1. Show that in a pure birth process $\sum_{n=0}^{\infty} P_n(t) = 1$ for all t if and only if $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$. Come up with an "explosive" process, that is a birth process which cannot be normalized for all finite times. Write a program to simulate this process. You will have to be careful in the implementation as realizations can(and typically will) "blow up" in finite time.

First Part:
We have the system of differential equations satisfied by $P_n(t)$ for $t \geq 0$ for pure birth process is given by;

$$P_0'(t) = -\lambda_0 P_0(t),$$
$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \quad for \quad n \geq 1 \tag{1}$$

with initial conditions;

$$P_0(0) = 1, \quad P_n(0) = 0, \quad n > 0.$$

$\implies$ Given, $\sum_{n=1}^{\infty} P_n(t) = 1$. We prove that $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$.
Let,

$$S_k(t) = P_0(t) + P_1(t) + \ldots + P_k(t) \tag{2}$$

since $S_k(t)$ is monotonic and bounded above by 1, and take,

$$g(t) = \lim_{k\to\infty} [1 - S_k(t)] \tag{3}$$

which exits. Now, differentiating $S_k(t)$, we have

$$S_k'(t) = P_0'(t) + P_1'(t) + \ldots + P_k'(t)$$

or,

$$S_k'(t) = -\lambda_0 P_0(t) + \{-\lambda_1 P_1(t) + \lambda_0 P_0(t)\} + \{-\lambda_2 P_2(t) + \lambda_1 P_1(t)\} + \ldots + \{-\lambda_k P_k(t) + \lambda_{k-1} P_{k-1}(t)\}.$$

or,

$$S_k'(t) = -\lambda_k P_k(t).$$

Integrating and using initial conditions, $P_0(0) = 1, \quad P_n(0) = 0, \quad n > 0.$

$$S_k(t) = -\lambda_k \int_0^t P_k(u)du + 1$$

$$1 - S_k(t) = \lambda_k \int_0^t P_k(u)du \tag{4}$$

RHS of equation (4) lies between $g(t)$ and 1.

$$g(t) \leq \lambda_k \int_0^t P_k(u)du \leq 1$$

$$\frac{1}{\lambda_k} g(t) \leq \int_0^t P_k(u)du \leq \frac{1}{\lambda_k}$$

finding the sum of these inequality for $k = i, \ldots, n$

$$g(t)\{\frac{1}{\lambda_i} + \ldots + \frac{1}{\lambda_n}\} \leq \int_0^t S_n(u)du \leq \frac{1}{\lambda_i} + \ldots + \frac{1}{\lambda_n} \tag{5}$$

If $\sum_n \frac{1}{\lambda_n}$ is finite, then the integral is bounded when $n \to \infty$. So S_n can not tend to 1 for all t.
Hence $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$ when $\sum_{n=0}^{\infty} P_n(t) = 1$ for all t.

$\Longleftarrow$ Conversely, if $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$, then by first part of inequality (5) $g(t)$ must be equal to 0 for all t. Hence by equation (3),

$$0 = \lim_{k\to\infty} [1 - S_k(t)]$$

or,

$$\sum_{n=0}^{\infty} P_n(t) = 1$$

Second Part:
Next we consider a fast growing population model, $\lambda_n = n^2\lambda$ then;

$$\sum_{i=1}^{\infty} \frac{1}{i^2\lambda} < \infty$$

the population grows to an infinite value in finite time which is called explosion.

The following is the simulation to this process:

```
fix lambda=2; N=100; t=zeros(1,N); t(1)=0; n(1)=1;
for i=1:N-1
n(i+1)=n(i)+1;
end
%%%%%%%%%%%% find time increment vector t
for i=1:N-1
    a= exprnd(1/((n(i)^2)*lambda));
    t(i+1)=t(i)+a;
end
plot(t,n,'linewidth',1.5)
```

