Inverse Trigonometric Functions

There are three sides to any argument: your side, my side and the right side.

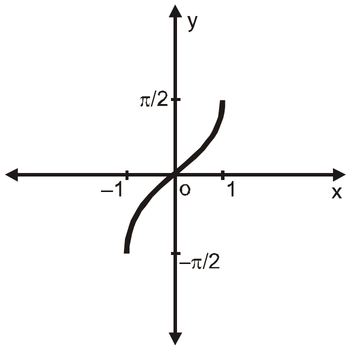
Introduction : The student may be familiar about trigonometric functions viz sin x, cos x, tan x, cosec x, sec x, cot x with respective domains R, R, R – {(2n + 1) π/2}, R – {nπ}, R – {(2n + 1) π/2}, R – {nπ} and respective ranges [–1, 1], [–1, 1], R, R – (–1, 1), R – (–1, 1), R.

Correspondingly, six inverse trigonometric functions (also called inverse circular functions) are defined.

$\sin^{-1}x$: The symbol $\sin^{-1}x$ or arcsinx denotes the angle θ so that $\sin\theta = x$. As a direct meaning, $\sin^{-1}x$ is not a function, as it does not satisfy the requirements for a rule to become a function. But by a suitable choice [–1, 1] as its domain and standardized set [–π/2, π/2] as its range, then rule $\sin^{-1}x$ is a single valued function.

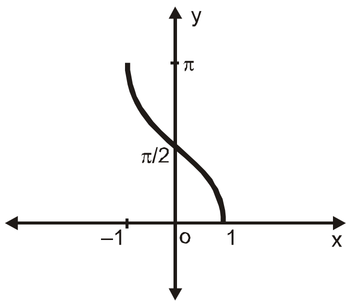
Thus $\sin^{-1}x$ is considered as a function with domain [–1, 1] and range [–π/2, π/2].

The graph of $y = \sin^{-1}x$ is as shown below, which is obtained by taking the mirror image, of the portion of the graph of y = sin x, from x = –π/2 to x = π/2, on the line y = x.

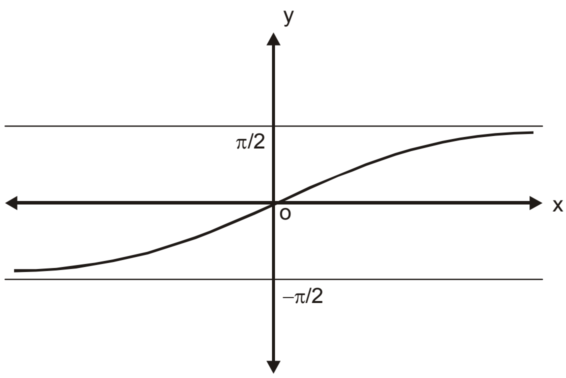


$\cos^{-1}x$: By following the discussions, similar to above, we have $\cos^{-1}x$ or arccos x as a function with domain [–1, 1] and range [0, π].

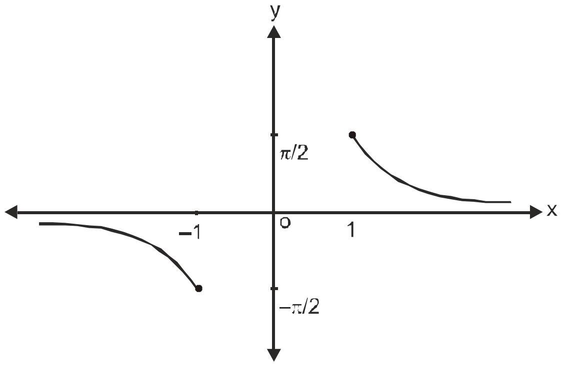
The graph of $y = \cos^{-1}x$ is similarly obtained as the mirror image of the portion of the graph of y = cos x from x = 0 to x = π.



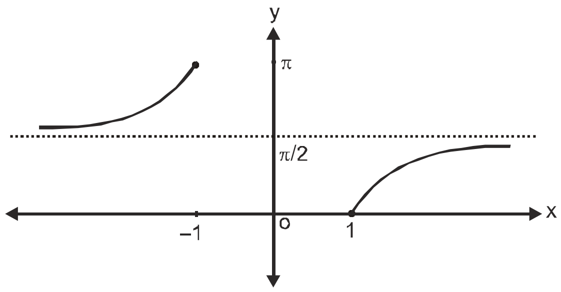
$\tan^{-1}x$: We get $\tan^{-1}x$ or arctanx as a function with domain R and range (–π/2, π/2).
Graph of $y = \tan^{-1}x$



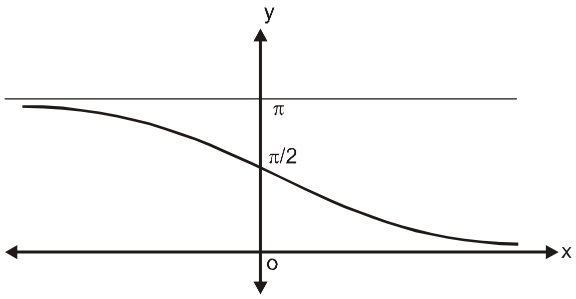
$cosec^{-1}x$: $cosec^{-1}x$ or arccosec x is a function with domain R – (–1, 1) and range [–π/2, π/2] – {0}.
Graph of y = $cosec^{-1}x$



$sec^{-1}x$: $sec^{-1}x$ or arcsec x is a function with domain R – (–1, 1) and range [0, π] – {π/2}.
Graph of y = $sec^{-1}x$



cot^{-1} x : $cot^{-1}x$ or arccot x is a function with domain R and range (0, π)
Graph of y = $cot^{-1}x$



Example # 1 : Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

Solution : $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \tan\left[\frac{\pi}{3}+\left(-\frac{\pi}{6}\right)\right] = \tan\left(\frac{\pi}{6}\right) = \mathbf{\frac{1}{\sqrt{3}}}$.

Example # 2 : Find domain of $\sin^{-1}(2x^2 - 1)$
Solution : Let $y = \sin^{-1}(2x^2 - 1)$
For y to be defined $-1 \le (2x^2 - 1) \le 1$
$\Rightarrow$ $0 \le 2x^2 \le 2 \Rightarrow 0 \le x^2 \le 1 \Rightarrow$ $\mathbf{x \in [-1, 1]}$.

