Board Level Exercise

Type (I) : Very Short Answer Type Questions : **[01 Mark Each]**

1. Write the principal value of $\sec^{-1}(-2)$.

2. If $\tan^{-1}\left(\sqrt{3}\right) + \cot^{-1}(x) = \frac{\pi}{2}$, find x.

Type (II) : Short Answer Type Questions : **[02 Marks Each]**

3. If $\sin^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then find x.

4. Solve for x : $\cos(2\sin^{-1}x) = \frac{1}{9}$, $x > 0$

Type (III) : Long Answer Type Questions: **[04 Mark Each]**

5. Solve the following for x : $\tan^{-1}\left[\frac{1+x}{1-x}\right] = \frac{\pi}{4} + \tan^{-1} x$, $0 < x < 1$.

6. Solve for x : $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$.

7. Prove the following : $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

8. Prove the following : $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

Type (IV) : Very Long Answer Type Questions: **[06 Mark Each]**

9. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, prove that $x + y + z = xyz$.

10. Prove that : $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$.

11. Solve the following for x : $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$

12. Solve for x : $\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}$; $\sqrt{6} > x > 0$.

13. Prove that : $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\tan^{-1}\frac{4}{3}$.

14. Prove that : $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$.

15. Solve for x : $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$, $-1 < x < 1$.

16. Solve for x : $\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right)$; $x > 0$.

17. Prove that : $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$.

18. Prove the following : $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$

Exercise # 1

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Definition, graphs and fundamentals

A-1. Find the simplified value of each of the following inverse trigonometric terms :

(i) $\sin^{-1}\left(-\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iii) $\text{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ (iv) $\sec^{-1}(-\sqrt{2})$

(v) $\cos^{-1}\left(-\frac{1}{2}\right)$

A-2. Find the simplified value of the following expressions :

(i) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ (ii) $\tan\left[\cos^{-1}\frac{1}{2} + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

(iii) $\sin^{-1}\left[\cos\left\{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]$

A-3. Draw the graph of the following functions :

(i) $y = \sin^{-1}(x+1)$ (ii) $y = \cos^{-1}(3x)$

(iii) $y = \tan^{-1}(2x-1)$

A-4. Solve the following inequalities :

(i) $\sin^{-1} x > -1$ (ii) $\cos^{-1} x < 2$

(iii) $\cot^{-1} x < -\sqrt{3}$

A-5_. (i) If $\sum_{i=1}^{n} \cos^{-1} \alpha_i = 0$, then find the value of $\sum_{i=1}^{n} i \cdot \alpha_i$

(ii) If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$, then show that $\sum_{i=1}^{2n} x_i = 2n$

Section (B) : Trig. (trig⁻¹x), trig⁻¹ (trig x) trig (–x)

B-1. Evaluate the following expressions :

(i) $\sin\left(\cos^{-1}\frac{3}{5}\right)$

(ii) $\tan\left(\cos^{-1}\frac{1}{3}\right)$

(iii) $\operatorname{cosec}\left(\sec^{-1}\frac{\sqrt{41}}{5}\right)$

(iv) $\tan\left(\operatorname{cosec}^{-1}\frac{65}{63}\right)$

(v) $\sin\left(\frac{\pi}{6}+\cos^{-1}\frac{1}{4}\right)$

(vi) $\cos\left(\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{3}\right)$

(vii) $\sec\left(\tan\left\{\tan^{-1}\left(-\frac{\pi}{3}\right)\right\}\right)$

(viii) $\cos \tan^{-1} \sin \cot^{-1}\left(\frac{1}{2}\right)$

B-2. Evaluate the following inverse trigonometric expressions :

(i) $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

(ii) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

(iii) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$

(iv) $\sec^{-1}\left(\sec\frac{7\pi}{4}\right)$

B-3. Find the value of the following inverse trigonometric expressions :

(i)_ $\sin^{-1}(\sin 4)$

(ii) $\cos^{-1}(\cos 10)$

(iii) $\tan^{-1}(\tan(-6))$

(iv) $\cot^{-1}(\cot(-10))$

(v) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\left(\cos\frac{9\pi}{10}-\sin\frac{9\pi}{10}\right)\right)$

B-4. Express $\sin^{-1}(\sin\theta)$, $\cos^{-1}(\cos\theta)$, $\tan^{-1}(\tan\theta)$ and $\cot^{-1}(\cot\theta)$ in terms of linear expression of θ for $\theta \in \left[\frac{3\pi}{2}, 3\pi\right]$

Section (C) : Property " $\frac{\pi}{2}$ " , Addition and subtraction rule, miscellaneous formula , summation of series

C-1. Find the value of following expressions :

(i) $\cot(\tan^{-1} a + \cot^{-1} a)$

(ii) $\sin(\sin^{-1}x + \cos^{-1}x)$, $|x| \le 1$

(iii) $\tan\left[\cos^{-1}\left(\frac{3}{4}\right)+\sin^{-1}\left(\frac{3}{4}\right)-\sec^{-1}3\right]$

C-2. **Prove that**

(i) $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\frac{77}{85}$

(ii) $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

(iii) $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cot^{-1} 3 = \frac{\pi}{4}$

(iv) $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

C-3. Simplify $\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$, if $x > y > 1$.

C-4. Find the value of $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$

Section (D) : Inverse trigonometric function Equations

D-1. Solve for x

(i) $\cos(2\sin^{-1}x) = \frac{1}{3}$ (ii) $\cot^{-1}x + \tan^{-1}3 = \frac{\pi}{2}$

D-2. **Solve the following equations :**

(i) $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$ (ii) $\sin^{-1}x + \sin^{-1}2x = \frac{2\pi}{3}$

D-3. **Solve the following equations :**

(i) $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$, $(x > 0)$

(ii) $3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{3}\right)$

PART - II : OBJECTIVE QUESTIONS

*** Marked Questions may have more than one correct option.**

Section (A) : Definition, graphs and fundamentals

A-1. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is equal to

(A) 75° (B) 105° (C) $\frac{5\pi}{12}$ (D) $\frac{3\pi}{5}$

A-2. Domain of $f(x) = \cos^{-1}x + \cot^{-1}x + \mathrm{cosec}^{-1}x$ is
(A) $[-1, 1]$ (B) R (C) $(-\infty, -1] \cup [1, \infty)$ (D) $\{-1, 1\}$

A-3. Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is

(A) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (B) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (C) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (D) none of these

A-4. $\operatorname{cosec}^{-1}(\cos x)$ is real if

(A) $x \in [-1, 1]$ (B) $x \in R$

(C) x is an odd multiple of $\frac{\pi}{2}$ (D) x is a multiple of π

A-5. If $\cos[\tan^{-1}\{\sin(\cot^{-1}\sqrt{3})\}] = y$, then

(A) $y = \frac{4}{5}$ (B) $y = \frac{2}{\sqrt{5}}$ (C) $y = -\frac{2}{\sqrt{5}}$ (D) $y^2 = \frac{10}{11}$

A-6*. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then

(A) $x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} = 0$ (B) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$

(C) $x^{50} + y^{25} + z^{5} = 0$ (D) $\frac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$

A-7*. If α satisfies the inequation $x^2 - x - 2 > 0$, then a value exists for

(A) $\sin^{-1}\alpha$ (B) $\cos^{-1}\alpha$ (C) $\sec^{-1}\alpha$ (D) $\operatorname{cosec}^{-1}\alpha$

Section (B) : Trig. (trig^{-1}x), trig^{-1} (trig x), trig (–x)

B-1. If $\pi \le x \le 2\pi$, then $\cos^{-1}(\cos x)$ is equal to

(A) x (B) $\pi - x$ (C) $2\pi + x$ (D) $2\pi - x$

B-2. The numerical value of $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$ is

(A) $\frac{-7}{17}$ (B) $\frac{7}{17}$ (C) $\frac{17}{7}$ (D) $-\frac{2}{3}$

B-3. The value of $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right)$ is

(A) $\frac{6}{17}$ (B) $\frac{22}{7}$ (C) $\frac{19}{9}$ (D) $\frac{17}{6}$

B-4. The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is

(A) $-\frac{31}{32}$ (B) $\frac{3}{4}$ (C) $\frac{\sqrt{7}}{4}$ (D) $-\frac{3}{4}$

Section (C) : Property " $\frac{\pi}{2}$ " , Addition and subtraction rule, miscellaneous formula , summation of series

C-1. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y$ is equal to

(A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) π

C-2. If $x \geq 0$ and $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$, then

(A) $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$ (B) $0 \leq \theta \leq \frac{\pi}{4}$ (C) $0 \leq \theta < \frac{\pi}{2}$ (D) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

C-3. If $x < 0$ then value of $\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ is equal to

(A) $\frac{\pi}{2}$ (B) $-\frac{\pi}{2}$ (C) 0 (D) none of these

C-4. The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(A) $\frac{6}{17}$ (B) $\frac{7}{16}$ (C) $\frac{5}{7}$ (D) $\frac{17}{6}$

C-5. $\tan^{-1} a + \tan^{-1} b$, where $a > 0$, $b > 0$, $ab > 1$, is equal to

(A) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (B) $\tan^{-1}\left(\frac{a+b}{1-ab}\right) - \pi$ (C) $\pi + \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (D) $\pi - \tan^{-1}\left(\frac{a+b}{1-ab}\right)$

C-6. $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) none of these

C-7. $\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{5}{13}\right)$ is equal to

(A) $\cos^{-1}\left(\frac{33}{65}\right)$ (B) $\cos^{-1}\left(-\frac{33}{65}\right)$ (C) $\cos^{-1}\left(\frac{64}{65}\right)$ (D) none of these

Section (D) : Inverse trigonometric function Equations

D-1. The equation $\sin^{-1} x - \cos^{-1}x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has :

(A) no solution (B) unique solution
(C) infinite number of solutions (D) none of these

D-2. If $\sin^{-1}x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

(A) 0 (B) $\frac{1}{\sqrt{5}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{3}}{2}$

D-3. The solution of the equation $\sin^{-1}\left(\tan\frac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\frac{3}{x}}\right) - \frac{\pi}{6} = 0$ is

(A) x = 2 (B) x = – 4 (C) x = 4 (D) none of these

D-4*. If $6\sin^{-1}\left(x^2 - 6x + \frac{17}{2}\right) = \pi$, then

(A) x = 1 (B) x = 2 (C) x = 3 (D) x = 4

PART - III : ASSERTION / REASONING

1. **STATEMENT-1 :** If α, β are roots of $6x^2 + 11x + 3 = 0$ then $\cos^{-1}\alpha$ exist but not $\cos^{-1}\beta$, $(\alpha > \beta)$.
STATEMENT-2 : Domain of $\cos^{-1} x$ is [– 1, 1].
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

2. **STATEMENT-1 :** $\tan^2(\sec^{-1} 2) + \cot^2(\text{cosec}^{-1} 3) = 11$.
STATEMENT-2 : $\tan^2\theta + \sec^2\theta = 1 = \cot^2\theta + \text{cosec}^2\theta$.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

3. **STATEMENT-1 :** If a > 0, b > 0, $\tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) = \frac{\pi}{2}. \Rightarrow x = \sqrt{ab}$.

STATEMENT-2 : If $m, n \in N$, $n \geq m$, then $\tan^{-1}\left(\frac{m}{n}\right) + \tan^{-1}\left(\frac{n-m}{n+m}\right) = \frac{\pi}{4}$.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

Exercise # 2

PART - I : SUBJECTIVE QUESTIONS

1. Solve the following inequalities:
(i) $\cos^{-1} x > \cos^{-1} x^2$
(ii) $\tan^{-1} x > \cot^{-1} x$.
(iii) $\text{arccot}^2 x - 5\ \text{arccot}\, x + 6 > 0$

2. If X = cosec $\tan^{-1}\cos\cot^{-1}\sec\sin^{-1}$ a & Y = sec $\cot^{-1}\sin\tan^{-1}\text{cosec}\cos^{-1}$ a; where $0 \le a < 1$. Find the relation between X & Y. Express them in terms of 'a'.

3. Prove each of the following relations :

(i) $\tan^{-1} x = -\pi + \cot^{-1}\dfrac{1}{x} = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = -\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$ when $x < 0$.