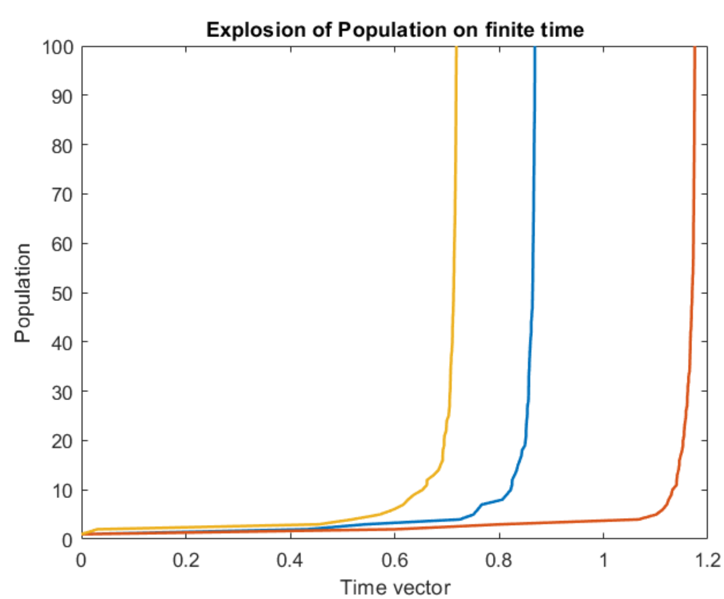

Figure 1: Explosions of Population on finite time, three different simulations:

2. In the simple birth and death process $\lambda_n = \lambda n$ and $\mu_n = \mu n$ for some constants λ and μ. We can add immigration to this process, by assuming that during each interval of length h, there an individual will move into the population with probability νh Here ν is constant independent of the population size. The deterministic model of the system has the form

$$\frac{dn}{dt} = (\lambda - \mu)n + \nu \tag{6}$$

Show that the mean of the of the corresponding stochastic model, $\overline{n}$, equals the solution of this differential equation, with an appropriate initial condition. Give an example to show that this is not true for more general birth and death processes with immigration, and illustrate the example numerically.

Here, we are using simple birth-death model with immigration:

$$\lambda_n = \lambda n + \nu. \quad \lambda > 0, \quad \nu > 0$$
$$\mu_n = \nu n \quad constant \quad death \quad rate$$

we will find $M(t) = \mathbb{E}[X(t)]$ by showing that $M(t)$ satisfy the ODE equation (6) and solving it,

$$M(t) = \sum_{n=1}^{\infty} nP_n(t), \quad P_n(t) = P(X(t) = n)$$

We have, in general

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - \{\lambda_n + \mu_n\}P_n(t) + \mu_{n+1}P_{n+1}(t), \quad n > 1$$
$$P_1'(t) = \lambda_0 P_0(t) - \{\lambda_1 + \mu_1\}P_1(t) + \mu_2 P_2(t)$$

.

$$M'(t) = \sum_{0}^{\infty} nP_n'(t).$$
$$= [\lambda_0 P_0(t) - \{\lambda_1 + \mu_1\}P_1(t) + \mu_2 P_2(t)] + 2[\lambda_1 P_1(t) - \{\lambda_2 + \mu_2\}P_2(t) + \mu_3 P_3(t)] + 3[\ldots$$
$$= P_0(t) + \{\lambda_1 - \mu_1\}P_1(t) + \{\lambda_2 - \mu_2\}P_2(t) + \{\lambda_3 - \mu_3\}P_3(t) + \ldots$$
$$= \sum_{n=0}^{\infty} \{\lambda_n - \mu_n\}P_n(t)$$

we know $\lambda_n = n\lambda + \nu$ and $\mu_n = n\mu$,

$$M'(t) = \sum_{n=0}^{\infty} \{n(\lambda - \mu) + \nu\}P_n(t)$$

$$= \sum_{n=0}^{\infty} \{n(\lambda - \mu)\}P_n(t) + \nu \sum_{n=0}^{\infty} P_n(t)$$

$$= (\lambda - \mu)\sum_{n=0}^{\infty} nP_n(t) + \nu, \quad since \quad \sum_{n=0}^{\infty} P_n(t) = 1$$

$$M'(t) = (\lambda - \mu)M(t) + \nu \Longrightarrow M'(t) - (\lambda - \mu)M(t) = \nu.$$

Integrating Factor(IF) $= e^{-(\lambda-\mu)t}$

$$\frac{d}{dt}[M(t)e^{-(\lambda-\mu)t}] = \nu e^{-(\lambda-\mu)t}$$

$$M(t)e^{-(\lambda-\mu)t} = \frac{\nu M(t)e^{-(\lambda-\mu)t}}{-(\lambda-\mu)} + C$$

$M(0) = 1 \quad so \quad 1 = \frac{\nu}{\mu-\lambda} + C \Longrightarrow C = 1 - \frac{\nu}{\mu-\lambda}$

Hence,

$$M(t) = (\ 1 - \frac{\nu}{\mu-\lambda})\ e^{(\lambda-\mu)t} + \frac{\nu}{\mu-\lambda}$$

Second Part:[1]
For an example to show that this is not true for more general birth and death processes with immigration, we take simple birth, death, immigration, emigration model where the rates are;

birth $i \longrightarrow i+1$ at rate $i\lambda$.

death $i \longrightarrow i-1$ at rate $i\mu$.

immigration $i \longrightarrow i+1$ at rate $i\beta$.

emigration $i \longrightarrow i-1$ at rate $i\rho$.