Self practice problems :

(1) Find the value of $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$

(2) Find the value of cosec $[\sec^{-1}(-\sqrt{2}) + \cot^{-1}(-1)]$

(3) Find the domain of $y = \sec^{-1}(x^2 + 3x + 1)$

(4) Find the domain of $y = \cos^{-1}\left(\frac{x^2}{1+x^2}\right)$

(5) Find the domain of $y = \tan^{-1}(\sqrt{x^2-1})$

Answers : (1) 1 (2) –1
(3) $(-\infty, -3] \cup [-2, -1] \cup [0, \infty)$ (4) R (5) $(-\infty, -1] \cup [1, \infty)$

Property 1 : "–x"

The graphs of $\sin^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$ are symmetric about origin.

Hence we get
$$\sin^{-1}(-x) = -\sin^{-1}x$$
$$\tan^{-1}(-x) = -\tan^{-1}x$$
$$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x.$$

Also the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \pi/2)$. From this, we get
$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$
$$\sec^{-1}(-x) = \pi - \sec^{-1}x$$
$$\cot^{-1}(-x) = \pi - \cot^{-1}x.$$

Property 2 : $T(T^{-1})$

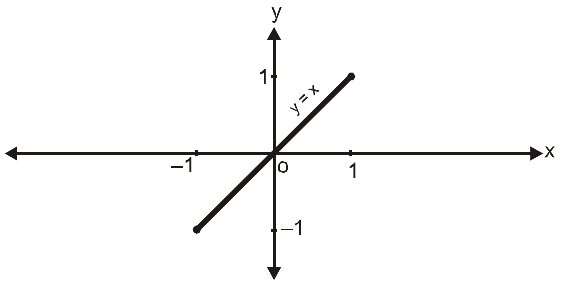
(i) $\sin(\sin^{-1}x) = x$, $\quad -1 \le x \le 1$

Proof : Let $\theta = \sin^{-1}x$. Then $x \in [-1, 1]$ & $\theta \in [-\pi/2, \pi/2]$.
$\Rightarrow$ $\sin\theta = x$, by meaning of the symbol
$\Rightarrow$ $\sin(\sin^{-1}x) = x$

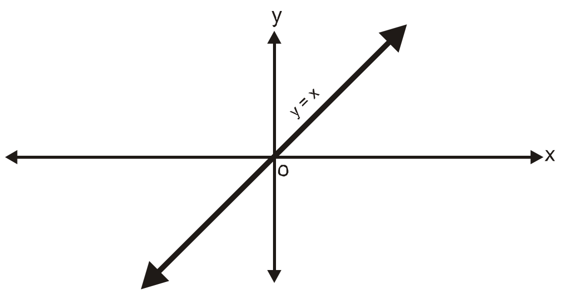
Similar proofs can be carried out to obtain

(ii) $\cos(\cos^{-1}x) = x$, $\quad -1 \le x \le 1$
(iii) $\tan(\tan^{-1}x) = x$, $\quad x \in R$
(iv) $\cot(\cot^{-1}x) = x$, $\quad x \in R$
(v) $\sec(\sec^{-1}x) = x$, $\quad x \le -1, x \ge 1$
(vi) $\text{cosec}(\text{cosec}^{-1}x) = x$, $\quad |x| \ge 1$

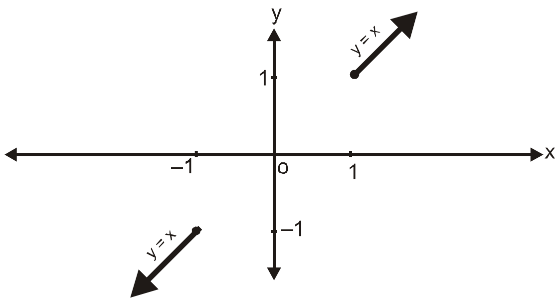
The graph of $y = \sin(\sin^{-1}x) \equiv \cos(\cos^{-1}x)$



The graph of $y = \tan(\tan^{-1}x) \equiv \cot(\cot^{-1}x)$



The graph of y = cosec (cosec^{-1}x) ≡ sec (sec^{-1}x)

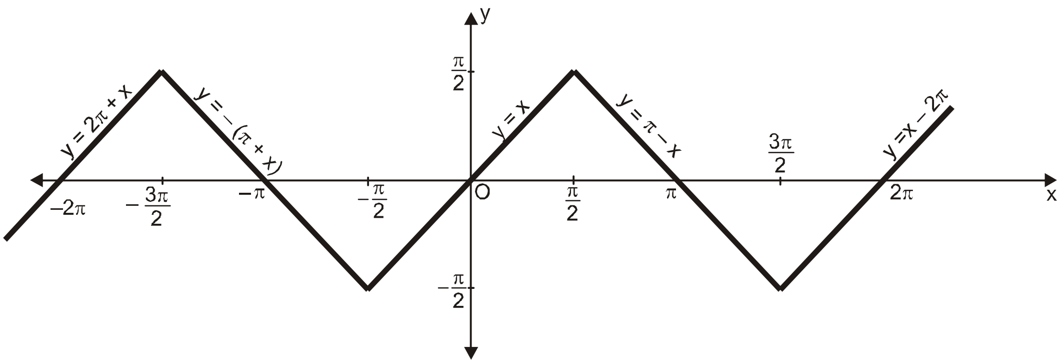


Property 3 : T^{-1}(T)

(i) $\sin^{-1}(\sin x) = \begin{cases} -2n\pi + x, & x \in [2n\pi - \pi/2, 2n\pi + \pi/2] \\ (2n+1)\pi - x, & x \in [(2n+1)\pi - \pi/2, (2n+1)\pi + \pi/2], n \in Z \end{cases}$

