University of Liège
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Investment Portfolio Management

MATH0461-2: Introduction to numerical optimization

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Linear Model

1. Data: stocks $i = 1, .., 462$; weeks $t = 1, .., 568$; sectors $s = 0, ..., 9$

   total capital $C = €500,000$

   $m_{s,i} = 1$ if the stock i belongs to sector s; 0 otherwise

   $p_{i,t}$ = closing price of stock i at week t [€]

   $r_{i,t}$ computed as $\frac{p_{i,t} - p_{i,t-1}}{p_{i,t-1}} \times 100$ = weekly return for a stock i at week t [%]

   $\hookrightarrow r_i$ computed as $\frac{1}{567} \sum_{t=2}^{568} r_{i,t}$ = mean weekly return for a stock i [%/week]

   Variables: $x_i \in [0, 1]$ = fraction of the capital invested in stock i $\quad \forall i = 1, .., 462$

   Objective function: maximize the historical average weekly return : $\max \sum_{i=1}^{462} r_i x_i$

   Constraints: $\sum_{i=1}^{462} x_i \leq 1$ the amount invested is the total capital

   $\sum_{i=1}^{462} x_i m_{s,i} \leq 0.2 \quad \forall s = 0, ..., 9$ no more than 20% of C allocated to a sector

2. We implemented this problem formulation in a Julia file `model.jl`. Here are the results that we obtained:

Stock	Sector	Capital invested [%]	Capital invested [€]	Mean of historical return [%/week]
ENPH	6	20	100000	1.13744
TSLA	7	20	100000	1.1126
NVDA	0	20	100000	1.05678
AMD	2	20	100000	0.919205
NFLX	8	20	100000	0.820564
others		0	0	

   Intuitively, the best investment would be to invest €100,000 (20% of the capital) in each of the five stocks that have the highest mean of historical return, making sure to take only one stock by sector. This would maximize the expected return of the invested amount of money. If we check the mean of the historical return computed, the five selected are indeed the five with the highest value.

   We found an objective value of 1.0093178124053863%/week which represents the historical average weekly return of the portfolio.

   From a financial point of view, we can see in the table that the performance of a diversified portfolio is almost as good as investing everything in the maximum return stock, but this approach helps limiting the risks in case the sector crashes. The mean return for the diversified portfolio is 1.0093178, whereas the mean return for the single-stock portfolio would be 1.13744 %/week.

   Note: In the basis, there is also the stock AVGO, but there is €0 invested in it. This will interest us for the sensitivity analysis.

3. Let $q \in \mathbb{R}_+$ be the dual variable associated with the capital constraint of the primal problem. Let $p_0, ...p_9 \in \mathbb{R}_+$ be dual variables associated with the sector constraints of the primal problem. Let's dualize the constraints:

$$\begin{aligned} d(q, p_0, ..., p_9) &= \max \sum_{i=1}^{462} r_i x_i + q\left(1 - \sum_{i=1}^{462} x_i\right) + \sum_{s=0}^{9} p_s \left(0.2 - \sum_{i=1}^{462} x_i m_{s,i}\right) \\ &= \max q + 0.2 \sum_{s=0}^{9} p_s + \sum_{i=1}^{462}\left(r_i - Cq - \sum_{s=0}^{9} p_s m_{s,i}\right) x_i \end{aligned}$$

We obtain the dual:

$$\begin{aligned} \min \quad & q + 0.2 \sum_{s=0}^{9} p_s \\ \text{s.t.} \quad & q + \sum_{s=0}^{9} p_s m_{s,i} \geq r_i \quad \forall i = 1, ..., 462 \\ & q, p_0, p_1, ..., p_9 \geq 0 \end{aligned}$$

In a general way, provided that the optimal basis remains the same, a dual variable of a constraint represents the marginal increase of the primal objective function if the right-hand side of this constraint is increased by 1.

Thus, the dual variable of a given constraint can be interpreted as the marginal amount of expected return which may be gained or lost as the right-hand side coefficient is updated, e.g. as more total capital is made available or as the capital limit for sector $0, .., 9$ respectively is relaxed. Therefore, q represents the shadow price for the capital constraint and $p_0, ..., p_9$ represents the shadow price for the sectors $0, ..., 9$ constraint respectively.