(ii) $\cos^{-1}x = \sec^{-1}\dfrac{1}{x} = \pi - \sin^{-1}\sqrt{1-x^2} = \pi + \tan^{-1}\dfrac{\sqrt{1-x^2}}{x} = \cot^{-1}\dfrac{x}{\sqrt{1-x^2}}$ when $-1<x<0$

4. If $f(x) = \cos^{-1}x + \cos^{-1}\left\{\dfrac{x}{2}+\dfrac{1}{2}\sqrt{3-3x^2}\right\}$, then find the value of

(i) $f\left(\dfrac{2}{3}\right)$ (ii) $f\left(\dfrac{1}{3}\right)$:

5. If $a\sin^{-1} x - b\cos^{-1} x = c$, then find the value of $a\sin^{-1} x + b\cos^{-1} x$

6. Solve the following equation :

$\sec^{-1}\dfrac{x}{a} - \sec^{-1}\dfrac{x}{b} = \sec^{-1}b - \sec^{-1}a$ $a \ge 1$; $b \ge 1$, $a \ne b$.

7. Find the number of values of x satisfying the equation $\sin^2(2\cos^{-1}(\tan x)) = 1$.

8. Find the sum of each of the following series :

(i) $\tan^{-1}\dfrac{1}{x^2+x+1} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \tan^{-1}\dfrac{1}{x^2+5x+7} + \tan^{-1}\dfrac{1}{x^2+7x+13}$ .... upto n terms.

(ii) $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{2}{9} + \ldots + \tan^{-1}\dfrac{2^{n-1}}{1+2^{2n-1}} + \ldots$ upto infinite terms

(iii) $\sin^{-1}\dfrac{1}{\sqrt{2}} + \sin^{-1}\dfrac{\sqrt{2}-1}{\sqrt{6}} + \ldots + \sin^{-1}\dfrac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + \ldots$ upto infinite terms

9. (i) Find all positive integral solutions of the equation, $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3$.
(ii) If 'k' be a positive integer, then show that the equation:
$\tan^{-1} x + \tan^{-1} y = \tan^{-1} k$ has no non–zero integral solution.

10. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, where $-1 \le x, y, z \le 1$, then find the value of $x^2 + y^2 + z^2 + 2xyz$

11. Determine the integral values of 'k' for which the system, $(\tan^{-1} x)^2 + (\cos^{-1} y)^2 = \pi^2 k$ and $\tan^{-1} x + \cos^{-1} y = \dfrac{\pi}{2}$ possess solution and find all the solutions.

PART - II : OBJECTIVE QUESTIONS

Single choice type

1. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right) + \tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)$, $x \neq 0$ is equal to

(A) x (B) 2x (C) $\frac{2}{x}$ (D) $\frac{x}{2}$

2. The value of $\sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}]$, where $x \in \left(\frac{\pi}{2}, \pi\right)$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $-\frac{\pi}{4}$ (D) $-\frac{\pi}{2}$

3. If $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = 4°$, then:

(A) x = tan 2° (B) x = tan 4° (C) x = tan (1/4)° (D) x = tan 8°

4. The value of $\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\}$, where $\frac{\pi}{2} < x < \pi$, is:

(A) $\pi - \frac{x}{2}$ (B) $\frac{\pi}{2} + \frac{x}{2}$ (C) $\frac{x}{2}$ (D) $2\pi - \frac{x}{2}$

5. The number of solution(s) of the equation, $\sin^{-1}x + \cos^{-1}(1 - x) = \sin^{-1}(-x)$, is/are

(A) 0 (B) 1 (C) 2 (D) more than 2

6. The smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right)$, $0 \le x \le 1$ are

(A) $0, \pi$ (B) $0, \frac{\pi}{4}$ (C) $-\frac{\pi}{4}, \frac{\pi}{4}$ (D) $\frac{\pi}{4}, \frac{\pi}{2}$

7. If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of 'n' is:

(A) 1 (B) 5 (C) 9 (D) none of these

8. The complete solution set of the inequality $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$, where [.] denotes greatest integer function, is

(A) $(-\infty, \cot 3]$ (B) $[\cot 3, \cot 2]$ (C) $[\cot 3, \infty)$ (D) none of these

9. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \frac{\pi}{4}$, then $\tan\theta$ is equal to

(A) 1/3 (B) 3 (C) 1 (D) – 1

10. The set of values of 'x' for which the formula $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ is true, is

(A) (– 1, 0) (B) [0, 1] (C) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ (D) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

11. The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds for

(A) $x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$ (B) $x > 2 + \sqrt{9-2\pi}$

(C) $x < 2 - \sqrt{9-2\pi}$ (D) None of these

12. The number of real solutions of equation $\sqrt{1+\cos 2x} = \sqrt{2}\ \sin^{-1}(\sin x),\ -\pi \le x \le \pi$, is

(A) 1 (B) 2 (C) 3 (D) 4

More than one choice type

13. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left\{\cos\left(-\frac{14\pi}{5}\right)\right\}\right]$ is:

(A) $\cos\left(-\frac{7\pi}{5}\right)$ (B) $\sin\left(\frac{\pi}{10}\right)$ (C) $\cos\left(\frac{2\pi}{5}\right)$ (D) $-\cos\left(\frac{3\pi}{5}\right)$

14. $\sin^{-1} x > \cos^{-1} x$ holds for

(A) all values of x (B) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$ (C) $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ (D) $x = 0.75$

15. If $0 < x < 1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to:

(A) $\frac{1}{2}\cos^{-1}x$ (B) $\cos^{-1}\sqrt{\frac{1+x}{2}}$ (C) $\sin^{-1}\sqrt{\frac{1-x}{2}}$ (D) $\frac{1}{2}\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

16. If $\cos^{-1}x = \tan^{-1}x$, then

(A) $x^2 = \left(\frac{\sqrt{5}-1}{2}\right)$ (B) $x^2 = \left(\frac{\sqrt{5}+1}{2}\right)$

(C) $\sin(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$ (D) $\tan(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

17. $\sum_{n=1}^{\infty} \tan^{-1}\frac{4n}{n^4-2n^2+2}$ is equal to:

(A) $\tan^{-1}2 + \tan^{-1}3$ (B) $4\tan^{-1}1$ (C) $\pi/2$ (D) $\sec^{-1}\left(-\sqrt{2}\right)$

PART - III : MATCH THE COLUMN

1. Match the column

Column - I	**Column - II**
(A) Let a, b, c be three positive real numbers $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$ then θ equal	(p) π
(B) The value of the expression $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$ for $0 < A < (\pi/4)$	(q) $-\frac{\pi}{2}$
(C) If $x < -1$, then $\sin^{-1}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1}x$	(r) $-\pi$
(D) The value of $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{16}{65}\right)$	(s) $\frac{\pi}{2}$

PART - IV : COMPREHENSION

Comprehension # 1

Let the domain and range of inverse circular functions are defined as follows

	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

1. $\sin^{-1}x < \frac{3\pi}{4}$ then solution set of x is

(A) $\left(\frac{1}{\sqrt{2}}, 1\right]$ (B) $\left(-\frac{1}{\sqrt{2}}, -1\right]$ (C) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (D) none of these

2. $\sin^{-1}x + \text{cosec}^{-1}x$ at $x = -1$ is

(A) π (B) 2π (C) 3π (D) $-\pi$

3. If $x \in [-1, 1]$, then range of $\tan^{-1}(-x)$ is

(A) $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$ (B) $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ (C) $[-\pi, 0]$ (D) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Exercise # 3

PART - I : IIT-JEE PROBLEMS (PREVIOUS YEARS)

* **Marked Questions may have more than one correct option.**

1. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$ is:

[IIT-JEE – 1999, Part-1, (2, 0), 80]

(A) zero (B) one (C) two (D) infinite

2. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\ldots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\ldots\right) = \frac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then x equals

[IITJEE-2001, Scr. (1, 0), 35]

(A) 1/2 (B) 1 (C) – 1/2 (D) –1

3. Prove that, $\cos \tan^{-1} \sin \cot^{-1} x = \sqrt{\frac{x^2+1}{x^2+2}}$. **[IIT-JEE-2002, Main (5, 0), 60]**

4. The value of x for which $\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1} x)$ is **[IIT-JEE-2005, Scr. (3, –1), 84]**

(A) 1/2 (B) 1 (C) 0 (D) – 1/2

5. Match the column **[IIT-JEE-2007, Paper-2, (6, 0), 81]**

Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$

	Column – I		**Column – II**
(A)	If a = 1 and b = 0, then (x, y)	(p)	lies on the circle $x^2 + y^2 = 1$
(B)	If a = 1 and b = 1, then (x, y)	(q)	lies on $(x^2 - 1)(y^2 - 1) = 0$
(C)	If a = 1 and b = 2, then (x, y)	(r)	lies on y = x
(D)	If a = 2 and b = 2, then (x, y)	(s)	lies on $(4x^2 - 1)(y^2 - 1) = 0$

6. If 0 < x < 1, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ **[IIT-JEE 2008, Paper-1, (3, –1), 82]**

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

7. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[IIT-JEE 2011, Paper-1, (4, 0), 80]

8. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

(A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

9. Match List I with List II and select the correct answer using the code given below the lists :

	List - I		**List - II**
P	$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)^2 + y^4\right)^{1/2}$ takes value	1.	$\frac{1}{2}\sqrt{\frac{5}{3}}$
Q.	If cos x + cos y + cox z = 0 = sin x + sin y + sin z then possible value of $\cos\frac{x-y}{2}$ is	2.	$\sqrt{2}$
R.	If $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$ then possible value of sec x is	3.	$\frac{1}{2}$
S.	If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \neq 0$, then possible value of x is	4.	1

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

Codes :

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

PART - II : AIEEE PROBLEMS (PREVIOUS YEARS)

1*. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ is equal to - **[AIEEE-2002]**

(1) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (2) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (3) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (4) $\tan^{-1}\frac{1}{2}$

2. $\cot^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$. then sin x is equal to - **[AIEEE-2002]**

(1) $\tan^2\left(\frac{\alpha}{2}\right)$ (2) $\cot^2\left(\frac{\alpha}{2}\right)$ (3) $\tan\alpha$ (4) $\cot\left(\frac{\alpha}{2}\right)$

3. The Inverse trigonometric equation $\sin^{-1}x = 2\sin^{-1}\alpha$, has a solution for **[AIEEE-2003]**

(1) $-\frac{1}{2} < \alpha < \frac{1}{2}$ (2) all real values of α (3) $|\alpha| \le \frac{1}{\sqrt{2}}$ (4) $|\alpha| \ge \frac{1}{\sqrt{2}}$

4. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to- **[AIEEE-2005, (3, 0)/225]**

(1) $2\sin 2\alpha$ (2) 4 (3) $4\sin^2\alpha$ (4) $-4\sin^2\alpha$

5. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$ then a value of x is- **[AIEEE-2007, (3, –1), 120]**

(1) 1 (2) 3 (3) 4 (4) 5

6. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is **[AIEEE 2008 (3, –1), 105]**

(1) $\frac{3}{17}$ (2) $\frac{2}{17}$ (3) $\frac{5}{17}$ (4) $\frac{6}{17}$

7. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[AIEEE - 2013, (4, –¼),360]**

(1) x = y = z (2) 2x = 3y = 6z (3) 6x = 3y = 2z (4) 6x = 4y = 3z

PART - III : CBSE PROBLEMS (PREVIOUS YEARS)

1. Find the principal value of $\cot^{-1}(-\sqrt{3})$. **[CBSE 2004, 2000]**

2. Prove the following : $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$. **[CBSE 2004]**

3. Write the following functions in the simplest form : $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$, $0 < x < \pi$. **[CBSE 2005]**

4. Show that $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$. **[CBSE 2005]**

5. Show that : $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$. **[CBSE 2005]**

6. Simplify : $\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$ **[CBSE 2006, 2005]**

7. Prove that : $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ **[CBSE 2007]**

8. Write into the simplest form : $\cot^{-1}(\sqrt{1+x^2} - x)$. **[CBSE 2007, 2003]**

9. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$; $x > 0$ **[CBSE 2009, 2008, 2006]**

10. Prove the following : $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$ **[CBSE 2010, 2009]**

11. Solve for x : $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\tan^{-1} x = 0,\ x > 0$ **[CBSE 2010, 2009, 2008]**

12. Find the value of $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$. **[CBSE 2010, 2008]**

13. Prove the following : $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$. **[CBSE 2010, 2008]**

14. Solve for x : $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$ **[CBSE 2010, 2009, 2008, 2005]**

15. Prove the following : $\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ |x| < \frac{1}{\sqrt{3}}$. **[CBSE 2010, 2001, 2000]**

16. Prove the following : $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$ **[CBSE 2010]**

17. Prove that following : $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in (0, 1)$. **[CBSE 2010]**

18. Find the value of the following : $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$. **[CBSE 2010, 2007]**

19. Prove the following ; $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$ **[CBSE 2010]**

20. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. **[CBSE 2011, 2010, 2009]**

21. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$. **[CBSE 2011, 2010, 2008, 2004, 2003, 2002, 2001, 2000]**

22. Prove the following $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] = \frac{x}{2},\ x \in \left(0, \frac{\pi}{4}\right)$ **[CBSE 2011, 2009, 2007, 2006]**

23. Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ **[CBSE 2011]**

24. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$. **[CBSE 2011, 2008]**

25. What is the principle value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ **[CBSE 2011, 2009, 2008]**

26. Prove that : $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}} \le x \le 1$ **[CBSE 2011, 2010, 2006]**

27. Prove the following : $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$ **[CBSE 2011, 2009, 2008, 2006]**

28. Prove that : $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{1}\left(\frac{1}{5}\right) + \tan^{1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$ **[CBSE 2011, 2008]**

29. Using the principal values, evaluate the following : $\tan^{-1} 1 + \sin^{-1}\left(-\frac{1}{2}\right)$ **[CBSE 2012, 2009]**

30. Solve for x : $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$. **[CBSE 2012, 2009, 2006]**

31. Prove that $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **[CBSE 2012, 2002]**

32. Prove that $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$. **[CBSE 2012, 2010]**

33. Prove that $\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$ **[CBSE 2012, 2009, 2006]**

34. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ **[CBSE 2013, 1]**

35. Write the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$. **[CBSE 2013, 1]**

36. Find the value of the following : **[CBSE 2013, 4]**

$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$,

$|x| < 1$, $y > 0$ and $xy < 1$

or

$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Answers

BOARD LEVEL SOLUTIONS

1. We have $\sec^{-1}(-2) = \pi - \sec^{-1}(2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

2. $\tan^{-1}\left(\sqrt{3}\right) + \cot^{-1}(x) = \frac{\pi}{2}$

$\Rightarrow \cot^{-1} x = \frac{\pi}{2} - \tan^{-1}\left(\sqrt{3}\right)$

$\Rightarrow \cot^{-1}x = \cot^{-1}\sqrt{3} \quad \therefore\ x = \sqrt{3}$

3. We have $\sin^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$\sin^{-1}(x) = \frac{\pi}{2} - \cos^{-1}\left(\frac{1}{2}\right)$

$\sin^{-1}x = \sin^{-1}\left(\frac{1}{2}\right) \qquad \therefore\ x = \frac{1}{2}$

4. The given equation is

$\cos(2\sin^{-1}x) = \frac{1}{9} \qquad (x > 0) \quad ...(i)$

Put : $\sin^{-1}x = \theta \ \Rightarrow\ x = \sin\theta$

$\therefore$ Equation (i) $\Rightarrow \cos 2\theta = \frac{1}{9}$

$\Rightarrow\ 1 - 2\sin^2\theta = \frac{1}{9}$

$\Rightarrow 2\sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$

$\Rightarrow x^2 = \frac{4}{9} \ \Rightarrow\ x = \frac{2}{3} \qquad (\because x > 0)$

5. Here $\tan^{-1}\left[\frac{1+x}{1-x}\right] = \frac{\pi}{4} + \tan^{-1}x$

$\tan^{-1}\left[\frac{1+x}{1-x}\right] - \tan^{-1} x = \frac{\pi}{4}$

$[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}]$

$\tan^{-1}\left[\frac{\frac{1+x}{1-x} - x}{1 + \frac{1+x}{1-x}\cdot x}\right] = \frac{\pi}{4}$

$\frac{\frac{1+x-x(1-x)}{1-x}}{\frac{(1-x)+x(1+x)}{1-x}} = \tan\frac{\pi}{4}$

$\frac{1+x-x+x^2}{1-x} \times \frac{1-x}{1-x+x+x^2} = 1$

$\frac{1+x^2}{1+x^2} = 1$

$1 + x^2 = 1 + x^2$

$1 = 1$

$\therefore$ Equation has infinitely many solutions

6. We have $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$

$\cos^{-1}x = \frac{\pi}{6} - \sin^{-1}x$

$\Rightarrow x = \cos\left(\frac{\pi}{6} - \sin^{-1}\frac{x}{2}\right)$

$= \cos\frac{\pi}{6}\cos\left(\sin^{-1}\frac{x}{2}\right) + \sin\frac{\pi}{6}\sin\left(\sin^{-1}\frac{x}{2}\right)$

$\Rightarrow x = \frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1-\frac{x^2}{4}}\right) + \frac{1}{2}\cdot\frac{x}{2}$

$\Rightarrow x = \frac{\sqrt{3}}{2}\sqrt{1-\frac{x^2}{4}} + \frac{x}{4}$

$\Rightarrow x - \frac{x}{4} = \frac{\sqrt{3}}{2}\sqrt{1-\frac{x^2}{4}}$

$\Rightarrow \frac{3x}{4} = \frac{\sqrt{3}}{2}\sqrt{1-\frac{x^2}{4}} \qquad \Rightarrow \frac{9x^2}{16} = \frac{3}{4}\left(1-\frac{x^2}{4}\right)$

$\Rightarrow \frac{3x^2}{4} = 1 - \frac{x^2}{4} \qquad \Rightarrow \frac{3x^2}{4} + \frac{x^2}{4} = 1$

$\Rightarrow \frac{4x^2}{4} = 1$

$\Rightarrow x^2 = 1 \qquad \therefore\ x = \pm 1 \qquad$ But $x = 1$

7. L.H.S. $= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2} + \tan^{-1}\frac{1}{7} \ [\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)]$

$= \tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right) + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right) + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$

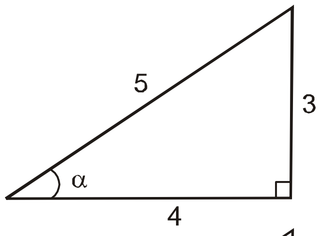
$= \tan^{-1}\left(\dfrac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right) = \tan^{-1}\left(\dfrac{\frac{21+4}{28}}{\frac{28-3}{28}}\right)$

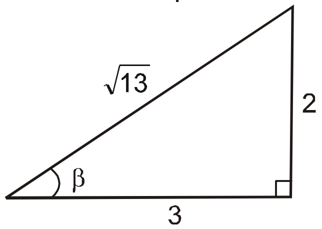
$= \tan^{-1}\left(\frac{25}{28}\times\frac{28}{25}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

8. Consider L.H.S. $= \cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$ and $\beta = \cot^{-1}\frac{3}{2}$

$\Rightarrow \sin\alpha = \frac{3}{5}$ and $\cot\beta = \frac{3}{2}$





$\Rightarrow \cos\alpha = \frac{4}{5}$, $\sin\beta = \frac{2}{\sqrt{13}}$, $\cos\beta = \frac{3}{\sqrt{13}}$

L.H.S.$= \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{4}{5}\times\frac{3}{\sqrt{13}} - \frac{3}{5}\times\frac{2}{\sqrt{13}} = \frac{12-6}{5\sqrt{13}} = \frac{6}{5\sqrt{13}} =$ R.H.S.

9. We have $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

$\tan^{-1}x + \tan^{-1}y = \pi - \tan^{-1}z$

$\tan^{-1}\frac{x+y}{1-xy} = \tan^{-1}z$

$[\because \tan^{-1}(-\theta) = \pi - \tan^{-1}\theta]$

$\frac{x+y}{1-xy} = -z$

$x + y = -z + xyz$

$x + y + z = xyz$

10. Let $x = \sin^{-1}\frac{4}{5} \Rightarrow \sin x = \frac{4}{5}$

and $y = \sin^{-1}\frac{5}{13} \Rightarrow \sin y = \frac{5}{13}$

$\therefore \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\because \cos(x + y) = \cos x\cos y - \sin x\sin y$

$\therefore \cos(x + y) = \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13}$

$\cos(x + y) = \frac{36}{65} - \frac{20}{65}$

$\cos(x + y) = \frac{16}{65}$

$\therefore (x + y) = \cos^{-1}\left(\frac{16}{65}\right)$

$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{16}{65}\right)$

$\therefore \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$

11. $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$

$\tan^{-1}x + 2\tan^{-1}\frac{1}{x} = \frac{2\pi}{3}$

$\tan^{-1}x + \tan^{-1}\left[\dfrac{2.\frac{1}{x}}{1-\left(\frac{1}{x}\right)^2}\right] = \frac{2\pi}{3}$

$\tan^{-1}x + \tan^{-1}\left[\dfrac{\frac{2}{x}}{\frac{x^2-1}{x^2}}\right] = \frac{2\pi}{3}$

$\tan^{-1}x + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$

$\tan^{-1}\left[\dfrac{x+\frac{2x}{x^2-1}}{1-x.\frac{2}{x^2-1}}\right] = \frac{2\pi}{3}$

$\tan^{-1}\left[\dfrac{x^3-x+2x}{x^2-1-2x^2}\right] = \frac{2\pi}{3}$

$\Rightarrow \tan^{-1}\left(\dfrac{x^3+x}{-x^2+1}\right) = \frac{2\pi}{3} \Rightarrow \tan^{-1}\left[\dfrac{x(x^2+1)}{-(x^2+1)}\right]$

$= \frac{2\pi}{3}$

$\Rightarrow \tan^{-1}(-x) = \frac{2\pi}{3}$

$\Rightarrow -x = \tan\frac{2\pi}{3} \Rightarrow -x = -\sqrt{3} \qquad \therefore\ x = \sqrt{3}$

12. We have $\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{x}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x}{2}+\frac{x}{3}}{1-\frac{x}{2}.\frac{x}{3}}\right] = \frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{3x+2x}{6}}{\frac{6-x^2}{6}} = \tan\frac{\pi}{4} \Rightarrow \frac{5x}{6-x^2} = 1$$

$\Rightarrow 5x = 6 - x^2 \Rightarrow x^2 - 5x + 6 = 0$
$\Rightarrow (x - 1)(x + 6) = 0 \Rightarrow x = 1, x = -6$
As $x > 0 \quad \therefore x = 1$

13. To prove : $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\tan^{-1}\frac{4}{3}$

$$\Leftrightarrow 2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right) = \tan^{-1}\frac{4}{3}$$

Now L.H.S. $= 2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right)$

$$= 2\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right) = 2\tan^{-1}\frac{17}{34}$$

$$= 2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}\times\frac{1}{2}}\right) = \tan^{-1}\frac{4}{3} = \text{R.H.S.}$$

14. L.H.S. $= 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{4}}{1-\frac{3}{4}\cdot\frac{3}{4}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{3/2}{1-\frac{9}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}.\frac{17}{31}}\right) = \tan^{-1}\left(\frac{24\times31-17\times7}{7\times31+24\times17}\right)$$

$$= \tan^{-1}\left(\frac{744-119}{217+408}\right) = \tan^{-1}\left(\frac{625}{625}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

= R.H.S.

15. The given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}, -1 < x < 1$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$$

$$\left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \frac{2x}{1-x^2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$\Rightarrow 2\sqrt{3}\,x = 1 - x^2$

$\Rightarrow x^2 + 2\sqrt{3}\,x - 1 = 0$

$$\Rightarrow x = \frac{-2\sqrt{3}\pm\sqrt{12+4}}{2} = -\sqrt{3} + 2$$

$= 2-\sqrt{3}$ (Reject $-\sqrt{3} - 2$ as $-1 < x < 1$)

16. We have $\tan^{-1}(x + 2) + \tan^{-1}(x - 2) = \tan^{-1}\left(\frac{8}{79}\right)$

$$\Rightarrow \tan^{-1}\left[\frac{(x+2)+(x-2)}{1-(x+2)(x-2)}\right] = \tan^{-1}\left(\frac{8}{79}\right)$$

$$\Rightarrow \frac{2x}{1-(x^2-4)} = \frac{8}{79} \Rightarrow \frac{2x}{1-x^2+4} = \frac{8}{79}$$

$$\Rightarrow \frac{x}{5-x^2} = \frac{4}{79} \Rightarrow 79x = 20 - 4x^2$$

$\Rightarrow 4x^2 + 79x - 20 = 0$
$\Rightarrow 4x^2 + 80x - x - 20 = 0$
$\Rightarrow 4x(x + 20) - 1(x + 20) = 0$
$\Rightarrow (x + 20)(4x - 1) = 0$

$\Rightarrow x = -20$ or $x = \dfrac{1}{4}$

Since $x > 0 \;\therefore\; x = \dfrac{1}{4}$

17. L.H.L. $= \tan^{-1} + \tan^{-1}2 + \tan^{-1}3$

$$= \frac{\pi}{4} + \frac{\pi}{2} - \cot^{-1}2 + \frac{\pi}{2} - \cot^{-1}3$$

$$= \frac{\pi + 2\pi + 2\pi}{4} - \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}$$

$$= \frac{5\pi}{4} - \left[\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}\right] = \frac{5\pi}{4} - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \frac{5\pi}{4} - \tan^{-1}(1) = \frac{5\pi}{4} - \frac{\pi}{4} = \frac{4\pi}{4} = \pi = \text{R.H.S.}$$