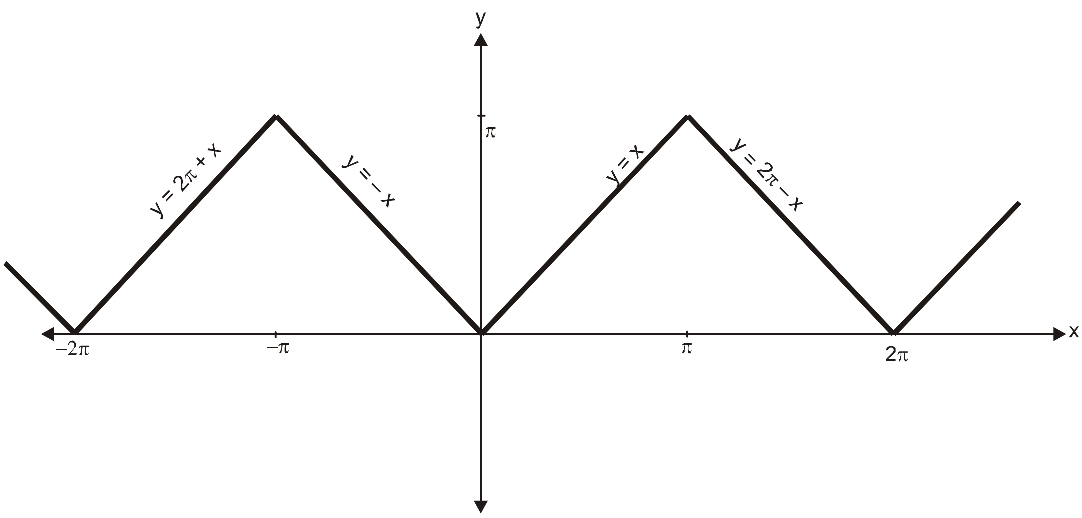
Proof : If x ∈ [2nπ – π/2, 2nπ + π/2], then –2nπ + x ∈ [–π/2, π/2] and sin (–2nπ + x) = sin x.
Hence sin^{-1} (sin x) = –2nπ + x for x ∈ [2nπ – π/2, 2nπ + π/2].
Proof of 2nd part is left for the students.

Graph of y = sin^{-1} (sin x)



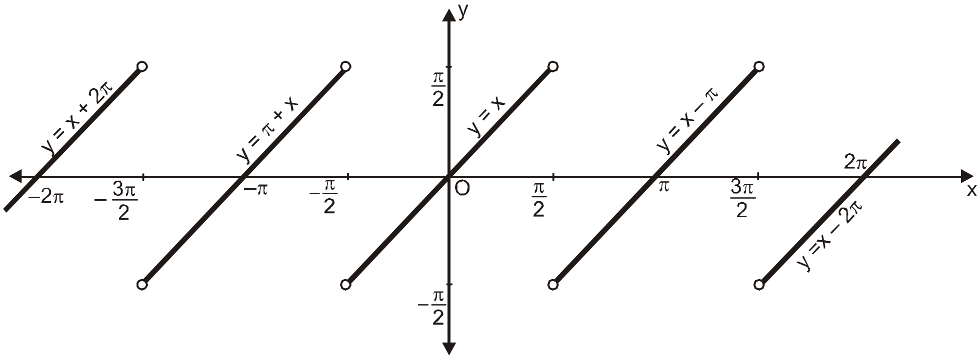
(ii) $\cos^{-1}(\cos x) = \begin{cases} -2n\pi + x, & x \in [2n\pi, (2n+1)\pi] \\ 2n\pi - x, & x \in [(2n-1)\pi, 2n\pi], n \in I \end{cases}$

Graph of y = cos^{-1} (cos x)



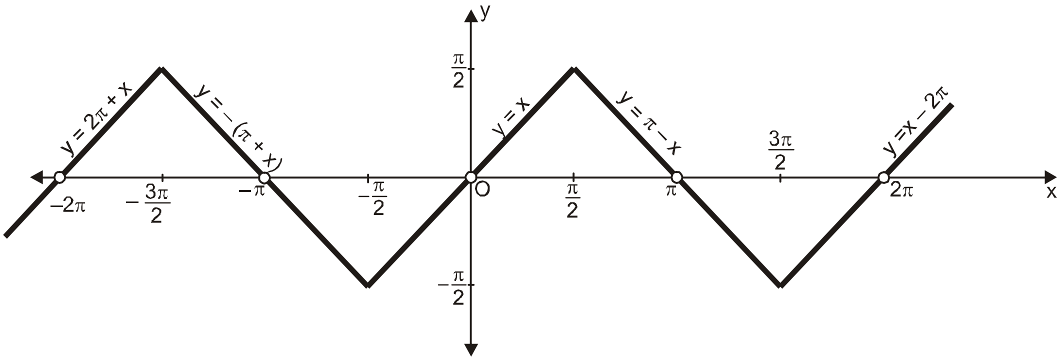
(iii) tan^{-1} (tan x) = – nπ + x, nπ – π/2 < x < nπ + π/2, n ∈ Z

Graph of y = tan^{-1} (tan x)



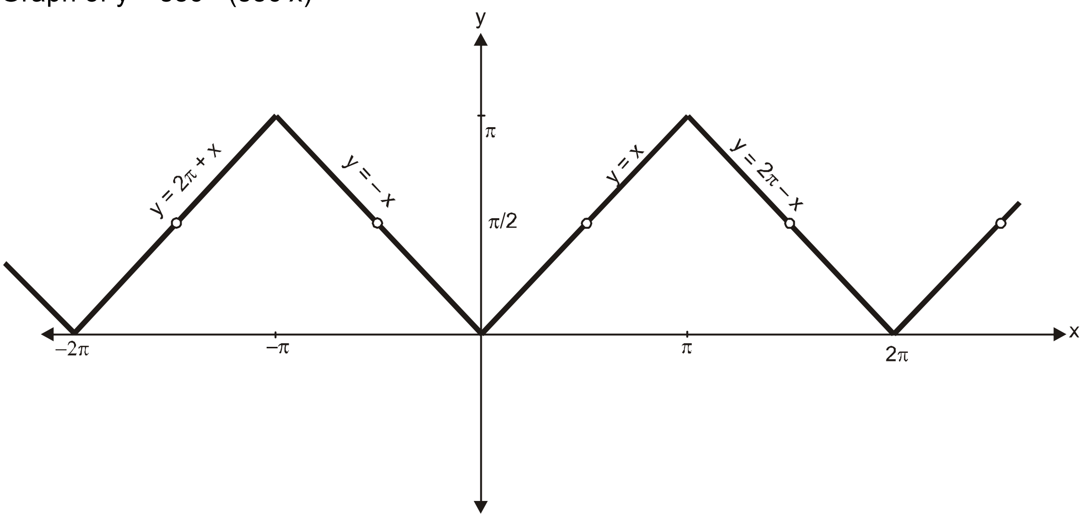
(iv) cosec^{-1} (cosec x) is similar to sin^{-1} (sin x)

Graph of y = cosec^{-1} (cosec x)