In this case the mean is given by;

$$m(t) = \begin{cases} \frac{\rho(N\mu-N\lambda-\beta)+\beta}{\mu-\lambda} & \text{if}\, \lambda \neq \mu \\ \beta t + N, & \text{if}\, \mu = \lambda \end{cases}$$

For the cases $\lambda > \mu$ and $\lambda = \mu$, the process is explosive, where when $\lambda > \mu$, the mean increases exponentially in time and when $\lambda = \mu$, the mean increases linearly in time. However, the process is nonexplosive in the case $\lambda < \mu$, where the mean approaches a constant.

$$m\,(\infty) = \frac{\beta}{\mu-\lambda}, \quad \lambda < \mu$$

So this mean does not match with the solution of differential equation.

[1]Dr V. Andasari, Continuous-Time Birth and Death Processes

3. The transition matrix for a four-state Markov chain is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

(a) Draw the directed graph of the chain.
(b) Show that the chain is irreducible, positive recurrent, and periodic. What is the period?
(c) Find the unique stationary probability distribution.

(a) Let, four states given by transition matrix P are state 1, state 2, state 3 and state 4.

$$P = \begin{array}{c} \times \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \end{array}$$

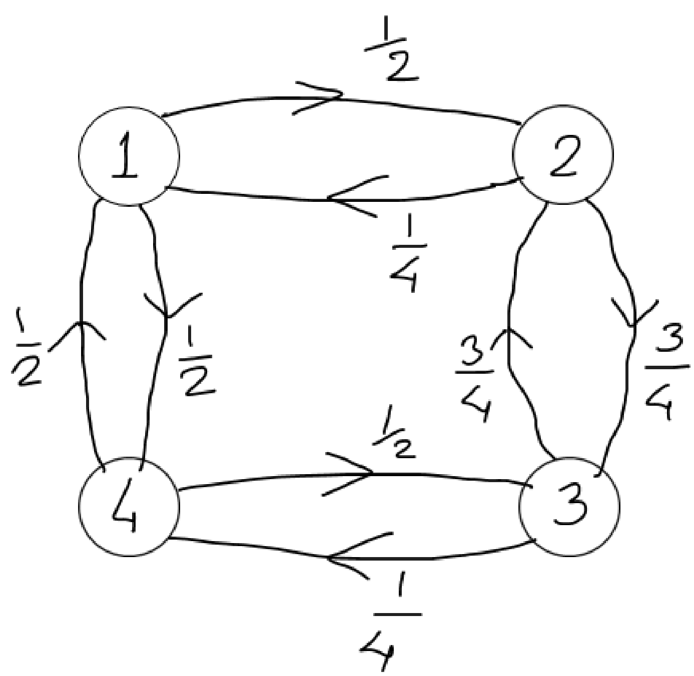

Figure 2: (a) Directed Graph of the chain

(b) A Markov chain is said to be irreducible if all states communicate with each other. Here each state communicates with other. In other words, there is only one communicating class.We can go from any state to any other in a finite number of steps. Hence it is irreducible.
Let $T_i = min\{n \geq 1 | X_n = i\}$ be the time of first return to i. A state i is known as positive recurrent if $\mathbb{E}(T_i|X_0 = i) < \infty$. So in our case, this is finite. Hence it is positive recurrent. Also it is periodic with period 2. If a state is periodic, it is positive recurrent.

(c) A probability distribution π on $S = \{1, 2, 3, 4\}$ which is a vector $\pi = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ where,

$$\pi_i \in [0,1] \quad and \quad \sum_{i \in S} \pi_i = 1$$

is said to be stationary distribution of $X(t)$ if

$$\pi = \pi P(t) \quad for \quad all \quad t \geq 0$$

$$\pi_1 = \frac{1}{4}\pi_2 + \frac{1}{2}\pi_4$$
$$\pi_2 = \frac{1}{2}\pi_1 + \frac{3}{4}\pi_3$$
$$\pi_3 = \frac{3}{4}\pi_2 + \frac{1}{2}\pi_4$$
$$\pi_4 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_3$$

solving this with $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ we get

$$[\pi_1, \pi_2, \pi_3, \pi_4] = [\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}]$$

Thus we conclude that, $\pi = [\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}]$ is the stationary probability distribution.