18. L.H.S. $= \tan^{-1}\left(\dfrac{3}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) - \tan^{-1}\left(\dfrac{8}{19}\right)$

$$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\cdot\frac{3}{5}}\right] - \tan^{-1}\left(\frac{8}{19}\right)$$

$$= \tan^{-1}\left[\frac{\frac{15+12}{20}}{1-\frac{9}{20}}\right] - \tan^{-1}\left(\frac{8}{19}\right)$$

$$= \tan^{-1}\left[\frac{27}{20}\times\frac{20}{11}\right] - \tan^{-1}\left(\frac{8}{19}\right)$$

$$= \tan^{-1}\frac{27}{11} - \tan^{-1}\frac{8}{19} = \tan^{-1}\left[\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\cdot\frac{8}{19}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right] = \tan^{-1}\left[\frac{425}{209}\times\frac{209}{425}\right]$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

EXERCISE # 1

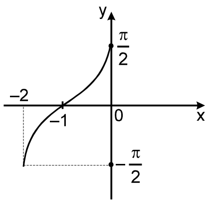
PART - I

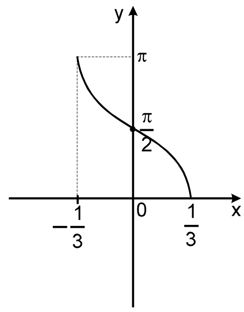
Section (A) :

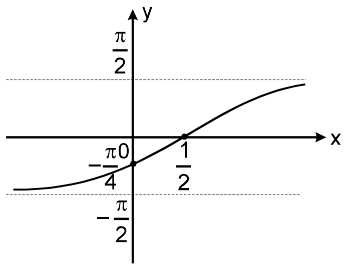
A-1. (i) $-\dfrac{\pi}{6}$ (ii) $\dfrac{\pi}{6}$ (iiii) $-\dfrac{\pi}{3}$ (iv) $\dfrac{3\pi}{4}$

(v) $\dfrac{2\pi}{3}$

A-2. (i) 1 (ii) $\dfrac{1}{\sqrt{3}}$ (iii) $\dfrac{\pi}{6}$

A-3. (i)


(ii)


(iii)


A-4. (i) $-\sin 1 < x \le 1$ (ii) $\cos 2 < x \le 1$
(iii) no solution

A-5_. (i) $n\left(\dfrac{n+1}{2}\right)$

Section (B) :

B-1. (i) $\dfrac{4}{5}$ (ii) $2\sqrt{2}$ (iii) $\dfrac{\sqrt{41}}{4}$ (iv) $\dfrac{63}{16}$

(v) $\dfrac{1+3\sqrt{5}}{8}$ (vi) $\dfrac{6-4\sqrt{5}}{15}$

(vii) 2 (viii) $\dfrac{\sqrt{5}}{3}$

B-2. (i) $-\dfrac{\pi}{6}$ (ii) $-\dfrac{\pi}{3}$ (iii) $\dfrac{3\pi}{4}$ (iv) $\dfrac{\pi}{4}$

B-3. (i) $\pi - 4$ (ii) $4\pi - 10$ (iii) $2\pi - 6$

(iv) $4\pi - 10$ (v) $\dfrac{17\pi}{20}$

B-4. $\sin^{-1}(\sin\theta) = \begin{cases} \theta - 2\pi, & \dfrac{3\pi}{2} \le \theta \le \dfrac{5\pi}{2} \\ 3\pi - \theta, & \dfrac{5\pi}{2} < \theta \le 3\pi \end{cases}$;

$\cos^{-1}(\cos\theta) = \begin{cases} 2\pi - \theta, & \dfrac{3\pi}{2} \le \theta < 2\pi \\ \theta - 2\pi, & 2\pi \le \theta \le 3\pi \end{cases}$;

$\tan^{-1}(\tan\theta) = \begin{cases} \theta - 2\pi, & \dfrac{3\pi}{2} < \theta < \dfrac{5\pi}{2} \\ \theta - 3\pi, & \dfrac{5\pi}{2} < \theta \le 3\pi \end{cases}$;

$\cot^{-1}(\cot\theta) = \begin{cases} \theta - \pi, & \dfrac{3\pi}{2} \le \theta < 2\pi \\ \theta - 2\pi, & 2\pi < \theta < 3\pi \end{cases}$

Section (C) :

C-1. (i) 0 (ii) 1 (iii) $\dfrac{1}{2\sqrt{2}}$

C-3. $\dfrac{1+xy}{x-y}$ **C-4.** $\dfrac{\pi}{2}$

Section (D) :

D-1. (i) $\pm\dfrac{1}{\sqrt{3}}$ (ii) $x = 3$

D-2. (i) $\pm\dfrac{1}{\sqrt{2}}$ (ii) $x = \dfrac{1}{2}$

D-3. (i) $x = \dfrac{1}{\sqrt{3}}$ (ii) $x = 2$

PART - II

Section (A) :

A-1. (B) **A-2.** (D) **A-3.** (C) **A-4.** (D)

A-5. (B) **A-6*.** (AB) **A-7*.** (CD)

Section (B) :

B-1. (D) **B-2.** (A) **B-3.** (D) **B-4.** (B)

Section (C) :

C-1. (B) **C-2.** (D) **C-3.** (B) **C-4.** (D)

C-5. (C) **C-6.** (A) **C-7.** (B)

Section (D) :

D-1. (B) **D-2.** (B) **D-3.** (C) **D-4*.** (BD)

PART - II

1. (A) **2.** (D) **3.** (B)

EXERCISE # 2

PART - I

1. (i) $[-1, 0)$ (ii) $x > 1$
(iii) $(-\infty, \cot 3) \cup (\cot 2, \infty)$

2. $X = Y = \sqrt{3-a^2}$ **4.** (i) $\dfrac{\pi}{3}$ (ii) $2\cos^{-1}\dfrac{1}{3} - \dfrac{\pi}{3}$

5. $\dfrac{\pi ab + c(a-b)}{a+b}$ **6.** $x = ab$ **7.** Infinite

8. (i) $\tan^{-1}(x+n) - \tan^{-1}x$ (ii) $\dfrac{\pi}{4}$ (iii) $\dfrac{\pi}{2}$

9. (i) Two solutions (1, 2) (2, 7)

10. 1

11. $k = 1$, $x = \tan(1-\sqrt{7})\dfrac{\pi}{4}$, $y = \cos(\sqrt{7}+1)\dfrac{\pi}{4}$

PART - II

1. (C) **2.** (D) **3.** (D) **4.** (B) **5.** (B) **6.** (B)

7. (B) **8.** (A) **9.** (B) **10.** (D) **11.** (A) **12.** (B)

13. (BCD) **14.** (CD) **15.** (ABC)

16. (AC) **17.** (AD)

PART - III

1. (A) $\rightarrow$ (p), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s)

PART - IV

1. (A) **2.** (C) **3.** (B)

EXERCISE # 3

PART - I

1. (C) **2.** (B) **4.** (D)

5. (A) $\rightarrow$ (p), (B) $\rightarrow$ (q), (C) $\rightarrow$ (p), (D) $\rightarrow$ (s)

6. (C) **7.** 1 **8.** (B) **9.** (B)

PART - II

1. (1,4) **2.** (1) **3.** (3) **4.** (3) **5.** (2) **6.** (4) **7.** (1)

PART - III

1. Let $x = \cot^{-1}(-\sqrt{3})$

$\cot x = -\sqrt{3} \Rightarrow \cot x = -\cot\frac{\pi}{3}$

$\cot x = \cot\left(\pi - \frac{\pi}{3}\right)$ $[\because \cot(\pi - \theta) = -\cot\theta]$

$\cot x = \cot\frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}$

$\therefore$ principal value of $\cot^{-1}(-\sqrt{3})$ is $\frac{2\pi}{3}$.

2. RHS = $\cos^{-1}(4x^3 - 3x)$
Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$
$\therefore$ RHS = $\cos^{-1}(4\cos^3\theta - 3\cos\theta)$
$= \cos^{-1}\cos 3\theta = 3\theta$
RHS = $3\cos^{-1}x$
$\therefore$ RHS = LHS

3. Let $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

$y = \tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}} = \tan^{-1}|\tan x/2|$

$\left[\because 0 < \frac{x}{2} < \frac{\pi}{2}\right]$

4. Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$

$\therefore \sin x = \frac{3}{5}$ $\sin y = \frac{8}{17}$

Now $\cos x$	$\cos y$
$= \sqrt{1-\sin^2 x}$	$\sqrt{1-\sin^2 y}$
$= \sqrt{1-\left(\frac{3}{5}\right)^2}$	$\sqrt{1-\left(\frac{8}{17}\right)^2}$
$= \sqrt{1-\left(\frac{9}{25}\right)}$	$\sqrt{1-\left(\frac{64}{289}\right)}$
$= \sqrt{\frac{16}{25}} = \frac{4}{5}$	$\sqrt{\frac{225}{289}} = \frac{15}{17}$

$\therefore \cos(x-y) = \cos x\cos y + \sin x\sin y$

$\cos(x-y) = \frac{4}{5}\cdot\frac{15}{17} + \frac{3}{5}\cdot\frac{8}{17} = \frac{60}{85} + \frac{24}{85}$

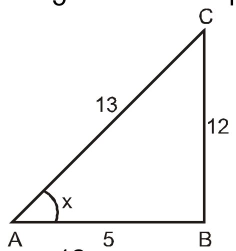
$\cos(x-y) = \frac{84}{85}$

$x - y = \cos^{-1}\frac{84}{85}$

$\therefore \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

5. Let $\sin^{-1}\frac{12}{13} = x$, $\cos^{-1}\frac{4}{5} = y$, and $\tan^{-1}\frac{63}{16} = z$

Then $\sin x = \frac{12}{13}$, $\cos y = \frac{4}{5}$ and $\tan z = \frac{63}{16}$



consider $\sin x = \frac{12}{13}$
In rt ΔABC, we have
$AC^2 = AB^2 + BC^2$
$169 = AB^2 + 144$
$\Rightarrow AB = 5$

$\therefore \cos x = \frac{5}{13}$ and $\tan x = \frac{12}{5}$

Similarly when $\cos y = \frac{4}{5}$ then $\sin y = \frac{3}{5}$ and $\tan y = \frac{3}{4}$

We have

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y} = \frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}$

$\tan(x+y) = \left[\frac{\frac{48+15}{20}}{\frac{20-36}{20}}\right]$

$\tan(x+y) = \frac{63}{20}\times\frac{20}{-16} = \frac{-63}{16} = -\tan z$

$\therefore \tan(x+y) = \tan(\pi - z)$
$x + y = \pi - z$
$x + y + z = \pi$

$\therefore \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

6. Let $y = \tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$

$= \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{a\cos x}}\right] = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$

$= \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$

$$\left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

$= \tan^{-1}\frac{a}{b} - x$

7. L.H.S. $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$

$= \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right] = \frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right)$

$$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$= \frac{9}{4}\sin^{-1}\sqrt{1-\left(\frac{1}{3}\right)^2}$

[$\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$ for $0 \le x \le 1$]

$= \frac{9}{4}\sin^{-1}\sqrt{1-\frac{1}{9}} = \frac{9}{4}\sin^{-1}\sqrt{\frac{8}{9}} = \frac{9}{4}\sin^{-1}\left(2\frac{\sqrt{2}}{3}\right)$

8. Let $y = \cot^{-1}(\sqrt{1+x^2} - x)$
Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$y = \cot^{-1}(\sqrt{1+\tan^2\theta} - \tan\theta)$
$y = \cot^{-1}(\sec\theta - \tan\theta)$

$y = \cot^{-1}\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)$

$y = \cot^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)$

$y = \cot^{-1}\left[\frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)}\right]$

$y = \cot^{-1}\left[\frac{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]$

$y = \cot^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$y = \cot^{-1}\cot\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}\right) \Rightarrow y = \frac{\pi}{4}+\frac{\theta}{2}$

$\therefore\ y = \frac{\pi}{4}+\frac{1}{2}\tan^{-1}x$

9. We have, $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

$\tan^{-1}\left(\frac{2x+3x}{1-2x\cdot 3x}\right) = \frac{\pi}{4}$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = 1$
$5x = 1 - 6x^2 \Rightarrow 6x^2 + 6x - x - 1 = 0$
$6x^2 + 6x - x - 1 = 0$
$6x(x+1) - 1(x+1) = 0$

$x = -1$, or $x = \frac{1}{6}$

Since $x = -1$ does not satisfy the equation as the L.H.S. of equation becomes negative.

