The Three Valued Semantics of Phenesthe+

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1 Three valued semantics

In this report, we define the three valued semantics of Phenesthe+, the extension of Phenesthe [2]. While the original version of Phenesthe, allowed future formulae—i.e., their evaluation at an instant t depends on information after t—for dynamic temporal phenomena, it didn't allowed future formulae, for events and states. Phenesthe+ allows the definition of future formulae for all phenomena categories. A *stream*, at any instant t can be represented by the finite model $\mathcal{M}_t = \langle T_t, I_t, <, V^{\bullet}, V^{-}, V^{=} \rangle$ where $T_t = \{0, 1, \cdots, t\}$, $I_t = T_t \times T_t \cup \{[ts, \infty) : ts \in T_t\}$, and $V^{\bullet} : \Phi^{\cdot}_o \to 2^T_t$, $V^{-} : \Phi^{-}_o \to 2^I_t$, $V^{=} : \Phi^{=}_o \to 2^I_t$ are valuations of atomic formulae.

A *stream processor*, in symbols $\mathcal{SP}_t$, is defined by the triplet $\langle \Lambda^{\bullet}_t, \Lambda^{-}_t, \Lambda^{=}_t \rangle$ where $t \in T$, $\Lambda^{\bullet}_t : \Phi^{\bullet} \times T_t \to \{\top, \bot, ?\}$, $\Lambda^{-}_t : \Phi^{-} \times (I^c_t \cup I^+_T) \to \{\top, \bot, ?\}$, and $\Lambda^{=}_t : \Phi^{-} \times I^c_t \cup I^+_t) \to \{\top, \bot, ?\}$, are formulae valuation functions assigning truth values on formulae-instants/intervals pairs and $I^c_t = T_t \times T_t$ and $I^+_t = \{[ts, t+] : ts, t \in T_t\}$. Intervals of I^c_t ([ts,te]) denote that a phenomenon started at ts and ended at te, while intervals of I^+_t ($[ts, t+]$) denote that a phenomenon started at ts, and continues to be true/unknown at t but does not end at t. We assume that all input phenomena are true and ordered. Where appropriate we will use the connectives $\wedge^u, \vee^u$ and $\neg^u$ of Kleene's strong logic of indeterminacy [1].

1.1 Formulae of $\Phi^{\bullet}$

The semantics for formulae of $\Phi^{\bullet}$ using $\Lambda^{\bullet}_{t_q} : \Phi^{\bullet} \times T \to \top, \bot, ?$ are as follows:

- $\Lambda^{\bullet}_{t_q}(P(a_1, ..., a_n), t) = \begin{cases} \top & t \in V^{\bullet}(P(a_1, ..., a_n)) \\ \bot & t \notin V^{\bullet}(P(a_1, ..., a_n)) \\ ? & \text{never.} \end{cases}$

  where P is an n-ary event predicate symbol.
- $\Lambda^{\bullet}_{t_q}(\neg\phi, t) = \neg^u \Lambda^{\bullet}_{t_q}(\phi, t)$ where $\phi \in \Phi^{\bullet}$.
- $\Lambda^{\bullet}_{t_q}(\phi \wedge \psi, t) = \Lambda^{\bullet}_{t_q}(\phi, t) \wedge^u \Lambda^{\bullet}_{t_q}(\psi, t)$, where $\phi, \psi \in \Phi^{\bullet}$.
- $\Lambda^{\bullet}_{t_q}(\phi \vee \psi, t) = \Lambda^{\bullet}_{t_q}(\phi, t) \vee^u \Lambda^{\bullet}_{t_q}(\psi, t)$, where $\phi, \psi \in \Phi^{\bullet}$.
- $\Lambda^{\bullet}_{t_q}(\mathsf{start}(\phi), t) = \begin{cases} \top, \bot & \exists te. \Lambda^{-}_{t_q}(\phi, [t, te]) = \top, \bot \\ ? & \exists_{\leq t} ts. \exists^{>t} te.\ \Lambda^{-}_{t_q}(\phi, [ts, te]) = ? \end{cases}$

