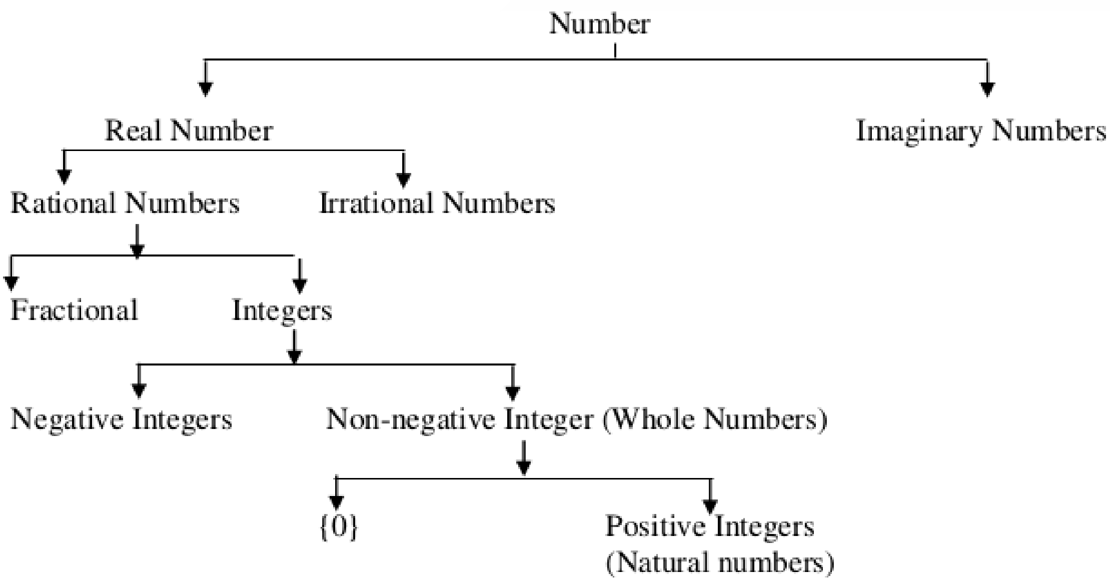
Number System

1 Classification of Numbers

Undoubtedly numbers are the cornerstone of mathematics. There is a huge literature on the development of numbers but that is beyond our scope. We directly jump to the classification of numbers. Remember this classification looks so obvious or trivial today but humans took centuries to reach here. Reader should appreciate this.



Here are some notations to denote the set of numbers :

$\mathbb{N}$ - Set of Natural Numbers = $\{1, 2, 3, ...\}$
$\mathbb{N}_0$ or $\mathbb{W}$ - Set of whole numbers = $\{0, 1, 2, ...\}$
$\mathbb{Z}$- Set of integers = $\{.... -5, -4, ... -1, 0, 1, 2...\}$
$\mathbb{Q}$ - Set of rational numbers = $\{... -5, \frac{-4}{7}, \frac{-1}{2}, 0, \frac{1}{2}, \frac{1}{3}, \frac{5}{7}, 4, ...\}$

$\mathbb{Q}$' or $\mathbb{Q}^c$ - Set of irrational Numbers = $\{... -\sqrt{11}, -\sqrt{5}, -\sqrt{7}, -\frac{1}{\sqrt{3}} .... \sqrt{5}, ...\}$
$\mathbb{R}$ - Set of real Number.

A number is said to be rational if it can be written as $\frac{p}{q}$ form where p, q are integers. For example $\frac{1}{2}, \frac{-1}{5}, \frac{1}{-5}, -\frac{2}{7}, -1, 0, 4, -\frac{7}{8}$ are rational numbers. A real number is said to be irrational if it is not rational. For example $\sqrt{2}, \sqrt{3}, \sqrt{5}, -\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}$ are irrational number. (It is very interesting to prove that $\sqrt{2}, \sqrt{3}$ etc cannot be written as $\frac{p}{q}$ form. We shall prove this shortly).

