldsymbol{\omega} + \nu\Delta\tilde{\boldsymbol{\omega}}.$$

Thus,

$$\iiint_R (\nabla \times (\nu\Delta(v+\tilde{v}) - \nabla p))\, d\boldsymbol{x} = \nu \iiint_R (\Delta\boldsymbol{\omega} + \Delta\tilde{\boldsymbol{\omega}})\, d\boldsymbol{x}.$$

Joining both sides:

$$\iiint_R \left(\frac{\partial\boldsymbol{\omega}}{\partial t} + \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial t} + [(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\omega}]^t + [(\tilde{\boldsymbol{v}}\cdot\nabla)\,\tilde{\boldsymbol{\omega}}]^t + [(\boldsymbol{v}\cdot\nabla)\,\tilde{\boldsymbol{\omega}}]^t + [(\tilde{\boldsymbol{v}}\cdot\nabla)\,\boldsymbol{\omega}]^t\right) d\boldsymbol{x} = \nu \iiint_R \Delta\,(\boldsymbol{\omega}+\tilde{\boldsymbol{\omega}})\, d\boldsymbol{x}. \tag{10}$$

We can remove the triple integral through the Intermediate Value Theorem [43, p. 124], knowing there is at least one point $\boldsymbol{x}_0 = \left(x_1^0, x_2^0, x_3^0\right) \in R \subset \mathbb{R}^3$ such that, putting together the expressions for the left and right hand sides, we obtain Eq. (4):

$$\left[\frac{\boldsymbol{\partial\omega}}{\partial t} + \frac{\boldsymbol{\partial\tilde{\omega}}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\,\tilde{\boldsymbol{\omega}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\,\boldsymbol{\omega} + (\tilde{\boldsymbol{v}}\cdot\nabla)\,\tilde{\boldsymbol{\omega}}\right](\boldsymbol{x}_0) = \nu\Delta\,(\boldsymbol{\omega}+\tilde{\boldsymbol{\omega}})\,(\boldsymbol{x}_0)\,.$$

□

3 Energy Bounds

Once the curl operator is applied to each equation in a Navier-Stokes incompressible system to yield the sum of the $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ fields. Subsequently, energy bounds for the fluctuation's curl are computed. In contrast to alternative approaches, this method does not presuppose any additional properties on the fluctuation beyond the premise of zero divergence. Consequently, it is anticipated that any additional property on the fluctuation will serve as a post-condition for determining boundary cases that are governed by more general laws and bounds; that is, from a unified perspective.

If $\boldsymbol{f} = (f_1, f_2, f_3),\ \boldsymbol{g} = (g_1, g_2, g_3) \in L^2(R; \mathbb{R}^3)$, we denote its inner product as:

$$(\boldsymbol{f}, \boldsymbol{g})_0 = \iiint_R (\boldsymbol{f}\,(\boldsymbol{x}) \cdot \boldsymbol{g}\,(\boldsymbol{x}))\, d\boldsymbol{x} = \iiint_R (f_1 g_1\,(\boldsymbol{x}) + f_2 g_2\,(\boldsymbol{x}) + f_3 g_3\,(\boldsymbol{x}))\, d\boldsymbol{x}.$$

Similarly, the inner product of an element $\boldsymbol{f} \in L^2(R; \mathbb{R}^2)$ with itself defines a norm [4]:

$$(\boldsymbol{f}, \boldsymbol{f})_0 = \|\boldsymbol{f}\|_0^2 = \iiint_R \left(f_1^2\,(\boldsymbol{x}) + f_2^2\,(\boldsymbol{x}) + f_3^2\,(\boldsymbol{x})\right) d\boldsymbol{x}.$$

Now, let $D_\alpha^k\,(\boldsymbol{f})$ express the partial derivative of $\boldsymbol{f}$ with absolute order k for the different spatial variables x_1, x_2, x_3:

$$D_\alpha^k\,(\boldsymbol{f}) = \frac{\partial^k \boldsymbol{f}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} = \left(\frac{\partial^k f_j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}\right)_{j=1}^{3},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, and $\alpha_1 + \alpha_2 + \alpha_3 = k$. Consider two smooth fields $\boldsymbol{f}, \boldsymbol{g} \in C^1\left([0,T), C^\infty(\bar{R})\right)$ for a finite time $0 < T < \infty$. Assume that they satisfy null boundary conditions, $\boldsymbol{f}|_{\partial R} = \boldsymbol{g}|_{\partial R} = (0,0,0)$, and that one of them is incompressible, $\nabla \cdot \boldsymbol{f} = 0$. Then, direct calculation, integration by parts, and the application of the Schwarz theorem to interchange the order of the partial derivatives lead to each of the next identities:

$$\left(\frac{\partial \boldsymbol{f}}{\partial t}, \boldsymbol{f}\right)_0 = \frac{1}{2}\frac{\partial}{\partial t}\left(\boldsymbol{f}, \boldsymbol{f}\right)_0 = \frac{1}{2}\frac{\partial}{\partial t}\|\boldsymbol{f}\|_0^2; \tag{11}$$

$$\left(\left(\boldsymbol{f} \cdot \nabla\right)\boldsymbol{f}, \boldsymbol{f}\right)_0 = 0; \tag{12}$$

$$\left(\left(\boldsymbol{f} \cdot \nabla\right)\boldsymbol{g}, \boldsymbol{f}\right)_0 = -\left(\left(\boldsymbol{f} \cdot \nabla\right)\boldsymbol{f}, \boldsymbol{g}\right)_0; \tag{13}$$

$$\left(\left(\boldsymbol{f} \cdot \nabla\right)\boldsymbol{f}, \boldsymbol{g}\right)_0 = -\left(\left(\boldsymbol{f} \cdot \nabla\right)\boldsymbol{g}, \boldsymbol{f}\right)_0; \tag{14}$$

$$\left(D_\alpha^k\left(\Delta \boldsymbol{f}\right), D_\alpha^k\left(\boldsymbol{g}\right)\right)_0 = -\left(\nabla D_\alpha^k\left(\boldsymbol{f}\right), \nabla D_\alpha^k\left(\boldsymbol{g}\right)\right)_0 \quad \forall k = \alpha_1 + \alpha_2 + \alpha_3, \text{ and } \alpha \in \mathbb{N}^3. \tag{15}$$

For example:

$$\begin{aligned}\left(D_\alpha^k\left(\Delta \boldsymbol{f}\right), D_\alpha^k\left(\boldsymbol{g}\right)\right)_0 &= \iiint_R \left(f_1 g_1\left(\boldsymbol{x}\right) + f_2 g_2\left(\boldsymbol{x}\right) + f_3 g_3\left(\boldsymbol{x}\right)\right) d\boldsymbol{x} \\ &= \iiint_R \left(f_1 g_1\left(\boldsymbol{x}\right) + f_2 g_2\left(\boldsymbol{x}\right) + f_3 g_3\left(\boldsymbol{x}\right)\right) d\boldsymbol{x} \\ &= \iiint_R \left(f_1 g_1\left(\boldsymbol{x}\right) + f_2 g_2\left(\boldsymbol{x}\right) + f_3 g_3\left(\boldsymbol{x}\right)\right) d\boldsymbol{x}\end{aligned}$$

Corollary 1. *Assume the same conditions as the Theorem 1. Then:*

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left\|D_\alpha^k\left(\tilde{\boldsymbol{w}}\right)\right\|_0^2 + \left(D_\alpha^k\left(\frac{\partial \boldsymbol{\omega}}{\partial t}\right), D_\alpha^k \tilde{\boldsymbol{\omega}}\right)_0 + \left(\left(D_\alpha^k\left(\left(\boldsymbol{v} \cdot \nabla\right)\boldsymbol{\omega}\right), D_\alpha^k \tilde{\boldsymbol{\omega}}\right)_0 + \right.\\ + \left(\left(D_\alpha^k\left((\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k \tilde{\boldsymbol{\omega}}\right)_0 + \left(D_\alpha^k\left((\boldsymbol{v} \cdot \nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k \tilde{\boldsymbol{\omega}}\right)_0 + \right.\\ + \left(\left(D_\alpha^k\left((\tilde{\boldsymbol{v}} \cdot \nabla)\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \nu\left(D_\alpha^k \nabla \boldsymbol{\omega}, D_\alpha^k \nabla \tilde{\boldsymbol{\omega}}\right)_0 + \nu\|D_\alpha^k\left(\nabla \tilde{\boldsymbol{\omega}}\right)\|_0^2 = 0,\right.\end{aligned} \tag{16}$$

$\forall k \in \mathbb{N}$ *and* $\forall \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ *such that* $k = \alpha_1 + \alpha_2 + \alpha_3$. *If* $k = 0$*, then the following relation holds:*

