Semiparametric estimation for time series: a frequency domain approach based on optimal transportation theory

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May 2021

Abstract

In this master thesis, we develop a novel methodology for estimating parameters in time series models based on optimal transportation results. The key idea is to use the Wasserstein distance and Sinkhorn divergence to derive minimum distance (or divergence) estimators for short- and long-memory time series models. More precisely, thanks to the frequency approach, we can compute the distance/divergence between the empirical distribution of the standardized periodogram ordinates and their theoretical distribution. To determine the properties of these new estimators, we perform several Monte-Carlo simulations. Our numerical results suggest that we have a novel class of root-n consistent minimum distance estimators. The performance, in terms of Mean Squared-Error, is similar to the one yielded by the state-of-the-art estimation method (Whittle's estimator) in the case of short- and long-memory Gaussian process. Furthermore, when the underlying innovation density of a long-memory process is skewed, our estimators overperform the Whittle's estimator.

Contents

1 Introduction

1.1 Motivation

The aim of this master thesis is to combine our knowledge of the optimal transport theory and the time series analysis in the frequency domain to develop a novel methodology for estimating the parameters of a process. We shift away from information divergence-based methods, among which the standard maximum likelihood estimator approach, and consider instead the mathematical theory of optimal measure transportation. The optimal transport theory has been applied in many research areas, especially machine learning (see e.g., Panaretos and Zemel (2019)). The Wasserstein distance has become popular specially for inference in generative adversarial networks (see e.g., Arjovsky, Chintala, and Bottou (2017)). To the best of our knowledge, only a limited number of papers has been investigating the use of the optimal transport theory in the statistical analysis of time series analysis (see Ni et al. (2020)). Our purpose is to fill this gap in the literature and study the applicability of the Wasserstein distance (or, more generally, of some results from optimal transportation theory) for the statistical analysis of time series, via frequency domain techniques. The key argument for moving from the time domain to the frequency domain is that we are dealing with data that are independent and identically distributed (i.i.d). The assumption of i.i.d. data facilitates, as it is often the case in statistics, the estimation of parameters in a model.

We propose a novel class of minimum distance estimator (see Basu, Shioya, and Park (2019)) by minimizing the distance between the theoretical and empirical distribution of the standardized periodogram ordinates (SPOs). This program is supported by the fact that the method to replace the maximum likelihood estimator with minimum Wasserstein distance has already been applied, for instance in astronomy and climate science (see Bernton et al. (2019)). Additionally, consistency properties of estimators based on the Wasserstein distance has already been studied by Bassetti, Bodini, and Regazzini (2006) and Bernton et al. (2019). In our case, we study the properties (bias, variance, consistency, etc.) of our new estimators by means of Monte-Carlo experiments and compare them to the state-of-the-art estimator, the Whittle's estimator (see Whittle (1953)). We analyze our results for several types of distribution (standard, heavy-tailed and skewed) and focus mainly on large sample sizes to satisfy the i.i.d conditions. Our results are promising and open the possibility of further research.

1.2 Organization

In the first chapter, we review the main concepts of the optimal transport theory and provide all the distance definitions necessary to understand the thesis. In the second chapter, we refresh the Whittle's estimator and all the theoretical results corresponding. Then, we present the different estimators that we have developed during our research and compare them to the state-of-art estimator. We end with a conclusion that contains all the possible research's areas opened up by this thesis. All the code to reproduce the plots/tables included in this thesis is available on Github: https://github.com/ManonFelix/Semidiscrete_estimation_ts.

2 Measure transportation

This chapter aims to explain the main principles behind the theory of optimal transport. The original formulation of the optimal transport problem was given by Monge (1781). He proposed a way to calculate the most effective strategy for moving a large amount of sand from one place to another with the least amount of effort required. In mathematical terms, given a source measure μ, target measure ν supported on sets X and Y respectively and a transportation cost function $c(x, y)$ the goal is to find a transport map $T : X \to Y$ such as

$$\min_{\nu=T_{\#}\mu} \int_X c(x, T(x))\mathrm{d}\nu(x)$$

where the constraint $\mu(T^{-1}(A) = \nu(A) \implies \nu = T_{\#}\mu \ , \forall A$ ensures that all the mass from μ is transported to ν by the map T. The notation $\nu = T_{\#}\mu$ means that the map T pushes forward μ to a new measure ν and therefore $T_{\#}$ is called the pushforward operator. A generalization of this problem was proposed by Kantorovich (1942). In his reformulation, he seeks for a transport plan and allows mass splitting. We therefore compute how much mass gets moved from x to y and store the results in a measure $\pi \in \mathcal{P}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ which satisfies for all $A, B \in \mathcal{B}\left(\mathbb{R}^d\right)$: $\pi\left(A \times \mathbb{R}^d\right) = \mu(A),\ \pi\left(\mathbb{R}^d \times B\right) = \nu(B)$. We denote by $\Pi(\mu, \nu)$ the set of transport plans between μ and ν (i.e. couplings). Then, the Kantorwich or p-Wasserstein distance is defined as

$$W_p(\mu, \nu) = \left(\min_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y)\right)^{\frac{1}{p}}. \tag{1}$$

The case of one dimensional probability densities, say $f_S(x)$ and $f_T(y)$ with cumulative distribution functions $F_S(x)$ and $F_T(x)$, is specifically interesting as the Wasserstein distance (a.k.a the Earth Mover's Distance (EMD)) has a closed-form solution

$$\mathcal{W}_1(\mu, \nu) = \int_{\mathbb{R}} |F_\mu(t) - F_\nu(t)| \, dt \tag{2}$$

The possibility of using this closed formed solution to conduct inference on time series motivated the thesis.

Solving the Eq. 2 can be computationally expensive. To prevent this inconvenient problem, Cuturi (2013) introduced a modified version of the Wasserstein distance:

$$W_\lambda(\mu, \nu) = \int_{x,y} C(x, y) \mathrm{d}\pi(x, y) + \varepsilon \int \log\left(\frac{\pi(x, y)}{\mathrm{d}\mu(x)\mathrm{d}\nu(y)}\right) \mathrm{d}\pi(x, y). \tag{3}$$

Minimizing Eq. 3 leads to the so called Sinkhorn divergence. This divergence is obtained adding to the original optimal transportation problem an entropic regularization term (right part). When λ is small, the Sinkhorn divergence approximates the Wasserstein distance. In contrast to the Wasserstein distance, the regularized Sinkhorn divergence is differentiable and smooth.

In this thesis, we use all the concepts presented in this section in order to establish new minimum distance estimators in time series analysis.

3 Inference in the frequency domain

Before introducing our estimators, let us first provide an overview of the theory that is commonly applied to conduct inference in the frequency domain.

Consider a stationary process $\{Y_t\}$ of n observations $y_{1:n} = y_1, \ldots, y_n$. During our research project, we study linear stochastic process $\{Y_t\}$ satisfying

$$\phi(L)(1 - L)^d Y_t = \varphi(L)\epsilon_t$$

where $LX_t = X_{t-1}$ (back shift operator). $\phi(z)$ and $\varphi(z)$ are the auto-regressive and moving average polynomial of order p and q respectively. The time series $\{Y_t\}$ may or may not have long memory

depending on the value of d. When $0 < d < 0.5$ the process is called a long-memory process and are extensively applied in finance (see e.g Tsay (2005)) . In the literature, we often rewrite d as $H = d - 0.5$. In our research, we do not assume any distribution for the innovation term ϵ_t. We present our results for the case when $\epsilon_t \sim N(0, \sigma_\epsilon^2 = 1)$ but also underlying innovation densities with fatter tails (like e.g. Skew t (see Azzalini and Capitanio (2003)) and Student t).