1.1 Exercises

Q.1 Find five rational numbers between

a. 1 and 10 b. 11 and 12 c. $\frac{1}{3}$ and $\frac{1}{2}$ d. $-\frac{1}{5}$ and $-\frac{1}{7}$ e. $-\frac{3}{5}$ and $\frac{3}{5}$

Q.2 State true or false

a. Every rational number is an integer.

b. Every natural number is an integer.

c. Every integer is an rational number.

d. 5 is an non-negative number.

e. 0 is an non-negative integer.

f. Every irrational number is an rational number.

g. $-\frac{1}{7}$ and 7 are rational numbers.

Q.3 Express the following in decimal by log division

a. $\frac{19}{5}$ c. $-\frac{19}{2}$ e. $-\frac{17}{8}$

b. $\frac{3}{15}$ d. $\frac{2157}{625}$ f. $\frac{327}{500}$

Q.4 Show that the decimal expansion of following rational numbers are non-terminating and repeating.

a. $\frac{8}{3}$ b. $\frac{2}{11}$ c. $\frac{1}{7}$ d. $-\frac{16}{45}$

2 Non-terminating and recurring rational number

There are rational numbers such that when we try to express them in decimal form by division method, we find that no matter how long we divide there is always a remainder. In other words, the division process never comes to an end. This is due to the reason that in the division process the remainder starts repeating after a certain number of steps. In such cases, a digit of a block of digits repeats itself. For example, 0.3333...,0.1666666..., 0.123123123...., 1.2692307692307692307...etc. Such decimals are called non-terminating repeating decimals or non-terminating recurring decimals. These decimal numbers are represented by putting a bar over the first block of the repeating part and omit the other repeating blocks. Thus, we write 0.33333... $= 0.3$, 0.16666... $= \overline{0.16}$, 0.123123123 ..... $= \overline{0.123}$ and 1.26923076923076292307... $= \overline{1.2692307}$.

Fact : Every non terminating and recurring number is a rational number.

1) Express the following rational numbers as decimals

(i) $\frac{2}{3}$ (iii) $-\frac{2}{15}$ (v) $\frac{437}{999}$

(ii) $-\frac{4}{5}$ (iv) $-\frac{22}{13}$ (vi) $\frac{33}{26}$

2) Look at several examples of rational numbers in the form $\frac{p}{q} (\neq 0)$, where p and q are integers with no common factors other than 1 and having terminating decimal representations. Can you guess what property q must satisfy?

3 Conversion and non-terminating recurring rational numbers in the form $\frac{p}{q}$

1) Express $0.\overline{4}$ in the form $\frac{p}{q}$

2) Express each of the following decimals in the form

(i) $0.\overline{35}$ (ii) $0.\overline{585}$

3) Express the following decimals in the form $\frac{p}{q}$

(i) $0.3\overline{2}$ (ii) $0.12\overline{3}$ (iii) $0.003\overline{52}$

4) Express each of the following decimals in the form $\frac{p}{q}$

(i) $0.\overline{4}$ (ii) $0.\overline{37}$ (iii) $0.\overline{54}$ (iv) $0.\overline{621}$ (v) $125.\overline{3}$ (vi) $4.\overline{7}$ (vii) $0.\overline{47}$

*** **Tricky Tip** ***

$$\frac{p}{q} = \frac{CompleteNumber - Numberformedbynon - repeatingdigits}{No.of9asno.ofrepeatingdigitsafterthatwriteno.of0asno.ofnon - repeatingdigits}$$

(i) $0.\overline{57} = \dfrac{57-0}{99} = \dfrac{57}{99}$

(ii) $0.\overline{347} = \dfrac{347-3}{990} = \dfrac{349}{990}$

(iii) $0.53\overline{763} = \dfrac{53763-53}{99900} = \dfrac{53710}{99900} = \dfrac{5371}{9990}$

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4 Irrational Numbers (finding the square root by division method)

A number which is not rational is known as a irrational number. In other words, a number which cant be written as $\frac{p}{q}$ form.
Fact : Every non-terminating and non-recurring number is irrational number.