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\tilde{\boldsymbol{\omega}}\|_0^2 + \left(\frac{\partial \boldsymbol{\omega}}{\partial t}, \tilde{\boldsymbol{\omega}}\right)_0 + \left((\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega}, \tilde{\boldsymbol{\omega}}\right)_0 + \left((\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\omega}}\right)_0 + \left((\boldsymbol{v} \cdot \nabla)\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\omega}}\right)_0 + \\ + \left((\tilde{\boldsymbol{v}} \cdot \nabla)\boldsymbol{\omega}, \tilde{\boldsymbol{\omega}}\right)_0 + \nu\left(\nabla \boldsymbol{\omega}, \nabla \tilde{\boldsymbol{\omega}}\right)_0 + \nu\left\|\nabla \tilde{\boldsymbol{\omega}}\right\|_0^2 = 0.\end{aligned} \tag{17}$$

In particular, for planar axisymmetric fields that depend solely on radius, we have the following: $\boldsymbol{v} = (v_1, v_2, 0)$, $\tilde{\boldsymbol{v}} = (\tilde{v}_1, \tilde{v}_2, 0)$*:*

$$\frac{1}{2}\frac{\partial}{\partial t}\|\tilde{\boldsymbol{\omega}}\|_0^2 + \left(\frac{\partial \boldsymbol{\omega}}{\partial t}, \tilde{\boldsymbol{\omega}}\right)_0 = 0. \tag{18}$$

Proof. First, consider Eq. (3), and apply the differential operator D_α^k to each side. This is:

$$D_\alpha^k\left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} + (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{\omega}} + (\boldsymbol{v} \cdot \nabla)\tilde{\boldsymbol{\omega}} + (\tilde{\boldsymbol{v}} \cdot \nabla)\boldsymbol{\omega} - \nu\Delta(\boldsymbol{\omega} + \tilde{\boldsymbol{\omega}})\right) = D_\alpha^k\left(\mathbf{0}\right) = \mathbf{0},$$

for all $k \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ such that $k = \alpha_1 + \alpha_2 + \alpha_3$, where $\mathbf{0} = (0,0,0)$. Second, the inner product in the Lebesgue space $L_2(R;\mathbb{R}^3)$ with the field $D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)$ of each side of the last equation is:

$$\begin{aligned}\left(D_\alpha^k\left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} + (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{\omega}} + (\boldsymbol{v} \cdot \nabla)\tilde{\boldsymbol{\omega}} + (\tilde{\boldsymbol{v}} \cdot \nabla)\boldsymbol{\omega} - \nu\Delta(\boldsymbol{\omega} + \tilde{\boldsymbol{\omega}})\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 &= \left(\mathbf{0}, D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 \\ &= 0.\end{aligned}$$

According to the linearity of the differential operator, and the properties recalled in Eq. (11), the left side of the last equation is developed as:

$$
\begin{aligned}
&\left(D_\alpha^k\left(\frac{\partial \boldsymbol{\omega}}{\partial t}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \left(D_\alpha^k\left(\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \left(D_\alpha^k\left((\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \\
&\quad + \left(D_\alpha^k\left((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \left(D_\alpha^k\left((\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \\
&\quad + \left(D_\alpha^k\left((\tilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 - \left(D_\alpha^k\left(\nu\Delta(\boldsymbol{\omega}+\tilde{\boldsymbol{\omega}})\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 = \\
&= \left(D_\alpha^k\left(\frac{\partial \boldsymbol{\omega}}{\partial t}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \frac{1}{2}\frac{\partial}{\partial t}\left\|D_\alpha^k\left(\tilde{\boldsymbol{w}}\right)\right\|_0^2 + \left(D_\alpha^k\left((\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \\
&\quad + \left(D_\alpha^k\left((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \left(D_\alpha^k\left((\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \\
&\quad + \left(D_\alpha^k\left((\tilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 - \nu\left(D_\alpha^k\left(\Delta\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 - \nu\left(D_\alpha^k\left(\Delta\tilde{\boldsymbol{\omega}}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 = \\
&= \left(D_\alpha^k\left(\frac{\partial \boldsymbol{\omega}}{\partial t}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \frac{1}{2}\frac{\partial}{\partial t}\left\|D_\alpha^k\left(\tilde{\boldsymbol{w}}\right)\right\|_0^2 + \left(D_\alpha^k\left((\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \\
&\quad + \left(D_\alpha^k\left((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \left(D_\alpha^k\left((\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{\omega}}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \\
&\quad + \left(D_\alpha^k\left((\tilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{\omega}\right), D_\alpha^k\left(\tilde{\boldsymbol{\omega}}\right)\right)_0 + \nu\left(D_\alpha^k\nabla\boldsymbol{\omega}, D_\alpha^k\nabla\tilde{\boldsymbol{\omega}}\right)_0 + \nu\|D_\alpha^k\left(\nabla\tilde{\boldsymbol{\omega}}\right)\|_0^2 = 0.
\end{aligned}
$$

□

The next section provides an insight into how this analytical structure can be simplified further using tools from differential algebra.

