#2| Derive the internal force vector and tangent stiffness matrix for a non-linear truss element based on the rotated engineering strain.

$$\varepsilon_E = \frac{L_n - L_0}{L_0} \quad (1) \longrightarrow \delta\varepsilon_E = \frac{\delta L_n}{L_0} \quad (2)$$

$$x_{21} = x_2 - x_1 \qquad d = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix}$$

$$y_{21} = y_2 - y_1$$

$$u_{21} = u_2 - u_1$$

$$v_{21} = v_2 - v_1$$

Initial config: $L_0 = \sqrt{x_{21}^2 + y_{21}^2}$ (3)

Def. config: $L_n = \sqrt{(x_{21}+u_{21})^2 + (y_{21}+v_{21})^2}$ (4)

$$\delta\varepsilon_E = \frac{\left[\sqrt{(x_{21}+u_{21})^2+(y_{21}+v_{21})^2}\right]'}{L_0} =$$

$$= \frac{1}{L_0} \cdot \frac{1}{2} \cdot \frac{2x_{21}\,\delta u_{21} + 2u_{21}\,\delta u_{21} + 2y_{21}\,\delta v_{21} + 2v_{21}\,\delta v_{21}}{\sqrt{(x_{21}+u_{21})^2 + (y_{21}+v_{21})^2}} =$$

$$= \frac{\cancel{2}}{L_0} \cdot \frac{1}{\cancel{2}} \cdot \frac{x_{21}(\delta u_2 - \delta u_1) + u_{21}(\delta u_2 - \delta u_1) + y_{21}(\delta v_2 - \delta v_1) + v_{21}(\delta v_2 - \delta v_1)}{L_n}$$

$$= \frac{1}{L_0 L_n} \cdot \left[\delta u_2(x_{21}+u_{21}) - \delta u_1(x_{21}+u_{21}) + \delta v_2(y_{21}+v_{21}) - \delta v_1(y_{21}+v_{21})\right]$$

$$= \underbrace{\frac{1}{L_0 L_n}\left[-(x_{21}+u_{21});\ (x_{21}+u_{21});\ -(y_{21}+v_{21});\ (y_{21}+v_{21})\right]}_{b(d) = c(d)\cdot\frac{1}{L_0 L_n}} \delta d$$

$$\boxed{\delta\varepsilon_E = \frac{1}{L_0 L_n}\, c(d)\, \delta d} \quad (5)$$

$$c = \left[-(x_{21}+u_{21}) ; (x_{21}+u_{21}) ; -(y_{21}+v_{21}) ; (y_{21}+v_{21}) \right]$$

The element is derived based on the initial config using the virtual work:

$$\int_{V_0} \delta \varepsilon^T \sigma \, dV = \delta d^T q$$

$$A_0 L_0 \delta \varepsilon^T \sigma = \delta d^T q$$

$$A_0 L_0 \cdot \frac{1}{L_0 L_n} \cdot c^T \cdot \delta d^T \sigma = \delta d^T q$$

$$\Rightarrow q(d) = \frac{A_0}{L_n} \sigma c^T(d) \quad (6)$$

Tangent stiffness matrix.

$$K = \frac{\partial q(d)}{\partial d} = A_0 L_0 \, b^T \frac{\partial \sigma}{\partial d} + A_0 L_0 \, \sigma \frac{\partial b^T}{\partial d}$$

$$\frac{\partial \sigma}{\partial d} = E \frac{\partial \varepsilon}{\partial d} = E \, b(d)$$

$$K = A_0 L_0 \left(b^T E b + \sigma \frac{\partial b^T}{\partial d} \right)$$

$$= A_0 L_0 \left[b^T E b + \sigma \left(\frac{1}{L_0} \left(\frac{K_A}{L_n} - \frac{c c^T}{L_n^3} \right) \right) \right]$$

$$\frac{\partial b^T}{\partial d} = \frac{1}{L_0 L_n} \frac{\partial c^T}{\partial d} + \frac{1}{L_0 L_n{}'_d} c^T = \frac{1}{L_0} \left[-\frac{1}{L_n^2} \frac{c}{L_n} \cdot c^T + \frac{\partial c^T}{\partial d} \cdot \frac{1}{L_n} \right] \quad (7)$$

$$\frac{\partial c^T}{\partial d} = \begin{bmatrix} -\partial u_{21} \\ \partial u_{21} \\ -\partial v_{21} \\ \partial v_{21} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{K_a} \begin{bmatrix} \partial u_1 \\ \partial u_2 \\ \partial v_1 \\ \partial v_2 \end{bmatrix} \quad (8)$$

$$(7)+(8) \Rightarrow \frac{db^T}{dd} = \frac{1}{L_0}\left(-\frac{1}{L_M^2}\frac{cc^T}{L_M} + \frac{1}{L_M}K_a\right) = \frac{1}{L_0}\left[\frac{K_a}{L_M} - \frac{cc^T}{L_M^3}\right] \quad (9)$$

$$K = A_0 L_0 \left[b^T E b + \frac{\sigma}{L_0}\left(\frac{K_a}{L_M} - \frac{cc^T}{L_M^3}\right)\right] =$$

$$= A_0 \cancel{L_0}\left(\frac{1}{\cancel{L_0} L_M} c^T E \frac{1}{L_0 L_M} c + \frac{\sigma K_a}{\cancel{L_0} L_M} - \frac{\sigma cc^T}{\cancel{L_0} L_M^3}\right) =$$

$$K = \left(\frac{A_0 E}{L_0 L_M^2} - \frac{A_0 \sigma}{L_M^3}\right) cc^T + \frac{A_0 \sigma K_a}{L_M}. \quad (10)$$