To conduct inference on the model parameters $\theta = (\sigma_\epsilon^2, d, \phi_1, \ldots, \phi_p, \varphi_1, \ldots, \varphi_q)$ of long-memory processes, we could assume that ϵ_t is normally distributed. Thanks to this assumption, we can write the likelihood of the process and optimize it to find $\hat{\theta}$. Nevertheless, this approach is extremely time-consuming and can even be unfeasible due to the strong dependence and long-memory properties of the process. Instead, we can approach the problem in the frequency domain and work on Fourier frequencies rather than time data. The frequency domain approach represents a time series into combination of sinusoids.

The main tool utilized in the frequency domain is the spectral density. The spectral density $f(\lambda_j, \theta)$ of Y_t

$$f(\lambda_j, \theta) = \left|1 - e^{i\lambda}\right|^{-2d} \frac{\sigma_\epsilon^2}{2\pi} \frac{|\varphi(\exp\{-i\omega\})|^2}{|\phi(\exp\{-i\omega\})|^2}$$

where $\varphi(x) = 1 - \sum_{k=1}^{p} \varphi_k x^k$ and $\phi(x) = 1 + \sum_{k=1}^{q} \phi_k x^k$. λ_j are the fundamental Fourier frequencies where $\lambda_j = 2\pi(j/n), j \in \mathcal{J} = \{1, 2, \ldots, [(n-1)/2]\}$. The spectrum of a time series can be estimated by the method of moment. Its sample analogue is called the periodogram $I(\lambda_j) = \frac{1}{2\pi n} \left|\sum_{t=1}^{n} \left(Y_t - \bar{Y}_n\right) e^{it\lambda_j}\right|^2$. The periodogram is asymptotically unbiased. Nevertheless, it is an inconsistent estimator. An important and key result showed by Priestley (1981) and Brillinger (2001) is that the periodogram ordinates are asymptotically independent and exponentially distributed with rate equal to the spectral density. In other words, the standardized periodogram ordinates are asymptotically independent and have an exponential distribution with rate one. Therefore, in 1953, Whittle (1953) had the idea to minimize the Whittle approximated likelihood:

$$L_W(\theta) = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \ln f(\lambda, \theta) d\lambda + \int_{-\pi}^{\pi} \frac{I(\lambda)}{f(\lambda, \theta)} d\lambda\right] \tag{4}$$

which is derived from the fact that the SPOs are asymptotically identically distributed according to an exponential distribution. Eq. 4 can be rewritten by separating the variance component from the

rest of the parameters vector as

$$L_W(\theta^*) = \sum_{j \in \mathcal{J}} \frac{I(\lambda_{j:n})}{f(\lambda_{j:n}, \theta^*)} \tag{5}$$

where $f(\lambda_{j:n}, \theta^*) = 2\pi\sigma_\epsilon^2 f(\lambda_{j:n}, \theta^*)$ and $\theta^* = (1, \eta = (d, \phi_1, \ldots, \phi_p, \varphi_1, \ldots, \varphi_q))$.

Finally, Beran (1994) suggests the following minimization problem. Firstly, minimize $\arg\min_\eta L_W(\theta^*) = \arg\min_\eta L_W(\eta)$ which yields to $\hat{\eta}$. Secondly, set $\hat{\sigma}_\epsilon^2 = 2\pi L_W(\hat{\eta})$. The author demonstrates the consistency of the parameter $\hat{\theta}^*$. Additionally, the parameter is $\sqrt{n}$-consistent and converges to a normal distribution. In the case of underlying Gaussian innovation terms, $\hat{\theta}$ achieves the Cramer-Rao lower bound.

The Whittle's estimator can also be applied for ARIMA(p, q) process and remains $\sqrt{n}$-consistent and normally distributed. We therefore use this parameter as our reference to compare our results as it is still the state-of-the-art methodology.

4 Methodology

4.1 Problem settings

Our goal is to find the parameter $\eta = \theta^*$ of a time series model in the parameter space $\theta^* \in \mathcal{H}$ with dimension $\mathcal{H} \subset \mathbb{R}$ that minimizes the distance between the empirical and theoretical cumulative distributions of the SPOs. We denote the distance or divergence used by $\mathcal{D}$ and write our minimum distance estimator such as

$$\hat{\theta^*} = \underset{\theta^* \in \mathcal{H}}{\operatorname{argmin}} \, \mathcal{D}(\mu, \nu).$$

where μ is the theoretical exponential distribution and ν is the empirical distribution of the SPOs. In our study, several estimators are proposed and therefore $\mathcal{D}$ is redefined for each optimization problem. For instance, when $\mathcal{D}$ is the Wasserstein distance, we denote the corresponding minimum Wasserstein estimator (MWE) as $\hat{\theta}^*_{MWE}$. During our research, we always assumed the variance of the innovation term σ_ϵ^2 to be known and equal to one. Hence, our parameter vector to be estimated is $\theta^* = (d, \phi_1, \ldots, \phi_p, \varphi_1, \ldots, \varphi_q)$. In addition to that, we focus first on processes with underlying Gaussian distribution and then extend to other distributions whith fatter tails.

4.2 Estimation methods

4.2.1 Minimum Wasserstein Estimator

Recap that the Wasserstein distance when $p = 1$ is given by:

$$\mathcal{W}_1(\mu, \nu) = \int_{\mathbb{R}} |F_\mu(t) - F_\nu(t)| \, dt. \tag{6}$$

F_ν being the empirical cumulative distribution of the SPOs, we estimate it by $\hat{F}_\nu(x) = \frac{1}{m} \sum_{j=1}^{m} \mathbf{1}_{X_j \leq x}$ where x_j are the SPOs of a time series process asymptotically exponentially distributed. To compute F_μ, as it is common in machine learning literature about generative models, we initially thought to generate exponential random variables and stack then in a vector. In mathematical terms, for a sample size n, we generate a vector of length $(n-1)/2 = m$ containing random variables following an exponential distribution with rate one. Since $p = 1$, the Wasserstein distance can be approximated by

$$\mathcal{D}(\mu, \nu) = \frac{1}{m} \sum_{j=1}^{m} |x_j - z_j| \tag{7}$$

where x_j are the SPOs of a time series process asymptotically exponentially distributed and z_j are the observations generated according to $Z \sim Exp(1)$. Minimizing Eq. 7 leads to the minimum Wasserstein estimator noted $\hat{\theta}^*_{MWE} = argmin \ \mathcal{D}(\mu, \nu)$.

Figure 1 displays the Wasserstein loss function of two FARIMA$(0, d, 0)$ processes. The top plot shows a smooth and concave loss function with a global minimum that is the same value as the Whittle's estimator's. On the other hand, the lower shows a wiggly function around the true value of the parameter. Thus, if one uses the Wasserstein loss to estimate a parameter, these two phenomena arise: we end up with either a smooth function containing a global minimum or a function that fluctuates and has several local minima. Through this thesis, we present several estimators that aim to provide more reliable loss functions.

Our first finding is that the computed distance might vary widely around the true parameter value and its value depends heavily on the sample $z_1, ..., z_m$ simulated from an Exp(1). As a consequence, the estimated model parameter(s) $\hat{\theta}_{MWE}$ showed a strong dependence on the random exponential variables generated. In Figure 2 we continue with the process used to plot the second line on Figure

1 and simply change the seed with which the vector Z_j is generated. We remark that we are now dealing with a loss function that is smooth and has a global minimum that is precisely the true value of the parameter $\theta = H = 0.8$. Concretely, by modifying the vector Z_j, we can obtain a more appropriate loss function. However, by definition a random vector cannot be controlled.