  where $\phi \in \Phi^{-}$ and $\Lambda^{-}_{t_q}(\phi, [t, te])$ denotes denotes the valuation of formula $\phi \in \Phi^{-}$ at an interval $[t, te]$ as defined below.
- $\Lambda^{\bullet}_{t_q}(\mathsf{end}(\phi), t) = \begin{cases} \top, \bot & \exists ts. \Lambda^{-}_{t_q}(\phi, [ts, t]) = \top, \bot \\ ? & \exists_{<t} ts. \exists^{\geq t} te.\ \Lambda^{-}_{t_q}(\phi, [ts, te]) \ = ? \end{cases}$

  where $\phi \in \Phi^{-}$.
- $\Lambda^{\bullet}_{t_q}(\phi \in \psi, t) = \exists^{\leq t} ts. \exists^{\geq t} te.\ \Lambda^{\bullet}_{t_q}(\phi, t) \wedge^u \Lambda^{-}_{t_q}(\psi, [ts, te])$.

1.2 Formulae of $\phi \in \Phi^{-}$

The semantics for formulae $\phi \in \Phi^{-}$ given by means of the valuation function $\Lambda^{\bullet}_{t_q} : \Phi^{\bullet} \times I \to \{\top, \bot, ?\}$ are defined as follows:

- $\Lambda^{-}_{t_q}(P(a_1, ..., a_n), [ts, te]) = \begin{cases} \top & [ts, te] \in V^{-}(P(a_1, ..., a_n)) \\ \bot & [ts, te] \notin V^{-}(P(a_1, ..., a_n)) \\ ? & \text{never.} \end{cases}$

  where P is an n-ary state predicate symbol.

- $$\Lambda^-_{t_q}(\phi \rightarrowtail \psi, [ts,te]) = \begin{cases} \top & \Lambda^\bullet_{t_q}(\phi, ts) = \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top \wedge \\ & \forall^{\leq te}_{>ts} t. \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) = \bot \wedge \\ & \forall^{<ts} ts'. [\Lambda^\bullet_{t_q}(\phi, ts') \neq \bot \rightarrow \exists^{\leq ts}_{>ts'} te'. \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te') = \top] \\ \bot & \text{otherwise.} \\ ? & \Big[\Big[\big[\Lambda^\bullet_{t_q}(\phi, ts) =? \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top\big] \\ & \vee \big[\Lambda^\bullet_{t_q}(\phi, ts) = \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \wedge\ \exists^{\leq te}_{>ts} t.\ \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) =?\big]\Big] \wedge\ \big[\forall^{\leq te}_{>ts} t. \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) \neq \top\big]\Big] \\ & \wedge\ \forall^{<ts} ts'. [\Lambda^\bullet_{t_q}(\phi, t') \neq \bot \rightarrow \exists^{\leq ts}_{>ts'} te'. \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top] \end{cases}$$

  where $\phi, \psi \in \Phi^-$. Essentially, $\phi \rightarrowtail \psi$ holds for the disjoint maximal intervals that start at the earliest instant ts where ϕ is true (conditions 1,3) and end at the earliest instant $te, te > ts$ where ψ is true and ϕ is false.

- $$\Lambda^-_{t_q}(\phi \rightarrowtail \psi, [ts, tq+]) = \begin{cases} \top & \Lambda^\bullet_{t_q}(\phi, ts) = \top \wedge \forall_{>ts} t\ \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) = \bot \wedge \\ & \forall^{<ts} ts'. [\Lambda^\bullet_{t_q}(\phi, t') \neq \bot \rightarrow \exists^{\leq ts}_{>ts'} te'. \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top] \\ \bot & \text{otherwise.} \\ ? & \Big[\Lambda^\bullet_{t_q}(\phi, ts) =? \vee \big[\Lambda^\bullet_{t_q}(\phi, ts) = \top) \wedge \exists_{>ts} t.\ \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) =?\big]\Big] \\ & \wedge\ \forall_{>ts} t.\ \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) \neq \top \wedge \\ & \forall^{<ts} ts'. [\Lambda^\bullet_{t_q}(\phi, ts') \neq \bot \rightarrow \exists^{\leq ts}_{>ts'} te'. \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te') = \top] \end{cases}$$

  In this case, the maximal range operator holds for an interval with a known start but unknown end. For true valuations, this means that a state has started being true, and will continue to be true up to at least tq. For a valuation with unknown status it means that the formula might hold for an interval starting in the range [ts,tq+].