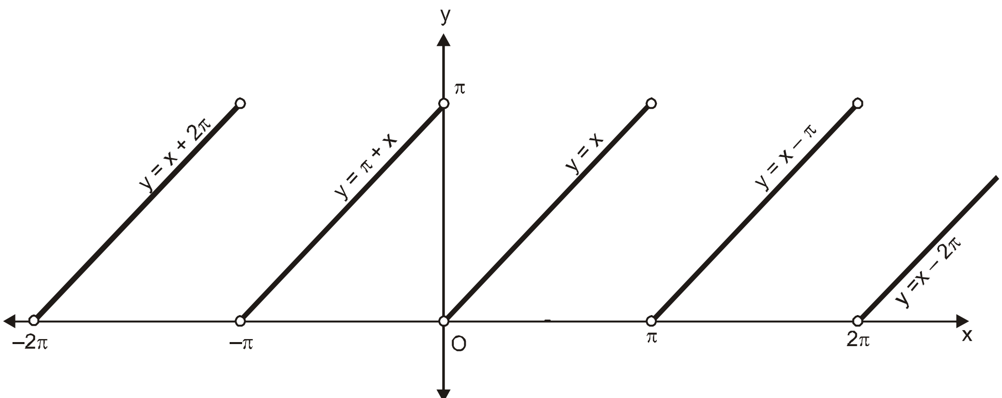
(v) sec^{-1} (sec x) is similar to cos^{-1} (cos x)

Graph of y = sec^{-1} (sec x)



(vii) cot^{-1} (cot x) = $-n\pi + x$, $x \in (n\pi, (n+1)\pi)$, $n \in \mathbb{Z}$

Graph of y = cot^{-1} (cot x)



Remark : $\sin(\sin^{-1}x)$, $\cos(\cos^{-1}x)$, ... $\cot(\cot^{-1}x)$ are aperiodic (non periodic) functions where as $\sin^{-1}(\sin x)$, ..., $\cot^{-1}(\cot x)$ are periodic functions.

Property 4 : "1/x"

(i) $\text{cosec}^{-1}(x) = \sin^{-1}(1/x)$, $|x| \geq 1$

Proof : Let $\text{cosec}^{-1} x = \theta$

$\Rightarrow$ $1/x = \sin\theta$

$\Rightarrow$ $\sin^{-1}(1/x) = \sin^{-1}(\sin\theta)$

$= \theta$ (as $\theta \in [-\pi/2, \pi/2] - \{0\}$)

$= \text{cosec}^{-1}x$

(ii) $\sec^{-1} x = \cos^{-1}(1/x)$, $|x| \geq 1$

(iii) $$\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$$

Property 5 : "π/2"

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$, $-1 \leq x \leq 1$

Proof : Let $A = \sin^{-1}x$ and $B = \cos^{-1}x$

$\Rightarrow$ $\sin A = x$ and $\cos B = x$

$\Rightarrow$ $\sin A = \cos B$

$\Rightarrow$ $\sin A = \sin(\pi/2 - B)$

$\Rightarrow$ $A = \pi/2 - B$, because A and $\pi/2 - B \in [-\pi/2, \pi/2]$

$\Rightarrow$ $A + B = \pi/2$.

Similarly, we can prove

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$, $x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2}$, $|x| \geq 1$

Example # 3 : Find the value of $\text{cosec}\left\{\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right)\right\}$.

Solution :

$\because$ $\cot(\cot^{-1} x) = x$, $\forall x \in R$

$\therefore$ $\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right) = \dfrac{3\pi}{4}$

$$\text{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\} = \text{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}.$$

Example # 4 Find the value of $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$.

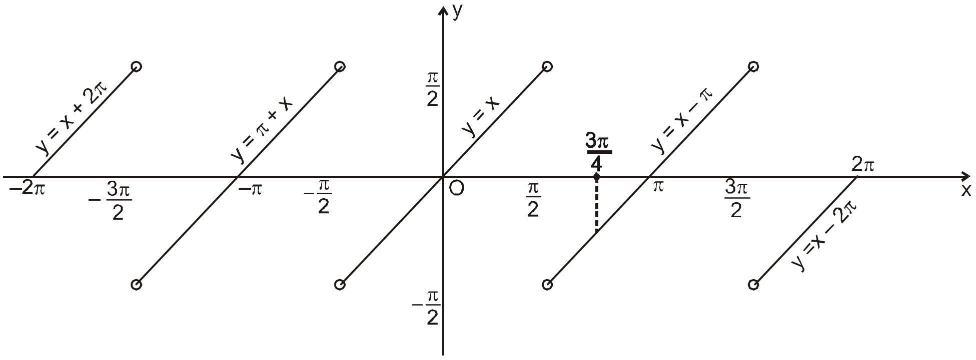
Solution :

$\because$ $\tan^{-1}(\tan x) = x$ if $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

As $\dfrac{3\pi}{4} \notin \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ $\therefore$ $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) \neq \dfrac{3\pi}{4}$

$\therefore$ $\dfrac{3\pi}{4} \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

graph of $y = \tan^{-1}(\tan x)$ is as :



$\because$ from the graph we can see that if $\frac{\pi}{2} < x < \frac{3\pi}{2}$,

then $\tan^{-1}(\tan x) = x - \pi$

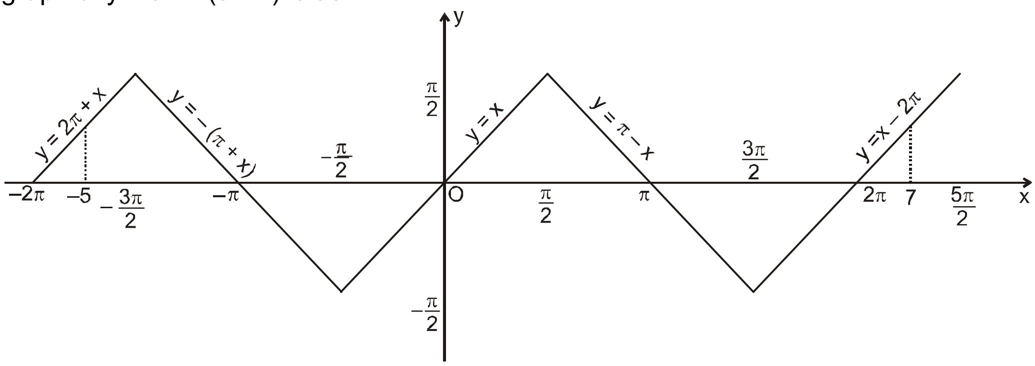
$\therefore$ $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4} - \pi = -\frac{\pi}{4}$

Example # 5 : Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Solution. Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad\because\quad 2\pi < 7 < \frac{5\pi}{2}$

graph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \leq x \leq \frac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as :

$y = x - 2\pi$

$\therefore$ $\sin^{-1}(\sin 7) = 7 - 2\pi$

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because$ $-2\pi < -5 < -\frac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that