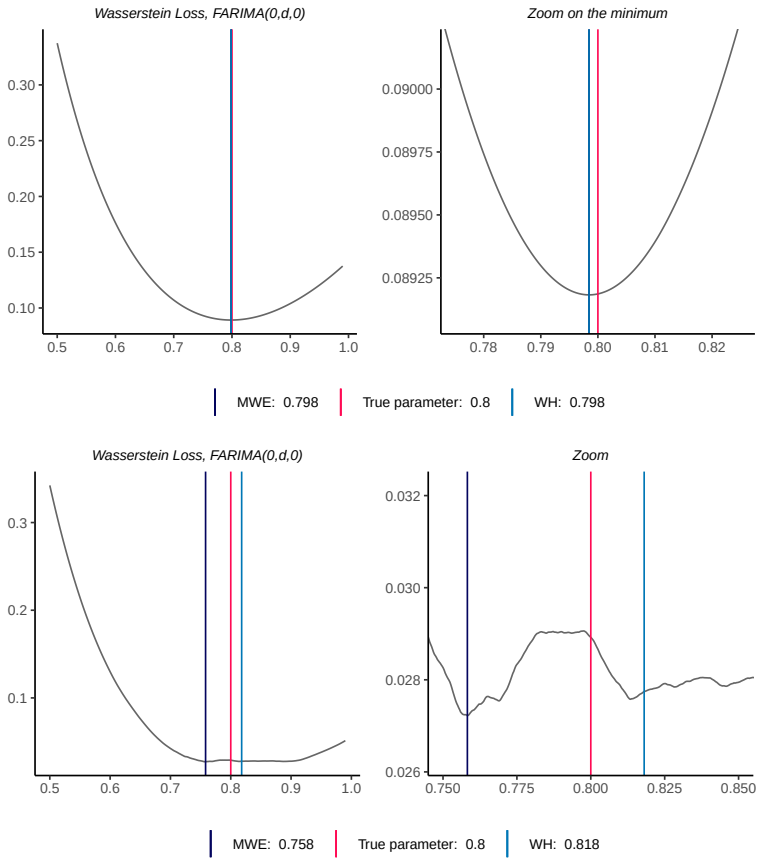


Figure 1: Wasserstein loss functions of two FARIMA(0,d,0) processes where H = 0.8 (d = 1.2). The sample size is 3001. The left column display the entire loss functions for all possible parameter values that a long-memory process can take ($0.51 < \mathrm{H} < 0.99$). The right column is a zoom on the functions.

In order to get a better overview of the behavior of the MWE when the vector Z_j changes, we compute $k = 200$ times $\hat{\theta}^*_{MWE}$ for a given process. Then, we plot them for different sample sizes in Figure 3. We can observe that, for a small sample size, the estimated parameter depends heavily on the random vector Z_j. Nevertheless, the mean remains relatively close to the true value. As the sample size increases, the mean of the minimum Wasserstein estimators $\hat{\theta}^*_{MMWE}$ concentrates around the true parameter value.

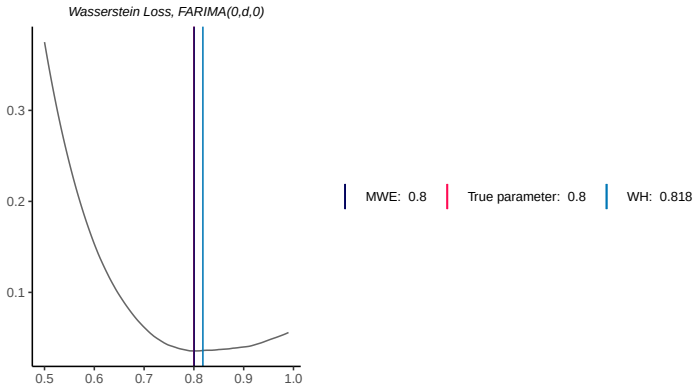


Figure 2: Wasserstein loss function of the FARIMA(0,d,0) process (bottom one) of Figure 1 computed with another random vector.

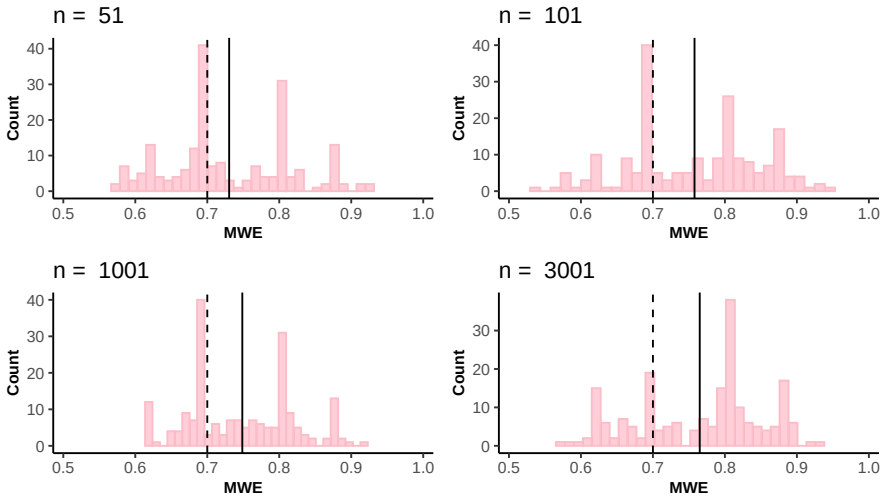


Figure 3: Four different FARIMA(0,d,0) processes for different sample sizes (n = 51, 101, 1001 and 3001). For the same process, we simulate 200 vectors following an exponential distribution and then compute the MWE. The figure represents the four histograms of the MWE. The dashed line is the true parameter H = 0.7 value and the full line is the mean of the MWE.

To cope with this problem of dependence between the random vector Z_j and the parameter estimate, we are going to explore two options.

4.2.2 Mean of Minimum Wasserstein Estimators

Option A: for the simulated times series, we generate several exponential random variables and stack them in vectors. Then, we estimate the model parameter for each of the simulated vector and report the mean of the estimated parameter. For illustration, based on the same process than in Figure 3 with $n = 3001$, we generate $k = 10, 20, 50, 100, 200, 500, 1000$ random vectors, estimate the k parameters and then report the mean. Thus, the mean becomes our estimator and we note it $\hat{\theta}^*_{MMWE}$. The results are listed in Table 1. As k increases, the average becomes progressively closer

to the true parameter.

k	1	10	20	50	100	200	500	1000
MWE	0.807	0.73	0.727	0.726	0.721	0.719	0.714	0.714

Table 1: Mean of the minimum Wasserstein estimators for a FARIMA$(0, d, 0)$ of size $n = 3001$ by varying the value of k, i.e. the number of exponential random vectors generated. The true value is 0.7.

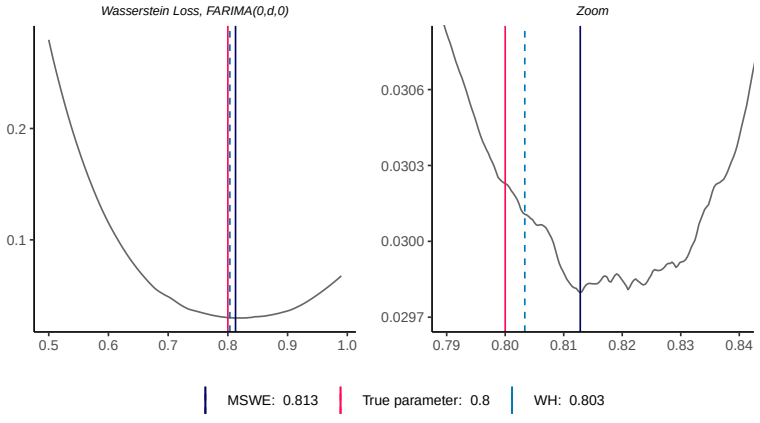
4.2.3 Minimum Semidiscrete Wasserstein Estimator

Option B: instead of using the empirical cumulative distribution function (c.d.f) of exponential random variables generated from a computer, we plan to use the c.d.f of exponential variables with rate one for the SPOs, namely $F(x) = 1 - e^{-x}$. Therefore, the Wasserstein distance becomes:

$$\int_{\mathcal{X}} \left| \hat{F}(x) - (1 - e^{-x}) \right| \mathrm{d}x \tag{8}$$

where $\hat{F}(x) = \frac{1}{m} \sum_{j=1}^{m} 1_{X_j \leq x}$ and x_j are the SPOs of a process. To compute this distance, we replace $\hat{F}_\mu(z) = \frac{1}{m} \sum_{j=1}^{m} \mathbf{1}_{Z_j \leq z}$ by $F_\mu = 1 - e^{-x}$. We use a trapezoidal integration to approximate the integral. Thanks to this second option, there is no longer randomness in our process estimation.