- $$\Lambda^-_{t_q}(\phi \looparrowright \psi, [ts,te]) = \begin{cases} \top & \Lambda^\bullet_{t_q}(\phi, ts) = \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \wedge\ \forall^{\leq te}_{>ts} t. \big[\Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) = \bot\ \wedge \Lambda^\bullet_{t_q}(\phi, t) = \bot\ \big] \\ \bot & \text{otherwise.} \\ ? & \Big[\Lambda^\bullet_{t_q}(\phi, ts) = \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \wedge\ \forall^{\leq te}_{>ts} t. \big[\Lambda^\bullet_{t_q}(\phi, t) \neq \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big]\Big] \\ & \vee \Big[\Lambda^\bullet_{t_q}(\phi, ts) =? \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \wedge\ \forall^{\leq te}_{>ts} t \big[\Lambda^\bullet_{t_q}(\phi, t) \neq \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big] \\ & \wedge\ \forall^{<ts} t.\ \Lambda^\bullet_{t_q}(\phi, t) \neq \top \rightarrow \exists^{\leq ts}_{>t} t' \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t') = \top\Big] \end{cases}$$

- $$\Lambda^-_{t_q}(\phi \looparrowright \psi, [ts, tq+]) = \begin{cases} \top & \text{never.} \\ \bot & \text{otherwise.} \\ ? & \Big[\Lambda^\bullet_{t_q}(\phi, ts) = \top \wedge\ \forall_{>ts} t. \big[\Lambda^\bullet_{t_q}(\phi, t) \neq \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big]\Big] \\ & \vee \Big[\Lambda^\bullet_{t_q}(\phi, ts) =? \wedge \forall_{>ts} t. \big[\Lambda^\bullet_{t_q}(\phi, t) \neq \top \wedge \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big] \\ & \wedge\ \forall^{<ts} t.\ \Lambda^\bullet_{t_q}(\phi, t) \neq \top \rightarrow \exists^{\leq ts}_{>t} t' \Lambda^\bullet_{t_q}(\psi \wedge \neg\phi, t') = \top\Big] \end{cases}$$

- For the semantics of temporal union first we define sequences of overlapping intervals. A sequence of k ($k > 0$) overlapping intervals $i^o_j = [ts^o_j, te^o_j]$ is denoted by formulae $EI^k_o(...)$ which are defined as follows:

$$EI^k_o = \exists i^o_1.\ ...\ \exists i^o_k.\ \big[\forall^{\leq k}_{\geq 1} j.\ te^o_j \in i^o_{j+1} \wedge ts^o_j < ts^o_{j+1} \wedge te^o_j < te^o_{j+1} \wedge (...)\big] \qquad (k > 1)$$
$$EI^k_o = \exists i^o_1.\ (...) \qquad (k = 1)$$

  Below we define the three valued semantics[1] of temporal union for $k > 0$.

[1] A first order definition of the semantics is possible, but the formula is not intuitive and lengthy.