$\sin^{-1}(\sin(-5)) = 2\pi + (-5) = 2\pi - 5$

Example # 6 : Find the value of $\cos^{-1}\{\sin(-5)\}$

Solution : Let $y = \cos^{-1}\{\sin(-5)\}$

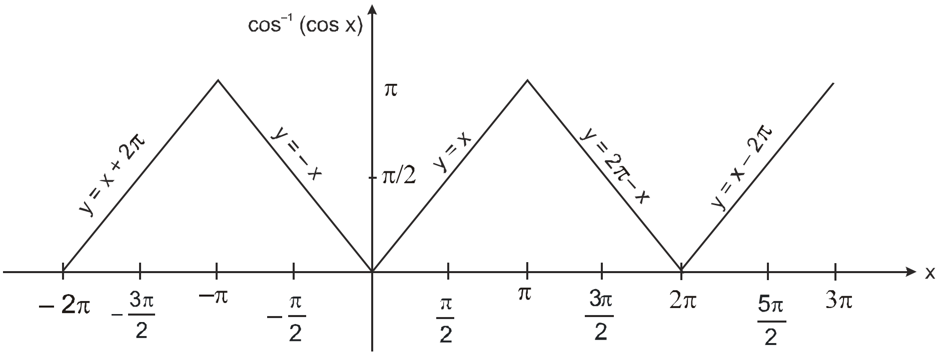
$= \cos^{-1}(-\sin 5)$

$= \pi - \cos^{-1}(\sin 5)$ $\qquad (\cos^{-1}(-x) = \pi - \cos^{-1}x, |x| \leq 1)$

$= \pi - \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\}$ ....(i)

Note that : $-2\pi < \left(\frac{\pi}{2} - 5\right) < -\pi$

graph of $\cos^{-1}(\cos x)$ is as :



From the graph we can see that if $-2\pi \le x \le -\pi$, then $\cos^{-1}(\cos x) = x + 2\pi$

$\therefore$ from the graph $\cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\} = \left(\frac{\pi}{2} - 5\right) + 2\pi = \left(\frac{5\pi}{2} - 5\right)$

$\therefore$ from (i), we get

$\therefore$ $y = \pi - \left(\frac{5\pi}{2} - 5\right) \quad \Rightarrow \quad y = 5 - \frac{3\pi}{2}$.

Example # 7 : Find the value of $\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Solution : Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ....(i)

$\because$ $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$
(i) can be written as

$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$

$\because$ $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ if $x > 0$

$\therefore$ $y = -\tan\left(\tan^{-1}\frac{3}{2}\right) \quad \Rightarrow \quad y = -\frac{3}{2}$

Example # 8 : Find the value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Solution : $\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$

Example # 9 : Find the value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Solution : Let $y = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ ....(i)

Let $\cos^{-1}\dfrac{\sqrt{5}}{3} = \theta$ $\Rightarrow$ $\theta \in \left(0, \dfrac{\pi}{2}\right)$ and $\cos\theta = \dfrac{\sqrt{5}}{3}$

$\therefore$ (i) becomes $y = \tan\left(\dfrac{\theta}{2}\right)$ ....(ii)

$$\because \quad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}} = \frac{3-\sqrt{5}}{3+\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$$

$$\tan\frac{\theta}{2} = \pm\left(\frac{3-\sqrt{5}}{2}\right) \qquad ....(iii)$$

$$\frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right) \quad \Rightarrow \quad \tan\frac{\theta}{2} > 0$$

$\therefore$ from (iii), we get $y = \tan\dfrac{\theta}{2} = \left(\dfrac{3-\sqrt{5}}{2}\right)$

Example # 10 : Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ when $x = \dfrac{1}{5}$

Solution : $\cos\left(2\cos^{-1}\dfrac{1}{5} + \sin^{-1}\dfrac{1}{5}\right) = \cos\left(\cos^{-1}\dfrac{1}{5} + \sin^{-1}\dfrac{1}{5} + \cos^{-1}\dfrac{1}{5}\right)$

$$= \cos\left(\frac{\pi}{2} + \cos^{-1}\frac{1}{5}\right) = -\sin\left(\cos^{-1}\left(\frac{1}{5}\right)\right) \qquad ....(i)$$

$$= -\sqrt{1-\left(\frac{1}{5}\right)^2} = -\frac{2\sqrt{6}}{5}.$$

***Aliter* :** Let $\cos^{-1}\dfrac{1}{5} = \theta$ $\Rightarrow$ $\cos\theta = \dfrac{1}{5}$ and $\theta \in \left(0, \dfrac{\pi}{2}\right)$

$\therefore$ $\sin\theta = \dfrac{\sqrt{24}}{5}$

$\therefore$ $\sin^{-1}(\sin\theta) = \sin^{-1}\left(\dfrac{\sqrt{24}}{5}\right)$ ....(ii)

$\because$ $\theta \in \left(0, \dfrac{\pi}{2}\right)$ $\Rightarrow$ $\sin^{-1}(\sin\theta) = \theta$

$\therefore$ equation (ii) can be written as

$\theta = \sin^{-1}\left(\dfrac{\sqrt{24}}{5}\right)$ $\because$ $\theta = \cos^{-1}\left(\dfrac{1}{5}\right)$

$\Rightarrow$ $\cos^{-1}\left(\dfrac{1}{5}\right) = \sin^{-1}\left(\dfrac{\sqrt{24}}{5}\right)$

Now equation (i) can be written as $y = -\sin\left\{\sin^{-1}\left(\dfrac{\sqrt{24}}{5}\right)\right\}$ ....(iii)

$\because$ $\dfrac{\sqrt{24}}{5} \in [-1, 1]$ $\therefore$ $\sin\left\{\sin^{-1}\left(\dfrac{\sqrt{24}}{5}\right)\right\} = \dfrac{\sqrt{24}}{5}$

$\therefore$ from equation (iii), we get $y = -\dfrac{\sqrt{24}}{5}$

Example # 11 : Solve $\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Solution : $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2} \Leftrightarrow f(x) = g(x)$ and $-1 \le f(x), g(x) \le 1$

$x^2 - 2x + 1 = x^2 - x \Leftrightarrow x = 1$, accepted as a solution

Self practice problems :

(6) Find the value of $\cos\left\{\sin\left(\sin^{-1}\frac{\pi}{6}\right)\right\}$

(7) Find the value of $\sin\left\{\cos\left(\cos^{-1}\frac{3\pi}{4}\right)\right\}$