Still, another problem persists. The Wasserstein loss, even for large sample size, is often not well-shaped (i.e smooth and concave): it may contain several local minima (see e.g. Figure 1). This concern leads to biased estimates with large variance. It should also be noted that the loss shape degenerates even more when n decreases (see Figure 5). So far, we are not able to explain why there is such diversity in the shape of the loss functions.



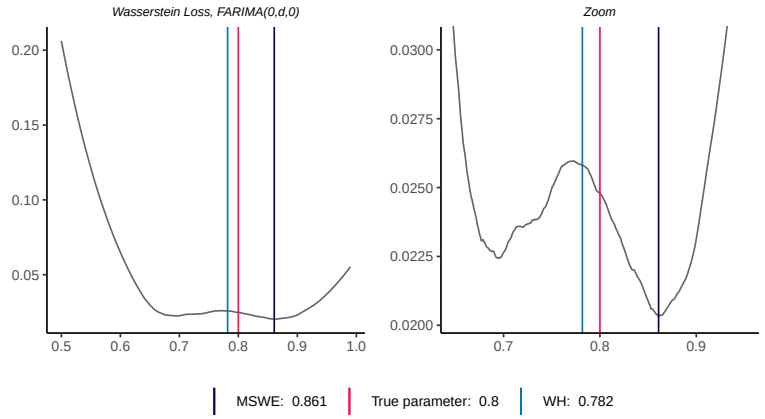


Figure 4: Semidiscrete Wasserstein loss functions for two FARIMA(0,d,0) processes. The sample size is 3001 and the true parameter value is 0.8.

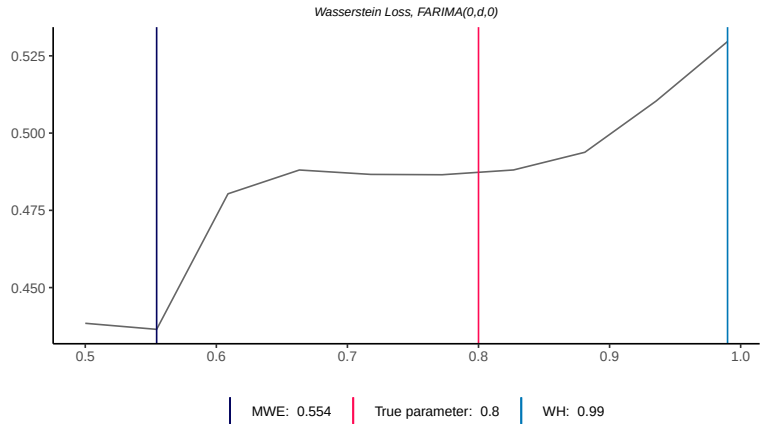


Figure 5: Wasserstein loss function for a small sample size (n = 21) FARIMA(0,d,0) process.

4.2.4 Minimum Weighted Wasserstein Estimator

After searching a way to fix this problem, we found that by putting some weights in the loss function defined by the Wasserstein distance, we obtain a much more regular optimization problem. Therefore, we seek the parameter that minimizes

$$\mathcal{W}_1(\mu,\nu) = \int_{\mathbb{R}} |F_\mu(t) - F_\nu(t)|\, dt \tag{9}$$

where the empirical cumulative distribution of the SPOs is $\hat{F}_\nu(x) = \sum_{i=1}^m w_j 1\{X_i \leq x\}$ and the weights w_j are

$$w_j = \frac{\frac{I(\lambda_j)}{f(\lambda_j;\theta)}}{\sum_{j=1}^m \frac{I(\lambda_j)}{f(\lambda_j;\theta)}} \tag{10}$$

The employment conditions of R packages to calculate the distances used in this thesis required that the weights sum to 1 and that are comprised between 0 and 1. Therefore, our first intuition is to use the weights proposed in Eq. 10.

Figure 6 shows the same process and vector Z_j as Figure 1 and the same vector Z_j but applies the weights to calculate our weighted Wasserstein distance. We can observe that the weighted Wasserstein loss function is immediately smoother and contains a minimum which is even closer to the true parameter than the Whittle's estimator.

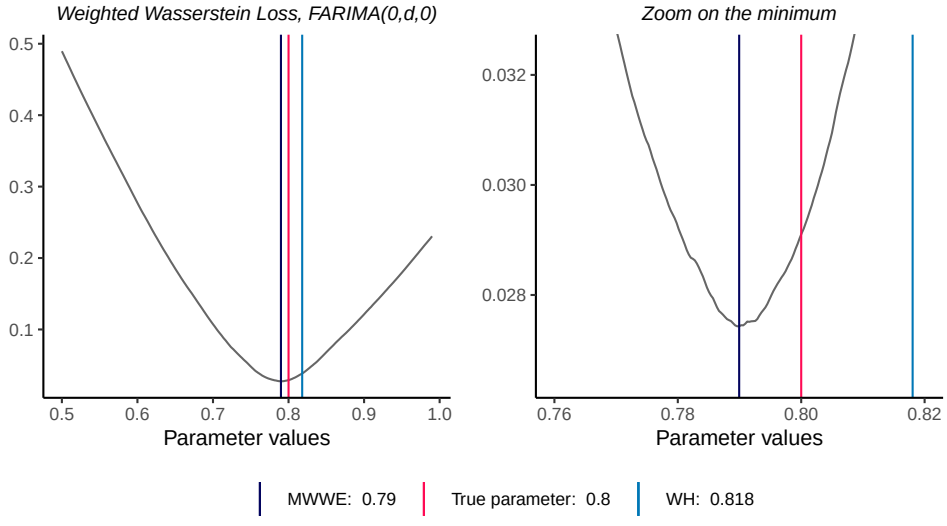


Figure 6: Weighted Wasserstein loss function of the FARIMA(0,d,0) process on Figure 1 (bottom).

It is important to note that the weights applied here are not optimal and, therefore, this question remains open and subject to further analysis. However, the weights presented in this section work well especially for ARMA(p, q) processes as illustrated on Figure 7.

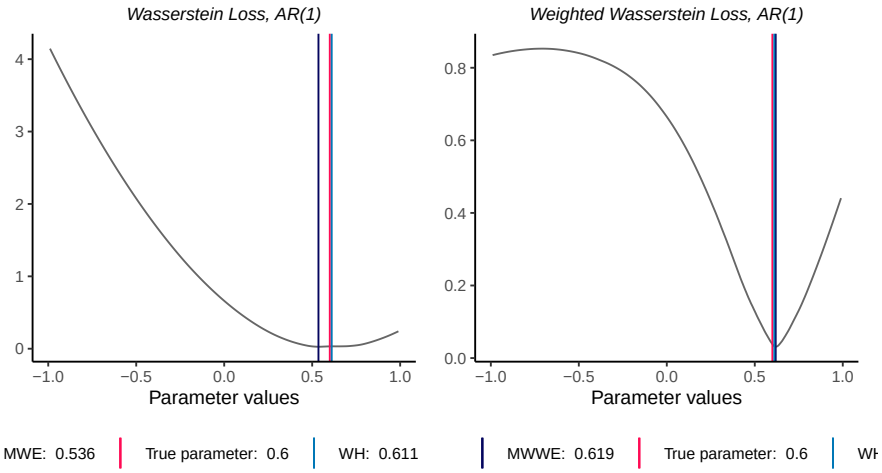


Figure 7: Wasserstein loss and weighted Wasserstein loss functions of a Gaussian AR(1) process. The sample size is 3001 and the true parameter is 0.6.