$$\Lambda^-_{t_q}(\phi \sqcup \psi, [ts, te]) = \begin{cases} \top & EI^k_o \Big[ts^o_1 = ts \wedge te^o_k = te \wedge \forall^{\leq k}_{>0} j.\ \big[\Lambda^-_{t_q}(\phi, i^o_j) = \top \\ & \vee\ \Lambda^-_{t_q}(\psi, i^o_j) = \top\big]\Big] \wedge \forall^{\leq ts} ts'.\forall_{\geq te} te'.\ \Big[[ts, te] \neq [ts', te'] \rightarrow \\ & \neg\Big[EI^z_\mu.\ \Big[ts^\mu_1 = ts' \wedge te^\mu_z = te' \wedge \forall^{\leq z}_{>1} j.\ \big[\Lambda^-_{t_q}(\phi, i^\mu_j) \neq \bot \\ & \vee\ \Lambda^-_{t_q}(\psi, i^\mu_j) \neq \bot\big]\Big]\Big]\Big] \\ \bot & \text{otherwise.} \\ ? & EI^k_o \Big[ts^o_1 = ts \wedge te^o_k = te \wedge \forall i \in I^k_o.\ \big[\Lambda^-_{t_q}(\phi, i^o) \neq \bot \\ & \vee\ \Lambda^-_{t_q}(\psi, i^o) \neq \bot\big] \wedge \exists^{\leq k}_{>0} j.\ \Big[\big[\Lambda^-_{t_q}(\phi, i^o) =?\ \wedge\ \Lambda^-_{t_q}(\psi, i^o) = \bot\big] \\ & \vee\ \big[\Lambda^-_{t_q}(\phi, i^o) = \bot\ \wedge\ \Lambda^-_{t_q}(\psi, i^o) = \bot\big]\Big]\Big] \wedge\ \forall^{\leq ts} ts'.\forall_{\geq te} te'. \\ & \Big[[ts, te] \neq [ts', te'] \rightarrow \neg\Big[EI^z_\mu\ \Big[ts^\mu_1 = ts' \wedge te^\mu_z = te' \wedge \\ & \forall^{\leq z}_{>1} j.\ \big[\Lambda^-_{t_q}(\phi, i_j) \neq \bot \vee\ \Lambda^-_{t_q}(\psi, i_j) \neq \bot\big]\Big]\Big]\Big] \end{cases}$$

where ϕ, ψ are formulae of Φ^-. In simple terms, the temporal union $\phi \sqcup \psi$ holds true for the intervals where at least one of ϕ or ψ hold true. The temporal union $\phi \sqcup \psi$ holds for an interval with unknown truth value if in there is at least one interval in I^k_o at which ϕ or ψ exclusively holds unknown.

- $$\Lambda^-_{t_q}(\phi \sqcap \psi, [ts, te]) = \begin{cases} \top & \forall^{\leq te}_{\geq ts} t.\exists^{\leq t} ts'_\phi.\exists^{\leq t} ts'_\psi.\exists_{\geq t} te'_\phi.\exists_{\geq t} te'_\psi. \\ & \big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) = \top \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) = \top\big] \\ & \wedge\neg\Big[\exists^{\leq ts} ts'.\exists_{\geq te} te'.[ts', te'] \neq [ts, te] \\ & \wedge\Big[\forall^{\leq te'}_{\geq ts'} t.\exists^{\leq t} ts''_\phi.\exists^{\leq t} ts''_\psi.\exists_{\geq t} te''_\phi.\exists_{\geq t} te''_\psi. \\ & \big[\Lambda^-_{t_q}(\phi, [ts''_\phi, te''_\phi]) \neq \top \wedge\ \Lambda^-_{t_q}(\psi, [ts''_\psi, te''_\psi]) \neq \top\big]\Big]\Big] \\ \bot & \text{otherwise.} \\ ? & \forall^{\leq te}_{\geq ts} t.\exists^{\leq t} ts'_\phi.\exists^{\leq t} ts'_\psi.\exists_{\geq t} te'_\phi.\exists_{\geq t} te'_\psi. \\ & \Big[\big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) = \top \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) =?\big] \\ & \vee\big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) =? \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) = \top\big] \\ & \vee\big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) =? \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) =?\big]\Big] \\ & \wedge\neg\Big[\exists^{\leq ts} ts'.\exists_{\geq te} te'.[ts', te'] \neq [ts, te] \\ & \wedge\Big[\forall^{\leq te'}_{\geq ts'} t.\exists^{\leq t} ts''_\phi.\exists^{\leq t} ts''_\psi.\exists_{\geq t} te''_\phi.\exists_{\geq t} te''_\psi. \\ & \big[\Lambda^-_{t_q}(\phi, [ts''_\phi, te''_\phi]) \neq \top \wedge\ \Lambda^-_{t_q}(\psi, [ts''_\psi, te''_\psi]) \neq \top\big]\Big]\Big] \end{cases}$$

where $\phi, \psi \in \Phi^-$. In other words, the temporal intersection of two formulae of Φ^- holds true for the maximal sub-intervals of the intervals where both formulae hold true. The temporal intersection holds unknown for an interval if both formulae hold for intervals with true or unknown status, or one holds for an interval with unknown status and the other holds true.