(8) Find the value of $\cos^{-1}(\cos 13)$

(9) Find $\sin^{-1}(\sin\theta)$, $\cos^{-1}(\cos\theta)$, $\tan^{-1}(\tan\theta)$, $\cot^{-1}(\cot\theta)$ for $\theta \in \left(\frac{5\pi}{2}, 3\pi\right)$

(10) Find the value of $\cos^{-1}(-\cos 4)$

(11) Find the value of $\tan^{-1}\left\{\tan\left(-\frac{7\pi}{8}\right)\right\}$

(12) Find the value of $\tan^{-1}\left\{\cot\left(-\frac{1}{4}\right)\right\}$

(13) Find the value of $\sec\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$

(14) Find the value of $\operatorname{cosec}\left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$

(15) Find the value of $\sin(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

(16) Solve the following equations (i) $5\tan^{-1}x + 3\cot^{-1}x = 2\pi$ (ii) $4\sin^{-1}x = \pi - \cos^{-1}x$

(17) Evaluate $\tan\left(\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}\right)$

(18) Evaluate $\sec\left(\cot^{-1}\frac{16}{63}\right)$

(19) Evaluate $\sin\left\{\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right\}$

(20) Evaluate $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$

(21) Solve $\sin^{-1}(x^2 - 2x + 3) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Answers :

(6) $\frac{\sqrt{3}}{2}$ (7) not defined

(8) $13 - 4\pi$ (9) $3\pi - \theta$, $\theta - 2\pi$, $\theta - 3\pi$, $\theta - 2\pi$

(10) $4 - \pi$ (11) $\frac{\pi}{8}$ (12) $\left(\frac{1}{4} - \frac{\pi}{2}\right)$

(13) $\frac{3}{2}$ (14) $-\sqrt{3}$ (15) $\frac{1}{5}$

(16). (i) $x = 1$ (ii) $x = \frac{1}{2}$

(17) $\frac{4}{5}$ (18) $\frac{65}{16}$ (19) $\frac{2\sqrt{5}}{5}$ (20) $\frac{-7}{17}$

(21) No solution

Property 6 : Identities on addition and subtraction:

(i) $$\sin^{-1}x + \sin^{-1}y \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), x \ge 0, y \ge 0 & \& \ (x^2+y^2) \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), x \ge 0, y \ge 0 & \& \ x^2+y^2 \ge 1 \end{cases}$$

Proof : Let A = $\sin^{-1} x$ and B = $\sin^{-1} y$ where x, y $\in$ [0, 1].

$$\sin(A + B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\Rightarrow \quad \sin^{-1}\sin(A + B) = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$= \begin{cases} A+B & \text{for } 0 \le A+B \le \pi/2 \\ \pi-(A+B) & \text{for } \pi/2 \le A+B \le \pi \end{cases} = \begin{cases} \sin^{-1}x + \sin^{-1}y, & x^2+y^2 \le 1 \\ \pi - (\sin^{-1}x + \sin^{-1}y), & x^2+y^2 \ge 1 \end{cases}$$

(ii) $$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right); \ x, y \in [0, 1]$$

(iii) $$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right); \ x, y \in [0, 1]$$

(iv) $$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & 0 \le x < y \le 1 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & 0 \le y < x \le 1 \end{cases}$$

(v) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \pi/2 & \text{if } x, y > 0 \ \& \ xy = 1 \\ -\pi/2 & \text{if } x, y < 0 \ \& \ xy = 1 \\ \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } x, y \ge 0 \ \& \ xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } x, y \ge 0 \ \& \ xy > 1 \end{cases}$$

(vi) $$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \ x \ge 0, y \ge 0$$

Notes :(i) $x^2 + y^2 \le 1$ & x, y ≥ 0 $\Rightarrow$ $0 \le \sin^{-1}x + \sin^{-1}y \le \dfrac{\pi}{2}$

and $x^2 + y^2 \ge 1$ & x, y ≥ 0 $\Rightarrow$ $\dfrac{\pi}{2} \le \sin^{-1}x + \sin^{-1}y \le \pi$

(ii) xy < 1 and x, y ≥ 0 $\Rightarrow 0 \le \tan^{-1}x + \tan^{-1}y < \dfrac{\pi}{2}$; xy > 1 and x, y $\ge 0 \Rightarrow \dfrac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi$

(iii) For x < 0 or y < 0 these identities can be used with the help of property "– x"
i.e. change x or y to – x or – y which are positive .

Example # 12 : Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\frac{84}{85}$

Solution : $\because \quad \frac{3}{5} > 0,\ \frac{15}{17} > 0$ and $\left(\frac{3}{5}\right)^2 + \left(\frac{15}{17}\right)^2 = \frac{8226}{7225} > 1$

$$\therefore \quad \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{225}{289}} + \frac{15}{17}\sqrt{1-\frac{9}{25}}\right)$$

$$= \pi - \sin^{-1}\left(\frac{3}{5}\cdot\frac{8}{17} + \frac{15}{17}\cdot\frac{4}{5}\right) = \pi - \sin^{-1}\left(\frac{84}{85}\right)$$

Example # 13 : Evaluate $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

Solution : Let $z = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

$\because \quad \sin^{-1}\frac{4}{5} = \frac{\pi}{2} - \cos^{-1}\frac{4}{5}$

$\therefore \quad z = \cos^{-1}\frac{12}{13} + \left(\frac{\pi}{2} - \cos^{-1}\frac{4}{5}\right) - \tan^{-1}\frac{63}{16}.$

$z = \frac{\pi}{2} - \left(\cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13}\right) - \tan^{-1}\frac{63}{16}$ ....(i)

$\because \quad \frac{4}{5} > 0,\ \frac{12}{13} > 0$ and $\frac{4}{5} < \frac{12}{13}$

$$\therefore \quad \cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13} = \cos^{-1}\left[\frac{4}{5}\times\frac{12}{13} + \sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right] = \cos^{-1}\left(\frac{63}{65}\right)$$

$\therefore$ equation (i) can be written as

$z = \frac{\pi}{2} - \cos^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right)$

$z = \sin^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right)$ ....(ii)

$\because \quad \sin^{-1}\left(\frac{63}{65}\right) = \tan^{-1}\left(\frac{63}{16}\right)$

$\therefore$ from equation (ii), we get

$\therefore \quad z = \tan^{-1}\left(\frac{63}{16}\right) - \tan^{-1}\left(\frac{63}{16}\right) \quad \Rightarrow \quad z = 0$

Example # 14 : Evaluate $\tan^{-1} 9 + \tan^{-1}\frac{5}{4}$

Solution : $\because \quad 9 > 0,\ \frac{5}{4} > 0$ and $\left(9\times\frac{5}{4}\right) > 1$

$$\therefore \quad \tan^{-1} 9 + \tan^{-1}\frac{5}{4} = \pi + \tan^{-1}\left(\frac{9+\frac{5}{4}}{1-9.\frac{5}{4}}\right) = \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Example # 15 : Define $y = \cos^{-1}(4x^3 - 3x)$ in terms of $\cos^{-1} x$ and also draw its graph.