4.2.5 Minimum Sinkhorn Estimator

A second idea is to employ the Sinkhorn divergence (see Eq. 3) to estimate our parameter based on Cuturi (2013). The regularization term should make the loss function smoother. On Figure 8, we compare the loss function when we employ the Wasserstein distance or the Sinkhorn divergence to estimate our parameter. Indeed, we end up with a smooth and concave function. The reached minimum is very close to the true value. A good property with the Sinkhorn divergence is that it remains smooth even for a very small sample size.

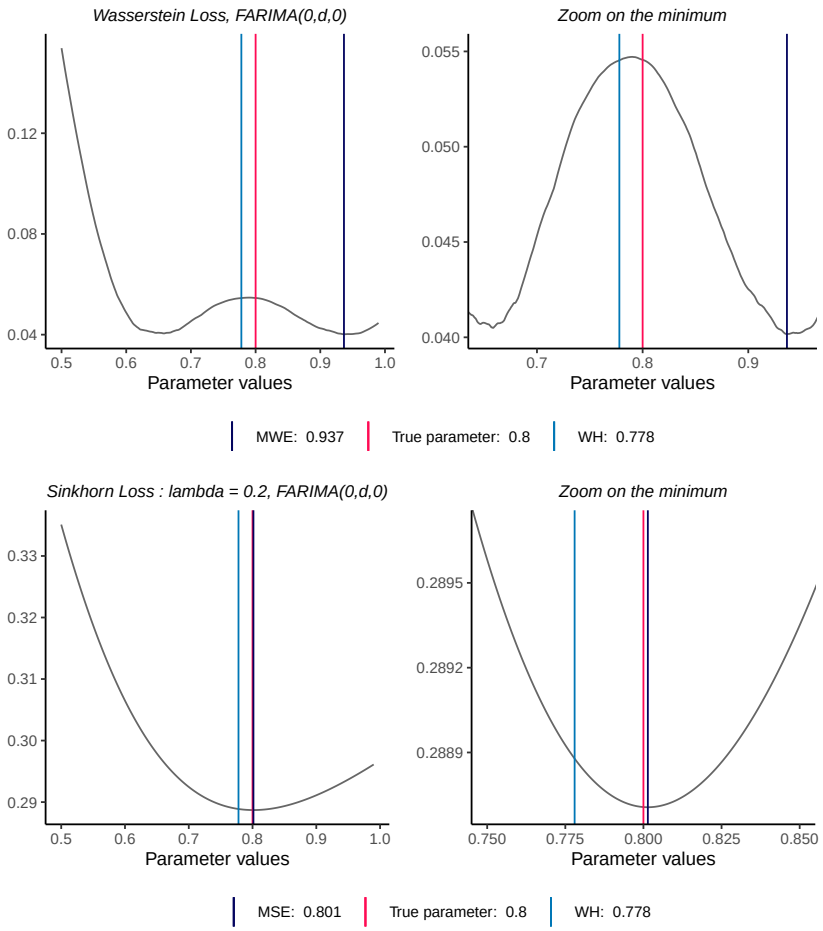


Figure 8: Top: Wasserstein loss function of a FARIMA(0,d,0) process. Bottom: Sinkhorn loss function of the same process. The sample size is 1801.

Following this, we are confronted to an important choice: which values to select for λ? As a reminder, when λ is very small, the Sinkhorn divergence approximates the Wasserstein distance. In order to choose the optimal lambda we suggest to implement classical machine learning techniques to perform model selection such as cross validation, leave-one-out, etc. For more information see for example Friedman et al. (2001) Chapter 7.

For instance, we randomly divide a time series into 2 groups C_1 (80%), C_2 (20%) also called folds. We treat the first group as train and the second group as validation/test. For a selection of λ, we estimate our MSE parameter on the train set and then use the corresponding $\hat{\theta}_{MSE}$ to predict the time data of our test set. For simplicity, we demonstrate the procedure with an AR(1) process, $Y_t = \phi_1 Y_{t-1} + \epsilon_t$ where $\epsilon_t \sim N(0,1)$ and $\theta^* = \phi_1 = 0.6$. After estimating the parameter thanks to the train set, we substitute its value in $\hat{\epsilon}_t = Y_t - \hat{\theta}^*_{MSE} Y_{t-1}$ where $t = 2, ..., l$. l is the length of the testing vector and depends on which ratios we choose to split our time process Y_t in our case

$80\% - 20\%$. Then, we use the predictions of the error terms to compute the Mean Squared Errorfor a given λ:

$$MSE_\lambda \ of \ the \ test \ set = \frac{1}{l}\sum_{t=2}^{l}\hat{\epsilon}_t = \frac{1}{l}\sum_{t=2}^{l}(Y_t - \hat{\theta}^*_{MSE}Y_{i-t})$$

We repeat this method for several lambda values and plot the results on Figure 9. The minimum testing error is achieved when $\lambda = 0.1$ given an estimate $\hat{\theta}^*_{MSE_{\lambda=0.1}} = 0.572$ which is close to the true parameter value $\theta^* = 0.6$.

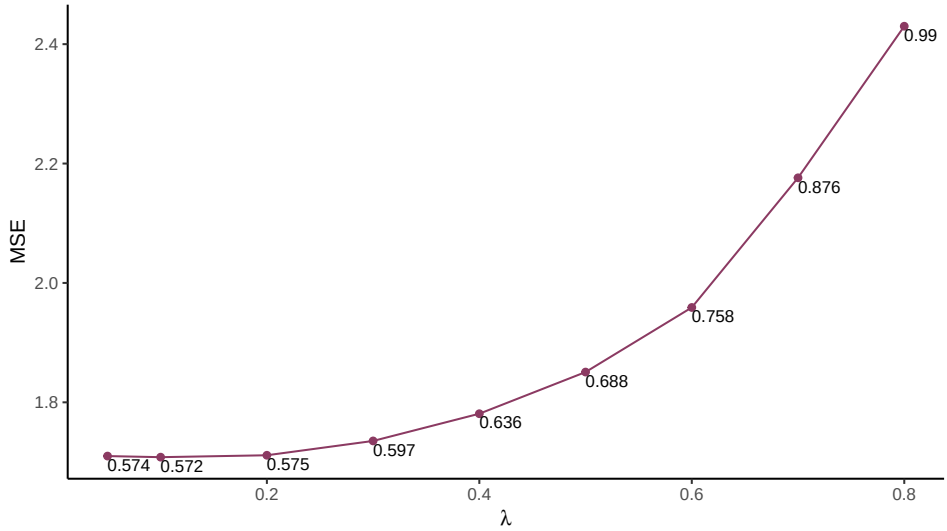


Figure 9: Testing MSE vs lambda values for an AR(1) process. The sample size is 4001 and the true parameter value is 0.6. For this process, the minimum MSE is achieved when lambda = 0.1.

5 Results

5.1 Monte Carlo Simulations

Our criterion for evaluating the performance of each of the estimators is, as it is often the case in machine learning, the Mean Squared Error (MSE)

$$MSE(\hat{\theta}^*, \theta^*) = \sum^{mt}(\hat{\theta}^* - \theta^*)^2 \approx \text{Var}(\hat{\theta^*}) + \text{Bias}^2(\hat{\theta^*})$$

where mt is the number of Monte Carlo simulations, i.e. the number of simulated processes. The MSE represents the bias-variance trade-off which typically emerges in statistics when it comes to model selection.