Hence $x = \frac{1}{6}$ is the required solution.

10. Let $x = \cos^{-1}\frac{4}{5} \Rightarrow \cos x = \frac{4}{5}$

$\therefore\ \sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

and $y = \cos^{-1}\frac{12}{13} \Rightarrow \cos y = \frac{12}{13}$

$\therefore\ \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$= \frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$

$\therefore\ x + y = \cos^{-1}\left(\frac{33}{65}\right)$

Hence $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

11. Given Equation is $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\tan^{-1}x = 0 ; x > 0$

$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$ [$\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$]

$\tan^{-1}\left[\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right] = \tan^{-1}x \Rightarrow \frac{2\left(\frac{1-x}{1+x}\right)}{1-\frac{(1-x)^2}{(1+x)^2}} =$

$x = \frac{2\left(\frac{1-x}{1+x}\right)}{\frac{(1+x)^2-(1-x)^2}{(1+x)^2}} = x$

$$\frac{2(1-x)}{(1+x)} \times \frac{(1+x)^2}{1+x^2+2x-1-x^2+2x} = x$$

$$\frac{2(1-x)(1+x)}{4x} = x$$

$2x^2 = (1-x)(1+x) \Rightarrow 2x^2 = 1 - x^2$

$x^2 = \frac{1}{3}$ $\quad\therefore x = \pm\frac{1}{\sqrt{3}}$

12. We have $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$

$$= \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{5}\right)\right] = \sin^{-1}\left(\sin\frac{\pi}{5}\right) = \frac{\pi}{5}$$

13. $\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

L.H.S. $= \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$

$$= \tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\cdot\frac{1}{7}}\right] + \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\cdot\frac{1}{8}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{7+5}{35}}{1-\frac{1}{35}}\right] + \tan^{-1}\left[\frac{\frac{8+3}{24}}{1-\frac{1}{24}}\right]$$

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right) = \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left[\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\cdot\frac{11}{23}}\right] = \tan^{-1}\left[\frac{\frac{138+187}{17\times 23}}{1-\frac{66}{17\times 23}}\right]$$

$$= \tan^{-1}\left[\frac{325}{325}\right] = \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$$

14. We have,

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\cdot\frac{x+1}{x+2}}\right] = \frac{\pi}{4}$$

$$\left[\frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}}\right] = \tan\frac{\pi}{4}$$

$$\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1} = 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\frac{2x^2-4}{-3} = 1 \Rightarrow 2x^2 - 4 = -3$$

$2x^2 = -3 + 4 = 1 \Rightarrow 2x^2 = 1$

$x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

15. L.H.S. $= \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$= \tan^{-1}\left[\frac{x+\frac{2x}{1-x^2}}{1-x\frac{2x}{1-x^2}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x-x^3+2x}{1-x^2}}{\frac{1-x^2-2x^2}{1-x^2}}\right] = \tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] = \text{R.H.S.}$$

16. L.H.S. $= \cos[\tan^{-1}\sin(\cot^{-1}x)]$

Put $\cot^{-1}x = t \Rightarrow \cot t = x$

$\therefore \operatorname{cosec}^2 t = 1 + \cot^2 t$

$\operatorname{cosec} t = \sqrt{1+x^2}$

$\therefore \sin t = \frac{1}{\sqrt{1+x^2}}$

$\therefore$ L.H.S. $= \cos[\tan^{-1}\sin t] = \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right]$

Let $\tan^{-1}\frac{1}{\sqrt{1+x^2}} = z \Rightarrow \tan z = \frac{1}{\sqrt{1+x^2}}$

$\therefore \sec^2 z = 1 + \tan^2 z = 1 + \frac{1}{1+x^2} = \frac{2+x^2}{1+x^2}$

$\sec z = \sqrt{\frac{2+x^2}{1+x^2}} \Rightarrow \cos z = \sqrt{\frac{1+x^2}{2+x^2}}$

Hence L.H.S. $= \cos z = \sqrt{\frac{1+x^2}{2+x^2}}$ = R.H.S.

17. R.H.S. $= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

Put $x = \tan^2\theta \Rightarrow \tan\theta = \sqrt{x}$

$\therefore \theta = \tan^{-1}\sqrt{x}$

R.H.S. $= \frac{1}{2}(2\theta) = \theta = \tan^{-1}\sqrt{x}$

18. Here $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \left[\cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \left[\cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{4} + \frac{\pi}{2} \qquad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$= \frac{\pi + 2\pi}{4} = \frac{3\pi}{4}$$

19. Let $x = \cos^{-1}\frac{12}{13} \Rightarrow \cos x = \frac{12}{13}$

$$\therefore \sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

and let $y = \sin^{-1}\frac{3}{5} \Rightarrow \sin y = \frac{3}{5}$

$$\therefore \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$= \frac{5}{13}\cdot\frac{4}{5} + \frac{12}{13}\cdot\frac{3}{5} = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$

$$\therefore x + y = \sin^{-1}\frac{56}{65}$$

Hence, $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

20. We have $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

$$= \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right] = \cos^{-1}\cdot\cos\frac{5\pi}{6} = \frac{5\pi}{6}$$

21. Since $\sin^{-1}(-x) = -\sin^{-1}x$

$\therefore$ The principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

$$= -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}.$$

(The principal value of $\sin^{-1}x$ must lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

22. L.H.S. $= \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left(\frac{\sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} + \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} - \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}\right)$$

$$\because \left[\sin^2\frac{x}{2} + \cos\frac{x}{2}\right] = 1 \text{ and } \left[\sin x = 2\sin\frac{x}{2}\cdot\cos^2\frac{x}{2}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$$

$$= \cot^{-1}\left[\frac{\sin\frac{x}{2} + \cos\frac{x}{2} + \cos\frac{x}{2} - \sin\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2} - \cos\frac{x}{2} + \sin\frac{x}{2}}\right]$$

$$= \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] = \cot^{-1}\cdot\cot\frac{x}{2} = \frac{x}{2} = \text{R.H.S.}$$

23. L.H.S. $= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left[\frac{\frac{x}{y} - \left[\frac{x-y}{x+y}\right]}{1+\frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right]$$

$$= \tan^{-1}\left[\frac{x^2 + xy - xy + y^2/(x+y)y}{xy + y^2 + x^2 - xy/(x+y)y}\right]$$

$$= \tan^{-1}\left[\frac{x^2+y^2}{x^2+y^2}\right] = \tan^{-1}1 = \tan^{-1}\cdot\tan\frac{\pi}{4} = \frac{\pi}{4}$$

24. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\sin\frac{\pi}{6}\right)\right)$

$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{3\pi}{6} = \sin\frac{\pi}{2} = 1$$

25. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$$= \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right] + \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[-\cos\frac{\pi}{3}\right] + \sin^{-1}\left[\sin\frac{\pi}{3}\right]$$

[$\because$ $(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$]

$$= \pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right) + \frac{\pi}{3} = \pi - \frac{\pi}{3} + \frac{\pi}{3} = \pi$$

26. LHS $= \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

Let $\cos^{-1}x = \theta$, so that $x = \cos\theta$ and $0 \le \theta \le \frac{3\pi}{4}$

$\therefore \quad \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$= \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$

$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right]$

($\because$ $1 + \cos\theta = 2\cos^2(\theta/2)$ and $1 - \cos\theta = 2\sin^2(\theta/2)$)

$= \tan^{-1}\left[\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right]$

Inside the bracket divide numerator and denomerator by $\cos\frac{\theta}{2}$.

$= \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$

$= \frac{\pi}{4}-\frac{\theta}{2}\left(0 \le \theta \le \frac{3\pi}{4} \Rightarrow \frac{\pi}{4} \ge \frac{\pi}{4}-\frac{\theta}{2} \ge -\frac{\pi}{4}\right)$

$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$ = RHS

27. L.H.S. $= 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$= \tan^{-1}\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{1-\frac{1}{4}} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} \quad = \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}}\right]$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$

$= \tan^{-1}\left[\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right] = \tan^{-1}\left(\frac{31}{21}\times\frac{21}{17}\right)$

$= \tan^{-1}\frac{31}{17}$ = R.H.S.

28. L.H.S. $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}\right] + \tan^{-1}\left(\frac{1}{8}\right)$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$

$= \tan^{-1}\left[\frac{\frac{5+2}{10}}{\frac{10-1}{10}}\right] + \tan^{-1}\frac{1}{8} = \tan^{-1}\left[\frac{7}{10}\times\frac{10}{9}\right] + \tan^{-1}\frac{1}{8}$

$= \tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8} \quad = \tan^{-1}\left[\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}\right]$

$= \tan^{-1}\left[\frac{\frac{56+9}{72}}{\frac{72-7}{72}}\right] \quad = \tan^{-1}\left[\frac{65}{72}\times\frac{72}{65}\right] = \tan^{-1}(1) = \frac{\pi}{4}$

29. We have $\tan^{-1} 1 + \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{4} - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi-2\pi}{12} = \frac{\pi}{12}$

30. We have $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$\therefore \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$

$\frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$

$2\sin^2 x = 2\sin x\cos x$

$2\sin^2 x - 2\sin x\cos x = 0$

$2\sin x(\sin x - \cos x) = 0$

$\therefore \quad \sin x = 0$ or $\sin x - \cos x = 0$

$\Rightarrow \quad \sin x = \sin 0$ or $\sin x = \cos x$

$\Rightarrow\quad x = 0 \text{ or } \tan x = 1 = \tan\frac{\pi}{4}$

$\therefore\quad x = 0 \text{ or } x = \frac{\pi}{4}$

31. $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right\}$

$= \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}$

$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \frac{\pi}{4}-\frac{x}{2}$

32. $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$= \sin^{-1}\left[\frac{8}{17}\sqrt{1-\frac{9}{25}}+\frac{3}{5}\sqrt{1-\frac{64}{289}}\right]$

$= \sin^{-1}\left[\frac{8}{17}\times\frac{4}{5}+\frac{3}{5}\times\frac{15}{17}\right] = \sin^{-1}\left[\frac{32+45}{85}\right]$

$= \sin^{-1}\left[\frac{77}{85}\right]$

$= \cos^{-1}\sqrt{1-\left(\frac{77}{85}\right)^2} \qquad [\because \sin^{-1}x \cos^{-1}\sqrt{1-x^2}\,]$

$= \cos^{-1}\sqrt{\frac{7225-5929}{(85)^2}} = \cos^{-1}\sqrt{\frac{1296}{(85)^2}} = \cos^{-1}\left(\frac{36}{85}\right)$

33. Consider, R.H.S. $= \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

$= \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{4}{5}\right) \quad [\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\,]$

$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{16}{25}}+\frac{4}{5}\sqrt{1-\frac{25}{169}}\right]$

$[\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]]$

$= \sin^{-1}\left[\frac{5}{13}\times\frac{3}{5}+\frac{4}{5}\times\frac{12}{13}\right] = \sin^{-1}\left(\frac{63}{65}\right)$ = L.H.S.

34. Since $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$ for $|x| < 1$

so, $2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right) = \tan^{-1}\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right)$

$= \tan^{-1}\left(\frac{5}{12}\right)$

$\therefore\quad \tan\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{12}$

35. $\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right)$

$= \frac{\pi}{4}+\frac{2\pi}{3} = \frac{3\pi+8\pi}{12} = \frac{11\pi}{12}$

36. $\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$,

$= \tan\frac{1}{2}[2\tan^{-1}x + 2\tan^{-1}y] = \tan(\tan^{-1}x + \tan^{-1}y)$

$= \tan\left\{\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\} = \frac{x+y}{1-xy}$

OR

L.H.S. $= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right)+\tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{7}{10}}{1-\frac{1}{10}}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}\right) = \tan^{-1}\left(\frac{\frac{65}{72}}{\frac{65}{72}}\right) = \tan^{-1}(1)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$ = R.H.S.