Solution : **Part - 1:** Let $y = \cos^{-1}(4x^3 - 3x)$

$\because$ Domain : $[-1, 1]$ and range : $[0, \pi]$

Let $\cos^{-1} x = \theta \Rightarrow \theta \in [0, \pi]$ and $x = \cos\theta$

$\therefore$ $y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$y = \cos^{-1}(\cos 3\theta)$ ....(i)

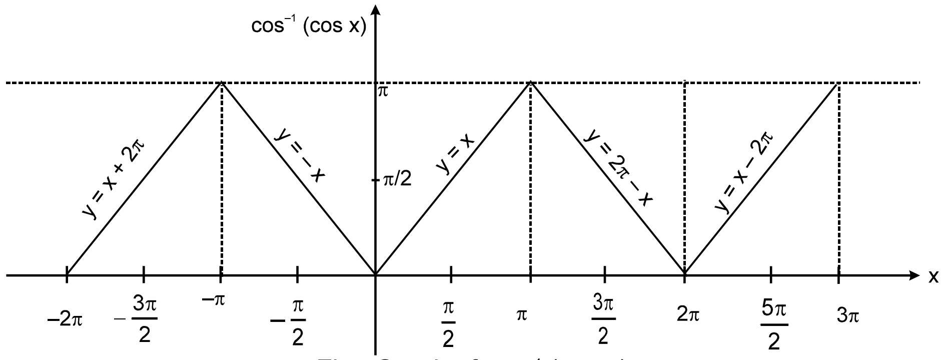


Fig.: Graph of $\cos^{-1}(\cos x)$

$\because$ $\theta \in [0, \pi]$

$\therefore$ $3\theta \in [0, 3\pi]$

$\therefore$ to define $y = \cos^{-1}(\cos 3\theta)$, we consider the graph of $\cos^{-1}(\cos x)$ in the interval $[0, 3\pi]$. Now, from the above graph we can see that

(i) if $0 \le 3\theta \le \pi \Rightarrow \cos^{-1}(\cos 3\theta) = 3\theta$

$\therefore$ from equation (i), we get

$y = 3\theta$ if $0 \le 3\theta \le \pi$

$\Rightarrow y = 3\theta$ if $0 \le \theta \le \frac{\pi}{3}$

$\Rightarrow y = 3\cos^{-1}x$ if $\frac{1}{2} \le x \le 1$

(ii) if $\pi < 3\theta \le 2\pi \Rightarrow \cos^{-1}(\cos 3\theta) = 2\pi - 3\theta$

$\therefore$ from equation (i), we get

$y = 2\pi - 3\theta$ if $\pi < 3\theta \le 2\pi$

$\Rightarrow y = 2\pi - 3\theta$ if $\frac{\pi}{3} < \theta \le \frac{2\pi}{3}$

$y = 2\pi - 3\cos^{-1} x$ if $-\frac{1}{2} \le x < \frac{1}{2}$

(iii) $2\pi < 3\theta \le 3\pi \Rightarrow \cos^{-1}(\cos 3\theta) = -2\pi + 3\theta$

$\therefore$ from equation (i), we get

$\Rightarrow y = -2\pi + 3\theta$ if $2\pi < 3\theta \le 3\pi$

$\Rightarrow y = -2\pi + 3\theta$ if $\frac{2\pi}{3} < \theta \le \pi$

$\Rightarrow y = -2\pi + 3\cos^{-1} x$ if $-1 \le x < -\frac{1}{2}$

$\therefore$ from (i), (ii) & (iii), we get

$$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x & ; \quad \frac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1}x & ; \quad -\frac{1}{2} \le x < \frac{1}{2} \\ -2\pi + 3\cos^{-1}x & ; \quad -1 \le x < -\frac{1}{2} \end{cases}$$

Part-2 : For $y = \cos^{-1}(4x^3 - 3x)$

domain : $[-1, 1]$

range : $[0, \pi]$

(i) if $\frac{1}{2} \le x \le 1$, $y = 3\cos^{-1}x$.

$\Rightarrow \quad \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} = -3(1-x^2)^{-1/2}$ ....(i)

$\Rightarrow \quad \frac{dy}{dx} < 0$ if $x \in \left[\frac{1}{2}, 1\right)$

$\Rightarrow$ decreasing if $x \in \left[\frac{1}{2}, 1\right)$

again if we differentiate equation (i) w.r.t. 'x', we get

$\frac{d^2y}{dx^2} = -\frac{3x}{(1-x^2)^{3/2}}$

$\Rightarrow \quad \frac{d^2y}{dx^2} < 0$ if $x \in \left[\frac{1}{2}, 1\right)$ $\Rightarrow$ concavity downwards if $x \in \left[\frac{1}{2}, 1\right)$

(ii) if $-\frac{1}{2} \le x < \frac{1}{2}$, $y = 2\pi - 3\cos^{-1}x$.

$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$ $\Rightarrow$ $\frac{dy}{dx} > 0$ if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

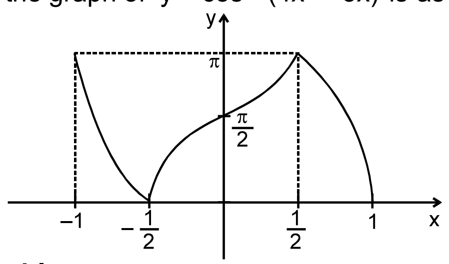
$\Rightarrow$ increasing if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ and $\frac{d^2y}{dx^2} = \frac{3x}{(1-x^2)^{3/2}}$

(a) if $x \in \left[-\frac{1}{2}, 0\right)$ then $\frac{d^2y}{dx^2} < 0$ $\Rightarrow$ concavity downwards if $x \in \left[-\frac{1}{2}, 0\right)$

(b) if $x \in \left(0, \frac{1}{2}\right)$ then $\frac{d^2y}{dx^2} > 0$ $\Rightarrow$ concavity upwards if $x \in \left(0, \frac{1}{2}\right)$

(iii) Similarly if $-1 \le x < -\frac{1}{2}$ then $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$.