5.1.1 Long-memory Process

Firstly, we simulate $mt = 200$ stationary FARIMA$(0, d, 0)$ processes of size $n = 3201$ according to

$$(1 - L)^{1.3} Y_t = \epsilon_t.$$

For each process, we compute the Whittle's estimator $\hat{\theta}^*_{WH}$, the minimum Wasserstein estimator $\hat{\theta}^*_{MWE}$, the mean of the minimum Wasserstein estimators $\hat{\theta}^*_{MMWE}$, the minimum semidiscrete Wasserstein estimator $\hat{\theta}^*_{MSWE}$, the minimum weighted Wasserstein estimator $\hat{\theta}^*_{MWWE}$ and the minimum Sinkhorn estimator $\hat{\theta}^*_{MSE}$. Figure 10 reports the results. We can observe that, due to the random form of the Wasserstein loss function, we have an estimator with a very large variance. As expected, by using either the mean of the minimum Wasserstein estimators or the true cumulative distribution function, we can reduce the variance. The most important results concern the MWWE and MSE. Indeed, both new estimators have small variance and no bias (at least, similar to the Whittle's estimator).The MWWE estimator is very close to the Whittle estimator in terms of MSE (see Table 2). Both new estimators have small variance and no bias.

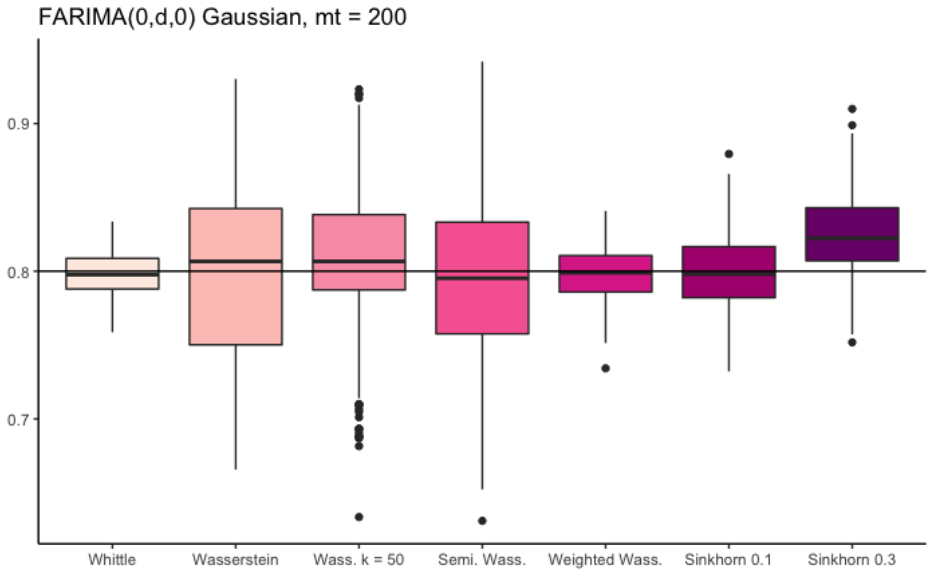


Figure 10: Boxplots of all the estimators presented during this thesis. The sample size of the 200 simulated FARIMA(0,d,0) is 3201.

An important point to note is that the estimation depends a lot on the vector Z_j. On Figure 10 each process is compared to a new vector Z_j. So we have a total of 200 time series and 200 vectors Z_j. For the sake of the example, we now simulate 200 FARIMA$(0, d, 0)$ processes again and compare

them all to the same unique vector Z_j. This leads to the graph in Figure 11 and the corresponding MSE in Table 2. With this vector, for example, the MWWE and the MSE ($\lambda = 0.1$) overperform the Whittle's estimator in terms of MSE.

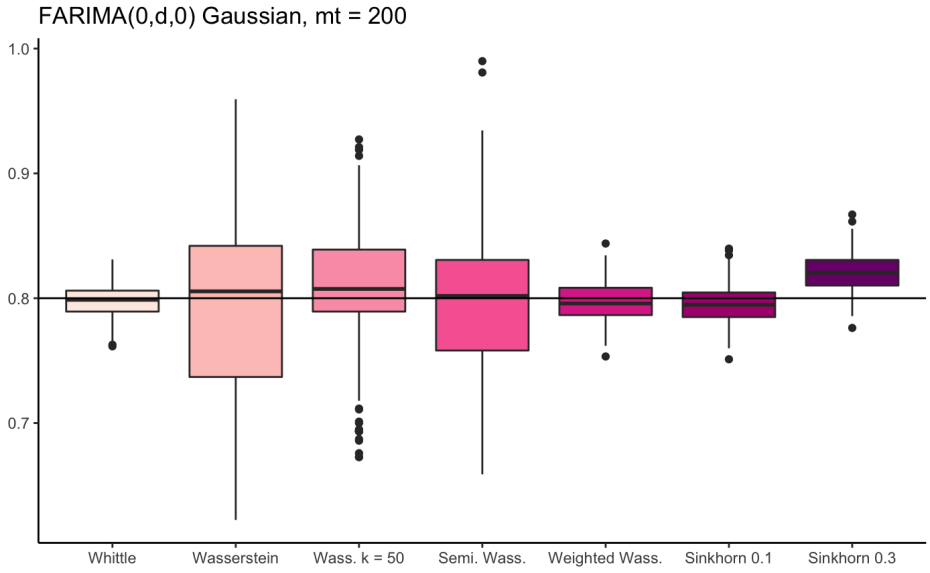


Figure 11: Boxplots of all the estimators presented during this thesis. The sample size of the 200 simulated FARIMA(0,d,0) is 3201. All the estimators are computed using a unique random vector.

MSE	***d***	***d***
Distribution	**Gaussian with 200 vectors Z**	**Gaussian with unique vector Z**
Whittle	7.921	7.861
MWE	8.692	8.322
MMWE, k = 50	9.191	9.281
MSWE	8.326	8.313
MWWE	7.933	7.822
MSE 0.1	8.095	7.699
MSE 0.3	10.729	9.8352

Table 2: Mean Squared Errors of Figure 10. and 11.

5.1.1.1 Heavy-tailed Distribution

Mikosch et al. (1995) showed that the fatter the tails of the innovation distributions, the faster the Whittle's estimator converges to the true parameter value. Regarding the Wasserstein loss function, it becomes smooth and concave when the error distribution is heavy-tailed (see Figure 12), even for small sample sizes. Therefore, the Whittle's estimator and the MWE are often very close, unbiased and with small variances.

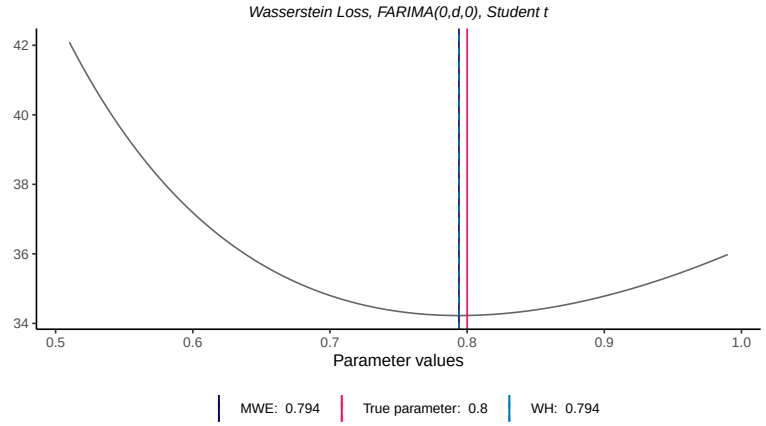


Figure 12: Wasserstein loss function of a FARIMA(0,d,0) process distributed according to a Student t distribution with degree of freedom equal to 2. The sample size is equal to 3001.

We simulate again $mt = 200$ FARIMA(0,d,0) processes with, this time, a Student t underlying distribution with degree of freedom equal to 2. On Figure 13, we note that all estimators (apart from those based on the Sinkhorn distance) have indeed a very small variance and are mostly unbiased. The weighted wasserstein distance is irrelevant and all the minimum Sinhorn estimators (see Table 3) are similar.