Advanced Level Problems

PART - I : OBJECTIVE QUESTIONS

Single choice type

1. $\sin^{-1}\left(\frac{x^2}{4}+\frac{y^2}{9}\right) + \cos^{-1}\left(\frac{x}{2\sqrt{2}}+\frac{y}{3\sqrt{2}}-2\right)$ equals to :

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{\pi}{\sqrt{2}}$ (D) $\frac{3\pi}{2}$

2. If $a = \frac{1}{4} + i\,\frac{\sqrt{3}}{4}$ and $z = x + iy$, then $\sin^{-1}|z|^2 + \cos^{-1}(a\bar{z}+\bar{a}z-2)$ equals to :

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{2}$

Comprehension # 1 (Q.3 to 5)

(i) For any angle θ , there is an integer n such that

$$\sin^{-1}(\sin\theta) = \begin{cases} \theta-2n\pi & \text{if} \quad 2n\pi-\frac{\pi}{2}\le\theta\le 2n\pi+\frac{\pi}{2} \\ (2n+1)\pi-\theta & \text{if} \quad (2n+1)\pi-\frac{\pi}{2}\le\theta\le(2n+1)\pi+\frac{\pi}{2} \end{cases}$$

and $$\cos^{-1}(\cos\theta) = \begin{cases} \theta-2n\pi & \text{if} \quad 2n\pi\le\theta\le(2n+1)\pi \\ 2n\pi-\theta & \text{if} \quad (2n-1)\pi\le\theta\le 2n\pi \end{cases}$$

(ii) When $\theta \ne p\pi + \frac{\pi}{2}$, $p \in I$, we have $\tan^{-1}(\tan\theta) = \theta - n\pi$ if $n\pi - \frac{\pi}{2} < \theta < n\pi + \frac{\pi}{2}$

(iii) When $\theta \ne p\pi$, $p \in I$, we have $\cot^{-1}(\cot\theta) = \theta - n\pi$ if $n\pi < \theta < n\pi + \pi$

Read the above passage and answer the following

3. $\sin^{-1}(\sin 100) + \cos^{-1}(\cos 100) + \tan^{-1}(\tan 100) + \cot^{-1}(\cot 100)$ equals to :

(A) $100 - 31\pi$ (B) $100 - 32\pi$ (C) $200 - 63\pi$ (D) $200 - 32\pi$

4. if $\theta \in \left(\frac{15\pi}{2}, 8\pi\right)$, then $\sin^{-1}(\sin\theta) + \cos^{-1}(\cos\theta)$ equals to :

(A) 0 (B) $2\theta - 16\pi$ (C) 16π (D) 2θ

5. If $\theta \in \left(7\pi, \frac{15\pi}{2}\right)$, then $\sin^{-1}(\sin\theta) + \cos^{-1}(\cos\theta) + \tan^{-1}(\tan\theta) + \cot^{-1}(\cot\theta)$ equals to :

(A) 0 (B) π (C) $7\pi - \theta$ (D) $7\pi + \theta$

Comprehension # 2 (Q.6 to 10)

$$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} -\pi - 2\tan^{-1}x & \text{if} \quad x < -1 \\ 2\tan^{-1}x & \text{if} \quad -1 \le x \le 1 \\ \pi - 2\tan^{-1}x & \text{if} \quad x > 1 \end{cases}$$

$$\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases} \quad \text{and} \quad \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & \text{if} \quad x < -1 \\ 2\tan^{-1}x & \text{if} \quad -1 < x < 1 \\ -\pi + 2\tan^{-1}x & \text{if} \quad x > 1 \end{cases}$$

Using the above information solve each of the following

6. If $0 < x < 1$, then number of solutions of $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$ is

(A) 0 (B) 1 (C) 2 (D) 3

7. If $0 < x < 1$, then number of solutions of $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = -2\pi$ is

(A) 0 (B) 1 (C) 2 (D) 3

8. If $0 < x < 1$, then number of solutions of $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = -\pi$ is

(A) 0 (B) 1 (C) 2 (D) 3

9. If $-1 < x < 0$, then number of solutions of $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = \pi$ is

(A) 0 (B) 1 (C) 2 (D) 3

10. If $x > 1$, then number of solutions of $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = -\pi$ is

(A) 0 (B) 1 (C) 2 (D) 3

Comprehension # 3 (Q. NO. 11 to 13)

$$\tan^{-1}(\tan\theta) = \begin{cases} \pi + \theta & , \quad -\frac{3\pi}{2} < \theta < -\frac{\pi}{2} \\ \theta & , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\pi + \theta & , \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}, \quad \sin^{-1}(\sin\theta) = \begin{cases} -\pi - \theta & , \quad -\frac{3\pi}{2} \le \theta < -\frac{\pi}{2} \\ \theta & , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \pi - \theta & , \quad \frac{\pi}{2} < \theta \le \frac{3\pi}{2} \end{cases},$$

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \quad -\pi \le \theta < 0 \\ \theta & , \quad 0 \le \theta \le \pi \\ 2\pi - \theta & , \quad \pi < \theta \le 2\pi \end{cases}$$

Based on the above results, answer each of the following :

11. $\cos^{-1} x$ is equal to

(A) $\sin^{-1}\sqrt{1-x^2}$ if $-1 < x < 1$ (B) $-\sin^{-1}\sqrt{1-x^2}$ if $-1 < x < 0$

(C) $\sin^{-1}\sqrt{1-x^2}$ if $-1 < x < 0$ (D) $\sin^{-1}\sqrt{1-x^2}$ if $0 < x < 1$

12. $\sin^{-1} x$ is equal to

(A) $\cos^{-1}\sqrt{1-x^2}$ if $-1 < x < 0$ (B) $\cos^{-1}\sqrt{1-x^2}$ if $-1 < x < 1$

(C) $\cos^{-1}\sqrt{1-x^2}$ if $0 < x < 1$ (D) $-\cos^{-1}\sqrt{1-x^2}$ if $0 < x < 1$

13. $\cos^{-1} x$ is equal to

(A) $-\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$ (B) $\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$

(C) $-\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ if $0 < x < 1$ (D) $\pi + \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$

More than one choice type

14. $\tan^{-1}x + \tan^{-1}\dfrac{2x}{1-x^2}$ equals to :

(A) $\pi + 3\tan^{-1}x$ if $x < -1$ (B) $\pi - 3\tan^{-1}x$ if $x > 1$

(C) $3\tan^{-1}x$ if $-1 < x < 0$ (D) $-\pi + 3\tan^{-1}x$ if $0 < x < 1$

15. If $\sin^{-1} x + 2\cot^{-1}(y^2 - 2y) = 2\pi$, then

(A) $x + y = y^2$ (B) $x^2 = x + y$ (C) $y = y^2$ (D) $x^2 - x + y = y^2$

PART - II : SUBJECTIVE QUESTIONS

1. Find the sum of all the solutions of $\cot^{-1}(x-2) + \cot^{-1}(3-x) = \cot^{-1}(x-12)$.

2. If $\cos\theta = \dfrac{2}{3}$, where $\theta \in [31\pi, 32\pi]$, then find the value of θ.

3. If $x < 0$, then prove that $\cot^{-1}x = \pi + \tan^{-1}\dfrac{1}{x}$

4. Express $\cot(\text{cosec}^{-1}x)$ as an algebraic function of x.

5. Express $\sin^{-1}x$ in terms of (i) $\cos^{-1}\sqrt{1-x^2}$ (ii) $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ (iii) $\cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

6. If $\alpha = 2\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$. What the value of $\alpha + \beta$ will be if $x > 1$?

7. If $f(x) = e^{\cos^{-1}\sin\left(x+\frac{\pi}{3}\right)}$ then the value of $f\left(-\dfrac{7\pi}{4}\right)$ is

8. Solve $\{\cos^{-1}x\} + [\tan^{-1}x] = 0$ for real values of x. Where { . } and [.] are fractional part and greatest integer functions respectively.

9. Find the set of all real values of x satisfying the inequality $\sec^{-1}x > \tan^{-1}x$.

10. Prove that $\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) & \text{if } x+y \ge 0 \\ 2\pi - \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) & \text{if } x+y < 0 \end{cases}$.

11. By substituting $x = \cos\theta$, $0 \le \theta \le \pi$, express $\sin^{-1} 2x\sqrt{1-x^2}$ in terms of $\sin^{-1}x$

12. Express $\tan^{-1}x + \tan^{-1}\dfrac{2x}{1-x^2}$ in terms of $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

13. If $y - x < 0$, then prove that $\cot^{-1}x - \cot^{-1}y = -\pi + \cot^{-1}\dfrac{xy+1}{y-x}$.

14. Find the solution of $\sin^{-1}\sqrt{\dfrac{x}{1+x}} - \sin^{-1}\dfrac{x-1}{x+1} = \sin^{-1}\dfrac{1}{\sqrt{1+x}}$ is equal to

Answers

PART - I

1.	(D)	**2.**	(D)	**3.**	(C)	**4.**	(A)	**5.**	(B)	**6.**	(B)	**7.**	(A)
8.	(A)	**9.**	(A)	**10.**	(A)	**11.**	(D)	**12.**	(C)	**13.**	(D)	**14.**	(AC)
15.	(CD)												

PART - II

1. 4

2. $32\pi - \cos^{-1}\frac{2}{3}$

4. $\cot(\operatorname{cosec}^{-1}x) = \begin{cases} -\sqrt{x^2-1} & \text{if } x \le -1 \\ \sqrt{x^2-1} & \text{if } x \ge 1 \end{cases}$

5. (i) $\sin^{-1}x = \begin{cases} -\cos^{-1}\sqrt{1-x^2}, & \text{if } -1 \le x < 0 \\ \cos^{-1}\sqrt{1-x^2} & \text{if } 0 \le x \le 1 \end{cases}$

(ii) $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, for all $x \in (-1, 1)$

(iii) $\sin^{-1}x = \begin{cases} \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0 \\ \cot^{-1}\frac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1 \end{cases}$

6. $-\pi$

7. $e^{\pi/12}$

8. $\{1, \cos 1\}$

9. $\{x : x \in (-\infty, -1)\}$

11. $\sin^{-1} 2x\sqrt{1-x^2} = \begin{cases} \pi - 2\sin^{-1}x & \text{if } \frac{1}{\sqrt{2}} < x \le 1 \\ 2\sin^{-1}x & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ -\pi - 2\sin^{-1}x & \text{if } -1 \le x < -\frac{1}{\sqrt{2}} \end{cases}$

12. $\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} -\pi + \tan^{-1}\frac{3x-x^3}{1-3x^2} & \text{when } -1 < x < -\frac{1}{\sqrt{3}} \\ \tan^{-1}\frac{3x-x^3}{1-3x^2} & \text{when } |x| > 1 \text{ or } |x| < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\frac{3x-x^3}{1-3x^2} & \text{when } \frac{1}{\sqrt{3}} < x < 1 \end{cases}$

14. $x \ge 0$

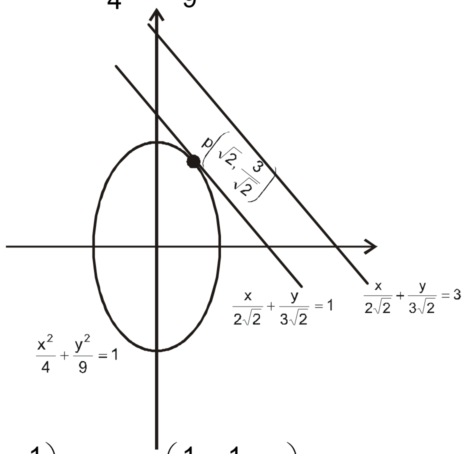
ALP Solutions

PART - I

1. $-1 \le \frac{x^2}{4} + \frac{y^2}{9} \le 1$ represents interior and the boundary of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ....(i)

Also $-1 \le \frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2 \le 1$

i.e. $\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \ge 1$ and $\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \le 3$



$\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \ge 1$ represents the portion of xy plane

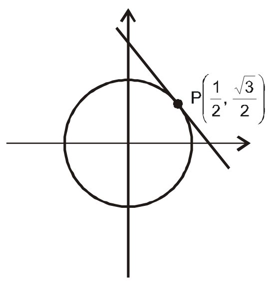
which contains only one point viz : $\left(\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ of $\frac{x^2}{4} + \frac{y^2}{9} \le 1$

$\therefore \quad \sin^{-1}\left(\frac{x^2}{4} + \frac{y^2}{9}\right) + \cos^{-1}\left(\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2\right) = \sin^{-1}\left(\frac{1}{2} + \frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2} + \frac{1}{2} - 2\right)$

$= \sin^{-1} 1 + \cos^{-1}(-1) = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$

2. $-1 \le |z|^2 \le 1$ is $0 \le |z|^2 \le 1$ represents interior and the circumference of the circle $|z| = 1$. If $a = \frac{1}{4} + i\frac{\sqrt{3}}{4}$, then

$a\bar{z} + \bar{a}z - 2 = \frac{1}{2}x + \frac{\sqrt{3}}{2}y - 2$



$\therefore \quad -1 \le \frac{1}{2}x + \frac{\sqrt{3}}{2}y - 2 \le 1$

i.e. $\frac{x}{2} + \frac{\sqrt{3}y}{2} \ge 1$ and $\frac{x}{2} + \frac{\sqrt{3}y}{2} \le 3$