$\therefore$ the graph of $y = \cos^{-1}(4x^3 - 3x)$ is as



Self practice problems:

(22) Evaluate $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

(23) If $\tan^{-1}4 + \tan^{-1}5 = \cot^{-1}\lambda$, then find 'λ'

(24) Prove that $2\cos^{-1}\frac{3}{\sqrt{13}} + \cot^{-1}\frac{16}{63} + \frac{1}{2}\cos^{-1}\frac{7}{25} = \pi$

(25) Solve the equation $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

(26) Solve the equation $\sin^{-1}x + \sin^{-1}2x = \frac{2\pi}{3}$

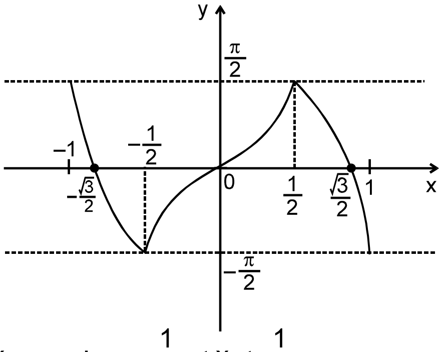
(27) Define $y = \sin^{-1}(3x - 4x^3)$ in terms of $\sin^{-1}x$ and also draw its graph.

(28) Define $y = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$ in terms of $\tan^{-1}x$ and also draw its graph.

Answers. (22) $\frac{\pi}{2}$ (23) $\lambda = -\frac{19}{9}$ (25) $x = \frac{1}{6}$ (26) $x = \frac{1}{2}$

(27) $y = \sin^{-1}(3x - 4x^3) = \begin{cases} 3\sin^{-1}x & ; \ -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & ; \ \frac{1}{2} < x \le 1 \\ -\pi - 3\sin^{-1}x & ; \ -1 \le x < -\frac{1}{2} \end{cases}$

$\therefore$ graph of $y = \sin^{-1}(3x - 4x^3)$



(28) $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} 3\tan^{-1}x & ; \ -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & ; \ -\infty < x < -\frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & ; \ \frac{1}{\sqrt{3}} < x < \infty \end{cases}$

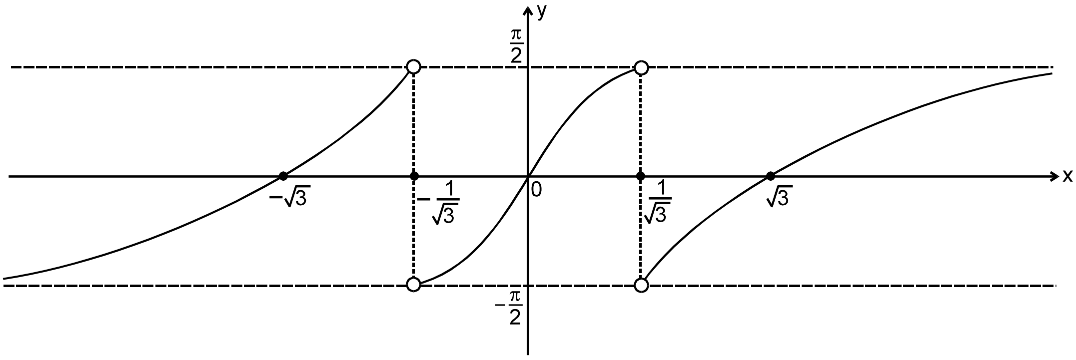
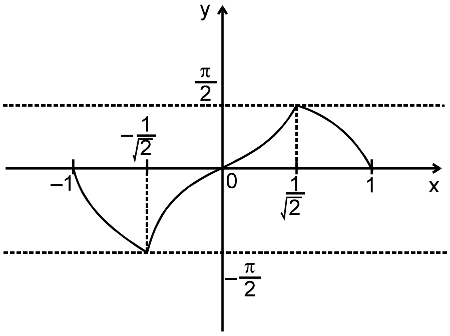


Fig.: Graph of $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Property 7 : Miscellaneous identities

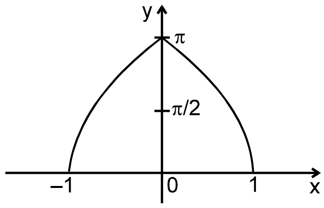
(i) $\sin^{-1}\left(2x\sqrt{1 - x^2}\right) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } \frac{1}{\sqrt{2}} < x \le 1 \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if } -1 \le x < -\frac{1}{\sqrt{2}} \end{cases}$

graph of $y = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$



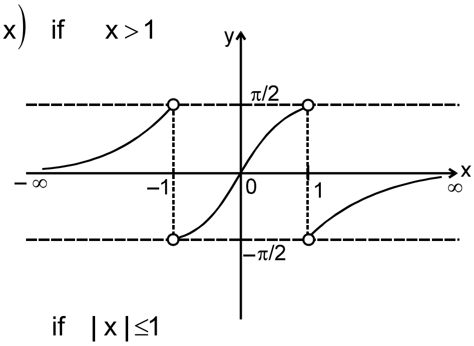
(ii) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$

graph of $y = \cos^{-1}(2x^2 - 1)$



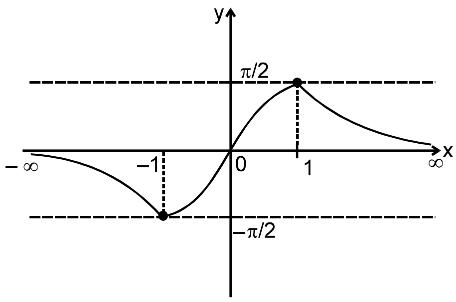
(iii) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{cases}$

graph of $y = \tan^{-1}\dfrac{2x}{1-x^2}$



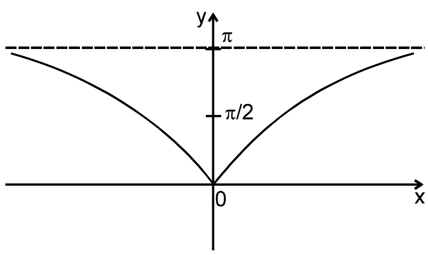
(iv) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & \text{if } x < -1 \end{cases}$

graph of $y = \sin^{-1}\dfrac{2x}{1+x^2}$



(v) $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

graph of $y = \cos^{-1}\dfrac{1-x^2}{1+x^2}$



(vi) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$

(vii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$, then $xy + yz + zx = 1$

(viii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(ix) $\tan^{-1}1 + \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$