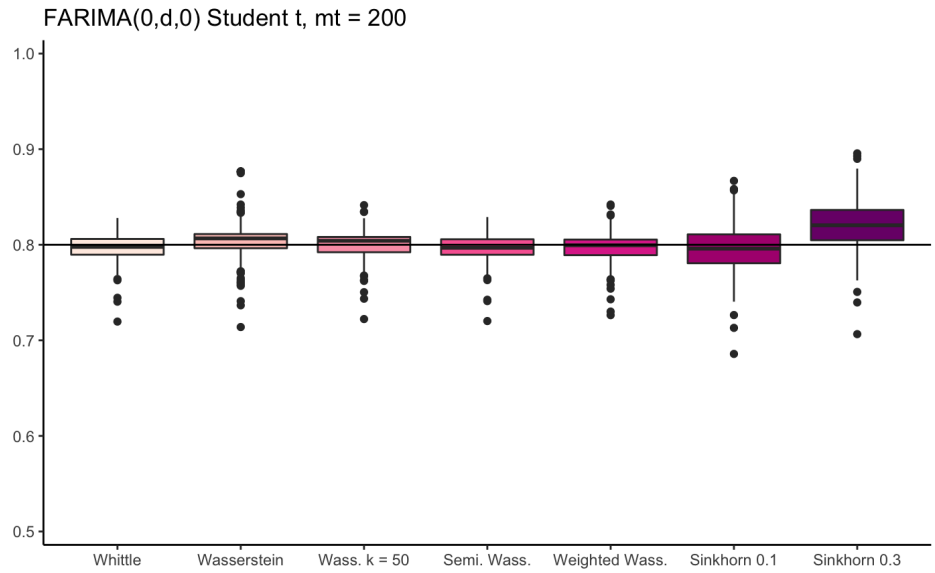


Figure 13: Boxplots of all the estimators presented during this thesis. The sample size of the 200 simulated FARIMA(0,d,0) is 3201 and the underlying distribution is a Student t with degree of freedom equal to 2.

Let us now focus on the case where the distribution of the innovation terms is skewed. The skew t distribution was recently developed by Azzalini and Capitanio (2003). It is related to a standard skew normal random variable Z and a random variable W following a chi-squared distribution with

ν degree of freedom by the equation:

$$Y = \frac{Z}{\sqrt{\frac{W}{\nu}}}.$$

Then the linear transformation $X = \mu + \sigma Y$ has a skew-t distribution with parameters μ, σ, α, and ν and the corresponding notation $ST(\mu = 0, \sigma = 1, \gamma, \nu)$ to denote the skew t random variable X. For example, we consider the underlying distribution of the process as a skew t distribution with degree of freedom equal to 2 and skewness parameter γ equal to 4 (see Figure 14).

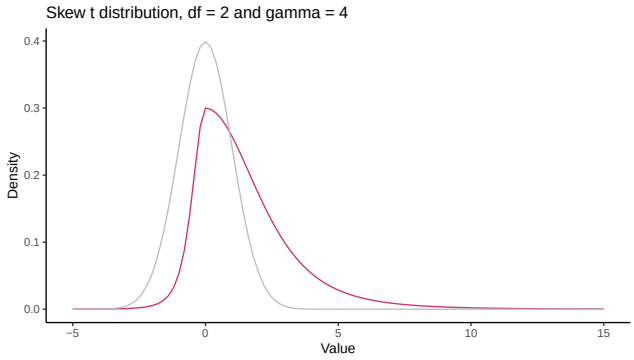


Figure 14: Skew t distribution with degree of freedom = 2 and gamma = 4.

The Wasserstein loss function is still smooth and concave (see Figure 15).

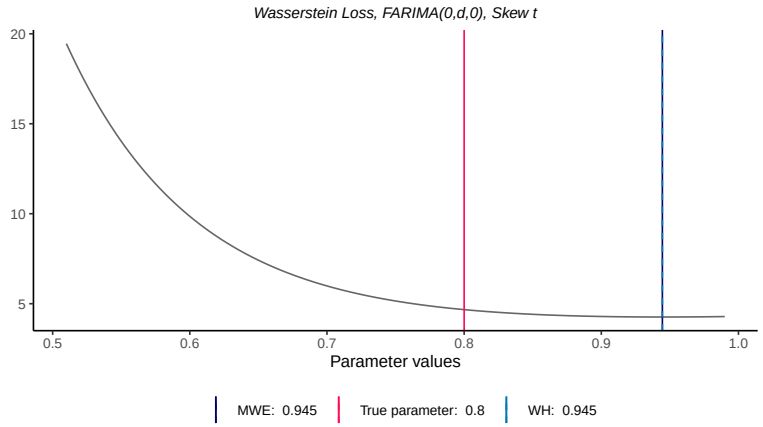


Figure 15: Wasserstein loss function of a FARIMA(0,d,0) process distributed according to a skew t distribution with degree of freedom = 4 and gamma = 2. The sample size is equal to 3001.

As we can see on Figure 16, all estimators are biased and overestimate the value of the parameter. Regarding the MSE values listed in Table 3, all our new estimators surpass the Whittle's estimator except for the MWWE which is irrelevant. The gain is principally in terms of bias.

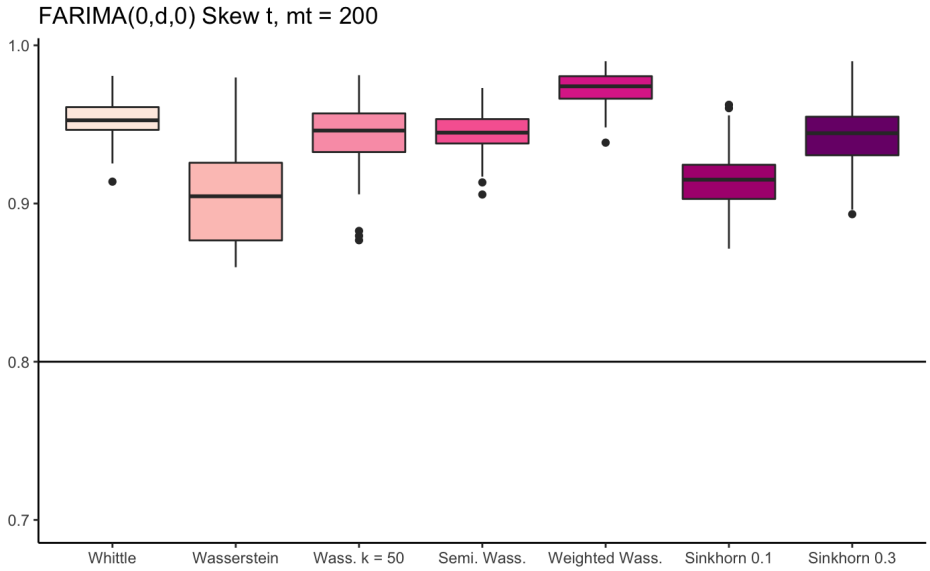


Figure 16: Boxplots of all the estimators presented during this thesis. The sample size of the 200 simulated FARIMA(0,d,0) is 3201 and the underlying distribution is a skew t with df = 4 and gamma = 2.

MSE	***d***	***d***
Distribution	**Student t**	**Skew t**
Whittle	7.768	24.906
MWE	7.850	18.708
MMWE, k = 50	8.058	23.761
MSWE	7.754	23.826
MWWE	8.485	27.889
MSE 0.1	7.815	19.848
MSE 0.3	9.944	23.590

Table 3: Mean Squared Error of Figure 13 and 16.