<u>Quick Check:</u>

$$\left[\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right] P = \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right].$$

Hence the result is valid.

4. Assume that the arrival of proteins to a promoter can be described as a Poisson process with rate λ. If the promoter is not occupied, it will bind the protein. The time the proteins remain bound are independent random variables with mean μ. A protein that arrives when the promoter is occupied diffuses away. Show that the long time fraction of time that the promoter is unoccupied is $\frac{1}{1+\lambda\mu}$.

Let, Time between arrival of proteins to promoter $X(t) = x$ with rate λ.
Time between binding and diffusion of the proteins $Y(t) = y$.

Since he time the proteins remain bound are independent random variables with mean μ.

So here,

$$X \sim Exp(\lambda) \quad and \quad Y \sim Exp(\frac{1}{\mu})$$

Probability density functions:

$$f_X(x) = \lambda e^{-\lambda x}$$
$$f_Y(y) = \frac{1}{\mu} e^{-\frac{1}{\mu}y}$$

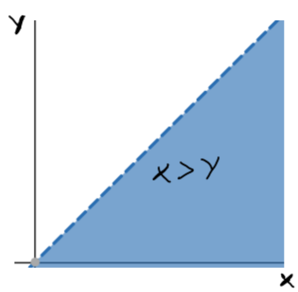

Figure 3: Joint Probability region

For long time fraction of time that he promoter is unoccupied is given by $P(x > y)$.

$$P(x > y) = \int_0^\infty \int_0^x f_{X,Y} \text{ d}y \text{ d}x$$
$$= \int_0^\infty \int_0^x \frac{\lambda}{\mu} e^{-\lambda x - \frac{1}{\mu}y} \text{d}y \text{ d}x$$
$$= \frac{\lambda}{\mu} \int_0^\infty \left| -\mu \, e^{-\lambda x - \frac{1}{\mu}y} \right|_{y=0}^{y=x} \text{d}x$$
$$= \lambda \left[\int_0^\infty e^{-\lambda x} \text{d}x - \int_0^\infty e^{-(\lambda + \frac{1}{\mu})x} \text{d}x \right]$$
$$= \lambda \left[\frac{1}{\lambda} - \frac{\mu}{\lambda\mu + 1} \right] = \frac{1}{1 + \lambda\mu}.$$

5. Assume that the sequence of action potentials (APs) fired by a neuron can be described as a Poisson process. For a fixed time t, let $T(t)$ be the time to the nearest AP in time. This could be an AP preceding or subsequent to the time t. What is the mean of $T(t)$? What is the probability density function of $T(t)$?

First Part:
Let, $T(t)$ be the time to the nearest AP in time,
The sequence of action potential(APs) $X \sim Pois(\lambda)$
let t_0 be the time of first AP fired by neuron,
Consider the interval $\mathbb{B}_\epsilon(t_\nu)$ containing t_0,
The number of points in $\mathbb{B}_\epsilon(t_\nu)$ belong to the Poisson process $N(t)$,

So,

$$P\left(N\left(\mathbb{B}_\epsilon\left(t_\nu\right)\right)\right) = e^{-\lambda\epsilon} \cdot \frac{(\lambda\epsilon)^n}{n!}$$

Now let R be the distance of any point in $N\left(\mathbb{B}_\epsilon\left(t_\nu\right)\right)$ to t_0,

$$P\left(R > \epsilon\right) = P\left(N\left(\mathbb{B}_\epsilon\left(t_\nu\right)\right) = 0\right) = e^{-\lambda\epsilon}.$$

Hence, mean time of $T\left(t\right) = \mathbb{E}\left(R\right) = \frac{1}{\lambda}$.

Second Part:
Also

$$f_{R|\epsilon}\left(\epsilon\right) = P\left(R \leq \epsilon\right) = 1 - e^{-\lambda\epsilon}$$

Hence, probability density function of $T(t) = f_R\left(\epsilon\right) = \lambda\ e^{-\lambda\epsilon}$.