$\sqrt{2}, \sqrt{3}, \sqrt{7}, \pi$ are rational number. Lets find the value of $\sqrt{2}, \sqrt{3}$ by division method.

1.

	1.4142135
1	2.00000000000000
	1
24	100
	96
28	400
	281
2824	11900
	11296
28282	60400
	56564
282841	383600
	282841
2828423	10075900
	8485269
28284265	159063100
	141421325
28284270	17611775

$\Rightarrow \sqrt{2} = 1.4142135$

2.

	1.732050807
1	3.00 00 00 00 00 00 00 00
	1
27	200
	189
343	1100
	1029
3462	7100
	6924
346405	1760000
	1732025
34641008	279750000
	277128064
3464101607	26219360000
	24248711249
	1970648751

$\Rightarrow \sqrt{3} = 1.732050807$

5 Proof by Contradiction (Irrationality of $\sqrt{2}$)

We will discuss this topic more thoroughly in the chapter Mathematical Logic. Currently our main motive is to prove the irrationality of $\sqrt{2} i.e \sqrt{2}$ cant be written as $\frac{p}{q}$ form. This method proof by contradiction is used frequently in calculus, one of the most interesting branch of mathematics. In this method we assume the negation of something which is to beproved then use deduce a contradiction which implies that our assumption was wrong. Now lets prove that $\sqrt{2}$ is an irrational number.

Assume that $\sqrt{2}$ is an rational number it means it can be written as $\frac{p}{q}$ form. Let
$\sqrt{2} = \frac{p}{q}$
Where p, q are co-prime (p, q have not common factor) Now we have
$\alpha = \frac{p^2}{q^2}$
$p^2 = \alpha q^2$

It shows that p^α is an even integer, since p is an integer it must be an even integer. So take $p = \alpha m$ for some integer m. Now we have

$4m^\alpha = 2\alpha^2$
$2m^\alpha = q^2$

It shows q^α is an even integer hence q is an even integer which contradicts our assumption that p, q are co-prime. Hence our assumption was wrong that $\sqrt{2}$ is an irrational number. $\sqrt{2}$ is an rational number.

5.5 Exercises

Q.1 Prove that $\sqrt{3}$ is an irrational number
Q.2 Prove that $\sqrt{5}$ is an irrational number
Q.3 Prove that $\sqrt{n}$ is an irrational number if n is not a perfect square.
Q.4 Prove that $\sqrt{3} + \sqrt{5}$ is an irrational number.
Q.5 Using the fact that negative of rational number is also a rational number prove that negative of irrational number is an irrational number.
Q.6 Prove that the sum of rational number and irrational number is an irrational number.
Q.7 Prove a disprove that sum of two irrational number is an irrational number.
Q.8 Prove that $\sqrt{ab}$ lies between two positive numbers between a and b ($a < b$). Hence find 2 irrational numbers between

(i) 2 and 3 (ii) $\sqrt{2}$ and $\sqrt{3}$ (iii) 2 and $\sqrt{5}$

6 Primes and Composites

Definition : A prime number (or simply prime) is a natural number $p > 1$ whose only positive divisors 1 and p.

Definition : A natural number greater than 1 which is not prime is called a composite number (or simply composite).

Fact : 1 is the only natural number which is neither prime nor composite.
Fact : If none of the prime number upto $\sqrt{n}$ divides n then n is a prime number. (Try to prove it yourself).

6.1 Exercises

Q.1 Identify which of the following is prime?

(i) 197
(ii) 297
(iii) 499
(iv) 553
(v) 773

Q.2 Find the smallest composite number that has no prime factors less than 10.
Q.3 What is the largest two-digit prime number whose digits are also each prime?
Q.4 How many prime number are perfect cubes?
Q.5 The number 13 is prime. If you reverse the digits you also obtain a prime number, 31. What is the larger of the pair of primes that satisfies this condition and has a sum of 110?
Q.6 What is the smallest prime divisor of 101729 + 729101?
Q.7 A group of 25 coins is arrange into three piles such that each pile contains a different prime number of coins. What is the greatest number of coins possible in any of the three piles?
Q.8 Is 9409 prime?
Q.9 What are the 5 smallest prime numbers greater than 1000?
Q.10 What is the first year in the twenty-first century that is a prime number?