4 Experiments on multiplicity and algebraic reductibility

4.1 Motivation and Theoretical Background

Over the past few years, algebraic methods such as Gröbner Basis [11, 10, 17, 29], Differential Algebra [6, 30, 21], and Tropical Differential Algebra [3, 18] have provided an alternative ways to study intricate differential systems. When a system of differential equations can be presented as a set of *differential polynomials*, certain tools from the differential algebra repertoire can be used. One of the most powerful theoretical results in this field is *Ritt's theorem*, which states that there exists a finite basis of the associated *differential ideal* [38]. This implies that there may exist equivalent restatements of a given system, potentially reducing its dimensionality or decoupling variables, allowing for the isolation of a single unknown. For example, this was achieved in the Modified Chua's circuit and the Rössler system [19, p. 719]. This particular framework opens the door to new formulations of classic fluid equations, a perspective that remains largely unexplored in the field. Simpler systems could not only provide new insights and uncover novel algebraic structures, but also significantly reduce computational costs by simplifying the problem.

Among recent developments, *differential elimination algorithms* aim to take advantage of Ritt´s theorem by reducing complex systems of differential equations into simpler, lower-dimensional, and *triangular* equivalent forms. Joseph Ritt was the first to develop a differential elimination algorithm, but with some limitations. Recent algorithms attempt to mimic the structure of Gröbner Basis techniques used in polynomial algebra but extended to the differential case. One of the most advanced and complete algorithms in this domain is the Rosenfeld-Gröbner algorithm, a method pioneered by Boulier and Lemaire [6, 7, 8]. This algorithm constructs a differential characteristic set that represents the differential ideal generated by the input system, under a given ranking of the variables and their derivatives. It is implemented in symbolic computation packages such as Maple and now also available in Python.

4.2 Application of the Rosenfeld-Gröbner Algorithm

Empirical research suggests that both the Navier-Stokes and Reynolds-Averaged Navier-Stokes (RANS) equations are algebraically irreducible in a sufficiently short time (after a certain number of iterations, the algorithm stops). As it is shown in the Table 1, after attempting to utilize Boulier's algorithm on either Navier-Stokes or RANS equations, the algorithm fails to converge unless the dimension is $n = 2$ and the problem is restated in terms of the *stream-function*.

This is likely due to the presence of the convective derivative, which from an algebraic perspective, introduces cross-coupled nonlinearities that propagate high-order derivatives of each variable across the system. This coupling obstructs the construction of a triangular set or characteristic decomposition, thus preventing reduction.

Model	Compressible	Incompressible	Stokes	Euler	Stationary
Busemann [12]	Irreducible	N/A	N/A	N/A	N/A
Prandtl [37]	N/A	Reducible	N/A	N/A	N/A
Navier-Stokes	Irreducible	Irreducible	Reducible	Irreducible	Irreducible
RANS [39]	Irreducible	Irreducible	Reducible	Irreducible	Irreducible
ω-RANS (1)	Irreducible	Irreducible	Reducible	Irreducible	Irreducible
ψ-NS	N/A	Reducible	N/A	N/A	Reducible

Table 1: Algebraic reductibility of various fluid models

4.3 Examples of Differential Polynomial Systems

We consider a system Σ_n of polynomial differential equations[3], where Σ_1 represents the set of differential equations of three-dimensional Navier Stokes for incompressible fluids under Stokes' condition $(\mathbf{v} \cdot \nabla)\mathbf{v} = 0$, where u, v, w are the velocity components, p is the pressure and ν is the kinematic viscosity:

$$\Sigma_1 = \begin{cases} u_x + v_y + w_z = 0, \\ u_t + p_x - \nu(u_{xx} + u_{yy} + u_{zz}) = 0, \\ v_t + p_y - \nu(v_{xx} + v_{yy} + v_{zz}) = 0, \\ w_t + p_z - \nu(z_{xx} + z_{yy} + z_{zz}) = 0. \end{cases} \tag{19}$$

For Σ_1 with a *ranking* $u > v > w > p$, we obtain a reduction:

$$u_x = -v_y - w_z, \tag{20}$$
$$p_{x,x} = -p_{y,y} - p_{z,z}, \tag{21}$$
$$w_{x,x} = -w_{y,y} - w_{z,z} + \nu^{-1}p_z + \nu^{-1}w_t, \tag{22}$$
$$v_{x,x} = -v_{y,y} - v_{z,z} + \nu^{-1}p_y + \nu^{-1}v_t, \tag{23}$$
$$u_{y,y} = -u_{z,z} + v_{x,y} + w_{x,z} + \nu^{-1}p_x + \nu^{-1}u_t. \tag{24}$$