- $$\Lambda^{-}_{t_q}(\phi \ \setminus \ \psi, [ts, te]) = \begin{cases} \top & \forall^{\leq te}_{\geq ts} t. \big[\exists^{\leq t} ts'.\exists_{\geq t} te'.\Lambda^{-}_{t_q}(\phi, [ts'_{\phi}, te'_{\phi}]) = \top \wedge \ \forall^{\leq t} ts'.\forall_{\geq t} te'. \\ & \Lambda^{-}_{t_q}(\psi, [ts', te']) = \bot\big] \wedge \neg \Big[\exists^{\leq ts} ts'.\exists_{\geq te} te'.\ [ts', te'] \neq [ts, te] \\ & \wedge \ \forall^{\leq te'}_{\geq ts'} t. \big[\exists^{\leq t} ts''.\exists_{\geq t} te''.\Lambda^{-}_{t_q}(\phi, [ts'', te'']) \neq \bot \\ & \wedge \ \forall^{\leq t} ts''.\forall_{\geq t} te''. \wedge \Lambda^{-}_{t_q}(\psi, [ts'', te'']) \neq \top\big]\Big] \\ \bot & \text{otherwise.} \\ ? & \forall^{\leq te}_{\geq ts} t. \ \Big[\big[\exists^{\leq t} ts'.\exists_{\geq t} te'.\Lambda^{-}_{t_q}(\phi, [ts', te']) = \ ? \wedge \ \forall^{\leq t} ts'.\forall_{\geq t} te'. \\ & \Lambda^{-}_{t_q}(\psi, [ts', te']) \neq \top\big] \vee \ \Big[\big[\exists^{\leq t} ts'.\exists_{\geq t} te'.\Lambda^{-}_{t_q}(\phi, [ts', te']) = \top \\ & \wedge \ \forall^{\leq t} ts'.\forall_{\geq t} te'.\Lambda^{-}_{t_q}(\psi, [ts', te']) \neq \top\big] \wedge \ \exists^{\leq t} ts'.\exists_{\geq t} te'. \\ & \Lambda^{-}_{t_q}(\psi, [ts', te']) =?\big]\Big] \wedge \neg \Big[\exists^{\leq ts} ts'.\exists_{\geq te} te'.\ \big[[ts', te'] \neq [ts, te] \\ & \wedge \forall^{\leq te'}_{\geq ts'} t.\ \big[\exists^{\leq t} ts''.\exists_{\geq t} te''.\Lambda^{-}_{t_q}(\phi, [ts'', te'']) \neq \bot \wedge \ \forall^{\leq t} ts'.\forall_{\geq t} te'. \\ & \Lambda^{-}_{t_q}(\psi, [ts'', te'']) \neq \top\big]\big]\Big] \end{cases}$$

  where $\phi, \psi \in \Phi^{-}$. In other words, the temporal intersection of two formulae of Φ^{-} holds for the maximal intervals at which both formulae hold.

- $$\Lambda^{-}_{t_q}(\phi \ \mathsf{filter}_{\{<,\geq,=\}} \ n, [ts, te]) = \begin{cases} \top & \Lambda^{-}_{t_q}(\phi, [ts, te]) = \top \wedge te - ts \ \{<, \geq, =\} \ n \\ \bot & \text{otherwise.} \\ ? & \text{if } \mathsf{filter}_{<} : \Lambda^{-}_{t_q}(\phi, [ts, te]) = \ ? \\ & \text{if } \mathsf{filter}_{\{\geq,=\}} : \Lambda^{-}_{t_q}(\phi, [ts, te]) = \ ? \wedge te - ts \ \geq n \end{cases}$$