5.1.1.2 Additive Outliers

In the presence of contamination in the time series (e.g. additive outliers). For in example, in the case of Gaussian FARIMA(0, d, 0) some of our estimators (in particular, the ones based on weighted Wasserstein distance and/or on the Sinkhorn divergence) seem to overperform Whittle's estimator in terms of MSE. To demonstrate this propriety we simulate $mt = 200$ FARIMA$(0, d, 0)$ contaminated by occasional isolated outliers. The processes $\{Y_t\}$ are distributed according to

$$Y_t = (1 - W_t)\, X_t + W_t\, (c \cdot V_t)$$

where $W_t \sim Bern(p)$, $V_t \sim t_2$ and c = 10. In Table 4, we report the ratio between the MSE of the Whittle's estimator and the minimum weighted Wasserstein estimator for different values of p. The results suggest that when the time series is contaminated, the MWWE overperform the Whittle's estimator in terms of MSE.

p	0	0.001	0.01	0.05
ratio	0.682	1.208	1.105	1.018939

Table 4: MSE of the Whittle's estimator divided by the MSE of the MWWE. The number of simulated time series is equal to 200 with sample size equal to 3001.

5.1.2 Short-memory Process

We also aim to demonstrate the performance of our estimators for short-memory processes. To do this, we simulate $mt = 200$ auto-regressive processes of order 2 according to:

$$Y_t = 0.75Y_{t-1} - 0.25Y_{t-2} + \epsilon_t.$$

The processes are stationary since the three stationary conditions are met:

1. $\phi_2 < 1 + \phi_1$
2. $\phi_2 < 1 - \phi_1$
3. $\phi_2 > -1$

where $\phi_1 = 0.75$ and $\phi_2 = -0.25$.

We cannot include the Sinkhorn divergence in our comparison because the function used on R requires too much time to calculate this divergence and fails to converge. The results when $\theta^* \subset \mathbb{R}^2$ are on Figure 17 with corresponding MSE in Table 5. Again, we consider several distributions for ϵ_t: $\epsilon_t \sim N(0,1), \epsilon_t \sim t_2$ and $\epsilon_t \sim ST(\mu = 0, \sigma = 1, \gamma = 2, \nu = 4)$

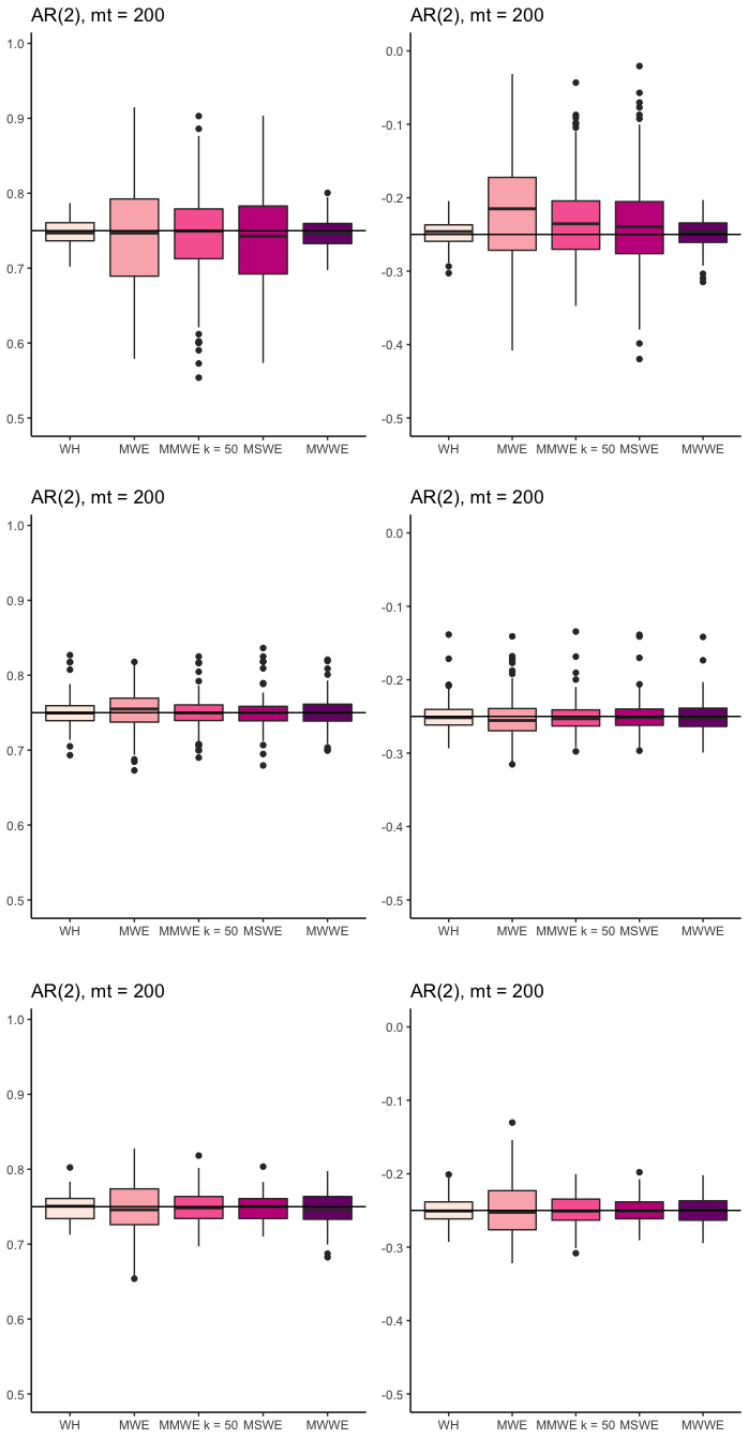


Figure 17: Boxplots of the Whittle's estimator, MWE, MSWE, MMWE, MWWE for 200 AR(2) processes. The innovation terms densities are (in the order of apparition): Gaussian, Student t, Skew t. The left column is the first parameter (0.75) of the process, the right one is for the second parameter (-0.25).

When the process is Gaussian, the MSE of the minimum weighted Wasserstein estimator is similar to Whittle's estimator. The other estimators have large variance. As observed for processes with

long-memory, when the tails of the error distributions are wider than those of the normal distribution, the MWE converges to Whittle's estimator. In this case, all estimators are relatively similar in terms of MSE (bias - variance). On the other hand, contrary to long-memory processes, when we work with short-memory processes we observe that the fact that the underlying distribution are skewed or not is not relevant during the estimation procedure. Indeed, the results for the Student t and the skew t distribution are very close.

MSE	ϕ_1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	ϕ_2
Distribution	**Gaussian**		**Student t**		**Skew t**	
Whittle	0.052	0.059	0.061	0.067	0.061	0.060
MWE	1.024	1.345	0.134	0.139	0.255	0.290
MMWE, k = 50	0.690	0.652	0.073	0.087	0.087	0.087
MSWE	0.885	0.897	0.073	0.083	0.062	0.061
MWWE	0.070	0.083	0.078	0.084	0.091	0.079

Table 5: Mean Squared Error of Figure 17.

6 Conclusion

To conclude, we introduce, in this thesis, five new estimators that are based on minimum distance estimation. Our results suggest that we can outperform the state-of-the art estimation procedure when we are dealing with long-memory processes that have skewed underlying distributions. Moreover, it seems that our minimum weighted Wasserstein estimator can also be more efficient when the process is contaminated by occasional outliers. In the case of short memory processes, we have similar results to Whittle's estimator in terms of MSE. Through this thesis, we open the possibility for further research. Indeed, the weights are certainly not optimal and therefore would be subject to further investigation. As well as the choice of the regularization parameter when using the Sinkhorn divergence. The shape of the Wasserstein loss function and why the estimation procedure behaves better for certain vector Z_j also remains an opened question. We can also extend our research to other distances such as the energy distance:

$$D^2(F, G) = 2 \int_{-\infty}^{\infty} (F(t) - G(t))^2 \, \mathrm{d}t.$$

Another important step is to compute the theory surrounding these estimators (consistency, robustness, etc.). To sum up, our results are promising and open the possibility of further researches.

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