Σ_2 represents the set of differential of three dimensional RANS equations for incompressible fluids under Stokes' condition $(\mathbf{v} \cdot \nabla)\mathbf{v} = 0$, where u, v, w are the averaged velocity components, u', v', w' are the fluctuations of the velocities, p is the pressure and ν is the kinematic viscosity:

$$\Sigma_2 = \begin{cases} u_x + v_y + w_z = 0, \\ p_x + u_t + u'_t - \nu(u_{xx} + u_{yy} + u_{zz} + u'_{xx} + u'_{yy} + u'_{zz}) = 0, \\ p_y + v_t + v'_t - \nu(v_{xx} + v_{yy} + v_{zz} + v'_{xx} + v'_{yy} + v'_{zz}) = 0, \\ p_z + w_t + w'_t - \nu(w_{xx} + w_{yy} + w_{zz} + w'_{xx} + w'_{yy} + w'_{zz}) = 0. \end{cases} \tag{25}$$

Furthermore, for Σ_2 with a *ranking* $u > v > w > u' > v' > w' > p$, we obtain a reduction:

$$u_x = -v_y - w_z; \tag{26}$$
$$\begin{aligned} u'_{x,x,x} = &-u'_{x,y,y} - u'_{x,z,z} - v'_{x,x,y} - v'_{y,y,y} - v'_{y,z,z} - w'_{x,x,z} - w'_{y,y,z} - w'_{z,z,z} \\ &+ \nu^{-1}(p_{x,x} + p_{y,y} + p_{z,z} + u'_{t,x} + v'_{t,y} + w'_{t,z}); \end{aligned} \tag{27}$$
$$w_{x,x} = -w'_{x,x} - w'_{y,y} - w'_{z,z} - w_{y,y} - w_{z,z} + \nu^{-1}(p_z + {w_1}_t + w_t); \tag{28}$$
$$v_{x,x} = -v'_{x,x} - v'_{y,y} - v'_{z,z} - v_{y,y} - v_{z,z} + \nu^{-1}(p_y + {v'}_t + v_t); \tag{29}$$
$$u_{y,y} = -u'_{x,x} - u'_{y,y} - u'_{z,z} - u_{z,z} + v_{x,y} + w_{x,z} + \nu^{-1}(p_x + {u'}_t + u_t). \tag{30}$$

[3]For section 4.3, subscript notation is used to represent partial derivatives. For example, $u_{x,x} = \frac{\partial^2 u}{\partial x^2}$.

5 Conclusions

Even if convective terms do not appear to reduce sufficiently quickly, the reducibility of Prandtl's incompressible boundary layer, which includes a nonlinear convective term, suggests that a case-by-case analysis should be performed. From a theoretical point of view, obtaining a triangular system allows us to focus on a single variable, resulting in a more efficient iterative process. It also raises the question of classifying RANS-type turbulent models algebraically. The algebraic structure in the differential ideal generated by a convective derivative becomes even more compelling when combined with classic energy methods. In particular, we are exploring the possibility of defining a functional over the differential ideal, suggesting the existence of a dual formulation where algebraic reducibility is reflected through energetic bounds. This opens the door to considering new functionals and variational structures derived from the ideal's generators.

Applying Differential Algebra methods, specifically the Rosenfeld-Gröbner algorithm developed by Boulier and Lemaire, opens up a new avenue for reduction and comparison in the turbulent regime. Although the resulting reduced systems are still mathematically and algebraically complex, the reduction offers a novel perspective that could potentially facilitate the development of new models and the exploration of new structures. For instance, reducibility appears to depend not on whether the model is turbulent or compressible, but on the absence of a convective term. Therefore, it is necessary to determine whether there is a dual ideal in terms of Energy Methods. Despite the complexity, the refined algebraic approach, when coupled with classical energy and numerical methods, has the potential to make groundbreaking advancements in the field.

Furthermore, specific classical solutions can be considered to deduce properties over the fluctuations. The symmetry found in the vorticity equations shows that the fluctuation itself depends on the original curl. From an existence and uniqueness perspective, we will review the maximum principles available for quasi-linear parabolic equations (for example, those given by Oleinik in 1963) to see if the result can be extended to n = 3. We would like to examine the impact of different classical vorticity ω frequencies over $\tilde{\omega}$. To this end, we will experiment with the algorithm to count rational solutions to differential polynomial systems, particularly periodic solutions.

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