- $$\Lambda^{-}_{t_q}(\phi \ \mathsf{filter}_{\{<,\geq,=\}} \ n, [ts, tq+]) = \begin{cases} \top & \text{if } \mathsf{filter}_{\geq} : \Lambda^{-}_{t_q}(\phi, [ts, tq+]) = \top \wedge tq - ts \ \geq \ n \\ \bot & \text{otherwise.} \\ ? & \text{if } \mathsf{filter}_{\{<,\geq,=\}} : \Lambda^{-}_{t_q}(\phi, [ts, tq+]) = \ ? \\ & \text{if } \mathsf{filter}_{<} : \Lambda^{-}_{t_q}(\phi, [ts, tq+]) = \top \wedge ts - tq \ < n \\ & \text{if } \mathsf{filter}_{\{\geq,=\}} : \Lambda^{-}_{t_q}(\phi, [ts, tq+]) = \top \wedge ts - tq \ \leq n \end{cases}$$

Given a model $\mathcal{M}$, the validity of a formula $\phi \in \Phi^{=}$ at a time interval $[ts, te] \in I$ (in symbols $\mathcal{M}, [ts, te] \models \phi$) is defined as follows[2]:

- $$\Lambda^{=}_{t_q}(P(a_1, ..., a_n), [ts, te]) = \begin{cases} \top & [ts, te] \in V^{=}(P(a_1, ..., a_n)) \\ \bot & [ts, te] \notin V^{=}(P(a_1, ..., a_n)) \\ ? & \text{never.} \end{cases}$$

  where P is n-ary dynamic temporal phenomenon predicate symbol.

- $$\Lambda^{=}_{t_q}(\phi \ \mathsf{before} \ \psi, [ts, te]) = \begin{cases} \top & \exists te' \exists_{>te'} ts' \Big[\Lambda^{\bullet/-/=}_{t_q}(\phi, [ts, te']) = \top \wedge \Lambda^{\bullet/-/=}_{t_q}(\psi, [ts', te]) = \top \\ & \wedge \forall^{ts'}_{>te'} te'' \forall^{\leq te'} ts'' \Lambda^{\bullet/-/=}_{t_q}(\phi, [ts'', te'']) = \bot \wedge \forall^{<ts'}_{>te'} ts'' \forall_{>ts''} te'' \Lambda^{\bullet/-/=}_{t_q}(\psi, [ts'', te'']) = \bot \\ & \wedge \forall^{<ts'}_{>te'} t. \nexists_{\leq t} ts'' \nexists_{\geq t} te'' \big[\Lambda^{\bullet/-/=}_{t_q}(\psi, [ts'', te'']) =? \vee \Lambda^{\bullet/-/=}_{t_q}(\phi, [ts'', te'']) =?\big]\Big] \\ \bot & \text{otherwise.} \\ ? & \exists te' \exists_{>te'} ts' \Big[\Lambda^{\bullet/-/=}_{t_q}(\phi, [ts, te']) = \top \wedge \Lambda^{\bullet/-/=}_{t_q}(\phi, [ts', te]) = \top \\ & \wedge \forall^{ts'}_{>te'} te'' \forall^{\leq te'} ts'' \Lambda^{\bullet/-/=}_{t_q}(\phi, [ts'', te'']) = \bot \wedge \forall^{<ts'}_{>te'} ts'' \forall_{>ts''} te'' \Lambda^{\bullet/-/=}_{t_q}(\psi, [ts'', te'']) = \bot \\ & \wedge \forall^{<ts'}_{>te'} t. \exists_{\leq t} ts'' \exists_{\geq t} te'' \big[\Lambda^{\bullet/-/=}_{t_q}(\psi, [ts'', te'']) =? \vee \Lambda^{\bullet/-/=}_{t_q}(\phi, [ts'', te'']) =?\big]\Big] \\ & \vee \exists te' \exists_{>te'} ts' \Big[\Lambda^{\bullet/-/=}_{t_q}(\phi, [ts, te']) =? \wedge \Lambda^{\bullet/-/=}_{t_q}(\psi, [ts', te]) = \top \\ & \wedge \forall^{ts'}_{>te'} te'' \forall^{\leq te'} ts'' \Lambda^{\bullet/-/=}_{t_q}(\phi, [ts'', te'']) \neq \top \wedge \forall^{<ts'}_{>te'} ts'' \forall_{>ts''} te'' \Lambda^{\bullet/-/=}_{t_q}(\psi, [ts'', te'']) \neq \top \\ & \wedge \forall^{<ts'}_{>te'} t. \exists_{\leq t} ts'' \exists_{\geq t} te'' \big[\Lambda^{\bullet/-/=}_{t_q}(\psi, [ts'', te'']) =? \vee \Lambda^{\bullet/-/=}_{t_q}(\phi, [ts'', te'']) =?\big]\Big] \end{cases}$$

  where $\phi, \psi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$. In short, before holds true if between two intervals/points at which ϕ ψ are true are contiguous, and there is no interval/point starting ending between them at which ϕ or ψ has unknown status.