Answer

Section 2

1)

a. $0.\overline{6}$ b. $-0.\overline{4}$ c. $-0.\overline{13}$ d. $-1.\overline{6692307}$ e. $0.\overline{437}$

Section 3

1) Solution :
Let $x = 0.\overline{4} = 0.444$
$10\ x = 4.444....$
$10\ x - x = 4$
$9x = 4 = x = \frac{4}{9}$

2) Solution :

(i) Let $x = 0.\overline{35}$
$x = 0.353535......(i)$

Here, we have two repeating digits after the decimal point. So, we multiply sides of (i) by $10^2 = 100$ to get
$100x = 35.3535...$ (ii)
Subtracting (i) from (ii), we get
$100xx = (35.3535...)(0.3535...)$
$99x = 35$

$X = \dfrac{35}{99}$

Hence, $0.\overline{35} = \dfrac{35}{99}$

(ii) Let $x = 0.\overline{585}$

$x = 0.585585585...$

Here, we have three repeating digits after the decimal point. So, we multiply both sides of (i) by $10^3 = 1000$ to get

$1000x = 585.585585...$

Subtracting (i) from (ii), we get

$1000xx = (585.585585...) - (0.585585585...)$

$1000xx = 585$

$999x = 585$

$x = \dfrac{585}{999}$

3) Solution

(i) Let $x = 0.3\overline{2}$

Clearly, there is just one digit on the right side of the decimal point which is without bar. So, we multiply both sides of x by 10 so that only the repeating decimal is left on the right side of the decimal point.

i.e $10x = 3.\overline{2}$

$\Rightarrow 10x = 3 + 0.\overline{2}$

$\Rightarrow 10x = 3 + \dfrac{2}{9}$

$\Rightarrow 10x = \dfrac{9 \times 3 + 2}{9} \Rightarrow 10x = \dfrac{29}{9} \Rightarrow x = \dfrac{29}{90}$

(ii) Let $x = 0.12\overline{3}$

Clearly, there are two digits on the righ side of the decimal point which are without bar. So, we multiply both sides of x by $102 = 100$ so that only the repeating decimal is left on the right side of the decimal point.

i.e $100x = 12.\overline{3}$

$\Rightarrow 100x = 12 + 0.\overline{3}$

$\Rightarrow 100x = 12 + \dfrac{3}{9}$

$\Rightarrow 100x = \dfrac{12 \times 9 + 3}{9}$

$\Rightarrow 100x = \dfrac{108 + 3}{9} \Rightarrow 100x = \dfrac{111}{9} \Rightarrow x = \dfrac{111}{900}$

(iii) Let $x = 0.003\overline{52}$

Clearly, there are three digits on the right side of the decimal point which are without bar. So, we multiply both sides of x by 103= 1000 so that only the repeating decimal is left on the right side of the decimal point.

i.e $1000x = 3.\overline{52}$

$\Rightarrow 1000x = 3 + 0.52$

$\Rightarrow 1000x = 3 + \dfrac{52}{99}$

$$\Rightarrow 1000x = \frac{3 \times 99 + 52}{99}$$

$$\Rightarrow 1000x = \frac{297 + 52}{99} \Rightarrow 1000x = \frac{349}{99} \Rightarrow x = \frac{340}{99000}$$

4) Solution

(i) $\frac{4}{9}$ (ii) $\frac{37}{99}$ (iii) $\frac{6}{11}$ (iv) $\frac{23}{37}$ (v) $\frac{376}{3}$ (vi) $\frac{43}{9}$ (vii) $\frac{43}{90}$