Computer lab I: Implementation of arc-length solvers with a total Lagrangian bar element

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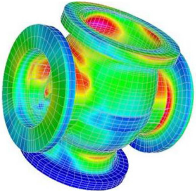
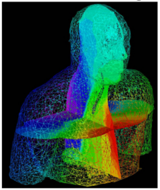




Outline

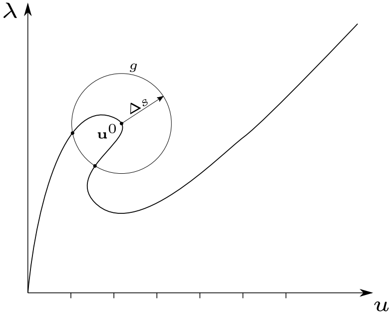
1. Introduction
2. Implementation of arc-length solvers
3. Implementation of a geometrically non linear bar element using the total Lagrangian formulation

Learning goals

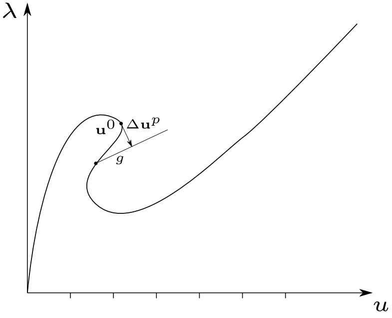
- Implementation of arc length solvers
- Implementation of a total Lagrangian bar element

Illustration of arc-length methods:

Crisfield:



Riks:



Path-following methods

The system to be solved at each step of a path-following method is:

$$\begin{bmatrix} \mathbf{K_T} & -\mathbf{f_{ext}} \\ \mathbf{h}^T & s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{R}^i \\ -g^i \end{bmatrix}$$

where:

- $\Delta\mathbf{u}$ is the nodal displacement increment
- $\Delta\lambda$ is the load factor increment
- g_i The value of the constraint function used at the previous increment
- $\mathbf{h}$ Is the gradient of g with respect to $\mathbf{u}$: $\mathbf{h} = \dfrac{\partial g}{\partial \mathbf{u}}$
- s is the derivative of g with respect to λ: $s = \dfrac{\partial g}{\partial \lambda}$

Block solution of the system:

$$\Delta \mathbf{u}^{I} = \mathbf{K_T}^{-1}\mathbf{f_{ext}}, \ \ \Delta \mathbf{u}^{II} = -\mathbf{K_T}^{-1}\mathbf{R}^{i}$$

$$\Delta\lambda = -\frac{g^{i} + \mathbf{h}^{T}\Delta \mathbf{u}^{II}}{s + \mathbf{h}^{T}\Delta \mathbf{u}^{I}}$$

$$\Delta \mathbf{u} = \Delta\lambda\, \Delta \mathbf{u}^{I} + \Delta \mathbf{u}^{II}$$

Predictor step incremental displacements:

$$\Delta \mathbf{u}^{p} = \mathbf{K_T}^{-1}\mathbf{f_{ext}}$$

Load factor increment:

$$\Delta \lambda^{p} = \pm \frac{\Delta s}{\|\Delta \mathbf{u}^{p}\|}$$

Sign is determined by:

$$\kappa = \frac{\mathbf{f_{ext}}^{T}\Delta \mathbf{u}}{\Delta \mathbf{u}^{T}\Delta \mathbf{u}}$$

Solution overview:

1	Initial values	$\mathbf{u}^0, \lambda^0$
2	Obtain internal force, tangent stiffness and external force	$\mathbf{R}, \mathbf{K_T}, \mathbf{f_{ext}}$
3	Predictor step incremental displacements	$\Delta\mathbf{u}^p = \mathbf{K_T}^{-1}\mathbf{f_{ext}}$
4	Predictor step incremental load factor	$\kappa = \frac{\mathbf{f_{ext}}^T \Delta\mathbf{u}}{\Delta\mathbf{u}^T \Delta\mathbf{u}}, \quad \Delta\lambda^p = \text{sign}(\kappa) \frac{\Delta s}{\lVert\Delta\mathbf{u}^p\rVert}$
5	Load factor and displacement update	$\lambda^1 = \lambda^0 + \Delta\lambda^p, \ \mathbf{u}^1 = \mathbf{u}^0 + \Delta\lambda^p \Delta\mathbf{u}^p$
6	Update residual and tangent stiffness	$\mathbf{R}, \mathbf{K_T}$
7	Iterations	For $i = 1, 2, 3, \ldots$
8	Evaluate constraint	g^i, $\mathbf{h}$, s
9	Solution of linear systems	$\Delta\mathbf{u}^I = \mathbf{K_T}^{-1}\mathbf{f_{ext}}, \quad \Delta\mathbf{u}^{II} = -\mathbf{K_T}^{-1}\mathbf{R}^i$
10	Displacement and load factor increments	$\Delta\lambda = -\frac{g^i + \mathbf{h}^T \Delta\mathbf{u}^{II}}{s + \mathbf{h}^T \Delta\mathbf{u}^I}, \quad \Delta\mathbf{u} = \Delta\lambda\Delta\mathbf{u}^I + \Delta\mathbf{u}^{II}$
11	Solution update	$\lambda^{i+1} = \lambda^i + \Delta\lambda, \quad \mathbf{u}^{i+1} = \mathbf{u}^i + \Delta\mathbf{u}$
12	Update residual and tangent stiffness	$\mathbf{R}, \mathbf{K_T}$
13	Convergence check	If $\lVert\mathbf{R}\rVert \leq tol$ converged, else go to 7

Path-following methods

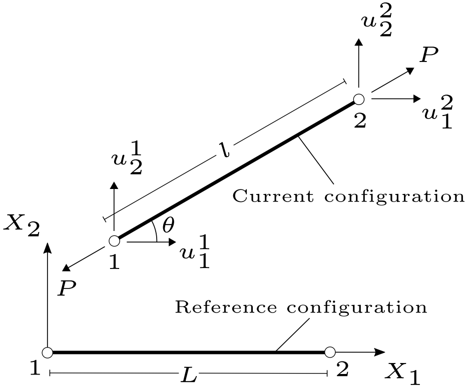
Constraint definitions:

Name	g	$\mathbf{h}$	s
Load control	$\lambda - \bar{\lambda}$	$\mathbf{0}$	1
Displacement control	$\mathbf{T} \cdot \mathbf{u} - \bar{u}$	$\mathbf{T}$	0
Arc length method	$\sqrt{\left(\mathbf{u}-\mathbf{u}^0\right)^T \cdot \left(\mathbf{u}-\mathbf{u}^0\right) + \left(\lambda-\lambda^0\right)^2} - \Delta s$	$\dfrac{\left(\mathbf{u}^i - \mathbf{u}^0\right)}{g}$	$\dfrac{\left(\lambda^i - \lambda^0\right)}{g}$
Riks method	$\left(\Delta \mathbf{u}^p\right)^T \left(\mathbf{u}-\mathbf{u}^1\right) + \Delta\lambda^p \left(\lambda - \lambda^1\right)$	$\Delta \mathbf{u}^p$	$\Delta\lambda^p$

where $\mathbf{u}^0$ and λ^0 are the displacements and load factor at the beginning of the step and:

$$\mathbf{u}^1 = \mathbf{u}^0 + \Delta\lambda^p \Delta\mathbf{u}^p, \;\; \lambda^1 = \lambda^0 + \Delta\lambda^p$$

Geometry of the truss element.



Where:

$$u_1^2 = u_1^1 + l\cos\theta - L$$

$$u_2^2 = u_2^1 + l\sin\theta$$

Linear part of the stiffness matrix:

$$\mathbf{K_L} = EA\frac{l^2}{L^3}\begin{bmatrix} cos^2\theta & cos\theta sin\theta & -cos^2\theta & -cos\theta sin\theta \\ & sin^2\theta & -sin\theta cos\theta & -sin^2\theta \\ & & cos^2\theta & sin\theta cos\theta \\ \text{Symm} & & & sin^2\theta \end{bmatrix}$$

where A is the cross section of the bar in the reference configuration.

The nonlinear part of the tangent stiffness matrix is:

$$\mathbf{K_{NL}} = EA\frac{l^2 - L^2}{2L^3}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The force vector is:

$$\mathbf{f_{int}} = EA\,\frac{l^2 - L^2}{2L^2}\begin{bmatrix} -cos\theta \\ -sin\theta \\ cos\theta \\ sin\theta \end{bmatrix}$$

Simulate the following arc truss with radii $R_1 = 49$, $R_2 = 51$, and angle $\phi = \pi/6$