$\frac{x}{2} + \frac{\sqrt{3}y}{2} \ge 1$ is represents portion of xy plane which contains exactly one point viz $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ of $|z| \le 1$

$\therefore \quad \sin^{-1}|z|^2 + \cos^{-1}(a\bar{z} + \bar{a}z - 2) = \sin^{-1} 1 + \cos^{-1}\left(\frac{1}{4} + \frac{3}{4} - 2\right) = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$

3. Since $32\pi - \frac{\pi}{2} < 100 < 32\pi$

$\therefore \quad \sin^{-1}(\sin 100) = 100 - 32\pi$
$\tan^{-1}(\tan 100) = 100 - 32\pi$
$\cos^{-1}(\cos 100) = 32\pi - 100$
$\cot^{-1}(\cot 100) = 100 - 31\pi$

4. Since $8\pi - \frac{\pi}{2} < \theta < 8\pi$

$\therefore \quad \sin^{-1}(\sin\theta) = \theta - 8\pi$ and $\cos^{-1}(\cos\theta) = 8\pi - \theta$
$\therefore \quad \sin^{-1}(\sin\theta) + \cos^{-1}(\cos\theta) = 0$

5. Since $7\pi < \theta < 7\pi + \frac{\pi}{2}$

$\therefore$ $\sin^{-1}(\sin\theta) = 7\pi - \theta$, $\cos^{-1}(\cos\theta) = 8\pi - \theta$
$\tan^{-1}(\tan\theta) = \theta - 7\pi$, $\cot^{-1}(\cot\theta) = \theta - 7\pi$

6. $\frac{\pi}{3} = 3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = 3(2\tan^{-1}x) - 4(2\tan^{-1}x) + 2(2\tan^{-1}x) = 2\tan^{-1}x$

Now $2\tan^{-1}x = \frac{\pi}{3} \Rightarrow \tan^{-1}x = \frac{\pi}{6} \Rightarrow x = \frac{1}{\sqrt{3}}$

$\therefore$ there is 1 solution.

7. $-2\pi = 3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = 3(2\tan^{-1}x) - 4(2\tan^{-1}x) + 2(2\tan^{-1}x) = 2\tan^{-1}x$

Now $2\tan^{-1}x = -2\pi \Rightarrow \tan^{-1}x = -\pi$ (not possible)

$\therefore$ Number of solution is 0.

8. $-\pi = 3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = 3(2\tan^{-1}x) - 4(2\tan^{-1}x) + 2(2\tan^{-1}x) = 2\tan^{-1}x$

Now $2\tan^{-1}x = -\pi \Rightarrow \tan^{-1}x = -\frac{\pi}{2}$

$\therefore$ Number of solution is 0.

9. $\pi = 3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = 3(2\tan^{-1}x) - 4(-2\tan^{-1}x) + 2(2\tan^{-1}x) = 18\tan^{-1}x$

Now $\tan^{-1}x = \frac{\pi}{18}$

$\therefore$ Number of solution is 0.

10. $-\pi = 3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = 3(\pi - 2\tan^{-1}x) - 4(2\tan^{-1}x) + 2(-\pi + 2\tan^{-1}x)$
$= \pi - 10\tan^{-1}x$

$\therefore$ $\tan^{-1}x = \frac{\pi}{5}$

$\therefore$ $x = \tan\frac{\pi}{5} < 1$

$\therefore$ Number of solution is 0.

11. Let $\cos^{-1}x = \theta$, then $x = \cos\theta$ and $0 \le \theta \le \pi$

$$\therefore \quad \sin^{-1}\sqrt{1-x^2} = \sin^{-1}(\sin\theta) = \begin{cases} \theta & \text{if } 0 \le \theta \le \frac{\pi}{2} \\ \pi - \theta & \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

$$= \begin{cases} \cos^{-1}x & \text{if } 0 \le x \le 1 \\ \pi - \cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$$

$\therefore$ $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$ if $0 < x < 1$ is true.

12. Let $\sin^{-1} x = \theta$, then $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\therefore$ $\cos^{-1}\sqrt{1-x^2} = \cos^{-1}(\cos\theta)$

$$= \begin{cases} -\theta, & -\frac{\pi}{2} \le \theta \le 0 \\ \theta, & 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} -\sin^{-1} x, & -1 \le x \le 0 \\ \sin^{-1} x, & 0 \le x \le 1 \end{cases}$$

$\therefore$ $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ if $0 < x < 1$ is true

13. Let $\cos^{-1} x = \theta$, then $x = \cos\theta$ and $0 \le \theta \le \pi$

$\therefore$ $\tan^{-1}\frac{\sqrt{1-x^2}}{x} = \tan^{-1}(\tan\theta)$

$$= \begin{cases} \theta, & 0 \le \theta < \frac{\pi}{2} \\ \theta - \pi, & \frac{\pi}{2} < \theta \le \pi \end{cases}$$

$$= \begin{cases} \cos^{-1} x, & 0 < x \le 1 \\ -\pi + \cos^{-1} x, & -1 \le x < 0 \end{cases}$$

i.e $\cos^{-1} x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$, $-1 < x < 0$ is correct.

14. Let $\tan^{-1}x = \theta$. Then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\theta \ne \pm\frac{\pi}{4}$ and $x = \tan\theta$

$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \theta + \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \theta + \tan^{-1}(\tan 2\theta)$, where $-\pi < 2\theta < \pi$, $2\theta \ne \pm\frac{\pi}{2}$

$$= \begin{cases} \theta + \pi + 2\theta & \text{when } -\pi < 2\theta < -\frac{\pi}{2} \\ \theta + 2\theta & \text{when } -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \\ \theta - \pi + 2\theta & \text{when } \frac{\pi}{2} < 2\theta < \pi \end{cases} = \begin{cases} \pi + 3\theta & \text{when } -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \\ 3\theta & \text{when } -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ -\pi + 3\theta & \text{when } \frac{\pi}{4} < \theta < \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} \pi + 3\tan^{-1} x & \text{when } x < -1 \\ 3\tan^{-1} x & \text{when } -1 < x < 1 \\ -\pi + 3\tan^{-1} x & \text{when } 1 < x \end{cases}$$

15. If $-1 \le x < 0$, then $-\frac{\pi}{2} \le \sin^{-1}x < 0$

Also $0 < 2\cot^{-1}(y^2 - 2y) < 2\pi$

$\therefore$ $-\frac{\pi}{2} < \sin^{-1} x + 2\cot^{-1}(y^2 - 2y) < 2\pi$

$\therefore$ there is no solution in this case.

thus x can not be negative ....(i)

Now if $x \geq 0$, then $0 \leq \sin^{-1}x \leq \frac{\pi}{2}$

$\Rightarrow \quad \frac{3\pi}{4} \leq \cot^{-1}(y^2 - 2y) < \pi$

$\Rightarrow \quad y^2 - 2y \leq -1$

$\Rightarrow \quad y = 1$

since for y = 1, we have $2\cot^{-1}(y^2 - 2y) = 2\cot^{-1}(-1) = \frac{3\pi}{2}$

$\therefore \quad \sin^{-1}x = \frac{\pi}{2}$ i.e. $x = 1$

$\therefore$ the solution is x = 1, y = 1

PART - II

1. $\cot\{\cot^{-1}(x-2) + \cot^{-1}(3-x)\} = \cot(\cot^{-1}(x-12))$

$\Rightarrow \quad \frac{(x-2)(3-x)-1}{(x-2)+(3-x)} = x - 12$

$\Rightarrow \quad x^2 - 4x - 5 = 0 \quad \Rightarrow \quad x = -1, x = 5$

verification :

for x = – 1 : LHS $= \cot^{-1}(-3) + \cot^{-1}4 = \pi - \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} = \pi + \tan^{-1}\left(\frac{\frac{1}{4}-\frac{1}{3}}{1+\frac{1}{4}\cdot\frac{1}{3}}\right) = \pi - \tan^{-1}\frac{1}{13}$

RHS $= \cot^{-1}(-13) = \pi - \tan^{-1}\frac{1}{13}$

$\therefore \quad x = -1$ is a solution

for x = 5 : LHS $= \cot^{-1}3 + \cot^{-1}(-2) = \pi + \cot^{-1}3 - \cot^{-1}2 = \pi - \cot^{-1}\frac{3\times 2+1}{3-2} = \pi - \cot^{-1}7$

RHS $= \cot^{-1}(-7) = \pi - \cot^{-1}7$

$\therefore \quad x = 5$ is a solution

$\therefore$ sum of the solutions = 4

2. $\cos\theta = \frac{2}{3} \Rightarrow \cos^{-1}\frac{2}{3} = \cos^{-1}(\cos\theta)$

Since $31\pi \leq \theta \leq 32\pi$

$\therefore \quad \cos^{-1}(\cos\theta) = 32\pi - \theta$

$\therefore \quad \cos^{-1}\frac{2}{3} = 32\pi - \theta$ i.e. $\theta = 32\pi - \cos^{-1}\frac{2}{3}$

3. Let $\cot^{-1}x = \theta$. Then $x = \cot\theta$ and $\frac{\pi}{2} < \theta < \pi$

i.e. $\tan\theta = \frac{1}{x}$, where $\frac{\pi}{2} < \theta < \pi$

$\therefore \quad \tan^{-1}\frac{1}{x} = \tan^{-1}(\tan\theta) = \theta - \pi \quad \{\text{because } \pi - \frac{\pi}{2} < \theta < \pi + \frac{\pi}{2}\}$

$= \cot^{-1}x - \pi$

$\therefore \quad \cot^{-1}x = \pi + \tan^{-1}\frac{1}{x}$

4. Let $\operatorname{cosec}^{-1} x = \theta$, then $x = \operatorname{cosec}\theta$ and $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$$\therefore \quad \cot(\operatorname{cosec}^{-1}x) = \cot\theta = \begin{cases} -\sqrt{\operatorname{cosec}^2\theta - 1} & \text{if } -\frac{\pi}{2} \le \theta < 0 \\ \sqrt{\operatorname{cosec}^2\theta - 1} & \text{if } 0 < \theta \le \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} -\sqrt{x^2 - 1} & \text{if } x \le -1 \\ \sqrt{x^2 - 1} & \text{if } x \ge 1 \end{cases}$$

5. (i) Let $\sin^{-1}x = \theta$. Then $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$$

$$\therefore \quad \cos^{-1}\sqrt{1-x^2} = \cos^{-1}(\cos\theta) = \begin{cases} -\theta & \text{if } -\frac{\pi}{2} \le \theta < 0 \\ \theta & \text{if } 0 \le \theta \le \frac{\pi}{2} \end{cases} = \begin{cases} -\sin^{-1}x & \text{if } -1 \le x < 0 \\ \sin^{-1}x & \text{if } 0 \le x \le 1 \end{cases}$$

$$\therefore \quad \sin^{-1}x = \begin{cases} -\cos^{-1}\sqrt{1-x^2} & \text{if } -1 \le x < 0 \\ \cos^{-1}\sqrt{1-x^2} & \text{if } 0 \le x \le 1 \end{cases}$$

(ii) Let $\sin^{-1}x = \theta$. Then $x = \sin\theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

{Note : $\theta \ne -\frac{\pi}{2}, \frac{\pi}{2}$ because $x \ne \pm 1$ }

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \quad \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}x$$

Thus $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, for all $x \in (-1, 1)$

(iii) Let $\sin^{-1}x = \theta$. Then $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

{ Note : $\theta \ne 0$, because $x \ne 0$ }

$$\therefore \quad \cot\theta = \frac{\sqrt{1-x^2}}{x}$$

$$\therefore \quad \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}(\cot\theta) = \begin{cases} \theta + \pi & \text{if } -\frac{\pi}{2} \le \theta < 0 \\ \theta & \text{if } 0 < \theta \le \frac{\pi}{2} \end{cases} = \begin{cases} \pi + \sin^{-1}x & \text{if } -1 \le x < 0 \\ \sin^{-1}x & \text{if } 0 < x \le 1 \end{cases}$$

$$\text{Thus } \sin^{-1}x = \begin{cases} \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0 \\ \cot^{-1}\frac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1 \end{cases}$$

6. $\alpha = 2\tan^{-1}\left(\dfrac{1-x}{1-x}\right)$

$\beta = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

put $x = \tan\theta$ $\quad \left\{\because\ x > 1 \Rightarrow \theta > \dfrac{\pi}{4}\right\}$