4. Using the complementary slackness, we can compute the optimal dual variables from the primal solution. This means that we must respect the following conditions:

$$\begin{aligned} p_i^*(a_i^T x^* - b_i) = 0 \qquad & \forall i = 1, ..., 11 \\ x_j^*(c_j - A_j^T p^*) = 0 \qquad & \forall j = 1, ..., 462 \end{aligned}$$

In Julia, the values can be found by using the function `shadow_price` for each constraint. We arrived to:

Dual variable	Value
q	0.7373045990342595
p_1, p_3, p_4	0.0
p_0	0.319474849833971
p_2	0.18190044262601446

Dual variable	Value
p_6	0.4001318037485162
p_7	0.37529922267470806
p_8	0.0832597479724243

The value of the shadow price of the capital is about 0.74, this means that if we increase the right-hand-side of the capital constraint by 1, being able to invest 200% of the capital, we will increase the expected return by about 0.74€/week. Since this value is positive, it could be great to invest more in order to increase our return.

The value of the shadow price for the sectors that are not in the basis, so that did not have any stocks selected, is 0. This is logical since allowing to invest more in them won't impact the expected return as they are not chosen.

For the other sectors, we have values which indicate by how much the expected return would decrease if we reduce 1 unit in these sectors. We can see that the values are higher for stocks with a higher mean of historical return. This was expected since investing less in a stock that has a high return would decrease more our expected return than for a smaller mean of historical return.

In order to verify our dual formulation, we implemented it in the file `dual_model.jl`. This model provides the same results indicating that we formulate it correctly.

In addition, we implemented the changes on the right-hand side to be sure of our interpretation and we indeed found that the value corresponded to the marginal change of the objective function.

5. The primal simplex algorithm starts from a feasible solution in the primal space, ensuring that all constraints are satisfied from the start. It iteratively moves along the vertices of the feasible region, improving the objective value at each step, until it reaches an optimal solution where no adjacent feasible vertex can improve the objective. This process guarantees that the solution will be both primal and dual feasible upon completion, as all reduced costs will be non-negative, satisfying dual feasibility.

   On the other hand, the dual simplex starts with a solution that is dual feasible (all dual constraints are satisfied) but may be primal infeasible. It progressively adjusts the solution to restore primal feasibility while maintaining dual feasibility, ensuring that the objective is non-decreasing. The dual simplex method is particularly effective for sensitivity analysis and re-optimization scenarios, where small changes to the problem can render the current solution primal-infeasible but still dual-feasible. By focusing on restoring primal feasibility without disrupting dual feasibility, the dual simplex method often converges to an optimal solution more quickly in these cases.

   In Gurobi, the dual simplex reached the optimal solution in 4 iterations, compared to 7 iterations for the primal simplex. This difference is common in cases where dual simplex can leverage its efficiency in handling primal infeasibilities. The final optimal solution is guaranteed to satisfy both primal and dual feasibility along with complementary slackness, thus ensuring optimality.

6. Using `lp_sensitivity_report`, we found that the interval for the value of l_6 in which the optimal basis stays the same is $[0; 0.2]$.

   Since initially $l_6 = 0.2$, this limit l_6 can be changed by a value in the interval $\Delta l_6 = €[-0.2; 0]$ without impacting the optimal basis.

   The new optimal solution in the interval can be found thanks to the dual since it represents the impact of the modification of the right-hand side on the optimal value. Therefore, the new optimal solution is simply objective value + $p_6 \times \Delta l_6 = 1.009318 + 0.40013 \times \Delta l_6$. The optimal solution ranges linearly from 0.92929 to 1.009318.

   The composition of the portfolio in this interval can be illustrated as follows:

Stock	Sector	Capital invested [%]	Capital invested [€]	Mean of historical return [%/week]
ENPH	6	$20 + \Delta l_6 \times 100$	$(0.2 + \Delta l_6) \times 500000$	1.13744
TSLA	7	20	100000	1.1126
NVDA	0	20	100000	1.05678
AMD	2	20	100000	0.919205
NFLX	8	20	100000	0.820564
AVGO	4	$\Delta l_6 \times 100$	$-\Delta l_6 \times 500000$	0.737305

   We can see that reducing the limit of capital invested in sector 6 displaces the investment to the stock AVGO since it is the 6^{th} stock with the highest mean of historical return. Since we invest less in the stock with the highest average return, the optimal solution reduces.