[2] For simplicity we omit the cases of intervals of the form [ts,tq+]. Moreover in order to avoid much lengthier semantics we assume that if $\Lambda^{=}_{t}([ts, te]) = \top \wedge \Lambda^{=}_{t}(\phi, [ts, te]) =?$, then $\Lambda^{=}_{t}(\phi, [ts, te]) = \top$

- $$\Lambda^{=}_{t_q}(\phi \textsf{ meets } \psi, [ts,te]) = \begin{cases} \top & \exists t' \Lambda^{-/=}_{t_q}(\phi,[ts,t']) = \top \wedge \Lambda^{-/=}_{t_q}(\phi,[t',te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists^{<te} te'.\exists^{\leq te'} ts'.\big[\Lambda^{-/=}_{t_q}(\phi,[ts,te']) = \top \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te]) =?\big] \\ & \vee \exists te'.\exists^{\leq te'}_{>ts} ts'.\big[\Lambda^{-/=}_{t_q}(\phi,[ts,te']) =? \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te]) = \top\big] \\ & \vee \exists te'.\exists^{\leq te'} ts'.\exists_{>ts} te''\big[\Lambda^{-/=}_{t_q}(\phi,[ts,te']) =? \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te'']) =? \\ & \wedge te = min(te',te'')\big] \end{cases}$$

  where $\phi, \psi \in \Phi^{-} \cup \Phi^{=}$.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ overlaps } \psi, [ts,te]) = \begin{cases} \top & \exists_{>ts} ts'.\exists^{\leq te}_{>ts} te'.\Lambda^{-/=}_{t_q}(\phi,[ts,te']) = \top \wedge \Lambda^{-/=}_{t_q}(\psi,[ts',te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists^{<te} te'.\exists^{\leq te'} ts'.\big[\Lambda^{-/=}_{t_q}(\phi,[ts,te']) = \top \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te]) =?\big] \\ & \vee \exists te'.\exists^{\leq te'}_{>ts} ts'.\big[\Lambda^{-/=}_{t_q}(\phi,[ts,te']) =? \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te]) = \top\big] \\ & \vee \exists te'.\exists^{< te'} ts'.\exists_{>ts} te''\big[\Lambda^{-/=}_{t_q}(\phi,[ts,te']) =? \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te'']) =? \\ & \wedge te = min(te',te'')\big] \end{cases}$$

  where $\phi, \psi \in \Phi^{-} \cup \Phi^{=}$.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ contains } \psi, [ts,te]) = \begin{cases} \top & \exists_{>ts} ts'.\exists^{<te} te'.\Lambda^{-/=}_{t_q}(\phi,[ts,te]) = \top \wedge \Lambda^{\bullet/-/=}_{t_q}(\psi,[ts',te']) = \top \\ \bot & \text{otherwise.} \\ ? & \exists te'.\exists^{<te'} ts' \exists_{>ts} te'.\ \big[\Lambda^{-/=}_{t_q}(\phi,[ts,te]) = \top \wedge \Lambda^{\bullet/-/=}_{t_q}(\psi,[ts',te']) =?\big] \\ & \vee \exists_{>ts} ts'.\exists^{<te} te'.\ \big[\Lambda^{-/=}_{t_q}(\phi,[ts,te]) =? \wedge \Lambda^{\bullet/-/=}_{t_q}(\psi,[ts',te']) = \top\big] \\ & \vee \exists te'.\exists^{<te'} ts' \exists_{>ts} te'.\ \big[\Lambda^{-/=}_{t_q}(\phi,[ts,te]) =? \wedge \Lambda^{\bullet/-/=}_{t_q}(\psi,[ts',te']) =?\big] \\ & \wedge te = min(te',te'')\big] \end{cases}$$