$\Rightarrow \quad \alpha = 2\tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

$\Rightarrow \quad \alpha = 2\tan^{-1}\left\{\tan\left(\dfrac{\pi}{4}+\theta\right)\right\} = 2\left\{\dfrac{\pi}{4}+\theta-\pi\right\} = 2\left\{\theta-\dfrac{3\pi}{4}\right\} = 2\theta - \dfrac{3\pi}{2}$ ......(i)

$\beta = 2\sin^{-1}\left\{\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right\} = \sin^{-1}(\cos 2\theta) = \sin^{-1}\left[\sin\left(\dfrac{\pi}{2}-2\theta\right)\right] = \dfrac{\pi}{2} - 2\theta$ ......(ii)

by (i) and (ii) $\quad \alpha + \beta = -\pi$

7. $f\left(-\dfrac{7\pi}{4}\right) = e^{\cos^{-1}\left(\sin\left(-\frac{17\pi}{12}\right)\right)} = e^{\cos^{-1}\left(\sin-\frac{5\pi}{12}\right)} = e^{\cos^{-1}\cos\frac{\pi}{12}} = e^{\pi/12}$

8. Since $-1 \le x \le 1$

$\therefore \quad -\dfrac{\pi}{4} \le \tan^{-1}x \le \dfrac{\pi}{4}$

$\therefore \quad [\tan^{-1}x] = -1, 0$

When $[\tan^{-1}x] = -1$, then $\{\cos^{-1}x\} = 1$ (not possible)

When $[\tan^{-1}x] = 0$, then $\{\cos^{-1}x\} = 0$

$\therefore \quad \cos^{-1}x$ is integer

Since $0 \le \cos^{-1}x \le \pi \quad \therefore \quad \cos^{-1}x = 0, 1, 2, 3$

$x = \cos 0, \cos 1, \cos 2, \cos 3$ but $x \ne \cos 2, \cos 3$

$\therefore$ the solution set is $\{1, \cos 1\}$

9. If $x \le -1$, then $\sec^{-1}x > \dfrac{\pi}{2}$ and $\tan^{-1}x \le -\dfrac{\pi}{4} < 0$

$\therefore \quad \sec^{-1}x > \tan^{-1}x$ for all $x \le -1$

If $x \ge 1$, suppose $\tan^{-1}x = \theta$, then $\dfrac{\pi}{4} \le \theta < \dfrac{\pi}{2}$ and $x = \tan\theta$

$\therefore \quad \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2}$

$\therefore \quad \sec^{-1}\sqrt{1+x^2} = \sec^{-1}(\sec\theta) = \theta = \tan^{-1}x$

thus the inequality becomes $\sec^{-1}x > \sec^{-1}\sqrt{1+x^2}$

$\therefore \quad x > \sqrt{1+x^2}$ i.e. $x^2 > 1 + x^2$ which is not possible

$\therefore \quad \{x : x \in (-\infty, -1)\}$ is the solution set

10. Let $\cos^{-1}x = \alpha$ and $\cos^{-1}y = \beta$. Then $x = \cos\alpha$, $y = \cos\beta$ and $0 \le \alpha, \beta \le \pi$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = xy - \sqrt{1-x^2}\sqrt{1-y^2}$

$\therefore \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}\cos(\alpha+\beta)$

Case -I : When $x \ge 0,\ y \ge 0$, then $0 \le \alpha, \beta \le \frac{\pi}{2}$ i.e. $0 \le \alpha + \beta \le \pi$

$$\therefore \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}\cos(\alpha+\beta) = \alpha+\beta$$

Thus $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

Case-II : When $x \ge 0,\ y \le 0$, then $0 \le \alpha \le \frac{\pi}{2}$, $\frac{\pi}{2} \le \beta \le \pi$ and so $\frac{\pi}{2} \le \alpha+\beta \le \frac{3\pi}{2}$

Now $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = y\sqrt{1-x^2} + x\sqrt{1-y^2} \ge 0$

Iff $x\sqrt{1-y^2} \ge -y\sqrt{1-x^2}$ iff $x^2 \ge y^2$ iff $x \ge -y$ iff $\frac{\pi}{2} \le \alpha+\beta \le \pi$

Further $\sin(\alpha+\beta) < 0$ if $x + y < 0$ i.e. $\pi < \alpha+\beta \le \frac{3\pi}{2}$

$$\therefore \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}\cos(\alpha+\beta) = \begin{cases} \alpha+\beta & \text{if } 0 \le \alpha+\beta \le \pi \\ 2\pi-(\alpha+\beta) & \text{if } \pi < \alpha+\beta \le \frac{3\pi}{2} \end{cases}$$

$$\text{Thus} \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \begin{cases} \cos^{-1}x+\cos^{-1}y & \text{if } x+y \ge 0 \\ 2\pi-(\cos^{-1}x+\cos^{-1}y) & \text{if } x+y < 0 \end{cases}$$

Case- III : When $x \le 0,\ y \ge 0$, then $\frac{\pi}{2} \le \alpha \le \pi$, $0 \le \beta \le \frac{\pi}{2}$ and so $\frac{\pi}{2} \le \alpha+\beta \le \frac{3\pi}{2}$

Now $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = y\sqrt{1-x^2} + x\sqrt{1-y^2} \ge 0$ iff $y - x \ge 0$

$\therefore \quad 0 \le \alpha+\beta \le \pi$ iff $y + x \ge 0$

$$\therefore \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \begin{cases} \alpha+\beta & \text{if } \frac{\pi}{2} \le \alpha+\beta \le \pi \\ 2\pi-(\alpha+\beta) & \text{if } \pi < \alpha+\beta \le \frac{3\pi}{2} \end{cases}$$

$$= \begin{cases} \cos^{-1}x+\cos^{-1}y & \text{if } x+y \ge 0 \\ 2\pi-(\cos^{-1}x+\cos^{-1}y) & \text{if } x+y < 0 \end{cases}$$

Case- IV : When $x \le 0,\ y \le 0$, then $\frac{\pi}{2} \le \alpha \le \pi$, $\frac{\pi}{2} \le \beta \le \pi$ and so $\pi \le \alpha+\beta \le 2\pi$

$$\therefore \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}\cos(\alpha+\beta) = 2\pi-(\alpha+\beta) = 2\pi-(\cos^{-1}x+\cos^{-1}y)$$

From all the four cases , we get

$$\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) & \text{if } x+y \ge 0 \\ 2\pi-\cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) & \text{if } x+y < 0 \end{cases}$$

11. Let $x = \cos\theta$, where $0 \le \theta \le \pi$, then

$\sin^{-1} 2x\sqrt{1-x^2} = \sin^{-1}(2\cos\theta\,|\sin\theta|) = \sin^{-1}(\sin 2\theta)$ $[\because \sin\theta \ge 0]$

$$= \begin{cases} 2\theta & \text{if} \quad 0 \le 2\theta < \dfrac{\pi}{2} \quad \text{i.e.} \quad 0 \le \theta < \dfrac{\pi}{4} \\ \pi - 2\theta & \text{if} \quad \dfrac{\pi}{2} \le 2\theta \le \dfrac{3\pi}{2} \quad \text{i.e.} \quad \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4} \\ 2\theta - 2\pi & \text{if} \quad \dfrac{3\pi}{2} < 2\theta \le 2\pi \quad \text{i.e.} \quad \dfrac{3\pi}{4} < \theta \le \pi \end{cases}$$

$$= \begin{cases} 2\cos^{-1}x & \text{if} \quad \dfrac{1}{\sqrt{2}} < x \le 1 \\ \pi - 2\cos^{-1}x & \text{if} \quad -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ 2\cos^{-1}x - 2\pi & \text{if} \quad -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases} = \begin{cases} \pi - 2\sin^{-1}x & \text{if} \quad \dfrac{1}{\sqrt{2}} < x \le 1 \\ 2\sin^{-1}x & \text{if} \quad -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ -\pi - 2\sin^{-1}x & \text{if} \quad -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

12. Case 1 If $x = 0$, then $\tan^{-1}x + \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

Case 2 If $x > 0$ and $\dfrac{2x}{1-x^2} < 0$ i.e. $x > 1$, then $\tan^{-1}x + \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

Case 3 If $x < 0$ and $\dfrac{2x}{1-x^2} > 0$ i.e. $x < -1$, then $\tan^{-1}x + \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

Case 4 If $x > 0$ and $\dfrac{2x}{1-x^2} > 0$ i.e. $0 < x < 1$, then

$$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} \tan^{-1}\dfrac{3x-x^2}{1-3x^2} & \text{when} \quad \dfrac{2x^2}{1-x^2} < 1 \\ \pi + \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{when} \quad \dfrac{2x^2}{1-x^2} > 1 \end{cases} = \begin{cases} \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{when} \quad 0 < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{when} \quad \dfrac{1}{\sqrt{3}} < x < 1 \end{cases}$$

Case 5 If $x < 0$ and $\dfrac{2x}{1-x^2} < 0$ i.e. $-1 < x < 0$, then

$$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} -\pi + \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{when} \quad -1 < x < -\dfrac{1}{\sqrt{3}} \\ \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{when} \quad |x| > 1 \text{ or } |x| < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\dfrac{3x-x^3}{1-3x^2} & \text{when} \quad \dfrac{1}{\sqrt{3}} < x < 1 \end{cases}$$

13. Let $\cot^{-1} x = \alpha$ and $\cot^{-1}y = \beta$. Then $x = \cot\alpha$, $y = \cot\beta$ and $0 < \alpha < \beta < \pi$

Now $\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha} = \dfrac{xy+1}{y-x}$

$\therefore \quad \cot^{-1}\dfrac{xy+1}{y-x} = \cot^{-1}\cot(\alpha - \beta)$

Case- I When $y < x < 0$, then $\frac{\pi}{2} < \alpha < \beta < \pi$

$\therefore \quad -\frac{\pi}{2} < \alpha - \beta < 0$

$\therefore \quad \cot^{-1}\frac{xy+1}{y-x} = \cot^{-1}\cot(\alpha-\beta) = \pi + \alpha - \beta = \pi + \cot^{-1}x - \cot^{-1}y$

$\therefore \quad \cot^{-1}x - \cot^{-1}y = -\pi + \cot^{-1}\frac{xy+1}{y-x}$

Case - II When $y < 0 < x$, then $0 < \alpha < \frac{\pi}{2} < \beta < \pi$

$\therefore \quad -\pi < \alpha - \beta < 0$

$\therefore \quad \cot^{-1}\frac{xy+1}{y-x} = \cot^{-1}\cot(\alpha-\beta) = \pi + \alpha - \beta = \pi + \cot^{-1}x - \cot^{-1}y$

$\therefore \quad \cot^{-1}x - \cot^{-1}y = -\pi + \cot^{-1}\frac{xy+1}{y-x}$

Case-III When $0 < y < x$, then $0 < \alpha < \beta < \frac{\pi}{2}$

$\therefore \quad -\frac{\pi}{2} < \alpha - \beta < 0$

$\therefore \quad \cot^{-1}\frac{xy+1}{y-x} = \cot^{-1}\cot(\alpha-\beta) = \pi + \alpha - \beta = \pi + \cot^{-1}x - \cot^{-1}y$

$\therefore \quad \cot^{-1}x - \cot^{-1}y = -\pi + \cot^{-1}\frac{xy+1}{y-x}$

From cases-I, II and III, we get

$$\cot^{-1}x - \cot^{-1}y = -\pi + \cot^{-1}\frac{xy+1}{y-x} \quad \text{for } y - x < 0$$

14. $\sin^{-1}\sqrt{\frac{x}{1+x}} - \sin^{-1}\frac{x-1}{x+1} = \sin^{-1}\frac{1}{\sqrt{1+x}}$

$\Rightarrow \quad \sin^{-1}\sqrt{\frac{x}{1+x}} - \sin^{-1}\frac{1}{\sqrt{1+x}} = \sin^{-1}\frac{x-1}{x+1}$

$\Rightarrow \quad \sin^{-1}\left\{\sqrt{\frac{x}{1+x}}\sqrt{1-\frac{1}{1+x}} - \frac{1}{\sqrt{1+x}}\sqrt{1-\frac{x}{1+x}}\right\} = \sin^{-1}\left(\frac{x-1}{x+1}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{x+1} - \frac{1}{1+x}\right) = \sin^{-1}\left(\frac{x-1}{x+1}\right) \qquad \forall\ x \in R$

But domain of $\sin^{-1}\sqrt{\frac{x}{1+x}} - \sin^{-1}\frac{x-1}{x+1} = \sin^{-1}\frac{1}{\sqrt{1+x}}$ is $x > 0$

Hence $x > 0$