Non-Linear Model: Risk Management

7. The problem is quite similar to the linear formulation, so we'll only state the differences.

   Data: Σ is the covariance matrix of the average historical return of each stock
   γ the risk-aversion coefficient

   Objective function: maximize the utility, representing the trade-off between the average historical return and the risk of the portfolio : $\max \sum_{i=1}^{462} r_i x_i - \gamma x^T \Sigma x$

   We want to express this as a SOCP. We'll introduce a new optimization variable $t \in \mathbb{R}_+$ and denote the quadratic form $f(x) = x^T \Sigma x$. Using the epigraph, we obtain the optimisation problem:

   $$\max \sum_{i=1}^{462} r_i x_i - \gamma t \qquad \text{s.t.} x^T \Sigma x \leq t; \qquad \sum_{i=1}^{462} x_i \leq 1; \qquad \sum_{i=1}^{462} x_i m_{s,i} \leq 0.2 \quad \forall s = 0, ..., 9$$

   Since Σ is a covariance matrix, it is semi-positive. Its Cholesky factorization is unique and can be written as $\Sigma = R^T R$ where R is a upper triangular matrix. Therefore,
   $x^T \Sigma x \leq t \ \Leftrightarrow \ x^T R^T R x \leq t \ \Leftrightarrow \ (xR)^T R x \leq t \ \Leftrightarrow \ \|xR\|_2^2 \leq t$

   We can express t as $\frac{(t+1)^2}{4} - \frac{(t-1)^2}{4}$ which gives $\|xR\|_2^2 + \frac{(t-1)^2}{4} \leq \frac{(t+1)^2}{4}$ and we obtain $\left(\frac{t+1}{2}, (xR)^T, \frac{t-1}{2}\right) \in \mathbb{L}^{n+1}$.

8. We implemented this problem formulation in the Julia files `non_linear_model.jl` and for the SOCP formulation in `non_linear_model_SOCP.jl`.

   The efficient frontier can be found in Figure 1. This efficient frontier represents the set of optimal portfolios that provide the highest expected return for a given level of risk. It therefore is impossible to find a portfolio above this curve. Portfolios that are below the curve are suboptimal since they do not achieve the maximum return for the level of risk. In our case, the shape is similar to a logarithmic function.

   As the risk-aversion coefficient γ increases, the portfolio allocation will be created in order to be safer. This can be seen in the Figure 1 as the risk reduces with the increase of gamma. Due to this risk decrease, the expected return also decreases. In addition, as the risk increases, the increase in expected return reduces. At a certain point, allowing to take more risk does not really impact the expected return. Due to this, if conservative investor (with large γ) agrees to take a little more risk, it is expected that they will increase their expected return much more compared to the situation where an aggressive investor (with small γ) does it.

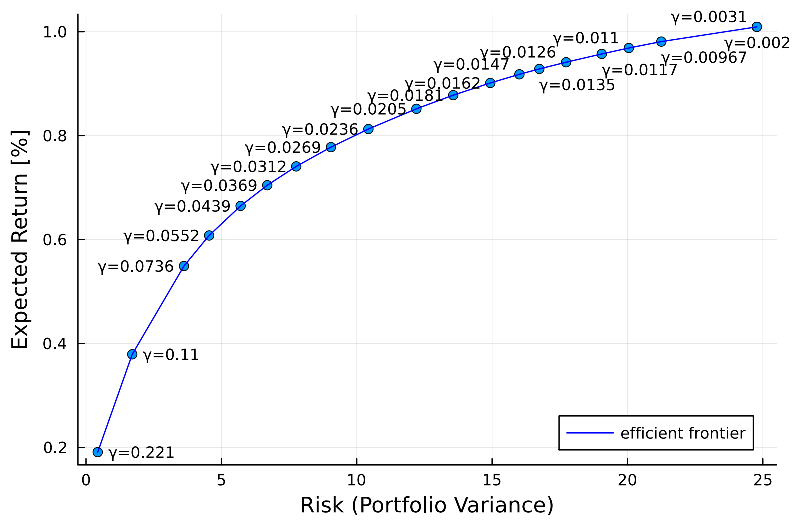

Figure 1. Efficient frontier