  where $\phi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$ and $\psi \in \Phi^{-} \cup \Phi^{=}$.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ finishes } \psi, [ts,te]) = \begin{cases} \top & \exists_{>ts} ts'.\Lambda^{\bullet/-/=}_{t_q}(\phi,[ts',te]) = \top \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists_{\geq te} te'.\exists^{<te} ts'.\big[ts' < te' \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te']) =? \wedge \Lambda^{-/=}_{t_q}(\phi,[ts,te]) = \top\big] \\ & \exists t.\big[t = te \wedge \Lambda^{\bullet}_{t_q}(\phi,[t,t]) =? \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te]) = \top\big] \\ & \exists te'.\exists^{<te'} ts'.\exists_{\geq te'} te''\big[\Lambda^{-/=}_{t_q}(\phi,[ts',te'']) = \top \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te']) =? \\ & \wedge te = min(te',te'')\big] \\ & \vee \exists_{\geq te} t'\big[\Lambda^{\bullet}_{t_q}(\phi,[te,te]) = \top \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te']) =?\big] \\ & \vee \exists te'.\exists_{\leq te'} ts'.\exists_{\geq ts} te''.\big[ts' < te' \wedge \Lambda^{\bullet/-/=}_{t_q}(\phi,[ts',te'']) =? \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te']) = \top \\ & \wedge te = min(te'',te')\big] \end{cases}$$

  where $\phi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$ and $\psi \in \Phi^{-} \cup \Phi^{=}$.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ starts } \psi, [ts,te]) = \begin{cases} \top & \exists_{<te} te'.\Lambda^{\bullet/-/=}_{t_q}(\phi,[ts,te']) = \top \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists_{\geq ts} te'.\exists^{\leq ts} ts'.\big[ts' < te' \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te']) =? \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te]) = \top\big] \\ & \exists t.\big[t = ts \wedge \Lambda^{-}_{t_q}(\phi,[t,t]) =? \wedge \Lambda^{\bullet}_{t_q}(\phi,[ts,te]) = \top\big] \\ & \exists te'.\exists_{<te} ts''.\exists^{\leq ts''} ts'.\big[\Lambda^{-/=}_{t_q}(\phi,[ts',te'']) = \top \wedge \Lambda^{-/=}_{t_q}(\phi,[ts,te']) =? \\ & \wedge ts = max(ts',ts'')\big] \\ & \vee \exists_{\leq ts} ts'.\ \big[\Lambda^{\bullet}_{t_q}(\phi,[te,te]) = \top \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te]) =?\big] \\ & \vee \exists ts'.\exists_{\geq} ts'.\exists^{\leq te} ts''.\big[\Lambda^{\bullet/-/=}_{t_q}(\phi,[ts'',te']) =? \wedge \Lambda^{-/=}_{t_q}(\phi,[ts',te]) =? \\ & \wedge ts = max(ts'',ts')\big] \end{cases}$$

  where $\phi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$ and $\psi \in \Phi^{-} \cup \Phi^{=}$.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ equals } \psi, [ts,te]) = \begin{cases} \top & \Lambda^{-/=}_{t_q}(\phi,[ts,te]) = \top \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists^{\leq ts} ts'.\exists_{\geq te'} te.\big[\Lambda^{-/=}_{t_q}(\psi,[ts,te]) = \top \wedge \Lambda^{-/=}_{t_q}(\psi,[ts',te']) =?\big] \\ & \vee \exists^{\leq ts} ts'.\exists_{\geq te'} te.\big[\Lambda^{-/=}_{t_q}(\psi,[ts',te']) =? \wedge \Lambda^{-/=}_{t_q}(\psi,[ts,te]) = \top\big] \\ & \vee \exists ts'.\exists te'.\exists^{\leq te'} ts''.\exists^{\geq ts'} te''.\big[\Lambda^{-/=}_{t_q}(\psi,[ts',te']) =? \wedge \Lambda^{-/=}_{t_q}(\psi,[ts'',te'']) =? \\ & \wedge ts = max[ts',ts''] \wedge te = min(te'',te')\big] \end{cases}$$

where $\phi, \psi \in \Phi^{-} \cup \Phi^{=}$.

References

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